Drainage Dynamics of a Conical Frustum Bucket Under Ideal Flow Assumptions

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Abstract

Lately a meeme have been circulating in social media asking the question whether a frustum bucket drains faster or slower if it is turned upside down. There have been long threads of conversation arguing for both alternative.

Here we derive the governing ODE as welll as the implicit solution and the closed expression for the time T it takes to drain a conical frustum bucket draining through a circular outlet under ideal flow conditions.

The analysis is based on Torricelli's law and the geometric properties of a truncated cone. A comparison is made between the emptying times when the bucket is used in its normal orientation and when it is inverted. A closed-form expression for the ratio of emptying times is obtained, showing that it only depends on the top and bottom radii of the frustum.

We also derive the closed form expression for time τ when the rate of change in the water level is at its highest.

Finally we derive the theorethically maximum gain limit possible when inverting a bucket when the ratio of the top radii to the bottom radii approaches infinity.

1 Geometry and variables

We will assume gravity g and use Torricelli's law for the exit velocity of the water at hte outlet. We will treat the outlet as a sharp-edged orifice of area A_o, with an ideal discharge coefficient $C = 1$ as we ignore the flow friction on the inside bucket and assume non-turbulent flow of water out of the bucket.

Let

$$
\begin{cases}
H = \text{bucket height} \\
r_1 = \text{top radii} \\
r_2 = \text{bottom radii} \\
h = \text{water height in bucket} \\
L = \text{water volume} \\
d = \text{outlet diameter}
\end{cases}
$$

The total water volume L in a frustum bucket is given by its geometry, specifically

$$L = \frac{\pi H}{3}\left(r_1^2 + r_1 r_2 + r_2^2\right)$$

equally we can express H in terms of L, r_1, r_2 as

$$H = \frac{3L}{\pi\left(r_1^2 + r_1 r_2 + r_2^2\right)}$$

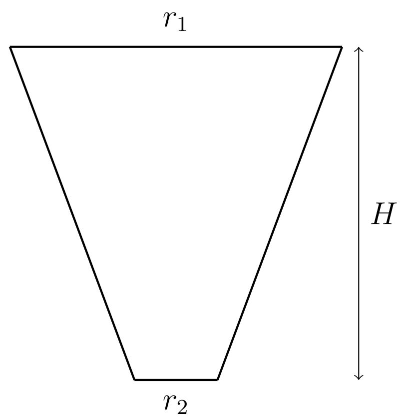

Figure 1: Geometry of the conical frustum bucket.

2 Cross-sectional area of water surface as a function of height

At a height h (from the bottom), the radius of the water surface is obtained by linear interpolation along the frustum.

$$r(h) = r_2 + kh$$

where

$$k = \frac{r_1 - r_2}{H}$$

The cross-sectional area $A(h)$ at height h, is thus

$$A(h) = \pi r(h)^2 = \pi\left(r_2 + kh\right)^2.$$

3 Torricelli's law and ODE for $h(t)$

Assuming ideal flow and negligible velocity of the free surface compared to exit velocity, the speed of water leaving the outlet is

$$v = \sqrt{2gh}$$

Volume flow rate out

$$Q = A_o v = A_o\sqrt{2gh}$$

Conservation of volume

$$\frac{dV}{dt} = -Q$$

But $V(h)$ is the volume of water from bottom up to height h. Its derivative with respect to h is the cross-sectional area:

$$\frac{dV}{dh} = A(h)$$

Chain rule:

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = A(h)\frac{dh}{dt}$$

So

$$A(h)\frac{dh}{dt} = -A_o\sqrt{2gh}$$

and we get

$$\frac{dh}{dt} = -\frac{A_o}{A(h)}\sqrt{2gh} = -\frac{A_o}{\pi\left(r_2 + kh\right)^2}\sqrt{2gh}$$

Hence, we have arrived at the wanted ODE (using $A_0 = \pi d^2/4$)

$$\frac{dh}{dt} = -\frac{d^2\sqrt{2g}}{4}\,\frac{\sqrt{h(t)}}{\left[\,r_2 + k\,h(t)\,\right]^2} \tag{1}$$

or alternatively with $k = (r_1 - r_2)H^{-1}$ expanded

$$\frac{dh}{dt} = -\frac{d^2\sqrt{2g}}{4}\,\frac{\sqrt{h(t)}}{\left[\,r_2 + h(t)\,(r_1 - r_2)H^{-1}\,\right]^2} \tag{2}$$

or rewritten in terms of bucket volume L as

$$\frac{dh}{dt} = -\frac{d^2\sqrt{2g}}{4}\,\frac{\sqrt{h(t)}}{\left[\,r_2 + h(t)\,\pi\,(r_1^3 - r_2^3)\,(3L)^{-1}\,\right]^2} \tag{3}$$

where

$$\begin{cases} h(t) = \text{water height} & H = \text{bucket height} \\ r_1 = \text{top radii} & d = \text{outlet diameter} \\ r_2 = \text{bottom radii} & L = \text{bucket volume} \end{cases}$$

3.1 Implicit solution of the ODE

As it happens this is a non-linear separable ODE so we cannot in general write an explicit solution but we can separate variables and write an implicit solution as follows.

To get a more tractable equation start by introducing

$$\begin{cases} K = -\dfrac{d^2\sqrt{2g}}{4} & \\ a = r_2 & b = \dfrac{\pi(r_1^3 - r_2^3)}{3L} \end{cases}$$

and the ODE becomes

$$\frac{dh}{dt} = -\frac{K\sqrt{h}}{(a+bh)^2}$$

This is clearly separable: the right-hand side depends only on h and we can rewrite the equation as

$$\frac{(a+bh)^2}{\sqrt{h}}dh = -Kdt$$

Sidenote: The separated form

$$\int \frac{A(h)}{\sqrt{h}}dh = -Ct + C_0$$

shows exactly how

- the cross-sectional area $A(h)$
- the Torricelli velocity $\sqrt{h}$
- and the geometry parameters

combine to determine the draining behavior. It allows us to immediately draw the following conclusions

- the draining time is finite, because the integral converges
- the shape of the bucket affects the exponent of h in the integrand
- the draining slows down near the bottom

We expand the square, integrate both sides and get

$$2a^2\sqrt{h} + \frac{4}{3}abh^{3/2} + \frac{2}{5}b^2h^{5/2} = -Kt + C_0$$

If we assume $h(0) = H$ (the height if the full bucket of volume L) we can determine the constant C_0 as

$$C_0 = 2r_2^2\sqrt{H} + \frac{4}{3}r_2bH^{3/2} + \frac{2}{5}b^2H^{5/2}$$

The full implicit solution now becomes

$$2r_2^2\sqrt{h(t)} + \frac{4}{3}r_2bh(t)^{3/2} + \frac{2}{5}b^2h(t)^{5/2} = 2r_2^2\sqrt{H} + \frac{4}{3}r_2bH^{3/2} + \frac{2}{5}b^2H^{5/2} - \frac{d^2\sqrt{2g}}{4}t \quad (4)$$

where

$$b = \frac{\pi(r_1^3 - r_2^3)}{3L}, \qquad H = \frac{3L}{\pi\left(r_1^2 + r_1r_2 + r_2^2\right)}.$$

Getting a implicit form is actually more usefull than it might seem at first glance. We can use this form to

- compute the draining time directly (see section 4.2)
- compute the time to reach any height by solving a single algebraic equation (and not have to solve the ODE numerically each time)
- compare different bucket geometries symbolically

4 Time to empty the bucket

We will determine this in two ways. First from the original ODE and then from the implicit solution we obtained above. They will of course yield the same result but is useful to verify we have not done any arithmetic mistake.

4.1 Using the original ODE to find the emptying time

We want the time T to go from $h(0) = H$ (full) to $h(T) = 0$ (empty). From

$$\frac{dh}{dt} = -\frac{A_o}{\pi\left(r_2 + kh\right)^2}\sqrt{2gh},$$

we write

$$dt = -\frac{\pi\left(r_2 + kh\right)^2}{A_o\sqrt{2g}}\frac{dh}{\sqrt{h}}$$

Integrate

$$T = \int_0^T dt = \int_{h=H}^{h=0} -\frac{\pi\left(r_2 + kh\right)^2}{A_o\sqrt{2g}}\frac{dh}{\sqrt{h}}$$

Flip the limits to get rid of the negative term

$$T = \frac{\pi}{A_o\sqrt{2g}}\int_0^H \frac{\left(r_2 + kh\right)^2}{\sqrt{h}}\,dh$$

Expand the square

$$\left(r_2 + kh\right)^2 = r_2^2 + 2kr_2h + k^2h^2$$

So

$$T = \frac{\pi}{A_o\sqrt{2g}}\int_0^H \left(r_2^2h^{-\frac{1}{2}} + 2kr_2h^{\frac{1}{2}} + k^2h^{\frac{3}{2}}\right)dh.$$

Integrate term by term

$$\int_0^H r_2^2h^{-\frac{1}{2}}\,dh = r_2^2\cdot 2H^{\frac{1}{2}} = 2r_2^2\sqrt{H}$$
$$\int_0^H 2kr_2h^{\frac{1}{2}}\,dh = 2kr_2\cdot\frac{2}{3}H^{\frac{3}{2}} = \frac{4}{3}kr_2H^{\frac{3}{2}}$$
$$\int_0^H k^2h^{\frac{3}{2}}\,dh = k^2\cdot\frac{2}{5}H^{\frac{5}{2}} = \frac{2}{5}k^2H^{\frac{5}{2}}$$

Some simplifictions then gives

$$T = \frac{2\pi\sqrt{H}}{A_o\sqrt{2g}}\left(r_2^2 + \frac{2}{3}kr_2H + \frac{1}{5}k^2H^2\right).$$

Now, recall $k = \dfrac{r_1 - r_2}{H}$, so

$$kH = r_1 - r_2$$
$$k^2H^2 = (r_1 - r_2)^2$$

Thus the bracket simplifies to a function of only r_1, r_2

$$r_2^2 + \frac{2}{3}r_2(r_1 - r_2) + \frac{1}{5}(r_1 - r_2)^2.$$

and the complete expression can be simplified to

$$T = \frac{2\pi\sqrt{H}}{A_o\sqrt{2g}} \cdot \frac{1}{15}\left(8r_2^2 + 4r_1r_2 + 3r_1^2\right).$$

Finally, plug in $A_o = \dfrac{\pi d^2}{4}$ and cancel π

$$T = \frac{2\pi\sqrt{H}}{\left(\frac{\pi d^2}{4}\right)\sqrt{2g}} \cdot \frac{1}{15}\left(8r_2^2 + 4r_1r_2 + 3r_1^2\right) = \frac{8\sqrt{H}}{d^2\sqrt{2g}} \cdot \frac{1}{15}\left(8r_2^2 + 4r_1r_2 + 3r_1^2\right).$$

So the time T in seconds to empty the bucket is

$$T = \frac{8\sqrt{H}}{15\,d^2\sqrt{2g}}\left(8r_2^2 + 4r_1r_2 + 3r_1^2\right)$$

we can then plug-in the expression for H which was

$$H = \frac{3L}{\pi\left(r_1^2 + r_1r_2 + r_2^2\right)}.$$

The time T in seconds it takes to empty the full bucket is then

$$T = \frac{4\sqrt{6}}{15\sqrt{g\,\pi}}\sqrt{L}\left(\frac{3\,r_1^2 + 4\,r_1r_2 + 8\,r_2^2}{d^2\sqrt{r_1^2 + r_1r_2 + r_2^2}}\right) \tag{5}$$

where

$$\begin{cases} r_1 = \text{top radii} \\ r_2 = \text{bottom radii} \\ L = \text{water (volume in } m^3\text{)} \\ d = \text{outlet diameter} \end{cases}$$

4.2 Using the implicit solution to find the emptying time

The implicit solution to our ODE is

$$2a^2\sqrt{h} + \frac{4}{3}abh^{3/2} + \frac{2}{5}b^2h^{5/2} = -Kt + C_0$$

where

$$a = r_2, \quad b = \left(\frac{\pi(r_1^3 - r_2^3)}{3L}\right), \quad H = \frac{3L}{\pi\left(r_1^2 + r_1 r_2 + r_2^2\right)}, \quad K = \frac{d^2\sqrt{2g}}{4}$$

Call the LHS $F(h)$ so that

$$F(h) = 2a^2\sqrt{h} + \frac{4}{3}abh^{3/2} + \frac{2}{5}b^2h^{5/2}$$

Then the implicit solution is

$$F(h(t)) = C_0 - Kt$$

GIven the initial condition $h(0) = H$

$$C_0 = F(H)$$

so the solution is

$$F\left(h(t)\right) = F(H) - Kt$$

Now, the draining time T is the time when the height reaches zero, i.e $h(T) = 0$. From the previous equation we get

$$F\left(h(T)\right) = F(0) = F(H) - KT$$

But $F(0) = 0$ (all terms have a positive power of h), so

$$0 = F(H) - Kt \implies T = \frac{F(H)}{K}$$

Explicitly

$$F(H) = 2a^2\sqrt{H} + \frac{4}{3}abH^{3/2} + \frac{2}{5}b^2H^{5/2}$$

so

$$\begin{aligned} T &= \frac{2a^2\sqrt{H} + \frac{4}{3}abH^{3/2} + \frac{2}{5}b^2H^{5/2}}{K} \\ &= \frac{4\left(2r_2^2\sqrt{H} + \frac{4}{3}r_2bH^{3/2} + \frac{2}{5}b^2H^{5/2}\right)}{d^2\sqrt{2g}} \end{aligned}$$

So in summary, the time T in seconds it takes to empty the full bucket is then

$$T = \frac{4\sqrt{H}}{d^2\sqrt{2g}}\left(2r_2^2 + \frac{4}{3}r_2 bH + \frac{2}{5}(bH)^2\right) \tag{6}$$

where

$$\begin{cases} r_1 = \text{top radii} & r_2 = \text{bottom radii} \\ H = \dfrac{3L}{\pi\left(r_1^2 + r_1 r_2 + r_2^2\right)} & b = \dfrac{\pi(r_1^3 - r_2^3)}{3L} \end{cases}$$

4.3 Comparing the two expressions for T

We determined T from the ODE as

$$T_{\text{ODE}} = \frac{4\sqrt{6L}}{15\,d^2\,\sqrt{g\,\pi}}\left(\frac{3\,r_1^2 + 4\,r_1 r_2 + 8\,r_2^2}{\sqrt{r_1^2 + r_1 r_2 + r_2^2}}\right) \tag{7}$$

When we started from the implicit solution to the ODE we found

$$T_{\text{IMP}} = \frac{4\sqrt{H}}{d^2\sqrt{2g}}\left(2r_2^2 + \frac{4}{3}r_2 bH + \frac{2}{5}(bH)^2\right) \tag{8}$$

It is not be obvious that they are the same expression and it indeed takes some tedious manipulation (which we will not spend an extra page here on deriving). This is instead left as an exercise to the reader.

Exercise 1:

Show equivalence between T_{IMP} and T_{ODE}.

Hint: To show that the expressions are equal start by introducing S

$$S = r_1^2 + r_1 r_2 + r_2^2, \;\; r_1^3 - r_2^3 = (r_1 - r_2)S \quad \Longrightarrow \quad bH = r_1 - r_2$$

We can then show that

$$2r_2^2 + \frac{4}{3}r_2 bH + \frac{2}{5}(bH)^2$$

is just a quadratic form in (r_1, r_2) which simplifies to

$$\frac{2}{15}(3r_1^2 + 4r_1 r_2 + 8r_2^2)$$

Finally we can match the prefactors by rewriting all square root in the same structure. The rest is just careful bookkeeping.
For someone truly stuck and in need of help, drop me a note and a proof of donation to a recognized charity working to help children and I will send my proof version.

5 Time for highest rate of change $|dh/dt|$

This is a two-step process that first finds the h where the maxima occurs and as he second step find at what time t the height is h.

Since we can ignore constants (they do not impact the derivate) we are looking for the h that maximizes

$$f(h) = \frac{\sqrt{h}}{(r_2 + Mh)^2}$$

Since it is easier to work with logarithms we write

$$\ln f(h) = \frac{1}{2} \ln h - 2 \ln(r_2 + Mh)$$

Differentiate and set to zero

$$\frac{d}{dh} \ln f(h) = \frac{1}{2h} - \frac{2M}{r_2 + Mh} = 0$$

gives

$$h_{\mathrm{m}} = \frac{r_2}{3M} = \frac{r_2\, L}{(r_1^3 - r_2^3)\, \pi}$$

Now recall that

$$T = \frac{\pi}{A_o \sqrt{2g}} \int_0^H \frac{(r_2 + kh)^2}{\sqrt{h}}\, dh$$

so the time τ to go from full bucket to height h_{m} can be written as

$$\tau = \frac{\pi}{A_o \sqrt{2g}} \int_{h_{\mathrm{m}}}^H (r_2 + ku)^2\, u^{-\frac{1}{2}}\, du$$

An antiderivate is

$$F(u) = \frac{2u^{\frac{5}{2}} k^2}{5} + \frac{4u^{\frac{3}{2}} k\, r_2}{3} + 2\sqrt{\mathrm{u}}\, r_2^2$$

which gives

$$\tau = \frac{\pi}{A_0 \sqrt{2g}} \left[F(H) - F(h)\right] = \frac{4}{d^2 \sqrt{2g}} \left[F(H) - F(h)\right]$$

This allows a closed expression for τ

$$
\begin{aligned}
\tau &= \frac{4}{d^2\sqrt{2g}}\left[F(H) - F(h_\mathrm{m})\right] \\
&= \frac{4}{d^2\sqrt{2g}}\left[\frac{4(r_1-r_2)r_2}{3H}\left(H^{\frac{3}{2}} - h_\mathrm{m}^{\frac{3}{2}}\right) + \frac{2(r_1-r_2)^2}{5H^2}\left(H^{\frac{5}{2}} - h_\mathrm{m}^{\frac{5}{2}}\right) + 2r_2^2\left(H^{\frac{1}{2}} - \mathrm{h}_\mathrm{m}^{\frac{1}{2}}\right)\right]
\end{aligned}
$$

We can now plugin the previous established values

$$
\begin{cases}
h_\mathrm{m} = \dfrac{r_2\,L}{(r_1^3 - r_2^3)\,\pi} \\[2ex]
H = \dfrac{3L}{\pi\,(r_1^2 + r_1 r_2 + r_2^2)}
\end{cases}
$$

and after some tedious but trivial algebraic manipulations we arrive at a closed expression.

The time τ for highest rate of change of the waterlevel is given by

$$
\tau = \frac{4\sqrt{2L}}{45\sqrt{g\pi}} \cdot \frac{3\sqrt{3}\,\sqrt{r_1 - r_2}\,\left(3\,r_1^2 + 4\,r_1 r_2 + 8\,r_2^2\right) - 56\,r_2^{5/2}}{d^2\,\sqrt{r_1^3 - r_2^3}} \tag{9}
$$

6 Inverted Bucket

If the bucket is inverted (flipped on its head), the emptying time instead become becomes

$$T^{'} = K\sqrt{L}\,\left(\frac{3\,r_2^2 + 4\,r_1 r_2 + 8\,r_1^2}{d^2\,\sqrt{r_2^2 + r_1 r_2 + r_1^2}}\right)$$

and the ratio of the emptying time between the non-inverted and inverted bucket is given as

$$R = \frac{T}{T'} = \frac{8r_2^2 + 4r_1 r_2 + 3r_1^2}{8r_1^2 + 4r_1 r_2 + 3r_2^2}.$$

7 Plots

7.1 Ratio of Emptying Times

Let $\alpha = r_1/r_2$ (and hence $r_1 = \alpha\, r_2$) we can write the ratio as

$$R(\alpha) = \frac{3\alpha^2 + 4\alpha + 8}{8\alpha^2 + 4\alpha + 3}.$$

Figure 2. shows the plot of $R(\alpha)$ vs. α.

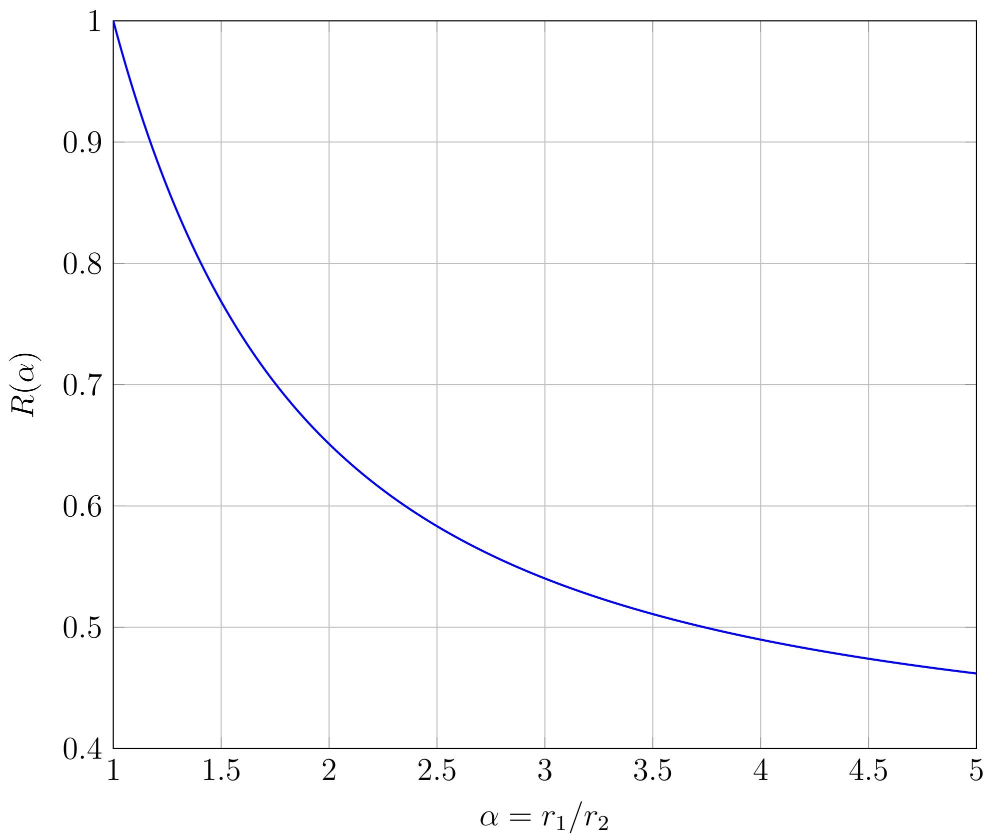

Figure 2: Ratio of emptying times $R(\alpha)$ for $\alpha = r_1/r_2$.

We also make the observation that

$$\lim_{\alpha \to \infty} R(\alpha) = \frac{3}{8} \approx 0.375$$

which tells us that regardless of the size difference between the top and bottom radii the flipped bucket can at most gain $\approx 62\%$ speed compared with the original orientation.

7.2 Water Height vs. Time (Drainage speed)

Choose some numeric radii and outlet diameter d

$$r_1 = 0.7, \quad r_2 = 0.3, \quad d = 0.02, \quad L = 1, \quad g = 9.81.$$

and we can solve the ODE numerically (this time with a Python program, see `https://github.com/johan162/frustum`) to show how the water level changes with time. See Figure 3

We can also calculate the theoretical τ for the time of the highest rate of change in the water level and see how good our simulation is. Referring to Eq. 5 we get

$$\tau = \frac{14600\sqrt{474}}{79\sqrt{\mathrm{g}\,\pi}} \approx 724.79$$

from the simulation plot (Figure 3.) we see that it gives $724.8s$ so it looks like our simulation is adequate.

The emptying time of a conical frustum bucket depends strongly on its orientation. If $r_1 > r_2$, then

$$R(\alpha) < 1,$$

meaning the bucket empties faster when the smaller radius is at the bottom. The ratio depends only on the radius ratio $\alpha = r_1/r_2$, not on the total volume, outlet size, or gravity.

Special Case: $r_1 = 2r_2$, Substituting $r_1 = 2r_2$ gives

$$R = \frac{28}{43} \approx 0.651.$$

Thus the bucket with twice as large top radii as bottom radii empties approximately 35% faster in the orientation where the smaller radius is at the bottom.

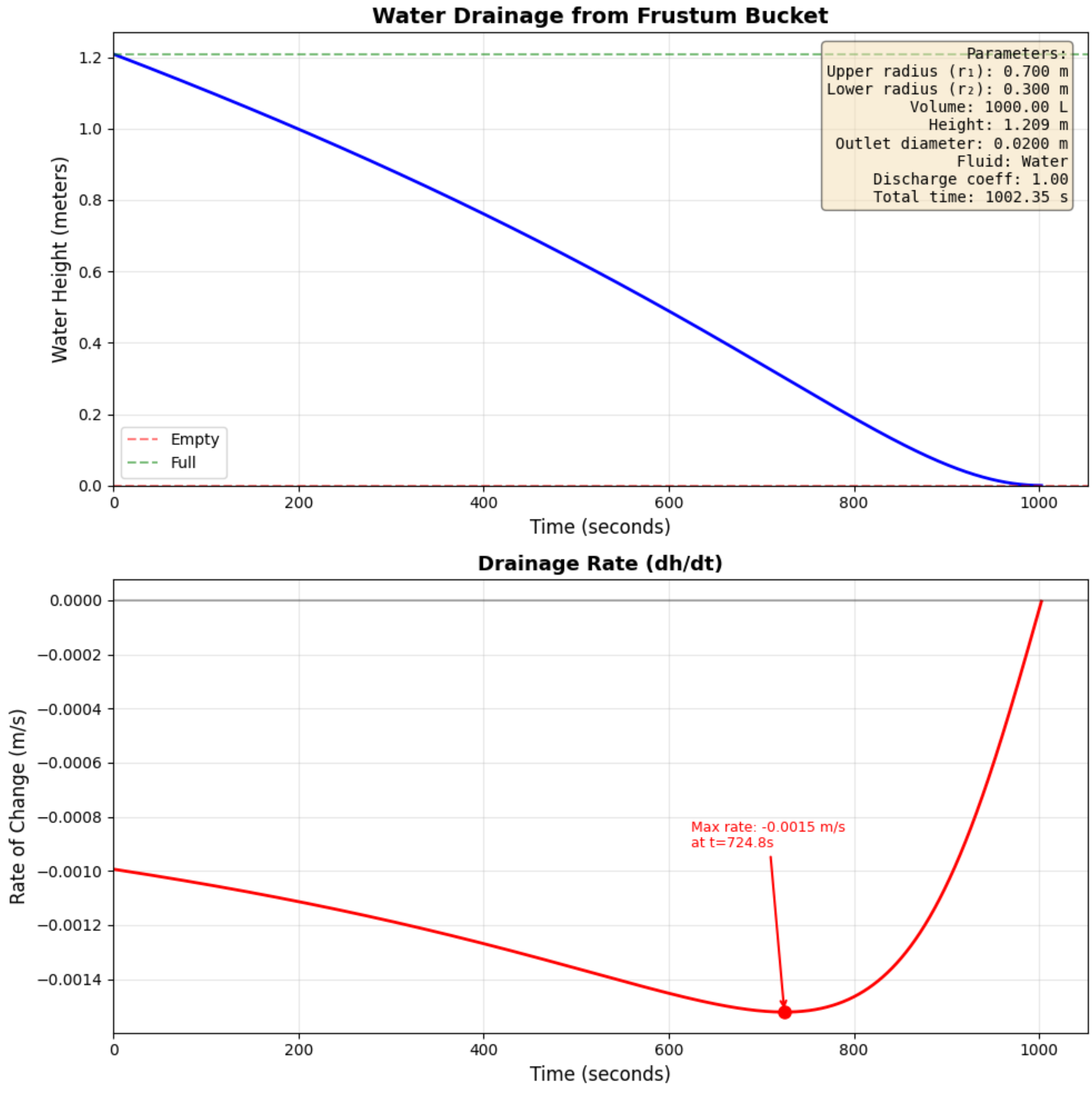

Figure 3: Emptying time and rate of change for a specific bucket

8 Summary

The governing ODE for the change in the water level $h(t)$ written in terms of the bucket height H is

$$\frac{dh}{dt} = -\frac{d^2\sqrt{2g}}{4}\,\frac{\sqrt{h(t)}}{\left[\,r_2 + h(t)\,(r_1 - r_2)H^{-1}\,\right]^2}$$

or in terms of bucket volume L as

$$\frac{dh}{dt} = -\frac{d^2\sqrt{2g}}{4}\,\frac{\sqrt{h(t)}}{\left[\,r_2 + h(t)\,\pi\,(r_1^3 - r_2^3)\,(3L)^{-1}\,\right]^2}$$

The general implicit solution (using the bucket height H version)can be written as

$$2r_2^2\sqrt{h(t)} + \frac{4}{3}r_2 b h(t)^{3/2} + \frac{2}{5}b^2 h(t)^{5/2} = 2r_2^2\sqrt{H} + \frac{4}{3}r_2 b H^{3/2} + \frac{2}{5}b^2 H^{5/2} - \frac{d^2\sqrt{2g}}{4}\,t$$

where

$$b = \frac{\pi(r_1^3 - r_2^3)}{3L}$$

The time T to empty a full bucket can be written

$$T = \frac{4\sqrt{6L}}{15\sqrt{g\,\pi}}\left(\frac{3\,r_1^2 + 4\,r_1 r_2 + 8\,r_2^2}{d^2\sqrt{r_1^2 + r_1 r_2 + r_2^2}}\right)$$

The maximum rate of change is at time τ

$$\tau = \frac{4\sqrt{2L}}{45\sqrt{g\,\pi}}\,\frac{3\sqrt{3}\,\sqrt{r_1 - r_2}\,(3\,r_1^2 + 4\,r_1 r_2 + 8\,r_2^2) - 56\,r_2^{5/2}}{d^2\sqrt{r_1^3 - r_2^3}}$$

where

$$\begin{cases} h(t) = \text{water height} & H = \text{bucket height} = \dfrac{3L}{\pi\,(r_1^2 + r_1 r_2 + r_2^2)} \\ r_1 = \text{top radii} & d = \text{outlet diameter} \\ r_2 = \text{bottom radii} & L = \text{bucket volume} \end{cases}$$

9 Conclusion

The drainage time of a conical frustum bucket under ideal flow conditions depends mainly on the orientation of the bucket. The derived expressions show that placing the smaller radius at the bottom always results in a shorter emptying time. The ratio of emptying times depends only on the radii r_1 and r_2, and not on the total volume, outlet size, or gravitational acceleration.