Drainage Dynamics of a Conical Frustum Bucket Under Ideal Flow Assumptions

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Abstract

Lately a meeme have been circulating in social media asking the question whether a frustum bucket drains faster or slower if it is turned upside down. There have been long threads of conversation arguing for both alternative.

Here we derive the governing ODE and the closed expression for the time T it takes to drain a conical frustum bucket draining through a circular outlet under ideal flow conditions.

The analysis is based on Torricelli's law and the geometric properties of a truncated cone. A comparison is made between the emptying times when the bucket is used in its normal orientation and when it is inverted. A closed-form expression for the ratio of emptying times is obtained, showing that it only depends on the top and bottom radii of the frustum.

Finally we derive the theorethically maximum gain limit possible when inverting a bucket when the ratio of the top radii to the bottom radii approaches infinity.

1 Geometry and variables

We will assume gravity g and use Torricelli's law for the exit velocity of the water at hte outlet. We will treat the outlet as a sharp-edged orifice of area A_o, with an ideal discharge coefficient $C = 1$ as we ignore the flow friction on the inside bucket and assume non-turbulent flow of water out of the bucket.

Let

$$\begin{cases} H = \text{bucket height} \\ r_1 = \text{top radii} \\ r_2 = \text{bottom radii} \\ h = \text{water height in bucket} \\ L = \text{water volume} \\ d = \text{outlet diameter} \end{cases}$$

The total water volume L in a frustum bucket is given by its geometry, specifically

$$L = \frac{\pi H}{3}\left(r_1^2 + r_1 r_2 + r_2^2\right)$$

equally we can express H in terms of L, r_1, r_2 as

$$H = \frac{3L}{\pi\left(r_1^2 + r_1 r_2 + r_2^2\right)}.$$

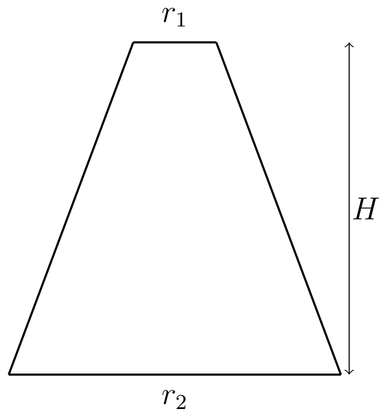

Figure 1: Geometry of the conical frustum bucket.

2 Cross-sectional area of water surface as a function of height

At a height h (from the bottom), the radius of the water surface is obtained by linear interpolation along the frustum.

$$r(h) = r_2 + kh$$

where

$$k = \frac{r_1 - r_2}{H}$$

The cross-sectional area $A(h)$ at height h, is thus

$$A(h) = \pi r(h)^2 = \pi\left(r_2 + kh\right)^2.$$

3 Torricelli's law and ODE for $h(t)$

Assuming ideal flow and negligible velocity of the free surface compared to exit velocity, the speed of water leaving the outlet is

$$v = \sqrt{2gh}$$

Volume flow rate out

$$Q = A_o v = A_o\sqrt{2gh}$$

Conservation of volume

$$\frac{dV}{dt} = -Q$$

But $V(h)$ is the volume of water from bottom up to height h. Its derivative with respect to h is the cross-sectional area:

$$\frac{dV}{dh} = A(h)$$

Chain rule:

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = A(h)\frac{dh}{dt}$$

So

$$A(h)\frac{dh}{dt} = -A_o\sqrt{2gh}$$

and we get

$$\frac{dh}{dt} = -\frac{A_o}{A(h)}\sqrt{2gh} = -\frac{A_o}{\pi\left(r_2 + kh\right)^2}\sqrt{2gh}$$

Hence, we have arrived at the wanted ODE

$$\boxed{\frac{dh}{dt} = -\sqrt{2gh}\,\frac{A_o}{\pi\left(r_2 + kh\right)^2}} \tag{1}$$

where

$$\begin{cases} h = \text{water height} & A_o = \dfrac{\pi d^2}{4} \\ r_1 = \text{top radii} & d = \text{outlet diameter} \\ r_2 = \text{bottom radii} & k = \dfrac{r_1 - r_2}{H} \\ H = \text{bucket height} & \end{cases}$$

which could also be rewritten as

$$\frac{dh}{dt} = -\left(dH\right)^2\,\frac{\sqrt{2gh}}{4\left(\left(r_1 - r_2\right)h + r_2 H\right)^2} \tag{2}$$

4 Time to empty the bucket

We want the time T to go from $h(0) = H$ (full) to $h(T) = 0$ (empty). From

$$\frac{dh}{dt} = -\frac{A_o}{\pi\left(r_2 + kh\right)^2}\sqrt{2gh},$$

we write

$$dt = -\frac{\pi\left(r_2 + kh\right)^2}{A_o\sqrt{2g}}\frac{dh}{\sqrt{h}}$$

Integrate

$$T = \int_0^T dt = \int_{h=H}^{h=0} -\frac{\pi\left(r_2 + kh\right)^2}{A_o\sqrt{2g}}\frac{dh}{\sqrt{h}}$$

Flip the limits to get rid of the negative term

$$T = \frac{\pi}{A_o\sqrt{2g}}\int_0^H \frac{\left(r_2 + kh\right)^2}{\sqrt{h}}\,dh$$

Expand the square

$$\left(r_2 + kh\right)^2 = r_2^2 + 2kr_2h + k^2h^2$$

So

$$T = \frac{\pi}{A_o\sqrt{2g}}\int_0^H \left(r_2^2h^{-\frac{1}{2}} + 2kr_2h^{\frac{1}{2}} + k^2h^{\frac{3}{2}}\right)dh.$$

Integrate term by term

$$\int_0^H r_2^2h^{-\frac{1}{2}}\,dh = r_2^2 \cdot 2H^{\frac{1}{2}} = 2r_2^2\sqrt{H}$$
$$\int_0^H 2kr_2h^{\frac{1}{2}}\,dh = 2kr_2 \cdot \frac{2}{3}H^{\frac{3}{2}} = \frac{4}{3}kr_2H^{\frac{3}{2}}$$
$$\int_0^H k^2h^{\frac{3}{2}}\,dh = k^2 \cdot \frac{2}{5}H^{\frac{5}{2}} = \frac{2}{5}k^2H^{\frac{5}{2}}$$

Some simplifictions then gives

$$T = \frac{2\pi\sqrt{H}}{A_o\sqrt{2g}}\left(r_2^2 + \frac{2}{3}kr_2H + \frac{1}{5}k^2H^2\right).$$

Now, recall $k = \dfrac{r_1 - r_2}{H}$, so

$$kH = r_1 - r_2$$
$$k^2H^2 = (r_1 - r_2)^2$$

Thus the bracket simplifies to a function of only r_1, r_2

$$r_2^2 + \frac{2}{3} r_2 (r_1 - r_2) + \frac{1}{5}(r_1 - r_2)^2.$$

and the complete expression can be simplified to

$$T = \frac{2\pi\sqrt{H}}{A_o\sqrt{2g}} \cdot \frac{1}{15}\left(8r_2^2 + 4r_1 r_2 + 3r_1^2\right).$$

Finally, plug in $A_o = \dfrac{\pi d^2}{4}$ and cancel π

$$T = \frac{2\pi\sqrt{H}}{\left(\frac{\pi d^2}{4}\right)\sqrt{2g}} \cdot \frac{1}{15}\left(8r_2^2 + 4r_1 r_2 + 3r_1^2\right) = \frac{8\sqrt{H}}{d^2\sqrt{2g}} \cdot \frac{1}{15}\left(8r_2^2 + 4r_1 r_2 + 3r_1^2\right).$$

So the time T in seconds to empty the bucket is

$$T = \frac{8\sqrt{H}}{15\, d^2\sqrt{2g}}\left(8r_2^2 + 4r_1 r_2 + 3r_1^2\right)$$

we can then plug-in the expression for H which was

$$H = \frac{3L}{\pi\left(r_1^2 + r_1 r_2 + r_2^2\right)}.$$

and we get the final form for the time it takes to empty the bucket T in seconds as

$$\boxed{T = K\sqrt{L}\,\left(\frac{3\,r_1^2 + 4\,r_1 r_2 + 8\,r_2^2}{d^2\,\sqrt{r_1^2 + r_1 r_2 + r_2^2}}\right)} \tag{3}$$

where

$$\begin{cases} K = \dfrac{4\sqrt{6}}{15\sqrt{g\,\pi}} \approx 0.1177 \\ r_1 = \text{top radii} \\ r_2 = \text{bottom radii} \\ L = \text{water (volume in } m^3) \\ d = \text{outlet diameter} \end{cases}$$

5 Inverted Bucket

If the bucket is inverted (flipped on its head), the emptying time instead become becomes

$$T' = \frac{8\sqrt{H}}{15\,d^2\sqrt{2g}}\left(8r_1^2 + 4r_2r_1 + 3r_2^2\right)$$

and the ratio of the emptying time between the non-inverted and inverted bucket is given as

$$R = \frac{T}{T'} = \frac{8r_2^2 + 4r_1r_2 + 3r_1^2}{8r_1^2 + 4r_1r_2 + 3r_2^2}.$$

6 Plots

6.1 Ratio of Emptying Times

Let $\alpha = r_1/r_2$ (and hence $r_1 = \alpha\, r_2$) we can write the ratio as

$$R(\alpha) = \frac{3\alpha^2 + 4\alpha + 8}{8\alpha^2 + 4\alpha + 3}.$$

Figure 2. shows the plot of $R(\alpha)$ vs. α.

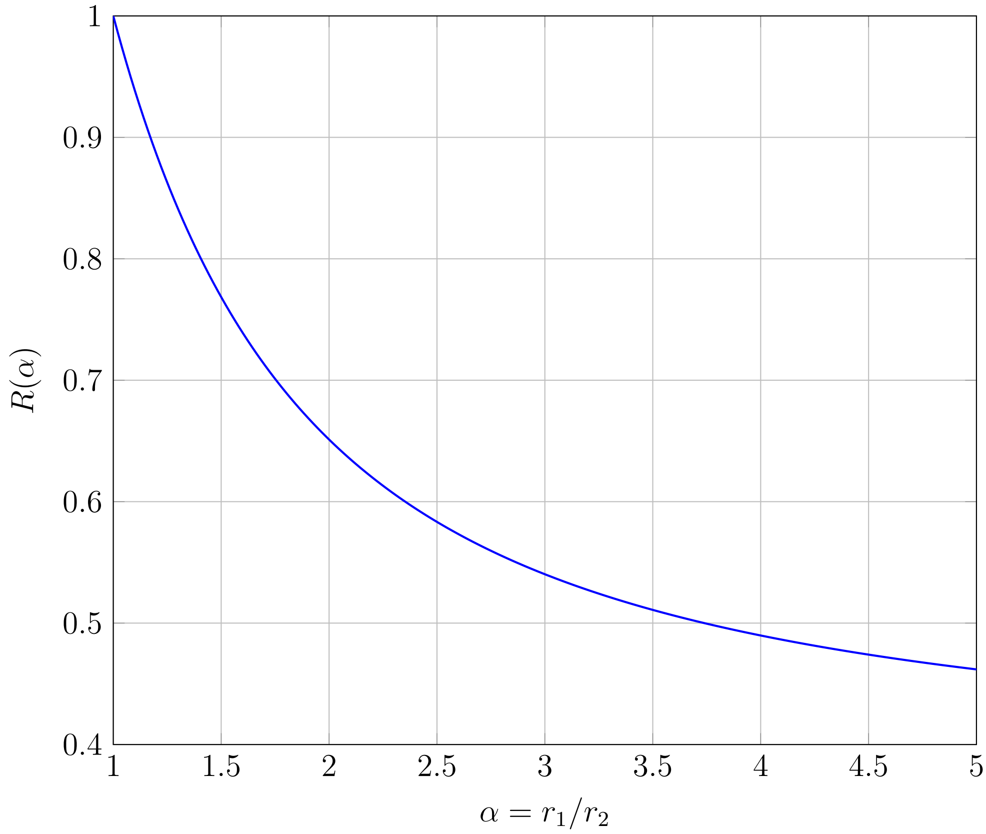

Figure 2: Ratio of emptying times $R(\alpha)$ for $\alpha = r_1/r_2$.

We also make the observation that

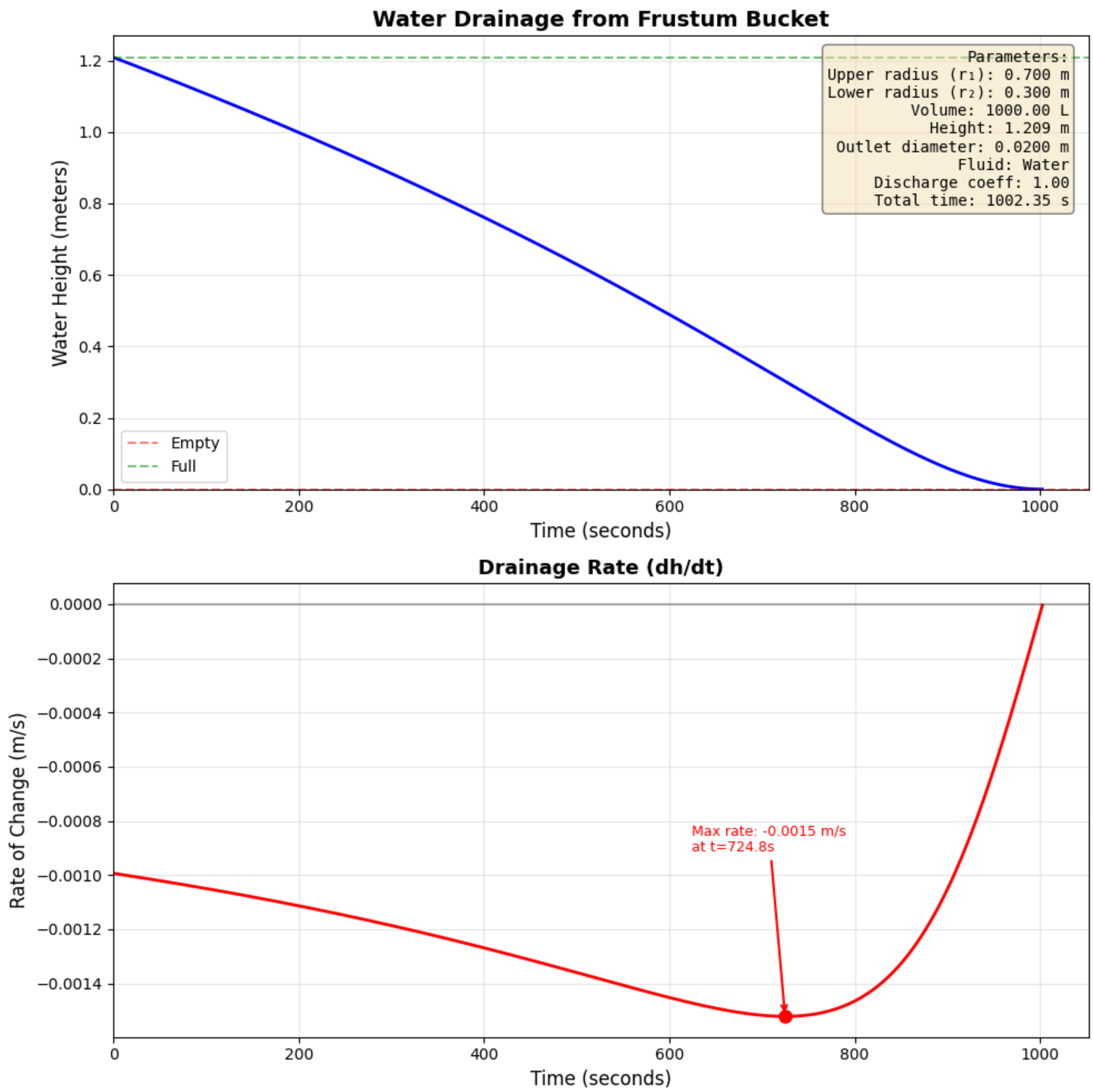

Figure 3: Emptying time and rate of change for a especific bucket

$$\lim_{\alpha \to \infty} R(\alpha) = \frac{3}{8} \approx 0.375$$

which tells us that regardless of the size difference between the top and bottom radii the flipped bucket can at most gain $\approx 62\%$ speed compared with the original orientation.

6.2 Water Height vs. Time (Drainage speed)

Choose some numeric radii and outlet diameter d

$$r_1 = 0.7, \quad r_2 = 0.3, \quad d = 0.02, \quad L = 1, \quad g = 9.81.$$

and we can plot the ODE numerically to show how the water level changes with time. See Figure 3

The emptying time of a conical frustum bucket depends strongly on its orientation. If $r_1 > r_2$, then

$$R < 1,$$

meaning the bucket empties faster when the smaller radius is at the bottom. The ratio depends only on the radius ratio $\alpha = r_1/r_2$, not on the total volume, outlet size, or gravity.

Special Case: $r_1 = 2r_2$, Substituting $r_1 = 2r_2$ gives

$$R = \frac{28}{43} \approx 0.651.$$

Thus the bucket with twice as large top radii as bottom radii empties approximately 35% faster in the orientation where the smaller radius is at the bottom.

7 Conclusion

The drainage time of a conical frustum bucket under ideal flow conditions depends mainly on the orientation of the bucket. The derived expressions show that placing the smaller radius at the bottom always results in a shorter emptying time. The ratio of emptying times depends only on the radii r_1 and r_2, and not on the total volume, outlet size, or gravitational acceleration.