# Método de Discos o Arandelas

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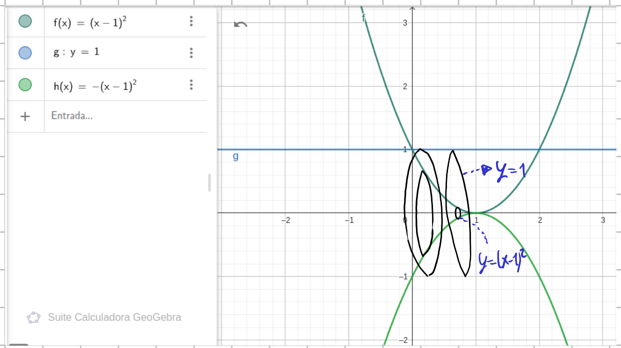
Respecto a $x$      Respecto a $y$

$V=\pi\int f(x)^2-g(x)^2\,dx \qquad V=\pi\int f(y)^2-g(y)^2\,dy$

Ejm: Calcular volumen del sólido al girar la región limitada por: $y=(x-1)^2$; $y=1$; $y=0$ alrededor de $x$

Por Arandelas:




$$V=\pi\int_{x_1}^{x_2}\underbrace{f(x)^2}_{R}-\underbrace{g(x)^2}_{r}\,dx$$

Los límites de integración son los pts. de intersección de las curvas

$y_1=(x-1)^2 \quad y_2=1$

$y_1=y_2;\quad (x-1)^2=1 \quad x=\pm\sqrt{1}+1$

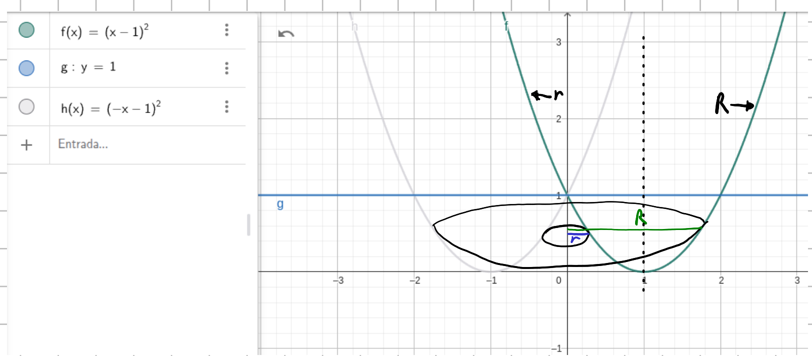
$\underline{x_1=2} \quad \underline{x_2=0}$

$$V=\pi\int_0^2 1^2-(x-1)^4\,dx \qquad \begin{aligned}u&=x-1\\ du&=dx\end{aligned}$$

$$V=\pi x\Big|_0^2-\pi\int_0^2 u^4\,du$$

$$V=2\pi-\frac{\pi}{5}(x-1)^5\Big|_0^2 \quad V=\pi\left(x-\frac{1}{5}(x-1)^5\right)\Big|_0^2 \quad V=\pi\left(2-0-\frac{1}{5}\left((2-1)^5-(0-1)^5\right)\right) \quad V=\left(2-\frac{2}{5}\right)\pi \quad \boxed{V=\frac{8}{5}\pi}$$

Ejm: 〃〃 Respecto a $y$




$$V=\pi\int_{y_1}^{y_2}R^2-r^2\,dy \qquad \text{Región: } 0\le y\le 1$$

$R=f(y) \quad r=g(y)$

$y=(x-1)^2$

$$x=\pm\sqrt{y}+1 \qquad \begin{aligned}x_1&=1+\sqrt{y}=f(y)\\ x_2&=1-\sqrt{y}=g(y)\end{aligned} \qquad \boxed{a^2-b^2=(a-b)(a+b)}$$

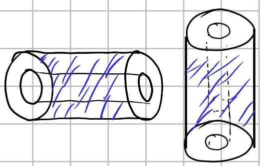
$$V=\pi\int_0^1 x_1^2-x_2^2\,dy \quad V=\pi\int_0^1\underbrace{(1+\sqrt{y})^2}_{a}-\underbrace{(1-\sqrt{y})^2}_{b}\,dy \quad V=\pi\int_0^1\overbrace{(\cancel{1}+\sqrt{y}-\cancel{1}+\sqrt{y})}^{2\sqrt{y}}\overbrace{(1+\cancel{\sqrt{y}}+1-\cancel{\sqrt{y}})}^{2}\,dy$$

$$V=\pi\int_0^1 4\sqrt{y}\,dy \quad V=4\pi\cdot\frac{2}{3}y^{3/2}\Big|_0^1 \quad V=\frac{8}{3}\pi$$

# Método de Cilindros

- Se integra respecto a la variable contraria a la del eje de giro

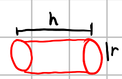
Cilindro SIN TAPAS



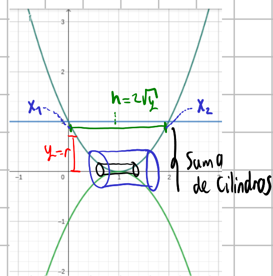
Si se hace ⊙
lo suficientemente pequeño tal que: ⊙ ≈ ◯ → $dx :=$ pequeño, infinitesimal

$V = 2\pi r h$

Ejem: '''' Eje x




$$V = 2\pi\int_a^b y\,(f(y) - g(y))\,dy \qquad 0 \le y \le 1$$




$x_2 = 1+\sqrt{y}$
$x_1 = 1-\sqrt{y}$
$h = x_2 - x_1$

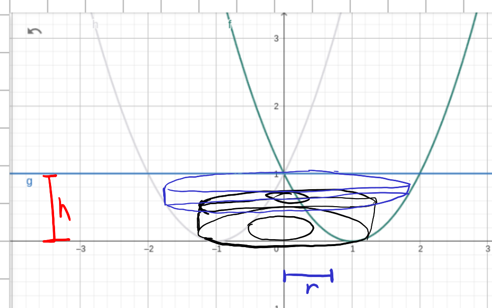
$\boxed{h = 2\sqrt{y}}$

$$V = 2\pi\int_0^1 y\,(x_2 - x_1)\,dy \qquad V = 2\pi\int_0^1 y\left[(1+\sqrt{y}) - (1-\sqrt{y})\right]dy$$

$$V = 2\pi\int_0^1 2y\sqrt{y} = 4\pi\int_0^1 \sqrt{y^3}\,dy = 4\pi\cdot\frac{2}{5}y^{5/2}\Big|_0^1 = \frac{8}{5}\pi$$

→ Ejem eje y '''

$$V = 2\pi\int_a^b x\left[f(x) - g(x)\right]dx$$


$h = f(x) - g(x)$

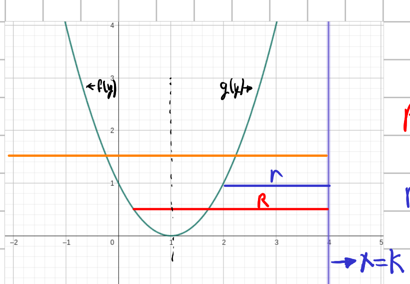
$$V = 2\pi\int_0^2 x\left[1-(x-1)^2\right]dx$$

$$V = 2\pi\int_0^2 x\left[\cancel{1} - x^2 + 2x - \cancel{1}\right]dx \qquad V = 2\pi\int_0^2 2x^2 - x^3\,dx$$

$$V = \frac{4}{3}\pi x^3\Big|_0^2 - \frac{1}{2}\pi x^4\Big|_0^2 \qquad V = \frac{32}{3}\pi - 8\pi = \boxed{\frac{8}{3}\pi}$$

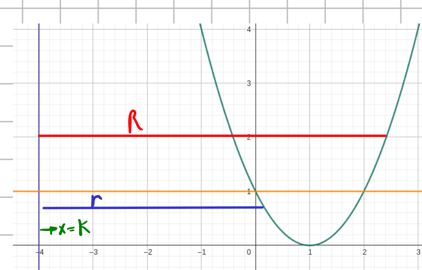
# Rotación Respecto a una Recta:

## Paralela a y:




$R = K - \underbrace{f(y)}$ Radio exterior

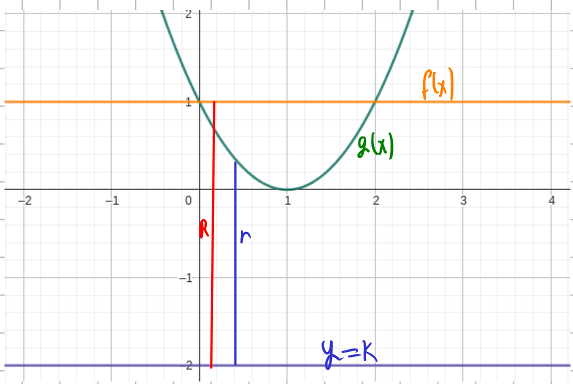
$r = K - \underbrace{g(y)}$ radio interior




$R = K + f(y)$

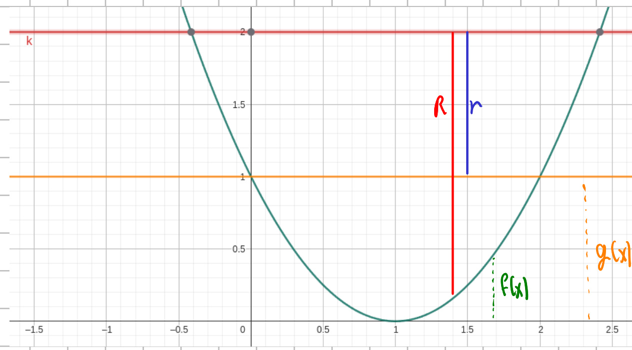
$r = K + g(y)$

# Paralela a X:




$R = K + f(x)$

$r = K + g(x)$




$R = K - f(x)$

$r = K - g(x)$

# Ejemplos:

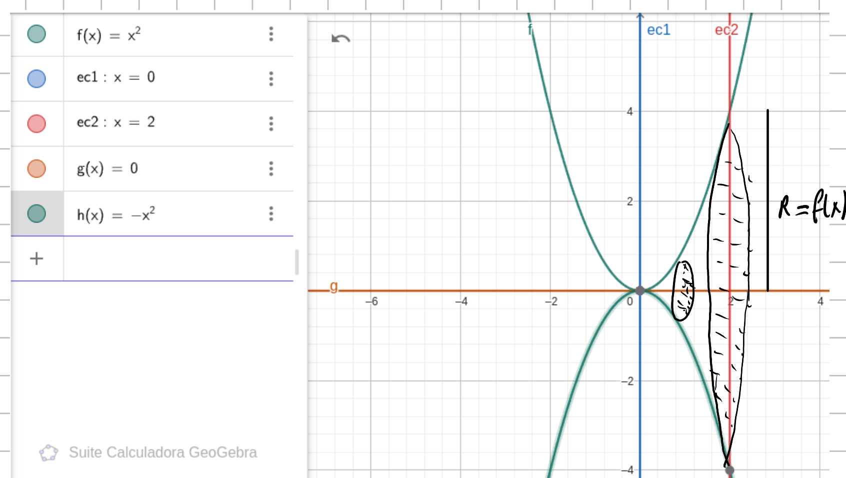
2. Encuentre el volumen del sólido generado por la rotación de la región delimitada por $y = x^2$ y $x = 2$ alrededor del eje $x$.

Región: $\{(x,y): 0 \leq x \leq 2,\ 0 \leq y \leq x^2\}$

$f(x) = x^2 \quad g(x) = 0 \quad x_1 = 0 \quad x_2 = 2$

$$V = \pi \int_{x_1}^{x_2} f(x)^2 - g(x)^2$$

$$V = \pi \int_0^2 x^4 = \frac{\pi}{5} x^5 \Big|_0^2 = \frac{32\pi}{5}$$

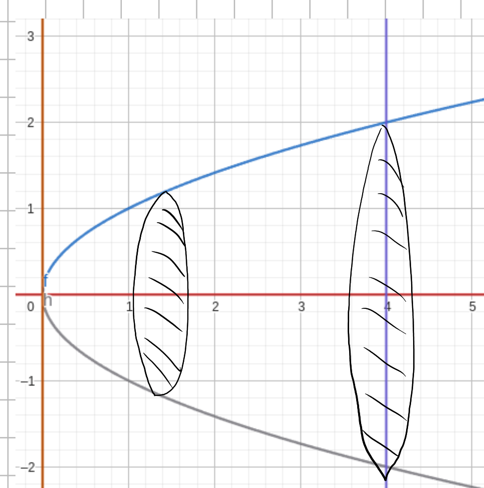



3. Calcule el volumen del sólido obtenido al rotar $y = \sqrt{x}$ en $[0, 4]$ alrededor del eje $x$.

Región: $\{(x,y): 0 \leq x \leq 4,\ 0 \leq y \leq \sqrt{x}\}$

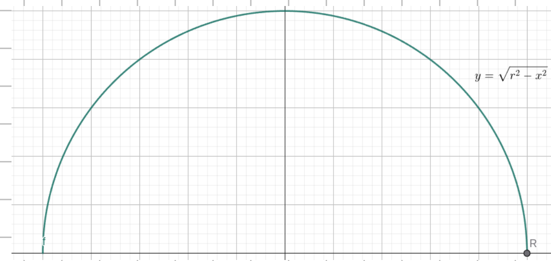
$f(x) = \sqrt{x} \quad g(x) = 0 \quad x_1 = 0 \quad x_2 = 4$

$$V = \pi \int_0^4 x = \frac{\pi}{2} x^2 \Big|_0^4 = 8\pi$$



## Ejercicios No Rutinarios

1. Use integración para demostrar la fórmula del área de un círculo.




$x^2+y^2=r^2$

$y=\sqrt{r^2-x^2}$

$A_0=4\int_0^r y$

$u=\text{arcsen}\left(\frac{x}{r}\right)$
$x=r\,\text{Sen}(u)$
$dx=r\,\text{Cos}(u)\,du$

$\text{Sen}^2(x)+\text{Cos}^2(x)=1$
$\text{Cos}^2(x)=1-\text{Sen}^2(x)$ Ambos lados por $r^2$
$(r\cos(x))^2=r^2-(r\,\text{Sen}(x))^2$
$r^2\text{Cos}^2(x)=r^2(1-\text{Sen}^2(x))$

$A=4\int_0^r\sqrt{r^2-x^2}\,dx$

$A=4\int_0^r\sqrt{r^2\underbrace{(1-\text{sen}^2(u))}_{\cos^2(u)}}\cdot r\,\text{Cos}(u)\,du$

$A=4\int_0^r r\,\text{Cos}(u)\cdot r\cos(u)\,du$

$A=4r^2\int_0^r\text{Cos}^2(u)\,du \qquad \cos^2(u)=\frac{1+\cos(2u)}{2}$

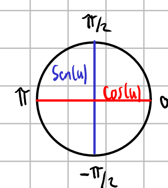
$A=4r^2\left[\frac{1}{2}\left(\int_0^r du+\int_0^r\text{Cos}(2u)\,du\right)\right]$

$A=2r^2\left[u+\frac{1}{2}\text{Sen}(2u)\right]_{x_1=0}^{x_2=r}$

$\text{Sen}(\pi)=0$ y $\text{Sen}(-\pi)=0$

$x_1=r\cos(u)$
Si $x_1=0$
$0=\cos(u)$
$u=-\pi/2$ ó $\pi/2$

$x_2=r\cos(u)$
Si $x_2=r$
$\cancel{r}=\cancel{r}\cos(u)$
$u=\text{arccos}(1)$
$u=0$




$A=2r^2\left[(0+\pi/2)+\left(\cancel{\frac{1}{2}\text{Sen}(2\cdot 0)}^{\,0}\right)-\frac{1}{2}\text{sen}(2\cdot\pi/2)\right]$

$A=2r^2\left[\frac{\pi}{2}+\cancel{\frac{1}{2}\text{Sen}(\pi)}^{\,0}\right]$

$\boxed{A=\pi r^2}$

$\int_1^\infty\frac{dx}{x^2}=\lim_{b\to\infty}\int_1^b x^{-2}=\lim_{b\to\infty}\left[\frac{-1}{x}\right]_1^b=\lim_{b\to\infty}\cancel{\frac{-1}{b}}^{\,0}+1=\boxed{1}$ Converge