COMPULSORY EXERCISE 1

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1. Problem 1

Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular triangular matrix with inverse $B = [\mathbf{b_1}\mathbf{b_2}\cdots\mathbf{b_n}]$. Then we have

$$AB = [A\mathbf{b}_1\ A\mathbf{b}_{2-}\cdots\ A\mathbf{b}_n]\ = I\,, \tag{1.1}$$

that is,

$$A\mathbf{b}_k = \mathbf{e}_k\,, \tag{1.2}$$

$\mathbf{e_k} \in \mathbb{R}^n$ denoting the k'th unit vector.
As usual, denote entry (i,j) of a matrix A by $a_{i,j}$.

Case 1: A lower triangular. For $k = 1$, there is nothing to prove. Let $1 < k \le n$. Define $A_{-[n-k+1]} \in \mathbb{C}^{n-k+1\times n-k+1}$ by

$$A_{-[n-k+1]} := \begin{pmatrix} a_{k,k} & \cdots & a_{k,n} \\ \vdots & \vdots & \vdots \\ a_{n,k} & \cdots & a_{n,n} \end{pmatrix}. \tag{1.3}$$

Then, for some matrix $L \in \mathbb{C}^{k-1\times n-k+1}$, we have

$$A\mathbf{b}_k = \begin{pmatrix} A_{[k-1]} & 0 \\ L & A_{-[n-k+1]} \end{pmatrix} \begin{pmatrix} \mathbf{b}_k^u \\ \mathbf{b}_k^l \end{pmatrix} = \begin{pmatrix} A_{[k-1]}\mathbf{b}_k^u \\ L\mathbf{b}_k^u + A_{-[n-k+1]}\mathbf{b}_k^l \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{e}_1 \end{pmatrix}, \tag{1.4}$$

where $\mathbf{b}_k^u \in \mathbb{R}^{k-1}$ denote the $k-1$ upper elements of $\mathbf{b}_k$ and $\mathbf{b}_k^l \in \mathbb{R}^{n-k+1}$ denote the remaining $n-k+1$ lower elements.
Since A is nonsingular and triangular, so is $A_{[k]}$ (because necessarily the diagonal elements of $A_{[k]}$ are then all nonzero), so we conclude that $\mathbf{b}_k^u \equiv 0$. Thus, we are left only with the system of equations

$$A_{-[n-k+1]}\mathbf{b}_k^l = \mathbf{e_1}\,, \tag{1.5}$$

or in MATLAB notation, $A(k:n,k:n)b(k:n,k:n) = I(k:n,k)$, as was to be shown. $\square$

Case 2: A upper triangular. Similarly, for some matrix U, we write

$$A\mathbf{b}_k = \begin{pmatrix} A_{[k]} & U \\ 0 & A_{-[n-k]} \end{pmatrix} \begin{pmatrix} \mathbf{b}_k^u \\ \mathbf{b}_k^l \end{pmatrix} = \begin{pmatrix} A_{[k]}\mathbf{b}_k^u + U\mathbf{b}_k^l \\ A_{-[n-k]}\mathbf{b}_k^l \end{pmatrix}, \tag{1.6}$$

leading us to conclude that $b_k^l \equiv \mathbf{0}$, so we need only solve

$$A_{[k]}\mathbf{b}_k^u = \mathbf{e}_k\,, \tag{1.7}$$

or in MATLAB notation, $A(1:k,1:k)\mathbf{b}_k(1:k) = I(1:k,k)$. □

As a final remark, we note that the nondiagonal nonzero elements of B may be stored in place of the zero elements of A. That is, we may simply store $b_{i,j}$ as $a_{j,i}$, for all elements of B not on the diagonal or on the zero part of B.

The entries on the diagonal are not worth storing, as $b_{i,i} = 1/(a_{i,i})$.

Problem 2

We have, for $j \geq k = 1, 2, \cdots, n$, if A is lower triangular,

$$A_{-[n-k+1]}\mathbf{b}_k^l = \mathbf{e_1} \tag{1.8}$$

which is solved by

$$(\mathbf{b}_k^l)_1 = b_{k,k} = \frac{-1}{a_{k,k}}\,,\ \text{1 division}\,,$$

$$(\mathbf{b}_k^l)_2 = b_{k,k+1} = -\frac{a_{k+1,k}b_{k,k}}{a_{k+1,k+1}}\,,\ \text{1 multiplication, 1 division;}$$

$$\cdots$$

$$(\mathbf{b}_k^l)_{n-k+1} = b_{k,n} = -\frac{1}{a_{n,n}}\sum_{j=n-k+1}^{n} a_{j,n}b_{k,j}\,,\ \text{2k-1 arithmetic operations.}$$

This gives j multiplications/divisions and $j-1$ additions/subtractions for the j'th element of the k'th column of B, so upon adding all operations for all k rows, we have that the total number of arithmetic operations is

$$\sum_{k=1}^{n}\sum_{j=1}^{n-k+1}(2j-1) = \sum_{k=1}^{n}(n-k+1)(n-k+2) - (n-k+1) = \sum_{k=1}^{n}(n-k+1)^2$$

$$\approx \int_1^n (n-k+1)^2\,\mathrm{d}k = -\frac{1}{3}[(n-k+1)^3]_{k=1}^n = -\frac{1}{9} + \frac{n^3}{3} \approx \frac{n^3}{3}\,.$$

Hence, the algorithm is of order $\mathcal{O}(n^3)$.

Problem 3

The code snippets compute the inverse of an upper diagonal matrix; it does this via a row-oriented back-solve very similar to Algorithm 2.7. Finally, the program prints the product AU, where U is the computed inverse matrix of A, hopefully giving the identity matrix as a result. The variables r and k denote the rows and columns of U respectively.