MANDATORY ASSIGNMENT - MAT4500

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1. Problem 1

$D^n := \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ unit ball in $\mathbb{R}^n$, $S^n := \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ unit sphere in $\mathbb{R}^{n+1}$. Upper and lower hemispheres are respectively denoted $S^n_U := \{x \in S^n : x_{n+1} \geq 0\}$, $S^n_L := \{x \in S^n : x_{n+1} \leq 0\}$.

a. Define $f_U : S^n_D \to S^n_U$ and $f_L : D^n \to S^n_U$ by

$$f_U(x) = (x, \sqrt{1 - ||x||^2}), \tag{1.1}$$

$$f_L(x) = (x, -\sqrt{1 - ||x||^2}). \tag{1.2}$$

Since the procedure for f_L in general is so similar, I omit it from the following procedure.
To show that f_U is a homeomorphism, I need to verify four things: injectivity, continuity, surjectivity and open mapping.

Injectivity: let π project S^n_U surjectively onto D^n. Then for $x \in D^n$, $\pi(f_U(x)) = x$, hence the composition $\pi \circ f_u$ is injective, which implies that f_U is injective.

To check for surjectivity, let $y \in S^n_U$. Then as $||y||^2 = 1$, we may always write, with $y = (x, x_{n+1})$,

$$\sum_{i=1}^{n} x_i^2 + x_{n+1}^2 = 1 \iff x_{n+1}^2 = 1 - ||x||^2 \iff x_{n+1} = \sqrt{1 - ||x||^2}, \tag{1.3}$$

with $x = (x_1, x_2, \cdots, x_n) \in D^n$ (because then $||x|| \leq 1$). This shows surjectivity.

For continuity, let $O \subset S^n_U$ be open. For $y \in O$, we write $y = (x, \sqrt{1 - ||x||^2})$, which implies that $f^{-1}(O) = \pi(O)$, which is open (because π is an open map).

It only remains to show that f_U and f_L are open maps. Let $O \subset D^n$ be an open set. Then

$$f_U(O) = \{(x, \sqrt{1 - ||x||^2}) : x \in O\} = S^n_U \cap \Big(O \times (-\epsilon, 1 + \epsilon)\Big) \tag{1.4}$$

for some $\epsilon > 0$; hence f_U is open.

This shows that f_U and f_L are indeed homeomorphisms. $\square$

b. For $x \in D^n \sqcup D^n$, write $x = (z, j)$, with $z \in D^n$ and $j \in \{U, L\}$.
We now form an equivalence relation $\sim$ on $D^n \sqcup D^n$ given by $x^1 \sim x^2$ iff either $x^1 = x^2$ or $||z^1|| = ||z^2|| = 1$ *and* $z^1 = z^2$.

Let $f : D^n \sqcup D^n \to S^n$ be given by $f(x) = f_j(z)$. To see that f is continuous, let $O \subset S^n$, and write $O = O_U \cup O_L$ with $O_U \subset S^n_U$ and $O_L \subset S^n_L$. Then

$$\begin{aligned} f^{-1}(O) &= f^{-1}(O_U) \cup f^{-1}(O_L) = (f_U^{-1}(O), \{U\}) \cup (f_L^{-1}(O_L), \{L\}) \\ &= \phi_U(f^{-1}(O)) \cup \phi_L(f^{-1}(O)) \,, \end{aligned}$$

where ϕ_U, ϕ_L are the canonical injections [1] . Hence, f is indeed continuous. Moreover, we have seen that for $y \in S^n$, either $y \in S^n_U$, or $y \in S^n_L$, or y is in both. In all cases there is an $x \in D^n \sqcup D^n$ such that $f(x) = y$.

Finally, let $f(x^1) = f(x^2)$. If $||x^1|| < 1$, we have that $f(x^1)$ is either in S^n_U or in S^n_L, but not in both. But then $f(x^1) = f_j(z^1) = f_j(z^2)$ where $j = j_1 = j_2$, and these maps have already been shown to be injective–thus $z^1 = z^2$ and so $x^1 = x^2$.
If, on the other hand $||x^1|| = ||x^2|| = 1$, then writing $y = f(x^1) = (z, z_{n+1})$, we have $z_{n+1} = 0$ (because $||y|| = ||z|| = 1$) and $z^1 = z^2 = z$; by definition, then $z^1 \sim z^2$. $\square$

c. Define $\hat{f} : (D^n \sqcup D^n)/\sim$ by $f([x]) = f(x)$, for some representative $x \in [x]$. I then claim that $\hat{f}$ is a homeomorphism onto S^n–this will follow readily from Corollary 22.3, pp. 140.
First, note that we do have $\sim$ defined in such a way that

$$(D^n \sqcup D^n)/\sim = \{f^{-1}(\{x\}) : x \in (D^n \sqcup D^n)\} \,. \tag{1.5}$$

Hence, $\hat{f}$ is a homeomorphism if and only if f is a quotient map. But from (a) we have that f_U and f_L are open maps. We have, for an $O \subset D^n \sqcup D^n$, writing $O = (O_U, \{U\}) \cup (O_L, \{L\})$, so that

$$\begin{aligned} f(O) &= f((O_U, \{U\}) \cup (O_L, \{L\}) = f((O_U, \{U\})) \cup f((O_L, \{L\})) \\ &= f_U(O_U) \cup f_L(O_L) \,. \end{aligned}$$

Hence f is an open map, and $\hat{f}$ is a homeomorphism. $\square$

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[1] At this point it bears noting that as I was unsure of a formal definition of a topology on a disjoint union, though an intuitive notion seemed clear enough. To see how I defined it, I (perhaps unwisely) direct the reader to the Wikipedia article on disjoint union topology. f is obviously continuous by this definition.

2. Problem 2

(a). Let (X, d) be a metric space. Let $C \subset X$ be compact.
First assume C is not closed–then there exists a series $\{x_n\}_{n=1}^{\infty}$ s.t. $x_n \to x \notin C$. Hence, for each $y \in C$, there is some neighbourhood $B(y; r_y)$ that fails to intersect with some neighbourhood of x intersecting with C. This family of neighbourhoods forms an open cover of C, from which we may take a finite subcover $\{B(y_i; r_i\}_{i=1}^{N}$ of C–but since each of these open balls fail to intersect with some neighbourhood of x in C, this cannot be a finite subcover after all, and so C is not compact–a contradiction.

Now to show that C is compact: fix $x \in C$ and let $\mathcal{O} := B(x, n)$ for $n \in \mathbb{N}$. Then clearly this is an open cover of C, and so there exists a finite subcover $\{B(x, r_i)\}_{i=1}^{N}$. Choose j such that $r_j \geq r_i$ for all $i \leq N$; then $C \subset B(x, r_j)$, and so $d(x, y) \leq r_j$ for all $x, y \in C$.
This shows that C is closed and bounded. $\square$

(b). For an example of a metric space where closed and bounded sets are not closed, we may take $(\mathbb{R}^n, \rho)$, with ρ being the bounded metric on $\mathbb{R}^n$ induced by $||\cdot||$. Then any subset of $\mathbb{R}^n$ is bounded, yet clearly not necessarily compact.

3. Problem 3

(a). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. Then f is clearly continuous, and so the preimage of a closed set is closed under f. Specifically, we have $\mathrm{Ker}(f) = f^{-1}(\{0\})$ closed because $\{0\}$ is closed. $\square$

(b). Let $\mathbf{SL}(2, \mathbb{R})$ denote the set of real-valued (2×2) matrices with determinant 1, with the subspace topology of $\mathbb{R}^4$.
That is,

$$\mathbf{SL}(2, \mathbb{R}) = \{(a, b, c, d) \in \mathbb{R}^4 : ad - bc = 1\}\,. \tag{3.1}$$

Consider the function $f : \mathbb{R}^4 \to \mathbb{R}$ given by

$$f(a, b, c, d) = ad - bc\,, \tag{3.2}$$

Then f is again clearly continuous. Hence, we have as in (a),

$$f^{-1}(\{1\}) = \{(a, b, c, d) \in \mathbb{R}^4 : ad - bc = 1\} = \mathbf{SL}(2, \mathbb{R})\,, \tag{3.3}$$

which shows that $\mathbf{SL}(2, \mathbb{R})$ is indeed closed. $\square$

(c). For $t \neq 0$, setting $b = c = 0$ and letting $a = t, d = 1/t$ gives $ad - bc = 1$. Letting $t \to 0$ or $t \to \infty$ then forces $||(a, b, c, d)||$ to grow out of any bounds; hence $\mathbf{SL}(2, \mathbb{R})$ is not bounded and so cannot be compact. $\square$

4. Problem 4

Let X be a topological space.

(a). To show that path-equivalence is an equivalence relation on X, I need to establish the three usual criteria.

For reflexivity, we have for $x \in X$, $\alpha(t) = x$ for $t \in [0,1]$. Thus x is path-equivalent to itself.

For symmetry, assume that x is path-equivalent to y, and let $\alpha(t)$ be the continuous map satisfying $\alpha(0) = x$, $\alpha(1) = y$.
Define $\beta : [0,1] \to X$ by $\beta(t) = \alpha(1-t)$. Hence y is path-equivalent to x.

For transivity, assume that x is path-equivalent to y with path-map α and that y is path-equivalent to z with path-map β. Define $\gamma : [0,1] \to X$ by

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 \le t \le 1/2\,, \\ \beta(2t-1), & 1/2 < t \le 1\,. \end{cases} \tag{4.1}$$

Then γ is a continuous map connecting x to z, as was to be shown. □

Henceforth, let $\pi_0(X)$ denote the set of equivalence classes of points in X under path-equivalence.

(b). Let $X = \mathbb{R}^n$. Then X is convex, and so for $x, y \in X$, the map $\alpha(t) = x + (y-x)t$ satisfies the criteria. Hence x and y are path-equivalent for all $x, y \in X$, that is, $\pi_0(\mathbb{R}^n)$ consists of one element.

(c). Let $X = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Then if x and y are both either positive or negative, the same map as in (b) will provide a path between x and y, and so they are path-equivalent.
Assume $x < 0$ and $y > 0$. Assume that there exists a path $\alpha : [0,1] \to X$ connecting x to y. Let $t^* = \sup\{t : \alpha(t) < 0\}$. Then either $\alpha(t^*) < 0$ or $\alpha(t^*) > 0$. In the first case, we have $\alpha^{-1}((x,0)) = (0,t^*]$ which is not open, and in the second case, we have $\alpha^{-1}((0,y)) = [t^*, 1)$, which is again not open. Hence α cannot be continuous, a contradiction. In conclusion, x and y cannot be path equivalent, and there exist exactly two partitions of $\mathbb{R} \setminus \{0\}$.

(d). Let $X = \mathbb{R}^n \setminus \{z\}$ for $n \ge 2$ and some $z \in \mathbb{R}^n$. Let $x, y \in X$. Then if z does not lie on the line between x and y, they are clearly path-equivalent. If z does lie between x and y, there exists some u, say $u = z + (1,0,0,\cdots,0)$ (replace if necessary), such that z does not lie between x and u or between u and y. Then x is path-equivalent to u, and u is path-equivalent to y–by transitivity, x is path-equivalent to y.
Hence $\pi_0(\mathbb{R}^n)$ consists of only one element.

(e). Note first that $\mathbb{R} \setminus \{0\}$ is not merely not path-connected, it is not connected, as $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$.
Assume that there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}$ with $f(z) = 0$ for some $z \in \mathbb{R}^n$. Since then $f(\mathbb{R}^n \setminus \{z\}) = \mathbb{R} \setminus \{0\}$, we have that f does not

preserve connectedness; hence f is not a homeomorphism, which is contradictory.
In conclusion, there cannot exist a homeomorphism between $\mathbb{R}$ and $\mathbb{R}^n$.