MANDATORY ASSIGNMENT - MAT-INF4300

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1. Problem 1

a. Assume $u \in C^3(\mathbb{R}^n \times (0,\infty))$ solves $u_t - \Delta u = 0$.
Let $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$, $\lambda \in \mathbb{R}$.
Then $(u_\lambda)_t(x,t) = \lambda^2 u_t(\lambda x, \lambda^2 x)$, and $(u_\lambda)_{x_i,x_i}(x,t) = \lambda^2 u_{x_i,x_i}(\lambda x, \lambda^2 t)$, and so

$$((u_\lambda)_t - \Delta u_\lambda)(x,t) = \lambda^2 (u_t(\lambda x, \lambda^2 t) - \Delta u(\lambda x, \lambda^2 t)) = 0\,. \tag{1.1}$$

□

b. Let $v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$.
(I am assuming now that u is three times continuously differentiable, not just two).
Differentiating $(x,t,\lambda) \mapsto u_\lambda(x,t)$ with regards to λ gives $x \cdot Du(\lambda x, \lambda^2 t) + 2t\lambda u(\lambda x, \lambda^2 t) =: w(x,t,\lambda)$; we then have $v(x,t) = w(x,t,1)$.
Since $(x,t,\lambda) \mapsto u_\lambda(x,t)$ solves the heat equation for all $\lambda \in \mathbb{R}$, so does $w(x,t,\lambda)$ for all $\lambda \in \mathbb{R}$, and in particular, for $\lambda = 1$. Thus v solves the heat equation.
Slightly more written out, continuity of the derivatives of u_λ allows us to interchange the order of differentiation, and so,

$$v_t - \Delta v = \left(\left(\frac{\partial u_\lambda}{\partial \lambda} \right)_t - \Delta \left(\frac{\partial u_\lambda}{\partial \lambda} \right) \right) \Bigg|_{\lambda=1} = \frac{\partial}{\partial \lambda} ((u_\lambda)_t - \Delta u_\lambda) \Bigg|_{\lambda=1} = 0\,. \tag{1.2}$$

□

c. Let $\eta : \mathbb{R} \to \mathbb{R}$ be convex and twice continuously differentiable, let u solve the heat equation, and set $v(x,t) := \eta(u)$. Then

$$v_t = \frac{\partial u}{\partial t} \eta'(u) = u_t \eta'(u)\,, \tag{1.3}$$

$$v_{x_i} = \frac{\partial u}{\partial x_i} \eta'(u) = u_{x_i} \eta'(u)\,, \tag{1.4}$$

$$v_{x_i x_i} = u_{x_i}^2 \eta''(u) + u_{x_i x_i} \eta'(u)\,. \tag{1.5}$$

Hence $v_t - \Delta v = \eta'(u)(u_t - \Delta u) - \eta''(u)|Du|^2 = -\eta''(u)|Du|^2 \le 0$ because $\eta''(u) \ge 0$. □

2. Problem 2

Problem:

$$(*)\qquad \begin{cases} u_t - \Delta u + cu = f\,, & (x,t) \in \mathbb{R}^n \times (0,\infty)\,, \\ u(x,0) = u_0(x)\,, & x \in \mathbb{R}^n\,, \end{cases}$$

for $c \in \mathbb{R}$, $f \in C_c^{2,1}(\mathbb{R}^n \times [0,\infty))$, $u_0 \in C_c(\mathbb{R}^n)$.

a. Assume there exists a v satisfying

$$(**)\qquad \begin{cases} v_t - \Delta v = e^{ct} f\,, & (x,t) \in \mathbb{R}^n \times (0,\infty)\,, \\ v(x,0) = u_0(x)\,, & x \in \mathbb{R}^n\,, \end{cases}$$

Then for $w := e^{-ct} v$ we have

$$w_t = e^{-ct}(-cv + v_t)\,, \tag{2.1}$$

$$\Delta w = e^{-ct} \Delta v\,. \tag{2.2}$$

And so

$$w_t - \Delta w + cw = e^{-ct}(-cv + v_t - \Delta v + cv) = e^{-ct}(v_t - \Delta v) = f\,, \tag{2.3}$$

so w satisfies the first half of (*).
A solution of (**) is

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) u_0(x)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) e^{cs} f(y,s)\,\mathrm{d}s\,. \tag{2.4}$$

This is valid because f and u_0 have compact support.
Hence, an explicit formula for u satisfying (*) is
(2.5)

$$u(x,t) = e^{-ct} \left(\int_{\mathbb{R}^n} \Phi(x-y,t) u_0(x)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) e^{cs} f(y,s)\,\mathrm{d}s \right)\,.$$

It remains to verify:

(1) that u really is in $C^{2,1}(\mathbb{R}^n \times (0,\infty)$,
(2) that u really does satisfy $u_t - \Delta u + cu = f$, and
(3) that $u(x,t) \to u_0(x_0)$ whenever $(x,t) \to (x_0, 0)$.

(1). Since $v \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$ and u is a product of a smooth function with v, it follows that u is also a member of $C^{2,1}(\mathbb{R}^n \times (0,\infty))$.

(2). Write u as

$$\begin{aligned} u(x,t) &= e^{-ct} \left(\int_{\mathbb{R}^n} \Phi(x-y,t) u_0(x)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) e^{cs} f(y,s)\,\mathrm{d}s \right) \\ &=: e^{-ct}(I(x,t) + J(x,t))\,. \end{aligned}$$

Then, from Theorem 1 on pp. 47 in Evans, I satisfies the homogenous heat equation; additionally, $I(x,t) \to u_0(x_0)$ as $x \to x_0$, $t \downarrow 0$ for $x_0 \in \mathbb{R}^n$.
Likewise, from Theorem 2 on pp. 50 in Evans, J satisfies $J_t - \Delta J = e^{ct} f$.
Additionally, $J(x,t) \to 0$ as $x \to x_0$, $t \downarrow 0$ for $x_0 \in \mathbb{R}^n$.

I omit any direct verification, as the calculations involved would necessarily just mirror those in the book.
Hence, we confirm

$$\begin{aligned} u_t &= e^{-ct}(-c(I+J) + (I_t + J_t))\,, \\ \Delta u &= e^{-ct}(\Delta I + \Delta J)\,, \\ u_t - \Delta u + cu &= e^{-ct}(-c(I+J) + (I_t - \Delta I) + (J_t - \Delta J) + c(I+J)) = f\,. \end{aligned}$$

□

(3). Again writing $u = e^{-ct}(I+J)$, for any $\epsilon > 0$, we may pick (x,t) with $t > 0$ s.t. $|I(x,t)| < \epsilon/3$ and $|J(x,t) - u_0(x)| < \epsilon/3$ and finally so that $1 - e^{-ct} < \epsilon/3/(|u_0(x_0)| + \epsilon/3)$.

$$\begin{aligned} |u(x,t) - u_0(x)| &= |e^{-ct}I(x,t) + (e^{-ct}J(x,t) - u_0(x_0)| \\ &\leq |e^{-ct}I(x,t) - u_0(x_0)| + e^{-ct}|J(x,t)|\,. \end{aligned}$$

Since $e^{-ct} = 1 - \delta$ for some $\delta > 0$, this can be written

$$\begin{aligned} (\cdots) &= |(1-\delta)I(x,t) - u_0(x_0)| + e^{-ct}|J(x,t)| \\ &\leq |I(x,t) - u_0(x_0)| + \delta|I(x,t)| + |J(x,t)|\,. \end{aligned}$$

Now by choice of (x,t), $\delta = 1 - e^{-ct} < \epsilon/3/(|u_0(x_0)| + \epsilon/3)$, and $|I(x,t)| < |u_0(x,t)| + \epsilon/3)$. Thus, finally,

$$< \epsilon/3 + \delta(|u_0(x_0)| + \epsilon/3) + \epsilon/3 < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon\,.$$

□

b. Note I failed to get the energy bound that was asked for, but I stand by my computations, and the result I got is more than sufficient.

Assume $f \equiv 0$, and that $u \to 0$ as $x \to \infty$. Now proceeding in the reverse direction, let $v := e^{ct}u$. Then as seen, v satisfies the homogenous heat equation, and so it is smooth; hence u is also smooth, and we can differentiate under the integral.
Define the energy $E(t)$ by

$$E(t) = ||u(\cdot,t)||_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x,t)^2\,\mathrm{d}x\,. \tag{2.6}$$

Then

$$E'(t) = \int_{\mathbb{R}^n} 2uu_t\,\mathrm{d}x = \int_{\mathbb{R}^n} -2cu^2\,\mathrm{d}x + \int_{\mathbb{R}^n} u\Delta t\,\mathrm{d}x\,, \tag{2.7}$$

whence, integrating the last term by parts, we otbain

$$(...) = \int_{\mathbb{R}^n} -2cu^2\mathrm{d}x - \int_{\mathbb{R}^n} |Du|^2\mathrm{d}x \;= -2cE(t) - \int_{\mathbb{R}^n} |Du|^2\mathrm{d}x \;\leq -2cE(t)\,. \tag{2.8}$$

This implies that $2cE(t) + E'(t) \leq 0$, and so, by Grönwall's inequality, $E(t) \leq e^{-2ct}E(0) = e^{-2ct}||u_0||_{L^2(\mathbb{R}^n)}$.

Assuming we have two solutions u_1, u_2 of (*) satisfying $u \to 0$ as $|x| \to \infty$, we let $w := u_1 - u_2$, $E_w := ||w||_{L^2(\mathbb{R}^n)}$. Then w satisfies

$$\begin{cases} w(x,t) = 0\,, & (x,t) \in \mathbb{R}^n \times (0,\infty)\,, \\ w(x,0) = 0\,, & x \in \mathbb{R}^n\,. \end{cases} \tag{2.9}$$

As has been shown, this implies $0 \leq E_w(t) \leq e^{-2ct}E_w(0) = 0$, so $w \equiv 0$; hence $u_1 \equiv u_2$, as was to be shown.

3. Problem 3

$$\begin{cases} u_t - \Delta u = -u^3\,, & (x,t) \in \Omega \times (0,\infty)\,, \\ u(x,) = u_0(x)\,, & x \in \mathbb{R}^n\,, \\ u(x,t) = 0\,, & (x,t) \in \partial\Omega \times (0,\infty)\,. \end{cases} \tag{***}$$

with $\Omega \subset \mathbb{R}^n$ is open and bounded, u_0 continuous.
Assume there exists a twice continuously differentiable u satisfying (***). Then as before, let

$$E(t) := ||u(\cdot,t)||_{L^2(\Omega)} = \int_\Omega u^2 \,\mathrm{d}x\,. \tag{3.1}$$

Justifying that I can differentiate under the integral sign is a bit finicky: let $v := u^2$ for simplicity of notation. Then v_t exists on Ω and is bounded wrt x, $||v_t(\cdot,t)||_{L^\infty(\Omega)} < \infty$. Fix $t \in (0,\infty)$, and $\epsilon > 0$ s.t. $t - \epsilon > 0$. Let $M = ||v||_{L^\infty(\Omega\times(t-\epsilon,t+\epsilon)} < \infty$. Then the function $g(x) = M$ is summable, and dominates v for all $(x,t) \in \Omega \times (t-\epsilon, t+\epsilon)$.
Take some sequence $\{t_n\}_{n\in\mathbb{N}}$ s.t. $t_n \to t$ and $|t_n - t| < \epsilon$. Then for $n \in \mathbb{N}$, by the mean value theorem,

$$\frac{v(x,t_n) - v(x,t)}{t_n - t} = \frac{\partial v(x,\zeta_n)}{\partial t} \leq M\,, \tag{3.2}$$

for some $\zeta_n \in (t_n, t)$.
Thus the sequence of functions $\{w_n\}_{n\in\mathbb{N}} := \frac{v(x,t_n)-v(x,t)}{t_n-t}$ is dominated by the summable function $g(x)$, and so by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_\Omega w_n(x,t)\mathrm{d}x = \int_\Omega \lim_{n\to\infty} w_n(x,t)\mathrm{d}x = \int_\Omega v_t(x,t)\mathrm{d}x\,. \tag{3.3}$$

Since this holds for any such sequence, for any $t \in (0,\infty)$, we are free to differentiate $v = u^2$ under the integral sign.
Phew. Now, let's finally do that.

$$E'(t) = \int_\Omega 2uu_t \,\mathrm{d}x = \int_\Omega 2u(\Delta u - u^3)\,\mathrm{d}x \ = \int_\Omega 2u\Delta u \,\mathrm{d}x - \int_\Omega 2u^4 \,\mathrm{d}x\ . \tag{3.4}$$

Now integrate the first by parts and use that $u \equiv 0$ on $\partial\Omega$ to obtain

$$(\cdots) = -\int_\Omega |Du|^2 + 2u^4 \,\mathrm{d}x \ \leq 0,. \tag{3.5}$$

Hence $E(t)$ is a nonincreasing function, and so $E(t) \leq E(0) = ||u_0||_{L^2(\Omega)}$. □

4. Problem 4

Let $\Omega \subset \mathbb{R}^n$ be bounded and open. Then the task is to show that the Hölder space $C^{0,\gamma}(\Omega)$ with exponent $\gamma \in [0,1)$ is a Banach space.

The space is equipped with the norm $||\cdot||_{C^{0,\gamma}(\overline{\Omega})}$ given by

$$||\cdot||_{C^{0,\gamma}(\overline{\Omega})} = ||\cdot||_{C(\overline{\Omega})} + [\cdot]_{C^{0,\gamma}(\overline{\Omega})} \,. \tag{4.1}$$

First, I verify that the γ'th Hölder seminorm is indeed a seminorm–then it follows that the $\gamma'th$ Hölder norm is a norm.
The two properties
$[u]_{C^{0,\gamma}(\overline{\Omega})} \geq 0$ and $[\lambda u]_{C^{0,\gamma}(\overline{\Omega})} = |\lambda|[u]_{C^{0,\gamma}(\overline{\Omega})}$ are trivial.
For the triangle inequality, we have

$$[u+v]_{C^{0,\gamma}(\overline{\Omega})} = \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ \left| \frac{u(x)-u(y)}{|x-y|^\gamma} + \frac{v(x)-v(y)}{|x-y|^\gamma} \right| \right\} , \tag{4.2}$$

but since

$$\left| \frac{u(x)-u(y)}{|x-y|^\gamma} + \frac{v(x)-v(y)}{|x-y|^\gamma} \right| \leq \frac{|u(x)-u(y)|}{|x-y|^\gamma} + \frac{|v(x)-v(y)|}{|x-y|^\gamma} \tag{4.3}$$

for all $x, y \in \overline{\Omega}, x \neq y$, the same goes for its supremum.
Hence $[\cdot]_{C^{0,\gamma}(\overline{\Omega})}$ is a seminorm as was to be shown.
Now, let $\{u_n\}_{n\in\mathbb{N}} \subset C^{0,\gamma}$ be a Cauchy sequence. Then necessarily it is also a Cauchy sequence in the supremum norm $||\cdot||_{C(\overline{\Omega})}$. Since $(C(\overline{\Omega}), ||\cdot||_{C(\overline{\Omega})})$ is complete, there then exists a continuous u s.t. $u_n \to u$ in the supremum norm.
Choose $N \in \mathbb{N}$ s.t. $||u-u_n||_{C(\overline{\Omega})} < \epsilon/2$ and $[u_n - u_m]_{C^{0,\gamma}(\overline{\Omega})} < \epsilon/2$ for all $n, m \in \mathbb{N}$. Then for all $x, y \in \Omega$, $x \neq y$,

$$\left| \frac{u_m(x) - u_m(y) + u_n(y) - u_n(x)}{|x-y|^\gamma} \right| < \epsilon/2 \,, \tag{4.4}$$

and so because $u_m \to u$ pointwise,

$$\left| \frac{u(x) - u(y) + u_n(y) - u_n(x)}{|x-y|^\gamma} \right| = \lim_{m\to\infty} \left| \frac{u_m(x) - u_m(y) + u_n(y) - u_n(x)}{|x-y|^\gamma} \right| < \epsilon/2 \,, \tag{4.5}$$

since the above holds for all $m \geq N$. Hence $[u-u_n]_{C^{0,\gamma}(\overline{\Omega})} \leq \epsilon/2$, and so we have $||u-u_n||_{C^{0,\gamma}(\overline{\Omega})} < \epsilon$.
It remains to show that $||u||_{C^{0,\gamma}(\overline{\Omega})} < \infty$. But $||u||_{C(\overline{\Omega})} < \infty$, and

$$[u]_{C^{0,\gamma}(\overline{\Omega})} = [u - u_n + u_n]_{C^{0,\gamma}(\overline{\Omega})} \leq [u-u_n]_{C^{0,\gamma}(\overline{\Omega})} + [u_n]_{C^{0,\gamma}(\overline{\Omega})} < \infty \,, \tag{4.6}$$

for some u_n satisfying $[u-u_n]_{C^{0,\gamma}(\overline{\Omega})} < \infty$.
This completes the proof. □

5. Problem 5

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with a C^1 boundary. Let V strictly contain Ω.[1]
The task is to show that there exists a bounded linear operator

$$E : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n) \tag{5.1}$$

satisfying, for all $w \in W^{1,\infty}(\Omega)$

(1) $Eu = u$ for almost all $x \in \Omega$.
(2) $\mathrm{spt}(Eu) \subset V$,
(3) $||Eu||_{W^{1,\infty}(\mathbb{R}^n)} \leq C||u||_{W^{1,\infty}(\Omega)}$, for some C not depending on u.

Note I believe for the case $p = \infty$, the procedure outlined in Evans for $u \in C^1(\Omega)$ (pp 268-270) will more or less hold directly, so I will roughly follow the book. Where I feel it is obvious, I will then simply refer to the book for the sake of brevity.
Now, let $u \in W^{1,\infty}$, and as in the book, assume $\partial\Omega$ is flat near x_0, lying in the plane $x_n = 0$.
Then there exists an open ball $B(x_0, r)$, which I split into

$$\begin{aligned} B^+ &:= B \cap \{x_n \geq 0\} \subset \overline{\Omega}\,, \\ B^- &:= B \cap \{x_n \leq 0\} \subset R^n \setminus \overline{\Omega}\,. \end{aligned}$$

Set $\overline{u}$ to be
(5.2)

$$\overline{u}(x) = \begin{cases} u(x)\,, & x \in B^+\,, \\ -3u(x_1, x_2, \cdots, x_{n-1}, -x_n) + 4u(x_1, x_2, \cdots, x_{n-1}, -\frac{x_n}{2})\,, & x \in B^-\,. \end{cases}$$

Moreover, define $\{v_j\}_{j=1}^n$ to be
(5.3)

$$v_j(x) = \begin{cases} u_{x_j}(x)\,, & x \in B^+\,, \\ 3u_{x_j}(x_1, x_2, \cdots, x_{n-1}, -x_n) + 4u_{x_j}(x_1, x_2, \cdots, x_{n-1}, -\frac{x_n}{2})\,, & x \in B^-, j \neq n\,, \\ 3u_{x_n}(x_1, x_2, \cdots, x_{n-1}, -x_n) - 2u_{x_n}(x_1, x_2, \cdots, x_{n-1}, -\frac{x_n}{2})\,, & x \in B^- \; j = n\,, \end{cases}$$

for $j = 1, \cdots, n$. Then the next step is to verify that v_j is a weak derivative of $\overline{u}$ in B. Let $\phi \in C_c^\infty(B)$. For ease of notation, let $x' = x_1, x_2, \cdots, x_{n-1}$.
Then for $j = n$,

$$\begin{aligned} \int_B \phi_{x_n} \overline{u} \mathrm{d}x &= \int_{B^+} \phi_{x_n} \overline{u}\, \mathrm{d}x + \int_{B^-} \phi_{x_n} \overline{u}\, \mathrm{d}x \\ &= \int_{B^+} \phi_{x_n} u\, \mathrm{d}x \\ &+ \int_{B^+} \phi_{x_n}(x', -x_n) - 3u(x', x_n) + 4u(x', \frac{x_n}{2}))\, \mathrm{d}x\,. \end{aligned}$$

[1]The text says "V strictly larger than Ω"–this is the only interpretation I can make sense of.

It is now safe to integrate by parts. Obtain

$$
\begin{aligned}
(\cdots) = & \cancel{\int_{x_n=0} \phi u \,\mathrm{D}s(x)} - \int_{B^+} \phi u_{x_n} \,\mathrm{d}x \\
& + \cancel{\int_{x_n=0} \phi(-3u + 4u)} \,\mathrm{D}s(x) - \int_{B^+} \phi(x', -x_n)(3u_{x_n}(x', x_n) + 2u_{x_n}(x', \frac{x_n}{2}))\,\mathrm{d}x \\
= & - \int_{B^+} \phi u_{x_n} \,\mathrm{d}x - \int_{B^-} \phi(3u_{x_n}(x', -x_n) - 2u_{x_n}(x', \frac{-x_n}{2}))\,\mathrm{d}x \\
= & \int_B \phi v \,\mathrm{d}x \,.
\end{aligned}
$$

The cases $1 \leq j \leq n$ are similar and omitted. Hence, for any multiindex α with $|\alpha| = 1$, letting $D^\alpha u = v_\alpha$, we have $\int_B D^\alpha \phi \, \overline{u} \,\mathrm{d}x = \int_B D^\alpha \overline{u} \, \phi \,\mathrm{d}x$.
As in the book, it is clear that $||\overline{u}B||_{C^{0,\gamma}(\overline{\leq})} C ||u||_{C^{0,\gamma}(\overline{B^+})}$.

The rest of the proof now goes exactly as in the book, so the esteemed reader may skip the rest: if $\partial\Omega$ is not flat near x_0, use a C^1 homeomorphism to straighten it out, then exploit compactness of $\partial\Omega$ to cover it with a finite number, say N, of open sets W_i in which to obtain extensions u_i of u. Let $u_0 = u$, choose W_0 such that $\bigcup_{i=0}^N W_i = \Omega$, and let $\{\zeta_i\}_{i=0}^N$ form an associated partition of unity. Finally, let $v = \sum_{i=0}^N \zeta_i u_i$. Then $||v||_{C^{0,\gamma}(\overline{\Omega})} \leq C ||u||_{C^{0,\gamma}(\overline{\Omega})}$ for some $C > 0$, and we may define E as the linear map mapping u to v. $\square$

6. Problem 6

Let $u : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$
u(x) := |x - x_0|^\alpha \,, \qquad x \in B(x_0; 1) =: \Omega \,. \tag{6.1}
$$

for some $\alpha > 0$. For $x \neq 0$, we have $u_{x_i} = -\alpha x_i |x - x_0|^{-\alpha-2}$, and so

$$
|Du| = |\alpha| |x - x_0|^{-\alpha-1} \,. \tag{6.2}
$$

For the notion of a weak derivative to make sense, we require that $\int_\Omega \phi_{x_i} u \mathrm{d}x = \int_\Omega u_{x_i} \phi : \mathrm{d}x$ for a test function ϕ, $j = 1, 2, 3$. Let $0 < \epsilon < 1$, and compute

$$
\int_{\Omega \setminus B(x_0, \epsilon} \phi_{x_i} u \,\mathrm{d}x = - \int_{\Omega \setminus B(x_0, \epsilon} \phi \, u_{x_i} \,\mathrm{d}x + \int_{\partial B(x_0, \epsilon)} \phi u \nu^i mboxd S(x) \,, \tag{6.3}
$$

ν denoting the inward pointing unit normal.
We have

$$
\left| \int_{\partial B(x_0, \epsilon)} \phi u \nu^i \mathrm{d}S(x) \right| \leq ||\phi||_{L^\infty(\Omega)} \int_{\partial B(x_0, \epsilon)} \epsilon^{-\alpha} \,\mathrm{d}S(x) \leq C \epsilon^{2-\alpha} \to 0 \tag{6.4}
$$

as $\epsilon \to 0$ so long as $\alpha < 2$.
For a bound on $||Du||_{W^{1,2}(\Omega)}$, we have

$$
\int_\Omega |Du|^2 \,\mathrm{d}x = \alpha^2 \int_\Omega |x - x_0|^{-2\alpha-2} \,\mathrm{d}x = \alpha^2 4\pi \int_0^1 r^{-2\alpha} \,\mathrm{d}r = \alpha^2 4\pi \frac{1}{1 - 2\alpha} \tag{6.5}
$$

which is finite iff $\alpha < 1/2$. This is consistent with the result in the book, pp. 260. $\square$