Solutions to exercise in part 2C

Contents

Usefull formulas and constants

Constants

The following constants may be used:

- Mass of the Earth: $M_{\text{Earth}} = 5.972 \cdot 10^{24} \, kg$

- Mass of the sun: $M_{\text{sun}} = 1.989 \cdot 10^{30} \, kg$

- Radius of the Earth: $r_{\text{Earth}} = 6371 \, km$

- Radius of the sun: $r_{\text{sun}} = 695\,508 \, km$

- Speed of light: $c = 299\,792\,458 \, m/s$

- Gravitational constant: $G = 6.67408 \cdot 10^{-11} \, \frac{m^3}{kg \, s^2}$

Line elements

The Schwawrzschild line element is given by:

$$\Delta s^2 = \left(1 - \frac{2M}{r}\right) \Delta t^2 - \frac{\Delta r^2}{1 - \frac{2M}{r}} - r^2 \Delta \phi^2, \tag{1}$$

where M is the mass of the central mass at $r = 0$ (usually a black hole, a star or a planet), r is the Schwarzschild radius (see lecture note 2C if you do not remember this), Δt is the difference in time, Δr is the difference in radial position and $\Delta \phi$ is the difference in the angular position between two events.
This equation tells us that the intervall Δs^2 between two events is allways equal for all frames of referance.

If we are dealing with a local inertial frame (see lecture note 2C if you do not remember what this is), we are allowed to use the Lorentz line element:

$$\Delta s^2 = \Delta t^2 - \Delta x^2 = \Delta t^2 - \Delta r^2 - r^2 \Delta \phi^2, \tag{2}$$

where Δx is the difference in x-position and all other variables are the same as in the Schwarzschild line element. As above, Δs^2 between two events is equal for all observers.

Change of units

In the theory of relativity we will use natural unit. That is, we want to measure time and mass in meters. We converte between mass in kilos and meters as follows:

$$\frac{M_m}{M_{kg}} = \frac{G}{c^2} \left(\approx \frac{6.67408 \cdot 10^{-11} \, m^3 kg^{-1} s^{-2}}{(299\,792\,458 \, m/s)^2} \approx 7.4259 \cdot 10^{-28} \right) \tag{3}$$

Here M_m and M_{kg} the mass of the object in question in meters and kilos respectively, G is the gravitational constant and c is the speed of light.

To converte between time in seconds and meters we do like this:

$$t_s \cdot c = t_m \tag{4}$$

Here t_s and t_m is time in seconds and meters respectively and c is the speed of light.

Time and length differance between observers

In lecture note 2C we have deduced the following relation between the time and length measured by the shell and far-away observer:

$$\Delta t_{\text{shell}} = \Delta t \sqrt{1 - \frac{2M}{r}}, \tag{5}$$

and

$$\Delta r_{\text{shell}} = \frac{\Delta r}{\sqrt{1 - \frac{2M}{r}}} \tag{6}$$

Here Δt_{shell} and Δt is the time difference measured by the shell and far-away observer respectively, Δr_{shell} and Δr is the difference in radial position measured by the shell and far-away observer respectively, M is the mass of the central mass and r is the Schwarzschild radius.

Conservation laws

In lecture note 2C we are given the following consevation laws:
Energy per mass

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = \text{constant}, \tag{7}$$

where E and m is the energy and mass of the object, M is the mass of the central mass, r is the Schwazscild radius, dt the time difference measured by the far-away observer and $d\tau$ is the proper time of the object.

Angular momentum per mass

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau} = \gamma_{\text{shell}}rv_\phi = \text{constant}, \tag{8}$$

where L is the angular momentum, m is the mass of the object, r is the Schwarzschild radius, $d\phi$ is the angular difference measured by the far-away observer, $d\tau$ is the proper time of the object, $\gamma_{\text{shell}} = \frac{1}{\sqrt{1-v_{\text{shell}}^2}}$ where v_{shell} is the velocity of the object measured by a shell observer and v_ϕ is the angular velocity of the object measured by the far-away observer.

3

Exercise 2C.1

We are observing a laser with frequency $\nu_{\text{shell}} = 1/\Delta t_{\text{shell}}$, measured by a shell observer at a distance r from a central mass M, and $\nu' = 1/\Delta t'$, measured by a far-away observer (when the laser reaches him). $\Delta t'$ and Δt_{shell} are the difference between two peaks of the electromagnetic wave of the laser.

1. We want to find the relation between the two time differences. In lecture note 2C we derived a relation between the time measured by a shell and far-away observer (equation 5). We derived this equation assuming $\Delta r = 0$, $\Delta \phi = 0$ and that the shell observer was in a local inertial frame so that $\Delta s^2 = \Delta \tau^2 = \Delta t_{\text{shell}}$. Since the distance between two peaks on an electromagnetic wave is very small (if we exclude radio waves), the wave is only moving in radial direction and we can think of an observer "on" the wave as a shell observer in a local inertial frame, we are allowed to use equation 5 which gives us

$$\Delta t' = \frac{\Delta t_{\text{shell}}}{\sqrt{1 - \frac{2M}{r}}}$$

2. We should now be able to show the gravitational "Doppler" formula. We know that the relation between wave length and frequency is $\lambda = \frac{c}{\nu} = \frac{1}{\nu}$, since $c = 1$ in our relativistic units (where λ and ν is the wave length and frequency of an electromagnetic wave). Using that frequency is given by $\nu = \frac{1}{\Delta t}$, we get (remember that want to find an expression for $\frac{\Delta \lambda}{\lambda}$)

$$
\begin{aligned}
\frac{\Delta \lambda}{\lambda_{\text{shell}}} &= \frac{\lambda' - \lambda_{\text{shell}}}{\lambda_{\text{shell}}} = \frac{\lambda'}{\lambda_{\text{shell}}} - 1 = \frac{1/\nu'}{1/\nu_{\text{shell}}} - 1 \\
&= \frac{\Delta t'}{\Delta t_{\text{shell}}} - 1 = \frac{\Delta t_{\text{shell}}}{\sqrt{1 - \frac{2M}{r}}} \frac{1}{\Delta t_{\text{shell}}} - 1 \\
&= \frac{1}{\sqrt{1 - \frac{2M}{r}}} - 1
\end{aligned}
$$

In the second line we use the relation we found above (equation5).

3. To show the relation $\frac{\Delta \lambda}{\lambda_{\text{shell}}} = \frac{M}{r}$ when $r \gg 2M$, we will create a Taylor expansion. We define $f(x) = \frac{1}{\sqrt{1-x}}$. In other words, if we choose $x = \frac{2M}{r}$ we have

$$\frac{\Delta \lambda}{\lambda_{\text{shell}}} = \frac{1}{\sqrt{1 - \frac{2M}{r}}} - 1 = f(x) - 1$$

We start by finding the derivative.

$$f'(x) = \left[(1-x)^{-1/2}\right]' = \frac{-1}{2}(1-x)^{-3/2}(-1) = \frac{1}{2}(1-x)^{-3/2} \quad \Rightarrow \quad f'(0) = \frac{1}{2}$$

4

This means that we can write the first order Taylor expasion for $f(x)$ (this will be a good approximation for $r \gg 2M$.

$$\frac{1}{\sqrt{1-x}} = f(x) \approx T_1 f(x) = f(0) + x f'(0) = 1 + \frac{1}{2}x$$

Inserting our newfound Taylor expansion into the relativistic "Doppler" formula we find

$$\frac{\Delta\lambda}{\lambda_{\text{shell}}} = \frac{1}{\sqrt{1 - \frac{2M}{r}}} - 1 \approx T_1 f(2M/r) - 1 = f(0) + \frac{2M}{r} f'(0) - 1 = 1 + \frac{1}{2}\frac{2M}{r} - 1$$
$$= \frac{M}{r}$$

4. We will now study how the relativistic doppler formula acts on the light from the sun. We will assume that the wave length of the light from the sun is at maximum $\lambda_{\text{max}} = 500\,nm$.

   (a) First we need to find the mass of our sun in meters. Equation 3 tells us how to converte from kilos to meters.

   $$M_m = \frac{G}{c^2} M_{kg} = 7.4259 \cdot 10^{-28}\,m/kg \cdot 2 \cdot 10^{30}\,kg \approx 1485.18\,m$$

   (b) we can now find the ratio M/r for the sun (the radius of the sun is listed under constants).

   $$\frac{M}{r} \approx \frac{1485.18\,m}{695\,508 \cdot 10^3\,m} \approx 2.1354 \cdot 10^{-6}$$

   (c) We now want to find the redshift measured by a far-away observer. Since $\frac{2M}{r} \ll 1$ we are allowed to use our Taylor expansion $\frac{\Delta\lambda}{\lambda} = \frac{M}{r} \approx 2.1354 \cdot 10^{-6}$.
   Alternatively we can use the full expression:

   $$\frac{\Delta\lambda}{\lambda} = \frac{1}{\sqrt{1 - \frac{2M}{r}}} - 1 \approx \frac{1}{\sqrt{1 - 2 \cdot 2.1354 \cdot 10^{-6}}} - 1 \approx 2.1354 \cdot 10^{-6}$$

   (d) The color of the sun observed by a far-away observer will be

   $$\frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda_{\text{shell}}}{\lambda_{\text{shell}}} = \frac{1}{\sqrt{1 - \frac{2M}{r}}} - 1$$
   $$\frac{\lambda'}{\lambda_{\text{shell}}} = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$$
   $$\lambda' = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \lambda_{\text{shell}} \approx \frac{1}{\sqrt{1 - 2 \cdot 2.1354 \cdot 10^{-6}}} \cdot 500 \cdot 10^{-9}\,m \approx 500.001\,nm$$

   The apparent color of the sun will therefore not change.

(e) We now want to study how light from a sun-like star is affected by the Earth's gravitational pull (the mass and radius of the Earth are listed under constants). We begin by finding Earth's mass in meters

$$M_{m,\text{Earth}} = \frac{G}{c^2} M_{kg,\text{Earth}} \approx 7.4259 \cdot 10^{-28}\, m/kg \cdot 5.972 \cdot 10^{24}\, kg \approx 4.4347 \cdot 10^{-3}\, m,$$

hence the mass to radius ratio becomes

$$\frac{M}{r} \approx \frac{4.4347 \cdot 10^{-3}\, m}{6\,371 \cdot 10^3\, m} \approx 6.9608 \cdot 10^{-10}$$

(f) We are now almost ready to find the Doppler shift caused by the Earth. Notice that we need to redefine who the far-away and shell observer is (the observer on Earth will be the shell observer and the one close to the sun will be the far-away observer). To be consistent, we will therefore switch the places in the Doppler formula (the ratio between the two wave lengths will be the same even if we do not switch the places).

$$\frac{\Delta\lambda}{\lambda'} = \frac{\lambda_{\text{shell}} - \lambda'}{\lambda'} = \frac{\Delta t_{\text{shell}}}{\Delta t'} - 1$$

$$= \frac{\Delta t_{\text{shell}}}{\Delta t_{\text{shell}}/\sqrt{1 - \frac{2M}{r}}} - 1 = \sqrt{1 - \frac{2M}{r}} - 1$$

$$\approx \sqrt{1 - 2 \cdot 6.9608 \cdot 10^{-10}} - 1 \approx -6.9608 \cdot 10^{-10}$$

Thus the observed wave length on earth is

$$\frac{\Delta\lambda}{\lambda'} = \frac{\lambda_{\text{shell}}}{\lambda'} - 1 = \sqrt{1 - \frac{2M}{r}} - 1$$

$$\lambda_{\text{shell}} = \lambda'\sqrt{1 - \frac{2M}{r}} \approx 500\,nm \cdot \sqrt{1 - 6.9608 \cdot 10^{-10}}$$

$$\approx 500.000\,nm$$

5. We are now looking at waves observed from a quasar and we want to find the distance r in terms of M. The shell observer will be the observer close to the quasar and the far-away observer will be us on Earth. We are given $\lambda' = 600\,nm$ and $\lambda_{\text{shell}} = 2150\,nm$. Rearranging the the relativistic Doppler formula and putting in numbers, we find:

$$\frac{\Delta\lambda}{\lambda_{\text{shell}}} = \frac{\lambda'}{\lambda_{\text{shell}}} - 1 = \frac{1}{\sqrt{1 - \frac{2M}{r}}} - 1$$

$$1 - \frac{2M}{r} = \left(\frac{\lambda_{\text{shell}}}{\lambda'}\right)^2$$

$$\frac{2M}{r} = 1 - \left(\frac{\lambda_{\text{shell}}}{\lambda'}\right)^2$$

$$r = \frac{2M}{1 - \left(\frac{\lambda_{\text{shell}}}{\lambda'}\right)^2}$$

$$r = \frac{2}{1 - \left(\frac{600\,nm}{2150\,nm}\right)^2} M \approx 2.1689M$$

We know from lecture note 2C that the Schwarzschild radius of a black hole is located at $r = 2M$. Since the light we measure is is emitted at a distance $r \approx 2M$, it would make sence that there is a black hole at the center of the quasar (light closer to the black hole then $r = 2M$ will just fall into it).

6. We are now situated at $r = 2.01M$ as a shell observer around a black hole with mass M. We want to find the Doppler shift from the stars around us. The Doppler formula gives us

$$\frac{\Delta\lambda}{\lambda'} = \frac{\lambda_{\text{shell}}}{\lambda'} - 1 = \sqrt{1 - \frac{2M}{r}} - 1$$

$$\lambda_{\text{shell}} = \lambda'\sqrt{1 - \frac{2M}{r}} = \lambda'\sqrt{1 - \frac{2M}{2.01M}} \approx 0.0705\lambda'$$

This means that we would need a telescope that can observe x-rays to look at the different stars.

Exercise 2C.2

We are looking at two shell observers: one close to the black hole far from the black hole. We want to study how they experience time differently.

1. First we need to find the mass of the black hole in our relativistic units. Equation 3 tells us how to convert mass from kilo into meters.

$$\frac{M_m}{M_{kg}} = \frac{G}{c^2}$$

(Check ... for solution)

2. Here you have to check MCast for solution. We suggest that you create a similar table to table 1. Note that all times are in your own frame.

Time	Wake up	Breakfast	Lunch	Dinner	Brush teeth	Bed
t (your schedule)	t_1	t_2	t_3	t_4	t_5	t_6
t' (partners schedule)	t_1'	t_2'	t_3'	t_4'	t_5'	t_6'

Table 1: A table for the different times in your frame.

3. Having found the time of your partners routine, you should now converte the time from seconds to meters. Equation 4 tells us how to do this.

$$t_s \cdot c = t_m$$

(Check ... for solution)

4. We know how to converte between time for the shell and far-away observer, but not between different shell observers. We can think of our situation as having two shells: shell 1 (closest to the black hole at a distance r_1) and shell 2 (furthest away at a distance r_2). Using the relation between shell and far-away time (equation 5) we find

$$\Delta t_{\text{shell 1}} = \Delta t \sqrt{1 - \frac{2M}{r_1}} \qquad \Rightarrow \qquad \Delta t = \frac{\Delta t_{\text{shell 1}}}{\sqrt{1 - \frac{2M}{r_1}}}$$

$$\Delta t_{\text{shell 2}} = \Delta t \sqrt{1 - \frac{2M}{r_2}} \qquad \Rightarrow \qquad \Delta t = \frac{\Delta t_{\text{shell 2}}}{\sqrt{1 - \frac{2M}{r_2}}}$$

Inserting one into the other we find

$$\Delta t_{\text{shell 2}} = \Delta t \sqrt{1 - \frac{2M}{r_2}} = \frac{\Delta t_{\text{shell 1}}}{\sqrt{1 - \frac{2M}{r_1}}} \sqrt{1 - \frac{2M}{r_2}} = \frac{\sqrt{1 - \frac{2M}{r_2}}}{\sqrt{1 - \frac{2M}{r_1}}} \Delta t_{\text{shell 1}}$$

5. It should now be easy to transforme between the different times with the above formula, but remember that you have the time in meters and not days and hours. To converte back to days and hours we do like this:

$$t_{\text{hours}} = \frac{t_s}{60 \cdot 60} = \frac{t_m \cdot c}{3600}$$

(Check ... for solution)

6. **Slå sammen 5 og 6**

7. Talk with your partner.

Exercise 2C.3

In this exercise we will show that angular momentum per mass is conserved in general relativity, using the principal of maximal aging. The deduction is very similar to those done in the lecture note.

1. First we want to find an expression for $\Delta\tau_{13} = \Delta\tau_{12} + \Delta\tau_{23}$. We will find $\Delta\tau_{12}$ and $\Delta\tau_{23}$ separately. We know that if we look at a very small movement in space, we can assume that the radius r in the Schwarzschild line element is constant. Since the proper time is always equal to Schwarzschild line element we find

$$\Delta\tau_{12}^2 = \Delta s_{12}^2 = \left(1 - \frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}^2}{1 - \frac{2M}{r_A}} - r_A^2 \Delta\phi_{12}^2$$

$$\Delta\tau_{23}^2 = \Delta s_{23}^2 = \left(1 - \frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}^2}{1 - \frac{2M}{r_B}} - r_B^2 \Delta\phi_{23}^2$$

Taking the square root we only get the positive solution here since we look at time moving in positive direction. The proper time between position 1 and 3 (since proper time is linear) must be $\Delta\tau_{13} = \Delta\tau_{12} + \Delta\tau_{23}$. Using the expressions we found above we get

$$\Delta\tau_{13} = \Delta\tau_{12} + \Delta\tau_{23}$$

$$= \sqrt{\left(1 - \frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}^2}{1 - \frac{2M}{r_A}} - r_A^2 \Delta\phi_{12}^2} + \sqrt{\left(1 - \frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}^2}{1 - \frac{2M}{r_B}} - r_B^2 \Delta\phi_{23}^2}$$

2. If we now take the derivative of the proper time $\Delta\tau_{13}$ with respect to the angle ϕ_2, the principle of maximal aging tells us that the object will choose the angle which gives the largest proper time. In other words, we want to find the maximum of $\tau_{13}(\phi_2)$, hence we want to find the value of ϕ_2 such that $\frac{d\tau_{13}}{d\phi_2} = 0$. **Hvorfor er dette toppunkt?**

$$\frac{d\tau_{13}}{d\phi_2} = \frac{d\tau_{12}}{d\phi_2} + \frac{d\tau_{23}}{d\phi_2} = \frac{\sqrt{\left(1 - \frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}^2}{1 - \frac{2M}{r_A}} - r_A^2 \Delta \phi_{12}^2}}{d\phi_2} + \frac{\sqrt{\left(1 - \frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}^2}{1 - \frac{2M}{r_B}} - r_B^2 \Delta \phi_{23}^2}}{d\phi_2}$$

$$= \frac{1}{2\sqrt{\left(1 - \frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}^2}{1 - \frac{2M}{r_A}} - r_A^2 \Delta \phi_{12}^2}} \frac{\left(1 - \frac{2M}{r_A}\right)\Delta t_{12}^2 - \frac{\Delta r_{12}^2}{1 - \frac{2M}{r_A}} - r_A^2(\phi_2 - \phi_1)^2}{d\phi_2}$$

$$+ \frac{1}{2\sqrt{\left(1 - \frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}^2}{1 - \frac{2M}{r_B}} - r_B^2 \Delta \phi_{23}^2}} \frac{\left(1 - \frac{2M}{r_B}\right)\Delta t_{23}^2 - \frac{\Delta r_{23}^2}{1 - \frac{2M}{r_B}} - r_B^2(\phi_3 - \phi_2)^2}{d\phi_2}$$

$$= \frac{1}{2}\frac{1}{d\tau_{12}}\left(\left(1 - \frac{2M}{r_A}\right)\frac{\Delta t_{12}^2}{d\phi_2} - \frac{1}{1 - \frac{2M}{r_A}}\frac{\Delta r_{12}^2}{d\phi_2} - r_A^2\frac{(\phi_2 - \phi_1)^2}{d\phi_2}\right)$$

$$+ \frac{1}{2}\frac{1}{d\tau_{23}}\left(\left(1 - \frac{2M}{r_B}\right)\frac{\Delta t_{23}^2}{d\phi_2} - \frac{1}{1 - \frac{2M}{r_B}}\frac{\Delta r_{23}^2}{d\phi_2} - r_B^2\frac{(\phi_3 - \phi_2)^2}{d\phi_2}\right)$$

$$= \frac{1}{2}\frac{1}{d\tau_{12}}(0 - 0 - 2r_A^2 d\phi_{12}) \cdot 1 + \frac{1}{2}\frac{1}{d\tau_{23}}(0 - 0 - 2r_B^2 d\phi_{23}) \cdot (-1)$$

$$= -\frac{1}{2}\frac{1}{d\tau_{12}}(2r_A^2 d\phi_{12}) + \frac{1}{2}\frac{1}{d\tau_{23}}(2r_B^2 d\phi_{23})$$

Setting this equal to zero and equation we find

$$\frac{r_A^2 d\phi_{12}}{d\tau_{12}} = \frac{r_B^2 d\phi_{23}}{d\tau_{23}}$$

3. What we have found is true for all intervals $[a_0, a_1]$ small enough. Thus if the interval was larger, for instance $[a = a_0, a_n = b]$, we could separate it to n small enough intervals so that the equation over holds for every intervall. By the principal of induction we get $\frac{d\phi_{ab}}{d\tau_{ab}}r_{ab/2}^2 = \text{constant}$. In other words,

$$\frac{r^2 d\phi}{d\tau} = \text{constant}$$

regardless of the intervall.

4. We remember from celest mechanics that angular velocity, measured by the shell observer, is given by $v_\phi = r\frac{d\phi}{dt_{\text{shell}}}$. We also remember from spesial relativity that $\frac{dt_{\text{shell}}}{d\tau} = \gamma_{\text{shell}}$. Using that $\frac{dt_{\text{shell}}}{dt_{\text{shell}}} = 1$ we therefore find

$$\frac{r^2 d\phi}{d\tau} = r\left(r\frac{d\phi}{dt_{\text{shell}}}\frac{dt_{\text{shell}}}{d\tau}\right) = rv_\phi \gamma_{shell}$$

5. We know from mechanics that spin is given by $L = |\vec{r} \times \vec{p}| = r \cdot v_\phi$. For small velocities $v_{\text{shell}} \ll 1$ we have $\gamma_{\text{shell}} = \frac{1}{1 - v_{shell}^2} \approx \frac{1}{1 - 0^2} = 1$. Thus

$$\frac{r^2 d\phi}{d\tau} = rv_\phi \gamma_{shell} = r\frac{mv_\phi}{m} \cdot 1 = \frac{L}{m}$$

Exercise 2C.4

1. In the lecture notes we have found that equation 7 can be written as

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1,$$

When the velocity $v = 0$ and the distance $r \to \infty$. This gives us

$$d\tau = \left(1 - \frac{2M}{r}\right)dt$$

2.

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{\frac{d\tau}{1-\frac{2M}{r}}}\right)^2 = \left(1 - \frac{2M}{r}\right)^2 \frac{dr^2}{\left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2M}{r}}}$$

$$= \left(1 - \frac{2M}{r}\right)\frac{dr^2}{dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}} = \left(1 - \frac{2M}{r}\right)\frac{dr^2/dt^2}{1 - \frac{dr^2/dt^2}{\left(1-\frac{2M}{r}\right)^2}}$$

$$\nu^2 = \alpha\frac{\nu^2}{1 - \frac{\nu^2}{\alpha^2}} = \frac{\alpha^3}{\frac{\alpha^2}{\nu^2} - 1}$$

$$\nu^2\left(\frac{\alpha^2}{\nu^2} - 1\right) = \alpha^3$$

$$\alpha^2 - \nu^2 = \alpha^3$$

$$\nu^2 = \alpha^2(1 - \alpha)$$

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right)^2\left[1 - \left(1 - \frac{2M}{r}\right)\right] = \left(1 - \frac{2M}{r}\right)^2\frac{2M}{r}$$

3. Velocity is the change in position over the change of time, in other words, $v = \frac{dr}{dt}$. Hence we have

$$v = \pm\sqrt{\left(\frac{dr}{dt}\right)^2} = \pm\sqrt{\left(1 - \frac{2M}{r}\right)^2\frac{2M}{r}} = -\left(1 - \frac{2M}{r}\right)\sqrt{\frac{2M}{r}}$$

The minus sign comes from the fact that the spaceship is traveling towards the black hole.

4. We have in the lecture notes derived equation 5 and 6 as the transformation between shell and far-away observers. Hence

$$v_{\text{shell}} = \frac{dr_{\text{shell}}}{dt_{\text{shell}}} = \frac{\frac{dr}{\sqrt{1-\frac{2M}{r}}}}{dt\sqrt{1 - \frac{2M}{r}}} = \frac{-\left(1 - \frac{2M}{r}\right)\sqrt{\frac{2M}{r}}}{1 - \frac{2M}{r}} = -\sqrt{\frac{2M}{r}} \tag{9}$$

Exercise 2C.5

1. The satellite positioned at $1\,AU$ from the black hole will be the shell observer and the "falling" spaceship will be the freely falling observer.

2. The equation for conservation of energy 7 and the tranformation between shell and far-away observers time 5 tells us

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right) dt \frac{1}{d\tau} = \left(1 - \frac{2M}{r}\right) \frac{dt_{\text{shell}}}{\sqrt{1 - \frac{2M}{r}}} \frac{1}{d\tau} = \sqrt{1 - \frac{2M}{r}} \frac{dt_{\text{shell}}}{d\tau} = \sqrt{1 - \frac{2M}{r}} \gamma_{\text{shell}}$$

3. (Check ... for solution)

4.

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$$

$$d\tau = \frac{1 - \frac{2M}{r}}{E/m} dt$$

5.

$$d\tau = \frac{1 - \frac{2M}{r}}{E/m} dt = \frac{1 - \frac{2M}{r}}{E/m} \frac{dt_{\text{shell}}}{\sqrt{1 - \frac{2M}{r}}} = \frac{1 - \frac{2M}{r}}{E/m} \frac{dt_{\text{shell}}}{\sqrt{1 - \frac{2M}{r}}} = \frac{\sqrt{1 - \frac{2M}{r}}}{E/m} dt_{\text{shell}}$$

6.

$$d\tau = \frac{\sqrt{1 - \frac{2M}{r}}}{E/m} dt_{\text{shell}}$$

$$\left(\frac{d\tau}{dt_{\text{shell}}} E/m\right)^2 = 1 - \frac{2M}{r}$$

$$r = \frac{2M}{1 - \left(\frac{d\tau}{dt_{\text{shell}}} E/m\right)^2}$$

(Check ... for solution)

7. (Check ... for solution)

8. We know $d\tau = \frac{\sqrt{1 - \frac{2M}{r}}}{E/m} dt_{\text{shell}}$, so when $r \to 2M$, $d\tau \to 0$ and $dt_{\text{shell}} \to \infty$. That is, the time difference increases for the shell observer and decreases for the freely falling observer.

9. When we get to close to the Schwarzschild horizon, we will see all of time passing by. This means that the signals from the shell observer will end up as a constant signal, just as in the video.

Exercise 2C.6

1. Using consevation of energy 7

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{\Delta t}{\Delta \tau}$$

$$\Delta t = \frac{E/m}{1 - \frac{2M}{r}}\Delta \tau$$

2. Using consevation of angular momentum 8

$$\frac{L}{m} = r^2\frac{\Delta \phi}{\Delta \tau}$$

$$\Delta \phi = \frac{L/m}{r^2}\Delta \tau$$

3. Using 1 we find

$$\Delta\tau^2 = \Delta s^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{1 - \frac{2M}{r}} - r^2\Delta\phi^2$$

$$= \left(1 - \frac{2M}{r}\right)\left(\frac{E/m}{1 - \frac{2M}{r}}\Delta\tau\right)^2 - \frac{\Delta r^2}{1 - \frac{2M}{r}} - r^2\left(\frac{L/m}{r^2}\Delta\tau\right)^2$$

$$= \frac{(E/m)^2}{1 - \frac{2M}{r}}\Delta\tau^2 - \frac{\Delta r^2}{1 - \frac{2M}{r}} - \frac{(L/m)^2}{r^2}\Delta\tau^2$$

$$\left(1 - \frac{2M}{r}\right)\Delta\tau^2 = (E/m)^2\Delta\tau^2 - \Delta r^2 - \left(1 - \frac{2M}{r}\right)\frac{(L/m)^2}{r^2}\Delta\tau^2$$

$$\Delta r^2 = (E/m)^2\Delta\tau^2 - \left(1 - \frac{2M}{r}\right)\frac{(L/m)^2}{r^2}\Delta\tau^2 - \left(1 - \frac{2M}{r}\right)\Delta\tau^2$$

$$= \left((E/m)^2 - \left[\frac{(L/m)^2}{r^2} + 1\right]\left(1 - \frac{2M}{r}\right)\right)\Delta\tau^2$$

$$\Delta r = \pm\sqrt{\left(\frac{E}{m}\right)^2 - \left[1 + \left(\frac{L/m}{r}\right)^2\right]\left(1 - \frac{2M}{r}\right)}\Delta\tau^2$$

Exercise 2C.7

1.

$$\Delta\tau^2 = \Delta(s')^2 = \left(1 - \frac{2M}{r + \Delta r}\right)\Delta t^2 - \overbrace{\frac{\Delta r^2}{1 - \frac{2M}{r+\Delta r}}}^{=0} - (r + \Delta r)^2\Delta\phi_{\text{plane}}^2$$

$$= \left(1 - \frac{2M}{r + \Delta r}\right) - (r + \Delta r)^2\frac{\Delta\phi_{\text{plane}}^2}{\Delta t^2} = 1 - \frac{2M}{r + \Delta r} - v_{\text{plane}}^2$$

$$\Delta t_{\text{earth}}^2 = \Delta s^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \overbrace{\frac{\Delta r^2}{1 - \frac{2M}{r}}}^{=0} - r^2\Delta\phi_{\text{earth}}^2$$

$$= \left(1 - \frac{2M}{r}\right) - r^2\frac{\Delta\phi_{\text{earth}}^2}{\Delta t^2} = 1 - \frac{2M}{r} - v_{\text{earth}}^2$$

$$\frac{\Delta\tau}{\Delta t_{\text{earth}}} = \sqrt{\frac{1 - \frac{2M}{r + \Delta r} - v_{\text{plane}}^2}{1 - \frac{2M}{r} - v_{\text{earth}}^2}}$$

2. We can change units with equations 3 and 4. This gives

$$M_m = \frac{G}{c^2}M_{kg} = \frac{6.67408 \cdot 10^{-11}\,\frac{m^3}{kg\,s^2}}{(299\,792\,458\,m/s)^2}5.972 \cdot 10^{24}\,kg \approx 4.435 \cdot 10^{-3}\,m$$

$$\frac{M_m}{r} = \frac{4.435 \cdot 10^{-3}\,m}{6371 \cdot 10^3\,m} \approx 6.961 \cdot 10^{-10}$$

$$v_{\text{plane}} = \frac{1000/3.6}{c} \approx 9.267 \cdot 10^{-7}$$

$$v_{\text{Earth}} = \frac{2\pi \cdot 6371 \cdot 10^3}{24 \cdot 60 \cdot 60}\frac{1}{c} \approx 1.545 \cdot 10^{-6}$$

3. We choose $x = -\left(\frac{2M}{r+\Delta r} + v_{\text{plane}}^2\right)$ and $y = -\left(\frac{2M}{r} + v_{\text{earth}}^2\right)$. We can now create a Taylor expansion of $f(x) = \sqrt{1 + x}$ and $g(y) = \frac{1}{\sqrt{1+y}}$.

$$f'(x) = \left[(1+x)^{1/2}\right]' = \frac{1}{2}(1+x)^{-1/2} \qquad \Rightarrow \qquad f'(0) = \frac{1}{2}$$

$$g'(y) = \left[(1+y)^{-1/2}\right]' = -\frac{1}{2}(1+y)^{-3/2} \qquad \Rightarrow \qquad g'(0) = -\frac{1}{2}$$

The Taylor expansion thus becomes $f(x) \approx T_1 f(x) = f(0) + xf'(0) = 1 + \frac{1}{2}x$ and $g(y) \approx T_1 g(y) = g(0) + xg'(0) = 1 - \frac{1}{2}y$. Hence we get

$$\frac{\Delta\tau}{\Delta t_{\text{earth}}} = \sqrt{\frac{1 - \frac{2M}{r+\Delta r} - v_{\text{plane}}^2}{1 - \frac{2M}{r} - v_{\text{earth}}^2}} \approx T_1 f(x) \cdot T_1 g(y) = 1 + \frac{1}{2}x - \frac{1}{2}y - \overbrace{\frac{1}{4}xy}^{\approx 0}$$

$$\approx 1 + \frac{1}{2}\left[-\left(\frac{2M}{r+\Delta r} + v_{\text{plane}}^2\right)\right] - \frac{1}{2}\left[-\left(\frac{2M}{r} + v_{\text{earth}}^2\right)\right]$$

$$= 1 + \frac{1}{2}\left[\frac{2M}{r} - \frac{2M}{r+\Delta r}\right] + \frac{1}{2}\left[v_{\text{earth}}^2 - v_{\text{plane}}^2\right]$$

$$= 1 + \frac{1}{2}\left[v_{\text{earth}}^2 - v_{\text{plane}}^2\right] + M\left[\frac{1}{r} - \frac{1}{r+\Delta r}\right]$$

4.

$$\frac{d\tau}{dt} = 1 + \frac{1}{2}\left[v_{\text{earth}}^2 - v_{\text{plane}}^2\right] + M\left[\frac{1}{r} - \frac{1}{r+\Delta r}\right]$$

$$\approx 1 + \frac{1}{2}\left[(1.545 \cdot 10^{-6})^2 - (9.267 \cdot 10^{-7})^2\right] + 4.435 \cdot 10^{-3}\left[\frac{1}{6371 \cdot 10^3} - \frac{1}{(6371 + 10) \cdot 10^3}\right] \approx 1$$

The differance is of order 10^{-12}, so almost nothing at all.

5. The difference is barely anything. A better reason would be to save the environment.

Exercise 2C.8

1.
$$|\vec{r}| = \sqrt{x^2 + y^2}$$

2. We remember Keplers' 2. law as

$$P^2 = \frac{4\pi^2}{G\underbrace{m_1 + m_2}_{=M}}a^3 \tag{10}$$

If we assume circular motion (that is $a = r$) and use that the period of the satellites orbits can be written as $P = t = \frac{s}{v} = \frac{2\pi r}{v_\theta}$, equation 10 becomes

$$\left(\frac{2\pi r}{v_\theta}\right)^2 = \frac{4\pi^2 r^3}{GM}$$

$$2\pi r = \pm 2\pi r\sqrt{\frac{r}{GM}}v_\theta$$

$$v_\theta = \pm\sqrt{\frac{GM}{r}}$$

15

3. **Trenger figur**

Law of cosine:

$$C^2 = A^2 + B^2 - AB\cos\theta \tag{11}$$

$$\vec{r}_{\text{sat}} = \begin{vmatrix} x_{\text{sat}} \\ y_{\text{sat}} \end{vmatrix} \qquad \text{and} \qquad \vec{r} = \begin{vmatrix} x \\ y \end{vmatrix} \qquad \Rightarrow \qquad \vec{r}_{\text{sat}} - \vec{r} = \begin{vmatrix} x_{\text{sat}} - x \\ y_{\text{sat}} - y \end{vmatrix}$$

$$(c\Delta t)^2 = |\vec{r}_{\text{sat}} - \vec{r}|^2 = |\vec{r}|^2 + |\vec{r}_{\text{sat}}|^2 - 2|\vec{r}||\vec{r}_{\text{sat}}|\cos\alpha$$

$$\alpha = \arccos\left(\frac{|\vec{r}|^2 + |\vec{r}_{\text{sat}}|^2 - (c\Delta t)^2}{2|\vec{r}||\vec{r}_{\text{sat}}|}\right)$$

$$\tan\theta = \frac{y_{\text{sat}}}{x_{\text{sat}}} \Rightarrow \theta = \arctan\left(\frac{y_{\text{sat}}}{x_{\text{sat}}}\right)$$

4.

$$\Delta t_{\text{sat}}^2 = \Delta(s')^2 = \left(1 - \frac{2M}{|\vec{r}_{\text{sat}}|}\right)\Delta t^2 - \overbrace{\frac{\Delta r^2}{1 - \frac{2M}{|\vec{r}_{\text{sat}}|}}}^{=0} - |\vec{r}_{\text{sat}}|^2 \Delta\theta_{\text{sat}}^2$$

$$= \left(1 - \frac{2M}{|\vec{r}_{\text{sat}}|}\right) - |\vec{r}_{\text{sat}}|^2 \frac{\Delta\theta_{\text{sat}}^2}{\Delta t^2} = 1 - \frac{2M}{|\vec{r}_{\text{sat}}|} - v_{\text{sat}}^2$$

$$\Delta t_{\text{earth}}^2 = \Delta s^2 = \left(1 - \frac{2M}{|r|}\right)\Delta t^2 - \overbrace{\frac{\Delta r^2}{1 - \frac{2M}{|r|}}}^{=0} - r^2 \Delta\phi_{\text{earth}}^2$$

$$= \left(1 - \frac{2M}{|r|}\right) - |r|^2 \frac{\Delta\phi_{\text{earth}}^2}{\Delta t^2} = 1 - \frac{2M}{r} - v_{\text{earth}}^2$$

$$\frac{\Delta t_{\text{sat}}}{\Delta t_{\text{earth}}} = \sqrt{\frac{1 - \frac{2M}{|\vec{r}_{\text{sat}}|} - v_{\text{sat}}^2}{1 - \frac{2M}{|r|} - v_{\text{earth}}^2}}$$

5. (Check ... for solution)

6. (Check ... for solution)

7. The more time that passes, the more wrong the expression $|\vec{r}_{\text{sat}} - r| = c\Delta t$ becomes since Δt is wrong. After a couple of days, the GPS will be totaly useless and just give wrong positions.
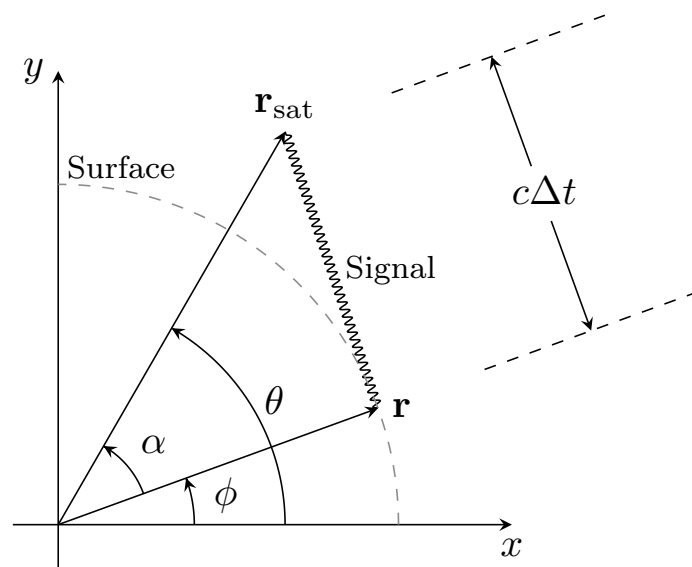
Figure 1: en Hei