Solutions to exercises in part 2A

Contents

Usefull formulas

Law of constant motion (in one dimention)

If an object moves a distance Δx with constant velocity v in the time interval Δt, the velocity is given by

$$v = \frac{\Delta x}{\Delta t} \tag{1}$$

The Lorentz transformation (in one dimention)

Assume that we have two coordinate systems: one moving along the x-axis with velocity v, and another one at rest. If the coordinates for an event is given by (x', t') in the moving frame and by (x, t) in the non-moving frame, then we can converte the coordinates between the two frames as follows:

$$t = v\gamma x' + \gamma t' \tag{2}$$
$$x = \gamma x' + v\gamma t' \tag{3}$$

$$t' = -v\gamma x + \gamma t \tag{4}$$
$$x' = \gamma x - v\gamma t \tag{5}$$

Here $\gamma = \frac{1}{\sqrt{1-v^2}}$.

Time dilation and length contraction

Assume that we have two coordinate systems: one moving along the x-axis with velocity v, and another one at rest. If the time for an event is given by $\Delta t'$ in the moving frame and by Δt in the non-moving frame, then we can find the time in one frame given the time in the other as follows:

$$\Delta t = \gamma \Delta t' \tag{6}$$

If we know that the distance between the two events are $\Delta x' = L_0$ in the moving frame and $\Delta x = L$ in the non-moving frame, then we can find the distance in one frame given the distance in the other as follows:

$$L = \frac{L_0}{\gamma} \tag{7}$$

Here $\gamma = \frac{1}{\sqrt{1-v^2}}$. Both of these equations can be derived from the Lorenz transformations.

Exercise 2A.1

1. Since both of the cylinders are centered around the same axis, cylinder B pass trough cylinder A if it has sufficiently large velocity (assuming that the radius of cylinder B shrinks with velocity along the x-axis).
2. For an observer positioned on cylinder B, cylinder A will from his point of view be moving towards him, while cylinder B is standing still. Thus the radius of sylinder A will shrink and therefore pass trough cylinder B.
3. The observations made in the earlier tasks shows that whichever cylinder passes through which depends on the referencesystem. Therefor z=z' and y=y' must be true to avoid contradictions.

Exercise 2A.2

1. Find answer in MCast
2. Find answer in MCast
3. The speed of the spaceship without acceleration is given by the law of constant velocity 1

$$v = \frac{s}{t}$$

In the spaceship frame the roles are changed and it is the planet that has velocity $v = \frac{s}{t}$. (The numerical answer can be found in document ...)

4. The space-time interval is defined as

$$\Delta s^2 = \Delta t^2 - \Delta x^2$$

where Δt is time difference between event A (spaceship enters the atmosphere) and G (spaceship hits the ground), while Δx is the distance between the two events. The spacetime interval thus becomes (remember that units of Δt and Δx must be equal)

$$\begin{aligned} \text{(spaceship frame:)} \qquad & \Delta(s'_{AG})^2 = (\Delta t'_{AG})^2 - 0^2 = (t'_G - t'_A)^2 \\ \text{(planet frame:)} \qquad & \Delta s^2_{AG} = \Delta t^2_{AG} - \Delta x^2_{AG} = (t_G - t_A)^2 - \frac{x_A^2}{c^2} \end{aligned}$$

where x_A is the distance from the ground to the beginning of the atmosphere.
(Remember that the spaceship is always centered at $x = 0$ in the rest frame of the spaceship, thus $\Delta x'_{AG} = 0$. Look up the definition of proper time in the lecture notes.)

(The numerical answer can be found in document ...)

5. Invariance of the space-time interval gives us (see pervious task for the other space-time interval)

$$
\begin{aligned}
(\Delta s'_{AG})^2 &= \Delta s_{AG}^2 \\
(\Delta t'_{AG})^2 &= \Delta t_{AG}^2 - \frac{\Delta x_{AG}^2}{c^2}
\end{aligned}
$$

Equation 1 (definition of constant velocity) gives us $\Delta x_{AG} = \Delta t_{AG} \cdot v$. Thus

$$
\begin{aligned}
(\Delta t'_{AG})^2 &= \Delta t_{AG}^2 - \Delta t_{AG}^2 \frac{v^2}{c^2} \\
(\Delta t'_{AG})^2 &= \Delta t_{AG}^2 \left(1 - (v/c)^2\right) \\
\Delta t'_{AG} &= \Delta t_{AG} \sqrt{1 - \nu^2} = \frac{\Delta t_{AG}}{\gamma}
\end{aligned}
\tag{8}
$$

which can be solved for the desired frame. Be aware that $\nu = v/c$. (The numerical answer can be found in document ...)

6. We are now looking at muons that are produced high up in the atmosphere, traveling at velocity $v = 0.999c$. We want to find the time a muon would use to arrive at the surface of the earth.
Assuming the muon has constant velocity, equation 1 gives us

$$
\Delta t_{AG} = \frac{\Delta x_{AG}}{v} = \frac{15km}{0.999c} \approx 5.005 \cdot 10^{-5}\, s.
$$

Hence the muon uses approximentely $50\,\mu s$ to reach the ground (in the planet frame).

7. Since

$$
50\,\mu s = \Delta t_{AG} > \Delta t_{\text{Muon}} = 2\,\mu s,
$$

where Δt_{Muon} is the mean lifetime of a muon, most muons will never reach the surface of the earth (when we do not take into account relativistic effects).

8. Thinking of the muon as the spaceship from above, we see that this situation is equivalent with the one above. In other words, solving task 1-5 symbolically, we find exactly the expression we want. Hence equation 8 (which is the equation for time dilation) gives us

$$
t' = \frac{t}{\gamma} = 5 \cdot 10^{-5} \sqrt{1 - 0.999^2} \approx 2.2\,\mu s
$$

Thus a large ammount of muons will be able to reach earth when relativistic effects are taken into account.

Exercise 2A.3

1. **For both frames:**

   (a) Let event Y be the emission of the yellow light, and event B be the emission of the blue light. The time and position for both events should be found in MCast. Here we will only use symbols. **Remember that you should know all the values in your own frame of referance.**

Time or position	Event Y	Event B
t	t_Y	t_B
x	x_Y	x_B
t'	t'_Y	t'_B
x'	$x'_Y = 0$	$x'_B = 0$

Table 1: The different times and positions for task 2A.3 1. (You should look up these in MCast.)

   (b) The time intervals between the two lights are $\Delta t_{YB} = t_Y - t_B$ and $\Delta t'_{YB} = t'_Y - t'_B$ (put in the numbers you found in MCast for numerical answer. Correct answers are given in document ...).

2. The solution to this task is given in the next task.

3. (frames are specified when needed below)

   (a) <u>**Spaceship frame:**</u> The lights are emitted from the spaceship. Therefore, since the spaceship is moving with a velocity v, the positions of the events (in the planet frame) are

$$x_Y = v \cdot t_Y$$
$$x_B = v \cdot t_B$$

   <u>**Planet frame:**</u> The lights are emitted from the spaceship. Therefore, since the spaceship always is centered at the origin in the spaceship frame, the positions of the events (in the spaceship frame) are

$$x'_Y = 0$$
$$x'_B = 0$$

   (b) The space-time interval for the <u>planet frame</u> is

$$\begin{aligned}
\Delta s_{YB}^2 &= \Delta t_{YB}^2 - \Delta x_{YB}^2 = (t_B - t_Y)^2 - (x_B - x_Y)^2 \\
&= (t_B - t_Y)^2 - (v \cdot t_B - v \cdot t_Y)^2 = (t_B - t_Y)^2 - v^2 (t_B - t_Y)^2 \\
&= (t_B - t_Y)^2 (1 - v^2) = \Delta t_{YB}^2 (1 - v^2) \\
&= \frac{\Delta t_{YB}^2}{\gamma^2}
\end{aligned}$$

The space-time interval for the spaceship frame is

$$\begin{aligned}(\Delta s'_{YB})^2 &= (\Delta t'_{YB})^2 - (\Delta x'_{YB})^2 \\ &= (t'_B - t'_Y)^2 - (x'_B - x'_Y)^2 \\ &= (t'_B - t'_Y)^2 - 0^2 \\ &= (\Delta t'_{YB})^2 = \Delta\tau^2\end{aligned}$$

(c) Using invarianse of the space-time interval we find that

$$\begin{aligned}(\Delta s'_{YB})^2 &= \Delta s^2_{YB} \\ (\Delta t'_{YB})^2 &= \frac{\Delta t^2_{YB}}{\gamma^2} \\ \Delta t'_{YB} &= \frac{\Delta t_{YB}}{\gamma}\end{aligned}$$

Which is exactly the expression for time dilation 6. (The numerical answer can be found in document ...)

4. From length contraction we know that $L = L_0/\gamma$, hence the length between the two lights should be largest in the rest frame. You should therefore observe a larger distance in the planet frame.

Exercise 2A.4

Part 1

We are studying two spaceships moving from left to right with velocity v in the planet frame (in the spaceship frame, the planet is moving from right to left). An observer M, right in between the two spaceships, also move from left to right with velocity v, thus it is in the same frame as the spaceships.
We let (x, t) and (x', t') denote the planet and spaceship frame respectively. In this exercise we are studying the four following events:

- Event A is when the left spaceship fires a deadly laser. This happens at $t_A = t'_A = 0$ and $x_A = x'_A = 0$.
- Event B is when the right spaceship fires a deadly laser. This happens simultaneously as event A in the spaceship frame.
- Event C is when the left spaceship explodes. This happens when the laser from the right spaceship hits the left spaceship.
- Event D is when the right spaceship explodes. This happens when the laser from the left spaceship hits the right spaceship.

In the first tasks we will pretend that the only thing we know about the theory of relativity is that the velocity of light is equal for all observers.

1. We want to show that the laser beams cross at observer M's position. The easiest way to see this is to understand that since nither the spaceships or observer M move in the spaceship frame, and the lasers have equal velocity, they will need to travel the same distance to reach observer M. Since they have equal absolute velocity, they must cross at observer M, at the same time.

   We can also deduce this answer more rigorously. Since we know that the velocity of light is equal for all observers, we know that light travels through space with velocity $c = 1$ (remember that velocity is unitless when working with relativity) in the spaceship frame. Thus the left laser travels with velocity $v_1 = c = 1$ and the right laser travels with velocity $v_2 = -c = -1$. The spaceships do not move with respect to the lasers in the spaceship frame, hence (since observer M is in the same frame of referance as the spaceships) observer M will always stay in the middle between the spaceships. Assuming that the distance between the two spaceships is L', the position of the lasers in the spaceship frame are (remember the definition of constant velocity 1)

   $$x_{\text{Left laser}} = c \cdot t = t \qquad \text{and} \qquad x_{\text{Right laser}} = L' - c \cdot t = L' - t$$

   The lasers will meet when $x_{\text{Left laser}} = x_{\text{Right laser}}$. Solving this equation with respect to time we find

   $$\begin{aligned} x_{\text{Left laser}} &= x_{\text{Right laser}} \\ t &= L' - t \\ t &= L'/2 \end{aligned}$$

   Inserting this time into the position for one of the lasers (or remebering that since time and position are measured in the same units) we get

   $$x_{\text{Left laser}} = L'/2 = x_{\text{Right laser}}$$

   Finally, since observer M's position is right in between the two spaceships, we must have

   $$x_M = x_{\text{Left laser}} = x_{\text{Right laser}} = L'/2$$

   In other words, the lasers will cross at observer M's position.

2. We want to explain why event A and B could not have been simultaneous in the planet frame. To explain this we need to explain why the lasers cross at the position of observer M in the planet frame too.
   The first axiom of relativity tells us that if an event occures in one frame of referance, it must occure in every frame of referance (this must be valid even though we do not know anything about relativity, if not, we can create paradoxes as explained in the lecture note). Since the lasers crossing at observer M can be regarded as an event, this must happen in every frame of referance, including the planet frame.
   We can now explain why the two events cannot be simultaneous in the planet frame.

Observer M moves with the spaceships at a velocity v, the left lasers move with velocity $c = 1$ and the right lasers move with velocity $-c = -1$ in the planet frame. This means that after a time Δt the left laser reaches the point where observer M was when it was fired, but observer M has now moved a distance $v\Delta t$ further. For the right laser we have an opposite case: before it reaches the point where observer M was when it was fired, observer M will have moved enough to reach the laser.
In other words, the left laser must travel a greater distance then the right laser. Since they are supposed to meet at M in all frames of referance and the two lasers have the same absolute velocity, the left laser must be fired first to give it enough time to reach observer M.

3. As explained above, the left laser must be emitted first.

4. We now move on to the explosions. We understand that these have to be simultaneous in the spaceship frame and we want to explain why they cannot be simultaneous in the planet frame.
In the pervious task we defined event A to be the origin of our spacetime coordinate system, but this need not be the origin. We now redefine the origin to be at the explosion of the left spaceship (we will only redefine the origin for this task to see the conclsion easier, but it is not needed and we will go back to our original definition of the origin after this task). Some of the light created from the explosion will travel towards M. This is true for both spaceships. Since the only thing that has changed from the pervious tasks is where the origin is and that we are observing light insted of a deadly laser, we have an identical case and can now think of event A and B as the explosion of the left and right spaceship respectively. Yet again the light from both explosion will cross at observer M's position and by our above deduction, we know that the explosions cannot be simultaneous in the planet frame (unless the velocity of the spaceships are $v = 0$).

5. It is not difficult to understand which explosion occures first in the planet frame now. Thinking of the light from the explosions as the lasers from event A and B, the above deduction tells us that the left spaceship must have exploded first.

6. The order of the events in the planet frame must be A, B, C and D:
We know that the left laser has a longer way to travel to reach observer M then the right laser, hence event A must occure first to give the left laser enough time to reach observer M simultaneos as the right laser. Since the two lasers are supposed to meet at observer M's position (before any of the spaceships explode), we know that event B must happen next.
We have explained that the light from the explosion must reach M simultaneously in the spaceship frame, thus event C (the left spaceship exploding) must occure next since the light from the explosion has a larger distance to travel to reach M. Finally event D happens last so that the light from both explosion meet at observer M.

7. We will solve task 7 and 8 in one go.

8. Let (x, t) and (x', t') denote the space-time coordinates in the planet and spaceship frame respectively. We are going to set up an event table for the four events A, B, C and D.

   We begin with some info. You should know the value of all the spaceship variables, that is, all marked variables should be known (look these up in MCast). We let L and L' be the distance between the spaceships in the planet and spaceship frame respectively. We are specifically asked to measure both time and distance in kilometers $[km]$. This means that when you find times in MCast, you will need to multiply them by the speed of light

$$\frac{t_m}{t_s} = c,$$

where t_m is time measured in meters and t_s is time measured in seconds. We are now ready to solve the task.

- We are told that event A happens at $t_A = t'_A = 0$ and $x_A = x'_A = 0$. Next we are told that event A and B are simultaneous in the spaceship frame, and we know that the distance between the the two spaceships are L and L' in the planet and spaceship frame respectively, therefore giving $t'_B = t'_A$ and $x'_B = L'$.
- Since both spaceships and observer M is at rest in the spaceship frame we must have $x'_C = 0$ and $x'_D = L'$. Because event C and D are simultaneous in the spaceship frame, $t'_D = t'_C$.
- We are only missing an expression for the position of event B, C and D in the planet frame. The pervious tasks show that non of these events are simultaneous in the planet frame, but we know that the spaceships travel with the same velocity v and are therefore always a distance L apart (L will be unknown for now). Since the events happen after times t_B, t_C and t_D we get (remember that the initial position of the spaceships are 0 and L, and they will have moved a distance $v \cdot \Delta t$ after a time Δt)

$$x_B = L + vt_B \qquad\qquad x_C = vt_C \qquad\qquad x_D = L + vt_D$$

We should aslo note that since time and distance is measured in the same units, the definition of constant velocity 1 for light tells us that $\Delta x = v \cdot \Delta t = \overbrace{c}^{=1} \cdot \Delta t = \Delta t$. This means that $L' = t'_C$ because the laser, which travel with velocity c, use t'_C time to reach the right spaceship. Now it is just a matter of putting the expressions in the table.

Time or position	Event A	Event B	Event C	Event D
t	$t_A = 0$	t_B	t_C	t_D
t'	$t'_A = 0$	$t'_B = t'_A = 0$	t'_C	$t'_D = t'_C$
x	$x_A = 0$	$x_B = L + vt_B$	$x_C = vt_C$	$x_D = L + vt_D$
x'	$x'_A = 0$	$x'_B = L' = t'_C$	$x'_C = 0$	$x'_D = L' = t'_C$

Table 2: Times and positions for events A, B, C and D. You should know the values of all variables in the spaceship frame. Look these up in MCast.

9. Let x_{Lb} denote the position of the left laser beam. Remember that the left laser has velocity $c = 1$ (in the positive direction).
Since event A occures at $x_A = 0$ and $t_A = 0$, the left lasers position must be

$$x_{Lb} = c \cdot t = t \tag{9}$$

We know that when $t = t_D$ the left laser and right spaceship collide, but this means that their position must be equal.

$$\begin{aligned} x_{Lb} &= x_D \\ t_D &= L + vt_D \\ t_D &= \frac{L}{1-v} \end{aligned}$$

Inserting the new expression for t_D into the position of event D we find

$$x_D = L + vt_D = L + v\frac{L}{1-v} = L\left(\frac{1-v}{1-v} + \frac{v}{1-v}\right) = \frac{L}{1-v}$$

10. We now want to find a new expression for the time t_C. This can be done using invariance of the space-time interval.

$$\begin{aligned} (\Delta s'_{AC})^2 &= \Delta s^2_{AC} \\ (\Delta t'_{AC})^2 - (\Delta x'_{AC})^2 &= \Delta t^2_{AC} - \Delta x^2_{AC} \\ \left(t'_C - \overbrace{t'_A}^{=0}\right)^2 - \left(\overbrace{x'_C}^{=0} - \overbrace{x'_A}^{=0}\right)^2 &= \left(t_C - \overbrace{t_A}^{=0}\right)^2 - \left(x_C - \overbrace{x_A}^{=0}\right)^2 \\ (t'_C)^2 &= t^2_C - x^2_C = t^2_C - (vt_C)^2 = (1-v^2)t^2_C = \frac{t^2_C}{\gamma^2} \\ t_C &= \gamma t'_C \end{aligned}$$

This is as expected the formula for time dilation. Inserting this into the position x_C we find

$$x_C = vt_C = v\gamma t'_C$$

You should recognize this from the Lorentz transformation $x = \gamma x' + v\gamma t'$ (remember that $x'_C = 0$, so $\gamma x'_C = \gamma \cdot 0 = 0$).

11. Let x_{Rb} denote the position of the right laser beam. Remember that the right laser has velocity $-c = -1$.
Event B occures at $x_B = L + vt_B$ at some time t_B. After event B the right laser travels to the left, thus after a time t the position for the laser is given by

$$x_{Rb} = L + vt_B - (t - t_B) = L + (v+1)t_B - t \tag{10}$$

The reason for $t - t_B$ is so that when $t = t_B$ the laser will be at event B. Since the laser moves in the negative direction, the distance it has traveled after it left the right spaceship is $-c(t - t_B) = -(t - t_B)$.
We know that when $t = t_C$ the right laser and left spaceship collide, but this means that their position must be equal. Since we have already found an expression for t_C and x_{Rb} contains the unknown time t_B, we can solve for the time t_B.

$$
\begin{aligned}
x_{Rb} &= x_C \\
L + (v+1)t_B - t_C &= v\gamma t'_C \\
L + (v+1)t_B - \gamma t'_C &= v\gamma t'_C \\
(v+1)t_B &= (v+1)\gamma t'_C - L \\
t_B &= \gamma t'_C - \frac{L}{v+1}
\end{aligned}
$$

We can now find the position x_B.

$$
x_B = L + vt_B = L + v\left(\gamma t'_C - \frac{L}{v+1}\right) = v\gamma t'_C + L\left(\frac{v+1}{v+1} - \frac{v}{v+1}\right) = v\gamma t'_C + \frac{L}{v+1}
$$

12. The only unknown variable that remains is L. We will use invariance of the space-time interval between event B and D to find a value for this last variable.

$$
\begin{aligned}
\left(\Delta s'_{BD}\right)^2 &= \Delta s^2_{BD} \\
\left(\Delta t'_{BD}\right)^2 - \left(\Delta x'_{BD}\right)^2 &= \Delta t^2_{BD} - \Delta x^2_{BD} \\
\left(\overbrace{t'_D}^{=t'_C} - \overbrace{t'_B}^{=0}\right)^2 - \left(\overbrace{x'_D}^{=L'} - \overbrace{x'_B}^{=L'}\right)^2 &= \left(t_D - t_B\right)^2 - \left(x_D - x_B\right)^2 \\
\left(t'_C\right)^2 - 0^2 &= \left(t_D - t_B\right)^2 - \left(x_D - x_B\right)^2 \qquad (11)
\end{aligned}
$$

Before we do any more calculations, lets take a closer look at $(t_D - t_B)$ and $(x_D - x_B)$.

$$
t_D - t_B = \frac{L}{1-v} - \left(\gamma t'_C - \frac{L}{v+1}\right) = L\left(\frac{v+1}{(1-v)(v+1)} + \frac{1-v}{(v+1)(1-v)}\right) - \gamma t'_C = \frac{2L}{1-v^2} - \gamma t'_C
$$

$$
x_D - x_B = \frac{L}{1-v} - \left(v\gamma t'_C + \frac{L}{v+1}\right) = L\left(\frac{v+1}{(1-v)(v+1)} - \frac{1-v}{(v+1)(1-v)}\right) - v\gamma t'_C = \frac{2vL}{1-v^2} - v\gamma t'_C = v\Delta t_{BD}
$$

We see that $\Delta x_{BD} = v\Delta t_{BD}$. We can now solve equation 11 easily.

$$
\begin{aligned}
\left(\Delta s'_{BD}\right)^2 &= \Delta s^2_{BD} \\
\left(t'_C\right)^2 &= \left(t_D - t_B\right)^2 - \left(x_D - x_B\right)^2 = \left(t_D - t_B\right)^2 - v^2\left(t_D - t_B\right)^2 \\
\left(t'_C\right)^2 &= \left(t_D - t_B\right)^2\left(1 - v^2\right) = \left(\frac{2L}{1-v^2} - \gamma t'_C\right)^2 \frac{1}{\gamma^2} \\
\gamma t'_C &= \gamma^2 2L - \gamma t'_C \\
t'_C &= \gamma L \\
L &= \frac{t'_C}{\gamma} = \frac{L'}{\gamma}
\end{aligned}
$$

As expected, we found the formula for length contraction.

Part 2

Assuming we do not know anything about relativity again, we want to see if we can deduce the formula for time dilation.

1. We begin by setting up functions for the positions of the spaceships, observer M and the laser beams. We have already found the position of the two lasers (see equation 9 and 10) and can therefore focus the spaceships and obeserver M.
   We know that they all move with velocity v in positive direction and that their initial positions are $x = 0$ (left spaceship), $x = L/2$ (observer M) and $x = L$ (right spaceship). It is now easy to find their positions using the definition of constant velocity 1.

$$\begin{aligned} &\text{Left spaceship:} && x_V = vt && \text{Left laser:} && x_{\text{Beam 1}} = x_{Lb} = t \\ &\text{Right spaceship:} && x_R = L + vt && \text{Right laser:} && x_{\text{Beam 2}} = x_{Rb} = L + (v+1)t_B - t \\ &\text{Observer M:} && x_M = L/2 + vt \end{aligned}$$

2. We will now let t_M be the time when the two lasers cross at observer M's position. This means that when $t = t_M$ the position of observer M and the left laser will be equal (you can find other expressions looking at $x_M = x_{\text{Beam 2}}$ and $x_{\text{Beam 1}} = x_{\text{Beam 2}}$, but to find the expression we want, we need to look at the following case).

$$\begin{aligned} x_M &= x_{\text{Beam 1}} \\ L/2 + vt_M &= t_M \\ t_M(1 - v) &= L/2 \\ t_M &= \frac{L/2}{1 - v} \end{aligned}$$

3. We now want to find an expression for t_B. Since the function for the right laser contains t_B and we know that when $t = t_M$ the right laser's and observer M's position are equal, we find

$$\begin{aligned} x_M &= x_{\text{Beam 2}} \\ L/2 + vt_M &= L + (v+1)t_B - t_M \\ (v+1)t_M &= L/2 + (v+1)t_B \\ t_B &= t_M - \frac{L/2}{v+1} \end{aligned}$$

4. We can now find the time t_C when the left spaceship explodes, since the position of the right laser must be equal to the left spaceship at this time.

$$
\begin{aligned}
x_V &= x_{\text{Beam 2}} \\
vt_C &= L + (v+1)t_B - t_C \\
(v+1)t_C &= L + (v+1)\left(t_M - \frac{L/2}{v+1}\right) \\
t_C &= \frac{L}{v+1} + t_M - \frac{L/2}{v+1} \\
t_C &= t_M + \frac{L/2}{v+1}
\end{aligned}
$$

5. We shall now find an expression for the position of the light from the explosion of the left spaceship. We know that the explosion happens at $x = vt_C$ (the position of the spaceship when the laser hits). After this the light moves with velocity $c = 1$ towards observer M, in other words, after a time t the light has traveled a distance $ct - ct_C = t - t_C$. Hence we know that the position of the light must be

   $x_{\text{Light}} = \text{Position of explosion} + \text{distance light has traveled} = vt_C + (t - t_C) = (v-1)t_C + t$

6. We should now be able to find the time t_{M2} when the light from both explosions meet at observer M.

$$
\begin{aligned}
x_M &= x_{\text{Light}} \\
L/2 + vt_{M2} &= (v-1)t_C + t_{M2} \\
(v-1)t_{M2} &= (v-1)\left(t_M + \frac{L/2}{v+1}\right) - L/2 \\
t_{M2} &= t_M + \frac{L/2}{v+1} - \frac{L/2}{v-1} = t_M + \frac{L}{2}\left(\frac{v-1}{(v+1)(v-1)} - \frac{v+1}{(v-1)(v+1)}\right) \\
&= t_M + \frac{L}{2}\frac{-2}{v^2-1} = t_M + \frac{L}{1-v^2} \\
t_{M2} &= t_M + \gamma^2 L
\end{aligned}
$$

7. We are now ready to find the time difference between the lasers and the light crossing at observer M, that is the difference $\Delta t = t_{M2} - t_M$.

$$
\Delta t = t_{M2} - t_M = t_M + \frac{L}{1-v^2} - t_M = \frac{L}{1-v^2} = \gamma^2 L
$$

8. We now want to find the corresponding time interval in the spaceship frame. It is now important to remember that the spaceships do not move in their rest frame. Since we only know that the velocity of light is equal for all observers, we assume that the distance between the two spaceships are equal for both frames, in other words $L = L'$. We have already found that $t'_C = L'$ because we measure time and distance in same units and light has velocity $v = c = 1$. We are interested in the time between the lasers crossing at M and the light from the explosions crossing at M. The distance between the spaceships are L (remember that we assume $L = L'$). Thus looking at only one laser at a time (for instance

the left laser. It does not matter which one, since they are both "equal" in the spaceship frame), the laser uses $\Delta t_1 = L$ to reach the other spaceship and the light uses $\Delta t_2 = L$ back to the spaceship from the explosion. Since observer M is located as $x'_M = L/2$ we know that the laser and the light must travel half the distance from is original spaceship. Thus the time difference become

$$\Delta t' = t'_{M2} - t'_M = \frac{L + L}{2} = L$$

9. If our assumptions where correct we should find γ as the ratio between the two time differences, but we find

$$\frac{\Delta t}{\Delta t'} = \frac{\gamma^2 L}{L} = \gamma^2$$

   We must have assumed something that is not correct, and that is $L = L'$. We know that we should get length contraction $L = L'/\gamma$. If we use the correct expression for the time difference $\Delta t' = L' = \gamma L$ we find

$$\frac{\Delta t}{\Delta t'} = \frac{\gamma^2 L}{L'} = \frac{\gamma^2 L}{\gamma L} = \gamma,$$

   which we recognize as the correct ratio.

Exercise 2A.5

We are studying a game of cosmic ping-pong. Two spaceships, moving from right to left with equal velocity v, reflect a laserbeam back and forth between eachother. At the origin of the planet frame, a space station, which is not moving with respect to the planet, is watching the intense battle. We let (x, t) and (x', t') denote the planet/space station frame and spaceship frame respectively. We will look at the following events:

- Event A, which is the left spaceship emitting a single laser, occures at $t_A = t'_A = 0$ and $x_A = x'_A = 0$.
- Event B is the right spaceship reflecting the laser.
- Event C is a space station at $x_C = 0$ exploding.
- Event D is the left spaceship reflecting the laser.

We want to understand how the time interval between the reflections are effected by relativity.

1. We begin with the time intervals $\Delta t'_{AB}$ and $\Delta t'_{BD}$. These time intervals are measured in the frame of the spaceships, which are not moving in their own frame. Hence the light must travel an equal distance from one ship to the other. That is, $\Delta t'_{AB} = \Delta t'_{BD}$ since $\Delta t'_{AB} = c \cdot \Delta x'_{AB} = \Delta x'_{BD} \cdot c = \Delta t'_{BD}$.
2. We try to visualize how the laser bounce back and forth in the planet's frame of referance. At first, the laser and the right spaceship will move in opposite directions: the laser move to the right, while the right spaceship move to the left. After event B, they move in the same direction: both laser and right spaceship move to the left. After event D, they will again move in opposite direction. This pattern will continue forever.

3. When the the laser and the right spaceship move in opposite directions, we know that it takes Δt_{AB} time before they meet. When they are moving in the same direction the laser must travel a greater distance, since the left spaceship is also traveling to the left. Thus, we must have $\Delta t_{BD} > \Delta t_{AB}$.

4. Visualization in MCast.

5. We want to understand what happens first in the planet frame: event B or C? We will sovle this as we did in exercise 2A.4.
We pretend that there exist an observer M in the middle between event B and C. That is, observer M's position is half way from the position of event B to the position of event C. This means that observer M will be in the middle of the two events, **when the events happen**, in both frames. We also let observer M have velocity v, hence it is in the same frame as the spaceships. We are now ready to explain what happens first.

   In the spaceship frame, only the planet is moving. We know that event B and C are simultaneous in this frame, and since the light from the explosion must travel an equal distance as the laser to reach observer M, they will meet at observer M's position. This can be regarded as an event and must therefore happen in all frames of referance. Thus the laser and the light from the explosion must meet at observer M in the planet frame. In the planet frame all spaceships, and observer M, are moving to the left, but the space station is at rest. Hence the laser reflected from the right spaceship must travel a greater distance to reach observer M then the light from the explosion. In other words, event B must occure before event C to give the laser enough time to meet the light from the explosion at observer M's position in the planet frame.

6. We now look at a different situation. The spaceships are moving with velocity $v = 50\,km/h$ with respect to the planet. They are passing a ping-pong ball between eachother, which always has the speed $v' = 80\,km/h$ in the spaceship frame. We want to know if the ball uses longer time in one of the directions. The answer is no. We are now in a non-relativistic setting and since the ball always travel with $v' = 80\,km/h$ with respect to the spaceships, we find that the ball should use an equal ammount of time between the two spaceships in both frames. This is because we now are allowed to add velocities as we please. In other words, the ball will at first have a velocity (in the planet frame) of $v = 80+50 = 130\,km/h$, and on the way back $v = 80-50 = 30\,km/h$. Hence it will use the same ammount of time in both directions.
On the other hand, light has the special property that every observer observs the same velocity. It is this fact that makes the time intervals differnt in the different frames.

7. You should write down the time and position of all event in the spaceship frame into a similar table as table 3. You should then be able to find $L' = x'_B = t'_B$, $\Delta t'_{AB} = t'_B - t'_A$ and $\Delta t'_{BD} = t'_D - t'_B$. Find the values in MCast.

   Every exercise is done analytically in this document. We will therefore find two other expressions to help us with the calculations. Since the spaceships do not move in the spaceship frame, we must have $x'_D = 0$. Since the planet, and therefore also the space station, is moving with velocity v in the spaceship frame, the position of event C must be $x'_C = v \cdot t'_C$.

8. We will now find expressions for the variables in table 3 such that only t_B, t_C, t_D and x_B are unknown.

We know that $t_A = t'_A = 0$ and $x_A = x'_A = 0$. The space station is always at the origin of the planet frame, thus $x_C = 0$. The spaceships are moving with velocity $-v$ to the left, hence $x_D = -v \cdot t_D$. Placing everything into a table, we are left with table 3.

Time or position	Event A	Event B	Event C	Event D
t	$t_A = 0$	t_B (unknown)	t_C (unknown)	t_D (unknown)
t'	$t'_A = 0$	$t'_B = t'_C$	t'_C (known)	t'_D (known)
x	$x_A = 0$	x_B (unknown)	$x_C = 0$	$x_D = -v \cdot t_D$
x'	$x'_A = 0$	$x'_B = t'_B$	$x'_C = v \cdot t'_C$	$x'_D = 0$

Table 3: Times and positions for the different events. You should find the values of all variables in the spaceship frame in MCast.

9. We will now use invariance of the spacetime interval to find a relation between x_B and t_B.

$$\begin{aligned}
(\Delta s'_{AB})^2 &= \Delta s_{AB}^2 \\
(\Delta t'_{AB})^2 - (\Delta x'_{AB})^2 &= \Delta t_{AB}^2 - \Delta x_{AB}^2 \\
(t'_B - t'_A)^2 - (x'_B - x'_A)^2 &= (t_B - t_A)^2 - (x_B - x_A)^2 \\
(t'_B - 0)^2 - (t'_B - 0)^2 &= (t_B - 0)^2 - (x_B - 0)^2 \\
0 &= t_B^2 - x_B^2 \\
x_B &= \pm t_B
\end{aligned}$$

Since both x_B and t_B are positive, we must have $x_B = t_B$.
This result should not come as a suprise. We know that the laser is traveling to the right from $x = 0$ with the speed of light. Hence after a time t_B it will reach the right spaceship, but since we measure time and distance in the same units, we must have $x_B = c \cdot t_B = t_B$. In other words, we could have found this answer without doing the calculation.

10. Next we want to find the value of t_C. We will use invariance of the spacetime interval.

$$\begin{aligned}
(\Delta s'_{AC})^2 &= \Delta s_{AC}^2 \\
(\Delta t'_{AC})^2 - (\Delta x'_{AC})^2 &= \Delta t_{AC}^2 - \Delta x_{AC}^2 \\
(t'_C - t'_A)^2 - (x'_C - x'_A)^2 &= (t_C - t_A)^2 - (x_C - x_A)^2 \\
(t'_C - 0)^2 - (v \cdot t'_C - 0)^2 &= (t_C - 0)^2 - (0 - 0)^2 \\
(t'_C)^2(1 - v^2) &= t_C^2 \\
t_C &= \pm \frac{t'_C}{\gamma}
\end{aligned}$$

Since both times are supposed to be positive, we must have $t_C = t'_C/\gamma$. The numerical answer can be found in document ...

11. We now need a number for t_B.

$$\begin{aligned}
(\Delta s'_{BC})^2 &= \Delta s^2_{BC} \\
(\Delta t'_{BC})^2 - (\Delta x'_{BC})^2 &= \Delta t^2_{BC} - \Delta x^2_{BC} \\
(t'_C - t'_B)^2 - (x'_C - x'_B)^2 &= (t_C - t_B)^2 - (x_C - x_B)^2 \\
\underbrace{(t'_C - t'_C)^2}_{=0} - (v \cdot t'_C - t'_B)^2 &= (t_C - t_B)^2 - (0 - t_B)^2
\end{aligned}$$

$$\begin{aligned}
-(vt'_C - t'_C)^2 &= t_C^2 - 2t_C t_B + \overbrace{t_B^2 - t_B^2}^{=0} \\
2t_C t_B &= t_C^2 + (v-1)^2(\gamma t_C)^2 \\
t_B &= \frac{t_C^2 + (v-1)^2 \frac{t_C^2}{(1-v)(1+v)}}{2t_C} = t_C \frac{1 + (v-1)\frac{-1}{(1+v)}}{2} \\
&= t_C \frac{\frac{1+v}{1+v} + \frac{1-v}{1+v}}{2} = \frac{t'_C}{\gamma} \frac{2}{2(1+v)} \\
t_B &= \frac{t'_B}{\gamma(1+v)}
\end{aligned}$$

The numerical answer can be found in document ...

12. Finally we want to find the time of event D.

$$\begin{aligned}
(\Delta s'_{AD})^2 &= \Delta s^2_{AD} \\
(\Delta t'_{AD})^2 - (\Delta x'_{AD})^2 &= \Delta t^2_{AD} - \Delta x^2_{AD} \\
(t'_D - t'_A)^2 - (x'_D - x'_A)^2 &= (t_D - t_A)^2 - (x_D - x_A)^2 \\
(t'_D - 0)^2 - (0 - 0)^2 &= (t_D - 0)^2 - (-v \cdot t_D - 0)^2 \\
(t'_D)^2 &= t_D^2(1 - v^2) \\
t_D &= \pm \gamma t'_D
\end{aligned}$$

Since both times are supposed to be positive, we must have $t_D = \gamma t'_D$. The numerical answer can be found in document ...

13. To find the time difference between the emission of the laser to the first reflection, we take $\Delta t_{AB} = t_B - t_A$. The numerical answer can be found in document ...

14. To find the time difference between the first and second reflection, we take $\Delta t_{BD} = t_D - t_B$. The numerical answer can be found in document ...

15. You should observe in the video that our reasoning in the first part of this exercise was correct: event B happens before event C. Alternatively you can compare t_B and t_C. Our deduction should still be correct.

16. Watch the video.

Exercise 2A.6

1. (This table will be used throughout the entire exercise. If you wonder where the numbers or relations in later tasks are coming from, check with this table.)

Time or position	Event G	Event P	Event B	Event Y
t	$t_G = 0$	t_P	$t_B = t_P$	t_Y
t'	$t'_G = 0$	t'_P	t'_B	$t'_Y = t'_P$
x	$x_G = 0$	x_P	x_B	$x_Y = 0$
x'	$x'_G = 0$	x'_P	$x'_B = 0$	x'_Y

Table 4: Times and positions for the different events. **All marked variables should be known in the spaceship frame and all unmarked variables should be known in the planet frame** (You should look up the values in MCast). NB! The person with the spaceship frame should not need event B and the person with the planet frame should not need event Y. Thus there is no reason to have the respective event in your own table.

2. (This and the next task are almost identical, therefore we have combined them here. The frame will be specified when needed.)

3. (a) **<u>Planet frame:</u>** The spacetime interval between event G and B in the different frames are

$$\Delta s_{GB}^2 = \Delta t_{GB}^2 - \Delta x_{GB}^2 = (t_B - \overbrace{t_G}^{=0})^2 - (x_B - \overbrace{x_G}^{=0})^2 = t_B^2 - x_B^2$$
$$(\Delta s'_{GB})^2 = (\Delta t'_{GB})^2 - (\Delta x'_{GB})^2 = (t'_B - \overbrace{t'_G}^{=0})^2 - (\overbrace{x'_B}^{=0} - \overbrace{x'_G}^{=0})^2 = (t'_B)^2$$

Hence by invariance of the space-time interval we obtain the time of event B in the spaceship frame:

$$\begin{aligned}(\Delta s'_{GB})^2 &= \Delta s_{GB}^2 \\ (t'_B)^2 &= t_B^2 - x_B^2 \\ t'_B &= \sqrt{t_B^2 - x_B^2}\end{aligned} \tag{12}$$

(Numerical answer can be found in document ...)

<u>Spaceship frame:</u> The spacetime interval between event G and Y in the different frames are

$$\Delta s_{GY}^2 = \Delta t_{GY}^2 - \Delta x_{GY}^2 = (t_Y - \overbrace{t_G}^{=0})^2 - (\overbrace{x_Y}^{=0} - \overbrace{x_G}^{=0})^2 = t_Y^2$$
$$(\Delta s'_{GY})^2 = (\Delta t'_{GY})^2 - (\Delta x'_{GY})^2 = (t'_Y - \overbrace{t'_G}^{=0})^2 - (x'_Y - \overbrace{x'_G}^{=0})^2 = (t'_Y)^2 - (x'_Y)^2$$

Hence by invariance of the space-time interval we obtain the time of event Y in the planet frame:

$$\begin{aligned}\Delta s_{GY}^2 &= (\Delta s'_{GY})^2 \\ t_Y^2 &= (t'_Y)^2 - (x'_Y)^2 \\ t_Y &= \sqrt{(t'_Y)^2 - (x'_Y)^2} \end{aligned} \tag{13}$$

(Numerical answer can be found in document ...)

(b) **Both frames:** The spacetime interval between event G and P in the different frames are

$$\Delta s_{GP}^2 = \Delta t_{GP}^2 - \Delta x_{GP}^2 = (t_P - \overbrace{t_G}^{=0})^2 - (x_P - \overbrace{x_G}^{=0})^2 = t_P^2 - x_P^2$$

$$(\Delta s'_{GP})^2 = (\Delta t'_{GP})^2 - (\Delta x'_{GP})^2 = (t'_P - \overbrace{t'_G}^{=0})^2 - (x'_P - \overbrace{x'_G}^{=0})^2 = (t'_P)^2 - (x'_P)^2$$

By invariance of the space-time interval we get

$$\begin{aligned}(\Delta s'_{GP})^2 &= \Delta s_{GP}^2 \\ (t'_P)^2 - (x'_P)^2 &= t_P^2 - x_P^2 \\ x'_P &= \sqrt{(t'_P)^2 - t_P^2 + x_P^2} \qquad (14) \\ x_P &= \sqrt{t_P^2 - (t'_P)^2 + (x'_P)^2} \qquad (15)\end{aligned}$$

Equation 14 corresponds to the unknown position for event P in the spaceship frame (needed by the person in the planet frame)
Equation 15 corresponds to the unknown position for event P in the planet frame (needed by the person in the spaceship frame).

(c) **Planet frame:** The spacetime interval between event P and B in the different frames are

$$\Delta s_{PB}^2 = \Delta t_{PB}^2 - \Delta x_{PB}^2 = (\overbrace{t_B}^{=t_P} - t_P)^2 - (x_B - x_P)^2 = -(x_B - x_P)^2 = -x_B^2 + 2x_Bx_P - x_P^2$$

$$(\Delta s'_{PB})^2 = (\Delta t'_{PB})^2 - (\Delta x'_{PB})^2 = (t'_B - t'_P)^2 - (\overbrace{x'_B}^{=0} - x'_P)^2 = (t'_B - t'_P)^2 - (x'_P)^2$$

Using equation 12 and 14, we can rewrite t'_B and x'_P, thus giving us

$$
\begin{aligned}
(\Delta s'_{PB})^2 = (t'_B - t'_P)^2 - (x'_P)^2 &= \left(\sqrt{t_B^2 - x_B^2} - t'_P\right)^2 - \left((t'_P)^2 - t_P^2 + x_P^2\right) \\
&= \left(\sqrt{t_B^2 - x_B^2}\right)^2 - 2t'_P\sqrt{t_B^2 - x_B^2} + \underbrace{(t'_P)^2 - (t'_P)^2}_{=0} + t_P^2 - x_P^2 \\
&= \overbrace{t_B^2}^{=t_P^2} - x_B^2 + t_P^2 - 2t'_P\sqrt{t_B^2 - x_B^2} - x_P^2 \\
&= 2t_P^2 - 2\sqrt{t_B^2 - x_B^2} \cdot t'_P - x_P^2 - x_B^2
\end{aligned}
$$

Invariance of the space-time interval then gives us

$$
\begin{aligned}
\Delta s_{PB}^2 &= (\Delta s'_{PB})^2 \\
-x_B^2 + 2x_B x_P - x_P^2 &= 2t_P^2 - 2t'_P\sqrt{t_B^2 - x_B^2} - x_P^2 - x_B^2 \\
2x_B x_P &= 2t_P^2 - 2t'_P\sqrt{t_B^2 - x_B^2} \\
t'_P &= \frac{t_P^2 - x_B x_P}{\sqrt{t_B^2 - x_B^2}} \qquad (16)
\end{aligned}
$$

(Numerical answer can be found in document ...)

Spaceship frame: The spacetime interval between event P and Y in the different frames are

$$
\Delta s_{PY}^2 = \Delta t_{PY}^2 - \Delta x_{PY}^2 = (t_Y - t_P)^2 - (\overbrace{x_Y}^{=0} - x_P)^2 = (t_Y - t_P)^2 - x_P^2
$$

$$
(\Delta s'_{PY})^2 = (\Delta t'_{PY})^2 - (\Delta x'_{PY})^2 = (\overbrace{t'_Y}^{=t'_P} - t'_P)^2 - (x'_Y - x'_P)^2 = -(x'_Y - x'_P)^2 = -(x'_Y)^2 + 2x'_Y x'_P - (x'_P)^2
$$

Using equation 13 and 15, we can rewrite t_Y and x_P, thus giving us

$$
\begin{aligned}
(\Delta s_{PY})^2 = (t_Y - t_P)^2 - (x_P)^2 &= \left(\sqrt{(t'_Y)^2 - (x'_Y)^2} - t_P\right)^2 - \left(t_P^2 - (t'_P)^2 + (x'_P)^2\right) \\
&= \left(\sqrt{(t'_Y)^2 - (x'_Y)^2}\right)^2 - 2t_P\sqrt{(t'_Y)^2 - (x'_Y)^2} + \underbrace{t_P^2 - t_P^2}_{=0} + (t'_P)^2 - (x'_P)^2 \\
&= \overbrace{(t'_Y)^2}^{=(t'_P)^2} - (x'_Y)^2 + (t'_P)^2 - 2t_P\sqrt{(t'_Y)^2 - (x'_Y)^2} - (x'_P)^2 \\
&= 2(t'_P)^2 - 2\sqrt{(t'_Y)^2 - (x'_Y)^2} \cdot t_P - (x'_P)^2 - (x'_Y)^2
\end{aligned}
$$

Invariance of the space-time interval then gives us

$$\begin{aligned}
\Delta s_{PB}^2 &= (\Delta s'_{PB})^2 \\
-(x'_Y)^2 + 2x'_Y x'_P - (x'_P)^2 &= 2(t'_P)^2 - 2\sqrt{(t'_y)^2 - (x'_Y)^2} \cdot t_P - (x'_P)^2 - (x'_Y)^2 \\
2x'_Y x'_P &= 2(t'_P)^2 - 2t_P\sqrt{(t'_Y)^2 - (x'_Y)^2} \\
t_P &= \frac{(t'_P)^2 - x'_Y x'_P}{\sqrt{(t'_Y)^2 - (x'_Y)^2}} \qquad (17)
\end{aligned}$$

(Numerical answer can be found in document ...)

(d) (Numerical answer can be found in document ...)

4. (Check with your partner)

5. We will need the following equation later on:

$$\gamma^2 - 1 = \frac{1}{1-v^2} - \frac{1-v^2}{1-v^2} = \frac{v^2}{1-v^2} = \gamma^2 v^2$$

$$\gamma^2 = \gamma^2 v^2 + 1 \qquad (18)$$

Planet frame: We know that the blue lightning hits the the spaceship simultaneously as event P in the planet frame (this means that $t_B = t_P$). This means that by the law of constant motion 1 we can write the position of event B as $x_B = vt_B = vt_P$. Equation 16 thus becomes:

$$\begin{aligned}
t'_P &= \frac{t_P^2 - x_B x_P}{\sqrt{t_B^2 - x_B^2}} = \frac{t_P^2 - vt_P x_P}{\sqrt{t_P^2 - (vt_P)^2}} \\
&= \frac{t_P(t_P - vx_P)}{t_P\sqrt{1-v^2}} = \gamma(t_P - vx_P)
\end{aligned}$$

$$t'_P = \gamma t_P - \gamma v x_P$$

Using the newly obtained result for t'_P and the transformation of γ from 18, equation 14 for x'_P thus becomes:

$$\begin{aligned}
(x'_P)^2 &= (t'_P)^2 - t_P^2 + x_P^2 = (\gamma t_P - \gamma v x_P)^2 - t_P^2 + x_P^2 \\
&= \gamma^2 t_P^2 - 2\cdot\gamma t_P \cdot \gamma v x_P + \gamma^2 v^2 x_P^2 - t_P^2 + x_P^2 \\
&= t_P^2(\gamma^2 - 1) - 2\cdot\gamma^2 t_P v x_P + x_P^2(\gamma^2 v^2 + 1) \\
\text{[equation (18)]} \rightarrow &= t_P^2 v^2 \gamma^2 - 2\cdot\gamma^2 t_P v x_P + x_P^2\gamma^2 \\
&= \gamma^2\left(t_P^2 v^2 - 2t_P v x_P + x_P^2\right) \\
&= \gamma^2\left(t_P v - x_P\right)^2
\end{aligned}$$

$$x'_P = \gamma v t_P - \gamma x_P^2$$

We now see that we have obtained the backwards Lorenz transformation.

Spaceship frame: We know that the yellow lightning hits the origin of the planet frame simultaneously as event P in the spaceship frame (this means that $t'_Y = t'_P$). This means that by the law of constant motion 1 we can write the position of event Y as $x'_Y = -vt'_Y = -vt'_P$ (remember that the planet has velocity $-v$ in the spaceship frame). Equation 17 thus becomes:

$$\begin{aligned} t_P &= \frac{(t'_P)^2 - x'_Y x'_P}{\sqrt{(t'_Y)^2 - (x'_Y)^2}} = \frac{(t'_P)^2 - (-vt'_P)x'_P}{\sqrt{(t'_P)^2 - (-vt'_P)^2}} \\ &= \frac{t'_P\,(t'_P + vx'_P)}{t'_P\sqrt{1-v^2}} = \gamma(t'_P + vx'_P) \end{aligned}$$

$$t_P = \gamma t'_P + \gamma v x'_P$$

Using the newly obtained result for t_P and the transformation of γ from 18, equation 15 for x_P thus becomes:

$$\begin{aligned} x_P^2 &= t_P^2 - (t'_P)^2 + (x'_P)^2 = (\gamma t'_P + \gamma v x'_P)^2 - (t'_P)^2 + (x'_P)^2 \\ &= (\gamma t'_P)^2 + 2\cdot\gamma t'_P \cdot \gamma v x'_P + (\gamma v x'_P)^2 - (t'_P)^2 + (x'_P)^2 \\ &= (t'_P)^2(\gamma^2 - 1) + 2\cdot \gamma t'_P \cdot \gamma v x'_P + (x'_P)^2(\gamma^2 v^2 + 1) \\ \text{[equation (18)]} \rightarrow &= (t'_P)^2 v^2\gamma^2 + 2\gamma^2 t'_P v x'_P + (x'_P)^2\gamma^2 \\ &= \gamma^2\left((t'_P)^2 v^2 + 2t'_P v x'_P + (x'_P)^2\right) \\ &= \gamma^2\left(t'_P v + x'_P\right)^2 \end{aligned}$$

$$x_P = \gamma v t'_P + \gamma x'_P$$

We now see that we have obtained the forwards Lorenz transformation.

Exercise 2A.7

1. We begin by copying table 3. It should contain everything we need. See exercise 2A.5 for deduction of the different variables.

Time or position	Event A	Event B	Event D
t	$t_A = 0$	t_B (unknown)	t_D (unknown)
t'	$t'_A = 0$	t'_B (known)	t'_D (known)
x	$x_A = 0$	x_B (unknown)	$x_D = -v \cdot t_D$
x'	$x'_A = 0$	$x'_B = t'_B$	$x'_D = 0$

Table 5: Times and positions for the different events. This is a copy of table 3 without event C. You should know all marked variables.

2. We need to use the forward Lorentz transformation (equation 2) to obtain t_B and t_D. We begin with t_B. Remember that we defined the velocity of the spaceships as $-v$.

$$t_B = (-v)\gamma x'_B + \gamma t'_B = -v\gamma t'_B + \gamma t'_B = t'_B\gamma(1-v)$$

This expression looks somewhat different from the one we found in exercise 2A.5, but a bit of algebra tells us that

$$\gamma(1-v) = \frac{1-v}{\sqrt{1-v^2}} = \frac{1-v}{\sqrt{1-v}\sqrt{1+v}} = \sqrt{\frac{1-v}{1+v}} = \sqrt{\frac{(1-v)(1+v)}{(1+v)(1+v)}} = \frac{\sqrt{1-v^2}}{1+v} = \frac{1}{\gamma(1+v)}$$

Thus we have

$$t_B = t'_B\gamma(1-v) = \frac{t'_B}{\gamma(1+v)},$$

exactly as in 2A.5. We find t_D exactly the same way:

$$t_D = (-v)\gamma x'_D + \gamma t'_D = -v\gamma \cdot 0 + \gamma t'_D = \gamma t'_D$$

You should also recognize this from exercise 2A.5.

3. All that is left now is to calculate the Δt_{AB} and Δt_{BD}. You can find the numerical answer in ...

Exercise 2A.8

Part 1:

1. From the Lorenz transformation vi can deduce the formula for time dilation. Assume that the marked frame is moving with relative velocity v. Then the difference in position of this frame, in the unmarked frame (the rest frame), can be written as $\Delta x = v\Delta t$. This gives us

$$\begin{aligned}\Delta t' &= \gamma\Delta t - \gamma v\Delta x = \gamma\left(\Delta t - v\cdot v\Delta t\right)\\ &= \gamma\Delta t\left(1-v^2\right) = \gamma\Delta t\frac{1}{\gamma^2}\\ \Delta t' &= \frac{\Delta t}{\gamma}\end{aligned}$$

2. Since the spaceship travels with constant velocity $v = 0.99c$ we know by equation 1 (definition of constant velocity) that the spaceship uses (remember that $t_A = 0$ and we therefore have $\Delta t_{AB} = t_B - t_A = t_B - 0 = t_B$)

$$t_B = \Delta t_{AB} = \frac{\Delta x_{AB}}{v} = \frac{200\,\text{years}}{0.99} \approx 202\,\text{years}$$

It will therefore take the spaceship approximately 202 years in the planet frame to travel to planet P2.

By the formula for time dilation it took (remember that $t'_A = 0$ and we therefore have $\Delta t'_{AB} = t'_B - t'_A = t'_B - 0 = t'_B$)

$$t'_B = \Delta t'_{AB} = \frac{\Delta t_{AB}}{\gamma} = \Delta t_{AB}\sqrt{1-v^2} = 202\,\text{years}\cdot\sqrt{1-0.99^2} \approx 28.5\,\text{years},$$

to reach planet P2 in the spaceship frame.

3. We now want to find the amount of time which passes on the return trip of the spacehip. Since the distance and absolute velocity is unchanged, we have an identical problem. The time the spaceship uses to travel back to P1 is therefore equal;

$$\Delta t = \Delta t_{AB} \approx 202\,\text{years}$$
$$\Delta t' = \Delta t'_{AB} \approx 28.5\,\text{years}$$

Part 2:

1. Switching the roles, we now observe planet P1 traveling away from the spaceship with velocity $-v = -0.99c$. Since the formula for time dilation is valid for all inertial frames, we now let t be the time on the spaceship (which is at rest for the hole journy) and t' be the time on P1. Thus we see that the time that passes on P1 is (remember that we already found that the spaceship uses $t = 28.5\,\text{years}$ in its own frame)

$$t'_{AB} = \frac{t_{AB}}{\gamma} = t_{AB}\sqrt{1-v^2} = 28.5\,\text{years}\cdot\sqrt{1-(-0.99)^2} \approx 4\,\text{years}$$

2. Again we want to find how much time passes on the return trip in the different frames. As in the pervious part, since the distance and absolute velocity is unchanged, we have an identical problem. The time planet P1 uses to travel back to the spaceship is therefore equal the time it took to travel away from the spaceship;

$$\Delta t' = \Delta t'_{AB} \approx 4\,\text{years}$$

3. The previous calculations are not all valid and this is because we have violated one of the principles of special relativity! The first principle of relativity states that

> The laws of physics are equal in all **inertial** reference frames.

For the spaceship to attain a speed comparable to the speed of light it must accelerate (which it does when it turns back), thus introducing fictional forces. Therefore the spaceship frame is no longer inertial. Since we are constantly changing frame of reference, the Lorentz transformations (which are deduced for constant speed) are no longer valid thus making our previous calculations invalid.

Part 3:

1. We know that the distace between the two planets is $L_0 = 200$ lightyears and that the spaceship is travelig with constant velocity $v = 0.99\,c$. Thus the equation of constant motion 1 gives
$$\Delta t_{AB} = \frac{\Delta x_{AB}}{v} = \frac{L_0}{v},$$
and since $\Delta t_{AB} = t_B - t_A = t_B - 0$ we get
$$t_B = L_0/v$$

2. We now want to find t'_B using Lorentz transformation. The Lorentz transformation is given by equation 4
$$t' = -v\gamma x + \gamma t$$
Inserting numbers we get
$$t'_B = -v\gamma x_B + \gamma t_B = \frac{-v}{1-v^2}x_B + \frac{1}{1-v^2}t_B = \frac{-0.99}{1-0.99^2}\cdot 200\,\text{years} + \frac{1}{1-0.99^2}\cdot(200/0.99)\,\text{years} \approx 28.5\,\text{years}$$
(Using $t_B = 202$ years we get 28.35 years not 28.5 years, due to roundoff errors since $200/0.99 \approx 202$.)

3. We want to show that $t_{B'}$ can be written as $t_{B'} = L_0/v - vL_0$. We will do this with the Lorenz transformation (equation 2)
$$t_{B'} = \gamma v x'_{B'} + \gamma t'_{B'},$$
but to do this we need both $t'_{B'}$ and $x'_{B'}$. We start by finding $t'_{B'}$.

   We know that $t'_{B'} = t'_B$ (check the information about event B' in the exercise if you do not remember this), and since we by time dilation have $t'_B = t_B/\gamma$ we thereby get
$$t'_{B'} = t'_B = t_B/\gamma = \frac{L_0/v}{\gamma} = \frac{L_0}{v\gamma} \tag{19}$$
Finding $x'_{B'}$ is just as simple. Event B' takes place at planet P1 and since the spaceship is always at the origin in its own system of referance, the position of event B' must be (remember that we get length contraction)
$$x'_{B'} = -x_B/\gamma = -L_0/\gamma \tag{20}$$
We now have everything we need. Inserting 19 and 20 into the Lorenz transformation we end up with
$$t_{B'} = \gamma v x'_{B'} + \gamma t'_{B'} = \gamma v\left(\frac{-L_0}{\gamma}\right) + \gamma\frac{L_0}{v\gamma} = L_0/v - vL_0 = (200/0.99)\,\text{years} - 0.99\cdot 200\,\text{years} \approx 4\,\text{years}$$

4. Compare your answers with the MCast videos

5. Everything up until now is consistent with the spesial relativity, as the above calculations show. This is because the events are **not** simultaneous;

   (a) The spaceship begins its travel towards P2 at $x = x' = 0$ and $t = t' = 0$ (event A).

   (b) After approximately 28.5 years the spaceship arrive at P2 in the spaceship frame (event B). At the same time as this happens (in the spaceship frame), another spaceship at P1 sends out a blue light and observe that the time on the clocks on P1 shows 4 years (event B'). The spaceship has not arrived at P2 in the planet frame (it still needs 198 years to reach the destination), but they observe the blue light and knows that the spaceship has arrived in the spaceship frame.

   (c) After another 198 years the spaceship arrives at P2 in the planet frame (event B).

   You should now understand that the events where not simultaneous. They happen at different times and places in space-time.

Part 4:

In this part we will let (x, t) denote the planet frame and (x'', t'') denote the returning elevator frame. Remember that event D is person P initiating his journey from P3 to P1, and event B'' is the returning elevator shining a blue light at P1 simulaneously as person P arrives at P2.
To make this part easier we set up an event table. Remember that (x'', t'') corresponds to the returning elevator frame of person P.

Time or position	Event B	Event D
t	$t_B = L_0/v$	$t_D = 0$
t''	$t''_B = t''_B$	$t''_D = 0$
x	$x_B = L_0$	$x_D = 2L_0$
x''	$x''_B = 0$	$x''_D = 0$

Table 6: Times and positions for event B and D.

Hopefully the values for the different variables does not suprise you. If it does, remember that event B is the spaceship arriving at P2, thus (since the distance between P1 and P2 is L_0 in the planet frame, and the spaceship is always at the origin in the spaceship frame) the positions of event B must be $x''_B = 0$ and $x_B = L_0$ and the time of event B must be $t_B = L_0/v$ and an unknown time t''_B. All information about event D is given in the exercise.

1. Table 6 gives us the following space and time intervals.

$$\begin{aligned}
\Delta x_{BD} &= x_D - x_B = 2L_0 - L_0 = L_0 \\
\Delta t_{BD} &= t_D - t_B = 0 - L_0/v = -L_0/v \\
\Delta x''_{BD} &= x''_D - x''_B = 0 - 0 = 0 \\
\Delta t''_{BD} &= t''_D - t''_B = 0 - t''_B = -t''_B
\end{aligned}$$

   This gives the corresponding space-time interval

$$(\Delta s''_{BD})^2 = \Delta s^2_{BD}$$
$$(\Delta t''_{BD})^2 - (\Delta x''_{BD})^2 = \Delta t^2_{BD} - \Delta x^2_{BD}$$
$$(-t''_B)^2 - 0^2 = \left(-\frac{L_0}{v}\right)^2 - L_0^2$$
$$(t''_B)^2 = \frac{L_0^2}{v^2} - L_0^2$$

We now see that we have

$$(t''_B)^2 = \frac{L_0^2}{v^2} - L_0^2 = L_0^2\left(\frac{1}{v^2} - \frac{v^2}{v^2}\right) = L_0^2\frac{1-v^2}{v^2} = \frac{L_0^2}{v^2\gamma^2}$$

Taking the square root, and remebering that the time must be positive, we see that we have the same expression as in equation 19, which means that $t''_B = t'_B = \frac{L_0}{v\gamma}$. Thus the returning and the outgoing elevator use the same amount of time to reach P2 in their own frame, traveling from their respective planets. (Check with MCast that your clocks show the same time and that you therefore agree on event B)

2. To find the space and time intervals between event D and B'' we again make an event table.
   Event D will be equal as in table 6 (we are looking at the same event from the two frames used in that table).
   Since event B'' happens at P1, we must have $x_{B''} = 0$ (unmarked is planet frame and the origin in the planet frame is at P1).
   Since the astronaut uses close to no time changing to the returning elevator, we must have

$$t''_{B''} = t''_B = L_0/(v\gamma)$$

   Event B'' takes place at P1 a distance L_0 from P2 in the planet frame. The returning elevator (with person P) is at this moment on P2 in the spaceship frame. Hence person P's position must be

$$x''_{B''} = L_0/\gamma$$

   (remember that we get length contraction). We are now ready to set up the following event table.

Time or position	Event B	Event D
t	$t_{B''} = t_{B''}$	$t_D = 0$
t''	$t''_{B''} = L_0/(v\gamma)$	$t''_D = 0$
x	$x_{B''} = 0$	$x_D = 2L_0$
x''	$x''_{B''} = L_0/\gamma$	$x''_D = 0$

Table 7: Times and positions for event D and B''.

It now only remains to set up the time and space intervals.

$$
\begin{aligned}
\Delta t''_{DB''} &= t''_{B''} - t''_D = L_0/(v\gamma) - 0 = L_0/(v\gamma) \\
\Delta t_{DB''} &= t_{B''} - t_D = t_{B''} - 0 = t_{B''} \\
\Delta x''_{DB''} &= x''_{B''} - x''_D = L_0/\gamma - 0 = L_0/\gamma \\
\Delta x_{DB''} &= x_{B''} - x_D = 0 - 2L_0 = -2L_0
\end{aligned}
$$

3. Using invariace of the space-time interval we find

$$
\begin{aligned}
(\Delta s''_{DB''})^2 &= \Delta s^2_{DB''} \\
(\Delta t''_{DB''})^2 - (\Delta x''_{DB''})^2 &= \Delta t^2_{DB''} - \Delta x^2_{DB''} \\
(L_0/(v\gamma))^2 - (L_0/\gamma)^2 &= t^2_{B''} - (-2L_0)^2 \\
t^2_{B''} - (2L_0)^2 = \frac{1}{v^2\gamma^2}\left(L_0^2 - v^2 L_0^2\right) &= \frac{L_0^2}{v^2\gamma^2}\overbrace{(1-v^2)}^{=1/\gamma^2} = \frac{L_0^2}{v^2}(1-v^2)^2
\end{aligned}
$$

Thus giving

$$
\begin{aligned}
t^2_{B''} &= \frac{L_0^2}{v^2}(1-v^2)^2 + (2L_0)^2 = L_0^2\frac{(1-2v^2+v^4)+4v^2}{v^2} \\
&= L_0^2\frac{1+2v^2+v^4}{v^2} = L_0^2\frac{(1+v^2)^2}{v^2} \\
t_{B''} &= L_0\frac{1+v^2}{v} = \frac{L_0}{v\gamma^2}
\end{aligned}
$$

We found exactly the formula we where looking for: $t_{B''} = L_0/v + vL_0$. Inserting numbers this gives

$$t_{B''} = (200/0.99)\,\text{years} + 0.99 \cdot 200\,\text{years} \approx 400\,\text{years}$$

4. We will now try to explain the solution to the twin paradox using the videos.

   (a) At $t = t' = 0$ and $x = x' = 0$ the yellow spaceship begins its journey from P1 to P2.

   (b) After $t = 4$ years a blue light is observed in the planet frame, signaling that the spaceship has reached its destination in the spaceship frame.

   (c) After $t' = 28.5$ years the spaceship reaches P2 and returns to P1 in the spaceship frame.

   (d) After $t = 202$ years the spaceship arrives at P2 and begins its return journey to P1 in the planet frame.

   (e) After $t = 400$ years a blue light is seen at P1 in the planet frame, signaling that the astronaut begins his return to P1 in the spaceship frame in the red spaceship.

   (f) After $t = 404$ and $t' = 57$ years the red spaceship arrives at P1 and the astronaut is back home again.

   We see now that the spaceship needed 396 years to accelerate and only 8 years to trave on the clock in the planet frame, as seen from the spaceship frame