Lecture 15

The structure of infinity

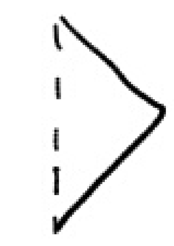
the infinity of flat plane is a point

conformal mappings

spacetime analogue : light rays $\longrightarrow$ lightrays

The Penrose diagram for Minkowski space

i^0 (a point!) , i^+ , i^- , $\mathscr{I}^+$, $\mathscr{I}^-$

correct way to draw the diagram : 

The Penrose diagram for the Kruskal extension of Schwarzsch.

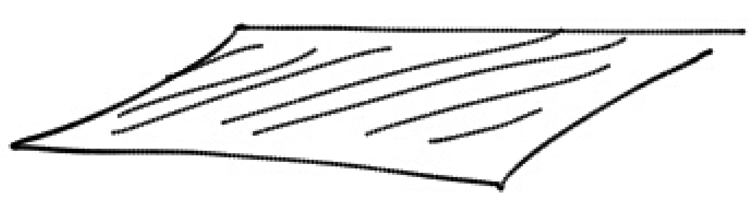
The Penrose diagram for a real black hole.

Definition of event horizon.

Examples : Kerr , Reissner-Nordström, evaporating black hole.

Penrose diagrams: Causal structure and the structure of infinity

Consider the flat 2 dimensional plane. What is its infinity like?

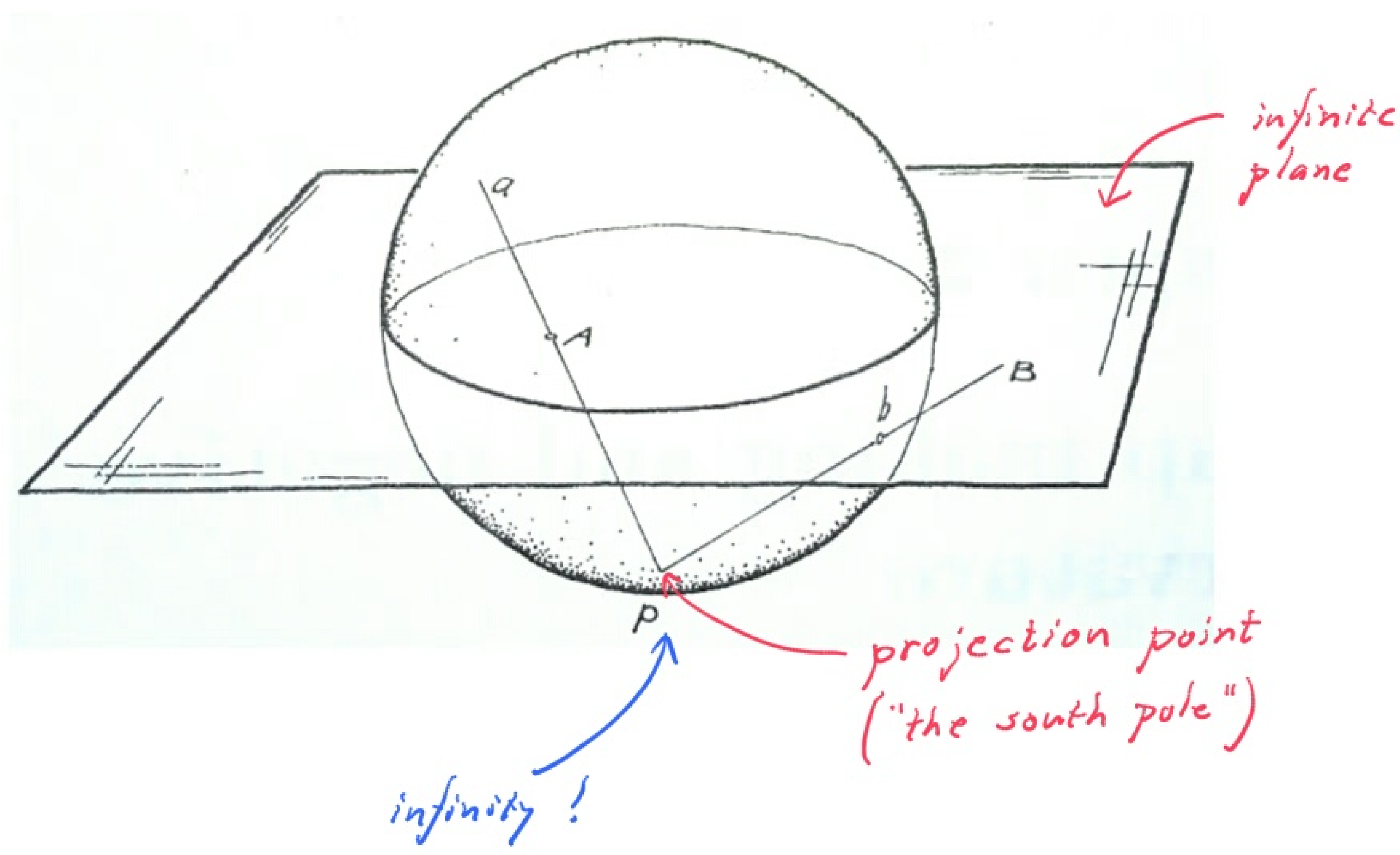


infinity?

— a point!

It is a point in this sense: if all points of the plane is mapped into a finite area, in such a way that all angles are preserved by the map — that is, if the map is conformal — then infinity is mapped into a single point.

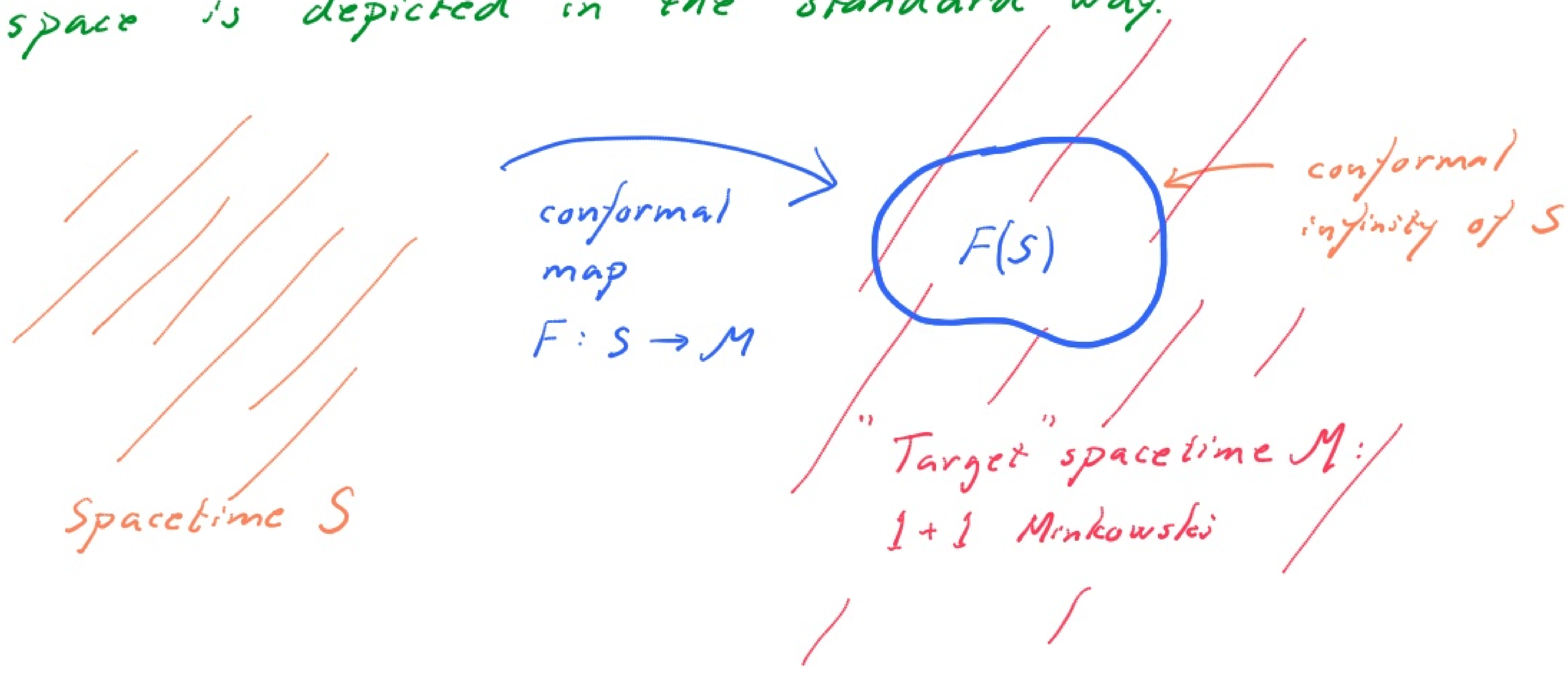
The simplest realization of such a map is the stereographic projection of the infinite plane onto the surface of a sphere:



Points far away will be mapped to points close to p, and all points infinitely far out will be mapped to p. Hence, infinity is a single point in this "conformal mapping"-sense.

Conformal map: preserves angles = acts locally as an isotropic rescaling

In a similar way we can discuss the structure of infinity also in spacetimes. Then the "target space" of the map is usually taken to be 1+1 Minkowski space, and it turns out that the requirement that the map should be conformal then is equivalent to requiring that all lightrays of the spacetime should be mapped onto the lightrays of 1+1 Minkowski, that is 45°-lines, if Minkowski space is depicted in the standard way.



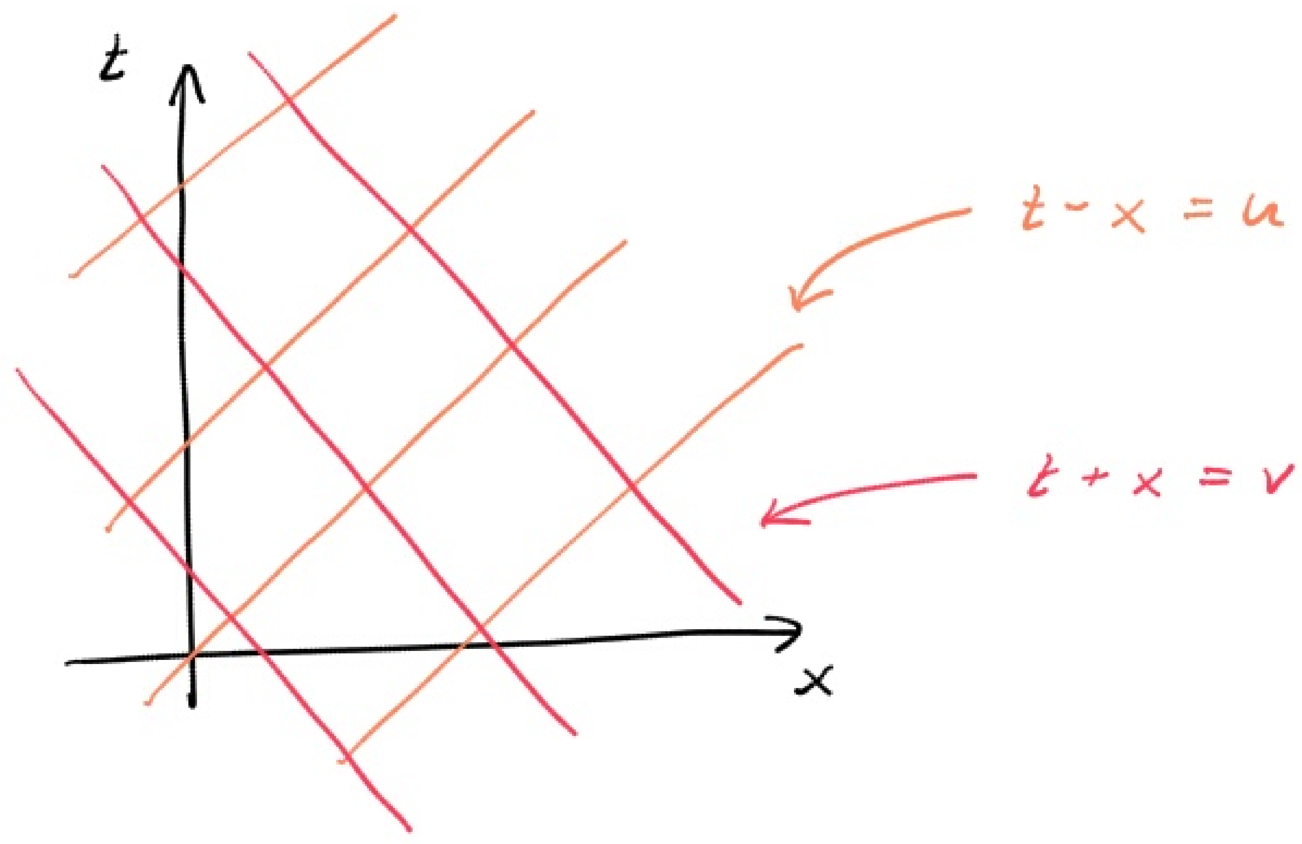
Conformal spacetime map: preserves relative speeds

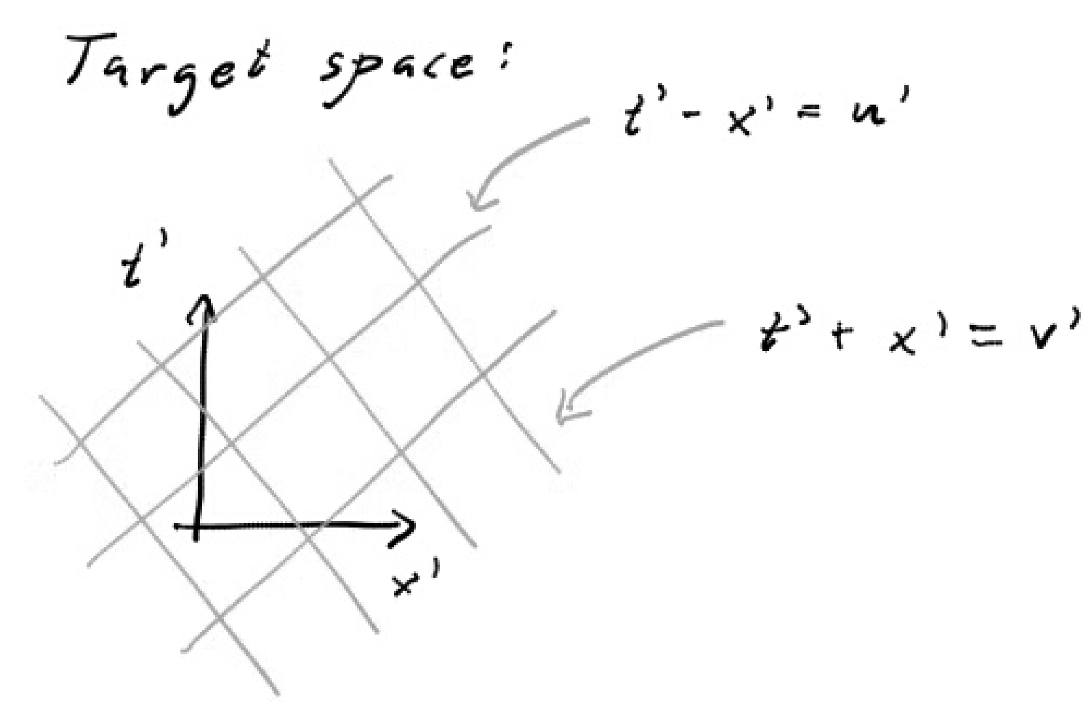
$\Longleftrightarrow$ lightrays $\rightarrow$ lightrays

To see how this works, let us consider 1 + 1 Minkowski space. What is its infinity like?

<u>1 + 1 Minkowski space</u>

There are two families of lightrays:





The target space also has two families of lightrays, described by lines of constant u' and v', respectively.

In order to fulfil the conformal requirement, we want lines of constant u to be mapped on lines of constant u', and lines of constant v to be mapped on lines of constant v'. That is, we are looking for a map of this kind:

$$u' = f(u)$$
$$v' = g(v)$$

Because of right-left symmetry, there is no point in having two different functions here. Hence, we put $g = f$.

$$u' = f(u)$$
$$v' = f(v)$$

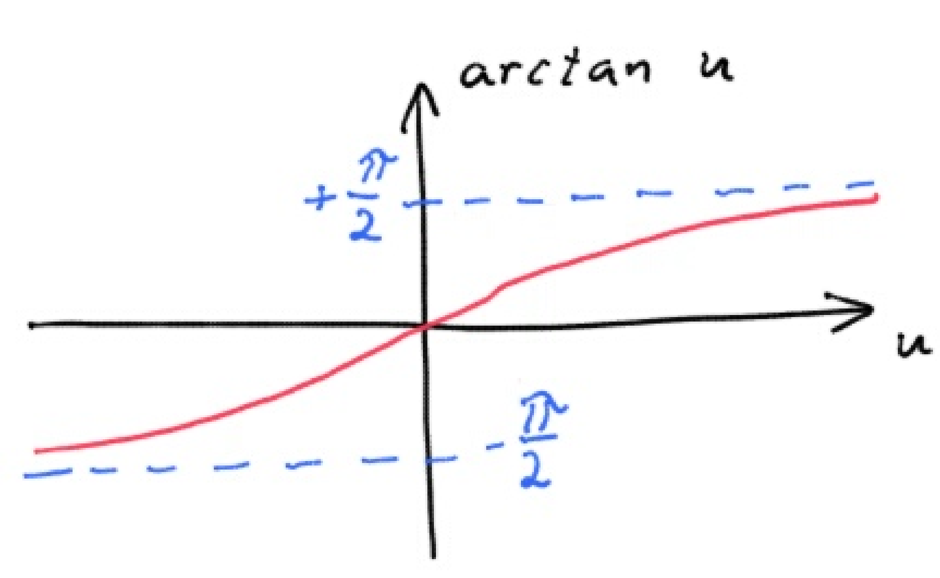
Now the point in all this was to map infinity into finite coordinate values. Hence we require

$$f(\pm\infty) = \pm\,[\text{finite value}]$$

The function f can of course be chosen in an infinite number of ways, and the exact choice makes no difference for the conclusions that we shall draw concerning the infinity of 1 + 1 Minkowski.

The conventional choice is

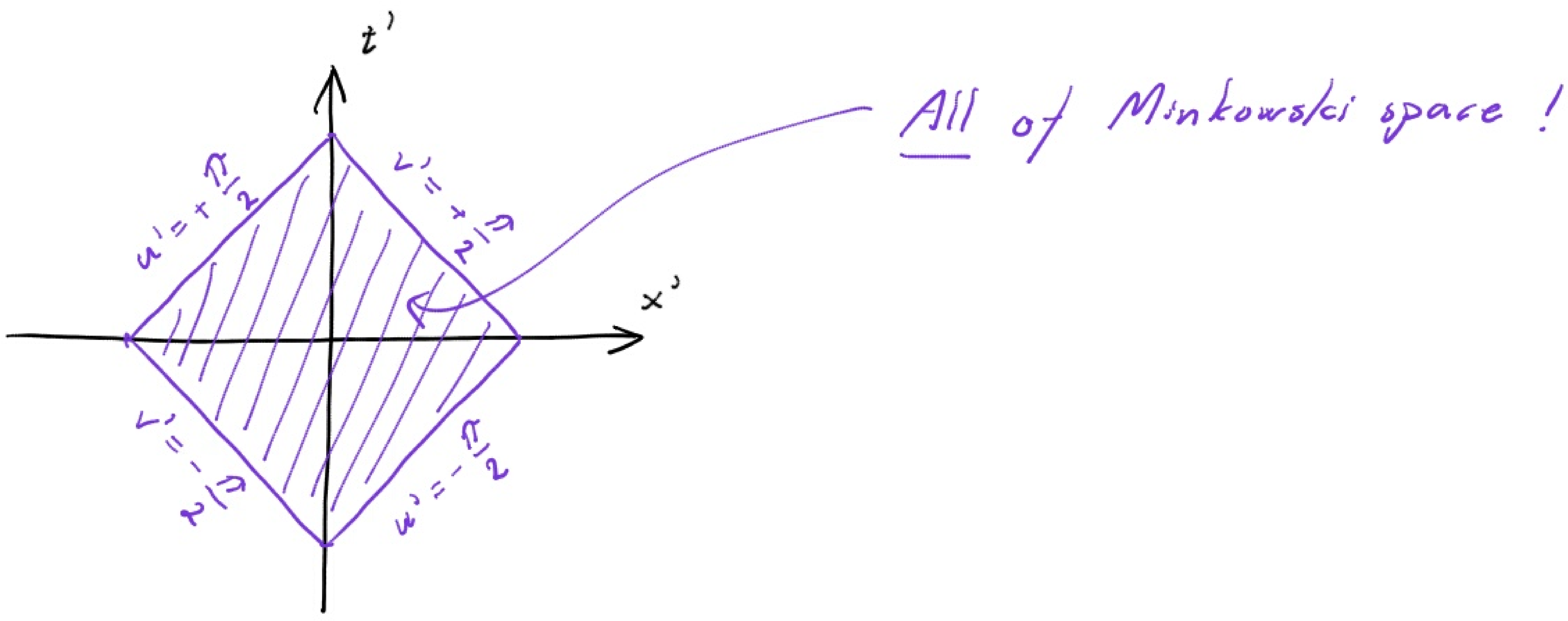
$$u' = \arctan u$$
$$v' = \arctan v$$



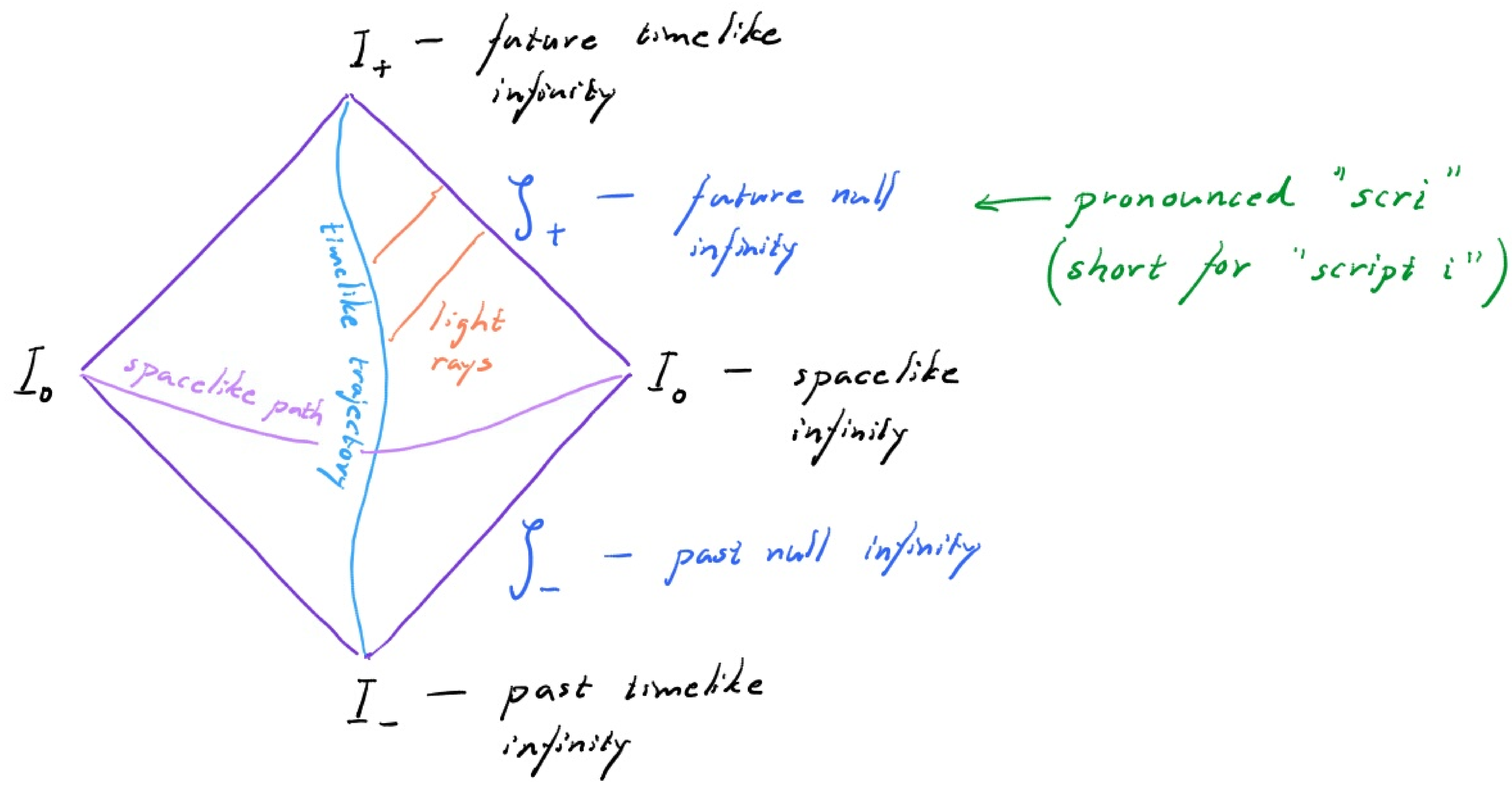
Since

$$\arctan(\pm\infty) = \pm\frac{\pi}{2}$$

and since the coordinate ranges for u, v that covers the full Minkowski space is from $-\infty$ to $+\infty$, we see that all Minkowski space by this function is mapped into the finite region with boundaries $u' = \pm\frac{\pi}{2}$ and $v' = \pm\frac{\pi}{2}$.

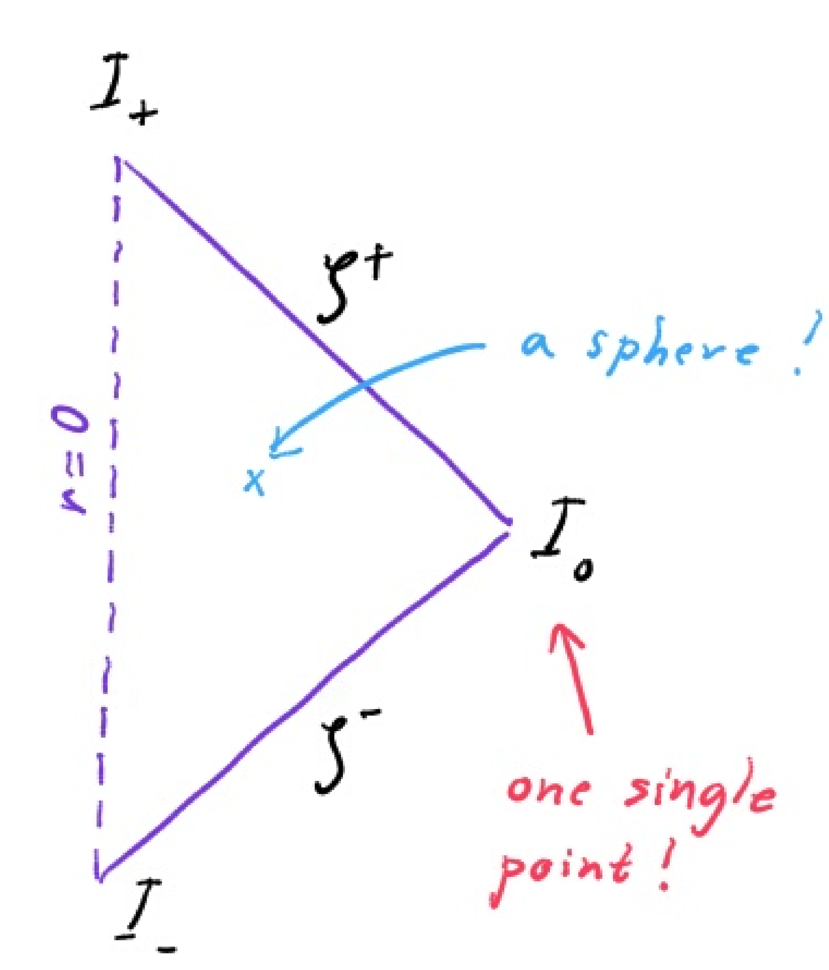


For the flat plane we saw that infinity was a point. Clearly, infinity of 1+1 Minkowski is much richer!



What about 3+1 Minkowski?

Because of spherical symmetry it is enough to depict the r, t-plane, remembering that to each value of r, t, there actually corresponds a full sphere of radius r. But then, because of the coordinate range of r (from 0 to $+\infty$) we should only draw one half of the diagram above.

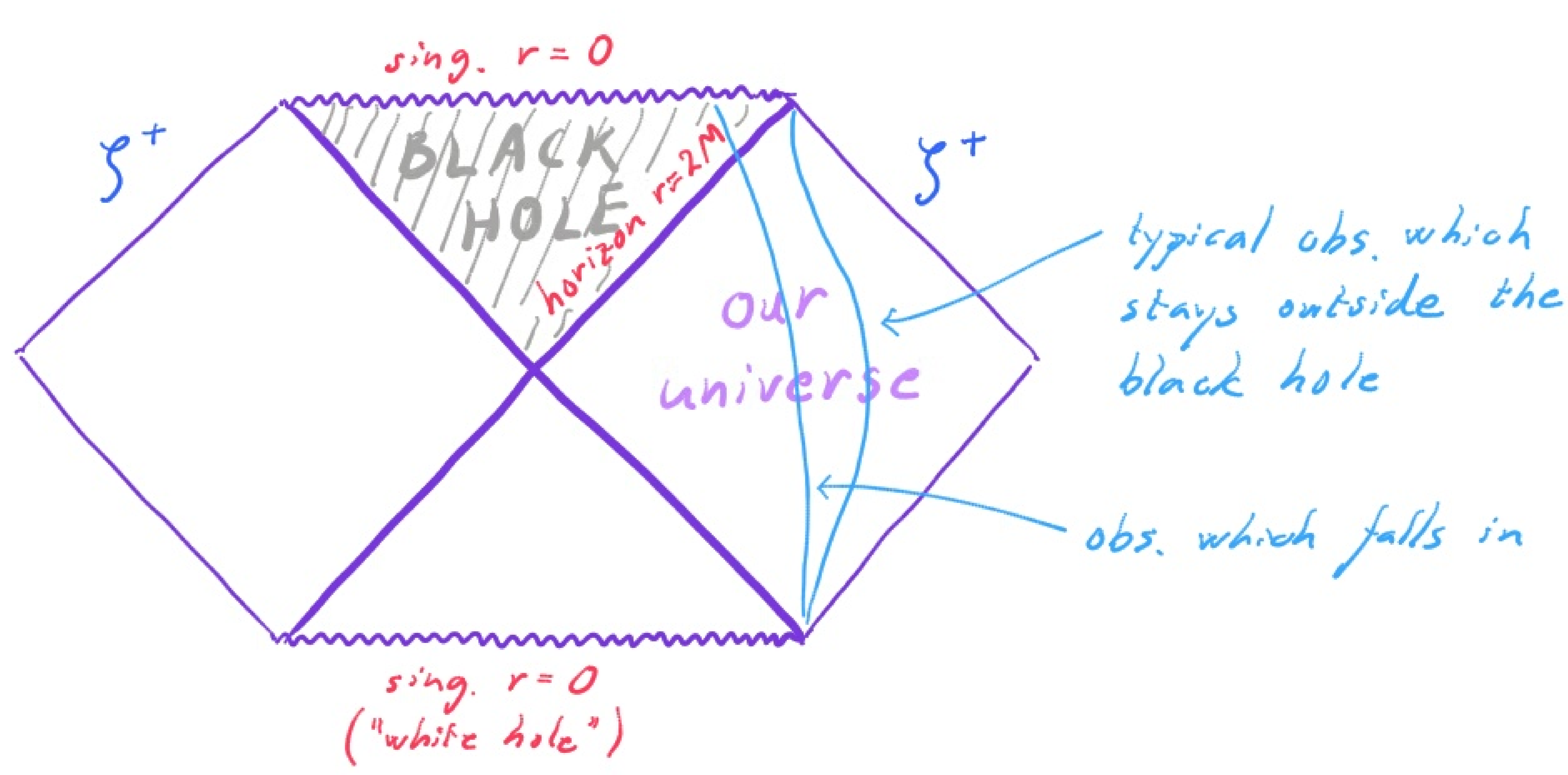


Penrose diagram of Minkowski space (any dim.)

Such conformal pictures of spacetime — or Penrose diagrams — thus makes it possible to discuss properties of infinity in a well-defined way. But their main value is that they also allows us to immediately read off the causal properties of a given spacetime.

Let us consider another example — the Kruskal extension of the Schwarzschild spacetime. We already introduced coordinates such that lightrays are 45°-lines. What remains in order to get a Penrose diagram is just a rescaling along these lightrays such that points infinitely far away are mapped to finite distances. This can be accomplished by the same function as we used for Minkowski space, that is, arctan. The result, after having considered the relevant coordinate ranges is this:

Penrose diagram for Kruskal extension of Schwarzschild



Remember that each point here is a sphere of radius r.

Note that there are two exterior regions, each with its own future null infinity.

We can now give the formal definition of event horizon:

Def: The event horizon is the boundary of the past of future null infinity.

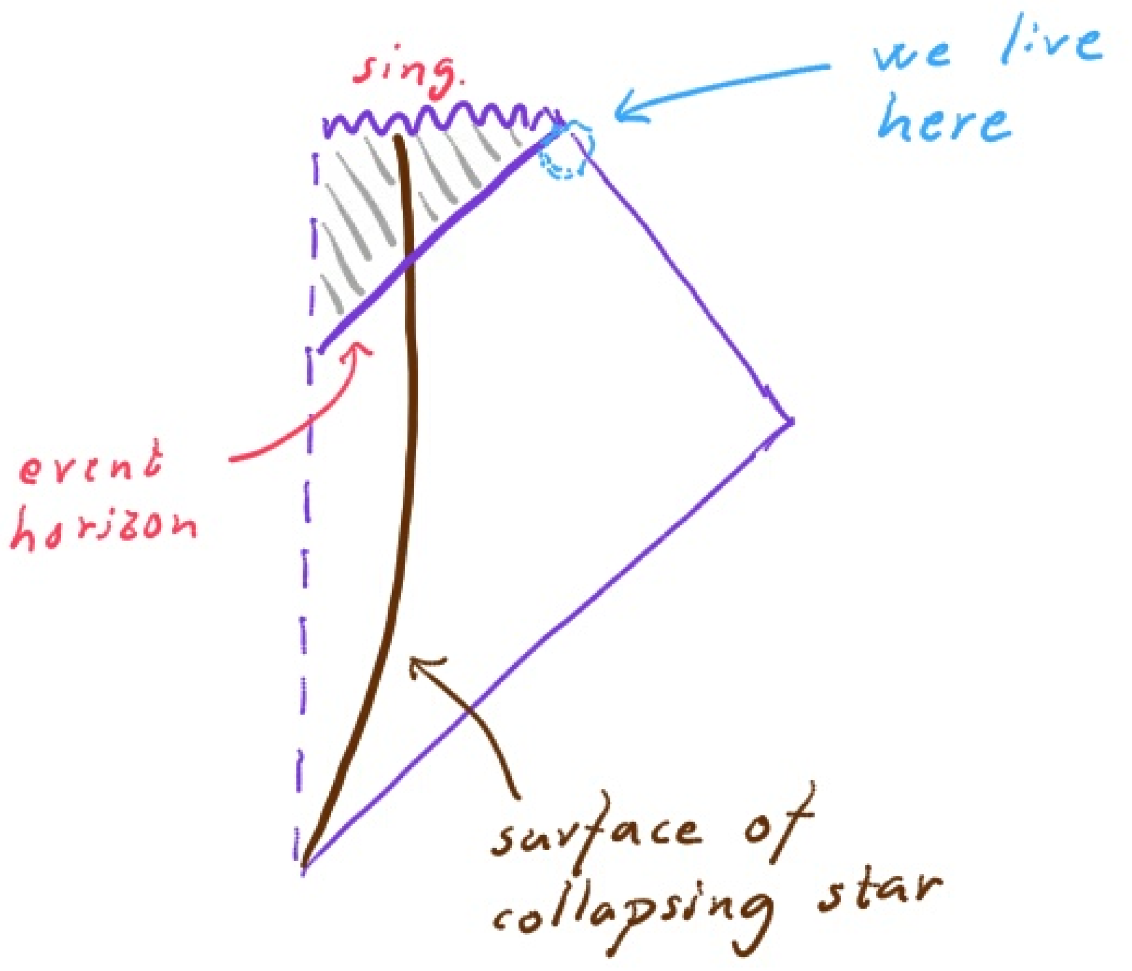
If there are more than one future null infinity, as here, then there is one event horizon for each of them.

Def: A black hole is the region of spacetime that does not belong to the past of any future null infinity.

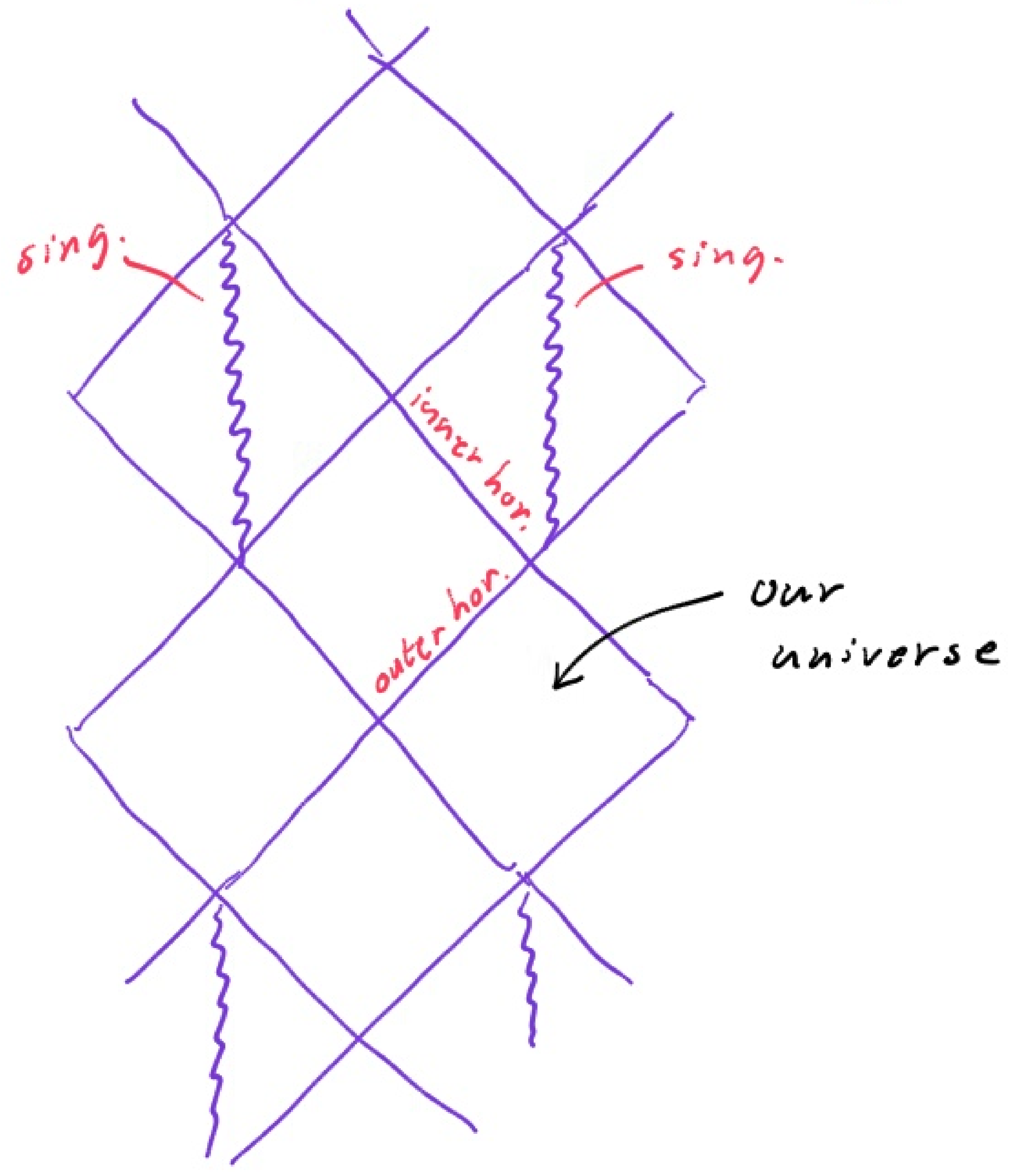
Sometimes, depending on context, it is defined instead with respect to a particular future null infinity.

Now, if we add a collapsing star to this, so that it becomes a realistic black hole, the picture changes drastically.

Penrose diagram of a Schwarzschild black hole

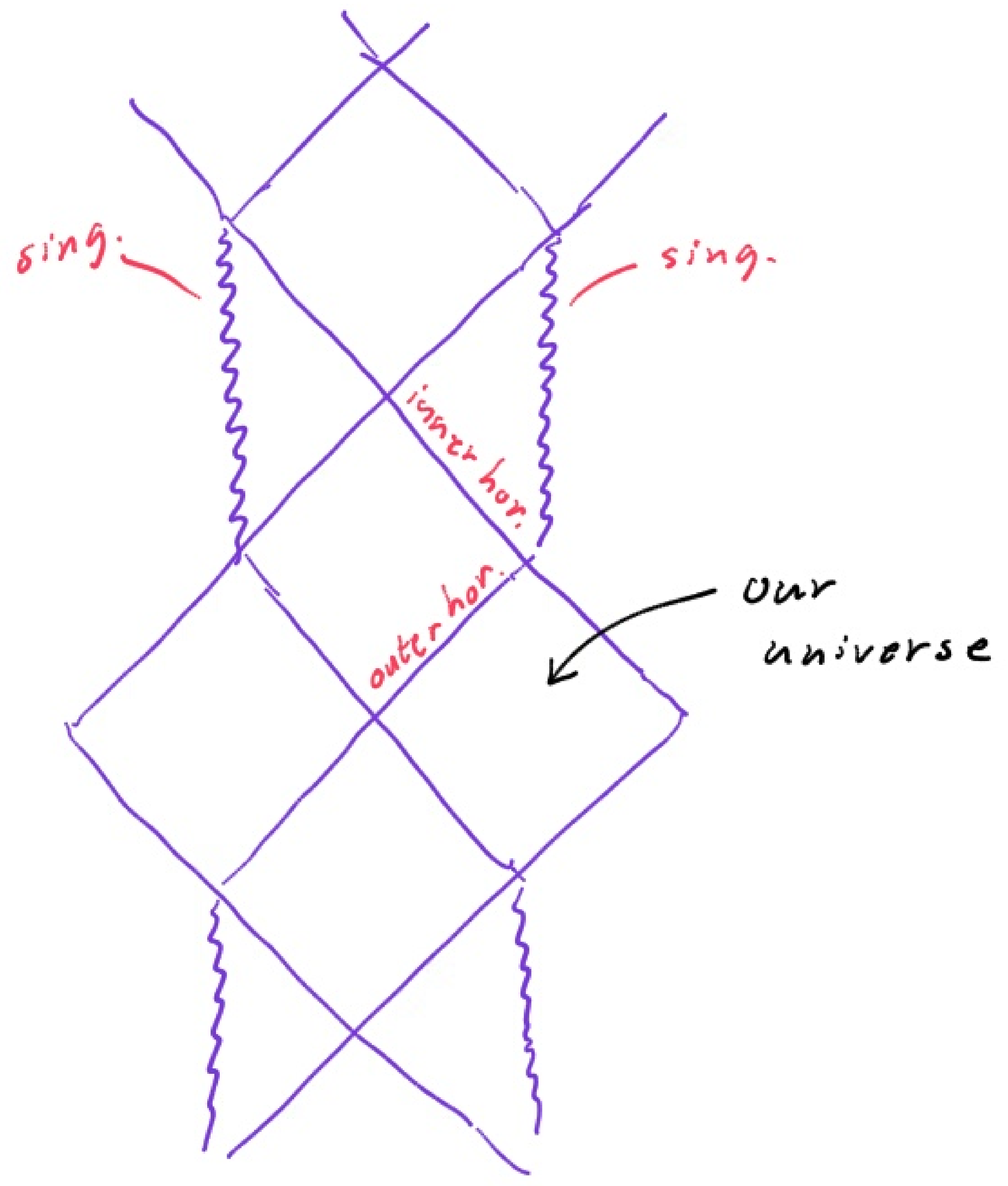


Just to give a couple of other examples:



Extended Kerr

(Rotating black hole, without collapsing star)



Reissner-Nordström black hole

(Charged black hole without collapsing star)