Lecture 11

Newtonian equation of deviation

- tidal acceleration tensor A_{ij}
- Newton's field equations

Einsteinian equation of deviation

- derivation
- the Riemann tensor — again!
- in freely falling ON-frame

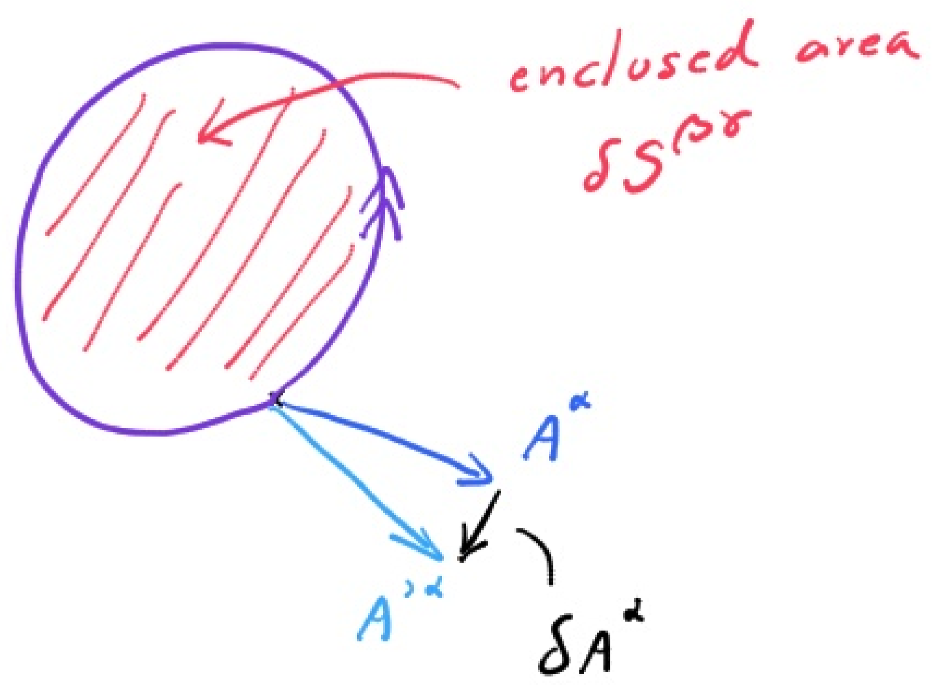
Comparison

- $R^{i}{}_{tjt} = A^{i}{}_{j}$

Einstein's vacuum equation

Last time we found out how to characterize curvature by means of parallel transport:

$$\delta A^{\alpha} = -R^{\alpha}{}_{\delta\beta\gamma}\, \delta S^{\beta\gamma} A^{\delta}$$



where

$$R^{\alpha}{}_{\delta\beta\gamma} A^{\delta} = \left(\nabla_{\beta}\nabla_{\gamma} - \nabla_{\gamma}\nabla_{\beta}\right) A^{\alpha} =$$

$$= \left(\partial_{\beta}\Gamma^{\alpha}_{\gamma\delta} - \partial_{\gamma}\Gamma^{\alpha}_{\beta\delta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\mu}_{\gamma\delta} - \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta}\right) A^{\delta}$$

Given the metric we can calculate the Christoffel symbols and then from them obtain the Riemann curvature tensor, which tells us the result of p.t. around any small loop.

But what does this say about the physics? When we think of gravitation as the expression of spacetime curvature, we are not thinking about the result of p.t. of vectors in spacetime. Rather we are thinking about the orbits of apples or planets, that is, the behaviour of the geodesics (and in particular the timelike ones). From a physicists point of view it would be much better to characterize curvature in terms of spacetime orbits.

As it turns out, the Riemann tensor does that too, and we will now show how.

What are the effects of gravity on geodesics that can be measured locally? If we consider just one geodesic there is no measurable effect — remember that a freely falling observer should not be able to detect any deviation from SR by local measurements. But if we consider a set of nearby geodesics, starting out parallel to each other, they will, because of the spacetime curvature, start to deviate from each other, either converge or diverge.

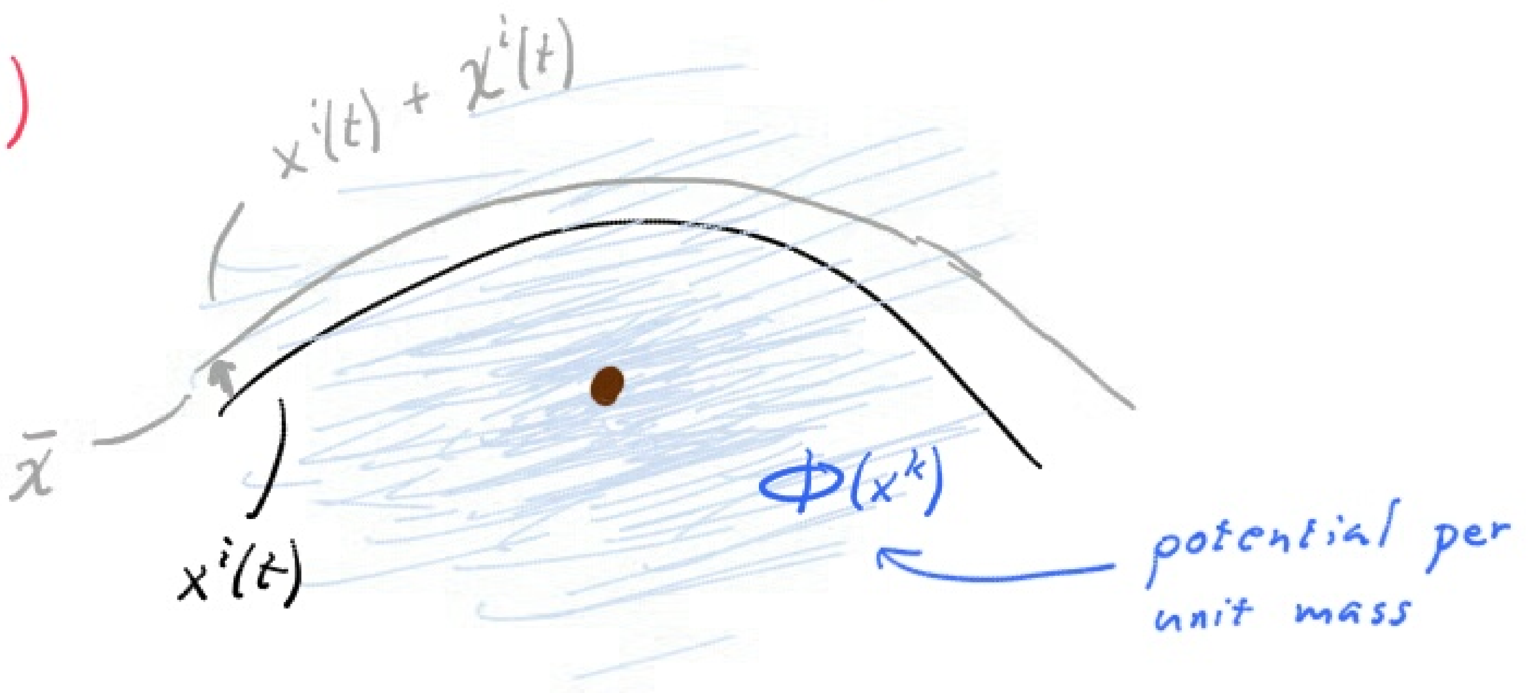
The amount of deviation is captured by the Riemann tensor and described by the so called "geodesic deviation equation".

To get the idea we start by deriving its Newtonian counterpart.

Newtonian equation of deviation

The path of a particle in potential ϕ is described by

$$\frac{d^2 x^i}{dt^2} = \frac{F^i}{m} = -\delta^{ij}\frac{\partial \phi(x^k)}{\partial x^j} \qquad (1)$$



A nearby path (separated from the first by the separation vector $\bar{\chi}$):

$$\frac{d^2(x^i + \chi^i)}{dt^2} = -\delta^{ij}\frac{\partial \phi(x^k + \chi^k)}{\partial x^j} \underset{\bar{\chi}\text{ small}}{\approx} -\delta^{ij}\left(\frac{\partial \phi(x^k)}{\partial x^j} + \chi^l \frac{\partial}{\partial x^l}\frac{\partial \phi(x^k)}{\partial x^j}\right) \qquad (2)$$

Take (2) − (1):

$$\frac{d^2 \chi^i}{dt^2} = -\delta^{ij}\left(\frac{\partial}{\partial x^l}\frac{\partial \phi}{\partial x^j}\right)\chi^l = -\delta^{ij}\underbrace{\frac{\partial^2 \phi}{\partial x^k \partial x^j}}_{\text{tidal acceleration tensor } A_{kj}}\chi^k$$

— the Newtonian deviation equation!

$$\boxed{\frac{d^2 \chi^i}{dt^2} = -A^i{}_k \chi^k}$$

The field equations of Newtons theory — giving the relation between the potential function and the matter distribution — could be expressed nicely using the tidal acceleration tensor.

Gravitational field: $\vec{g}(\vec{x}) = -\nabla \phi(\vec{x})$

"Gauss law": $\nabla \cdot \vec{g}(x) = -4\pi G \mu(\vec{x})$ — mass density

$$\Longrightarrow \underbrace{\nabla^2 \phi(\bar{x})}_{\boxed{?}\ \delta^{ij}A_{ij} = A^i{}_i} = 4\pi G \mu(\bar{x})$$

Hence:

$$\boxed{A^i{}_i = 4\pi G \mu}$$

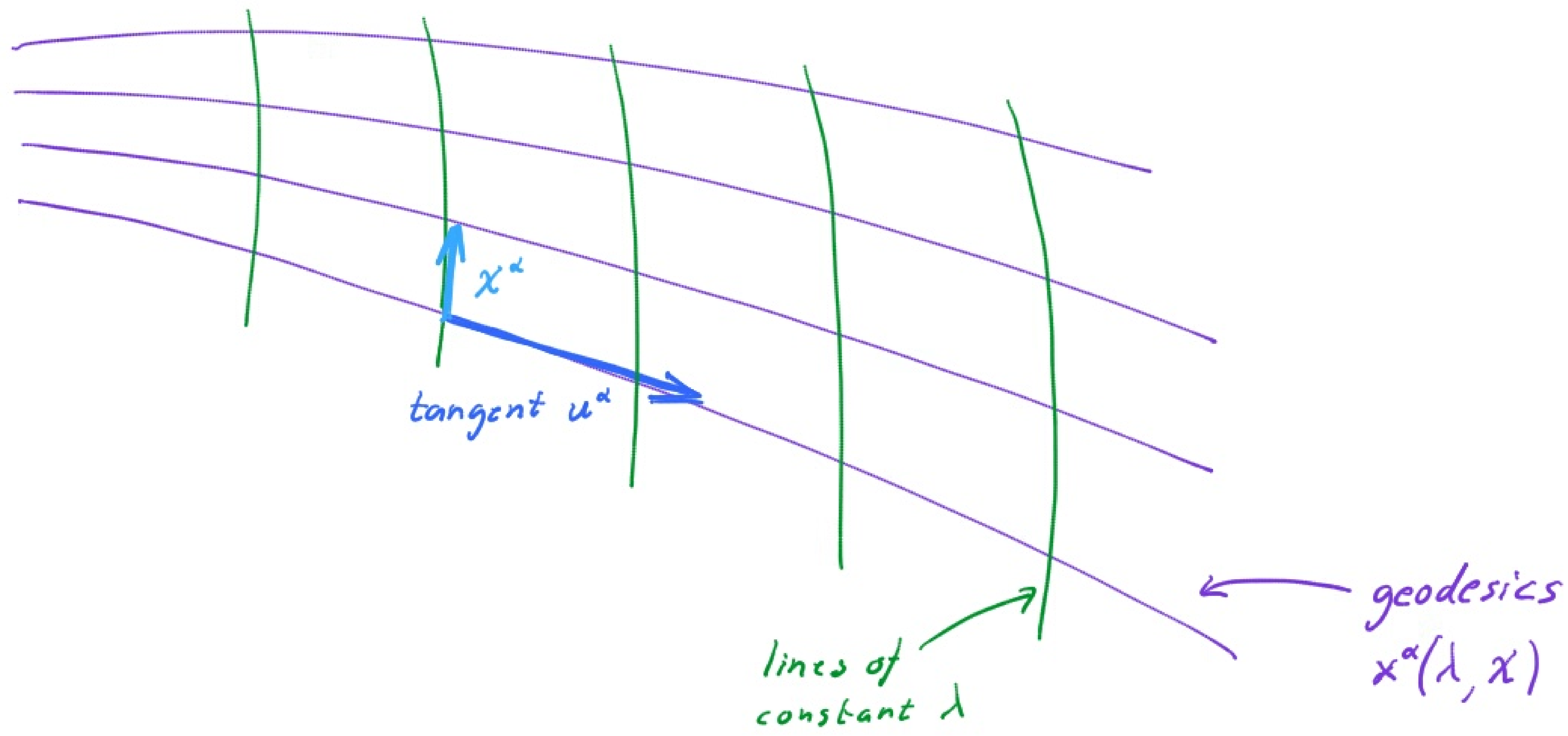
trace of tidal acc. tensor

matter

The corresponding relation in general relativity is closely analogous to this, as we will see.

Einsteinian equation of deviation

Consider a set of geodesics in a 2 dim. surface (embedded in spacetime). Each geodesic is parametrized by λ, and the set of geodesics is parametrized by χ.



Note that λ and χ then are a set of coordinates on this 2 dim. slice. The corresponding coord. basis vectors are $\bar{u}$ (the tangents to the geodesics) and $\bar{\chi}$ (pointing along the lines of constant parameter λ).

The change in $\bar{\chi}$ as we move along a geodesic says something about the behavior of nearby geodesics. For a set of parallel geodesics in flat space, for example, $\bar{\chi}$ would be constant, since such geodesics would stay parallel. For a set of diverging geodesics in flat space, $\bar{\chi}$ would increase, but it would do so linearly. That would also not be so interesting. But suppose we encountered a case where $\bar{\chi}$ increased non-linearly — that would be interesting, since it would mean that geodesics that started out parallel to each other would start to diverge or converge — a symptom of curvature.

Thus we expect

analogous to $\frac{d^2\bar{x}}{dt^2}$ in the Newtonian case

$$\nabla_{\bar{u}} \nabla_{\bar{u}} \bar{\chi} = \text{"acc. of separation"}$$

to have something to do with spacetime curvature.

Let us start from the geodesic equation with the goal of deriving an expression for this second covariant derivative of $\bar{\chi}$.

The geodesic equation: $\nabla_{\bar u}\bar u = 0$

$$u^\alpha \nabla_\alpha u^\beta = 0$$

where $u^\alpha = \dfrac{dx^\alpha}{d\lambda}$

Let us take the derivative in the $\bar{\chi}$-direction:

$$\chi^\gamma \nabla_\gamma \left(u^\alpha \nabla_\alpha u^\beta\right) = 0$$

$$\chi^\gamma\Big((\nabla_\gamma u^\alpha)(\nabla_\alpha u^\beta) + u^\alpha \underbrace{\nabla_\gamma \nabla_\alpha u^\beta}\Big) = 0$$

$$\nabla_\alpha \nabla_\gamma u^\beta - (\nabla_\alpha\nabla_\gamma - \nabla_\gamma\nabla_\alpha)u^\beta =$$

$$= \nabla_\alpha \nabla_\gamma u^\beta - R^\beta{}_{\delta\alpha\gamma}\, u^\delta$$

$$\chi^\gamma(\nabla_\gamma u^\alpha)(\nabla_\alpha u^\beta) + \underbrace{u^\alpha \chi^\gamma \nabla_\alpha \nabla_\gamma u^\beta}_{\nabla_\alpha(\chi^\gamma \nabla_\gamma u^\beta) - (\nabla_\alpha \chi^\gamma)(\nabla_\gamma u^\beta)} = R^\beta{}_{\delta\alpha\gamma}\, u^\delta u^\alpha \chi^\gamma$$

Now, remember that $\bar\chi$ and $\bar u$ are coordinate vectors, that is, we can write them as

$$u^\alpha = \frac{\partial x^\alpha}{\partial \lambda}, \qquad \chi^\alpha = \frac{\partial x^\alpha}{\partial \chi}$$

This leads to this relation:

$$u^\beta \nabla_\beta \chi^\alpha = u^\beta\left(\partial_\beta \chi^\alpha + \Gamma^\alpha_{\delta\beta}\chi^\delta\right) = \frac{\partial}{\partial\lambda}\frac{\partial x^\alpha}{\partial \chi} + \Gamma^\alpha_{\delta\beta}\chi^\delta u^\beta =$$

$$= \frac{\partial}{\partial \chi}\frac{\partial x^\alpha}{\partial\lambda} + \Gamma^\alpha_{\beta\delta} u^\delta \chi^\beta = \chi^\beta\left(\partial_\beta u^\alpha + \Gamma^\alpha_{\beta\delta}u^\delta\right) = \chi^\beta \nabla_\beta u^\alpha$$

Write out the equation above once more, and make use of this:

$$\cancel{\chi^\gamma(\nabla_\gamma u^\alpha)(\nabla_\alpha u^\beta)} + \underbrace{u^\alpha \nabla_\alpha(\underbrace{\chi^\gamma \nabla_\gamma u^\beta}_{u^\gamma \nabla_\gamma \chi^\beta})}_{\nabla_{\bar u}\nabla_{\bar u}\chi^\beta} - \cancel{u^\alpha(\underbrace{\nabla_\alpha \chi^\gamma)(\nabla_\gamma u^\beta}_{\chi^\alpha \nabla_\alpha u^\gamma})} = R^\beta{}_{\delta\alpha\gamma}\, u^\delta u^\alpha \chi^\gamma$$

Thus:

$$\boxed{\nabla_{\bar u}\nabla_{\bar u}\chi^\beta = R^\beta{}_{\delta\alpha\gamma}\, u^\delta u^\alpha \chi^\gamma = -R^\beta{}_{\delta\gamma\alpha}\, u^\delta \chi^\gamma u^\alpha} \qquad (*)$$

So the Riemann tensor characterizes geodesic deviation.

To facilitate the comparison to the Newtonian case, let us first write this in an orthonormal freely falling basis.

First, let me remind you how to go from one set of basis vectors to another.

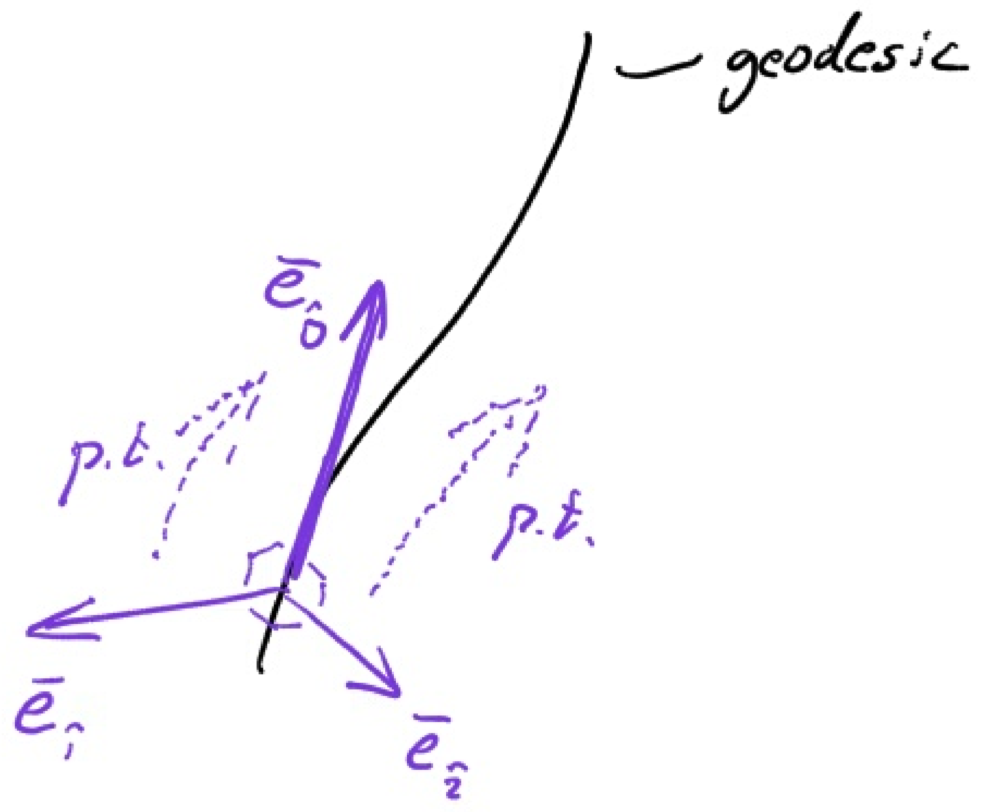
In general: $v^\alpha = \bar{e}^\alpha \cdot \bar{v}$

$\{\bar{e}_{\hat{\alpha}}\}$ ON basis : $\quad (e_{\hat{\alpha}})^\alpha = \bar{e}^\alpha \cdot \bar{e}_{\hat{\alpha}} \quad \left(= (e^\alpha)_{\hat{\alpha}} \right)$

$$a^{\hat{\alpha}} = \bar{e}^{\hat{\alpha}} \cdot \bar{a} = (e^{\hat{\alpha}})_\alpha \, a^\alpha$$

$$t^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}} = t^\alpha{}_{\beta\gamma} (e^{\hat{\alpha}})_\alpha (e_{\hat{\beta}})^\beta (e_{\hat{\gamma}})^\gamma$$

$\boxed{?}$ A freely falling ON-basis, along a geodesic with tangent $\bar{u}$, is one where $\bar{e}_{\hat{0}} = \bar{u}$ and the other basis vectors are p.t. along the geodesic.



Now since all basis vectors are p.t. along the geodesic we have

$$\nabla_{\bar{u}} \bar{e}_{\hat{\alpha}} = 0 \qquad \nabla_{\bar{u}} \bar{e}^{\hat{\alpha}} = 0$$

When translating (*) to the freely falling frame, we can therefore move the basis vectors inside the covariant derivative, for example:

$$(e^{\hat{\beta}})_\beta (\nabla_{\bar{u}} \bar{x})^\beta = \bar{e}^{\hat{\beta}} \cdot \nabla_{\bar{u}} \bar{x} = \nabla_{\bar{u}} (\bar{e}^{\hat{\beta}} \cdot \bar{x}) = \nabla_{\bar{u}} \left((e^{\hat{\beta}})_\beta \, x^\beta \right) = \nabla_{\bar{u}} x^{\hat{\beta}} = \frac{dx^{\hat{\beta}}}{d\tau}$$

Thus the left hand side will become

$$\frac{d^2 x^{\hat{\beta}}}{d\tau^2}$$

For the right hand side, we just put hats on all indices:

$$= -R^{\hat{\beta}}{}_{\hat{\delta}\hat{\gamma}\hat{\alpha}} u^{\hat{\delta}} x^{\hat{\gamma}} u^{\hat{\alpha}} = -R^{\hat{\beta}}{}_{\hat{\tau}\hat{\gamma}\hat{\tau}} x^{\hat{\gamma}}$$

$(e_{\hat{0}})^{\hat{\alpha}} = (1,0,0,0)$

In the Newtonian limit ($v \ll c$) we expect $\tau = t$ and the separation vector $\bar{\chi}$ to only have spatial components:

$$\underline{v \ll c}: \qquad \frac{d^2 \chi^i}{dt^2} = - R^i_{\ tjt}\, \chi^j$$

This has exactly the same form as the Newtonian expression:

$$\frac{d^2 \chi^i}{dt^2} = - A^i_{\ j}\, \chi^j$$

In a problem (problem 5) you will show that, indeed, for the weak-field line-element

$$ds^2 = -(1 + 2\Phi)\, dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2)$$

$$\Rightarrow \quad R^i_{\ tjt} = \delta^{ik} \frac{\partial^2 \Phi}{\partial x^k \partial x^j} = A^i_{\ j}$$

The next big question that we have to deal with, is what the connection is between "the field" R and the matter distribution in spacetime — the field equations of general relativity! For that we first need a relativistic way to express the matter (or energy) content. That is the subject of the next lecture.

But we can already try to answer the more modest question of what is the field equations for vakuum, when there is $\underline{\text{no}}$ matter.

The Newtonian vacuum field equation is

$$\text{Newton, vacuum:} \qquad A^i_{\ i} = 0 \qquad \text{where} \quad A_{ij} = \frac{\partial^2 \Phi}{\partial x^i \partial x^j}$$

The analogous equation in terms of $R^\alpha_{\ \beta\gamma\delta}$ would be

$$\underbrace{R^\gamma_{\ \alpha\gamma\beta}}_{\equiv R_{\alpha\beta}} = 0 \qquad \text{or} \qquad \boxed{R_{\alpha\beta} = 0} \quad \text{— Einstein's vacuum equations!}$$

$\equiv R_{\alpha\beta}$ — The Ricci tensor