Lecture 6

Local inertial frames and Riemann normal coordinates

The Schwarzschild metric

- compare "Newtonian e.p.-metric" and weak field metric
- symmetries
- meaning of coord. r
- the gravitational redshift

Review: Newtonian effective potential

Orbits in the Schwarzschild metric

- conserved quantities
- the effective potential
    - interpretation
    - circular orbits

How to construct a local inertial frame

Earlier I claimed that at a point P one can always choose coordinates such that

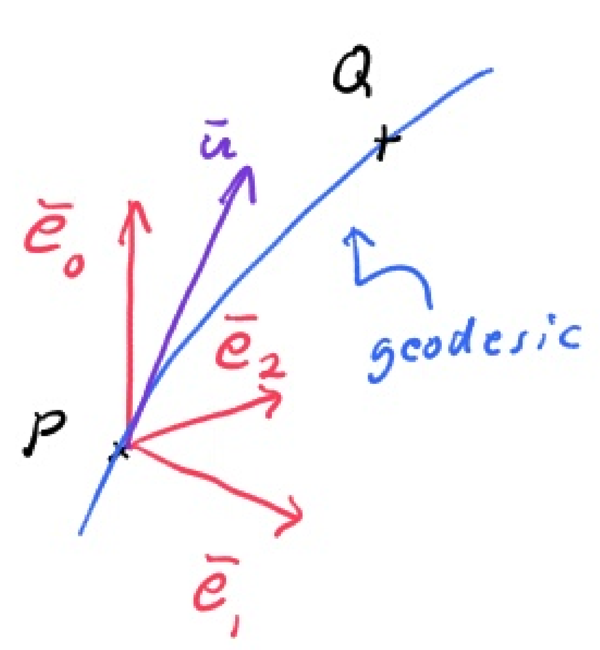
$$g_{\alpha\beta}(x_P) = \eta_{\alpha\beta} \quad (1)$$

$$\partial_\gamma g_{\alpha\beta}\Big|_{x_P} = 0 \quad (2)$$

Local inertial frame at P

With the geodesics at our hand we can see how to construct such coord. explicitly.

Start from a point P and an orthonormal basis at that point.



To construct a coordinate basis out of this orthonormal basis, consider any point Q, the geodesic passing through P and Q, and its unit tangent vector $\bar{u}$ at P. Assign the following coord. to Q:

$$x_Q^\alpha = s u^\alpha = s(u^0, u^1, u^2) = (su^0, su^1, su^2)$$ — Riemann normal coord.

s: proper length along the geodesic from P to Q

As long as there is one unique geodesic from P to Q, this provides Q with a unique coord.

Are conditions (1) and (2) fulfilled by these coord.?

(1) — OK, since $\{\bar{e}_\alpha\}$ is now a coord. basis.

(2): $x_Q^\alpha(s)$ — a geodesic as s varies.

$$\frac{d^2 x_Q^\alpha(s)}{ds^2} = 0 \implies \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

since $x_Q^\alpha = su^\alpha$; $\implies$: use the geodesic equation

$$\Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0$$ — Should hold for all $\bar{u}$ at point P!

$$\Gamma^\mu_{\alpha\beta} = 0 \implies \partial_\gamma g_{\alpha\beta} = 0$$

So condition (2) also OK at P.

So Riemann normal coordinates provide an explicit construction of a local inertial frame. But why can't we apply this same construction to the whole spacetime?

— Further from P there will <u>not</u> be a unique geodesic from P to Q. Stated differently: The geodesics starting out from P will start to cross each other because of the curvature when we are sufficiently far away from P.

The Schwarzschild metric

Earlier we found that e.p. demands that we modify the Minkowski spacetime:

$$ds^2 = -c^2\left(1 - \frac{2MG}{rc^2}\right)dt^2 + dr^2 + r^2 d\Omega^2$$

Magically, the geodesics of this metric turn out to be identical to the Newtonian trajectories, outside a spherically symmetric mass distribution of mass M.

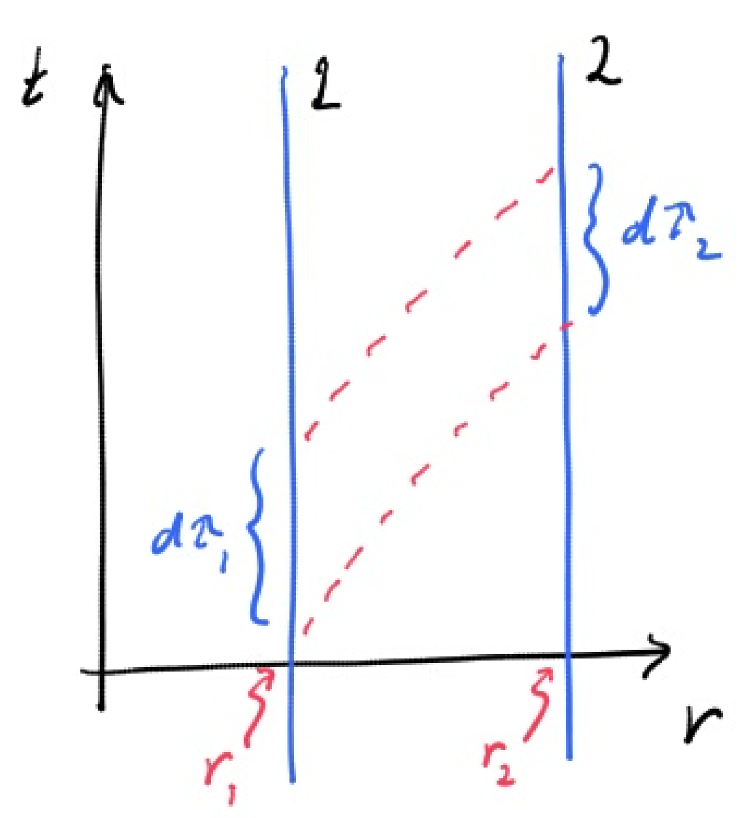
But this argument says nothing about the factor in front of dr^2 (only that it should be $= 1$ to zeroth order).

Later we will show (from Einstein's equations – the field equations of general relativity) that the true line element outside a spherically symmetric distribution of matter is this:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$ — The Schwarzschild metric

For now, we regard this as given, and shall investigate its consequences.

Note:
- × Something strange happens at $r = 2M$. But the star is always much larger than this, so for now we assume $r > 2M$.
- × The coord. r is not the distance to some center. Rather, it gives the area of the sphere surrounding the object at that distance: $A(r) = 4\pi r^2$
- $\boxed{?}$ × Symmetries: $\xi^\alpha = (1, 0, 0, 0)$ $\quad \eta^\alpha = (0, 0, 0, 1)$
- × Time is different at different radiuses, leading to a gravitational redshift. Consider two stationary observers, 1 and 2. Observer 1 sends two signals to 2:



$$d\tau_1 = \left(1 - \frac{2M}{r_1}\right)^{1/2} dt$$

$$d\tau_2 = \left(1 - \frac{2M}{r_2}\right)^{1/2} dt$$

the same because of time symmetry

Let $r_2 = \infty \Rightarrow d\tau_2 = dt = d\tau_\infty$

So $d\tau_1 = \left(1 - \frac{2M}{r_1}\right)^{1/2} d\tau_\infty$

or $d\tau_\infty = \left(1 - \frac{2M}{r_1}\right)^{-1/2} d\tau_1 > d\tau_1$

$$\omega_\infty = \left(1 - \frac{2M}{r_1}\right)^{1/2} \omega_1$$ — light from star will be redshifted

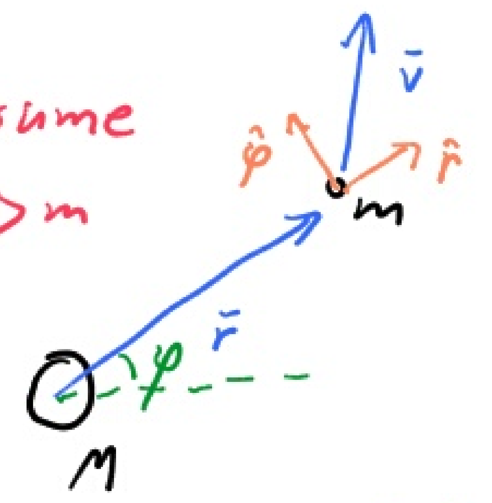
We already know a lot about the orbits in this spacetime: in the weak field-limit – that is, for large r – it reproduces the Newtonian orbits. Now we would like to know if and how its predictions differ from the Newtonian theory.

Let us first review a concept that is useful in Newton's theory when analyzing planetary motion.

Review: the effective potential in Newtonian gravitation

There are two constants of motion: $\boxed{?}$

Assume $M \gg m$



$\vec{v} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$

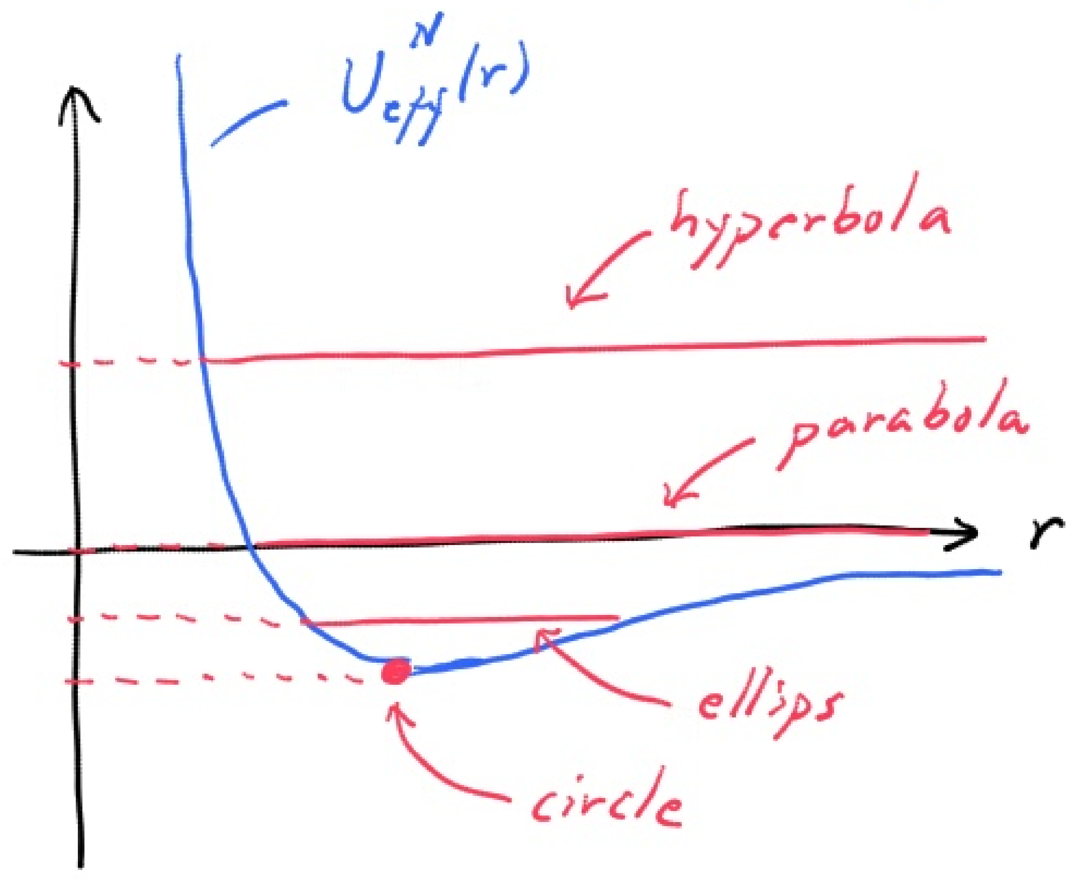
$$\text{Energy: } E = \frac{m|\vec{v}|^2}{2} + U(r) = \frac{m|\vec{v}|^2}{2} - \frac{GmM}{r}$$

$$\text{Angular momentum: } \quad L = |\vec{L}| = |m\vec{r} \times \vec{v}| = mr^2\dot{\varphi}$$

The idea $\boxed{?}$ is to write E in polar coordinates and then use the angular momentum to eliminate its $\dot{\varphi}$-dependence:

$$E = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\varphi}^2}{2} - \frac{GmM}{r} = \frac{m\dot{r}^2}{2} + \frac{\cancel{mr^2}}{2}\frac{L^2}{m^{\cancel{2}}r^{\cancel{4}\,2}} - \frac{GmM}{r} =$$

$$= \frac{m\dot{r}^2}{2} + \underbrace{\frac{L^2}{2mr^2} - \frac{GmM}{r}}_{\equiv\, U^N_{eff}(r)}$$



The exact shape depends on the value of L, but qualitatively $U_{eff}(r)$ looks like this for all $L \neq 0$.

Therefore we can from this graph read off the different possible qualitative behaviors, depending on the value of the energy.

Schwarzschild orbits

In order to analyze what possible qualitative behavior there is in the Schwarzschild spacetime we will use a very similar reasoning. First we need to write out in explicit form the two constants of motion corresponding to the Killing fields $\bar{\xi}$ and $\bar{\eta}$.

$$e \equiv -\bar{\xi}\cdot\bar{u} = -g_{\alpha\beta}\xi^{\alpha}u^{\beta} = -g_{00}u^{0} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}$$

— energy per unit mass (or rest energy)

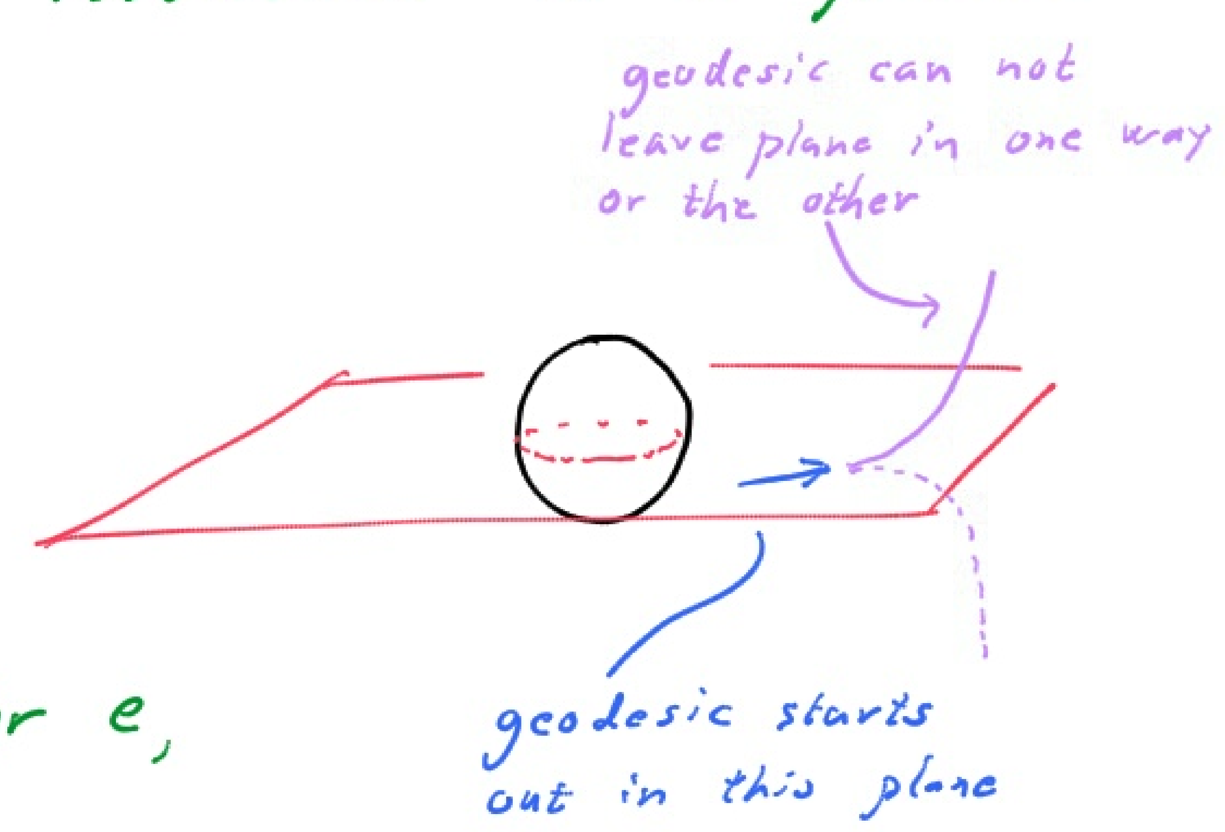
For large r this is just the energy per unit mass.

$$l \equiv \bar{\eta}\cdot\bar{u} = g_{\alpha\beta}\eta^{\alpha}u^{\beta} = g_{33}u^{3} = r^2\sin^2\theta\,\frac{d\varphi}{d\tau}$$

— angular momentum per unit mass

For small speeds this is just the ordinary angular momentum per unit mass.

As in Newtonian gravitation, symmetry (or conservation of l) directly implies that an orbit is restricted to a plane:



So we can choose

$$\theta = \frac{\pi}{2}\,,\quad u^{\theta} = 0$$

The goal now is to use the constants of motion to find an expression for e, only in terms of r and $\frac{dr}{d\tau}$.

By definition, $\bar{u}$ is normalized:

$$\bar{u}\cdot\bar{u} = g_{\alpha\beta}u^{\alpha}u^{\beta} = -1$$

Let us write this out (remembering that $g_{\alpha\beta}$ is diagonal and that $u^{\alpha} = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\varphi}{d\tau}\right)$):

$$-\underbrace{\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2}_{\left(1-\frac{2M}{r}\right)^{-2}e^2} + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{r^2\left(\frac{d\varphi}{d\tau}\right)^2}_{\frac{l^2}{r^4}} = -1$$

Multiply by $-\left(1-\frac{2M}{r}\right)$:

$$e^2 - \left(\frac{dr}{d\tau}\right)^2 - \frac{l^2}{r^2}\left(1-\frac{2M}{r}\right) = \left(1-\frac{2M}{r}\right)$$

$$e^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\left(1+\frac{l^2}{r^2}\right)$$

To make this easier to compare with the corresponding Newtonian expression, let us subtract -1 and divide by 2:

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{l^2}{r^2}\right)-1\right]$$

How can we compare a relativistic expression for energy with a Newtonian? Remember that in relativity the rest energy is included in the expression for energy. Thus, to make the comparison let us introduce

$$E_N = emc^2 - mc^2$$

So that $e = \dfrac{E_N + mc^2}{mc^2} = 1 + \dfrac{E_N}{mc^2}$

Then $\dfrac{e^2-1}{2} = \dfrac{1}{2}\left[\left(1+\dfrac{E_N}{mc^2}\right)^2 - 1\right] \approx \dfrac{1}{2}\left[\left(1+\dfrac{2E_N}{mc^2}\right)-1\right] = \dfrac{E_N}{mc^2}$

So in the non-relativistic limit $\dfrac{e^2-1}{2}$ is just the Newtonian energy per unit rest energy. The first term on the right hand side is the kinetic energy in this limit. Therefore the second term should be analogous to the effective potential:

$$\underbrace{\frac{e^2-1}{2}}_{\text{Constant for given } e} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

where $V_{eff}(r) = \dfrac{1}{2}\left[\left(1-\dfrac{2M}{r}\right)\left(1+\dfrac{l^2}{r^2}\right)-1\right] =$

$$= -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}$$

To compare this with the Newtonian expression, multiply with mc^2 and let

$$M \to \frac{MG}{c^2} \qquad l \to \frac{L}{cm}$$

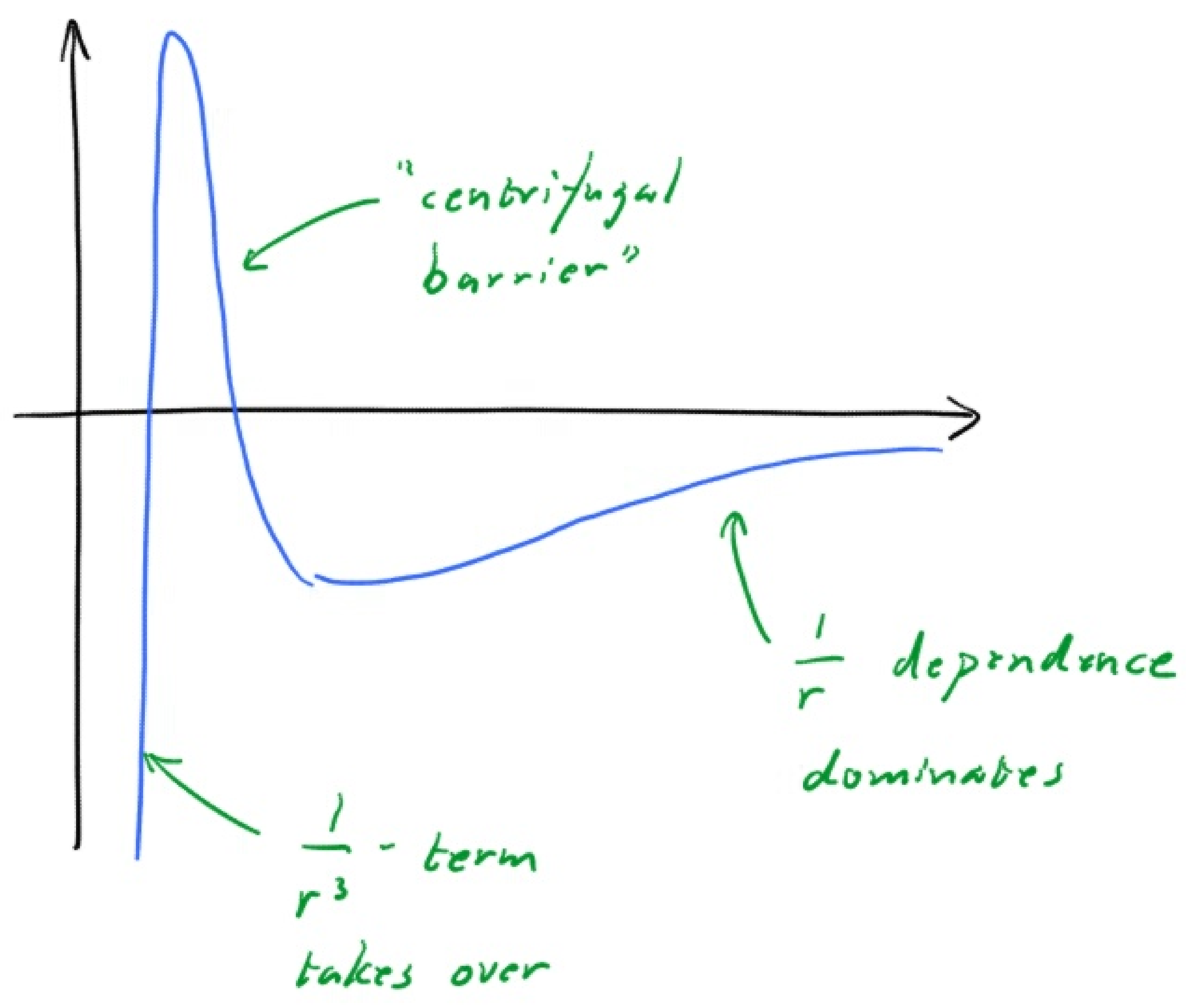
$$\Rightarrow mc^2 V_{eff}(r) = -\cancel{mc^2}\frac{MG}{\cancel{c^2}r} + \cancel{mc^2}\frac{L^2}{\cancel{c^2}m^{\cancel{2}}2r^2} - \cancel{mc^2}\frac{MG}{\cancel{c^2}}\frac{L^2}{c^2m^2}\frac{1}{r^3} =$$

$$= \underbrace{-\frac{mMG}{r} + \frac{L^2}{2mr^2}}_{= U^N_{eff}(r)} - \frac{MGL^2}{c^2mr^3}$$

New term!

The new term is small in the non-relativistic limit, and also negligible for large r. Note that it is attractive.

$$V_{eff}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}$$



Note that we can use this graph in the same way that we use its Newtonian counterpart: The particle can only be at those r where

$$\frac{e^2-1}{2} > V_{eff}(r)$$

and it will turn where

$$\frac{e^2-1}{2} = V_{eff}(r) \quad \text{since then} \quad \frac{dr}{d\tau} = 0 .$$

Thus we can read off the possible qualitative behaviour from the graph.

For large r the orbits are similar to the Newtonian one: we can have orbits that are almost elliptical. But the $\frac{1}{r^3}$-term leads to that these orbits don't quite close: the ellipses precesses.