Lecture 8

Manifolds without a metric

- functions, coordinates
- vectors as directional derivatives
- dual vectors as functions $\omega : V \to \mathbb{R}$
- transformation of vectors and dual vectors

Introducing a scalar product = metric

- correspondence between vectors and dual vectors
- the inverse metric
- basis vectors and dual basis vectors

The three roles of the metric

You already know a lot about how to describe the effects of gravity in terms of a curved spacetime. Given a line element you know how to calculate the proper distance or proper time along any curve. You know how to find the geodesics, and how Killing fields of the spacetime give rise to conserved quantities.

But how can we know what line element to use? For example, how do we know that the Schwarzschild line element really describes spacetime outside a star, as I have claimed.

More generally: What is the connection between the matter content in a spacetime and its geometry? The answer is given by Einstein's field equations. In order to formulate them, we first need some invariant measure of curvature. The line element in principle contains all information about the curvature. But, in itself, it is not a measure of curvature. Remember that curvature is a local concept (that is, it has some value at each point) but locally the line element can be made identical to the Minkowski line element. Also, as we have seen, the first derivatives can be made to vanish. Thus we expect a good measure of curvature to be made up from the second derivatives of the metric.

To find this curvature measure we have to do the mathematics of curved spaces more carefully. [Today: vectors and indices]

Why indices up and down?

To begin, let us first forget about the metric and the line element. Say that we just have a smooth manifold, that is, a continuous set of points. What mathematical structures are then meaningful?

Manifolds without a metric

We can think of the manifold as an elastic sheet. Since we have no metric, no distances are defined, and so we could stretch and bend

the sheet in whatever way we like without actually changing the manifold — it will still be the same set of points. Also note that orthogonality is not defined on such an elastic sheet.

What kind of structures could be defined on such an elastic sheet?

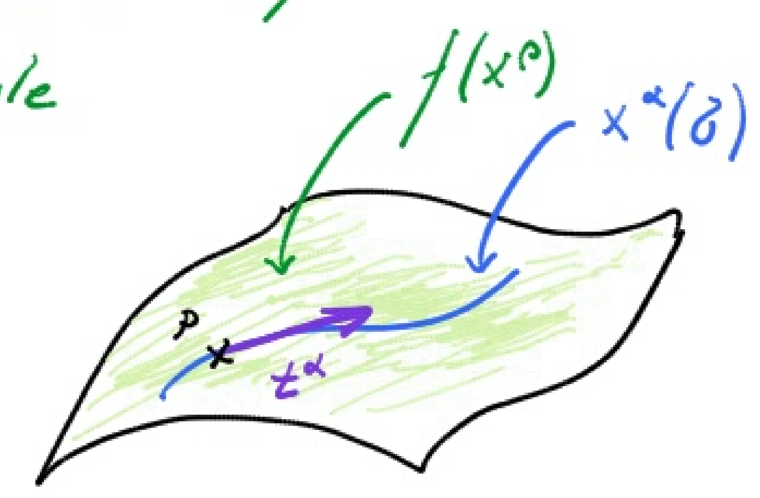
- functions $f : M \to \mathbb{R}$

We can associate a value to each point. Stretching will not change this association between points and values.

In particular this means that we can put coordinates on our manifold:

- coordinates x^α — a set of functions such that each point is associated with a unique set of values. (Several such coordinate patches may be needed to cover the whole manifold.)

- curves $x^\alpha(\lambda)$



- directional derivatives

Given a curve $x^\beta(\lambda)$ and a function $f(x^\beta)$:

$$\frac{d}{d\lambda} f(x^\beta)\Big|_P = \frac{dx^\alpha}{d\lambda}\Big|_P \partial_\alpha f(x^\beta)\Big|_P = \underbrace{t^\alpha \partial_\alpha f(x^\beta)\Big|_P}_{\substack{\text{invariant} \\ = \text{coord. independent}}}$$

— the change in f at P along the curve. This is also independent of how I stretch the surface.

The role of the "tangent vector" components t^α is to specify the directional derivative.

A directional derivative contains the same kind of information as a vector: A magnitude and a direction. Hence we can identify them!

- vectors = directional derivatives

$$\bar{t} \equiv t^\alpha \partial_\alpha \equiv t^\alpha \bar{e}_\alpha$$

components — coord. basis

This may sound a bit abstract, but think about it this way. Partial derivatives at a point P fulfil the same rules that we are used to apply to vectors: you can add or substract them and thereby obtain a new partial derivative for another direction. You add them by just adding their components. In this sense they span a tangent space at point P, and we might as well identify them with the vectors at point P.

Viewing vectors as directional derivatives also gives rise to the familiar transformation rules for changing coordinates:

$$a^\alpha \partial_\alpha = a^\alpha \underbrace{\frac{\partial}{\partial x^\alpha}}_{\text{chainrule}} = a^\alpha \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial}{\partial x^{\alpha'}} \equiv a^{\alpha'} \partial_{\alpha'} \implies \boxed{a^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} a^\alpha}$$

Note that, so far, we have no notion of scalar product, and thus "orthogonality" of two vectors have no meaning — stretching changes the angles!

– vector fields

Having introduced vectors at a point we may of course associate a vector to <u>each</u> point, thereby obtaining a vector field.
(But remember — there is no way of comparing vectors at different points.)

We can introduce one more object on our elastic surface.

– dual vectors

These are linear functions of the vectors (at p) into real numbers.

$$\omega(\bar{a}) = [\text{real number}]$$

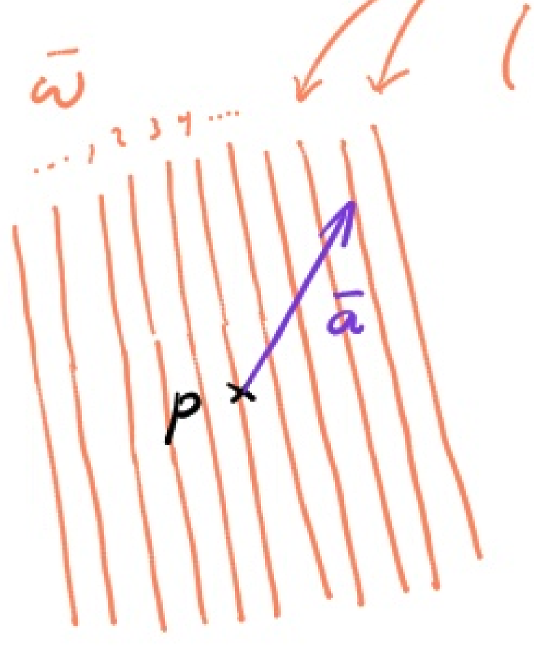
linearity: $\omega(r\bar{a} + s\bar{b}) = r\,\omega(\bar{a}) + s\,\omega(\bar{b})$

$$\Rightarrow \quad \omega(\bar{a}) = \underbrace{\omega_\alpha}_{\text{components of the dual vector}} a^\alpha$$

— most general linear map if $\omega(\bar{0}) = 0$.

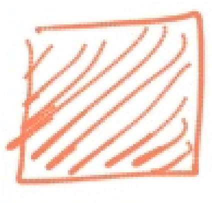
We are <u>not</u> allowed to view the dual vectors as small arrows.
So how should we understand them? Here is a pictorial way:

oriented set of hyperplanes (in tangent space of p)



$$\omega(\bar{a}) = 4$$

The set of hyperplanes at p defines a function of vectors at p in this sense:
The value that comes out when ω acts on $\bar{a}$ is the number of ω-surfaces that $\bar{a}$ penetrates.



You have actually met such dual vectors before, namely as gradients of functions:

$$t^\alpha \underbrace{\frac{\partial}{\partial x^\alpha} f(x^\beta)}_{\partial_\alpha f \,\text{— gradient of } f} = [\text{real number}]$$

Hence $\partial_\alpha f$ specify a linear map from vectors to real numbers.
$\partial_\alpha f$ is just the components of that map, or dual vector.
You may have imagined the gradient as a vector. That is wrong! The mental picture should rather be the level curves/surfaces of the function. Contracting with t^α then tells how many of these that $\bar{t}$ penetrates.

Since the number that the dual vector produces when acting on a vector should be independent of the particular coordinates being used, and since we know how the vector components transform, we immediately get the transformation rule also for the dual vector:

$$w(\bar{a}) = w_\alpha a^\alpha = w_\alpha \frac{\partial x^\alpha}{\partial x^{\alpha'}} a^{\alpha'} = w_{\alpha'} a^{\alpha'}$$

$$\Rightarrow \boxed{w_{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} w_\alpha}$$

Note the difference from the vector transformation formula.

The w_α are the components of $\bar{w}$. That means there is a set of basis dual vectors:

$$\bar{w} = w_\alpha \bar{e}^\alpha$$

What defines $\bar{e}^\alpha$?

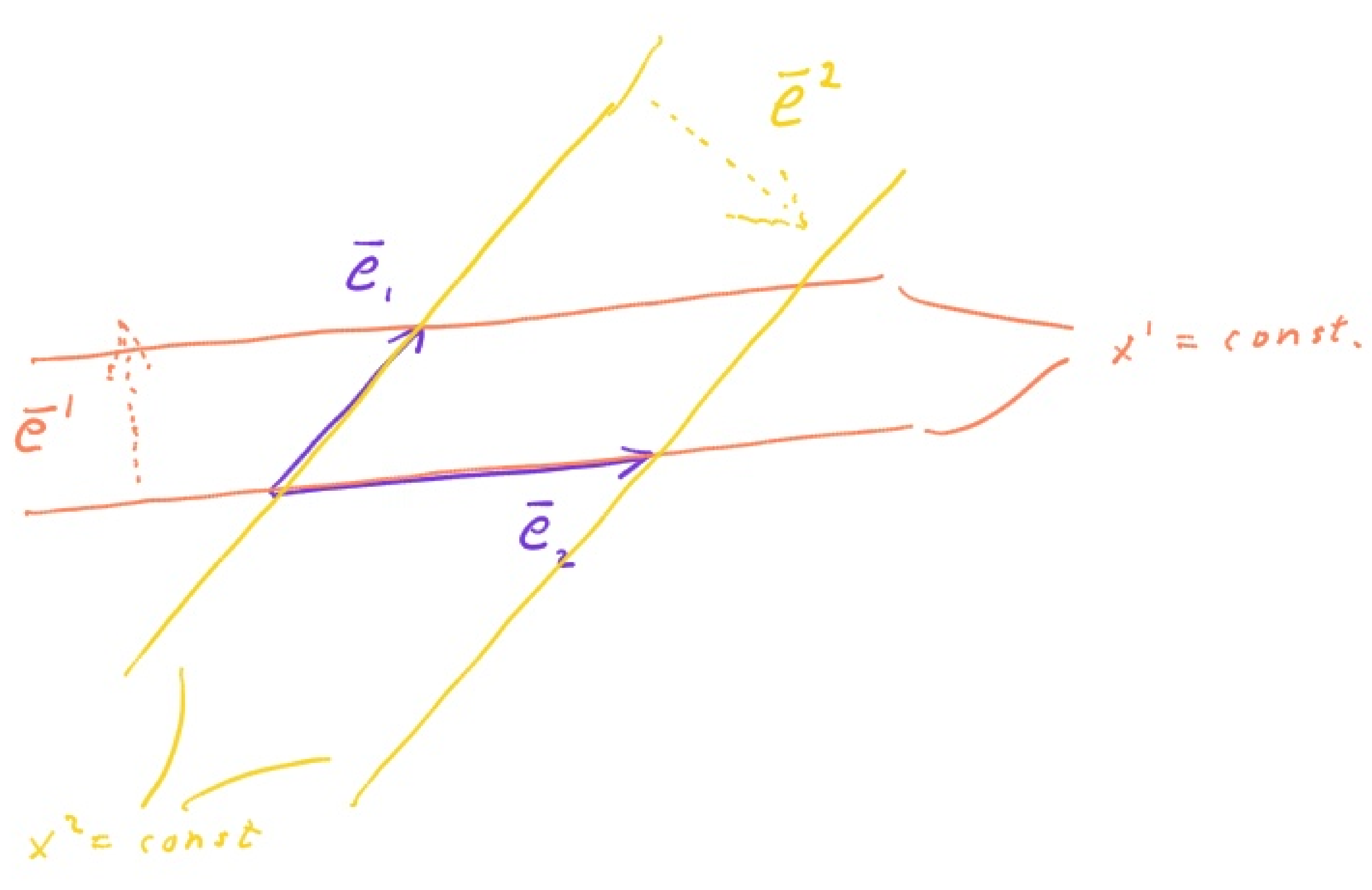
linearity

should be equal to

$$w(\bar{a}) = w_\alpha e^\alpha(a^\beta \bar{e}_\beta) = w_\alpha a^\beta e^\alpha(\bar{e}_\beta) = w_\alpha a^\alpha$$

$$\Rightarrow e^\alpha(\bar{e}_\beta) = \delta^\alpha{}_\beta$$

Consider a coordinate system:



$\bar{e}_1$ and $\bar{e}_2$ are the coordinate vectors.

$\bar{e}^1$ and $\bar{e}^2$ are the set of coordinate lines.

- dual vector fields

Of course, we can associate a dual vector to each point. (One way of doing this is just to take the gradient of a function.)

We could also introduce tensors, that is, objects with more than one index. And dual tensors. And mixed tensors.

So there is a lot of things that can be done with just a smooth manifold, without any further structure. Now, let us introduce a metric on our manifold.

Manifolds with a metric

To introduce a metric means to specify a scalar product. And the scalar product is specified by giving the result of the scalar product between all basis vectors:

Define a scalar product: $\bar{e}_\alpha \cdot \bar{e}_\beta = g_{\alpha\beta}$

In general: $\bar{a} \cdot \bar{b} = a^\alpha \bar{e}_\alpha \cdot b^\beta \bar{e}_\beta = a^\alpha b^\beta \, \bar{e}_\alpha \cdot \bar{e}_\beta = a^\alpha b^\beta g_{\alpha\beta}$

This gives a meaning to "orthogonality" $(\bar{a} \cdot \bar{b} = 0)$ and also to the norm (length) of a vector.

For example, let

$\bar{a} = dx^\alpha \, \bar{e}_\alpha$ — infinitesimal displacement

be the infinitesimal vector representing the displacement from one event to a neighbouring event. Then $\bar{a} \cdot \bar{a}$ should give the squared distance between the events:

$$ds^2 = \bar{a} \cdot \bar{a} = dx^\alpha \bar{e}_\alpha \cdot dx^\beta \bar{e}_\beta = dx^\alpha dx^\beta \, \bar{e}_\alpha \cdot \bar{e}_\beta = g_{\alpha\beta} dx^\alpha dx^\beta$$

Thus we see that, fixing the scalar product also fixes the line element. In our "elastic sheet"-representation, once a metric is introduced, we are not any longer allowed to stretch the sheet.

Introducing a metric also has another dramatic consequence:

Given a vector $\bar{a}$ the scalar product of that vector with other vectors clearly defines a linear map from vectors to real numbers. Hence, to each vector $\bar{a}$ there corresponds a dual vector:

$$\left.\begin{aligned} a(\bar{b}) &\equiv \bar{a} \cdot \bar{b} = a^\alpha b^\beta g_{\alpha\beta} \\ &= a_\beta b^\beta \end{aligned}\right\} \Rightarrow \quad a_\beta = g_{\alpha\beta} a^\alpha$$

So the metric provides a mapping between vectors and dual vectors! The mapping is 1-1 since we can introduce the inverse metric:

$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma$ — def. of inverse metric $g^{\alpha\beta}$

Then $g^{\gamma\beta} a_\beta = g^{\gamma\beta} g_{\alpha\beta} a^\alpha = \delta^\gamma{}_\alpha a^\alpha = a^\gamma$

Thus the metric plays three roles:

(1) $g_{\alpha\beta}$ defines the scalar product

(2) $g_{\alpha\beta}$ specifies the line element

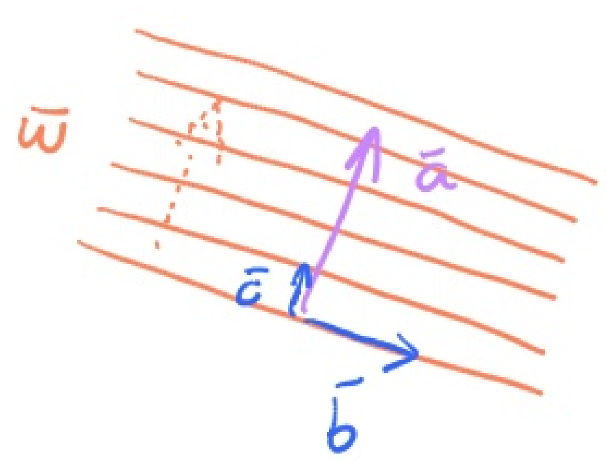
(3) $g_{\alpha\beta}$ $(g^{\alpha\beta})$ lowers (raises) indices

Now, since there is this 1-1 mapping we are free to view vectors and dual vectors as just different representations of the same thing. Since dual vectors are somewhat confusing, we will consider both of them as just vectors. Upstairs and downstairs components are then just different representations.

There is a pictorial way to explain these two representations.

First, what does it mean to view a dual vector as a vector?

$\boxed{?}$



$\bar{w}$ corresponds to $\bar{a}$ where

$a^\alpha = g^{\alpha\beta} w_\beta$ or $w_\alpha = g_{\alpha\beta} a^\beta$

$w(\bar{b}) = 0 \Rightarrow w_\alpha b^\alpha = 0 \Rightarrow g_{\alpha\beta} a^\beta b^\alpha = 0$

vector in a hypersurface of $\bar{w}$

$\Rightarrow \bar{a} \cdot \bar{b} = 0$

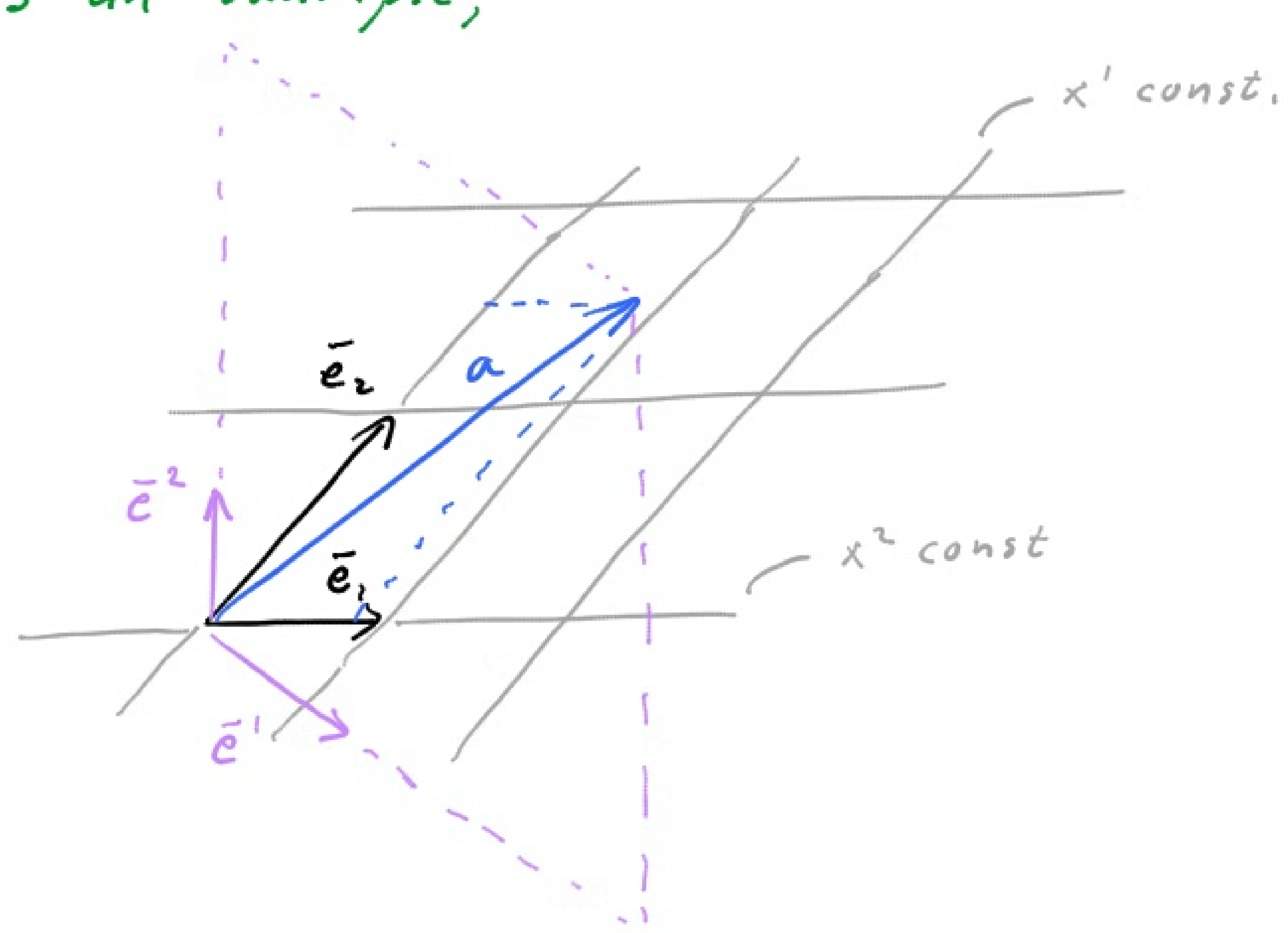
Hence, the vector $\bar{a}$ corresponding to $\bar{w}$ is orthogonal to the hypersurfaces of $\bar{w}$.

$w(\bar{c}) = 1 \Rightarrow w_\alpha c^\alpha = 1 \Rightarrow g_{\alpha\beta} a^\beta c^\alpha = 1 \Rightarrow \bar{a} \cdot \bar{c} = 1$

vector from one hypersurface of $\bar{w}$ to the next

Hence, the denser the surfaces of $\bar{w}$, the longer is $\bar{a}$.

As an example, consider a skew coordinate system in the flat plane:



There are now two ways to represent some vector $\bar{a}$:

$\bar{a} = a^1 \bar{e}_1 + a^2 \bar{e}_2$

$= a_1 \bar{e}^1 + a_2 \bar{e}^2$

So a^α and a_α are then just different representations of one and the same vector $\bar{a}$.

Note that $\bar{e}^\alpha \cdot \bar{e}_\beta = \delta^\alpha{}_\beta$