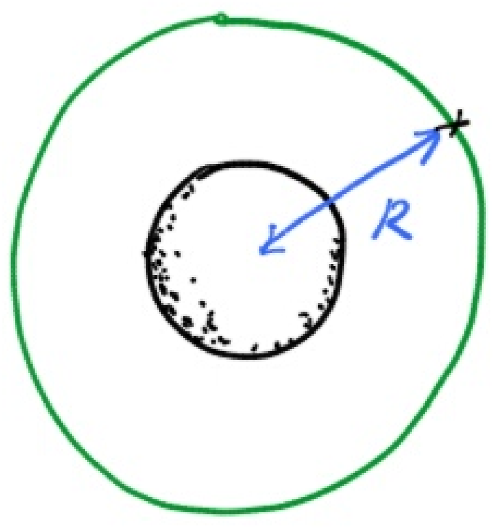
SOLUTIONS TO EXAM IN GENERAL RELATIVITY, JUNE 5, 2019

1. $$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 + \frac{2GM}{c^2 r}\right)dr^2 + r^2 d\Omega^2 \qquad (1)$$



Time for one turn around orbit at radius R, in coordinate time t, is P.

$$P^2 = \frac{4\pi^2}{GM} R^3 \qquad (2)$$

a) What is the proper time τ_a around the orbit?

Put $dr = 0$, $r = R$, $d\Omega^2 = d\varphi^2$ (choose $\theta = \frac{\pi}{2}$)

Then

$$d\tau^2 = -\frac{ds^2}{c^2} = \left(1 - \frac{2GM}{c^2 R}\right)dt^2 - \frac{R^2}{c^2} d\varphi^2$$

Use that $\frac{d\varphi}{dt} = \frac{2\pi}{P}$, that is, $d\varphi = \frac{2\pi}{P} dt$:

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 R} - \frac{4\pi^2 R^2}{c^2 P^2}\right) dt^2$$

$$d\tau = \left(1 - \frac{2GM}{c^2 R} - \frac{4\pi^2 R^2}{c^2 P^2}\right)^{1/2} dt \approx$$

first order in $\frac{1}{c^2}$

$$\approx \left(1 - \frac{GM}{c^2 R} - \frac{2\pi^2 R^2}{c^2 P^2}\right) dt$$

$$\tau_a = \int d\tau \approx \int_0^P \left(1 - \frac{GM}{c^2 R} - \frac{2\pi^2 R^2}{c^2 P^2}\right) dt =$$

$$= P\left(1 - \frac{GM}{c^2 R} - \frac{2\pi^2 R^2}{c^2 P^2}\right) \overset{(2)}{=} \left(\frac{4\pi^2 R^3}{GM}\right)^{1/2}\left(1 - \frac{GM}{c^2 R} - \frac{GM}{2c^2 R}\right) =$$

$$= \left(\frac{4\pi^2 R^3}{GM}\right)^{1/2}\left(1 - \frac{3GM}{2c^2 R}\right)$$

b) What is the proper time τ_b according to stationary observer at $r = R$?

Put $dr = 0$, $r = R$, $d\Omega = 0$.

$$(1) \Rightarrow \quad d\tau^2 = -\frac{ds^2}{c^2} = \left(1 - \frac{2GM}{c^2 R}\right) dt^2$$

$$\tau_b = \int d\tau = \int_0^P \left(1 - \frac{2GM}{c^2 R}\right)^{1/2} dt \approx$$

first order in $\frac{1}{c^2}$

(2)

$$= P\left(1 - \frac{GM}{c^2 R}\right) = \left(\frac{4\pi^2 R^3}{GM}\right)^{1/2} \left(1 - \frac{GM}{c^2 R}\right)$$

c) $P > \tau_b > \tau_a$

The time P measured by a stationary observer far away will be longer than the time τ_b measured by a stationary observer at $r = R$ because of "gravitational redshift": time is running slower closer to the Earth.

The proper time τ_a of the satelite is further slowed down because the satelite is in motion (time dilation in special relativity).

② a) $$\nabla_\gamma g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} - \Gamma^{\rho}_{\gamma\alpha} g_{\rho\beta} - \Gamma^{\rho}_{\gamma\beta} g_{\alpha\rho} =$$

$$= \partial_\gamma g_{\alpha\beta} - \underbrace{g_{\rho\beta}\left(\frac{g^{\rho\varepsilon}}{2}\right.}_{\delta^{\varepsilon}_{\beta}\cdot\frac{1}{2}}\left.\left(\partial_\gamma g_{\varepsilon\alpha} + \partial_\alpha g_{\varepsilon\gamma} - \partial_\varepsilon g_{\gamma\alpha}\right)\right) +$$

$$- \underbrace{g_{\alpha\rho}\left(\frac{g^{\rho\varepsilon}}{2}\right.}_{\delta^{\varepsilon}_{\alpha}\cdot\frac{1}{2}}\left.\left(\partial_\gamma g_{\varepsilon\beta} + \partial_\beta g_{\varepsilon\gamma} - \partial_\varepsilon g_{\gamma\beta}\right)\right) =$$

$$= \partial_\gamma g_{\alpha\beta} - \frac{1}{2}\left(\underline{\partial_\gamma g_{\beta\alpha}} + \cancel{\partial_\alpha g_{\beta\gamma}} - \cancel{\partial_\beta g_{\gamma\alpha}}\right) +$$

$$- \frac{1}{2}\left(\underline{\partial_\gamma g_{\alpha\beta}} + \cancel{\partial_\beta g_{\alpha\gamma}} - \cancel{\partial_\alpha g_{\gamma\beta}}\right) =$$

$$= \partial_\gamma g_{\alpha\beta} - \partial_\gamma g_{\alpha\beta} = 0$$

② b) In a local inertial frame (LIF): $\nabla_\alpha V^\beta = \partial_\alpha V^\beta$ (1)

General coord. transf: $t_{\alpha'}{}^{\beta'} = \dfrac{\partial x^\alpha}{\partial x^{\alpha'}} \dfrac{\partial x^{\beta'}}{\partial x^\beta} t_\alpha{}^\beta$ (2)

Introduce primed coordinates that are <u>not</u> a LIF. Then

$$\nabla_{\alpha'} V^{\beta'} \overset{(2)}{=} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^\beta} \nabla_\alpha V^\beta \overset{(1)}{=} \underbrace{\frac{\partial x^\alpha}{\partial x^{\alpha'}}} \frac{\partial x^{\beta'}}{\partial x^\beta} \underbrace{\frac{\partial}{\partial x^\alpha}} V^\beta =$$

$$= \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial V^\beta}{\partial x^{\alpha'}} = \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial}{\partial x^{\alpha'}} \left(\frac{\partial x^\beta}{\partial x^{\gamma'}} V^{\gamma'} \right) =$$

$$= \frac{\partial x^{\beta'}}{\partial x^\beta} \left(\frac{\partial x^\beta}{\partial x^{\gamma'}} \frac{\partial V^{\gamma'}}{\partial x^{\alpha'}} + V^{\gamma'} \frac{\partial}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\gamma'}} \right) =$$

$$= \underbrace{\frac{\partial x^{\beta'}}{\partial x^{\gamma'}}}_{\delta^{\beta'}_{\gamma'}} \partial_{\alpha'} V^{\gamma'} + V^{\gamma'} \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\gamma'}} =$$

$$= \partial_{\alpha'} V^{\beta'} + V^{\gamma'} \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\gamma'}}$$

Compare with

$$\nabla_{\alpha'} V^{\beta'} = \partial_{\alpha'} V^{\beta'} + \Gamma^{\beta'}{}_{\alpha'\gamma'} V^{\gamma'}$$

$$\Rightarrow \quad \Gamma^{\beta'}{}_{\alpha'\gamma'} = \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\gamma'}}$$

which clearly is symmetric in α', γ'.

(3) The Schwarzschild line element:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Killing field $\xi^\alpha = (1,0,0,0)$

Tangent to trajectory of radially infalling particle:

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right)$$

The conserved quantity $e = -\xi\cdot\bar{u} = -\xi^\alpha u^\beta g_{\alpha\beta} =$

$$= \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}$$

So $\frac{dt}{d\tau} = e\left(1-\frac{2M}{r}\right)^{-1}$

The stationary observer at $r=6M$ has tangent vector

$u^\alpha_{obs} = b(1,0,0,0)$ where b is determined by normalization:

$$-1 = u^\alpha_{obs}u_{obs\,\alpha} = -b^2\left(1-\frac{2M}{r}\right) \quad\Rightarrow\quad b = \left(1-\frac{2M}{r}\right)^{-1/2}$$

Note that the speed that observer u^α_{obs} sees for trajectory u^α can be obtained from

$$u^\alpha u_{obs\,\alpha} = -\gamma_v \qquad (*)$$

(as is seen by going to the LIF of the observer)

In this case

$$u^\alpha u_{obs\,\alpha} = u^\alpha u^\beta_{obs} g_{\alpha\beta} = \frac{dt}{d\tau}\, b\, g_{00} = e\left(1-\frac{2M}{r}\right)^{-1}\left(1-\frac{2M}{r}\right)^{-1/2}\left(-\left(1-\frac{2M}{r}\right)\right) =$$

$$= -e\left(1-\frac{2M}{r}\right)^{-1/2} \underset{r=6M}{=} -e\left(\frac{3}{2}\right)^{1/2}$$

Compare with $(*)$ $\Rightarrow$ $e\left(\frac{3}{2}\right)^{1/2} = \frac{1}{(1-v^2)^{1/2}}$

Solve for v: $v = \left(1-\frac{2}{3e^2}\right)^{1/2}$

So $\frac{v_{e=2}}{v_{e=1}} = \left(\frac{1-\frac{1}{6}}{1-\frac{2}{3}}\right)^{1/2} = \left(\frac{5}{2}\right)^{1/2} \approx 1{,}58$

④ Closed FRW with matter:

$$ds^2 = -dt^2 + a^2(t)\underbrace{\left[d\chi^2 + \sin^2\chi \, d\Omega^2\right]}_{\text{3-sphere}} \qquad (1)$$

$0 \leq \chi \leq \pi$, $\chi = 0$ and $\chi = \pi$ are anti-podal points

$$a(\eta) = C(1 - \cos\eta) \qquad (2)$$
$$t(\eta) = C(\eta - \sin\eta)$$

a) Note that

$$\frac{dt(\eta)}{d\eta} = C(1 - \cos\eta) = a(\eta)$$

Hence $dt = a\,d\eta$. Insert this into (1):

$$ds^2 = a^2(t)\left[-d\eta^2 + d\chi^2 + \sin^2\chi \, d\Omega^2\right]$$

b) Light rays in the (η, χ)-plane:

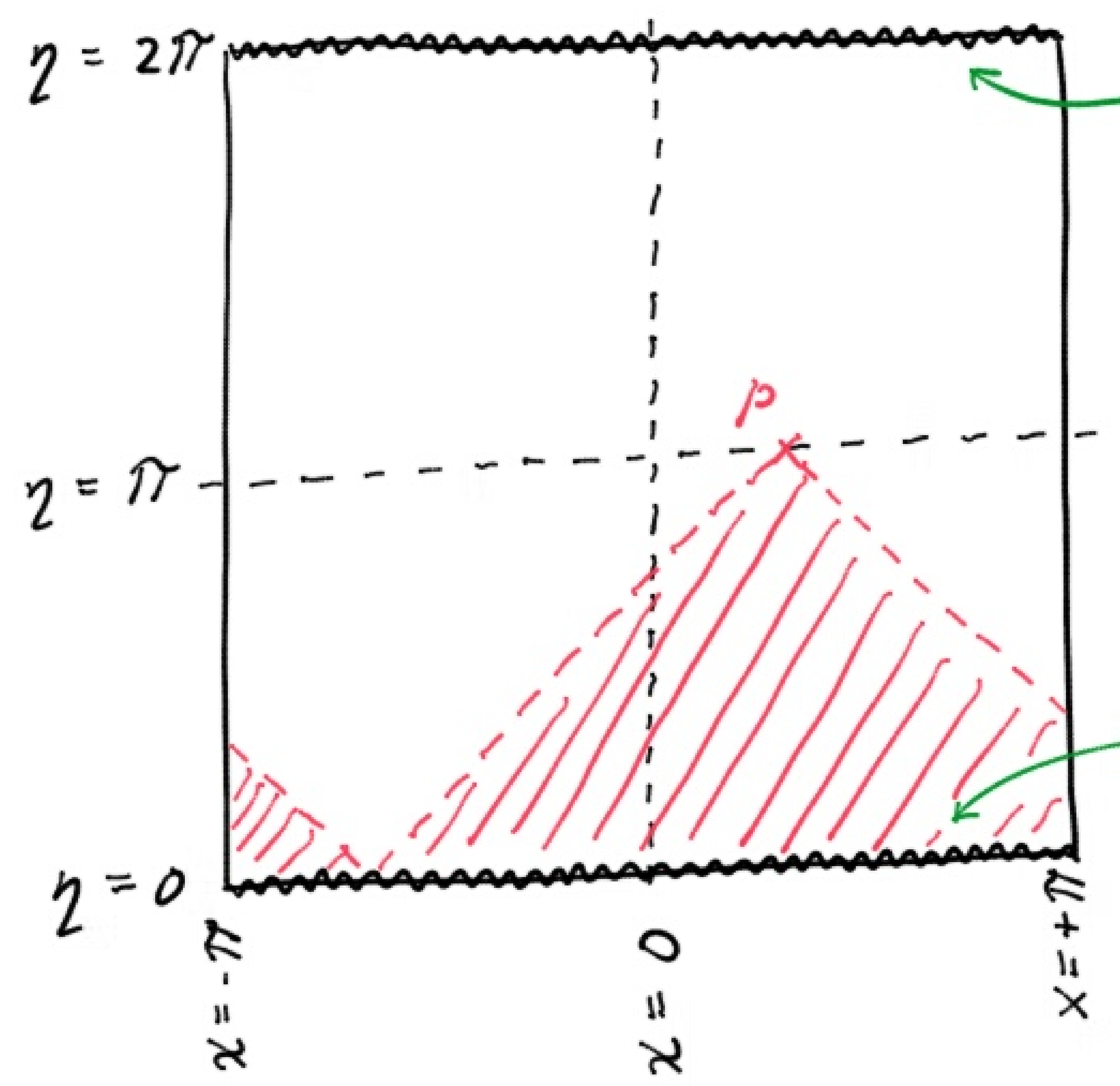
$$ds = 0, \quad d\Omega = 0 \quad \Rightarrow \quad d\eta^2 = d\chi^2$$
$$d\eta = \pm d\chi$$
$$\eta \mp \chi = \text{const}$$

Hence the light rays are just the $45°$ lines (that is, $\eta - \chi = \text{const}$ and $\eta + \chi = \text{const}$) in the (η, χ)-plane.

c) From (2) it follows that $a = 0$ for $\eta = 0$ and 2π (a is maximal for $\eta = -\pi$).
Hence the range of η is $[0, 2\pi]$, where $\eta = 0$ corresponds to Big Bang and where $\eta = 2\pi$ corresponds to Big Crunch.
The range of χ is $[0, \pi]$, but if we keep the other angular coordinates fixed and still want to include the whole circumference of the universe, we should let χ range from $-\pi$ to $+\pi$.

④ c) Hence, the (η, χ)-spacetime diagram:

(cont.)



$\chi = -\pi$ and $\chi = +\pi$ should be identified.

d) An observer at time $\eta = \pi$ has a past light cone that covers all space at $\eta = 0$ (red region in spacetime diagram above). Hence, an observer at any time later than $\eta = \pi$ can receive information from all parts of the universe.

e) As seen from the diagram above, a light ray just makes the full trip around the circumference in the time span from Big Bang to Big Crunch. Hence a timelike observer cannot make it.

5. $x^\alpha(\tau)$ is a timelike geodesic, parametrized in its proper time τ.

Tangent : $u^\alpha = \dfrac{dx^\alpha}{d\tau}$

It obeys the geodesic equation: $u^\alpha \nabla_\alpha u^\beta = 0$

$$\text{or} \quad u^\alpha\left(\frac{\partial u^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma} u^\gamma\right) = 0 \qquad (*)$$

Let $\lambda(\tau)$ be another parametrization (non-affine) : $x^\alpha(\lambda)$ with tangent $w^\alpha = \dfrac{dx^\alpha}{d\lambda}$

The relation between the tangent vectors then are

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \frac{d\lambda}{d\tau}\frac{dx^\alpha}{d\lambda} = \frac{d\lambda}{d\tau} w^\alpha$$

To find the equation for w^α, insert this into $(*)$:

$$\frac{d\lambda}{d\tau} w^\alpha\left(\frac{\partial}{\partial x^\alpha}\left(\frac{d\lambda}{d\tau} w^\beta\right) + \Gamma^\beta_{\alpha\gamma}\frac{d\lambda}{d\tau} w^\gamma\right) = 0$$

$$\frac{d\lambda}{d\tau} w^\alpha\left(w^\beta \frac{\partial}{\partial x^\alpha}\frac{d\lambda}{d\tau} + \underbrace{\frac{d\lambda}{d\tau}\frac{\partial w^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma}\frac{d\lambda}{d\tau} w^\gamma}_{\frac{d\lambda}{d\tau}\left(\frac{\partial w^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma} w^\gamma\right) = \frac{d\lambda}{d\tau}\nabla_\alpha w^\beta}\right) = 0$$

$$\left(\frac{d\lambda}{d\tau}\right)^2 w^\alpha \nabla_\alpha w^\beta = -w^\alpha w^\beta \frac{d\lambda}{d\tau}\frac{\partial}{\partial x^\alpha}\frac{d\lambda}{d\tau} = -w^\beta \frac{d\lambda}{d\tau}\underbrace{\underbrace{w^\alpha\frac{\partial}{\partial x^\alpha}}_{\frac{d}{d\lambda}}\frac{d\lambda}{d\tau}}_{\frac{d}{d\tau}}$$

$$\text{Hence} \quad w^\alpha \nabla_\alpha w^\beta = -w^\beta \left(\frac{d\lambda}{d\tau}\right)^{-2}\frac{d^2\lambda}{d\tau^2}$$

This is of the form $w^\alpha \nabla_\alpha w^\beta = K w^\beta$

$$\text{where} \quad K = -\left(\frac{d\lambda}{d\tau}\right)^{-2}\frac{d^2\lambda}{d\tau^2} = -\left(\frac{d\lambda}{d\tau}\right)^{-1}\underbrace{\frac{d\tau}{d\lambda}\frac{d}{d\tau}\frac{d\lambda}{d\tau}}_{\underbrace{\frac{d}{d\tau}\left(\frac{d\tau}{d\lambda}\frac{d\lambda}{d\tau}\right)}_{=0} - \frac{d\lambda}{d\tau}\frac{d}{d\tau}\frac{d\tau}{d\lambda}} = \frac{d}{d\tau}\frac{d\tau}{d\lambda} = \frac{d}{d\tau}\left(\frac{d\lambda}{d\tau}\right)^{-1}$$