Solutions to exam in General Relativity, May 31, 2018

① Line element for Schwarzschild geometry in Eddington-Finkelstein coordinates: $ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dv\,dr + r^2 d\Omega^2$

where $v = t + r + 2M \ln\left|\frac{r}{2M} - 1\right|$

a) $\xi^\alpha = (1,0,0,0)$ in Schwarzschild coord (t,r,θ,φ)

General transformation formula: $\xi^{\beta'} = \frac{\partial x^{\beta'}}{\partial x^\alpha}\xi^\alpha = \frac{\partial x^{\beta'}}{\partial x^t}\xi^t$

Only v depends on t and $\frac{\partial v}{\partial t} = 1$.

Hence $\xi^{\beta'} = (1,0,0,0)$ also in EF-coord.

b) In order to find K from $\xi^\alpha \nabla_\alpha \xi^\beta = K\xi^\beta$ we need

$$\xi^\alpha \nabla_\alpha \xi^\beta = \nabla_v \xi^\beta = \underbrace{\partial_v \xi^\beta}_{=0} + \Gamma^\beta{}_{v\gamma}\xi^\gamma = \Gamma^\beta{}_{vv} =$$

$$= \frac{g^{\beta\delta}}{2}\Big(\underbrace{\partial_v g_{v\delta}}_{=0} + \underbrace{\partial_v g_{v\delta}}_{=0} - \partial_\delta g_{vv}\Big) = -\frac{g^{\beta r}}{2}\underbrace{\partial_r g_{vv}}_{=-\frac{2M}{r^2}}$$

since no comp. depends on v

$\delta = r$ since g_{vv} only depends on r

We need the inverse metric for the v, r-components:

$$g_{\alpha\beta} = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 1 \\ 1 & 0 \end{pmatrix} \qquad g^{\alpha\beta} = (g_{\alpha\beta})^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1-\frac{2M}{r} \end{pmatrix}$$

Hence $\beta = v \Rightarrow g^{vr}\frac{M}{r^2} = \frac{M}{r^2}$

$\beta = r \Rightarrow g^{rr}\frac{M}{r^2} = \left(1-\frac{2M}{r}\right)\frac{M}{r^2}$

$\beta = \theta, \varphi \Rightarrow 0$

So $\xi^\alpha \nabla_\alpha \xi^\beta = \frac{M}{r^2}\left(1, 1-\frac{2M}{r}, 0, 0\right)$

Insert $r = 2M$:

$$\xi^\alpha \nabla_\alpha \xi^\beta\Big|_{r=2M} = \underbrace{\frac{1}{4M}(1,0,0,0)}_{\text{should be } K\xi^\beta} \Rightarrow K = \frac{1}{4M}$$

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General expressions:

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}{}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}{}_{\beta\gamma} + \Gamma^{\alpha}{}_{\gamma\varepsilon}\Gamma^{\varepsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\delta\varepsilon}\Gamma^{\varepsilon}{}_{\beta\gamma}$$

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{g^{\alpha\delta}}{2}\left(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}\right)$$

Consider a local inertial frame (LIF) at point p:

$$g_{\alpha\beta}\Big|_p = \eta_{\alpha\beta}$$

$$\partial_\gamma g_{\alpha\beta}\Big|_p = 0$$

Since $\Gamma^{\alpha}{}_{\beta\gamma}$ consists of first derivatives of $g_{\alpha\beta}$, it follows that

$$\Gamma^{\alpha}{}_{\beta\gamma}\Big|_p = 0$$

Hence the third and forth terms in $R^{\alpha}{}_{\beta\gamma\delta}$ vanish.
The first and second terms (the first derivatives of Γ) contain both first and second derivatives of $g_{\alpha\beta}$. Only those terms containing no first derivative survives in the LIF.

Hence:

$$R^{\alpha}{}_{\beta\gamma\delta}\Big|_p = \partial_\gamma \Gamma^{\alpha}{}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}{}_{\beta\gamma} =$$

$$= \frac{g^{\alpha\varepsilon}}{2}\Big(\underbrace{\partial_\gamma\partial_\delta g_{\varepsilon\beta}}_{\text{vanishes since sym. in } \gamma,\delta} + \partial_\gamma\partial_\beta g_{\varepsilon\delta} - \partial_\gamma\partial_\varepsilon g_{\beta\delta}\Big) - \left[\begin{array}{c}\text{same with} \\ \gamma \leftrightarrow \delta\end{array}\right] =$$

$$= \frac{g^{\alpha\varepsilon}}{2}\left(\partial_\gamma\partial_\beta g_{\varepsilon\delta} - \partial_\gamma\partial_\varepsilon g_{\beta\delta} - \partial_\delta\partial_\beta g_{\varepsilon\gamma} + \partial_\delta\partial_\varepsilon g_{\beta\gamma}\right)$$

So $$R_{\alpha\beta\gamma\delta}\Big|_p = \frac{1}{2}\left(\partial_\gamma\partial_\beta g_{\alpha\delta} - \partial_\gamma\partial_\alpha g_{\beta\delta} - \partial_\delta\partial_\beta g_{\alpha\gamma} + \partial_\delta\partial_\alpha g_{\beta\gamma}\right)$$

③ a) $T_{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$

Consider an arbitrary observer with 4-velocity

$u^\alpha = \gamma_v (1, v_x, v_y, v_z)$

This observer will measure the energy density

$\varepsilon = T_{\alpha\beta} u^\alpha u^\beta = \gamma_v^2 (\rho + p v^2)$ where $v^2 = v_x^2 + v_y^2 + v_z^2$

Require $\varepsilon > 0$

$\Rightarrow \quad \rho + p v^2 > 0$

$\rho > -p v^2$

v^2 can of course be anything between 0 and 1.

If $p > 0$ this requirement is strongest when $v = 0$.

Then it says that $\rho > 0$

If $p < 0$ this requirement is strongest when $v = 1$.

Then it says that $\rho + p > 0$

Hence we must require $\rho > 0$ and $\rho + p > 0$.

b) Vacuum energy:

$$T_{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 \\ 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & -\rho \end{pmatrix} = -\rho\, \eta_{\alpha\beta}$$

Energy density as measured by observer u^α:

$$\varepsilon = T_{\alpha\beta} u^\alpha u^\beta = -\rho \underbrace{\eta_{\alpha\beta} u^\alpha u^\beta}_{= \bar{u} \cdot \bar{u} = -1} = \rho$$

So all observers measures the same energy density ρ.

④ The Friedman equation: $\dot{a}^2 - \frac{8\pi\rho}{3} a^2 = -k$ (*)

a) Suppose that the energy density ρ is any combination of matter and radiation:

$$\rho = \rho_m + \rho_r = b_m a^{-3} + b_r a^{-4} \quad \text{where} \quad b_m > 0,\ b_r > 0$$

from (18.25) and (18.26)

Then (*) gives

Choose positive sign

$$\dot{a} = \left(\frac{8\pi\rho}{3} a^2 - k\right)^{1/2} = \left(\frac{8\pi}{3}\left(b_m a^{-1} + b_r a^{-2}\right) - k\right)^{1/2}$$

$$\frac{d^2 a}{dt^2} = \frac{d}{dt}(\dot{a}) = \frac{1}{2}\dot{a}^{-1}\,\frac{8\pi}{3}\left(-b_m a^{-2} - 2b_r a^{-3}\right)\cdot \dot{a} =$$

$$= -\frac{4\pi}{3}\left(b_m a^{-2} + 2b_r a^{-3}\right)$$

< 0 for all positive b_m and b_r, that is, the expansion slows down.

b) Now, suppose that we also have vacuum energy:

$$\rho = \rho_m + \rho_r + \rho_v = b_m a^{-3} + b_r a^{-4} + \frac{\Lambda}{8\pi}$$

from (18.28)

Then (*) gives

$$\dot{a} = \left(\frac{8\pi}{3}\left(b_m a^{-1} + b_r a^{-2} + \frac{\Lambda}{8\pi} a^2\right) - k\right)^{1/2}$$

and $$\frac{d}{dt}(\dot{a}) = -\frac{4\pi}{3}\left(b_m a^{-2} + 2b_r a^{-3} - \frac{\Lambda}{4\pi} a\right)$$

For b_m and b_r small enough, the Λ-term will dominate. Then $\frac{d}{dt}(\dot{a}) > 0$, so that the expansion speed will increase.

⑤ a) The longest time must be the time along a geodesic. Then (9.26) is applicable:

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]$$

$$\Rightarrow \quad \frac{d\tau}{dr} = -\left(e^2-\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)\right)^{-1/2}$$

ingoing geodesic means $\frac{d\tau}{dr} < 0$

$$\text{So} \quad \tau = \int d\tau = -\int_{r=2M}^{r=0}\left(e^2-\underbrace{\left(1-\frac{2M}{r}\right)}_{<0}\left(1+\frac{\ell^2}{r^2}\right)\right)^{-1/2} dr =$$

$$= \int_0^{2M}\left(e^2+\left(\frac{2M}{r}-1\right)\left(1+\frac{\ell^2}{r^2}\right)\right)^{-1/2} dr$$

The largest possible τ is obtained for $e=0$ and $\ell=0$:

$$\tau_{max} = \int_0^{2M}\left(\frac{2M}{r}-1\right)^{-1/2} dr = \left[\begin{array}{l} x = \frac{r}{2M} \\ dx = \frac{dr}{2M}\end{array}\right] = 2M\int_0^1\left(\frac{1}{x}-1\right)^{-1/2} dx =$$

$$= 2M \underbrace{\int_0^1\left(\frac{x}{1-x}\right)^{1/2} dx}_{=\frac{\pi}{2}} = \pi M$$

b) Changing to standard units: $c\tau_{max} = \frac{\pi M G}{c^2}$

$$\tau_{max} = \frac{\pi M G}{c^3}$$

$$M = 4\cdot 10^6 M_\odot \approx 8\cdot 10^{36}\ \text{kg}$$

$$\Rightarrow \quad \tau_{max} \approx \frac{\pi\cdot 8\cdot 10^{36}\cdot 6{,}67\cdot 10^{-11}}{(3\cdot 10^8)^3} \approx 62\ \text{s}$$