Lecture 5

Euler - Lagrange's equations

The shortest curve between two points in flat 2 dim. space from Euler-Lagrange.

The geodesic equation derived from Euler-Lagrange.

Riemann normal coordinates

Symmetries
- Killing vector fields
- conserved quantities

Geodesics

Geodesics are the curved space analogue of straight lines.
In a spacetime there are three kinds of geodesics: timelike, lightlike or spacelike, depending on the sign of ds^2 along the geodesic. Timelike geodesics are particularly important since they correspond to free fall trajectories. Thus, if we want to know how objects move in some spacetime geometry we have to know the geodesics.

$\boxed{?}$ A geodesic (of any kind) is defined as a curve of extreme length.

> A <u>geodesic</u> between A and B is a curve γ such that its length $S = \int_{A,\gamma}^{B} ds$ is extreme.
>
> That is, $\delta S = 0$ for any first order variation of the curve.

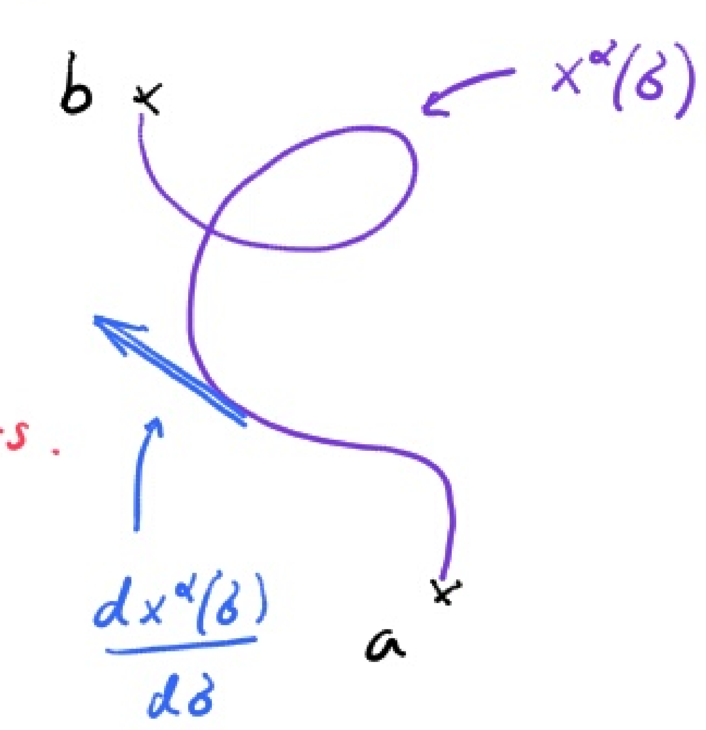
Such an extremum problem, where we look for a curve extremizing an integral, can be solved using Euler-Lagrange's equations.
In general:

σ is any parameter parametrizing the curve

$$S[x^\alpha(\sigma)] = \int_{\sigma_a}^{\sigma_b} L\left(x^\alpha(\sigma), \frac{dx^\alpha(\sigma)}{d\sigma}\right) d\sigma$$

A function of the curve

L takes values along the curve. Its values depend both on the positions and the tangent vectors.

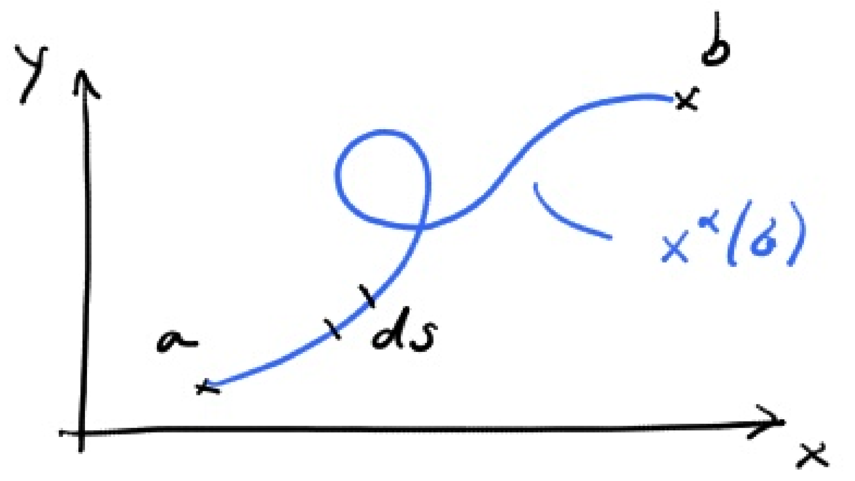


$$\delta S = 0 \iff \frac{\partial L}{\partial x^\alpha} - \frac{d}{d\sigma}\frac{\partial L}{\partial(dx^\alpha/d\sigma)} = 0$$

Our goal is to use these equations to find the equations for geodesics in a general curved spacetime (given its line element).

Let us, to get some practice, see how to get these equations for the flat 2-dimensional plane, in Cartesian coord., where we know the answer.

Flat 2 dim. space



$$ds^2 = dx^2 + dy^2$$

What is the curve of extreme distance between a and b?

$$\text{Curve length } S = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \underbrace{\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right]^{1/2}}_{L} d\sigma$$

$$L\left(x^\alpha(\sigma), \frac{dx^\alpha(\sigma)}{d\sigma}\right) = \left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right]^{1/2}$$

is the length per σ-unit along the trajectory: $ds = L\, d\sigma$

For E-L we need

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial (dx/d\sigma)} = \frac{1}{2} \cdot \frac{1}{L} \cdot 2 \frac{dx}{d\sigma} = \frac{1}{L} \frac{dx}{d\sigma}$$

$$\frac{\partial L}{\partial (dy/d\sigma)} = \frac{1}{L} \frac{dy}{d\sigma}$$

E-L then gives

$$\begin{cases} \dfrac{d}{d\sigma}\left(\dfrac{1}{L}\dfrac{dx}{d\sigma}\right) = 0 \\[2ex] \dfrac{d}{d\sigma}\left(\dfrac{1}{L}\dfrac{dy}{d\sigma}\right) = 0 \end{cases}$$

But note that $L\, d\sigma = ds$. So multiplying both equations with L^{-1} gives

$$\frac{d^2 x}{ds^2} = 0 \qquad \frac{d^2 y}{ds^2} = 0$$

$$\Longrightarrow \begin{cases} x = k_1 s + m_1 \\ y = k_2 s + m_2 \end{cases}$$

where s parametrizes the curve in its own length.

We can eliminate s from these two equations:

$$y(x) = k x + m$$

So the curves of extreme length are straight lines!

Now let us repeat the same steps, but for a general line element.

General space/spacetime

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$S = \int ds = \int \sqrt{g_{\alpha\beta}\, dx^\alpha dx^\beta} = \int \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}}\, d\sigma$$

For timelike curves there should be a minus sign here, but it will not matter in the end, so I will omit it

where $L = \left(g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)^{1/2}$ again is the length per σ-unit,

that is, $L\, d\sigma = ds$

Note that $g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma)$, so that L now has an explicit coord. dependence.

We get

Introduce notation: $\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}$

$$\partial_\gamma L = \frac{1}{2L} (\partial_\gamma g_{\alpha\beta}) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}$$

$$\frac{\partial L}{\partial \left(dx^\gamma / d\sigma \right)} = \frac{1}{2L} \left(g_{\gamma\beta} \frac{dx^\beta}{d\sigma} + g_{\alpha\gamma} \frac{dx^\alpha}{d\sigma} \right) = \frac{1}{L} g_{\gamma\beta} \frac{dx^\beta}{d\sigma}$$

E-L $\Rightarrow$

$$\frac{1}{2L} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \partial_\gamma g_{\alpha\beta} - \frac{d}{d\sigma} \left(\frac{1}{L} g_{\gamma\beta} \frac{dx^\beta}{d\sigma} \right) = 0$$

Multiply by $\frac{1}{L}$ and put $\frac{1}{L}\frac{d}{d\sigma} = \frac{d}{ds}$;

$$\frac{1}{2} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \partial_\gamma g_{\alpha\beta} - \frac{d}{ds} \left(g_{\gamma\beta} \frac{dx^\beta}{ds} \right) = 0$$

$$\frac{1}{2} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \partial_\gamma g_{\alpha\beta} - \Big(\underbrace{\frac{d}{ds}}_{= \frac{dx^\alpha}{ds} \frac{\partial}{\partial x^\alpha} \equiv \frac{dx^\alpha}{ds} \partial_\alpha} g_{\gamma\beta} \Big) \frac{dx^\beta}{ds} - g_{\gamma\beta} \frac{d^2 x^\beta}{ds^2} = 0$$

$$\frac{1}{2} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \partial_\gamma g_{\alpha\beta} - \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \partial_\alpha g_{\gamma\beta} - g_{\gamma\beta} \frac{d^2 x^\beta}{ds^2} = 0$$

Multiply by $-(g_{\mu\gamma})^{-1} \equiv - g^{\mu\gamma}$ and rearrange the terms :

Use $g^{\mu\gamma} g_{\gamma\beta} = \delta^\mu_\beta$

$$\frac{d^2 x^\mu}{ds^2} + g^{\mu\gamma} \underbrace{(\partial_\alpha g_{\gamma\beta})}_{= \frac{1}{2}(\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\gamma\alpha})} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} - \frac{1}{2} g^{\mu\gamma} (\partial_\gamma g_{\alpha\beta}) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0$$

since this is multiplied by something symmetric in α, β.

$$\frac{d^2 x^\mu}{ds^2} + \underbrace{\frac{g^{\mu\gamma}}{2} \left[\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \right]}_{\Gamma^\mu_{\alpha\beta}} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0$$

$\Gamma^\mu_{\alpha\beta}$ – Christoffel symbol

Hence $$\boxed{\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0}$$ – the geodesic equation !

The equation is the same for timelike curves, with the proper time τ replacing the proper distance s. But then we also may write the equation in terms of the 4-velocity

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

$$\Rightarrow \quad \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0$$

From its definition, note that $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$.

There is often a lot of algebraic work to find all Γ:s. Often it is actually easier to go through the derivation with the particular metric that one is interested in. (And then, from that derivation read off the Γ:s!)

But our general expression for Γ will turn out to be useful later.

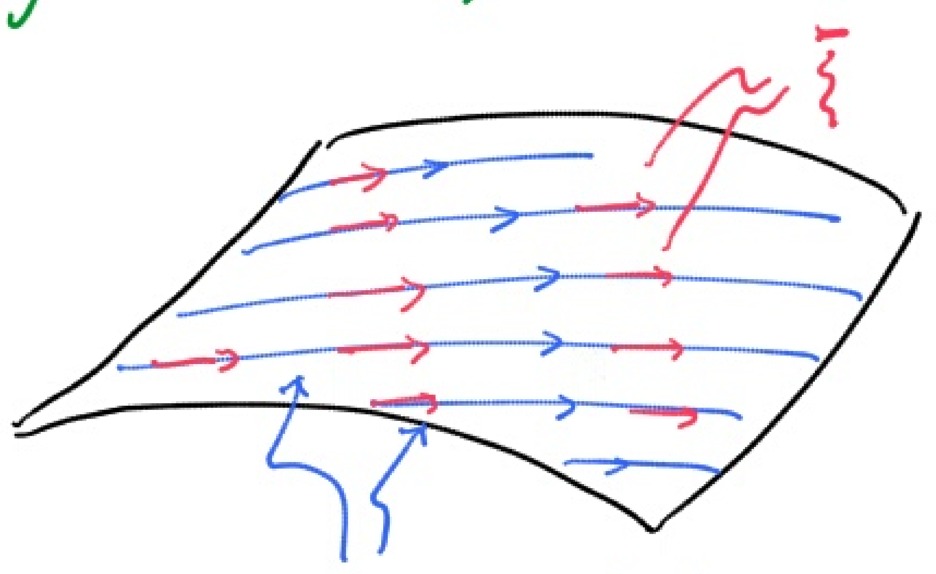
Symmetries and conserved quantities

As you remember from the analytical mechanics, there is a close relationship between symmetries and conservation laws. This connection is actually even easier to see in GR.

First, we have to talk a little about symmetries in GR.

Curved spacetimes can be more or less symmetric. But how can the symmetries be characterized?

Suppose we have the metric in some coordinates. If all its components are independent of one of these coord., say x^1, then that coordinate corresponds to a symmetry in this sense:
If all points are shifted $x^1 \to x^1 + c$ nothing has changed.



x^1-coord. lines are flow lines of ξ

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

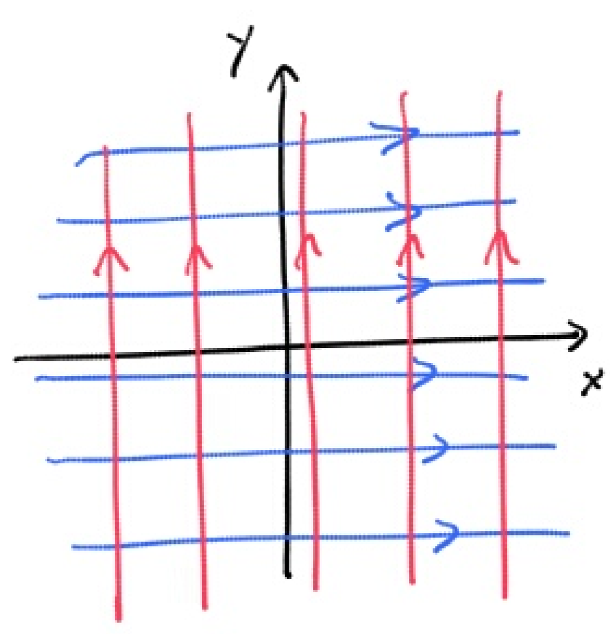
Suppose $g_{\alpha\beta}$ independent of x^1.

Define $\xi^\alpha = (0, 1, 0, 0)$ in these coord.

The Killing vector field corresponding to the x^1-transl. sym.

In this way we can always associate a vector field to a symmetry. And in the coord. where the symmetry is explicit, the Killing field always takes this simple form.

Examples:

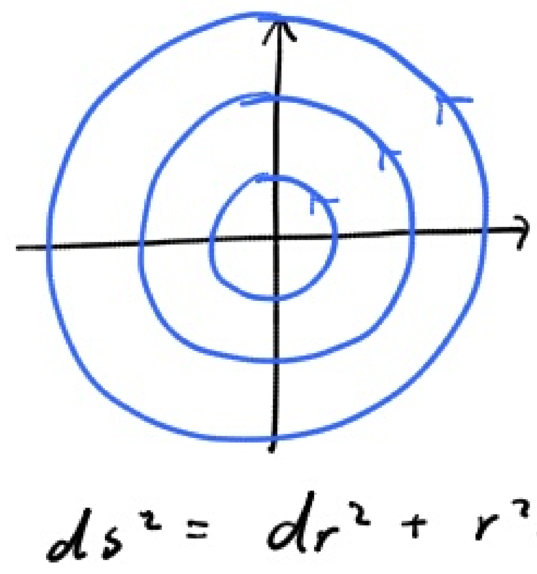


$$ds^2 = dx^2 + dy^2$$

$\xi_1^\alpha = (1, 0)$

$\xi_2^\alpha = (0, 1)$

Note that any linear comb. of these is a new Killing vector field.



$$ds^2 = dr^2 + r^2 d\varphi^2$$

$\xi^\alpha = (0, 1)$

Now, let us see how the existence of a Killing field implies a conserved quantity.

Suppose we have a metric which is independent of x^1, and therefore a Killingfield

$\xi^\alpha = (0, 1, 0, 0)$

Consider a geodesic $x^\alpha(\sigma)$. It must solve E-L-equations. But since the metric is independent of x^1, that must also be the case for the Lagrangian:

$$L = \left(g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)^{1/2}$$

$$\frac{\partial L}{\partial x^1} = 0 \underset{E-L}{\Longrightarrow} \frac{d}{d\sigma} \frac{\partial L}{\partial (dx^1/d\sigma)} = 0$$

$$\Longrightarrow \frac{\partial L}{\partial (dx^1/d\sigma)} = \text{const.} \quad \text{all along the geodesic.}$$

But $\displaystyle \frac{\partial L}{\partial (dx^1/d\sigma)} = \frac{1}{2} \cdot \frac{1}{L} 2 g_{\alpha 1} \frac{\partial x^\alpha}{\partial \sigma} = \underbrace{g_{\alpha 1}}_{g_{\alpha\beta}\xi^\beta \text{ in this frame}} \underbrace{\frac{\partial x^\alpha}{\partial \tau}}_{u^\alpha} = g_{\alpha\beta} u^\alpha \xi^\beta = \bar{u} \cdot \bar{\xi}$

u^α – the tangent to the geodesic

Frame independent expression!

Hence: $\boxed{\bar{u} \cdot \bar{\xi} = \text{const. along a geodesic}}$

Or for a freely falling particle with mass m:

$$\bar{p} \cdot \bar{\xi} = \text{const.}$$

Suppose, for example, that we have a static metric — a metric independent of some time coordinate. Then the time component of $\bar{p}$ — the energy measured in that frame — is conserved.

<u>Example</u>: $ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2 + dz^2$ — Minkowski in polar coord.

$\xi^\alpha = (0, 0, 1, 0)$ since metric independent of φ.

$u^\alpha = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\varphi}{d\tau}, \frac{dz}{d\tau} \right)$ — tangent to some geodesic

Conserved quantity:

$$\bar{\xi} \cdot \bar{u} = g_{\alpha\beta} \xi^\alpha u^\beta = r^2 \frac{d\varphi}{d\tau}$$ — angular momentum! (per unit mass)

The requirement that $\bar{\xi} \cdot \bar{u}$ should be constant along all geodesics obviously constrain the shape of the geodesics. When there are enough symmetries these kinds of constraints actually is enough to determine the geodesics.