Lecture 9

Tensors

- raising and lowering indices
- the metric
- general transformation rule

Non-coordinate bases

Parallel transport

- the global and the local definition of a geodesic
- comparing vectors through p. t. along geodesics

The covariant derivative

- defined to represent p.t. to nearby points
- derivation by comparison to the geodesic equation.
- acting on a^α
- the geodesic equation in the form $\nabla_{\bar{u}} \bar{u} = 0$
- acting on a_α
- acting on general tensors
- $\nabla_\alpha g_{\beta\sigma} = 0$

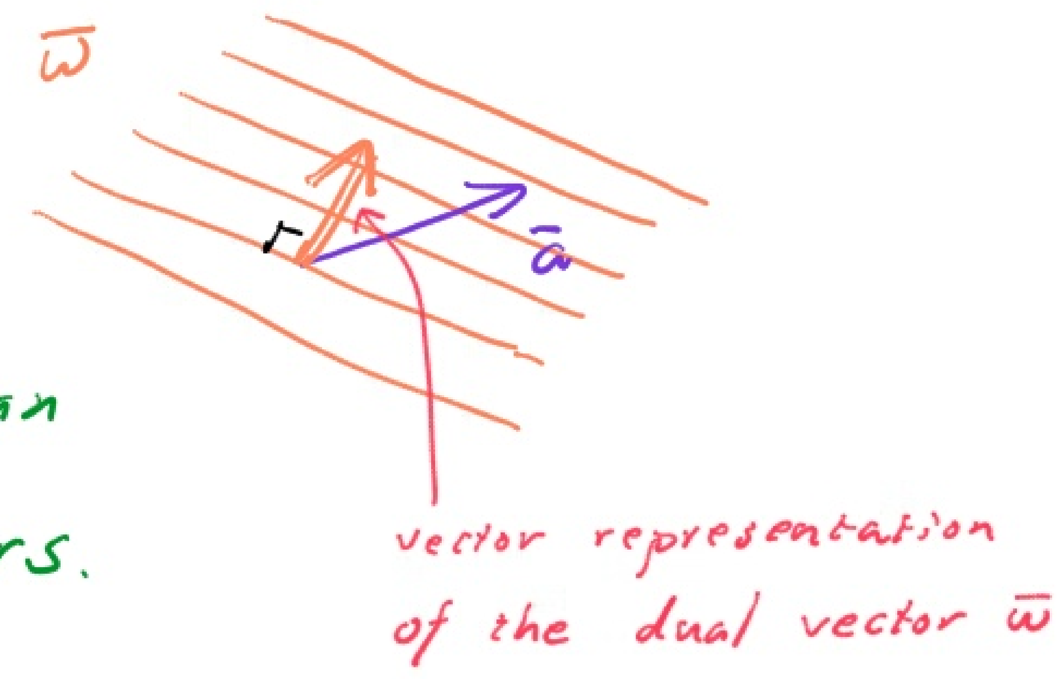
Review

Last time we identified vectors with directional derivatives

$$\bar{a} \equiv a^\alpha \partial_\alpha \equiv a^\alpha \bar{e}_\alpha$$

and defined dual vectors as functions of vectors to real numbers

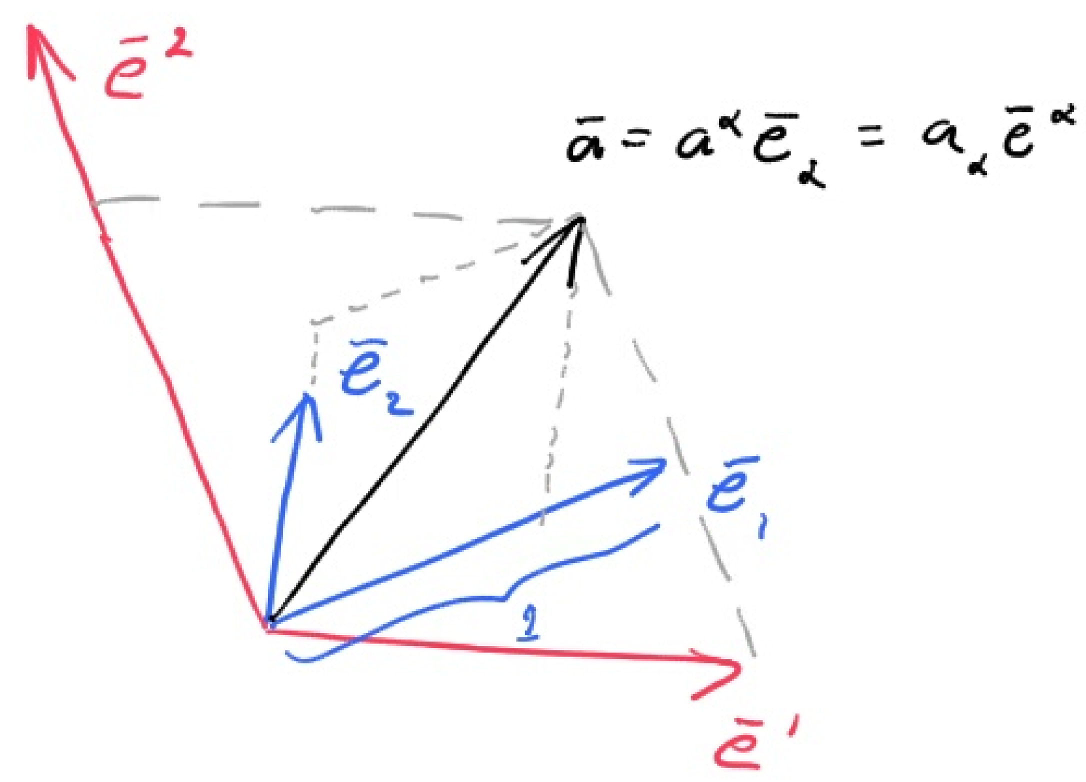
$$w(\bar{a}) = w_\alpha a^\alpha$$



Then we introduced a metric, and since that makes orthogonality well defined, we can agree to represent dual vectors as vectors.

In other words: Because of the metric we can consider a^α and a_α as two different representations of the same object: the vector $\bar{a}$.

$$\bar{a} = a^\alpha \bar{e}_\alpha = a_\alpha \bar{e}^\alpha$$



The dual basis vectors are defined by

$$\bar{e}^\alpha \cdot \bar{e}_\beta = \delta^\alpha_\beta$$

Note that we can get the components by projecting on the basis vectors:

$$\bar{e}^\alpha \cdot \bar{a} = \bar{e}^\alpha \cdot a^\beta \bar{e}_\beta = a^\beta \bar{e}^\alpha \cdot \bar{e}_\beta = a^\alpha$$

We also obtained the transformation laws for the two representations, index upstairs and downstairs:

$$a^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} a^\alpha$$

$$a_{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} a_\alpha$$

Tensors

It is now straightforward to generalize this to tensors, that is, objects with many indices, upstairs or downstairs or mixed.

Start by defining a tensor of rank r as a linear function from r vectors to the real numbers:

Tensor of rank r $\quad T : V^r \longrightarrow \mathbb{R}$

For example, for $r=3$:

$$T(\bar{a},\bar{b},\bar{c}) = T_{\alpha\beta\gamma}\, a^\alpha b^\beta c^\gamma$$ defined though scalar product

$$= T_{\alpha\beta}{}^{\gamma}\, a^\alpha b^\beta c_\gamma$$ using the downstairs representation of $\bar{c}$ instead.

$c_\gamma = c^\delta g_{\delta\gamma}$

$$= T_{\alpha\beta}{}^{\gamma}\, a^\alpha b^\beta c^\delta g_{\delta\gamma} = T_{\alpha\beta}{}^{\delta}\, a^\alpha b^\beta c^\gamma g_{\gamma\delta}$$

relabeling $\gamma \leftrightarrow \delta$

$$\Longrightarrow \quad T_{\alpha\beta\gamma} = T_{\alpha\beta}{}^{\delta}\, g_{\gamma\delta}$$

Hence, the metric can be ased to raise and lower index also on general tensors.

The metric itself is a tensor of rank 2:

$$g(\bar{a},\bar{b}) = \bar{a}\cdot\bar{b} = g_{\alpha\beta}\, a^\alpha b^\beta$$

What happens when we raise the indices on the metric itself?

$$g^{\gamma\alpha} g_{\alpha\beta} = g^{\gamma}{}_{\beta} \equiv \delta^{\gamma}_{\beta} \quad \text{by definition!}$$

$$g^{\delta\beta} g^{\gamma}{}_{\beta} = g^{\delta\beta}\delta^{\gamma}_{\beta} = g^{\delta\gamma}$$

So the notation is consistent!

The coordinate transformation rule for tensors follows directly from those for vectors, since a tensor may be formed by just combining vectors.

Transformation rule for tensors

For example: $T^{\alpha}{}_{\beta\gamma} = a^\alpha b_\beta c_\gamma$

$$T^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\, T^{\alpha}{}_{\beta\gamma}$$

Note: if a tensor vanish in one frame, it must vanish in all frames!

So far I have only discussed the components of tensors in coordinate bases. Such bases are the most natural ones given a coordinate system. But the base that we are using at a given point may have nothing to do with the coordinates at that point.

The most natural alternative to coordinate bases are orthonormal bases. But how do we go from one kind to the other?

Remember that

$$\bar{e}^{\alpha} \cdot \bar{e}_{\beta} = \delta^{\alpha}_{\beta}$$

$$\Rightarrow \quad \bar{e}^{\alpha} \cdot \bar{a} = \bar{e}^{\alpha} \cdot \left(a^{\beta} \bar{e}_{\beta} \right) = a^{\beta} \bar{e}^{\alpha} \cdot \bar{e}_{\beta} = a^{\beta} \delta^{\alpha}_{\beta} = a^{\alpha}$$

Similarly $a_{\alpha} = \bar{e}_{\alpha} \cdot \bar{a}$

So scalar product with the basis vectors project out the corresponding components of a vector, just as we are used to. This is true for any kind of basis:

ON-basis: $\quad a_{\hat{\alpha}} = \bar{e}_{\hat{\alpha}} \cdot \bar{a} \quad , \quad a^{\hat{\alpha}} = \bar{e}^{\hat{\alpha}} \cdot \bar{a}$

Suppose now that we know the coordinate basis components of the ON-basis vectors, for example

$(e^{\hat{\alpha}})_{\alpha}$

Then $\quad a^{\hat{\alpha}} = \bar{e}^{\hat{\alpha}} \cdot \bar{a} = (e^{\hat{\alpha}})_{\alpha} \, a^{\alpha}$

and, similarly $\quad a_{\hat{\alpha}} = \bar{e}_{\hat{\alpha}} \cdot \bar{a} = (e_{\hat{\alpha}})^{\alpha} \, a_{\alpha}$

These kind of expressions generalize to any kind of tensors. So in this way, we can move back and forth between different set of basis-vectors.

Parallel transportation

Let us, for a moment, return to the concept of geodesics. First, a straight line in flat Euclidean space can be defined in two ways, one global and one local:

Straight line, flat space:

- shortest path (global)
- "go straight" = keep tangent fixed (local)

The global definition we have generalized to curved spaces (or spacetimes) simply by replacing "shortest" by extremal. We then used that definition to derive the geodesic equation.

$$\frac{du^\alpha}{d\tau} + \Gamma^\gamma_{\alpha\beta} u^\alpha u^\beta = 0$$

But shouldn't there also be a <u>local</u> definition for geodesics in curved space, analogous to the local flat space definition?

Yes, but first we have to make precise the notion of "keeping the tangent fixed". That is, we have to introduce a way to parallel transport a vector along a curve. This is not trivial in a curved space, because it involves comparing vectors at different points, and remember that vectors at different points live in seperate vector spaces.

But there is a solution to this.

Remember that we can always, at any point P in a curved space, introduce a local inertial frame (LIF):

$$\text{LIF}: \quad g_{\alpha\beta}(x_P) = \eta_{\alpha\beta}$$

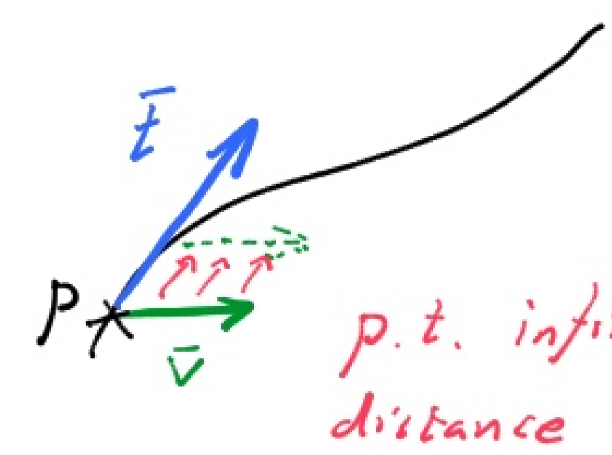
$$\partial_\gamma g_{\alpha\beta}\Big|_P = 0$$

$$\Rightarrow \quad \Gamma^\gamma_{\alpha\beta}\Big|_P = 0 \quad \Rightarrow \quad \text{geodesic equation} \quad \frac{du^\alpha}{d\tau}\Big|_P = 0$$

Hence the components u^α do not change as we move along the geodesic at P. In other words, the geodesic "look straight" at P, when drawn in the LIF-coordinates. This makes parallel transportation of its tangent vector $\bar{u}$ well-defined in the LIF.

We can use this to define parallel transportation for any vector:

> A vector $\bar{v}$ is parallel transported along a curve with tangent $\bar{t}$ at point P iff
> $$t^\alpha \frac{\partial v^\beta}{\partial x^\alpha}\bigg|_P = 0$$
> in a LIF at P.

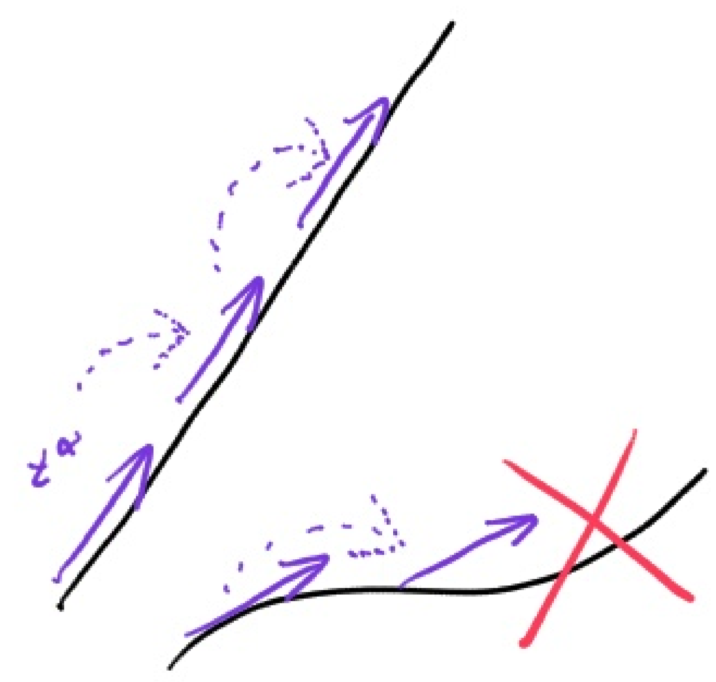


p.t. infinitesimal distance along curve, as drawn in a LIF at p.

If we were to parallel transport a vector along a whole curve the procedure according to this definition would be very unpractical: we would have to introduce an infinite sequence of LIF:s along the curve and make sure that the criteria in the definition were fulfilled in each of them!

But let us not worry about that. The important thing is that we now have a well defined procedure, whatever it is. Because this means that we now can write down a local definition of geodesics in curved spaces:

> A geodesic is such that its tangent vector at a point, when parallel transported along the geodesic, remains tangent to it.



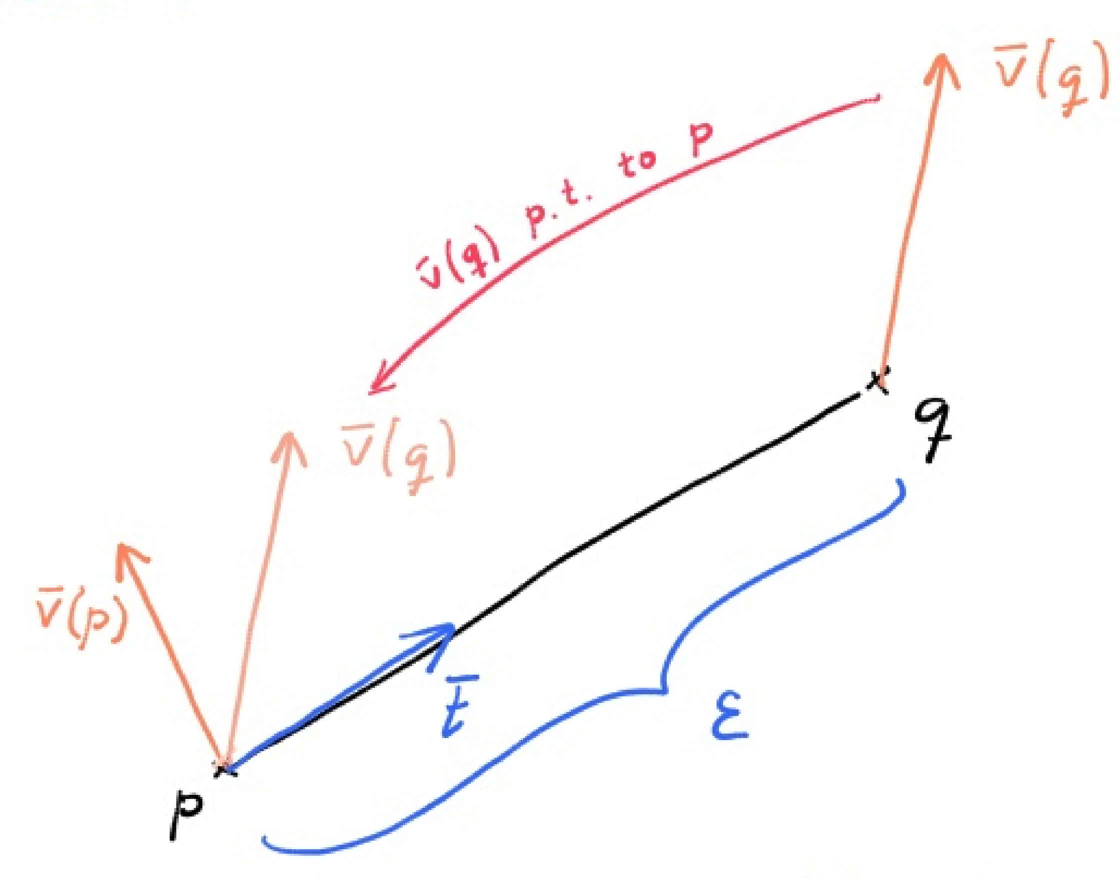
Our definition of parallel transportation also lets us compare vectors at different points, although they live in different vector spaces. If only a curve is specified between the points we can parallel transport any vector from one point to the other, letting us compare it to vectors at the second point. (But note that the result will in general depend on the curve!)

We can utilize this to define a derivative of vectors — the so called covariant derivative — which describes the change in a vectorfield $\bar{v}$ in direction $\bar{t}$:

The covariant derivative

$$\nabla_{\bar{t}}\, \bar{v}\Big|_p = \lim_{\varepsilon \to 0} \frac{\bar{v}(q)\big|_{\text{p.t. to } p} - \bar{v}(p)}{\varepsilon}$$

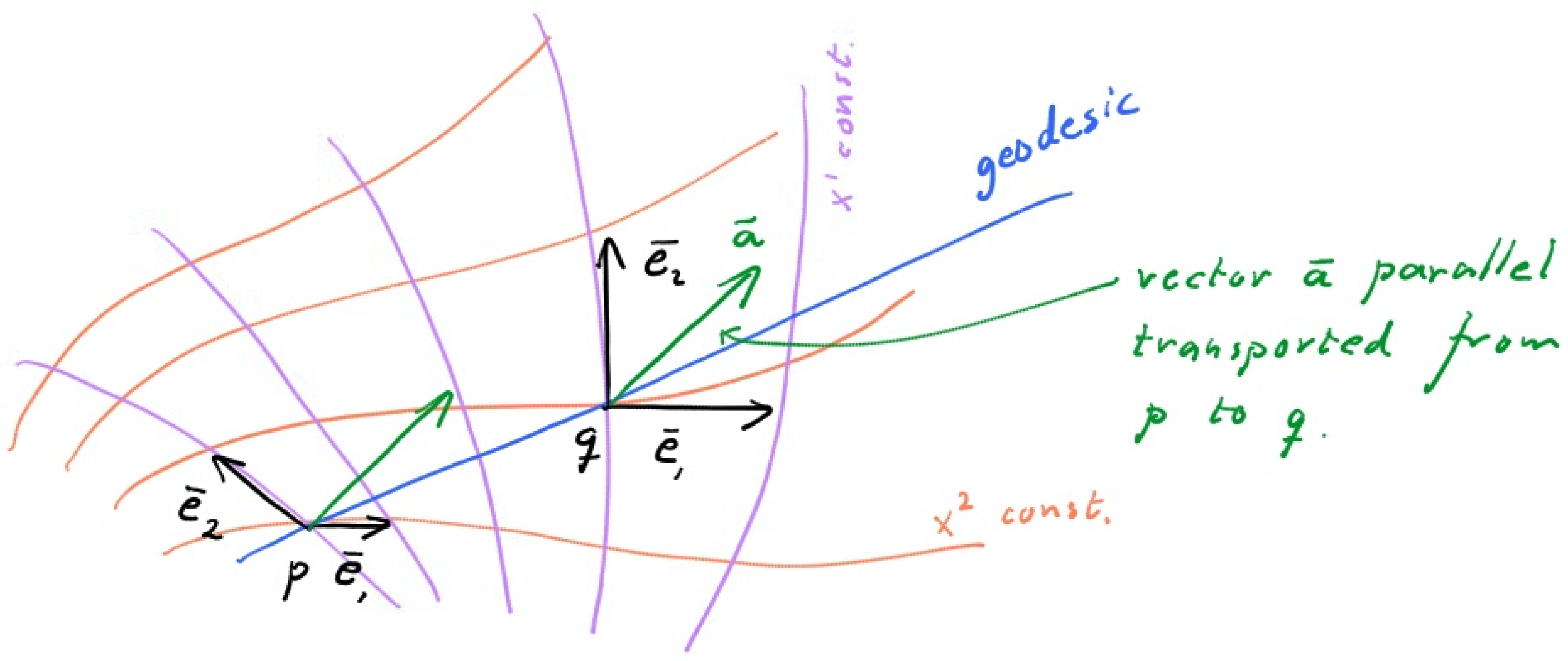
Cov. der. of vector field $\bar{v}$ at point p in direction $\bar{t}$



Using this we can write our local definition of geodesics like this:

$$\boxed{\nabla_{\bar{u}}\, \bar{u} = 0}$$ — local def. of geodesics

To see what the definition of such a derivative would involve, consider some curvilinear coord. system of the flat plane:



$$\bar{a}_p = a_p^1 \bar{e}_{p1} + a_p^2 \bar{e}_{p2} = a_p^\alpha \bar{e}_{p\alpha}$$

$$\bar{a}_q = a_q^1 \bar{e}_{q1} + a_q^2 \bar{e}_{q2} = a_q^\alpha \bar{e}_{q\alpha}$$

Clearly, $\bar{a}_p$ is equal to $\bar{a}_q$ (in the "p.t. along geodesic"-sense) but both the components and the basis vectors are different at the two points!

Now, suppose p and q are neighbouring points.
If $\bar{a}(x^\alpha)$ is a vector field which in points p and q are equal to $\bar{a}_p$ and $\bar{a}_q$, respectively, then the covariant derivative in direction $\bar{t}$, must fulfill:

tangent to the geodesic

$$\nabla_{\bar{t}}\, \bar{a} = 0 \qquad \text{or} \qquad t^\beta \nabla_\beta \bar{a} = 0$$

Since $\bar{a} = a^\alpha \bar{e}_\alpha$ we expect this derivative to split in two terms (assuming that it fulfills the product rule):

$$\nabla_\beta \bar{a} = \nabla_\beta (a^\alpha \bar{e}_\alpha) = \underbrace{\bar{e}_\alpha \nabla_\beta a^\alpha}_{\text{change in values of components}} + a^\alpha \underbrace{\nabla_\beta \bar{e}_\alpha}_{\text{change in vector basis}}$$

product rule

$$(\nabla_\beta \bar{a})^\gamma = \bar{e}^\gamma \cdot \nabla_\beta \bar{a} = \underbrace{\bar{e}^\gamma \cdot \bar{e}_\alpha}_{\delta^\gamma_\alpha} \underbrace{\nabla_\beta a^\alpha}_{\partial_\beta a^\alpha} + a^\alpha \bar{e}^\gamma \cdot \nabla_\beta \bar{e}_\alpha =$$

we now view the components a^α just as functions

$$= \partial_\beta a^\gamma + \underbrace{a^\alpha \bar{e}^\gamma \cdot \nabla_\beta \bar{e}_\alpha}_{?}$$

To find out what the second term is we can compare with a case where we already know the result: the tangent vector of the geodesic itself!

More precisely, for a geodesic where we know that its tangent vector must be p. t. into itself (along the geodesic) we must have

$$\nabla_{\bar{u}} \bar{u} = 0 \qquad \text{or} \qquad u^\beta \nabla_\beta \bar{u} = 0$$

where, for the same reason as above, we expect

$$(\nabla_\beta \bar{u})^\gamma = \partial_\beta u^\gamma + u^\alpha \, ?^\gamma{}_{\alpha\beta}$$

Let us compare this to the geodesic equation:

$$\frac{d^2 x^\gamma}{d\tau^2} + \Gamma^\gamma{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

written in terms of the tangent $u^\alpha = \dfrac{dx^\alpha}{d\tau}$:

$$\underbrace{\frac{du^\gamma}{d\tau}}_{\frac{dx^\beta}{d\tau}\frac{\partial u^\gamma}{\partial x^\beta} = u^\beta \frac{\partial u^\gamma}{\partial x^\beta}} + \Gamma^\gamma{}_{\alpha\beta} u^\alpha u^\beta = 0$$

$$u^\beta \left(\partial_\beta u^\gamma + \Gamma^\gamma{}_{\alpha\beta} u^\alpha \right) = 0$$

Comparing with $u^\beta \nabla_\beta \bar{u} = 0$ indicates that our question mark is Γ! That is

$$(\nabla_\beta \bar{u})^\gamma = \partial_\beta u^\gamma + \Gamma^\gamma{}_{\alpha\beta} u^\alpha$$

This is more commonly written as (in a more sloppy notation):

$$\boxed{\begin{gathered} \nabla_\beta u^\gamma = \partial_\beta u^\gamma + \Gamma^\gamma{}_{\beta\alpha} u^\alpha \\ \text{where } \Gamma^\gamma{}_{\alpha\beta} = \frac{g^{\gamma\delta}}{2} \left(\frac{\partial g_{\alpha\delta}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\delta} \right) \end{gathered}}$$

— the covariant derivative

symmetric in α, β!

Note: $\nabla_\beta u^\gamma$ should not be interpreted as the covariant derivative of the component u^γ (as it looks) but rather as the γ-component of the covariant derivative in direction β of the vector $\bar{u}$.

What are the downstairs components of the covariant derivative?

To find out, consider

$$\underbrace{\nabla_\beta (v_\gamma w^\gamma)}_{\partial_\beta (v_\gamma w^\gamma) =} = w^\gamma \nabla_\beta v_\gamma + v_\gamma \nabla_\beta w^\gamma = w^\gamma \nabla_\beta v_\gamma + \cancel{v_\gamma \partial_\beta w^\gamma} + v_\gamma \Gamma^\gamma_{\alpha\beta} w^\alpha$$

product rule

$$= w^\gamma \partial_\beta v_\gamma + \cancel{v_\gamma \partial_\beta w^\gamma}$$

$$\Longrightarrow \quad w^\gamma \nabla_\beta v_\gamma = w^\gamma \partial_\beta v_\gamma - \Gamma^{\alpha}_{\gamma\beta} w^{\gamma} v_{\alpha} = w^\gamma \left(\partial_\beta v_\gamma - \Gamma^\alpha_{\beta\gamma} v_\alpha \right)$$

should hold for any vector w^γ!

Hence: $$\boxed{\nabla_\beta v_\gamma = \partial_\beta v_\gamma - \Gamma^\alpha_{\beta\gamma} v_\alpha}$$

This is readily generalized to arbitrary tensors. One just adds one Γ-term for each index. For example:

$$\nabla_\alpha t^\beta{}_{\gamma\delta} = \partial_\alpha t^\beta{}_{\gamma\delta} + \Gamma^\beta_{\alpha\rho} t^\rho{}_{\gamma\delta} - \Gamma^\rho_{\alpha\gamma} t^\beta{}_{\rho\delta} - \Gamma^\rho_{\alpha\delta} t^\beta{}_{\gamma\rho}$$

An important property of the covariant derivative is that its action on the metric vanishes:

$$\nabla_\alpha g_{\beta\gamma} = 0$$

Why?

- Because this must be true in an inertial frame, where the first partial derivatives vanish. In such a frame, therefore, all the Christoffel symbols vanish.

  But if $\nabla_\alpha g_{\beta\gamma} = 0$ in one frame it has to vanish in any frame, since it is a tensor relation!

Note: The Christoffel symbols are <u>not</u> tensors! They vanish in some frames but not others! Also, $\partial_\alpha v^\beta$ is not a tensor.