Lecture 2

Four vectors (notation !)

- basis four vectors
- scalar product
- $\eta_{\alpha\beta}$
- spacelike, timelike, null vectors

4-velocity and 4-acceleration

Energy-momentum 4-vektor

- $\bar{p} = m\bar{u}$
- $p^{\alpha} = (m\gamma_u , m\gamma_u \vec{u}) \equiv (E , \vec{p})$
- $E = -\bar{p} \cdot \bar{u}_{obs}$

Equivalence principle

- free fall
- acceleration
- should be understood in a local sense

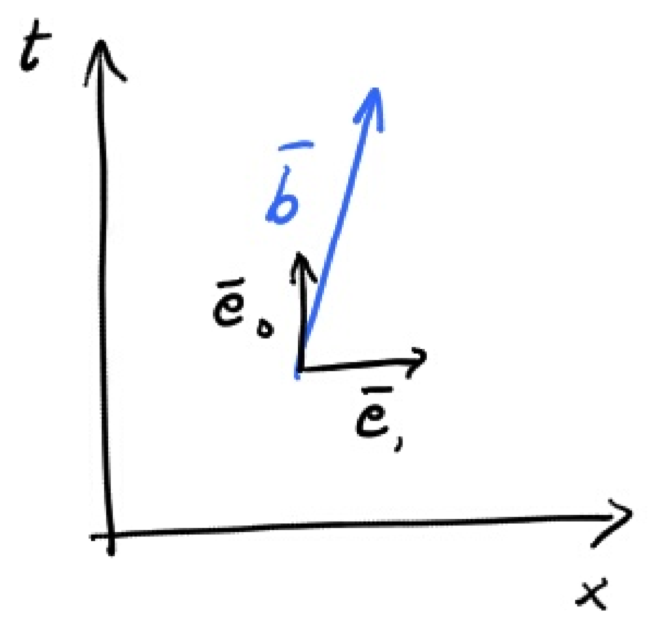
What is constant acceleration ?

- flow line of Lorentz boost !
- $a = \frac{1}{s}$
- The distance stays the same although the acc. is varying !
- the acc. time shift

E.p + SR $\longrightarrow$ gravitational time shift

We are going to need vectors in spacetime.

4-vectors



Basis vectors : $\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3$

$$\bar{b} = b^0\bar{e}_0 + b^1\bar{e}_1 + b^2\bar{e}_2 + b^3\bar{e}_3 =$$
$$= b^\alpha \bar{e}_\alpha$$

Einstein's summation conv.

Notation :

$\bar{b}$ — 4-vector

$\vec{b}$ — 3-vector

b^α — comp. of 4-vector

b^i — comp. of 3-vector

We can view the vector as a directed line segment in the spacetime. Hence its components will transform in the same way as the coord. under a Lorentz-transformation.

But this will not be true in curved spacetimes!

We can work with 4-vectors in the same way as we are used to work with 3-vectors. But because of the minus sign in the line-element, special care is needed when calculating the scalar product of two vectors.

$$\bar{a}\cdot\bar{b} = (a^\alpha \bar{e}_\alpha)\cdot(b^\beta \bar{e}_\beta) = a^\alpha b^\beta \, \bar{e}_\alpha \cdot \bar{e}_\beta$$

Thus we need to know $\bar{e}_\alpha \cdot \bar{e}_\beta$. In general, in any coordinates, whether or not space is curved this is called "the metric".

In SR, with Cartesian coord., we use the notation

$$\eta_{\alpha\beta} = \bar{e}_\alpha \cdot \bar{e}_\beta \qquad \text{so} \qquad \bar{a}\cdot\bar{b} = a^\alpha b^\beta \eta_{\alpha\beta}$$

What is $\eta_{\alpha\beta}$?

The notation should work for a displacement vector, and the squared length of that should just be $(\Delta s)^2$.

$$\Delta x^\alpha = (\Delta t, \Delta x, \Delta y, \Delta z)$$

$$\Delta\bar{x}\cdot\Delta\bar{x} = (\Delta s)^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

$$\Rightarrow \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

So when calculating the scalar product we must always remember to insert a minus sign in front of the time-term:

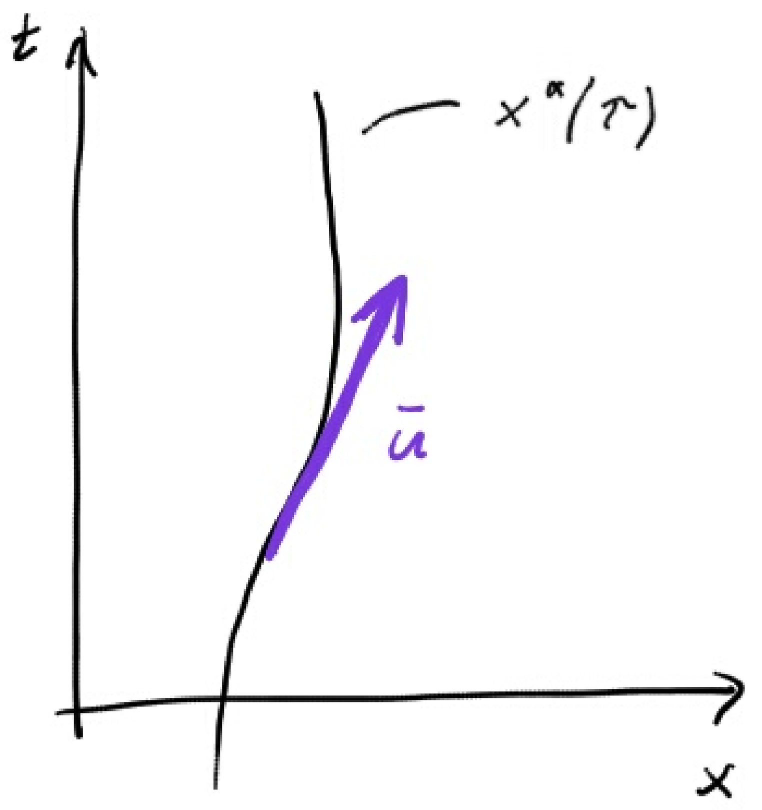
$$\bar{a}\cdot\bar{b} = -a^0 b^0 + \vec{a}\cdot\vec{b}$$

We will also use the notation $b_\alpha = \eta_{\alpha\beta} b^\beta$ so that $\bar{a}\cdot\bar{b} = a^\alpha b_\alpha$

(Hartle does not introduce this notation until chapter 20)

4-velocity and 4-acceleration

Introduce the 4-velocity $\bar{u}$, which is the unit tangent vector to an observer's worldline.



$$u^\alpha u_\alpha = -1 \qquad (1)$$

In the observers own frame:

$$u^\alpha = (1, 0, 0, 0)$$

L.-transf. with $\vec{v} = (v_x, 0, 0)$ $\Longrightarrow$

$$u^{\alpha'} = (\gamma_v, \gamma_v v_x, 0, 0)$$

L.-transf.

$$\begin{cases} t' = \gamma_v (t + v_x x) \\ x' = \gamma_v (x + v_x t) \\ y' = y \\ z' = z \end{cases}$$

In a general frame: (where the observer is moving with 3-velocity $\vec{u}$)

$$u^\alpha = \gamma_u (1, \vec{u})$$

Note that cond. (1) holds: $u^\alpha u_\alpha = \gamma_u^2(-1 + u^2) = -1$

Also note that we can write: $u^\alpha = \dfrac{dx^\alpha(\tau)}{d\tau}$ since $\dfrac{d}{d\tau} = \gamma_u \dfrac{d}{dt}$

Define the 4-acceleration as $\bar{a} = \dfrac{d\bar{u}}{d\tau}$ or $a^\alpha = \dfrac{du^\alpha}{d\tau}$.

(1) $\Longrightarrow$ $$0 = \frac{d}{d\tau}(u^\alpha u_\alpha) = 2 u_\alpha \frac{du^\alpha}{d\tau} = 2 u_\alpha a^\alpha$$

or $$\bar{a} \cdot \bar{v} = 0 \qquad (2)$$

that is, $\bar{a}$ is orthogonal to $\bar{v}$!

In the observers own frame, $\bar{a}$ is purely spatial, that is, it has no time component. Since in this frame $\tau = t$ we have

$$a^\alpha = (0, \vec{a})$$ where $\vec{a}$ is the ordinary 3-acceleration

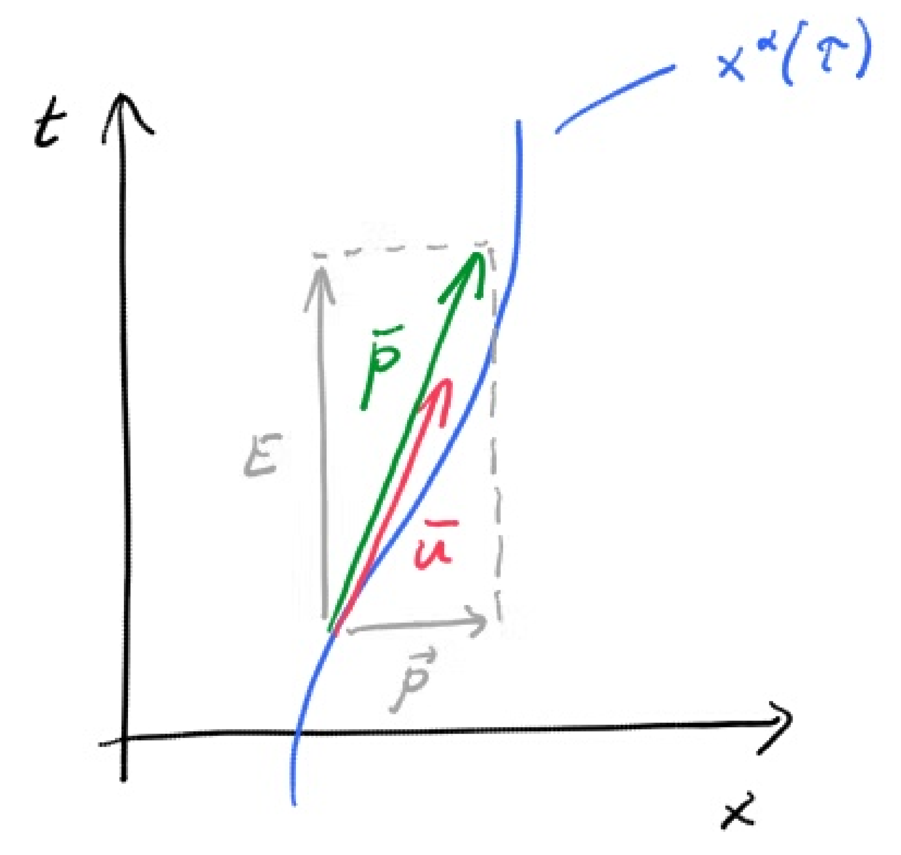
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}$$

Also: $$a^\alpha a_\alpha = |\vec{a}|^2 \qquad (3)$$

Thus, in <u>any</u> frame, $a^\alpha a_\alpha$ is the squared norm of the 3-acceleration measured by the observer herself.

Energy-momentum

Let us return to the 4-velocity:



$$u^\alpha = \frac{dx^\alpha(\tau)}{d\tau} = (\gamma_u , \gamma_u \vec{u})$$

Suppose $x^\alpha(\tau)$ is the worldline of a particle with mass m. Let us multiply u^α with m.

$$\bar{p} = m\bar{u}$$

$$p^\alpha = (m\gamma_u , m\gamma_u \vec{u})$$

$\boxed{?}$

The time component we recognize as the energy of the particle,

$$E = p^0 = m\gamma_u \qquad (\text{or } E = m\gamma_u c^2)$$

and the spatial components as 3-momentum, $\boxed{?}$

$$\vec{p} = m\gamma_u \vec{u}$$

So $\bar{p}$ is the energy-momentum vector (or 4-momentum).

As you know, the total energy-momentum vector of a system of particles is conserved. So what seems to be two different conservation laws in Newtonian physics, turns out to be just one.

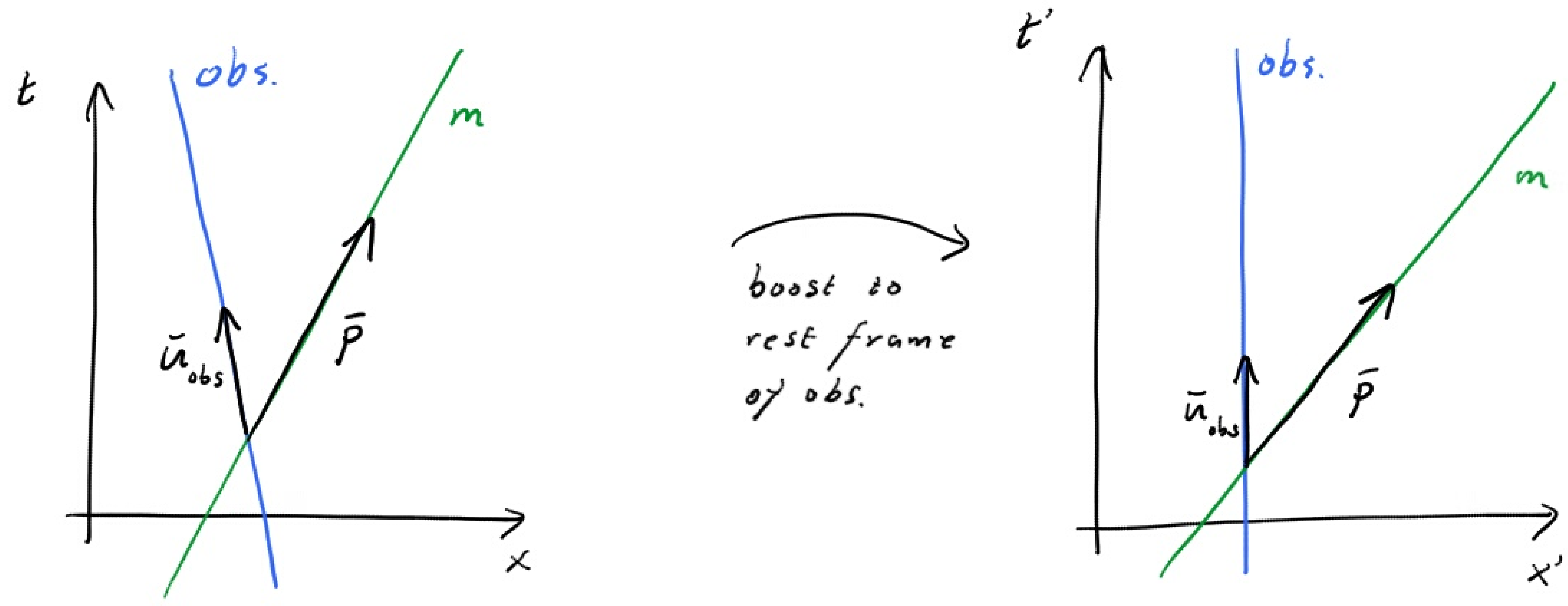
Since

$$p^\alpha p_\alpha = -m^2 \quad \Longrightarrow \quad -E^2 + |\vec{p}|^2 = -m^2$$

$\boxed{?}$

$$E = \sqrt{m^2 + |\vec{p}|^2}$$

Consider a particle with mass m that passes an observer with some speed. What energy will the observer ascribe to the particle?



Frame of obs.

$$p^{\alpha'} = (E', \vec{p}\,')$$

$$u^{\alpha'}_{obs} = (1, 0, 0, 0) \quad \boxed{?}$$

In the frame of the obs. it is clear that the energy E' of the particle seen by the observer can be written as

$$E' = -p^{\alpha'} u_{obs\,\alpha'} = -\bar{p} \cdot \bar{u}_{obs}$$

But this vector relation is frame-independent.

Hence:

$$\boxed{\text{Energy of particle with 4-momentum } \bar{p} \text{ as measured by obs. with 4-velocity } \bar{u}_{obs} \text{ is } E = -\bar{p} \cdot \bar{u}_{obs}}$$

Provided the measurement is local (that is, the obs. and the particle are at the same spacetime position when the measurement is performed) the same relation holds in general relativity (by the e.p.).

The equivalence principle

The first lecture I mentioned the weak e.p. : $\boxed{?}$

In Newtons theory all objects follow the same trajectories (if only influenced by gravity and when given the same initial conditions).

Einstein realized that this indicates that two pairs of situations are equivalent.

First, since all objects fall with the same acc :

[OH: e.p. I]

I) Free fall near gravitating object $\Longleftrightarrow$ Being weightless in outer space

So an obs. in a freely falling laboratory will never detect any gravitational forces.

When such forces are needed in the description they always are indistinguishable from the fictitious forces arising from uniform rectilinear acceleration :

[OH: e.p. II]

II) Standing on the surface of a gravitating object $\Longleftrightarrow$ Accelerating in outer space

Einstein declared that there are no experiments that can be used to distinguish between the two situations in each pair — this is called the strong e.p. (or just e.p.).

It is clear that if this equivalence should have any chance of being true, the e.p. has to be understood as a local statement. In a large enough freely falling laboratory the effect of gravitation will be detectable as tidal forces : there will be small relative motions of objects far apart.

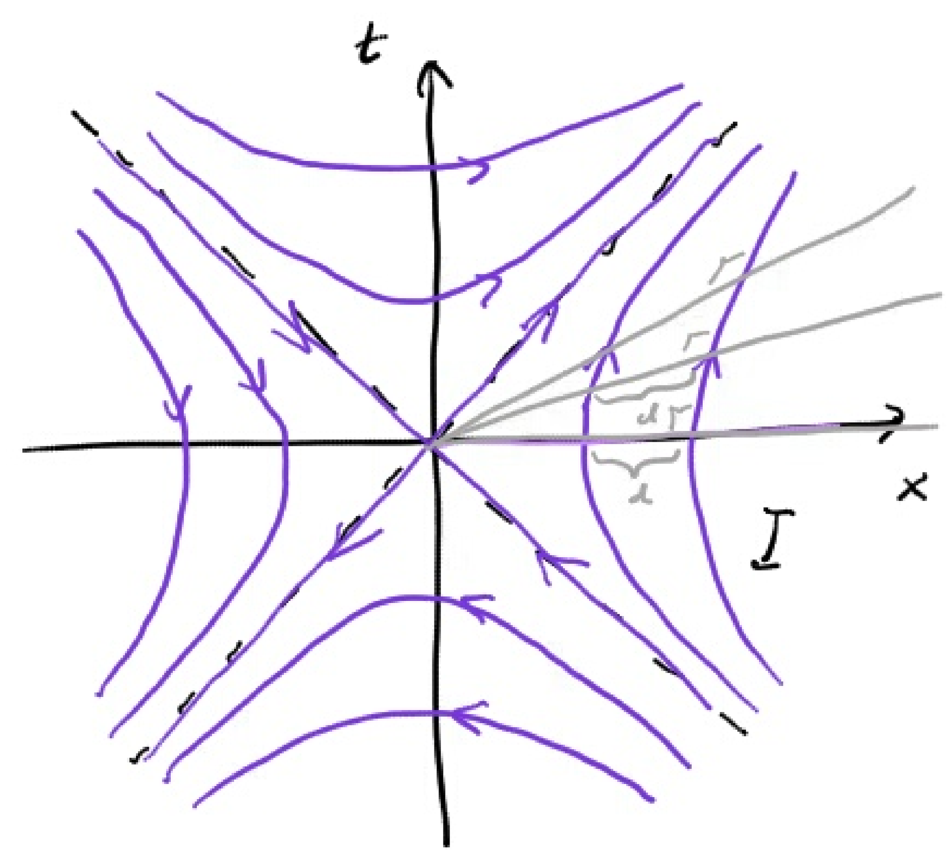
So to the statement of the e.p. we should add that the laboratories should be small. ("There will always be a laboratory small enough such that...")

In order to see the strength of the e.p. we first have to return to acceleration in SR.

Constant acc. in SR

Last time we discussed the flow lines of a Lorentz boost.

We found that they must be hyperbolas



$$x^2 - t^2 = s^2 \quad (\text{for region } I)$$

Reason:

The origin is left invariant, and the distance to it should also be left invariant for any point, that is, $x^2 - t^2 = \text{const.}$

There are three things to note about these flow lines:

(1) Straight lines through the origin are mapped onto straight lines through the origin. These lines must be everywhere orthogonal to the flow lines (since this is the case for the line along the x-axis).

(2) The distance between two flow lines, along the lines orthogonal to them, must be the same all along the flow lines (since such distances are mapped into each other by the transf.)

(3) Since each hyperbola is mapped into itself (just shifted) it must have the same properties all along it. In particular, an observer moving along it would move <u>uniformly</u>. Since such an observer obviously would be accelerating, she would move with constant acceleration!

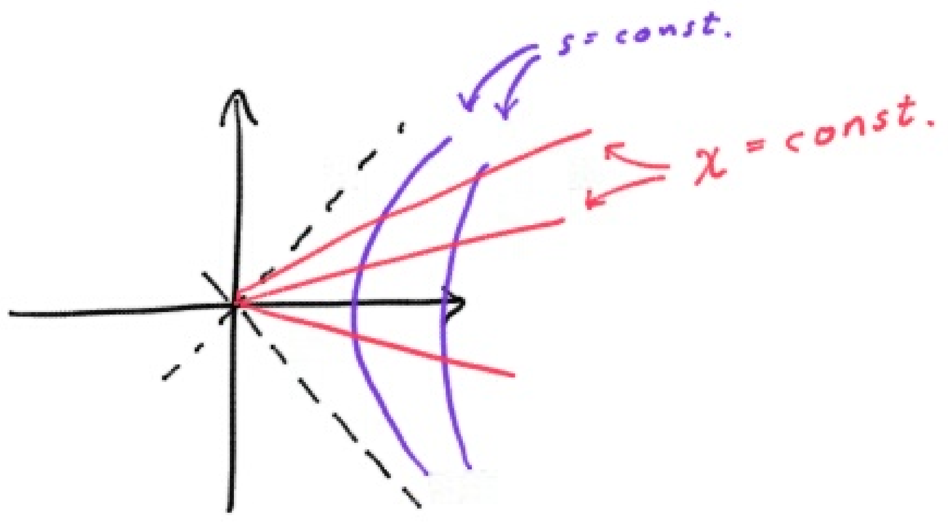
We can get the value of the acc. for hyperbola s from its value at $t = 0$:

$$x(t) = (t^2 + s^2)^{1/2} \qquad \frac{dx}{dt} = \frac{t}{(t^2+s^2)^{1/2}}$$

$$\left.\frac{d^2x}{dt^2}\right|_{t=0} = \left.\frac{(t^2+s^2)^{1/2} - t^2(t^2+s^2)^{-1/2}}{t^2+s^2}\right|_{t=0} = \frac{1}{s} \qquad \text{Hence } a = \frac{1}{s}$$

Note that neighbouring flowlines have slightly different acc. Yet, the distance between them stays the same! (Point (2) above.)

A natural parametrization of the hyperbola at distance s is



$$\begin{cases} x = s \cosh \chi \\ t = s \sinh \chi \end{cases}$$

(s and χ then forms coordinates on region I)

But we would prefer a parametrization in terms of proper time along the worldline. It is not difficult to show that

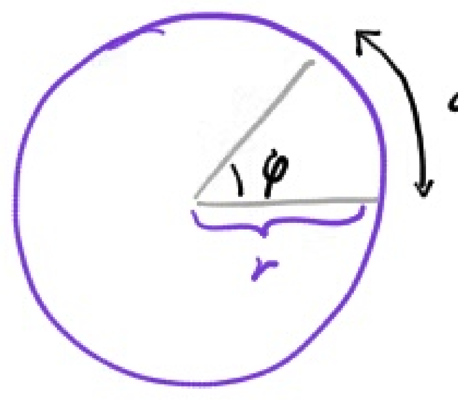
$\tau = s\chi$

proof

analogy

Note the analogy with the circle:

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases}$$



$d = r\varphi$

To summarize then:

A worldline of const. acc. a is described by

$$\begin{cases} x(\tau) = \dfrac{1}{a} \cosh \tau a \\ t(\tau) = \dfrac{1}{a} \sinh \tau a \end{cases}$$

(which you also will show together with Anders at the problem solving session in a very different way)

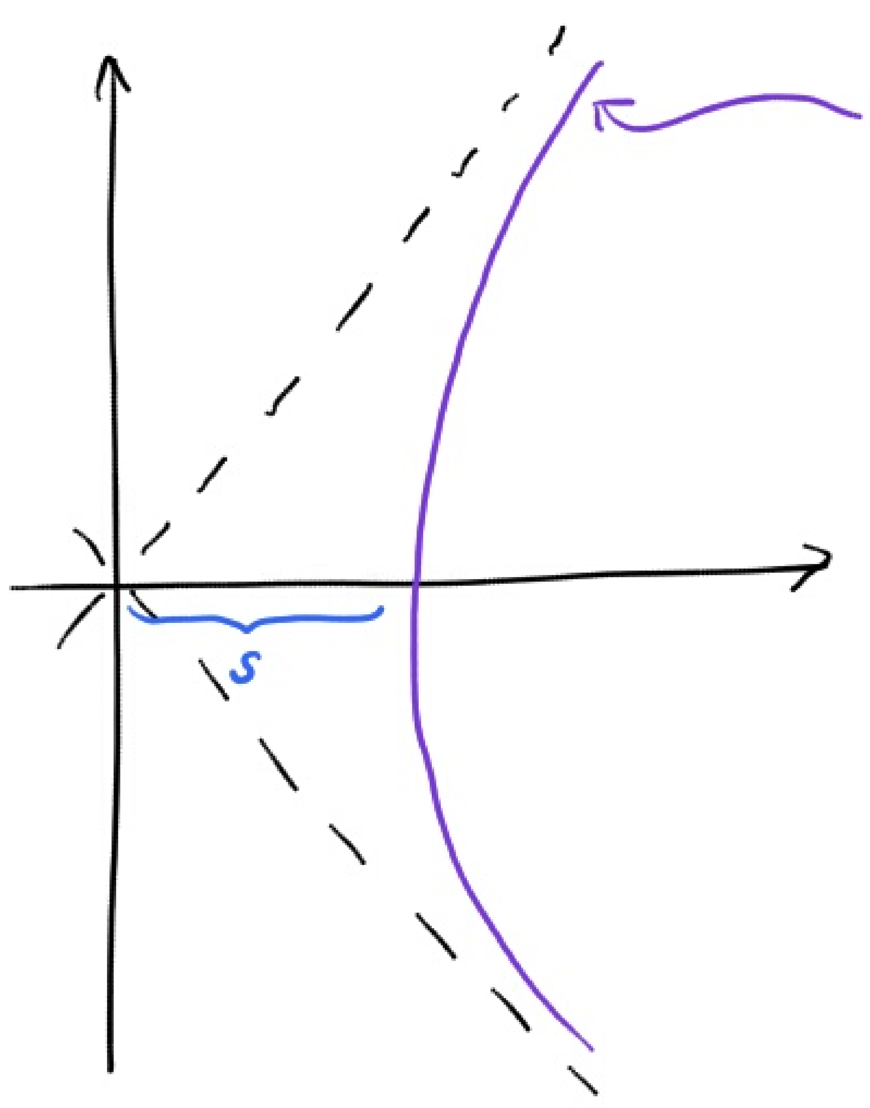
Consider the curve, parametrized by χ, and with constant s:

$$\begin{cases} x(\chi) = s \cosh \chi \\ t(\chi) = s \sinh \chi \end{cases} \Rightarrow \quad \begin{aligned} dx &= s\, d\chi \sinh \chi \\ dt &= s\, d\chi \cosh \chi \end{aligned}$$

$$d\tau = \sqrt{-ds^2} = \sqrt{dt^2 - dx^2} = \sqrt{s^2 d\chi^2 \cosh^2\chi - s^2 d\chi^2 \sinh^2 \chi} = s\, d\chi$$

$\Rightarrow \quad \tau = s\chi$

If time:



$$\begin{cases} x = s \cosh \dfrac{\tau}{s} \\ t = s \sinh \dfrac{\tau}{s} \end{cases}$$

How large is s for acc. g?

$g \simeq 10\ m/s^2 = \dfrac{10}{(3 \cdot 10^8)^2}\ m^{-1}$

$[1\ s = 3 \cdot 10^8\ m]$

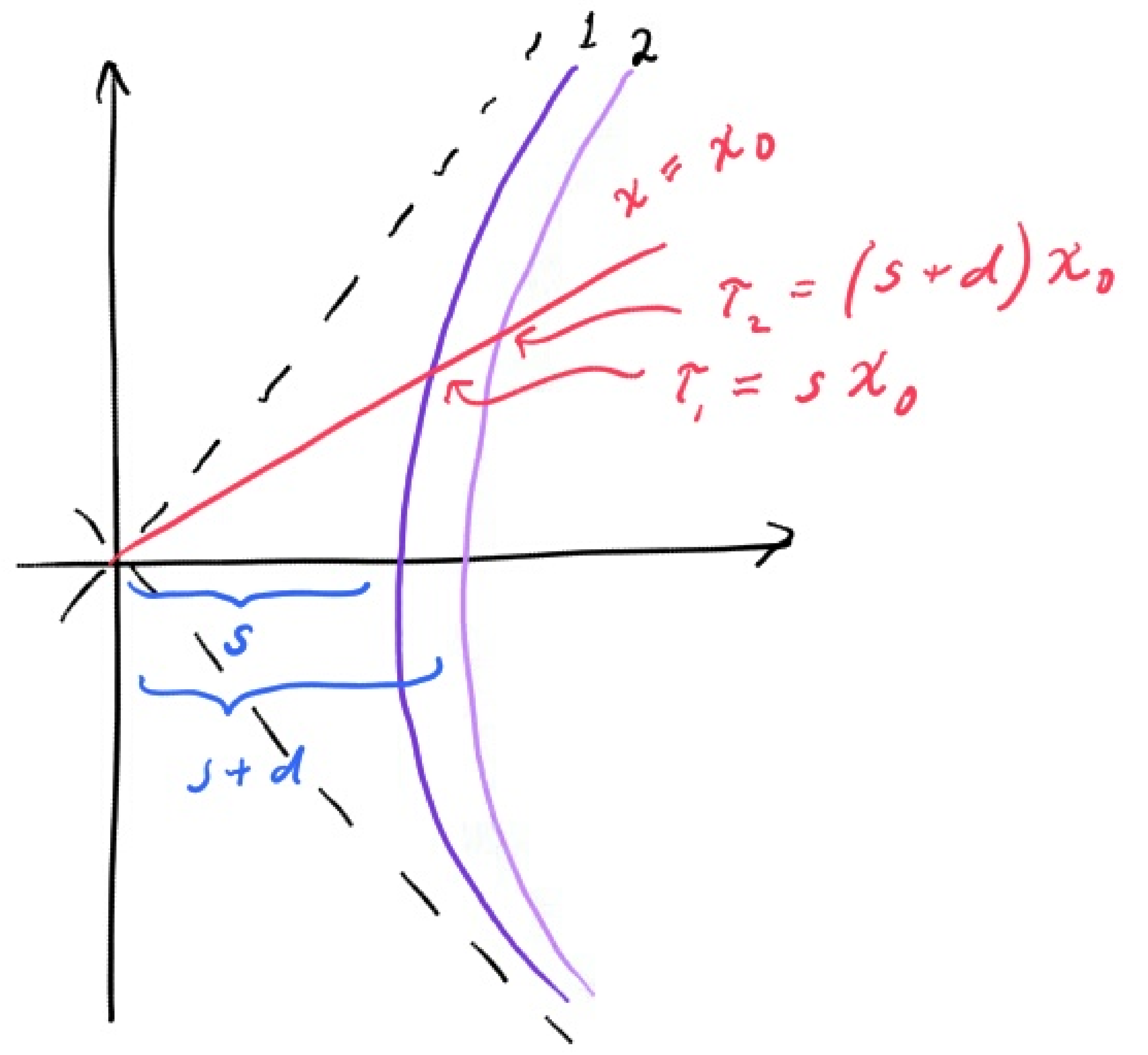
So $s = \dfrac{1}{g} \simeq 10^{16}\ m \sim 1$ light year

$[1 \text{ lightyear} = 3 \cdot 10^8 \cdot 3600 \cdot 24 \cdot 365 \sim 10^{16}\ m]$

Consider two flow lines, separated by distance d. These can be thought of as the worldlines of two observers in a spaceship of length d, one in the front and one in the back.

As the diagram suggests, the clocks of these two observers run at different rates, as measured by the observers themselves! (Since the red line in the diagram ($x = x_0$) is a line of simultaneity for the two observers, and thus the line along which they compare their clocks.)

What is the time difference?



$$\tau_2 - \tau_1 = d x_0 = d \frac{\tau_1}{s} = \tau_1 a d$$

$$\frac{\Delta \tau}{\tau} = a d$$

Or in ordinary units:

$$\frac{\Delta \tau}{\tau} = \frac{a d}{c^2}$$

— the acceleration time shift!

Let us now apply the e.p. to this result. According to that principle there should be no difference between a laboratory in constant acc. and one in a gravitational field with a corresponding grav. acc.

[OH: e.p. II]

But we have just shown that for the acc. laboratory, if we put one clock at the floor and one at the ceiling the one at the ceiling will run faster. So by the e.p. the same must be true of two clocks at different heights in a gravitational field!

$$\text{e.p} \Rightarrow \boxed{\frac{\Delta \tau}{\tau} = \frac{g d}{c^2}}$$ — the gravitational time shift

Note that this time shift (so far) is not the result of any spacetime curvature. Next time we will see that this result, together with the fact that g is not a constant, but varies from point to point, makes it necessary to introduce curved spacetime.

Also, note that it is clearly wrong to say that clocks at lower positions run slow "because they are in a stronger gravitational field". Rather, they run slow because they are further down in a grav. field.