Lecture 1

Equivalence principle (weak) $\longrightarrow$ curved spacetime

Curved 2-surfaces : Gauss curvature
- independent of embedding
- can be measured <u>in</u> the surface

The circumference of a circle on a sphere

$\Rightarrow$ 1) Curvature can be defined at each point by a limiting procedure

2) Given a measurement accuracy there is always a region small enough so that the surface seems flat.
— the Euclidean geometry the most important one!

Spacetime analogue of 2 $\longrightarrow$

Minkowski-space — the most important spacetime geometry

The Minkowski line-element
- timelike, spacelike, null
- the lightcone
- proper time along <u>arbitrary</u> curve

Lorentz transformation
- usual form
- lines of simultaneity
- flow lines

Gravitation

Gravitation has one property that singles it out among all other physical phenomena.

Newton: all objects follow the same paths!
only affected by gravity, and given the same initial conditions

If the moon is replaced by an apple, given the same initial speed, the apple will follow the same trajectory.
This is the characteristic property of gravitation, shared by no other known force.

— the "weak" equivalence principle

This suggests a different perspective on gravitation: consider the special trajectories traced out by any freely falling object, not as the result of a force acting on the object, but as a property of space itself. Consider them as straight lines in a curved spacetime.

$\xrightarrow[\text{(suggests)}]{}$ Freely falling trajectories are straight lines in a curved spacetime.

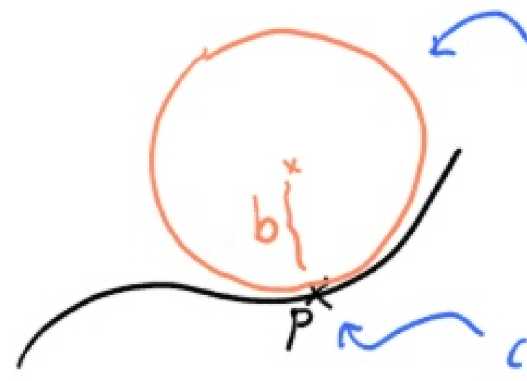
This is the main idea in GR.

This simple argument suggests that we can describe gravity as an expression of spacetime curvature. In a week we will see how to make the argument much stronger: it turns out that we are actually forced to describe gravity in this way.
(Assumptions? — A stronger version of the e.p. + SR!)

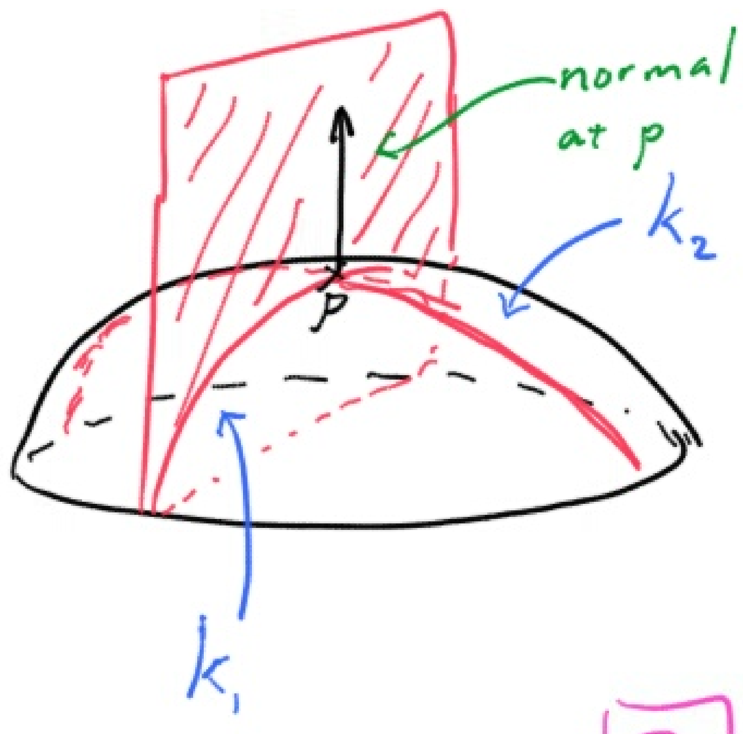
So our description will involve curved spaces, or spacetimes.
Therefore, today, as an introduction, I want to introduce some simple but important ideas about curvature.

Curvature

1-dimension:



2-dimensions:



k_1 and k_2 are the largest and smallest value of the curvature (the principal curvatures).

$k_1 k_2 = K =$ Gauss-curvature

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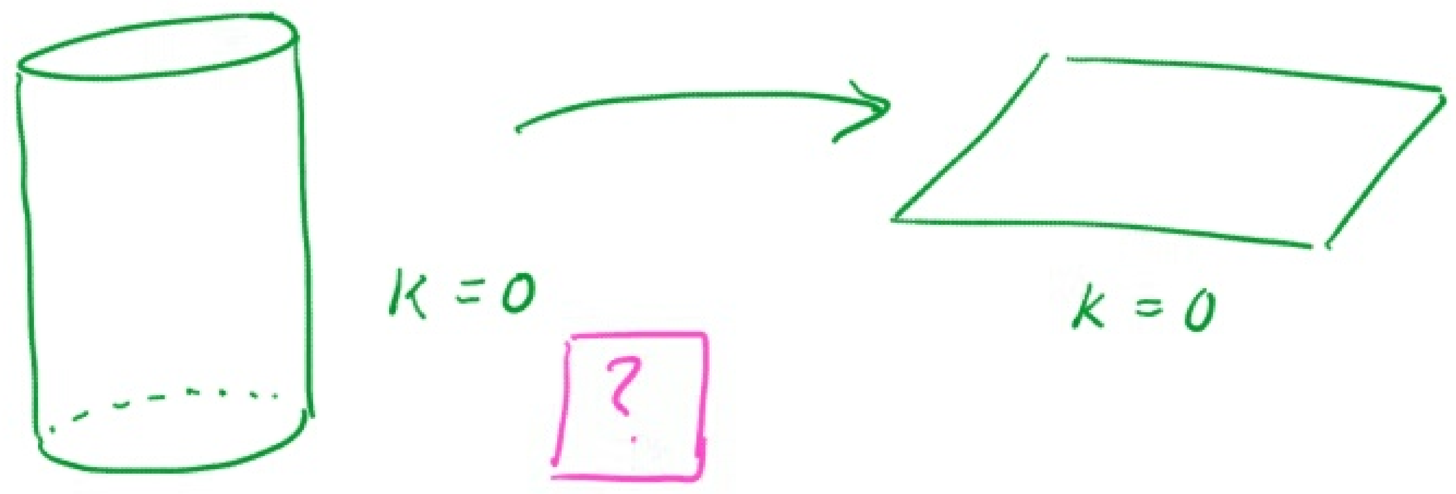
Gauss: Theorema Egregium: ("the remarkable theorem") K is independent of the embedding

That is: if the surface is deformed without stretching, that is, if the embedding is changed, the Gaussean curvature stays the same.



?

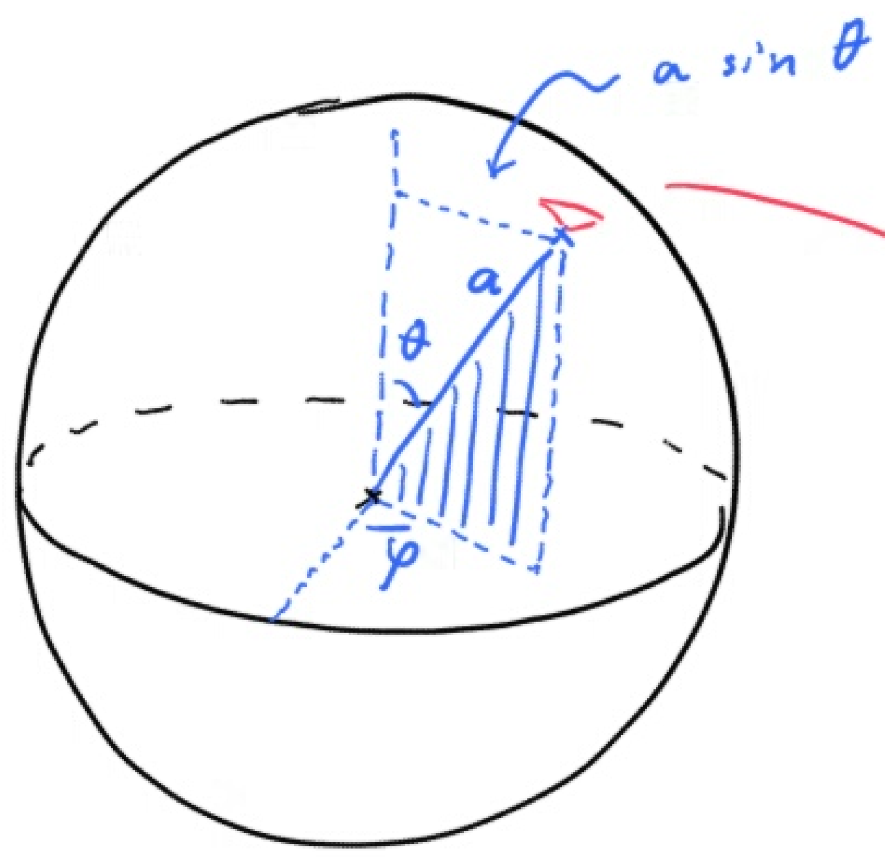
This also means that K could be determined solely by making measurements in the surface, since if the surface is not streched all interior distances must remain the same. When we speak of curved spacetimes in GR, we always refer to this kind of interior curvature, the Gaussian curvature or its counterparts in higher dimensions.

It is of no physical interest to ask for the embedding of a spacetime, because it does not affect any measurements that can be performed by someone living in the spacetime.

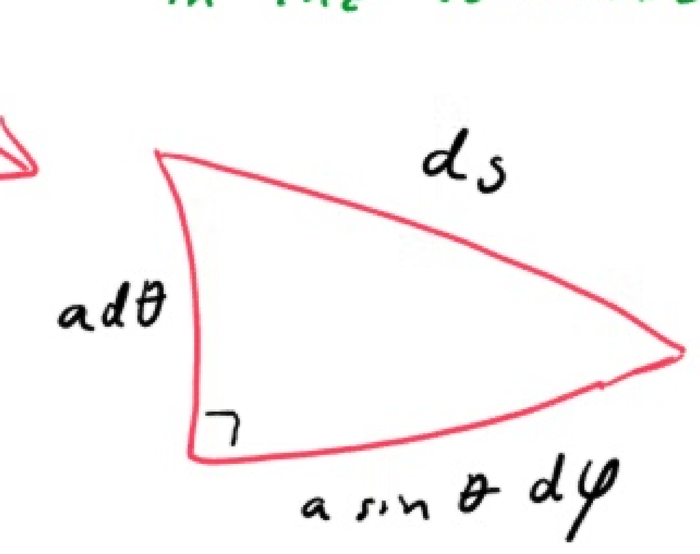
Let's imagine that we are ants living on a two dimensional curved surface. How can we measure its curvature?

As an example let's concentrate on the simplest of all curved surfaces: a sphere. A sphere with radius a has constant Gaussian curvature $\frac{1}{a^2}$. ?

A sphere with radius a



Let us express the infinitesimal distance ds between two points on the sphere in the coordinate distances $d\theta$ and $d\varphi$:

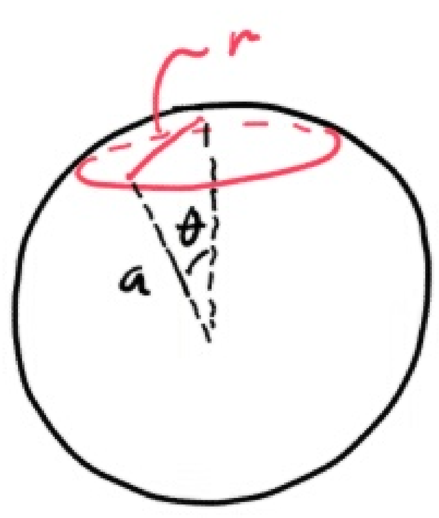


$$\Longrightarrow \quad ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\varphi^2$$

This "line-element" describes the infinitesimal distance between neighbouring points.

What is the circumference of a circle with radius r?

Consider, for simplicity, a circle at constant θ.



$$r = a\theta \, , \qquad \theta = \frac{r}{a}$$

$$d\theta = 0$$

$$C = \int_{\varphi=0}^{\varphi=2\pi} ds = a \sin\theta \int_0^{2\pi} d\varphi = 2\pi a \sin\frac{r}{a} \simeq$$

$$\overset{r \ll a}{\frown} \simeq 2\pi a \left(\frac{r}{a} - \frac{1}{6}\frac{r^3}{a^3} \right) = 2\pi r \left(1 - \frac{1}{6}\frac{r^2}{a^2} \right)$$

For small r this is just the usual $2\pi r$. But for larger r there is a deviation — the circumference is "too small"!

So in this way, if we were ants, we could measure the Gaussian curvature. And the same method of course would work if the surface was a general curved surface, where the curvature varied from point to point.

Note:

1) Curvature is a local concept. That is, it has a value at each point. But:

2) In a region small enough the geometry is indistinguishable from flat.

(Given some measurement accuracy.)

Therefore, Euclidean geometry is the most important of all spatial geometries!

These statements also hold for curved spaces of higher dimension than 2.

Such spaces are hard for us to imagine, since they cannot be embedded in 3 dimensions. But the embedding is unimportant, as we saw. We still, in principle, could determine that we lived in a curved 3 dim. space by making careful geometrical measurements. For example, we could draw circles, and could compare their radius and circumference.

There would be one difference, though: at each point we could draw several circles, oriented differently. We could obtain different results for these different circles. Thus, in higher dimensions than 2, we need more than one number to characterize the curvature at each point. (It turns out that in 3 dim. we need 6 numbers per point, and in 4 dim. we need 20 numbers!)

We saw that an observer in a curved space, with access to only a small region (small compared with the curvature) will not be able to detect the curvature. To her space will seem Euclidean.

The corresponding statement is true for spacetimes: at any point in any spacetime there is always a region small enough that the spacetime cannot be distinguished from Minkowski-space there.

Therefore, in GR, it is extremely helpful to know special relativity:

- SR is used to describe the outcome of all local measurements.
- SR + c.p. give rise to the gravitational time shift.
- SR even helps in understanding some aspects of black hole event horizons or cosmology.

The Minkowski spacetime is the most important of all spacetime geometries!

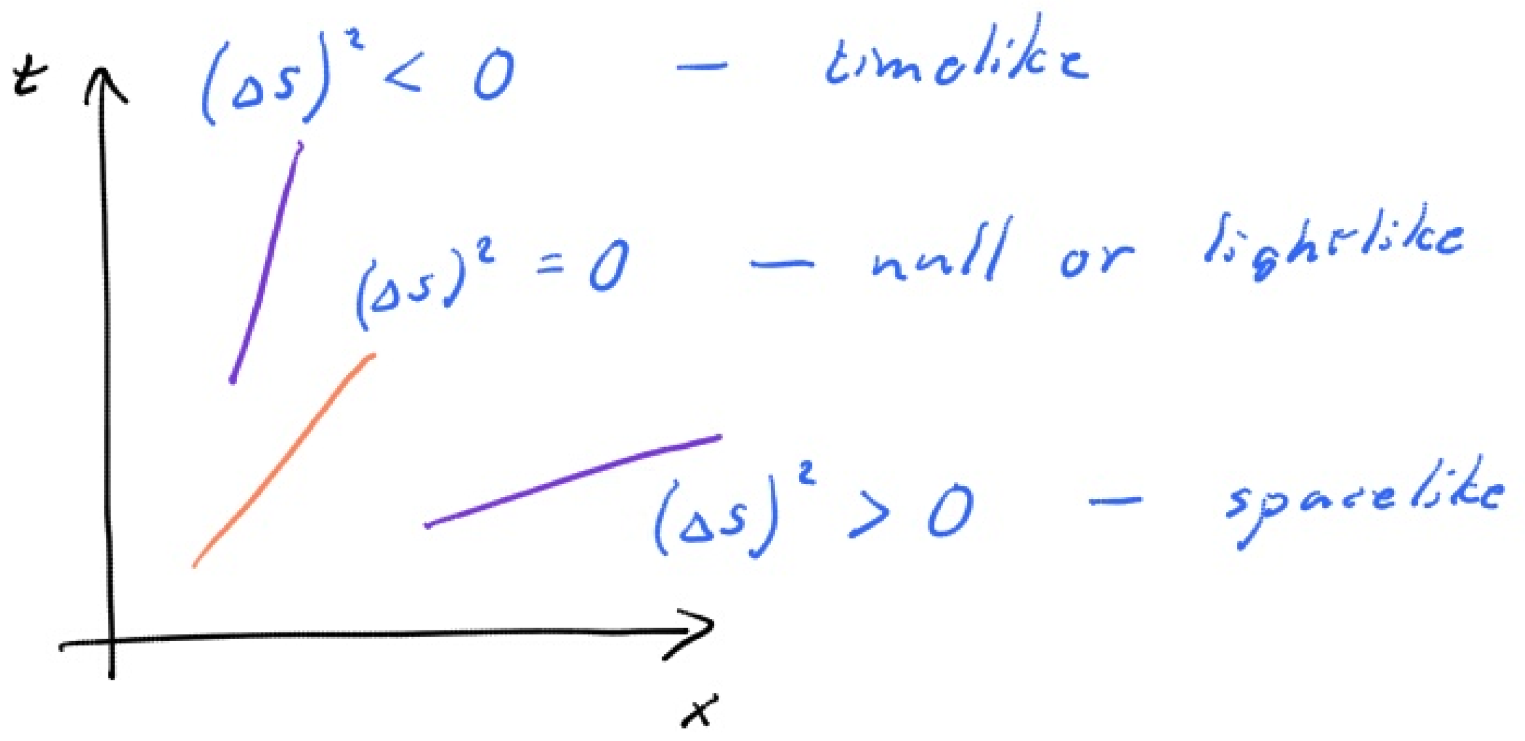
It is therefore important that we develop an intuition for the Minkowski space as deep as the one we have for Euclidean geometry.

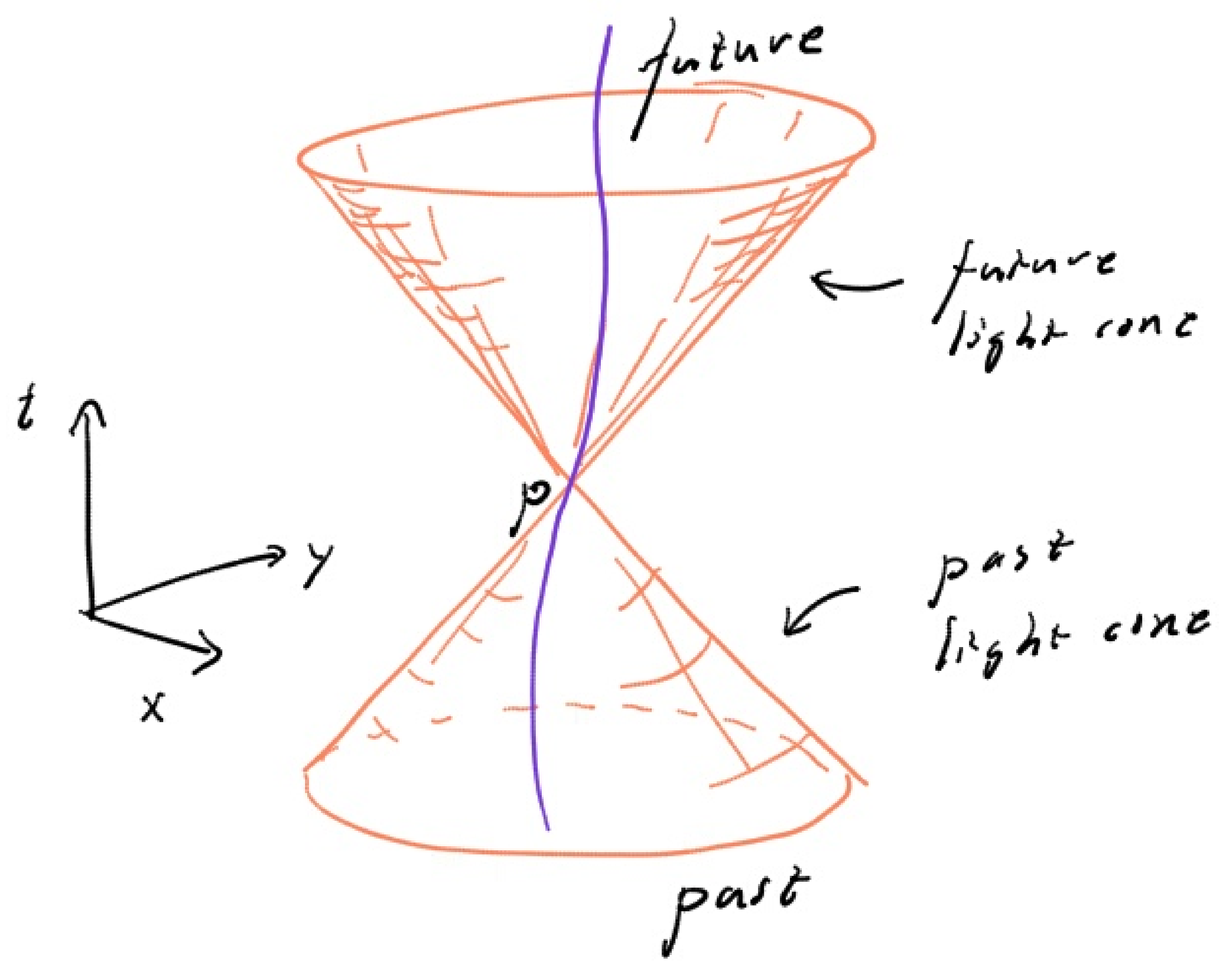
Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad \text{(and now I put } c=1\text{)}$$

Essentially everything in SR follows from this line-element.

We have three kinds of spacetime intervals:





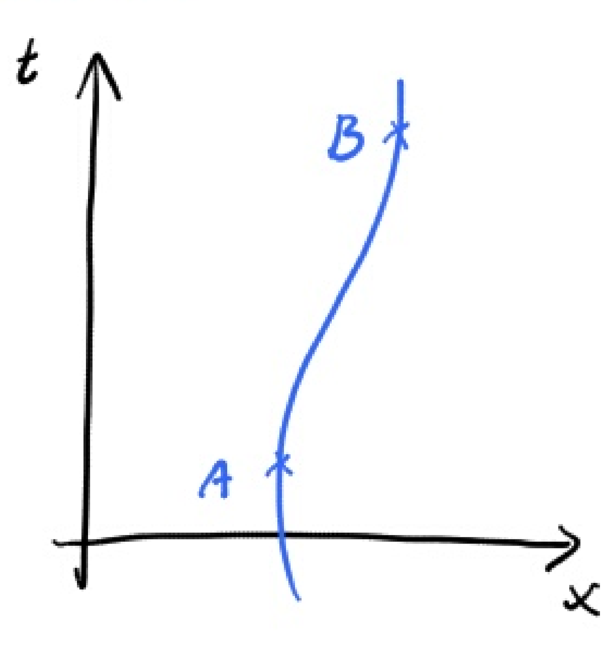
To each point p corresponds a light cone, that is, all points at zero spacetime distance to p.

The worldline of an observer, which is timelike, must be "inside" the light cone at each point that it passes.

The spacetime distance along an arbitrary curve can be calculated using the line-element. However, if the curve is timelike the proper time $d\tau$ along the curve is defined as

$$d\tau = \sqrt{-ds^2}$$

Hence



$$\tau_{AB} = \int_A^B \sqrt{-ds^2} = \int_A^B \sqrt{dt^2 - dx^2 - dy^2 - dz^2} =$$

$$= \int_A^B \sqrt{1 - \frac{dx^2}{dt^2} - \frac{dy^2}{dt^2} - \frac{dz^2}{dt^2}}\, dt =$$

$$= \int_A^B \sqrt{1 - v^2}\, dt \qquad \text{where } \vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

If $\vec{v}(t)$ is known it is thus easy to calculate the proper time along the curve. This also shows that

$$d\tau = \sqrt{1-v^2}\, dt \quad \text{or} \quad dt = \gamma_v\, d\tau \quad \text{where} \quad \gamma_v = \frac{1}{\sqrt{1-v^2}}$$

— time dilation!

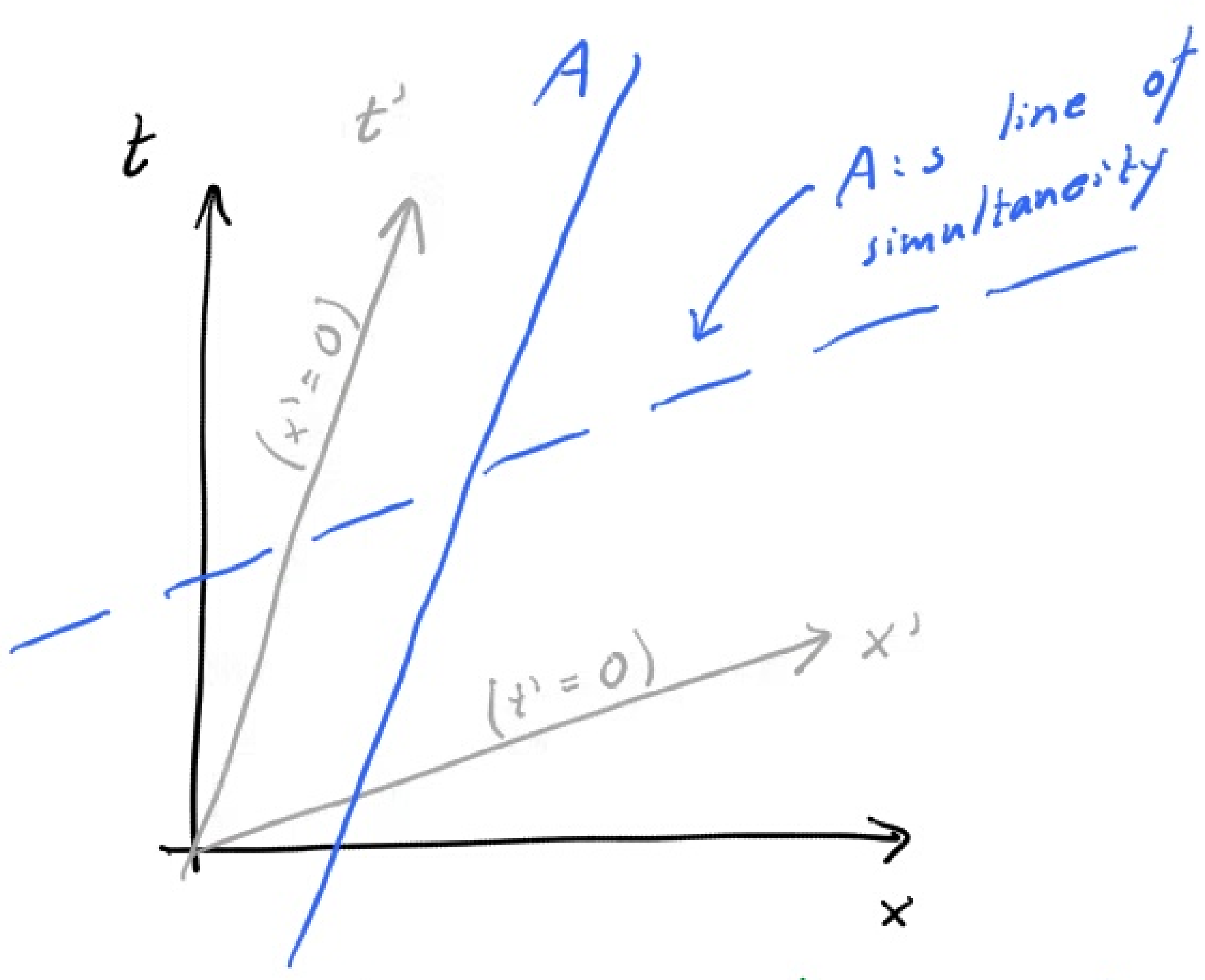
A common misconception about SR is that you cannot treat accelerated motion — that is clearly wrong as our integral for τ shows.

(Next lecture, we will return to accelerated motion in SR, which turns out to be a crucial part in taking the step to GR.)

The Lorentz-transformation takes us from the Cartesian coordinate system in one inertial frame to another.

The Lorentz-transformation

$$\begin{cases} t' = \gamma_v (t - vx) \\ x' = \gamma_v (x - vt) \\ y' = y \\ z' = z \end{cases}$$



It is easy to show that this transformation leaves the line element invariant: $ds^2 = ds'^2$

This reflects the fact that the line element ds represents the physical spacetime distance, as measured by real clocks and rulers. Such physical properties must, of course, be independent of the coordinates used.

Now, we can ask:

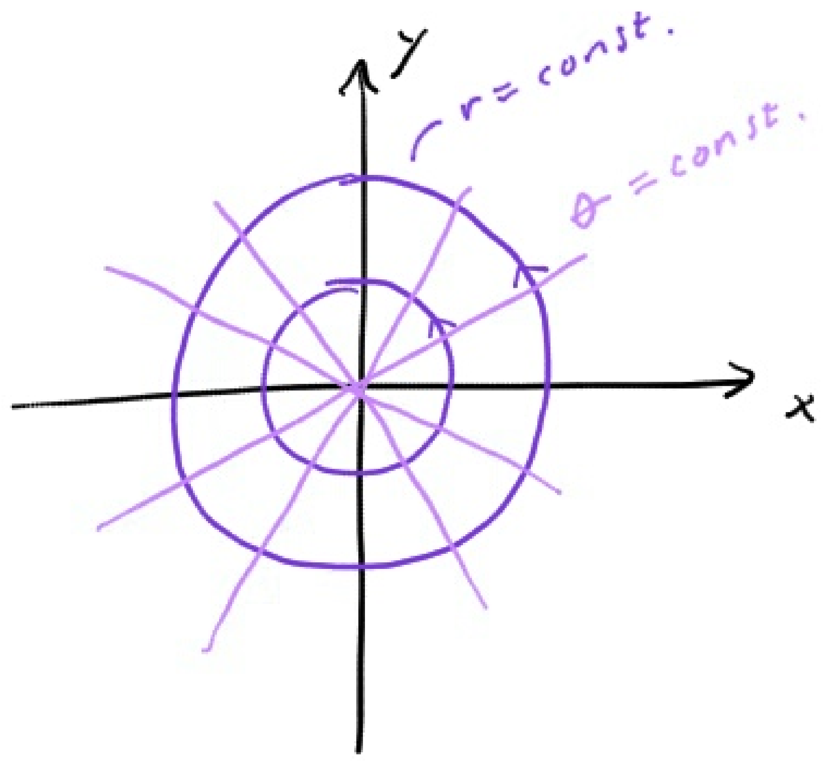
What are <u>the flowlines of a Lorentz-boost</u>?

That is, along what lines do the points in the diagram shift when we act with a Lorentz-transformation?

To better understand this concept of flowlines of a symmetry, let us first, as a warm up, consider rotations.

What does the flow lines of a rotation in the x-y plane look like?

- Circles centered around the origin!



A rotation with angle θ written as a coordinate transformation from (x, y) to (x', y'):

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{cases}$$

or in polar coordinates:

$$\begin{cases} r' = r \\ \varphi' = \varphi + \theta \end{cases}$$

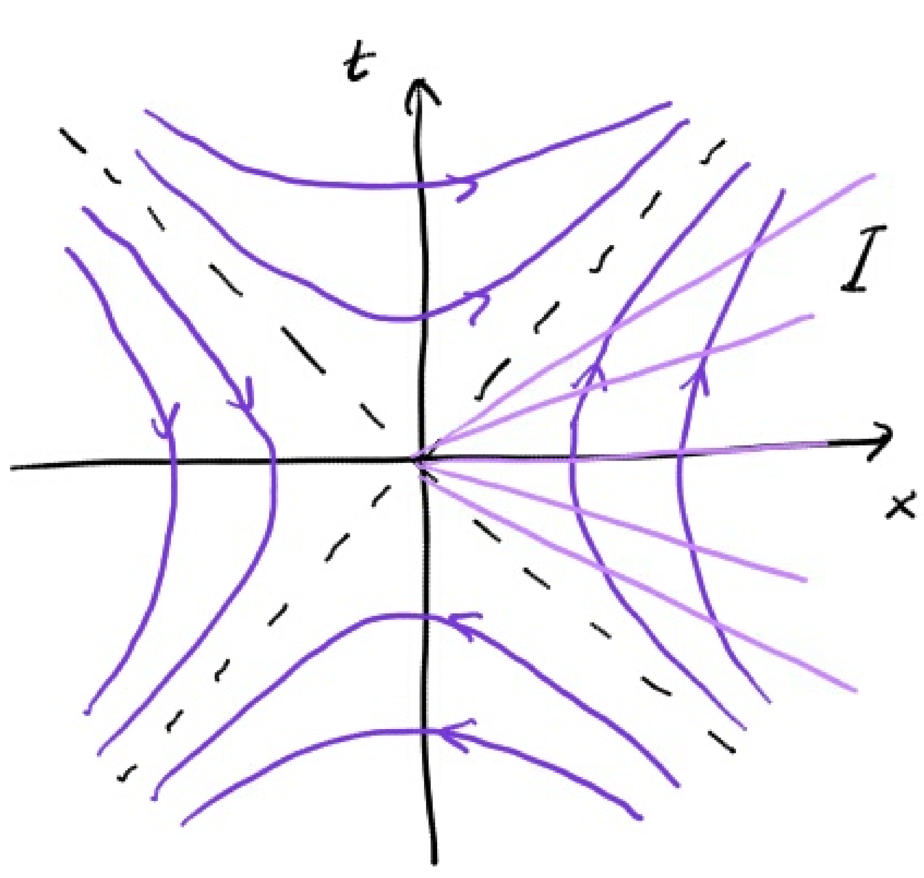
Note that it is much easier to describe rotations in polar coord.!

So, why, are the flow lines circles?

Because the origin is left invariant, and the distance to the origin (like all distances) must be left invariant. Hence, a point a distance r from the origin will end up a distance r from the origin, that is, somewhere on the same circle centered around the origin.

Now, let us return to the Lorentz transformation.

What are the flow lines of a boost in the t-x-plane?



- Hyperbolas:

$$x^2 - t^2 = s^2 \quad (\text{for region } I)$$

Why?

- Because the origin is left invariant, and the distance to it should be left invariant: $x^2 - t^2 = \text{const.}$