Lecture 16

Gravitational waves:
- vacuum
- weak

$\Big\} \Rightarrow$ solutions to linearized vacuum equation

Linearized vacuum equation — 10 linear equations!

Freedom in choosing coordinates

$\Rightarrow \begin{cases} \text{Wave equation} \\ \text{Lorenz gauge condition} \end{cases}$

solution $h_{\alpha\beta}(x) = a_{\alpha\beta}\, e^{i\bar{k}\cdot\bar{x}}$, $\bar{k} = (\omega, \vec{k})$

— wave with frequency ω travelling in direction $\vec{k}$ with the speed of light

More gauge freedom

$$\Rightarrow \quad h_{\alpha\beta}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)} \quad \text{— TT-gauge!}$$

A general superposition $f(t-z)$

a, b — two polarizations

Choose $a = 1$, $b = 0$:

$$ds^2 = -dt^2 + [1 + f(t-z)]dx^2 + [1 - f(t-z)]dy^2 + dz^2$$

Two test particles
- coordinate positions unchanged!
- distance between them changes!

A ring of test particles
- the two polarizations: $+$ and $\times$
- circular polarization

Gravitational waves

In Newton's theory gravitational information is mediated instantly through spacetime: a violent gravitational event (such as the collision of two stars) would affect the gravitational potential immediately, also far away.

That could not be true in a relativistic theory. So in general relativity we expect the information from such an event to be transfered through spacetime at some finite speed — we expect there to be gravitational waves.

These waves, and their properties, should of course be a consequence of Einstein's equations — we expect these to have wave solutions. How could we find them?

Two things simplify this problem:

(1) the waves should propagate through vacuum,

(2) the waves most relevant to us would be very weak.

Thus, we should look for solutions to the vacuum equation, and it should be possible to view the solutions as weak perturbations of flat Minkowski spacetime.

Therefore, we are looking for solutions to

The vacuum equation: $R_{\alpha\beta} = 0$

of the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

Minkowski in cartesian coord.

small perturbation

The vacuum equation could be expanded to different orders in $h_{\alpha\beta}$, and for small perturbations it should be enough to consider the equation to first order:

$$\delta R_{\alpha\beta} = 0$$

— 10 <u>linear</u> partial diff. equations for $h_{\alpha\beta}$!

first order perturbation of $R_{\alpha\beta}$

Write $R_{\alpha\beta}$ in terms of the Christoffel symbols, and the Christoffel symbols in terms of the metric. Keep only terms linear in $h_{\alpha\beta}$. The result is:

$$\delta R_{\alpha\beta} = \frac{1}{2}\left[-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha\right] \overset{!}{=} 0$$

— Linearized vacuum field equation

$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2$

$V_\alpha = \partial_\gamma h^\gamma{}_\alpha - \frac{1}{2}\partial_\alpha h^\gamma{}_\gamma$

where $h^\gamma{}_\alpha = \eta^{\gamma\delta} h_{\delta\alpha}$ ← to first order, indices can be raised and lowered with $\eta_{\alpha\beta}$

This equation can be simplified if we make use of the freedom in choosing coordinates. A small shift in coordinates will not change the form of the perturbation ($g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$). Consider

$$\underbrace{x^{\alpha'}}_{\text{new set of coord.}} = \underbrace{x^\alpha}_{\text{old set of coord.}} + \underbrace{\xi^\alpha(x)}_{\text{small shift}}$$

How does this change $h_{\alpha\beta}$?

In general:

$$g_{\alpha'\beta'}(x') = \frac{\partial x^\gamma}{\partial x^{\alpha'}}\frac{\partial x^\delta}{\partial x^{\beta'}} g_{\gamma\delta}(x)$$

$$\frac{\partial x^\gamma}{\partial x^{\alpha'}} = \frac{\partial}{\partial x^{\alpha'}}\left(x^{\gamma'} - \xi^\gamma(x)\right) = \underbrace{\delta_{\alpha'}^{\gamma'}}_{\text{same in all coord. systems}} - \partial_{\alpha'}\xi^\gamma(x) \overset{\text{first order}}{=} \delta_\alpha^\gamma - \partial_\alpha \xi^\gamma$$

Hence:

$$g_{\alpha'\beta'} = \left(\delta_\alpha^\gamma - \partial_\alpha\xi^\gamma\right)\left(\delta_\beta^\delta - \partial_\beta\xi^\delta\right) g_{\gamma\delta} \overset{\text{first order}}{=}$$

$$= g_{\alpha\beta} - \underbrace{g_{\gamma\beta}\partial_\alpha\xi^\gamma}_{\simeq\, \eta_{\gamma\beta}\partial_\alpha\xi^\gamma \,=\, \partial_\alpha\xi_\beta} - \underbrace{g_{\alpha\delta}\partial_\beta\xi^\delta}_{\simeq\, \eta_{\alpha\delta}\partial_\beta\xi^\delta \,=\, \partial_\beta\xi_\alpha} = \eta_{\alpha\beta} + \underbrace{h_{\alpha\beta} - \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha}_{\text{Perturbation in the new coordinates}}$$

So

$$h^{new}_{\alpha\beta} = h^{old}_{\alpha\beta} - \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha$$

Now, ξ^α is four arbitrary functions. It is always possible to choose them in a way so that

$$V_\alpha^{\text{new}} = 0$$

Then the field equation simplifies to

$$\Box h_{\alpha\beta} = 0 \qquad (1)$$

together with the conditions

$$V_\alpha = \partial_\beta h^\beta_{\ \alpha} - \frac{1}{2}\partial_\alpha h^\beta_{\ \beta} = 0 \qquad (2)$$ — Lorentz gauge condition

Equation (1) is just the ordinary wave equation for each component of the perturbation $h_{\alpha\beta}$. Its solution for a definite wave length described by wave vector $\vec{k}$ is

Solution : $h_{\alpha\beta}(x) = a_{\alpha\beta}\, e^{i\bar{k}\cdot\bar{x}} = a_{\alpha\beta}\, e^{i(-\omega t + \vec{k}\cdot\vec{x})}$

constants

$k^\alpha = (\omega, \vec{k})$

$\boxed{?}$ where $\omega = |\vec{k}| = \frac{2\pi}{\lambda}$

$\boxed{?}$ — Gravitational waves travel at the speed of light!

We also have to make sure that our solution fulfils the Lorentz gauge condition. But we still have some freedom left in our choice of coordinates — we can change coordinates in such a way that the Lorentz condition remains fulfilled.

I will not show the details here, but it turns out that by utilizing that gauge choice the Lorentz condition simplifies into a simple condition for the amplitudes $a_{\alpha\beta}$.

The final solution for a wave with a definite wave vector, travelling in the positive z-direction (that is, with $\vec{k} = (0, 0, \omega)$), then becomes

$$h_{\alpha\beta}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i\omega(t-z)}$$

"TT-gauge"
= Traceless, Transverse

Note that the waves are transverse: for this wave travelling in the z-direction, the interesting things happen in the x-y-plane.

Now, the wave equation is linear, so, as usual, we can obtain new solutions just by superposing these "pure" waves.

Each wave in such a superposition is characterized by its wave vector $\vec{k}$ and has its own values of a and b.

For simplicity let us just consider waves travelling in the z-direction, that is, with $\vec{k} = (0, 0, w)$. Then each wave is characterized by its frequency w, and the amplitudes $a(w)$ and $b(w)$.

Adding them together gives

$$h_{xx} = -h_{yy} = \int a(w)\, e^{-iw(t-z)} = f_+(t-z)$$

this notation will be explained soon...

So by choosing the amplitudes $a(w)$ we can obtain an arbitrary function for the diagonal elements. Similarly:

$$h_{xy} = h_{yx} = \int b(w)\, e^{-iw(t-z)} = f_x(t-z)$$

Hence, a wave moving in the z-direction is described by the metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+f_+(t-z) & f_x(t-z) & 0 \\ 0 & f_x(t-z) & 1-f_+(t-z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

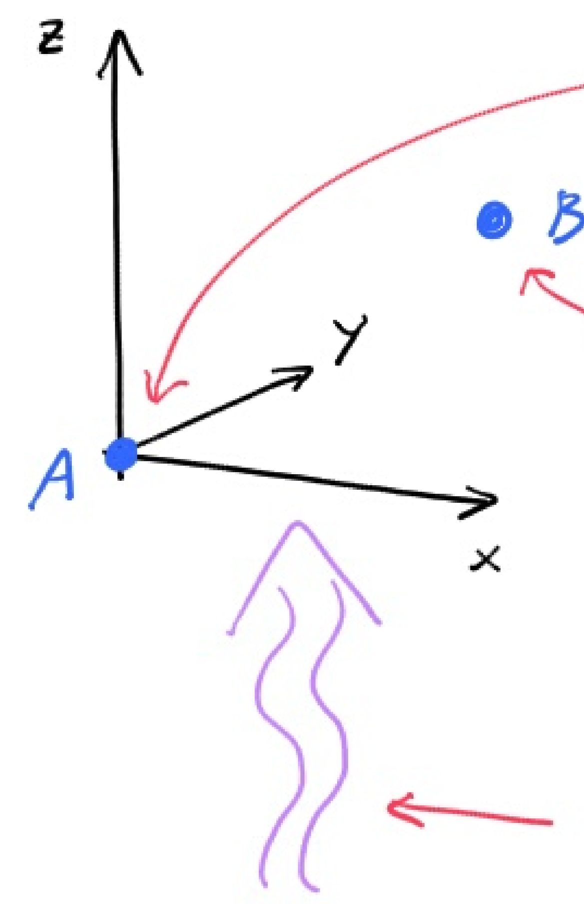
To see what this means, and what measurable consequences such a wave would have, let's first put $f_x(t-z) = 0$ and write out the corresponding line element:

$$ds^2 = -dt^2 + [1 + f_+(t-z)]\,dx^2 + [1 - f_+(t-z)]\,dy^2 + dz^2$$

The function $f_+(t-z)$ is arbitrary but small. It represents the wave packet or wave form. It could for example be a gaussian wave packet or a sinus-function. How could we detect such a wave if it were passing us?

As usual, to detect gravitational effects we have to use at least two test masses, and consider their relative motion.

Consider two test masses A and B:



A is initially at rest at the origin: $x^i_A(0) = (0,0,0)$

$u^\alpha_A(0) = (1,0,0,0)$

B is initially at rest at some arbitrary position: $x^i_B(0) = (x_B, y_B, z_B)$

$u^\alpha_B(0) = (1,0,0,0)$

Wave is travelling in positive z-direction

The geodesic equation should tell us what happens to the test masses as the wave passes. For the spatial coordinates it reads

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -\Gamma^i_{\alpha\beta}u^\alpha u^\beta$$

As everything here, this should only be evaluated to first order in the perturbation. The equation for the first order changes δx^i is

$$\frac{d^2(\delta x^i)}{d\tau^2} = -\delta\left(\Gamma^i_{\alpha\beta}u^\alpha u^\beta\right) = -\delta\Gamma^i_{\alpha\beta}u^\alpha u^\beta \underbrace{- 2\underbrace{\Gamma^i_{\alpha\beta}}_{{}^M\Gamma^i_{\alpha\beta} + \delta\Gamma^i_{\alpha\beta}} u^\alpha \delta u^\beta}_{\text{second order!}}$$

Γ for Minkowski vanishes

Hence:

$$\frac{d^2(\delta x^i)}{d\tau^2} = -\delta\Gamma^i_{\alpha\beta}u^\alpha u^\beta = -\delta\Gamma^i_{tt}$$

But $\Gamma^i_{tt} = \frac{1}{2}g^{i\mu}\left(\underbrace{\frac{\partial g_{t\mu}}{\partial x^t}}_{=0} + \underbrace{\frac{\partial g_{t\mu}}{\partial x^t}}_{=0} - \underbrace{\frac{\partial g_{tt}}{\partial x^\mu}}_{=0}\right) = 0 \quad \Rightarrow \quad \delta\Gamma^i_{tt} = 0$

So $\dfrac{d^2(\delta x^i)}{d\tau^2} = 0$

Initially there is no deviation and the particles are at rest:

$\delta x^i\big|_{\tau=0} = 0$

$\left.\dfrac{d(\delta x^i)}{d\tau}\right|_{\tau=0} = 0$

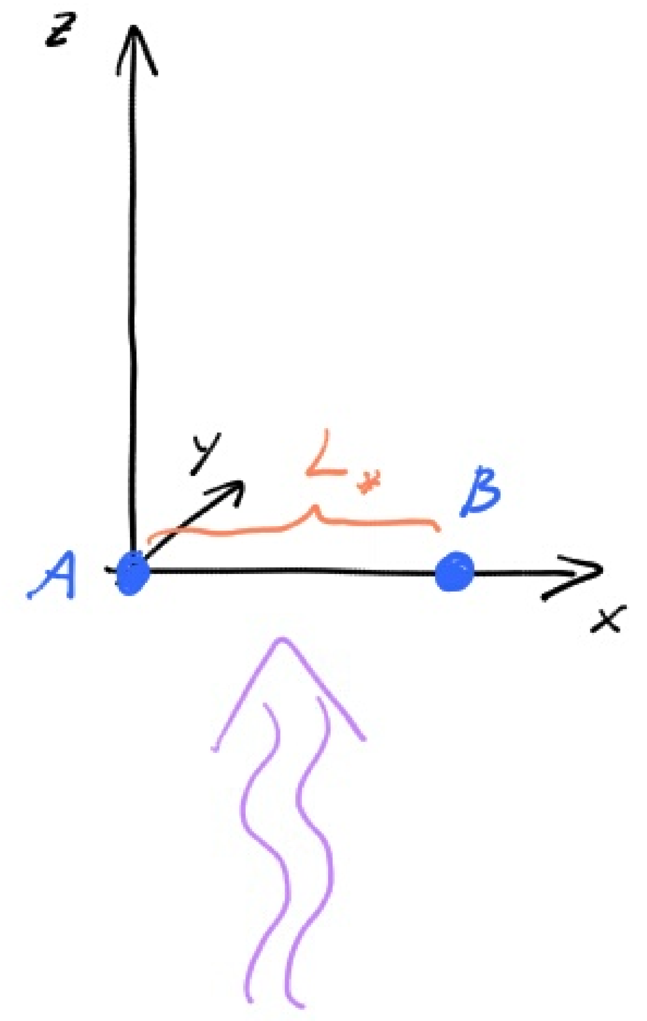
$\Longrightarrow \quad \delta x^i(\tau) = 0$ for all τ

The coordinate positions of the test particles remain unchanged as the wave passes!

Another way to express this is that straight lines in the Cartesian coordinate system of the flat background metric are geodesics of the full spacetime to first order.

It might seem that we have reached the disappointing conclusion that it is not possible to detect a first order wave perturbation by first order measurements of the positions of test particles. This, fortunately, is wrong!

Consider instead the distance between our two test masses. For simplicity, let us keep particle A at the origin, and put particle B at the position $x^i_B = (L_*, 0, 0)$ along the x-axis:



The distance between A and B along the x-axis at time t:

coord. position stays the same

$f_+(t) \equiv f_+(t-z)\big|_{z=0}$

$$L(t) = \int_0^{L_*} \left(1 + f_+(t)\right)^{1/2} dx \simeq$$

$f_+(t)$ small, and integrand independent of x

$$\simeq L_*\left(1 + \frac{1}{2} f_+(t)\right)$$

Hence the relative change in distance as the wave passes is

$$\frac{\delta L(t)}{L_*} = \frac{1}{2} f_+(t)$$ — variation in distance between test masses on x-axis

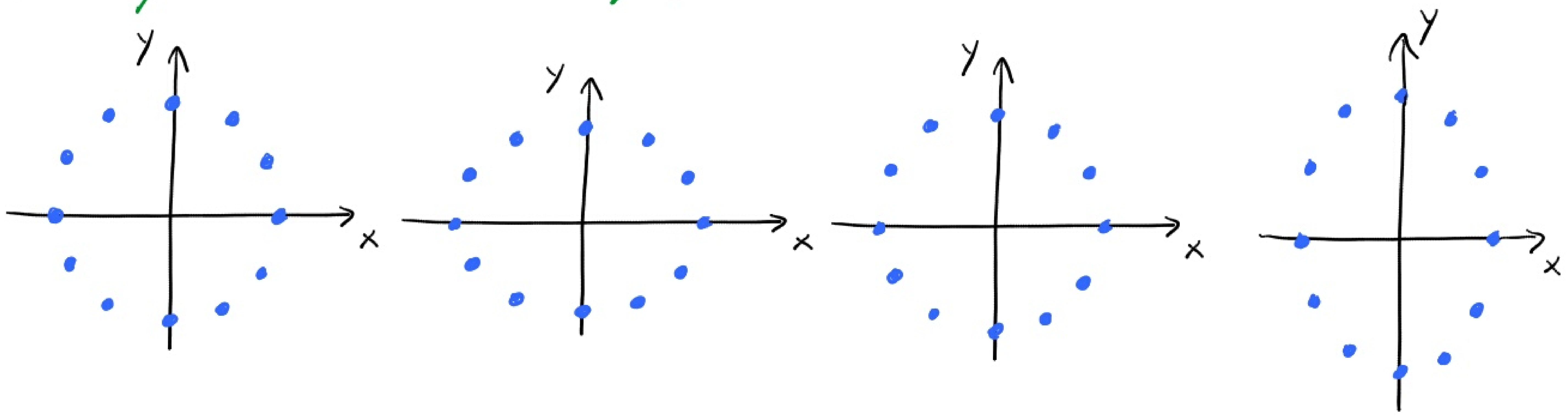
Eventhough the test masses stay at the same coordinate positions, the distance between them will vary as $\frac{1}{2}$ of the amplitude of the wave.

The corresponding calculation if the test masses are along the y-axis gives

$$\frac{\delta L(t)}{L_*} = -\frac{1}{2} f_+(t)$$ — variation in distance between test masses on y-axis

For example, say that the wave is just a sinus-wave with some definite frequency. Then, as the wave passes the $z=0$ plane, two particles, one at the x-axis and one at the y-axis, will oscillate along their axis, and with opposite phase.

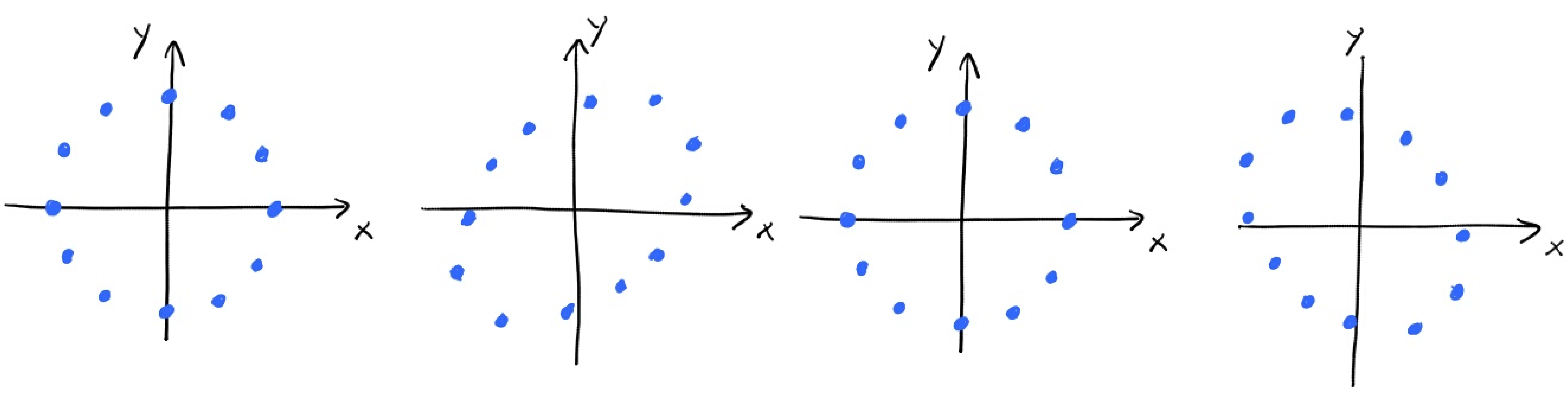
The picture gets clearer if we imagine a whole ring of test particles in the x-y-plane:



Now, remember that all this was for $f_\times(t-z)=0$.

What happens if we instead put $f_+(t-z)=0$?

It is easy to show that this is just the same thing but with the coordinate axes rotated 45°. Our ring of test particles thus would wobble in the "diagonal" instead:



Hence, f_+ and $f_\times$ corresponds to the two possible polarizations of the gravitational wave. (And by now the reason for the notation + and × should be obvious!)

Of course, as in electro-magnetism these two polarizations could be added to give more general polarizations. If the two added polarizations have the same phase we get a linearly polarized gravitational wave. If they are out of phase with $\frac{\pi}{2}$ we get a circularly polarized wave.

How would an ellipse of test particles behave if they were hit by a circularly polarized wave?

Each particle would rotate on a small circle, with the result that the ellipse shape would rotate!