Lecture 12

What is the connection between curvature and matter?

First task: finding a relativistic way to describe energy density

What kind of objects are "densities"?

- densities as 4-vectors
- number conservation

Definition of stress-energy-tensor

- dust
- interpretation of components
- perfect fluid
- conservation of stress-energy

Local conservation of stress-energy in curved spacetimes

Einstein's equation

- $R_{\alpha\beta} = \kappa T_{\alpha\beta}$? No: $\nabla R_{\alpha\beta} \neq 0$
- $\nabla G_{\alpha\beta} = 0$ — the Einstein tensor
  - the Bianchi identity
- $G_{\alpha\beta} = \kappa T_{\alpha\beta}$ with $\kappa = 8\pi G$ from Newtonian limit

We have seen how to describe the curvature of spacetime by means of the Riemann tensor, and how curvature is expressed as "geodetic deviation" – the tendency of initially parallel geodesics to start to converge or diverge, in curved regions.

The cause of this curvature is matter. But what exactly is the connection between curvature and matter? – the answer is given by Einstein's field equation.

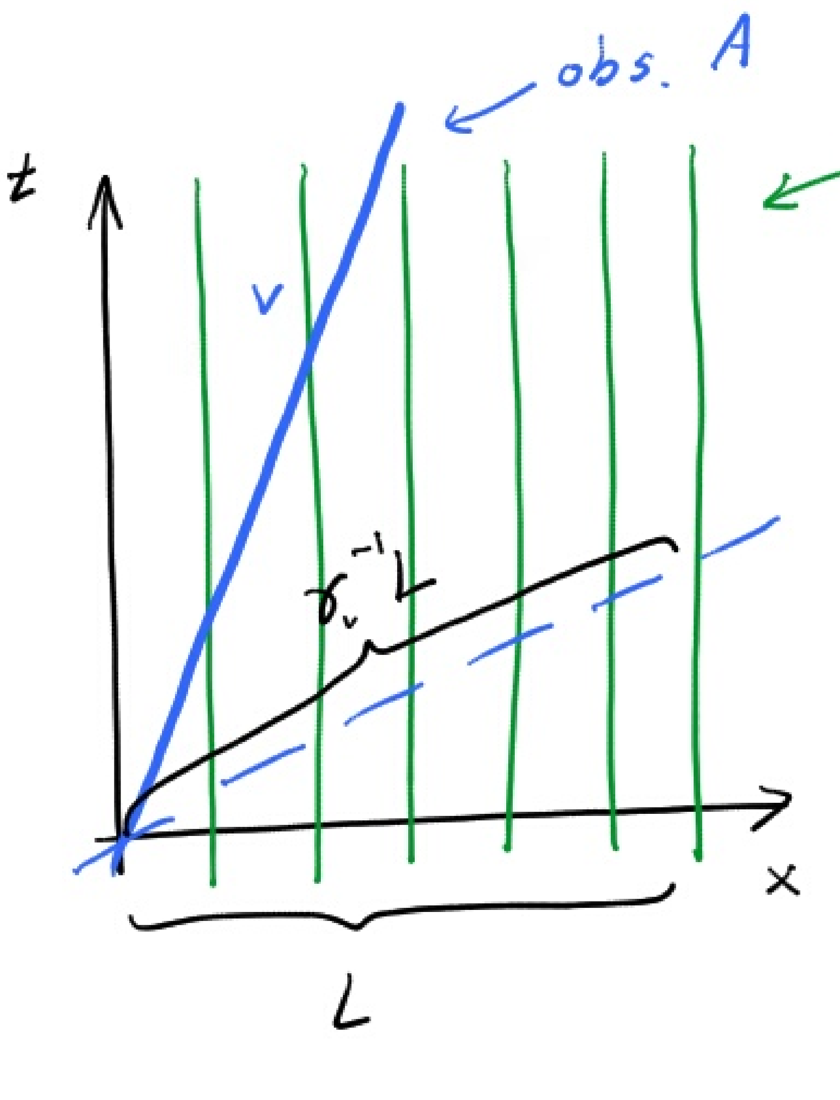
The first thing we need in order to formulate it, is a relativistic way of describing distributions of matter, or energy density. It will suffice to do that in Minkowski space. The generalization then will be straightforward.

But we will start with something even simpler.

Apple density

Suppose I have a lot of apples spread out over the ground. I could describe the distribution by giving the apple density – the number of apples per square meter – and how it varies from place to place.

But what kind of object is this apple density? One might believe that it is a scalar function. But it cannot be, since a density presupposes a volume (or an area or a length), and a volume, in turn, always refer to a particular inertial frame. In other words: the apple density will be different for different observers, and so cannot be a scalar.



apples at rest, density ρ

Density according to A:

$$\rho' = \gamma_v \rho$$

Transforms as the time component of a 4-vector!

If ρ is the time component of a 4-vector — what are then the space components? In the rest frame of the apples they must be zero by symmetry. (Since nothing is moving, no spatial direction is special.) Hence, let us introduce the apple density 4-vector:

$$\rho^\alpha = (\rho, 0, 0, 0) = \rho u^\alpha$$

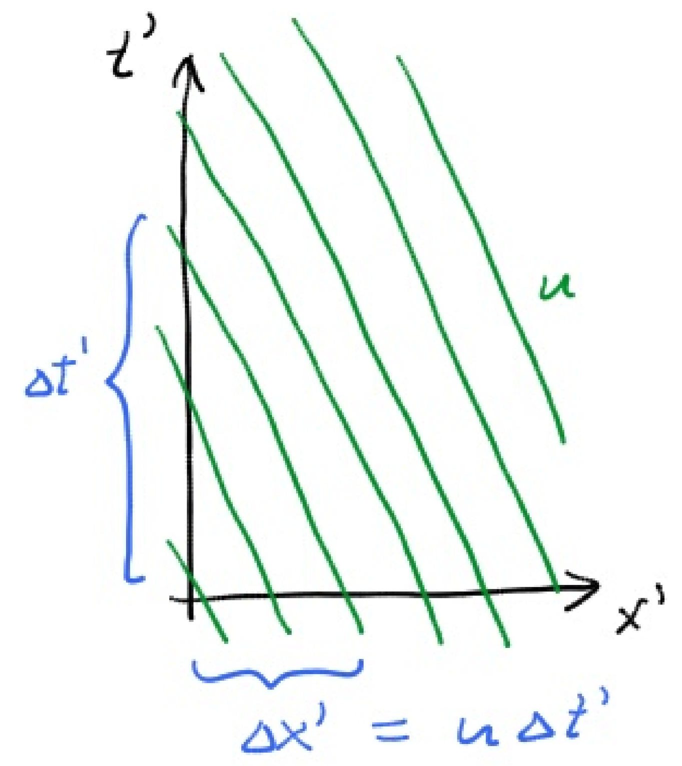
tangent to apple world lines

In A's frame this vector is

$$\rho^{\alpha'} = \rho u^{\alpha'} = (\rho \gamma_u , \rho \gamma_u \vec{u})$$

This is the flow according to A, since (with just one spatial dim. x):

$$\frac{\Delta n}{\Delta t'} = \frac{\Delta n \, u}{\Delta x'} = \rho' u = \rho \gamma_u u$$



Modifying ordinary language slightly into spacetime language, we can say:

$$\rho^{\alpha'} = (\rho \gamma_u , \rho \gamma_u u_x , \rho \gamma_u u_y , \rho \gamma_u u_z)$$

"flow" of apples through $t = \text{const.}$ hypersurface, that is, $\frac{\Delta n}{\Delta V'} = \frac{\Delta n}{\Delta x' \Delta y' \Delta z'} = \rho \gamma_u$

flow of apples through $z = \text{const.}$ hypersurface, that is, $\frac{\Delta n}{\Delta x' \Delta y' \Delta t'} = \frac{\Delta n \, u_z}{\Delta x' \Delta y' \Delta z'} = \rho' u_z = \rho \gamma_u u_z$

Given an apple density 4-vector $\bar{\rho}$, what density will an observer with 4 velocity $\bar{v}$ see?

$$\frac{\Delta n}{\Delta V} = - \bar{\rho} \cdot \bar{v} = - \rho^\alpha v_\alpha$$ — density acc. to obs. $\bar{v}$

To see this, evaluate the scalar product in the obs. rest frame:

$$\rho^\alpha = (\rho \gamma_u , \rho \gamma_u \vec{u})$$

$$v^\alpha = (1, 0, 0, 0) \implies - \rho^\alpha v_\alpha = \rho \gamma_u$$ as it should.

Apple conservation: $$\frac{\partial \rho^0}{\partial t} + \nabla \cdot \vec{\rho} = 0$$

$$\frac{d}{dt} \int_V \rho^0 \, d^3x + \int_{\partial V} \vec{\rho} \cdot d\vec{A} = 0$$ — integral version

$$\frac{\partial \rho^\alpha}{\partial x^\alpha} = 0$$ — in elegant notation!

Energy density

A number of apples is a scalar, but a number density is the time component of a 4-vector.

Energy is not a scalar, but already in itself the time component of a 4-vector — the energy-momentum.

Thus we need to consider energy-momentum density, and we expect that to be the time component of a rank 2 tensor, which we denote $T^{\alpha\beta}$ — the Stress-energy tensor.

We interpret its components in this way (in analogy with the apple density 4-vector):

$T^{\alpha\beta}$ — flow of component α of energy momentum through $x^\beta = \text{const.}$ hypersurface.

Then (still in analogy with the apple density case) the energy-momentum density as seen by obs. with 4-velocity $\vec{v}$ is

$$\frac{\Delta p^\alpha}{\Delta V} = - T^{\alpha\beta} v_\beta$$

Hartle writes this without minus sign, since he does not use the obs. 4 velocity, but instead a "normal" to a volume pointing backward in time.

The energy density, seen by an obs. at rest w.r.t. the inertial frame used $(v^\alpha = (1, 0, 0, 0))$, then is

energy density $\varepsilon = \dfrac{\Delta p^t}{\Delta V} = T^{tt}$ — flow of energy through $t = \text{const.}$ hypersurface

and the corresponding momentum density in direction i, is

3-momentum density $\pi^i = \dfrac{\Delta p^i}{\Delta V} = T^{it}$ — flow of p^i through $t = \text{const.}$ hypersurface

Also, note that

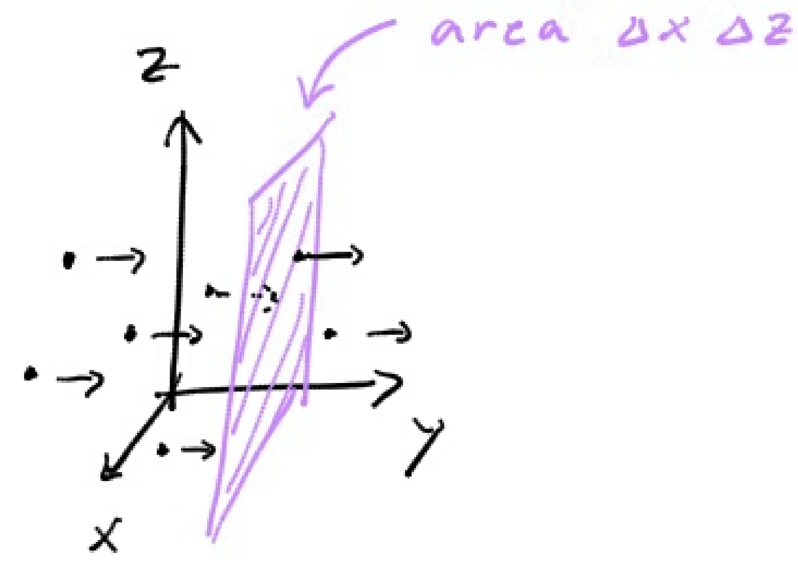
T^{tx} — flow of energy in x-direction:

$$\frac{\Delta E}{\Delta t \, \Delta y \, \Delta z} = \frac{\Delta E \, \Delta x}{\Delta V \, \Delta t} = \frac{\Delta E \, u_x}{\Delta V} = \frac{\Delta p_x}{\Delta V}$$

$=$ x-momentum density $= T^{xt}$

What then is T^{ij}? For example:

$$T^{yy} = \frac{\Delta p^y}{\Delta t\, \Delta x\, \Delta z} = \frac{F^y}{\Delta A_{xz}}$$ — pressure in y-direction



Components with $i \neq j$ represent shear stresses:

T^{ij} = i-component of force acting across surface with normal $\hat{j}$.

Let us summarize:

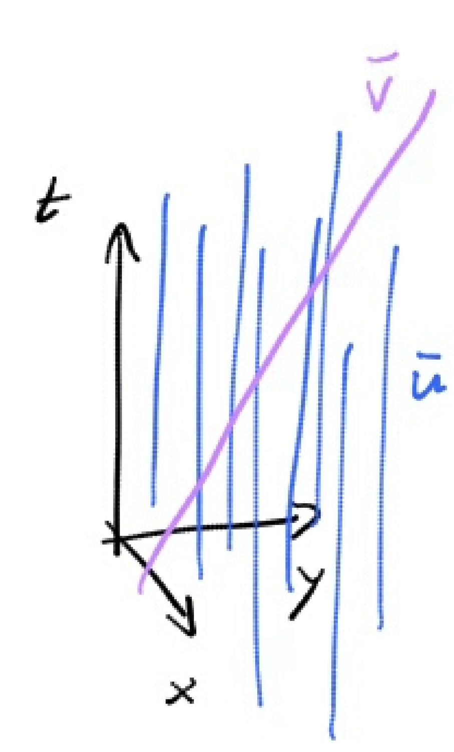
$$T^{\alpha\beta} = \left(\begin{array}{c|c} \text{energy density } \varepsilon & \text{momentum density } \pi^i \\ \hline \text{momentum density } \pi^i & \text{stress tensor } T^{ij} \end{array}\right)$$

Dust

The simplest example of an energy momentum tensor is when there is one frame in which its only non-zero component is T^{tt}. Then we can write

$$T^{\alpha\beta} = \mu u^\alpha u^\beta$$ — dust

where $u^\alpha = (1, 0, 0, 0)$ in that frame.



This corresponds to a situation where all particles are at rest, distributed with density μ, in that frame.

An observer with 4-velocity $\bar{v}$ will se the energy-mom. density:

$$\frac{\Delta p^\alpha}{\Delta V} = -T^{\alpha\beta} v_\beta = -\mu u^\alpha \underbrace{u^\beta v_\beta}_{-\gamma_u} = \mu \gamma_u u^\alpha = \frac{\gamma_u}{\Delta V} \underbrace{\Delta m\, u^\alpha}_{\text{4-mom. acc. to obs.}}$$

length contr. volume

Perfect fluid

The next simplest case is a gas of non-interacting particles (or where the interaction is so weak that heat conduction, viscosity et cetera are negligible) and with an isotropic velocity distribution.

Then in the rest frame of the gas there is no net momentum density (or flow of energy). But there is pressure, which by symmetry, is the same in all directions. Thus:

$$T^{\alpha\beta} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix} = \begin{pmatrix} \rho & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} + \begin{pmatrix} & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} =$$

$$= \rho u^\alpha u^\beta + p(\eta^{\alpha\beta} + u^\alpha u^\beta) = (\rho + p)u^\alpha u^\beta + p\eta^{\alpha\beta}$$

4-velocity of fluid $\quad \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

Or, in curved space, with metric $g^{\alpha\beta}$:

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p g^{\alpha\beta} \quad \text{— perfect fluid}$$

A perfect fluid is both a more narrow and broader concept than an "ideal gas". It is more narrow since all interactions are neglected – there is no heat flow, no viscosity. On the other hand, an ideal gas obeys the ideal gas law, which is a particular relation between density and pressure. For a perfect fluid no such particular relation is assumed.

Conservation of energy-momentum

Remember that each component of energy momentum is conserved. In analogy with "apple conservation" we can write this as

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0$$

reduces to this in a local inertial frame

Or in curved space: $\nabla_\beta T^{\alpha\beta} = 0$ — <u>local</u> conservation of energy momentum

– this does not imply conservation in the global SR sense!

What is Einstein's field equation?

We expect Einstein's field equation to be of the form

$$\begin{bmatrix} \text{Spacetime} \\ \text{curvature} \end{bmatrix} = G \begin{bmatrix} \text{matter} \\ \text{distribution} \end{bmatrix}$$

This is analogous to Newtons field equation which we have seen can be written in the form

$$\underbrace{A^i_{\ i}}_{\text{tidal acc.}} = 4\pi G \underbrace{\mu}_{\text{matter}} \quad \text{where} \quad A_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} \quad \text{— Newton}$$

We just saw that in order to describe a matter density in a relativistic way we have to use the stress energy tensor $T_{\alpha\beta}$. This is a symmetric tensor of rank 2. Since the Ricci tensor also is a symmetric tensor of rank 2 a natural guess for the field equations are:

First guess: $\quad R_{\alpha\beta} = \kappa T_{\alpha\beta} \quad (6) \qquad$ (10 equations)

This does not work. The reason is that the condition

Local energy-momentum conservation: $\nabla_\alpha T^{\alpha\beta} = 0$

isn't mirrored by a similar condition on $R_{\alpha\beta}$:

$$\cancel{\nabla_\alpha R^{\alpha\beta} =} \; 0 \qquad \text{(4 equations)}$$

That is, there is no geometrical reason that $R^{\alpha\beta}$ would have to fulfil this. So if we adopted (6) then we would have to add this as an extra condition. But then we would have too many equations: (6) is 10 $\boxed{?}$ equations (because of the symmetry in α and β) and $\nabla_\alpha R^{\alpha\beta} = 0$ is 4 $\boxed{?}$ equations. In total that means 14 equations.

But we are only solving for the 10 unknown functions in the metric. Furthermore, the freedom in choosing coordinates corresponds to 4 functions, one for each coordinate. Hence, we only expect to have 6 equations.

Therefore we would like to find an object to put to the left in our field equation, that in itself, automatically had divergence zero.

The number of equations would then be 10 minus those 4 identities = 6 independent equations. That would fit the expected number for the 6 unknown (non-gauge) functions determining the metric.

There is one object satisfying this requirement – the so called Einstein tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$ – Einstein tensor

where $R = R^{\gamma}{}_{\gamma}$ – Ricci curvature scalar ($= 2 \times$ Gauss curvature)

The identity

$$\nabla_\alpha G^{\alpha\beta} = 0$$ – the contracted Bianchi identity

follows from another identity for the full Riemann tensor called the Bianchi identity:

$$\nabla_\alpha R^{\mu}{}_{\nu\beta\gamma} + \nabla_\gamma R^{\mu}{}_{\nu\alpha\beta} + \nabla_\beta R^{\mu}{}_{\nu\gamma\alpha} = 0$$ – the Bianchi identity

Hence, the simplest guess for the field equations consistent with all this is

$$G_{\alpha\beta} = \kappa T_{\alpha\beta}$$

The constant κ is fixed by the requirement that this equation for the weak field metric must become the Newtonian field equation: $A^{i}{}_{;i} = 4\pi G \mu$.

It turns out that

$$\kappa = 8\pi G$$

Hence: $$\boxed{G_{\alpha\beta} = 8\pi G \, T_{\alpha\beta}}$$ – The Einstein equation

spacetime geometry

coupling between matter and geometry

matter distribution

These are 10 second order differential equations for the metric coefficients. They are coupled by the 4 conditions of local energy-momentum conservation (or the Bianchi identities).

Not so many exact solutions for interesting physical cases are known, especially not when there is matter involved. The reason is that one cannot <u>first</u> specify the matter on the right hand side and <u>then</u> solve the equations, since one cannot specify the distribution of matter without assuming some geometry for the space where the matter is. But that geometry is what the equations are supposed to give.

Note that for vacuum we get this:

$$T_{\alpha\beta} = 0 \quad \Rightarrow \quad G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0$$

Contract α, β :
$$R^{\beta}{}_{\beta} - \frac{1}{2} g^{\beta}{}_{\beta} R = 0$$

$$R - 2R = 0 \quad \Rightarrow \quad R = 0$$

Hence : $\boxed{R_{\alpha\beta} = 0}$ — Vacuum equation