Lecture 4

The line element

- coordinates arbitrary
- several patches may be needed
- $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$
- Local inertial frames

Vectors in curved space

- lives in tangent space
- coordinate bases
- orthonormal bases

Light cone structure in GR

- The Warp-drive spacetime

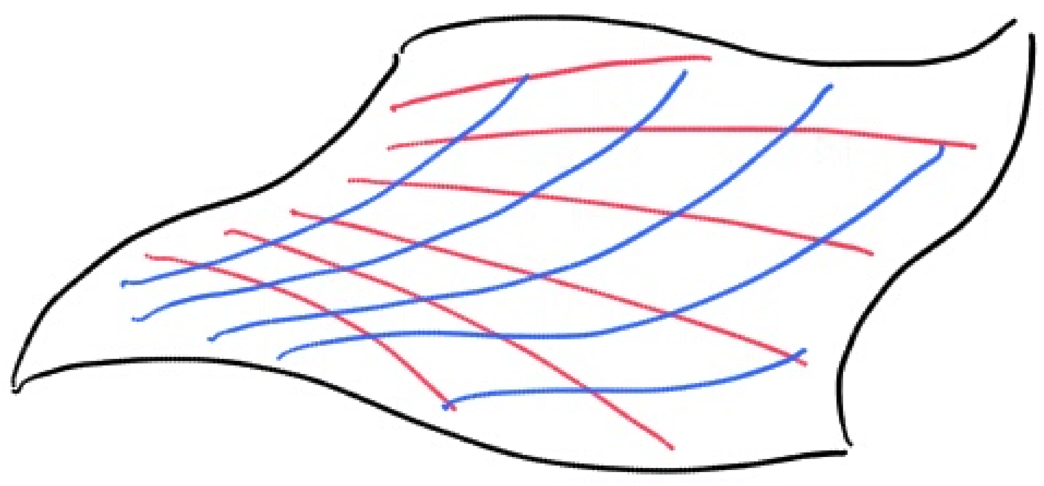
The Wormhole spacetime

Spacetime hypersurfaces

The e.p. forces us to describe gravity as the expression of spacetime curvature, that is, in terms of geometry. The most useful way to specify the geometry of a space or spacetime is by means of <u>the line element</u>, which gives all distances between neighbouring points.

Chapter 7 summarizes some things that we have to know about line elements, how to interpret them, how to handle them, and so on.

<u>The line element</u>



Suppose we have a space.

In order to give the distance between all pairs of neighbouring points we first have to label all points – that is, we have to put coordinates on our space.

The choice of coord. is of course arbitrary, but it will affect the form of the line element.

For example consider these two line elements:

$$ds^2 = dx^2 + dy^2$$
$$ds^2 = dx^2 + x^2 dy^2 \qquad \boxed{?}$$

Both represent the same geometry (2 dim. flat space) but in different coordinates (Cartesian and polar, respectively)

Note that, in order to fully specify a geometry it is enough to specify the line element in some coordinates. You don't have to say something else about what coord. you are using.

In general, it is a difficult problem to know if two line elements which look different, actually represents the same geometry.

Often one coord. system is not enough to cover the full space or spacetime in a non-singular way (that is, such that there is one unique coord. label to each point). Then one have to use several partly overlapping coordinate patches.

On the sphere, for example, it is not possible to find one single coordinate system that covers all points of the sphere in a non-singular way.

In general we can write

$$ds^2 = \underset{\text{the metric}}{g_{\alpha\beta}} dx^\alpha dx^\beta$$

where $g_{\alpha\beta}$ is a function of the coordinates called the metric.

For example, for Minkowski space in spherically polar coordinates we have

$$g_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & r^2 & \\ 0 & & & r^2 \sin^2\theta \end{pmatrix}$$

A diagonal metric means that the coordinate lines are orthogonal everywhere.

Note that $g_{\alpha\beta} = g_{\beta\alpha}$.

Suppose we have a non-diagonal spacetime metric.
At each point, for example point x^α_p, the metric is just a symmetric 4×4 matrix of numbers. Thus it can be diagonalized by some change of coord. $x^\alpha_p \to x^{\alpha'}_p$. Then, by a rescaling of each coord. we can transform the metric so that it becomes identical to the Minkowski metric there.

At a given point, we can always find coord. such that

$$g_{\alpha'\beta'}(x'_p) = \eta_{\alpha'\beta'} \qquad (1)$$

As we will show later, it is also possible to choose coord. such that the first derivatives of $g_{\alpha\beta}$ vanishes:

$$\left.\frac{\partial g_{\alpha'\beta'}}{\partial x^{\gamma'}}\right|_{x = x'_p} = 0 \qquad (2)$$

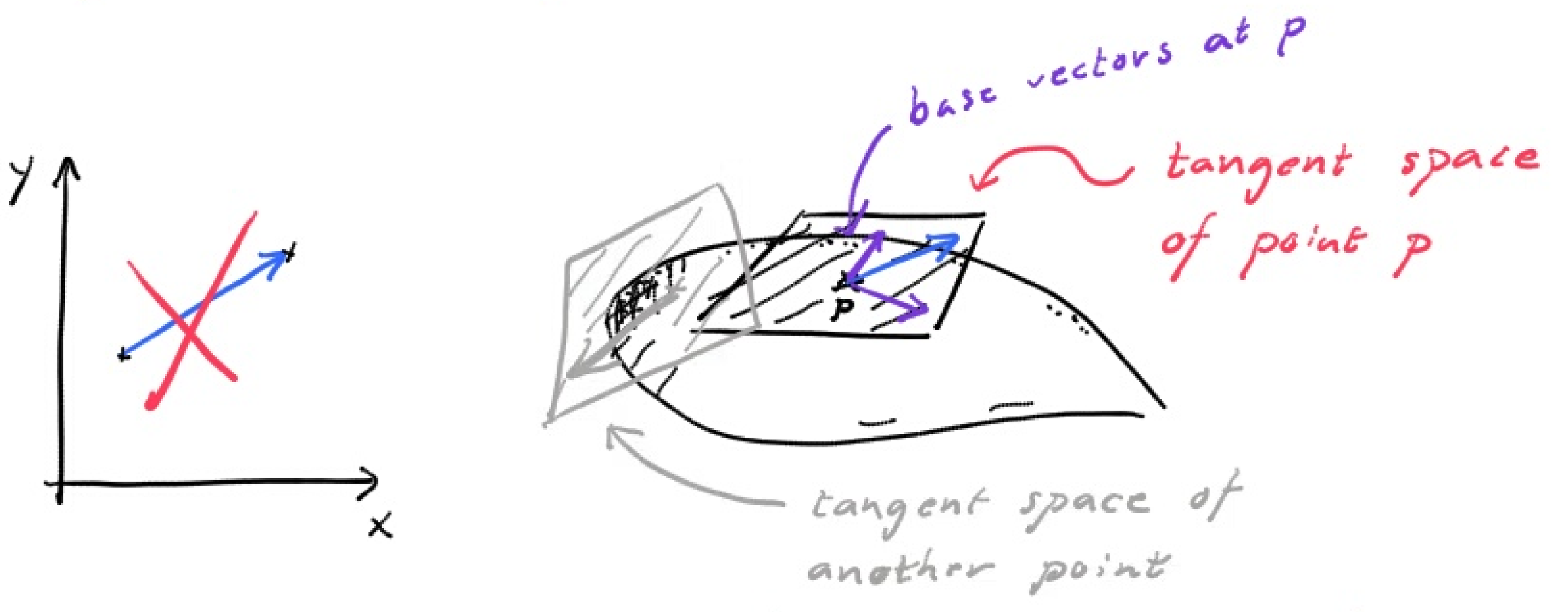
Local inertial frame

The coordinates satisfying (1) and (2) are called a local inertial frame. This is the mathematical expression of what we discussed earlier: that any curved spacetime locally is like Minkowski space.

Vectors in curved space

This we will do more carefully later (in Chapter 20). But some ideas about how we have to treat vectors in curved spaces is good to have already at this stage.

First, in a curved space it will not work any longer to consider vectors as line segments *in* the space as we are used to. Rather, a vector at a point p, is a direction with a magnitude at that point. We say that the vector lives in the tangent space of p.



If we assign a vector to each point x we get a vector field $\bar{a}(x)$. But note that the vectors at each point lives in different tangent spaces.

At each point we can introduce a basis $\bar{e}_\alpha(x_p)$.

Then $\bar{a}(x_p) = a^\alpha(x_p)\bar{e}_\alpha(x_p)$

and the scalar product $\quad \bar{a}\cdot\bar{b} = a^\alpha\bar{e}_\alpha\cdot b^\beta\bar{e}_\beta = a^\alpha b^\beta(\bar{e}_\alpha\cdot\bar{e}_\beta)$

Coordinate bases

In general $\{\bar{e}_\alpha\}_p$ can be chosen in any way at each point p. There may be no connection with the coordinates used to label points.

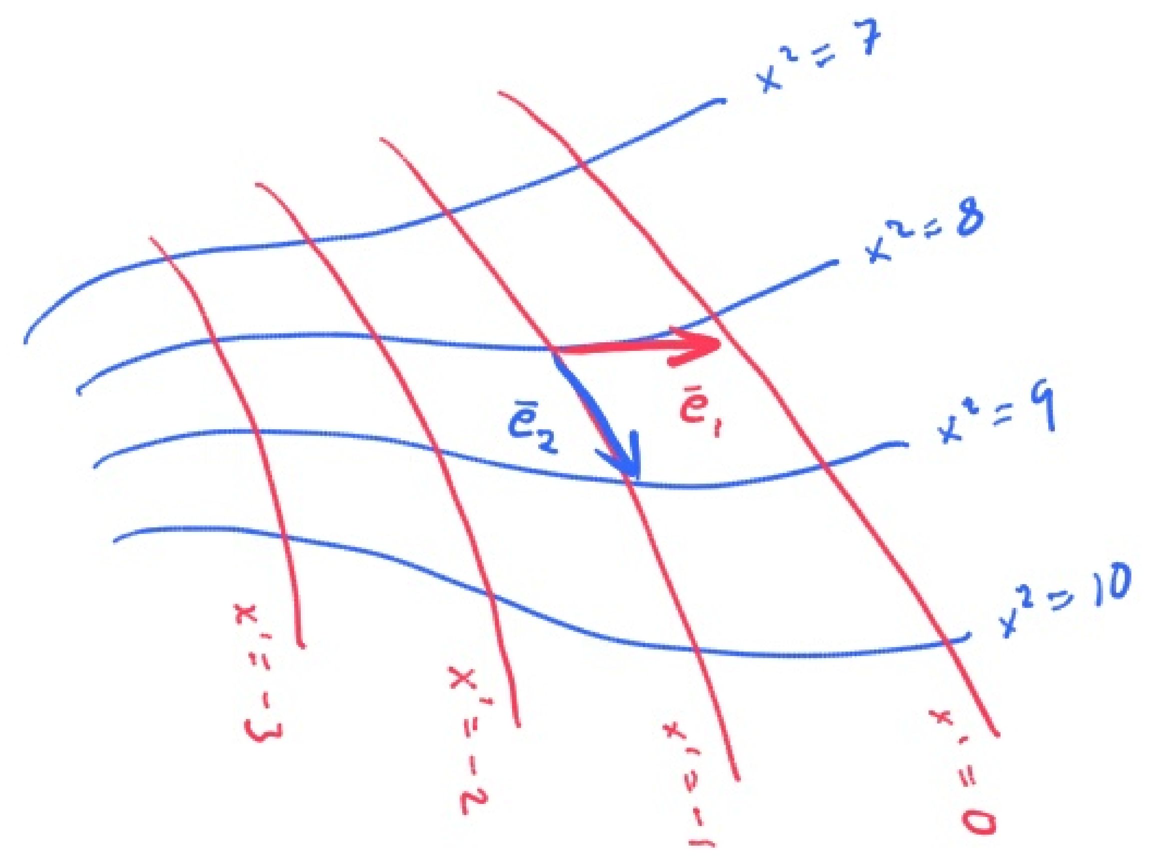
But the most natural and useful choice is when the basis vectors are adapted to the coordinates, in the sense that they point along the coordinate lines, and have a length reflecting how the coordinates change from point to point.

In that case

$$\bar{e}_\alpha\cdot\bar{e}_\beta = g_{\alpha\beta}$$

where the metric $g_{\alpha\beta}$ also gives the line element:

$$ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$$



A coordinate basis

Then $\quad \bar{a}\cdot\bar{b} = a^\alpha b^\beta g_{\alpha\beta} \equiv a^\alpha b_\alpha$

When performing calculations we will almost always use such a coordinate basis.

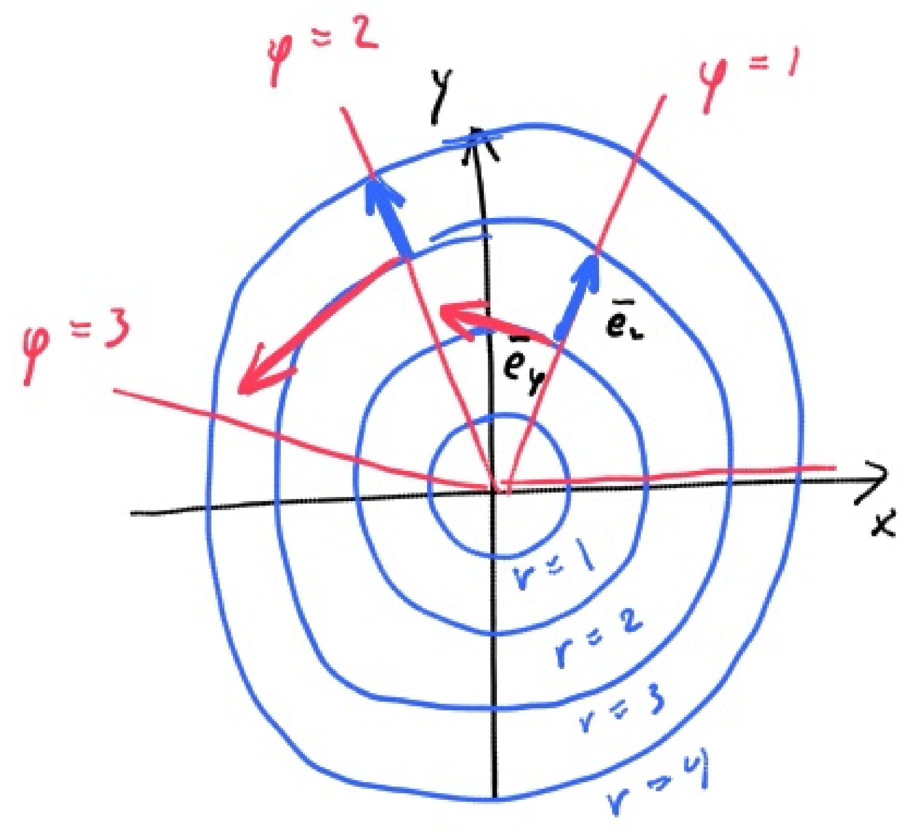
Orthonormal bases

But for an inertial obs., describing the local spacetime with the Minkowski line element, it is more natural to use an orthonormal frame (such as the usual Cartesian coord. describing Minkowski space).

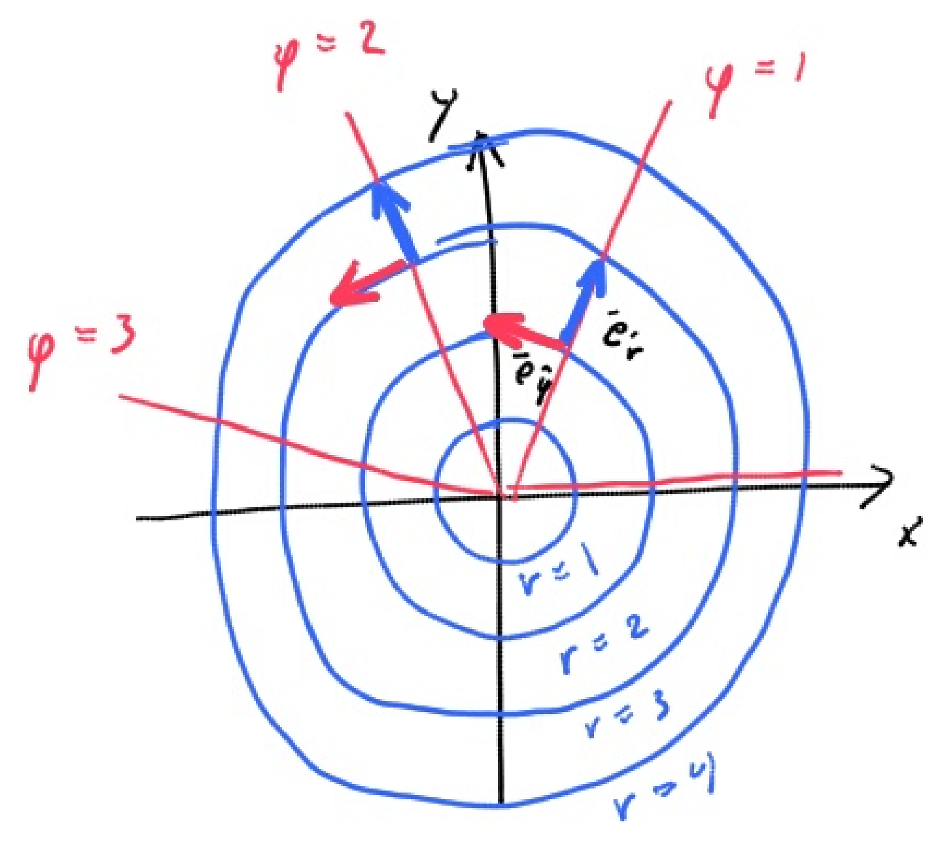
We will use a special notation for such bases:

$$\bar{a} = a^{\hat{\alpha}} \bar{e}_{\hat{\alpha}}$$

Then $\bar{a} \cdot \bar{b} = a^{\hat{\alpha}} \bar{e}_{\hat{\alpha}} \cdot b^{\hat{\beta}} \bar{e}_{\hat{\beta}} = a^{\hat{\alpha}} b^{\hat{\beta}} (\bar{e}_{\hat{\alpha}} \cdot \bar{e}_{\hat{\beta}}) = a^{\hat{\alpha}} b^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}}$



Coordinate basis for polar coord.



Orthonormal basis for polar coord.

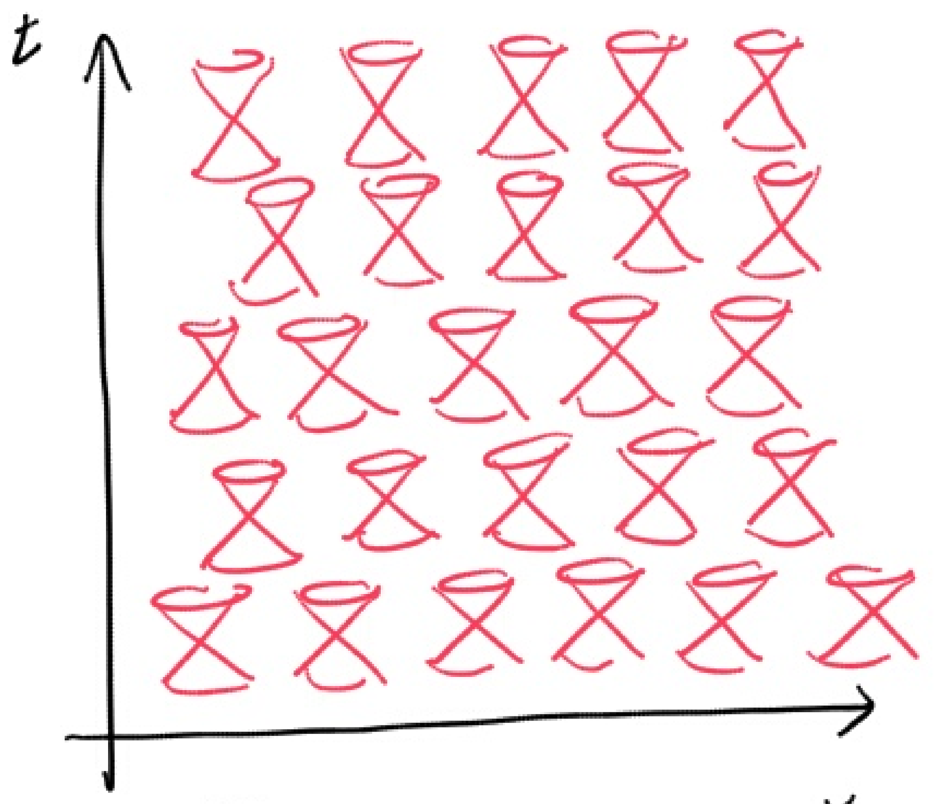
Later we will learn how to change between these two types of bases in general. But in the case where the coordinate lines are orthogonal (and the metric diagonal) it is particularly easy.

Then, in going from the coord. basis to the corresponding orthonormal frame, we just have to adjust the length of each basis-vector, that is, divide it by its own length:

$$\bar{e}_{\hat{1}} = \frac{\bar{e}_1}{(\bar{e}_1 \cdot \bar{e}_1)^{1/2}} = (g_{11})^{-1/2} \bar{e}_1 \qquad \boxed{?}$$

Causal structure

The causal structure in SR is trivial. All causal influences must be subluminal: they are transmitted from one spacetime point to another on timelike or lightlike lines, that is, inside the light cone. And since Minkowski space is flat it has a very simple light cone structure:



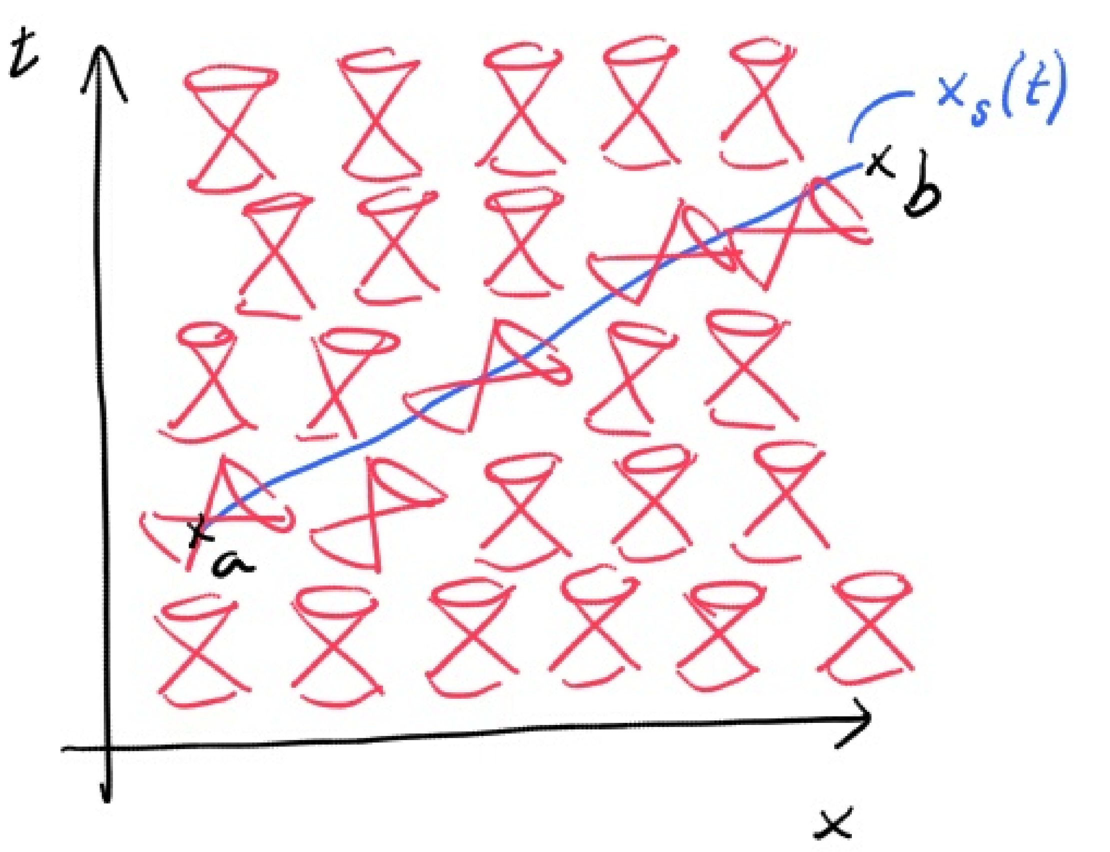
SR

Since the curved spacetimes of GR are locally like Minkowski space, GR inherits the local light cone structure of SR. For example, a timelike worldline is a curve where all segments have $ds^2 < 0$, that is, it lies inside the local light cone at every point that it passes.

And all causal influences must travel on lines inside or on the local light cone.

In a given set of coord. the local light cones may tilt with respect to one another. The statement that nothing can travel faster than light is thus degraded to a local statement. This opens the door to strange causal phenomena or situations.

BREAK



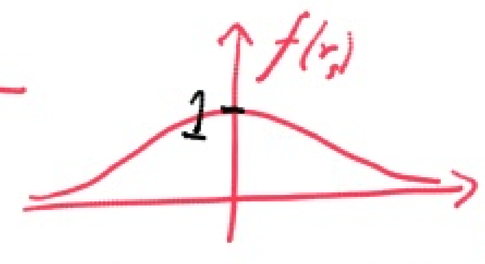
For example, suppose we have what in Minkowski space would be a spacelike curve $x_s(t)$ connecting two spacelike separated events a and b. Suppose we found a way to curve the spacetime near and along this line in such a way that the light cones became tilted, making the line (locally) timelike. Suppose that we had a spaceship that actually could cause this curvature in its neighbourhood. Then we could actually travel between a and b!

This is called a Warp-drive, and a metric accomplishing this is:

Warp-drive:

$$ds^2 = -dt^2 + \left[dx - V_s(t) f(r_s) dt\right]^2 + dy^2 + dz^2$$

velocity $V_s(t) = \frac{dx_s(t)}{dt}$



where $r_s = \sqrt{(x - x_s(t))^2 + y^2 + z^2}$

(spatial dist. from $x_s(t)$)

Note: Spacetime is Minkowski where $f(r_s) = 0$, that is, far from the space ship worldline. $\boxed{?}$

The lightcones are described by $\boxed{?}$

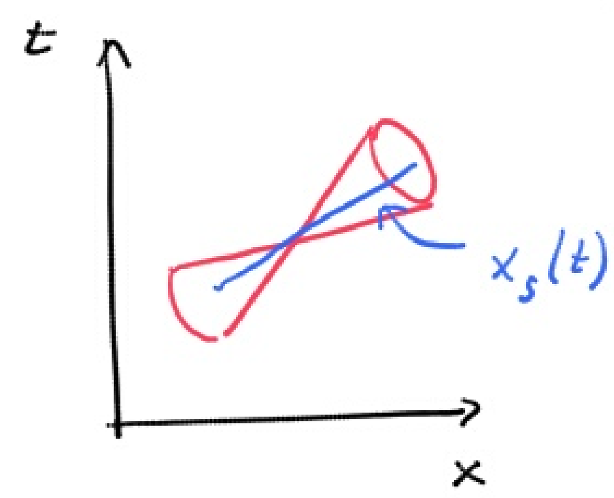
$$ds^2 = 0 \quad \Rightarrow \quad \pm dt = dx - V_s(t) f(r_s) dt$$

$$\frac{dx}{dt} = \pm 1 + V_s(t) f(r_s)$$

Where $f(r_s) = 0$ we again get the "normal" lightcones $\frac{dx}{dt} = \pm 1$

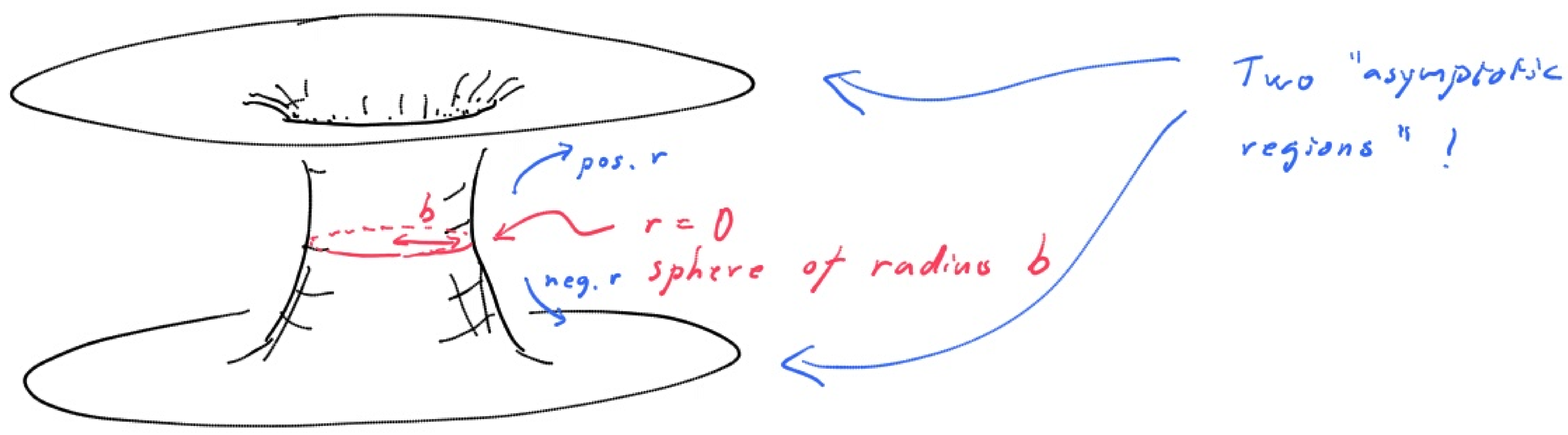
But the larger $V_s(t)$, the more tilted they are.

For example, for $r_s = 0$ and $V_s(t) = 2$ we get $\frac{dx}{dt} = \begin{cases} 3 \\ 1 \end{cases}$



In GR the simple causal structure of SR does not hold because of the curvature. Furthermore, having accepted that spacetime is curved, we also have to consider the possibility of different topologies. For example, otherwise distant parts of the universe may be connected, or two "different" universes may be connected.

This is what in science fiction is called a wormhole. A simple 2 dim. model of a wormhole would look like this:



The two planes represent two (otherwise) distant regions of the universe, or two different universes. But there is a connection between these — the wormhole.

A metric describing such a wormhole is

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)\,d\Omega^2$$

How can we see that this represents a wormhole?

The line element looks almost as that of Minkowski space in polar coord. The only difference is the factor (b^2+r^2). This makes the whole difference, since this makes the coordinates well-behaved at $r=0$. Hence, nothing stops us from continuing to negative values of r. Thus there are two infinities, $r \to +\infty$ and $r \to -\infty$.

Note that the waist of the wormhole, $r=0$, is a sphere of radius b.

(How would it be to travel through the wormhole?
— Not as going through a tunnel!)

1f time

Spacetime hypersurfaces

We differ between three different kinds of 3-dimensional subspaces of a 4-dim. spacetime. At each point in such a subspace there will be three orthogonal vectors spanning the subspace (or rather, the tangent space to the subspace at that point). There will be one vector in spacetime orthogonal to all of these three tangent vectors. That vector, $\bar{n}$, is the normal to the hypersurface.

$\bar{n} \cdot \bar{n} < 0$ ($\bar{n}$ timelike) — hypersurface spacelike

$\bar{n} \cdot \bar{n} = 0$ ($\bar{n}$ null) — hypersurface null

$\bar{n} \cdot \bar{n} > 0$ ($\bar{n}$ spacelike) — hypersurface timelike