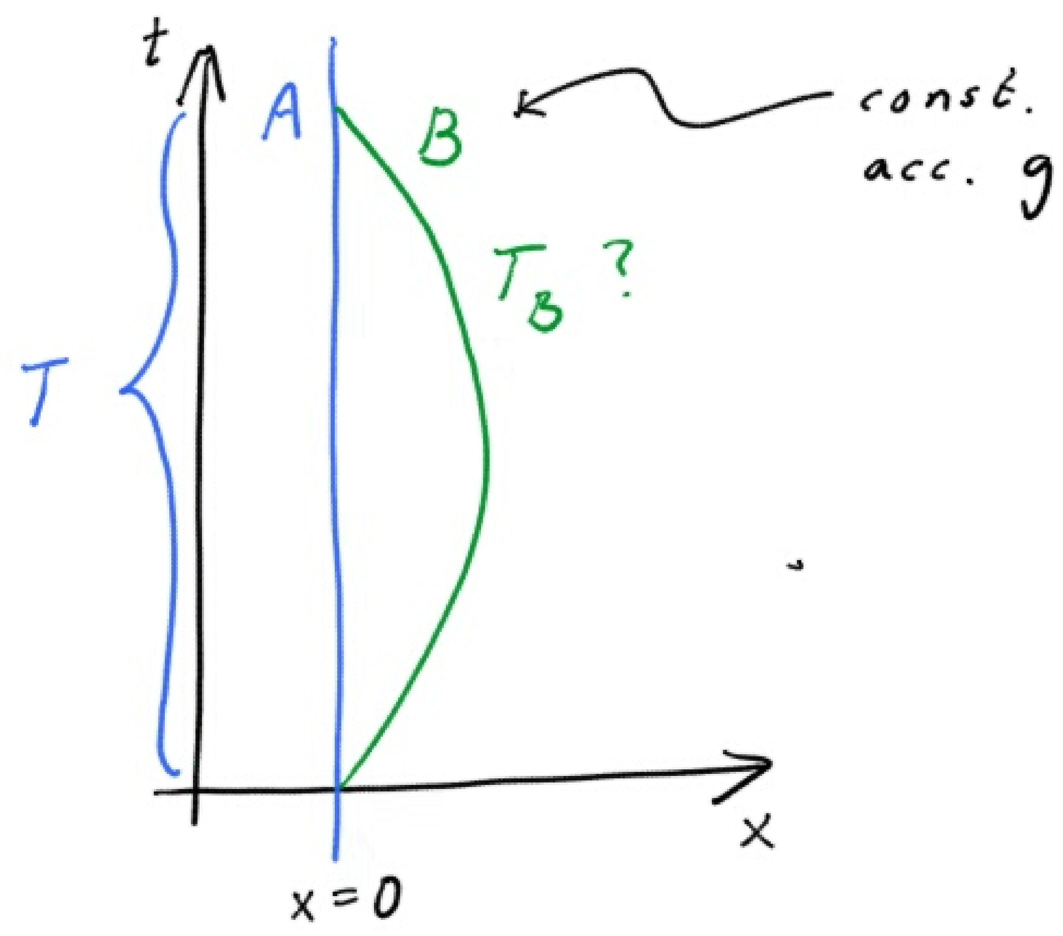
Solutions to Exam in General Relativity, 29/8 2018

1. a) Two observers at the surface of the earth:



Equation (6.26) gives the proper time for a clock moving in Newtonian potential $\Phi(x)$ to order $\frac{1}{c^2}$:

$$T_B = \int_0^{T_A} dt \left[1 - \frac{1}{c^2}\left(\frac{1}{2}\vec{v}\cdot\vec{v} - \Phi(x) \right) \right] \quad (*)$$

Let $\Phi(x) = gx$ (const. grav. acc g)

B:s trajectory: $x(t) = v_0 t - \frac{gt^2}{2}$

$$\frac{dx}{dt} = v_0 - gt$$

What is v_0 if total coord. time is T?

$$x(T) = v_0 T - \frac{gT^2}{2} \stackrel{!}{=} 0 \quad \Rightarrow \quad v_0 = \frac{gT}{2}$$

$$(*) \Rightarrow T_B = \int_0^T dt \left[1 - \frac{1}{c^2}\left(\frac{1}{2}(v_0 - gt)^2 - g\left(v_0 t - \frac{gt^2}{2}\right) \right) \right] =$$

$$= \int_0^T dt \left[1 - \frac{1}{c^2}\left(\frac{v_0^2}{2} + \frac{g^2t^2}{2} - v_0 g t - g v_0 t + \frac{g^2 t^2}{2} \right) \right] =$$

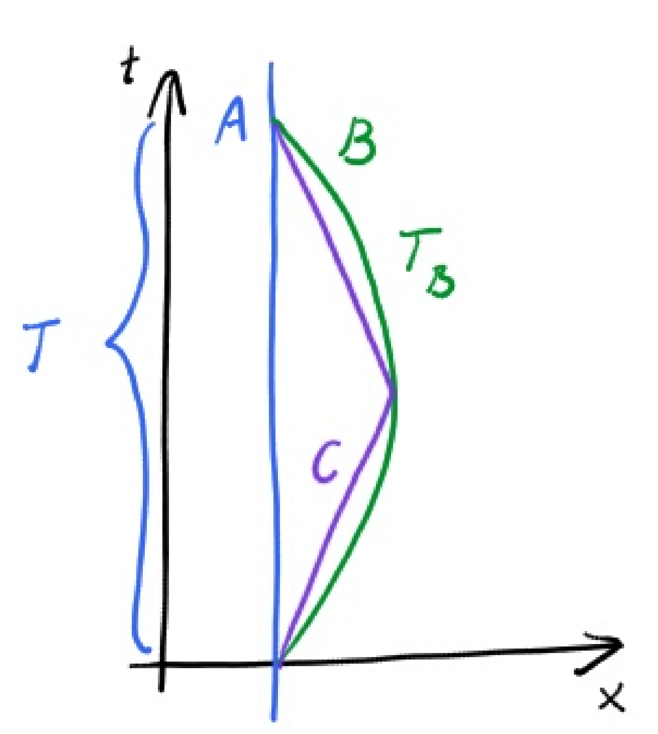
$$= \int_0^T dt \left[1 - \frac{1}{c^2}\left(\frac{v_0^2}{2} + g^2t^2 - 2v_0 g t \right) \right] =$$

$$= \left[t - \frac{1}{c^2}\left(\frac{v_0^2 t}{2} + \frac{g^2 t^3}{3} - v_0 g t^2 \right) \right]_0^T = \quad v_0 = \frac{gT}{2}$$

$$= T - \frac{1}{c^2}\left(\frac{g^2T^3}{8} + \frac{g^2T^3}{3} - \frac{g^2T^3}{2} \right) = \quad \left\{ \begin{array}{l} \frac{1}{8} + \frac{1}{3} - \frac{1}{2} = \\ = \frac{3+8-12}{24} = \frac{-1}{24} \end{array} \right.$$

$$= T + \frac{g^2T^3}{24c^2}$$

b)



Clock C will show a shorter time than B, since B is moving on a timelike geodesic, and timelike geodesics gives the longest proper time, when compared to neighbouring paths.

(2) $ds^2 = -x\,dv^2 + 2\,dv\,dx$

a) What is the "light cone" at point (v, x) ?

$ds^2 = 0 \Rightarrow 0 = -x\,dv^2 + 2\,dv\,dx$

This has two solutions: $dv = 0$

and

$0 = -x\,dv + 2\,dx$

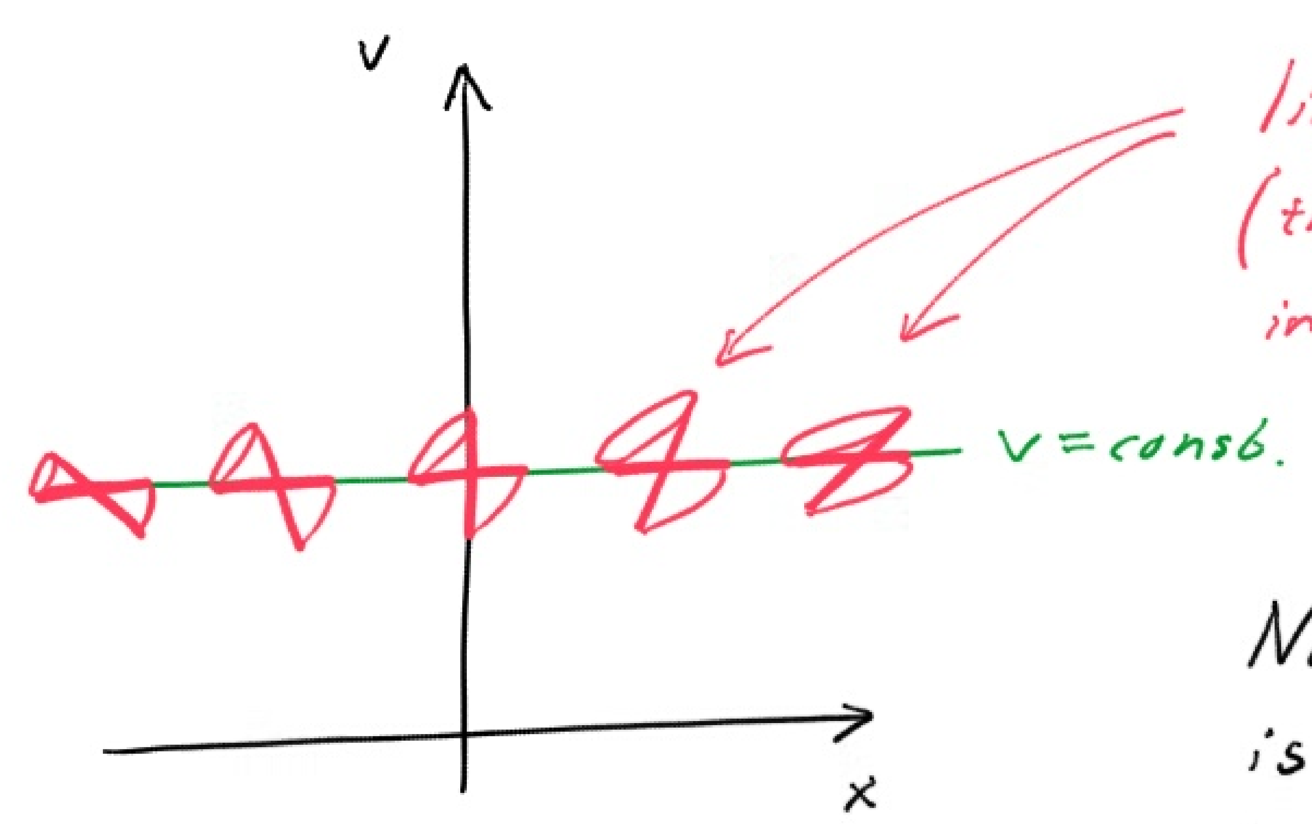
$$\Rightarrow \frac{dv}{dx} = \frac{2}{x}$$

So the slopes of the two lightrays at the point (v, x) is 0 and $\frac{2}{x}$. (The lightlike lines is thus described by $v = \text{const.}$ and $v(x) = 2 \ln x + \text{const.}$)

b) To find out what is "inside" the lightcone consider the "v-direction", described by the vector $z^\alpha = (1, 0)$. Its squared norm is

$$z^\alpha z_\alpha = g_{\alpha\beta} z^\alpha z^\beta = -x$$

Hence the v-direction is timelike for $x > 0$ but spacelike for $x < 0$. Hence:



Note that the line element is independent of v, so the light cone structure is the same along all $v = \text{const.}$ lines.

The line $x = 0$ can only be crossed from right to left.

③ Consider a Killing field $\bar{\xi}$ in a coordinate system where

$\xi^\alpha = (0, 1, 0, 0)$

Calculate

$\xi_\beta = g_{\beta\delta}\xi^\delta$

$$\nabla_\alpha \xi_\beta = \partial_\alpha \xi_\beta - \Gamma^\gamma_{\alpha\beta}\xi_\gamma =$$

$$= \xi^\delta \partial_\alpha g_{\beta\delta} + g_{\beta\delta}\underbrace{\partial_\alpha \xi^\delta}_{=0} - \Gamma^\gamma_{\alpha\beta} g_{\gamma\delta}\xi^\delta =$$

$$= \xi^\delta \partial_\alpha g_{\beta\delta} - \frac{1}{2} g^{\gamma\vartheta}\left(\partial_\alpha g_{\beta\vartheta} + \partial_\beta g_{\alpha\vartheta} - \partial_\vartheta g_{\alpha\beta}\right) g_{\gamma\delta}\xi^\delta =$$

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$$= \xi^\delta \partial_\alpha g_{\beta\delta} - \frac{1}{2}\left(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\right)\xi^\delta =$$

$$= \frac{1}{2}\left(\partial_\alpha g_{\beta\delta} - \partial_\beta g_{\alpha\delta} + \partial_\delta g_{\alpha\beta}\right)\xi^\delta =$$

$$= \frac{1}{2}\Big(\underbrace{\partial_\alpha g_{\beta 1} - \partial_\beta g_{\alpha 1}}_{\text{anti-symmetric in } \alpha,\beta} + \underbrace{\partial_1 g_{\alpha\beta}}_{=0}\Big)$$

$=0$ since $g_{\alpha\beta}$ is independent of x^1

Hence $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$

4 The Schwarzschild line element:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Killing field $\xi^\alpha = (1, 0, 0, 0)$

Tangent to trajectory of radially infalling particle:

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right)$$

The conserved quantity $e = -\xi \cdot \bar{u} = -\xi^\alpha u^\beta g_{\alpha\beta} =$

$$= \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}$$

So $\frac{dt}{d\tau} = e\left(1-\frac{2M}{r}\right)^{-1}$

The stationary observer at $r = 6M$ has tangent vector

$u^\alpha_{obs} = b(1, 0, 0, 0)$ where b is determined by normalization:

$$-1 = u^\alpha_{obs} u_{obs\,\alpha} = -b^2\left(1-\frac{2M}{r}\right) \Rightarrow b = \left(1-\frac{2M}{r}\right)^{-1/2}$$

Note that the speed that observer u^α_{obs} sees for trajectory u^α can be obtained from

$$u^\alpha u_{obs\,\alpha} = -\gamma_v \qquad (*)$$

(as is seen by going to the LIF of the observer)

In this case

$$u^\alpha u_{obs\,\alpha} = u^\alpha u_{obs}^\beta g_{\alpha\beta} = \frac{dt}{d\tau} b\, g_{00} = e\left(1-\frac{2M}{r}\right)^{-1}\left(1-\frac{2M}{r}\right)^{-1/2}\left(-\left(1-\frac{2M}{r}\right)\right) =$$

$$= -e\left(1-\frac{2M}{r}\right)^{-1/2} \underset{r=6M}{=} -e\left(\frac{3}{2}\right)^{1/2}$$

Compare with $(*)$ $\Rightarrow$ $e\left(\frac{3}{2}\right)^{1/2} = \frac{1}{(1-v^2)^{1/2}}$

Solve for v: $v = \left(1-\frac{2}{3e^2}\right)^{1/2}$

So $\frac{v_{e=2}}{v_{e=1}} = \left(\frac{1-\frac{1}{6}}{1-\frac{2}{3}}\right)^{1/2} = \left(\frac{5}{2}\right)^{1/2} \approx 1{,}58$

⑤ The Friedmann equation:

$$\dot{a}^2 - \frac{8\pi\rho}{3} a^2 = -k \qquad (1)$$

Matter density: $$\rho_m = \frac{\rho_{m0} a_0^3}{a^3} \qquad (2)$$

Vacuum density: $$\rho_\Lambda = \frac{\Lambda}{8\pi} \qquad (3)$$

a) Put $k=+1$ and consider $\rho = \rho_m + \rho_\Lambda$. (1), (2), (3) then gives

$$\dot{a}^2 \underbrace{- \frac{8\pi}{3}\left(\frac{\rho_{m0} a_0^3}{a} + \frac{\Lambda a^2}{8\pi}\right)}_{U_{eff}(a)} = -1 \qquad (4)$$

— introduce this effective potential

In order for there to exist a stationary solution with $\dot{a}=0$, we have to be at an extremum of the potential:

$$\frac{dU_{eff}(a)}{da} = 0 \quad \Longrightarrow \quad -\frac{\rho_{m0} a_0^3}{a^2} + \frac{\Lambda a}{4\pi} = 0$$

Since $\dot{a}=0$ and $\ddot{a}=0$ for all times we must have $a(t) = a_0$ and $\rho_m = \rho_{m0}$. This gives

$$-\rho_m a + \frac{\Lambda a}{4\pi} = 0 \quad \Longrightarrow \quad \underline{\rho_m = \frac{\Lambda}{4\pi}}$$

b) The spatial line element of any closed FRW-model is that of a 3-sphere with radius a:

$$dS^2 = a^2\left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right)$$

The volume is

$$V = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\pi} d\chi\, a^3 \sin^2\chi \sin\theta = 2\pi^2 a^3$$ (see Hartle eqn 18:55)

To get a, insert $a_0 = a$, $\rho_{m0} = \rho_m = \frac{\Lambda}{4\pi}$ into (4):

$$-\frac{8\pi}{3}\left(\frac{\Lambda}{4\pi}a^2 + \frac{\Lambda}{8\pi}a^2\right) = -1$$

$$\cancel{-}\frac{\cancel{8\pi}}{\cancel{3}} \cdot \frac{\cancel{3}\Lambda}{\cancel{8\pi}} a^2 = \cancel{-}1 \quad \Longrightarrow \quad a = \frac{1}{\sqrt{\Lambda}} \quad \Longrightarrow \quad \underline{V = \frac{2\pi^2}{\Lambda^{3/2}}}$$