Lecture 13

The Bianchi identity

geometric meaning

Black holes

Eddington-Finkelstein coordinates

- ingoing and outgoing light rays
- a black hole for $r < 2M$

Why do we expect black holes to actually exist?

- Birkhoff's theorem
- collapsing stars
- singularities are generic

Falling into a black hole

The Bianchi identity

$$\nabla_\alpha R^\mu{}_{\nu\beta\gamma} + \nabla_\gamma R^\mu{}_{\nu\alpha\beta} + \nabla_\beta R^\mu{}_{\nu\gamma\alpha} = 0$$

This has a simple geometrical meaning. To see what it is, let us consider a point p where we have introduced a local inertial frame, in usual Cartesian coordinates: t, x, y, z.

In these coordinates, at point p, we can write the identity:

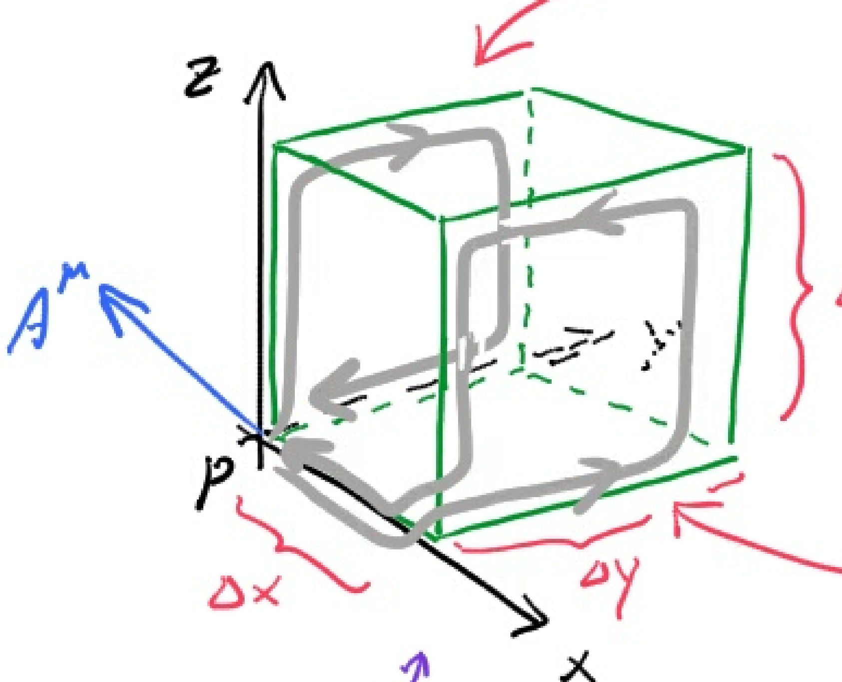
$$\partial_\alpha R^\mu{}_{\nu\beta\gamma} + \partial_\gamma R^\mu{}_{\nu\alpha\beta} + \partial_\beta R^\mu{}_{\nu\gamma\alpha} = 0$$

This only says something interesting if α, β, γ all are different, since if two of them are the same, the expression is zero already from the symmetries of $R^\mu{}_{\nu\beta\gamma}$ itself.

To see what it says when α, β, γ are all different, let us choose $\alpha = x$, $\beta = y$, $\gamma = z$:

$$\partial_x R^\mu{}_{\nu yz} + \partial_z R^\mu{}_{\nu xy} + \partial_y R^\mu{}_{\nu zx} = 0$$

General formula for p.t.: $\delta A^\mu = -R^\mu{}_{\nu\alpha\beta}\, \delta S^{\alpha\beta} A^\nu$



Riemann normal coord. at point p.

p.t. result:

$$\Delta A^\mu \Big|_{x=0} = R^\mu{}_{\nu yz}\Big|_{x=0} \Delta y \Delta z\, A^\nu$$

different signs because of opposite directions

p.t. result:

$$\Delta A^\mu \Big|_{x=\Delta x} = -R^\mu{}_{\nu yz}\Big|_{x=\Delta x} \Delta y \Delta z\, A^\nu$$

Combination of both p.t:s:

$$\Delta A^\mu \Big|_{\substack{x=0 \\ \text{and} \\ x=\Delta x}} = -\frac{\left(R^\mu{}_{\nu yz}\Big|_{x=\Delta x} - R^\mu{}_{\nu yz}\Big|_{x=0}\right)}{\Delta x} A^\nu \Delta y \Delta z \Delta x =$$

$$= -\left(\partial_x R^\mu{}_{\nu yz}\right) A^\nu \Delta x \Delta y \Delta z$$

Let us perform the p.t. also around the other four sides of the cube ($y=0$, $y=\Delta y$, $z=0$, $z=\Delta z$), always in a direction corresponding to outwards orientation from the center of the cube (according to the right hand rule).

P.t. once around each side of the cube:

$$\Delta A\Big|_{\substack{\text{all}\\ \text{sides}}} = -\left[\partial_x R^{\mu}{}_{\nu yz} + \partial_y R^{\mu}{}_{\nu zx} + \partial_z R^{\mu}{}_{\nu xy}\right] A^{\nu}\,\Delta x\,\Delta y\,\Delta z$$

Now, note that for this p.t., once around each of the six sides of the cube, each edge is traversed (at least) two times, in opposite directions.

Therefore, adding all those p.t. together, the result of them must cancel. That is:

$$\partial_x R^{\mu}{}_{\nu yz} + \partial_y R^{\mu}{}_{\nu zx} + \partial_z R^{\mu}{}_{\nu xy} = 0$$

Thus, the Bianchi identity is just the statement that this kind of p.t. around a cube must cancel.

It is an identity in the sense that it holds for <u>any</u> metric.

[In problem 22:13 you prove the contracted Bianchi identity. Instead of using the method outlined in the problem, prove it from the full Bianchi identity.]

Black holes

We have studied the Schwarzschild geometry using Schwarzschild coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

So far we have used this line element to describe the gravitational physics outside a star. This means that we have only been interested in radii much larger than $2M$. We did not bother that something strange happens to the line element for $r < 2M$.

The Schwarzschild coordinates are suitable as long as we stay outside $r = 2M$. Then they exhibit both the spherical symmetry and the time translation invariance. But they are not suitable for studying orbits passing $r = 2M$ — in fact the coord. t goes to infinity for such trajectories at $r = 2M$! Yet nothing dramatic happens to the Riemann tensor there. Thus, this must be just a coord. singularity.

Eddington-Finkelstein coordinates

Instead of the problematic Schwarzschild t, let us use a coord. v defined by

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right| \qquad (1)$$

This leads to (for both $r > 2M$ and $r < 2M$):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

This is not any longer singular for $r = 2M$, which shows that this was not a real singularity. The singularity for $r = 0$, on the other hand, will not go away by any change of coordinates.

To see the meaning of the new coord. v, let us investigate the light rays in a v-r-plane

Radial light rays

$$ds^2 = 0, \quad d\Omega = 0 \quad \Rightarrow \quad \left(1 - \frac{2M}{r}\right) dv^2 = 2\,dv\,dr \qquad (2)$$

Solution 1: $dv = 0$

$v = \text{const.}$ — ingoing light rays!

(as t increases r must decrease according to (1))

Thus our coord. v just labels the ingoing light rays!

Solution 2: $dv = 2\left(1 - \frac{2M}{r}\right)^{-1} dr$

> 0 for $r > 2M$ — outgoing

< 0 for $r < 2M$ — ingoing

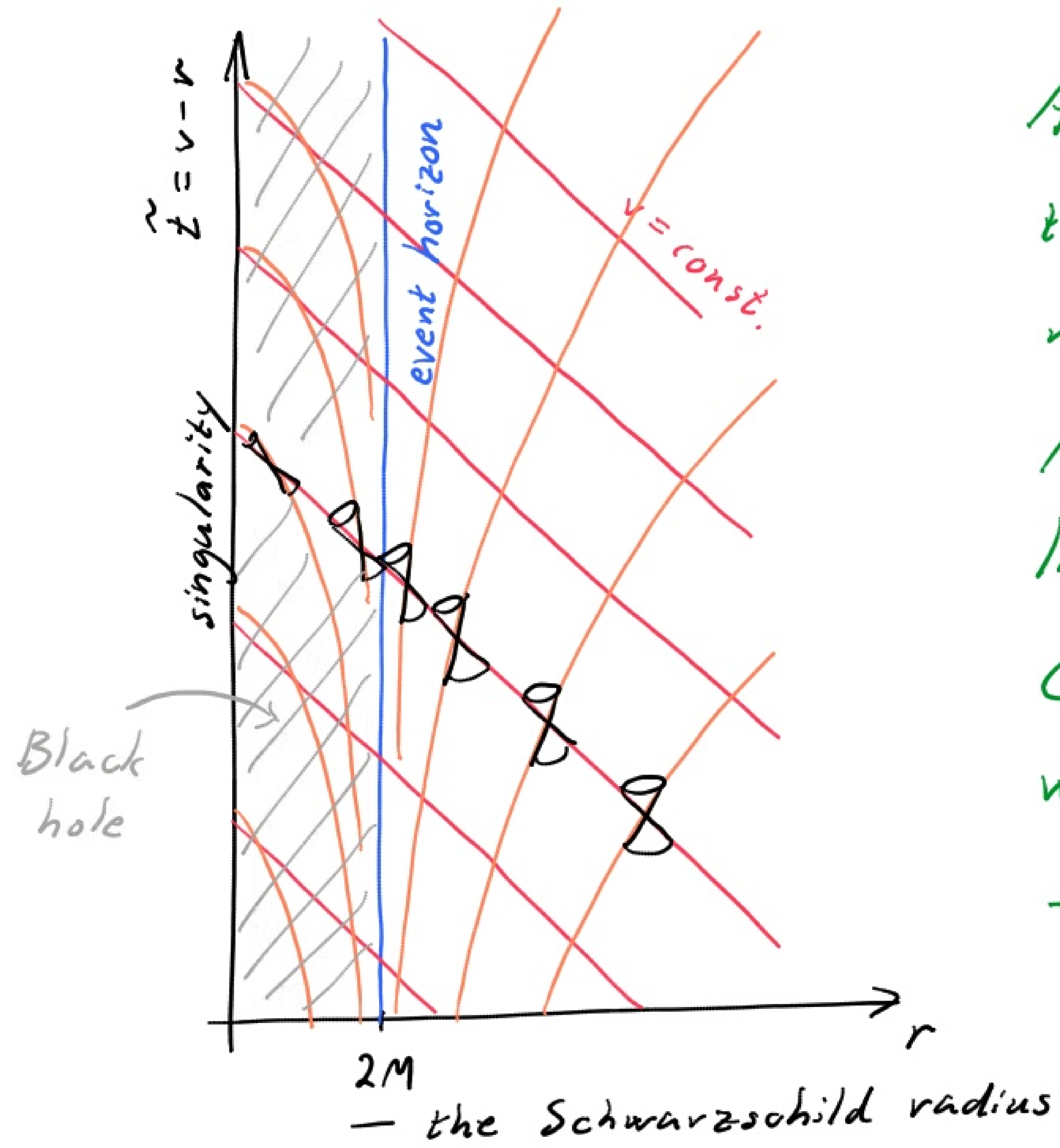
$$v = 2\left(r + 2M \ln\left|\frac{r}{2M} - 1\right|\right) + \text{const.}$$

This is the second set of light rays. Note that (2) is also solved by the limit case $r = 2M$:

Solution 3: $r = 2M$

One ray stays at constant radius!

Let us draw a picture of this. As "time" in the diagram I will not use v but $\tilde{t} = v - r$. Then the ingoing light rays $v = \text{const.}$ will be "45" degree lines sloping inwards.



As one gets closer to $r = 2M$ the light cones are sloping more and more inwards.

After one has passed $r = 2M$ all light rays are pointing inwards.

Clearly, no light can escape from within the surface $r = 2M$ — this is the black hole!

An observer falling into the black hole will not be able to escape from the region $r < 2M$. She will inexorably be pushed to the $r = 0$ line, called the singularity of the black hole, since here the curvature tensor diverges. She dies a moment before that from being torn apart by enormous tidal forces.

Note that the r-coord. gets timelike for $r < 2M$. (This is most obvious in the original Schwarzschild coord. line element, but also from the diagram above.) This means that the line $r = 0$ is actually a spacelike line. It is a moment in time, rather than a position in space. That is another way of saying that it is impossible to navigate away from it — the singularity just happens!

Now, do we have any reason to believe that this spacetime geometry corresponds to something real?

First (as you will show in the problem solving session), the Schwarzschild geometry is the unique spherically symmetric solution to the vacuum Einstein equation:

> Birkhoff's theorem: The Schwarzschild geometry is the only spherically symmetric vacuum solution.

This means that it must not only describe the geometry outside a spherically symmetric (that is, non-rotating) star, but also the geometry outside a spherically symmetric collapsing star.

And stars can collapse. They do so when they run out of fuel, so that their matter no longer is supported by the radiation pressure. The result of the collapse can be a

white dwarf — supported by Fermi pressure of electrons

neutron star — supported by Fermi pressure of neutrons

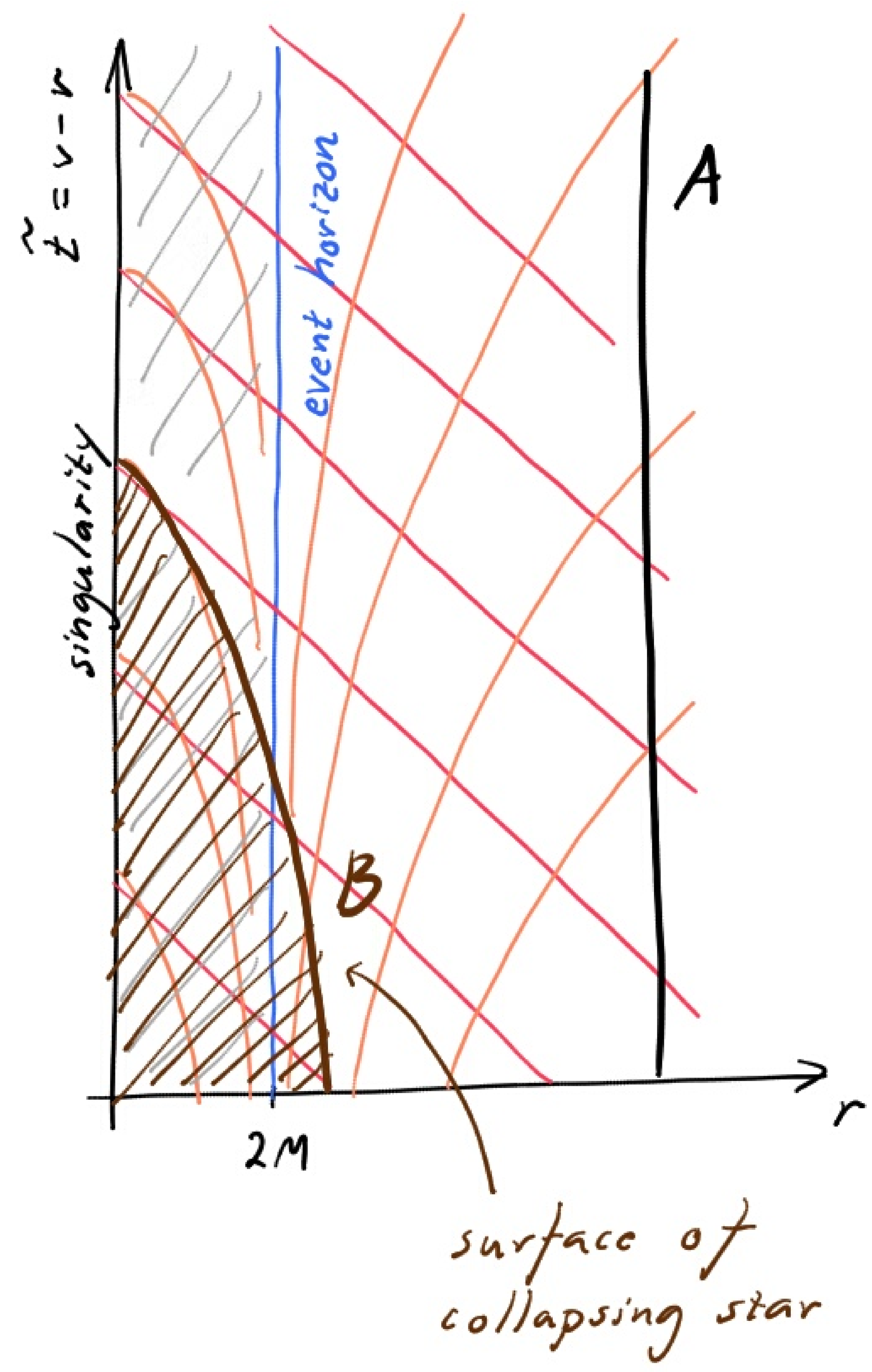
black hole — no support!

If the collapse into a black hole is spherically symmetric then the Schwarzschild geometry will be relevant all the way down to $r = 0$.

It is known that a neutron star is some times larger than the Schwarzschild radius: its surface is well outside $r = 2M$, but not so much. It is also known that the Fermi pressure of a neutron star cannot support more than $\sim 2M_\odot$. Some supernova explosions will leave behind a rest much larger than this. Then there is nothing to support them. And once the matter has collapsed to a region less than $2M$, the collapse is bound to continue — it is then a matter of causality.

For some time people argued that the black hole, and especially the singularity, was just an artefact of the unphysical assumption of spherical symmetry. But it has been shown that this is not so: the formation of singularities (and horizons) are generic in general relativity.

So there seems to be no way around black holes. They are today believed to be quite common in the universe.



Let us draw the surface of a collapsing star in the Eddington-Finkelstein diagram. Within that surface the diagram has no meaning, since there the spacetime geometry is different. But because of Birkhoff's theorem we know that the diagram must still be valid outside the star (assuming the collapse is spherically symmetric).

Consider two observers, A and B, one staying outside at some fixed coord. r (obs. A) and the other following the surface of the star (obs. B). Suppose B is continuously sending signals to A. What will A see?

As is clear from the diagram, the signals from B will reach A with longer and longer intervals. And the signal that B emits at the moment when she passes the horizon, will never reach A, since it will remain stuck at the horizon. This means that A will never see B passing the horizon. To A the picture of B gets more and more redshifted and quickly fades away.

(Hartle shows that this fading away is exponential.)