Lecture 3

The gravitational time shift
- for general potential ϕ
- comparing with clock at infinity

Replacement of Minkowski-metric to include time shift
- $ds^2 = -c^2 dt^2 \left(1 + \frac{2\phi(x^i)}{c^2}\right) + dr^2 + r^2 d\Omega^2$

- Measuring the time shift with light pulses

Equivalence principle $\Rightarrow$ Free fall path extremizes time
- (timelike) geodesic

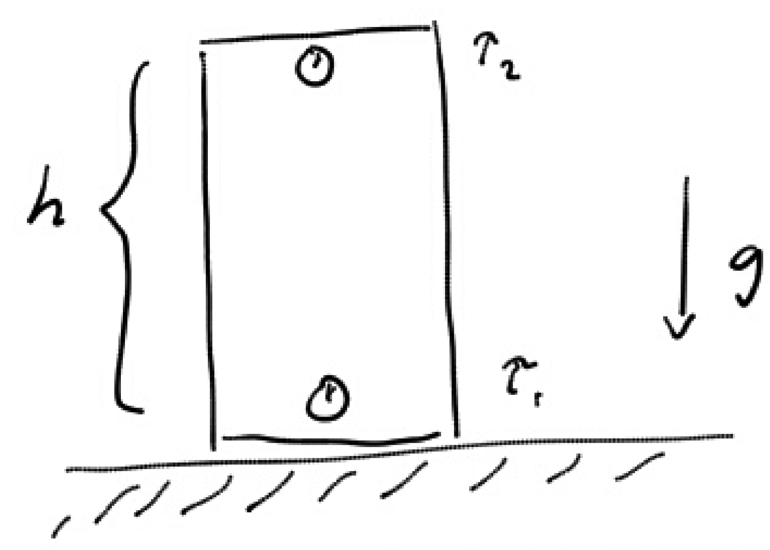
Extremizing ds^2
- the result: Newtonian trajectories
- Review: Newtonian action $\longrightarrow$ eq. of motion
- Review: Lagrange's equations

Correct mean-field theory
- Only time-part mattered in above calc.
- True metric
- The curvature "in time" is enough to produce Newtonian trajectories

Newtonian gravity described as the result of curved spacetime

The equivalence principle together with SR implies that two observers at different height h in a gravitational field g measures different time. The clock at the higher position runs faster:

$$E.p. + SR \implies \tau_2 - \tau_1 = hg\,\tau_1$$

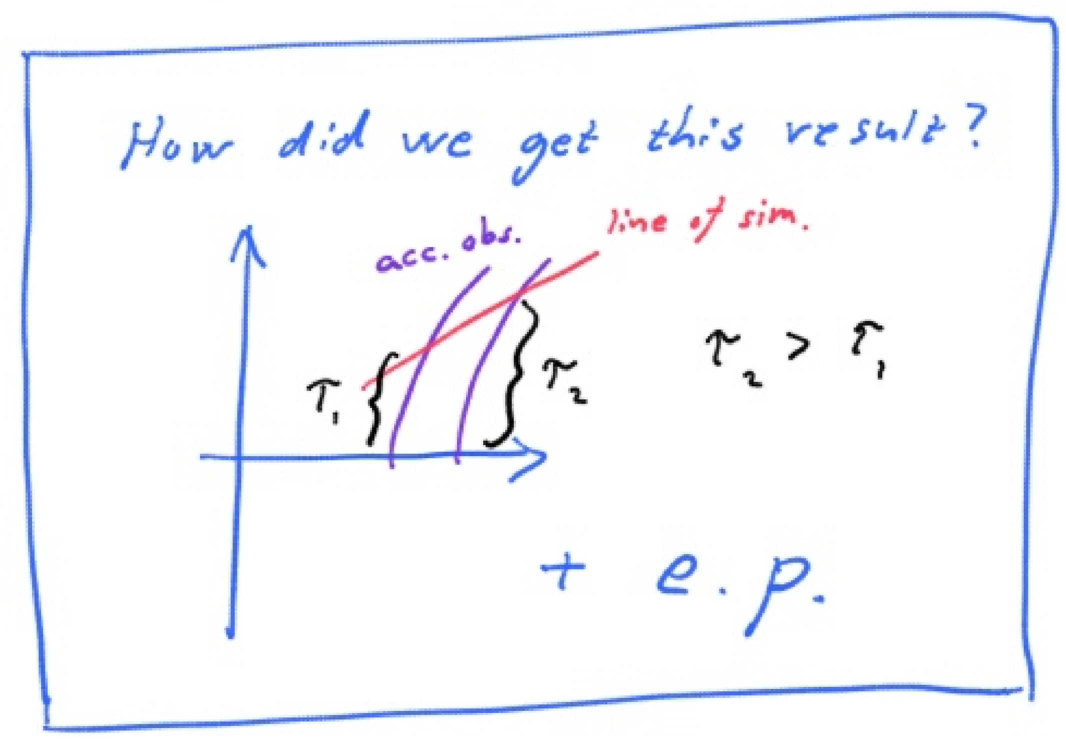


Let us rewrite this:

$$\tau_2 = \tau_1 \left(1 + hg\right)$$

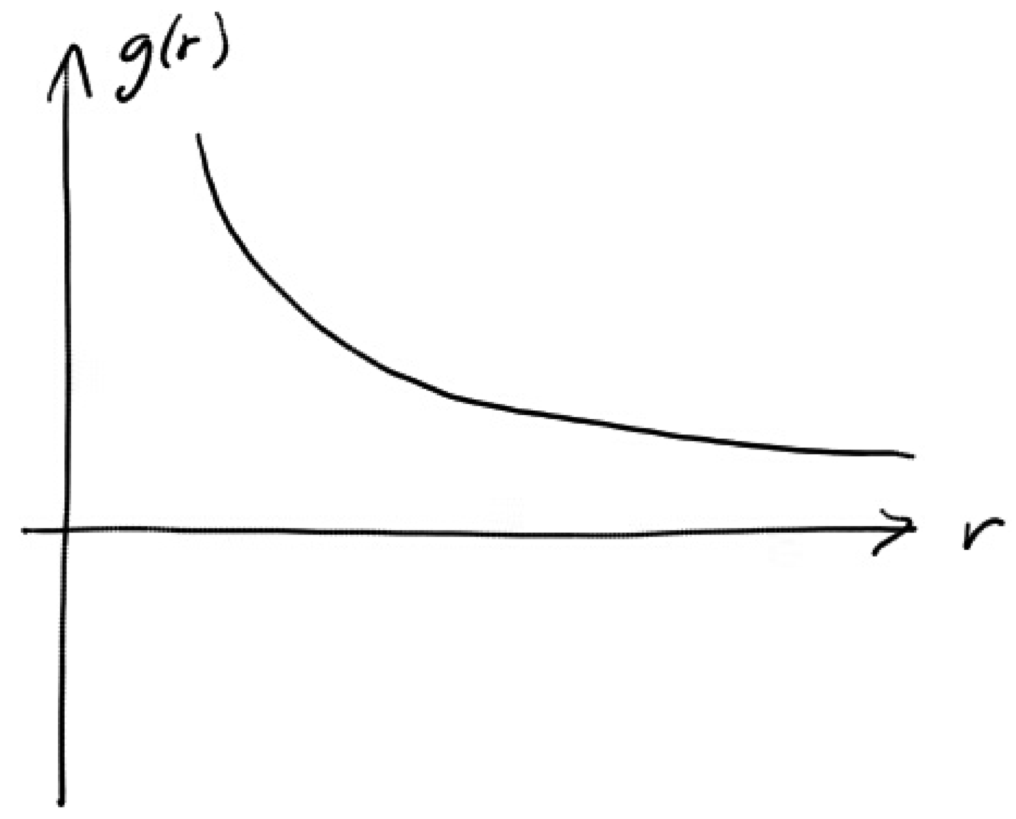
Or for infinitesimal intervals:

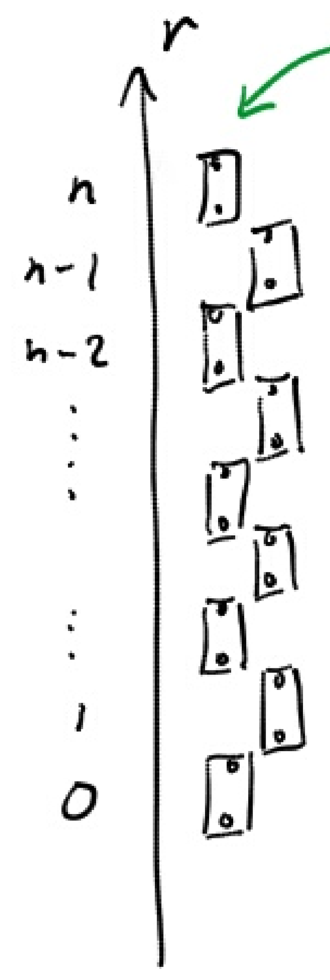
$$d\tau_2 = d\tau_1 \left(1 + hg\right)$$



In practice, we know that the gravitational acc. g is not constant. The gravitational field is varying. For example, at high altitudes g has a smaller value than at the surface of the Earth, and at large distances is goes to zero.

Can we draw any conclusion from the e.p. about the time shift between the surface of the Earth and far away from the Earth? Yes, but we have to remember that e.p. can be applied only locally. Thus, to mimic a situation with a varying grav. acc., $g(r)$, consider a series of (infinitesimally small) accelerating "elevators." We can arrange them so that they, at least for one moment, are lined up next to one another and all with zero speed, but with different acc. depending on their position. (The direction of acc. is in the r-direction.)





Series of "elevators", each with two clocks, and each given an individual acc. $g(r)$ corresponding to its position r. (The elevators are arranged in a zigzag pattern so that they will not immediately collide.)

The clock at the floor of elevator n is at the same r-coord. as the clock at the ceiling of elevator $n-1$. Thus the difference in time rate between the clocks at the ceilings of these two elevators is the same as the difference in time rate between the two clocks in elevator n:

$$d\tau_n = d\tau_{n-1}\left(1 + g(r_{n-1})\Delta r\right) =$$

$$= d\tau_{n-2}\left(1 + g(r_{n-1})\Delta r\right)\left(1 + g(r_{n-2})\Delta r\right) =$$

$$= d\tau_0 \prod_{i=1}^{n}\left(1 + \underbrace{g(r_{n-i})\Delta r}_{\text{small}}\right) \approx d\tau_0\left(1 + \sum_{i=1}^{n} g(r_{n-i})\Delta r\right)$$

$$\xrightarrow[\Delta \to 0]{} d\tau_0\left(1 + \int_{r_0}^{r_n} g(r)\,dr\right)$$

Remember that the gravitational potential (per unit mass) is

$$\Phi(r_b) - \Phi(r_a) = -\int_{r_a}^{r_b} \frac{F(r)}{m}\,dr = +\int_{r_a}^{r_b} g(r)\,dr \qquad \left(\text{since } \bar{F}(r) = -mg(r)\hat{r}\right)$$

$$\Rightarrow \quad d\tau_n = d\tau_0\left(1 + \left(\Phi(r_n) - \Phi(r_0)\right)\right)$$

Let us put $r_n = \infty$, choose $\Phi(r_n) = \Phi(\infty) = 0$

call the time measured at infinity t: $dt = d\tau_n$

and drop the "0" on $d\tau_0$ and r_0. Then:

$$dt = d\tau\left(1 - \Phi(r)\right) \quad \text{where } \Phi(r) = -\frac{MG}{r} \text{ outside spherically symmetric mass distribution}$$

Or to first order in Φ (since with c:s reinserted we have $\Phi \to \frac{\Phi}{c^2}$)

$$\boxed{d\tau = \frac{dt}{1 - \Phi(r)} \approx dt\left(1 + \Phi(r)\right)}$$

Since $\Phi < 0$ a clock in grav. pot. Φ runs slow compared to a clock at infinity.

Now, if this is true, then the ordinary Minkowski line element

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2 \quad , \quad d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

Now it will be better to write out the c:s

line element of sphere with radius 1

cannot be the right one when there is a gravitational potential Φ. The situation is serious: this line element is <u>inconsistent</u> with the equivalence principle!

But we can easily modify it so that it becomes consistent with the e.p. in this sense.

The simplest line element, with linear factors, accomplishing this is

$$ds^2 = -c^2 dt^2 \left(1 + \frac{2\Phi(x^i)}{c^2}\right) + dr^2 + r^2 d\Omega^2$$

Why?

Consider an observer at constant spatial coord. x^i, so that $dr = 0$ and $d\Omega = 0$.

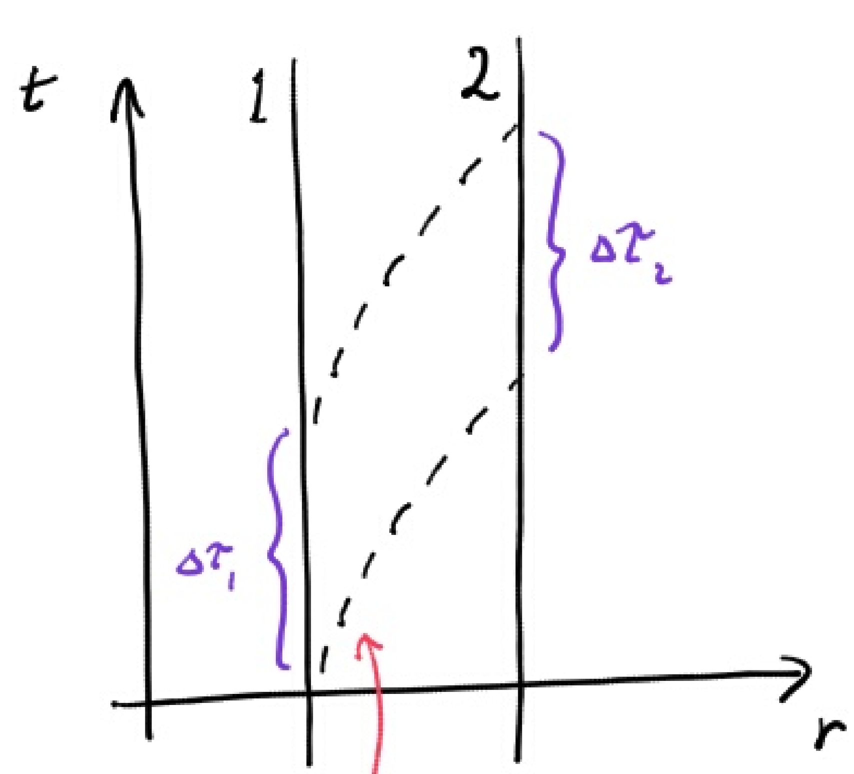
Stationary obs. $(dr = 0, \, d\Omega = 0)$ $\Rightarrow$

$$d\tau^2 = -\frac{ds^2}{c^2} = dt^2\left(1 + \frac{2\Phi}{c^2}\right)$$

$$d\tau = dt\sqrt{1 + \frac{2\Phi}{c^2}} \simeq dt\left(1 + \frac{\Phi}{c^2}\right)$$

How can two stationary observers, at different heights in this spacetime, determine their difference in proper time? $\boxed{?}$

— By exchanging light pulses!



Described by $\boxed{?}$

$$\frac{dr}{dt} = \sqrt{1 + \frac{2\Phi(r)}{c^2}} < 1$$

Observer 1 sends two signals to obs. 2 emitted with time interval $\Delta\tau_1$.
The pulses are received with time interval $\Delta\tau_2$. But since the metric is independent on coordinate time t, the corresponding coordinate time intervals must be the same. Hence:

$$\left.\begin{aligned} \Delta\tau_1 &= \Delta t\left(1 + \frac{\Phi_1}{c^2}\right) \\ \Delta\tau_2 &= \Delta t\left(1 + \frac{\Phi_2}{c^2}\right) \end{aligned}\right\} \Rightarrow \frac{\Delta\tau_1}{\Delta\tau_2} = \frac{\left(1 + \frac{\Phi_1}{c^2}\right)}{\left(1 + \frac{\Phi_2}{c^2}\right)} \simeq \left(1 + \frac{\Phi_1}{c^2}\right)\left(1 - \frac{\Phi_2}{c^2}\right) \simeq$$

$$\simeq 1 - \frac{\Phi_2 - \Phi_1}{c^2}$$

So the equivalence principle forces upon us a curved metric:

$$ds^2 = -c^2 dt^2 \left(1 + \frac{2\phi(x^i)}{c^2}\right) + dr^2 + r^2 d\Omega^2 \qquad (*)$$

This may sound a bit contradictory: By combining Minkowski space with the e.p. we are forced to adopt a curved line element — which is different from Minkowski space! But the only thing that we have required is that Minkowski space should hold locally. And, indeed, the metric $(*)$ is locally Minkowski, like all spacetime metrics.

This lead us to a new formulation of the e.p.:

> A local freely falling frame is indistinguishable from an inertial frame in SR.

Now, let's ask: What is a straight line?

In flat space: — The shortest path!

In flat spacetime, timelike line: — The longest path!

In general: — An extremal path!

Now, according to the e.p. any freely falling worldline must have the same properties as a non-accelerated worldline in Minkowski space.

Thus: $\boxed{\text{A freely falling worldline extremizes the proper time.}}$

or: $\boxed{\text{A timelike geodesic extremizes the proper time}}$

> In general a geodesic is defined as a curve which extremizes $\sqrt{|ds^2|}$

And now we are ready for some magic. We ask:

What do the freely falling worldlines in the metric $(*)$ look like?

Answer: Extremize

$$\tau = \int_A^B d\tau = \int_A^B \sqrt{\frac{-ds^2}{c^2}} = \int \left(dt^2 \left(1 + \frac{2\phi(r)}{c^2}\right) - \frac{1}{c^2}\left(dr^2 + r^2 d\Omega^2\right)\right)^{1/2} =$$

$$= \int_A^B \left[\left(1 + \frac{2\phi(r)}{c^2}\right) - \frac{1}{c^2} \underbrace{\left(\frac{dr^2}{dt^2} + r^2 \frac{d\Omega^2}{dt^2} \right)}_{\vec{v}\cdot\vec{v}} \right]^{1/2} dt =$$

where $\vec{v}$ is the ordinary 3-velocity

$$= \int_A^B \left[1 + \frac{1}{c^2}\left(2\Phi(r) - \vec{v}\cdot\vec{v}\right)\right]^{1/2} dt \simeq$$

weak potential Φ
and $v \ll c$

$$\simeq \int_A^B \left[1 + \frac{1}{c^2}\left(\Phi(r) - \frac{\vec{v}\cdot\vec{v}}{2}\right)\right] dt$$

This is extremized for the same curves as

$$S = \int_A^B \left(\frac{\vec{v}\cdot\vec{v}}{2} - \phi(r)\right) dt = \int_A^B \underbrace{\left(\frac{1}{2}\frac{d\vec{x}}{dt}\cdot\frac{d\vec{x}}{dt} - \phi(r)\right)}_{L} dt \qquad \boxed{?}$$

is extremized. The integrand is recognized as the usual Lagrangian for a particle moving in potential ϕ. From Euler-Lagrange's equations

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}$$

we get

$$-\nabla\phi = \frac{d}{dt}\left(\frac{d\vec{x}}{dt}\right) = \frac{d^2\vec{x}}{dt^2}$$

When multiplied by m this is just Newton's second law with $\vec{F} = -m\nabla\phi$.

So Newton's trajectories are straight lines (that is, lines with extreme proper time) in the metric $(*)$!

This shows that we can do without Newtons gravitational force. The same trajectories are described as straight lines in the curved metric $(*)$. And even more, the e.p. force us to this shift in perspective: because of the gravitational time shift, the Minkowski metric cannot be right – spacetime has to be curved. The simplest choice for this curved metric turns out to also describe the Newtonian trajectories.

We have assumed that Φ in the metric (*) is a weak field:

$$\frac{\Phi}{c^2} \ll 1$$

This weak field metric produces the right Newtonian orbits for non-relativistic speeds: $v \ll c$

But there are many similar weak field metrics that produce the right Newtonian orbits in this limit. They differ in their predictions for relativistic speeds. And what the correct predictions are we cannot know without the full theory of general relativity, which we do not yet have.

Is our metric (*) correct also for $v \sim c$?

As it turns out — No!

The correct weak field metric (applicable to any speed) outside a spherical mass distribution with $\Phi = -\frac{GM}{r}$:

$$ds^2 = -c^2 dt^2\left(1 + \frac{2\Phi(r)}{c^2}\right) + \left(1 - \frac{2\Phi(r)}{c^2}\right)dr^2 + r^2 d\Omega^2 =$$

$$= -c^2 dt^2\left(1 - \frac{2MG}{rc^2}\right) + \left(1 + \frac{2MG}{rc^2}\right)dr^2 + r^2 d\Omega^2$$

or in geometrized units $c = 1$, $G = 1$:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 d\Omega^2$$

(Weak field here means $r \gg 2M$)

Note that, because of the factor c^2 in front of dt^2, the factor $\left(1 + \frac{2MG}{rc^2}\right)$ in front of dr^2 will only contribute as a factor 1 in the calculation above, which led to the usual Lagrangian in Newtonian mechanics. (Se Hartle p. 128.)

Thus, for non-relativistic trajectories the curvature of space can be neglected. It is the curvature in timelike slices of spacetime that gives rise to the Newtonian orbits. Therefore, it is rather misleading to visualize the effects of general relativity as a "gravitational pit", or as a curved spatial surface.