General relativity: Bonustest II

May 16, 2019

Give your answers on this sheet. Unless stated in the question, you don't have to provide any reasoning or justifications for your answers. You can answer in english or in swedish.

Max: 14 p. At least 7 p gives 1 point to the exam. At least 11 p gives 2 points to the exam.

1. $A_\alpha^{\beta\gamma}$ is a rank 3 tensor. Insert indices below to obtain the correct transformation law to new coordinates α', β', γ'. (1 p)

$$A_{\alpha'}{}^{\beta'\gamma'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial x^{\gamma'}}{\partial x^{\gamma}} A_\alpha{}^{\beta\gamma}$$

2. The covariant derivative of a vector $\boldsymbol{a}$, expressed with index upstairs and index downstairs, is

$$\nabla_\alpha a^\beta = \partial_\alpha a^\beta + \Gamma^\beta{}_{\alpha\gamma} a^\gamma$$

$$\nabla_\alpha a_\beta = \partial_\alpha a_\beta - \Gamma^\gamma{}_{\alpha\beta} a_\gamma$$

What is the corresponding expression for the covariant derivative of the rank 3 tensor $A_\alpha^{\beta\gamma}$? (2 p)

$$\nabla_\alpha A_\beta{}^{\gamma\delta} = \partial_\alpha A_\beta{}^{\gamma\delta} - \Gamma^\mu{}_{\alpha\beta} A_\mu{}^{\gamma\delta} + \Gamma^\gamma{}_{\alpha\mu} A_\beta{}^{\mu\delta} + \Gamma^\delta{}_{\alpha\mu} A_\beta{}^{\gamma\mu}$$

3. What kind of object is each of the following quantities? Is it a tensor, one component of a tensor, or neither? Mark the correct alternative and fill in the appropriate rank! (All correct – 4 p; 6 correct – 3 p; 5 or 4 correct – 2 p. 3 or 2 correct – 1 p.)

(a) total energy of a particle

☐ tensor of rank __ ☒ comp. of tensor of rank 1 ☐ neither

(b) energy density

☐ tensor of rank __ ☒ comp. of tensor of rank 2 ☐ neither

(c) the Ricci tensor

☒ tensor of rank 2 ☐ comp. of tensor of rank __ ☐ neither

(d) partial derivative of a vector field: $\partial_\alpha a^\beta$

☐ tensor of rank __ ☐ comp. of tensor of rank __ ☒ neither

(e) partial derivative of a scalar field: $\partial_\alpha \Phi$

☒ tensor of rank 1 ☐ comp. of tensor of rank __ ☐ neither

(f) covariant derivative of a rank 2 tensor field: $\nabla_\alpha a^{\beta\gamma}$

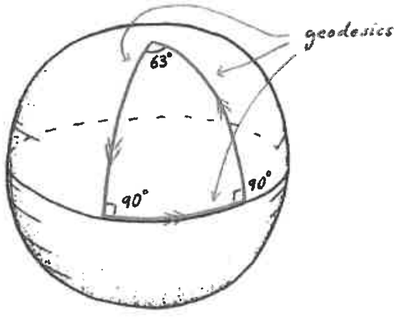
☒ tensor of rank 3 ☐ comp. of tensor of rank __ ☐ neither

(g) Christoffel symbol $\Gamma^\alpha{}_{\beta\gamma}$

☐ tensor of rank __ ☐ comp. of tensor of rank __ ☒ neither

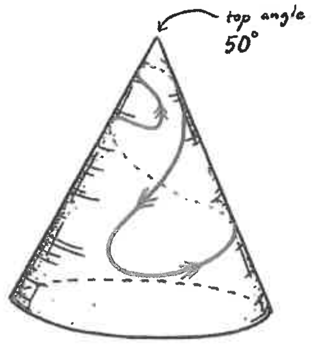
4. What are the results of parallel transporting a vector around the indicated loops on the following surfaces? Answer with the rotation angle (if any) as well as with the sense of rotation (clockwise or anti-clockwise). (3 p)

(a)



63° Counter-clockwise

(b)

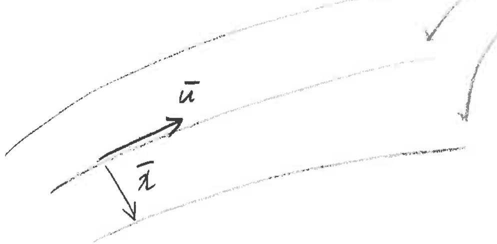


No rotation

5. The following two expressions ((a) and (b)) involving the Riemann tensor show its geometrical significance in two different ways. Explain the geometrical meaning of each of the expressions in a few words and with a simple drawing. Make sure that the meaning of each object involved is made clear. (4 p)

(a) $\nabla_u \nabla_u \chi^\alpha = -R^\alpha{}_{\beta\gamma\delta} u^\beta \chi^\gamma u^\delta$

Family of geodesics



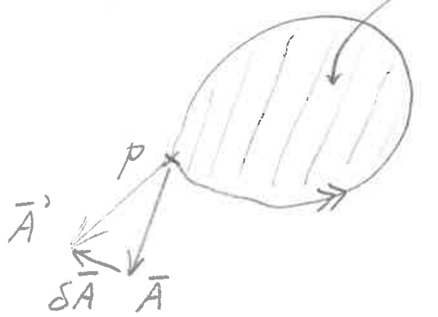
$\bar{u}$ – tangent to geodesics

$\bar{\chi}$ – separation vector between geodesics

$\nabla_{\bar{u}} \nabla_{\bar{u}} \bar{\chi}$ – relative acc. of neighbouring geodesics

(b) $\delta A^\mu = -R^\mu{}_{\nu\alpha\beta} \delta S^{\alpha\beta} A^\nu$

$\delta S^{\alpha\beta}$ – area element enclosed by loop



$\bar{A}'$ – result of parallel transporting vector $\bar{A}$ around closed loop

$\delta\bar{A} = \bar{A}' - \bar{A}$