Lecture 10

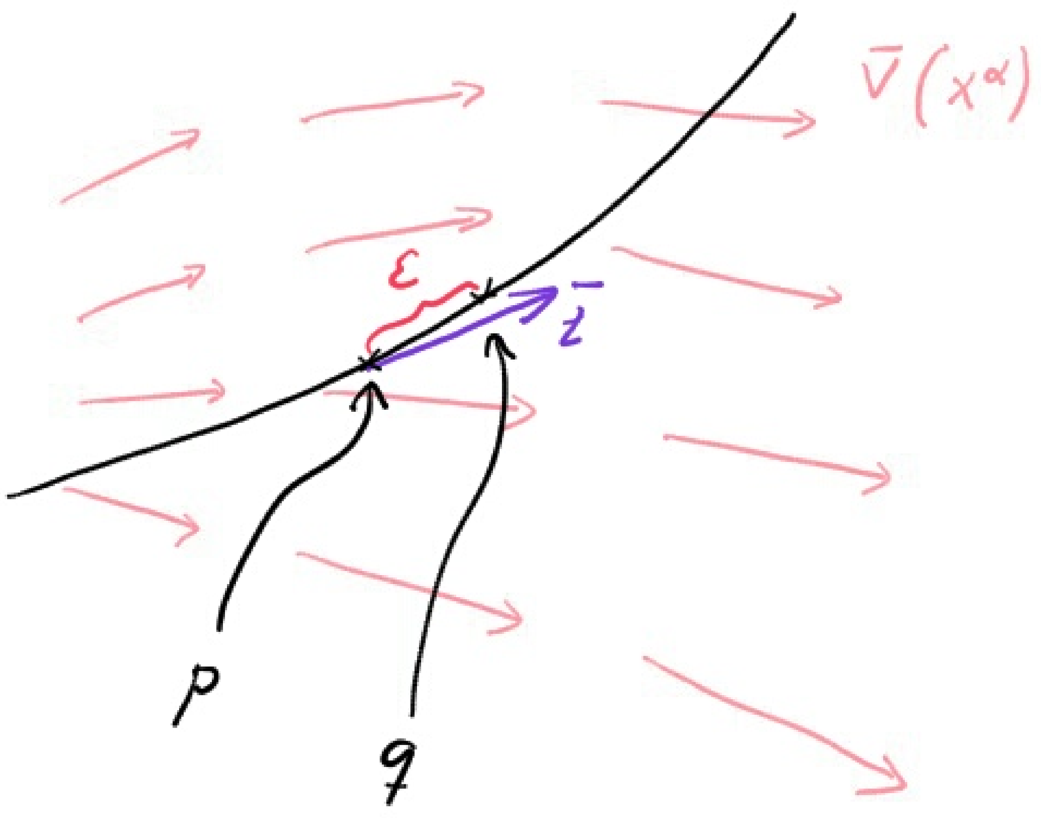
Measure of curvature : the result of p.t. around small loops

- Derivation of $R^{\delta}{}_{\alpha\beta\gamma}$

Expressing R in terms of Γ.

Last time we defined the covariant derivative through the notion of parallel transport:

$$\nabla_{\bar{t}}\, \bar{v}(p) = \lim_{\varepsilon \to 0} \frac{\bar{v}(q)\big|_{\text{p.t. to } p} - \bar{v}(p)}{\varepsilon}$$



For the tangent of a geodesic we know that

$$\nabla_{\bar{u}}\, \bar{u} = 0$$

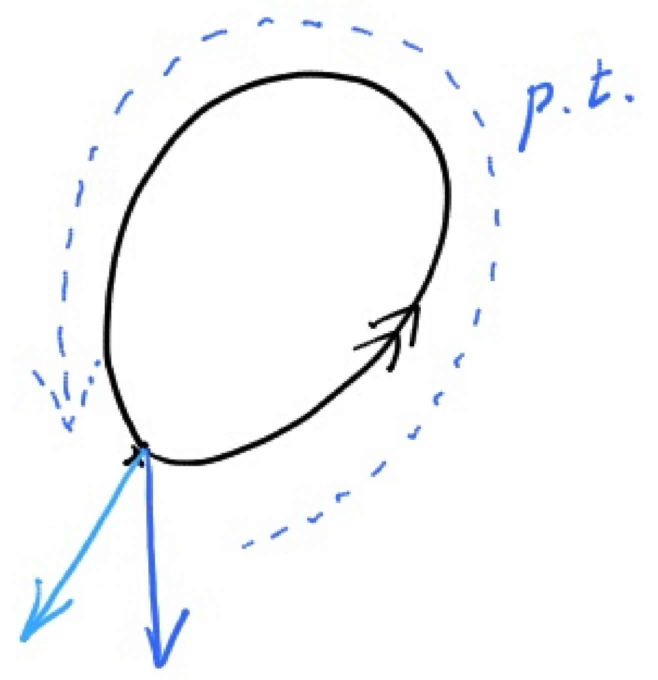
Using the already derived form of the geodesic equation, we could then conclude that

$$\nabla_\alpha v^\beta = \partial_\alpha v^\beta + \Gamma^\beta_{\alpha\gamma} v^\gamma \quad , \quad \nabla_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma^\gamma_{\alpha\beta} v_\gamma$$

where Γ is an expression containing the first partial derivatives of the metric.

We can now use this to define an invariant measure of curvature for generally curved spaces or spacetimes.

Qualitatively, this is how our measure will work:



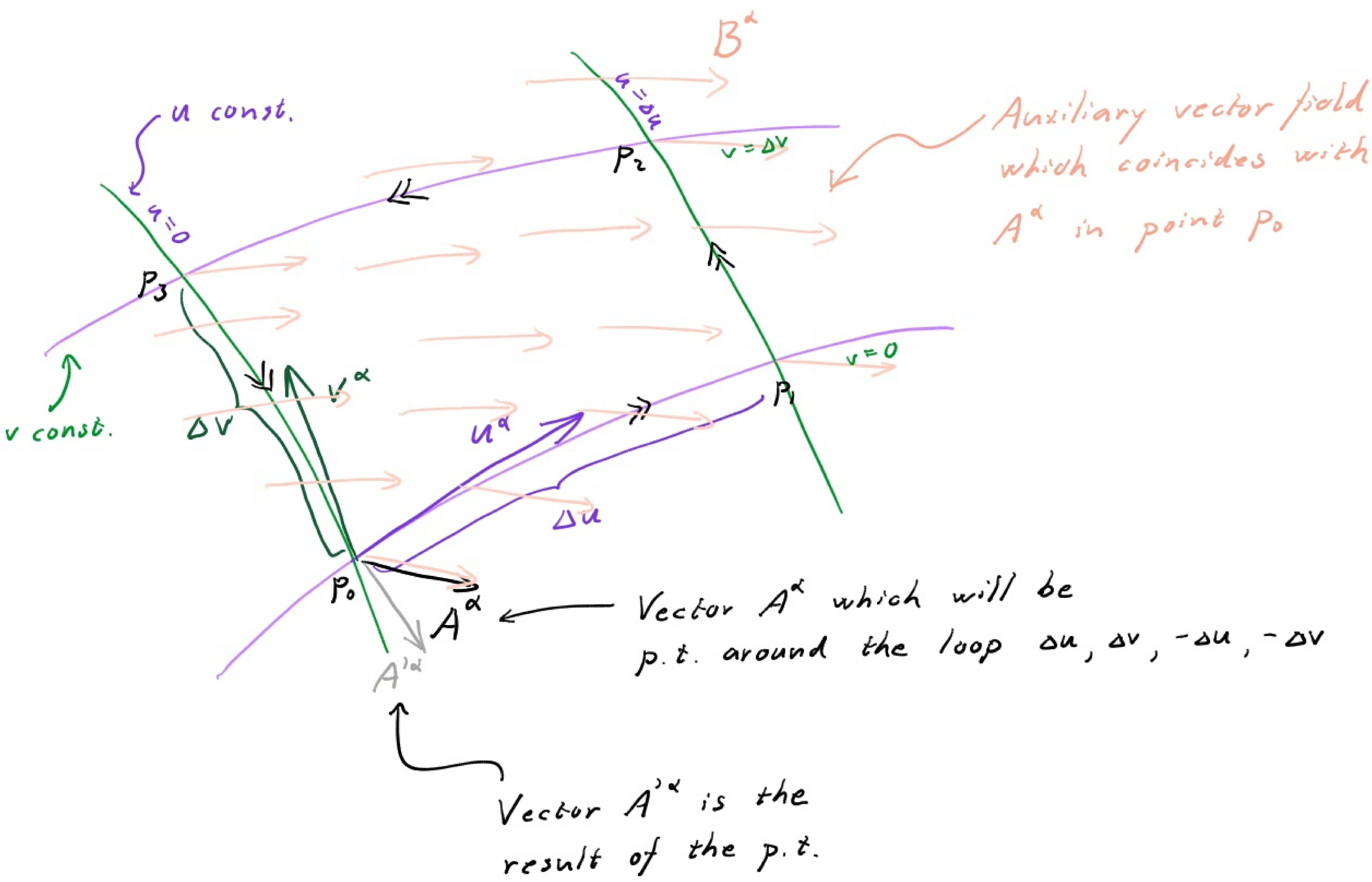
P.t. a vector around a closed loop. If space is curved the vector will not be the same when it returns: it will be rotated (or boosted) with respect to the original vector. The kind of transformation (and its size) is a measure of the curvature in the directions spanned by the loop.

Since the covariant derivative quantifies the result of p.t., we can use it to make this measure of curvature precise.

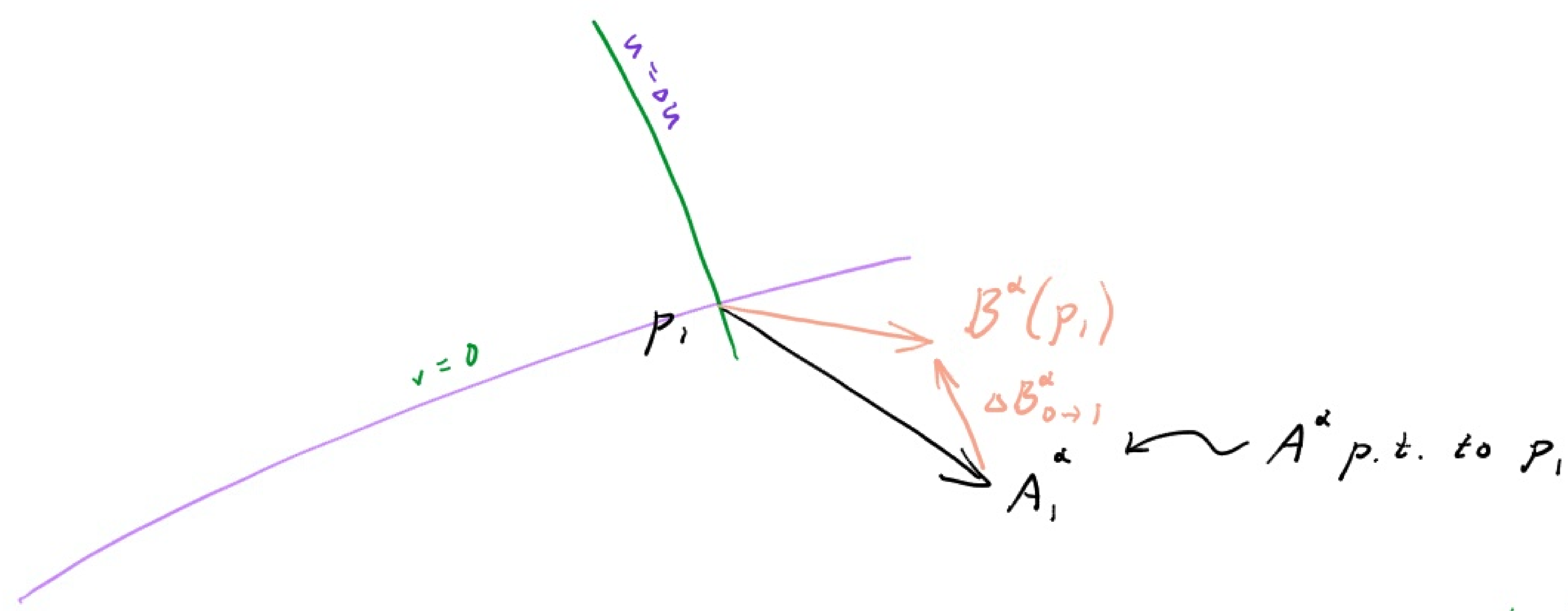
Consider a point p_0 and a 2-dimensional surface (the one in which the p.t. will take place). Introduce coordinates u and v in that surface such that $p_0^\alpha = (0, 0)$.

Consider a vector A^α at p_0. We will p.t. this around a small rectangular loop consisting of segments of the coordinate lines of u and v.

We also need to introduce an auxiliary vector field B^α.



As the first step, p.t. A^α along the u-coord. line a coordinate distance Δu to point p_1:



Now, note that the difference between A_1^α and the auxiliary field $B^\alpha(p_1)$ is the same as the covariant change in B^α when moving from p_0 to p_1 (since $B^\alpha(p_0) = A^\alpha$ at p_0): $\Delta B^\alpha_{0\to1} = \Delta u\, u^\beta \nabla_\beta B^\alpha \big|_{v=0}$

$$A_1^\alpha = B^\alpha(p_1) - \Delta B^\alpha_{0\to1} = B^\alpha(p_1) - \Delta u\, u^\beta \nabla_\beta B^\alpha \big|_{v=0}$$

By definition of covariant derivative!

Let us continue step by step along the remaining segments of the loop:

$$\Delta B^\alpha_{1\to 2} = \Delta v\, v^\beta \nabla_\beta B^\alpha \Big|_{u=\Delta u}$$

change in B^α when moving from point 1 to point 2

and $$A^\alpha_2 = B^\alpha(p_2) - \Delta B^\alpha_{0\to 1} - \Delta B^\alpha_{1\to 2}$$

the result of the p.t. of A^α at point 2, expressed in terms of B^α

$$\Delta B^\alpha_{2\to 3} = -\Delta u\, u^\beta \nabla_\beta B^\alpha \Big|_{v=\Delta v}$$

and $$A^\alpha_3 = B^\alpha(p_3) - \Delta B^\alpha_{0\to 1} - \Delta B^\alpha_{1\to 2} - \Delta B^\alpha_{2\to 3}$$

$$\Delta B^\alpha_{3\to 0} = -\Delta v\, v^\beta \nabla_\beta B^\alpha \Big|_{u=0}$$

and $$A'^\alpha = \underbrace{B^\alpha(p_0)}_{=A^\alpha} - \Delta B^\alpha_{0\to 1} - \Delta B^\alpha_{1\to 2} - \Delta B^\alpha_{2\to 3} - \Delta B^\alpha_{3\to 0}$$

Back at the starting point!

Hence we can express the result of the p.t., that is,

$$\Delta A^\alpha = A'^\alpha - A^\alpha$$

in terms of the covariant derivatives of the auxiliary field B^α!

Let us put all the pieces together:

$$\Delta A^\alpha = A'^\alpha - A^\alpha = -\Delta B^\alpha_{0\to 1} - \Delta B^\alpha_{1\to 2} - \Delta B^\alpha_{2\to 3} - \Delta B^\alpha_{3\to 0} =$$

$$= -\Delta u\, u^\beta \nabla_\beta B^\alpha \Big|_{v=0} - \Delta v\, v^\beta \nabla_\beta B^\alpha \Big|_{u=\Delta u} + \Delta u\, u^\beta \nabla_\beta B^\alpha \Big|_{v=\Delta v} + \Delta v\, v^\beta \nabla_\beta B^\alpha \Big|_{u=0}$$

$$= \Delta u\, \Delta v \Bigg[\underbrace{\frac{u^\beta \nabla_\beta B^\alpha \big|_{v=\Delta v} - u^\beta \nabla_\beta B^\alpha \big|_{v=0}}{\Delta v}}_{\xrightarrow[\Delta v \to 0]{\lim} \; v^\gamma \nabla_\gamma \left(u^\beta \nabla_\beta B^\alpha\right)} - \underbrace{\frac{v^\beta \nabla_\beta B^\alpha \big|_{u=\Delta u} - v^\beta \nabla_\beta B^\alpha \big|_{u=0}}{\Delta u}}_{\xrightarrow[\Delta u \to 0]{\lim} \; u^\gamma \nabla_\gamma \left(v^\beta \nabla_\beta B^\alpha\right)} \Bigg]$$

$\boxed{?}$

So

$$\delta A^\alpha = du\,dv\Big(v^\gamma(\nabla_\gamma u^\beta)(\nabla_\beta B^\alpha) + v^\gamma u^\beta \nabla_\gamma \nabla_\beta B^\alpha + - u^\gamma(\nabla_\gamma v^\beta)(\nabla_\beta B^\alpha) - u^\gamma v^\beta \nabla_\gamma \nabla_\beta B^\alpha \Big)$$

The first and third term vanish, since

$$v^\gamma(\nabla_\gamma u^\beta) = v^\gamma\left(\partial_\gamma u^\beta + \Gamma^\beta_{\ \gamma\delta}u^\delta\right) = \frac{\partial}{\partial v}\frac{\partial x^\beta}{\partial u} + v^\gamma u^\delta \underbrace{\Gamma^\beta_{\ \gamma\delta}}_{\text{sym.}} =$$

$$= \frac{\partial}{\partial u}\frac{\partial x^\beta}{\partial v} + u^\gamma v^\delta \Gamma^\beta_{\ \gamma\delta} = u^\gamma\left(\partial_\gamma v^\beta + v^\delta \Gamma^\beta_{\ \gamma\delta}\right) =$$

$$= u^\gamma \nabla_\gamma v^\beta$$

So what remains is

$$\delta A^\alpha = \underbrace{du\,dv\,u^\beta v^\gamma}_{\text{can be replaced by}}\left(\nabla_\gamma\nabla_\beta - \nabla_\beta\nabla_\gamma\right)B^\alpha$$

can be replaced by
$\frac{1}{2}\cdot du\,dv\left(u^\beta v^\gamma - u^\gamma v^\beta\right) \equiv \delta S^{\beta\gamma}$
since it is multiplied
with something that
is anti-sym. in β, γ.

Remember that the vector field was arbitrary. We only demanded that it should equal A^α at point p_0.

$\boxed{?}$ area-element spanned by $\bar{u}$ and $\bar{v}$

Mystery: How can the change in the vector A^α depend on an arbitrary field B^α?

To solve the mystery, let us evaluate the anti-sym. second covariant derivative on B^α.

$$\nabla_\gamma \nabla_\beta B^\alpha = \nabla_\gamma \left(\partial_\beta B^\alpha + \Gamma^\alpha_{\ \beta\delta} B^\delta \right) =$$

$$= \partial_\gamma \partial_\beta B^\alpha - \Gamma^\delta_{\ \beta\gamma}\, \partial_\delta B^\alpha + \Gamma^\alpha_{\ \delta\gamma} \partial_\beta B^\delta +$$

$$+ \partial_\gamma \left(\Gamma^\alpha_{\ \beta\delta} B^\delta \right) + \Gamma^\alpha_{\ \mu\gamma} \Gamma^\mu_{\ \beta\delta} B^\delta - \Gamma^\mu_{\ \beta\gamma} \Gamma^\alpha_{\ \mu\delta} B^\delta$$

The terms marked —×— are all symmetric in γ, β, so they will cancel against similar terms in $\nabla_\beta \nabla_\gamma B^\alpha$.

The forth term:

$$\partial_\gamma \left(\Gamma^\alpha_{\ \beta\delta} B^\delta \right) = \left(\partial_\gamma \Gamma^\alpha_{\ \beta\delta} \right) B^\delta + \Gamma^\alpha_{\ \beta\delta}\, \partial_\gamma B^\delta$$

This is symmetric together with the third term above. Thus they will also cancel similar terms in $\nabla_\beta \nabla_\gamma B^\alpha$.

We are left with two terms only. Hence:

$$\left(\nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma \right) B^\alpha =$$

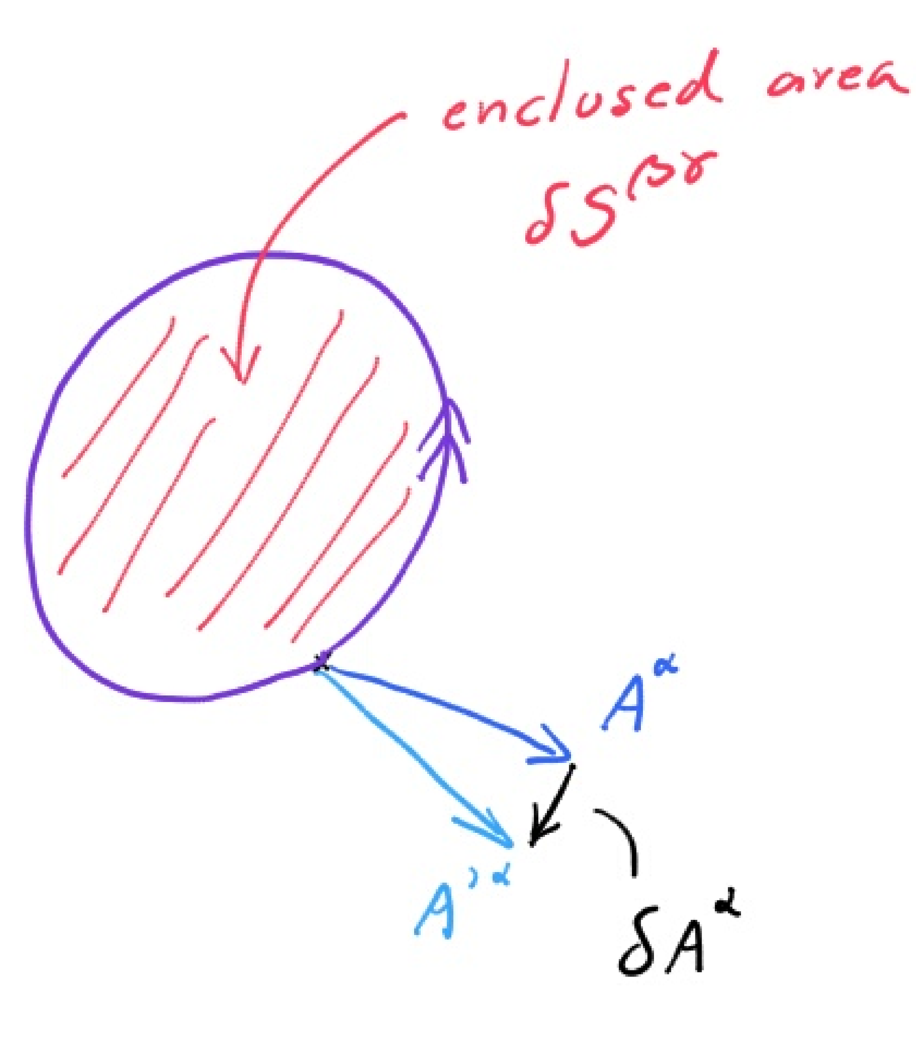
$$= \underbrace{\left(\partial_\gamma \Gamma^\alpha_{\ \beta\delta} - \partial_\beta \Gamma^\alpha_{\ \gamma\delta} + \Gamma^\alpha_{\ \mu\gamma} \Gamma^\mu_{\ \beta\delta} - \Gamma^\alpha_{\ \mu\beta} \Gamma^\mu_{\ \gamma\delta} \right)}_{R^\alpha_{\ \delta\gamma\beta}} B^\delta$$

So what looks like a second derivative operator acting on a field, actually just is a rank 4 tensor multiplying the vector at a point!

This is the Riemann tensor!

Hence we have shown:

$$\boxed{\delta A^\alpha = - R^\alpha_{\ \delta\beta\gamma}\, \delta S^{\beta\gamma} A^\delta}$$



$\boxed{?}$ Note that the area does not have to be rectangular.