Lecture 17

Einstein's cosmology :

* homogeneous
* closed (finite volume)
* static

— inconsistent with GR !

Adding a cosmological constant term

1929 : Hubble discovers that universe is expanding !

Assumptions : homogeneity
isotropy

$\Rightarrow$ RW line element

- meaning of $a(t)$
- redshift z
- Hubble "constant"
- Hubble's law

Matter content :

* galaxies — dust
* radiation
* vacuum energy

- how the energy changes during expansion

The Friedman equation, general k

- evaluated today $\Rightarrow$ H_0, ρ_0 determine geometry

The Friedman equation, $k = 0$

- matter
- radiation
- vacuum
- general case

Cosmology

The solutions to the field equation of general relativity are spacetimes, that is, spacetime geometries. Thus, it makes sense to apply the theory to the whole universe — GR is the foundation of modern cosmology. It should tell how the geometry of the universe depends on its content, and also, for given kinds of matter content, how the universe will evolve.

Soon after Einstein had completed the theory he thus wanted to apply it to the universe as a whole. Now, at this time, very little was known about the universe compared to what we know today. For example, only the stars within our own galaxy were known, and it was debated whether anything existed beyond Milky Way. Diffuse objects in the night sky that we today know are other galaxies, then were thought just to be gas clouds within our galaxy.

In order to apply his theory to the universe as a whole Einstein had to make some guesses. He was pretty sure about three things:

Einstein's cosmological assumptions:

- homogeneity, isotropy
- finite volume (closed)
- static

on a cosmological scale, all points are equal

all directions are equal

together with the previous assumption this means that space is a 3-sphere

universe is eternal

Unfortunately, Einstein soon found that his theory does not allow any solution of this kind! Not even if the finite volume assumption is given up. This made him modify his field equation, by adding a new term:

Modified theory: $$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G\, T_{\alpha\beta}$$

Cosmological constant

This new theory admitted a solution of the kind Einstein preferred: a closed static universe. However, it was later shown that this solution was unstable, and therefore could not account for our universe anyway.

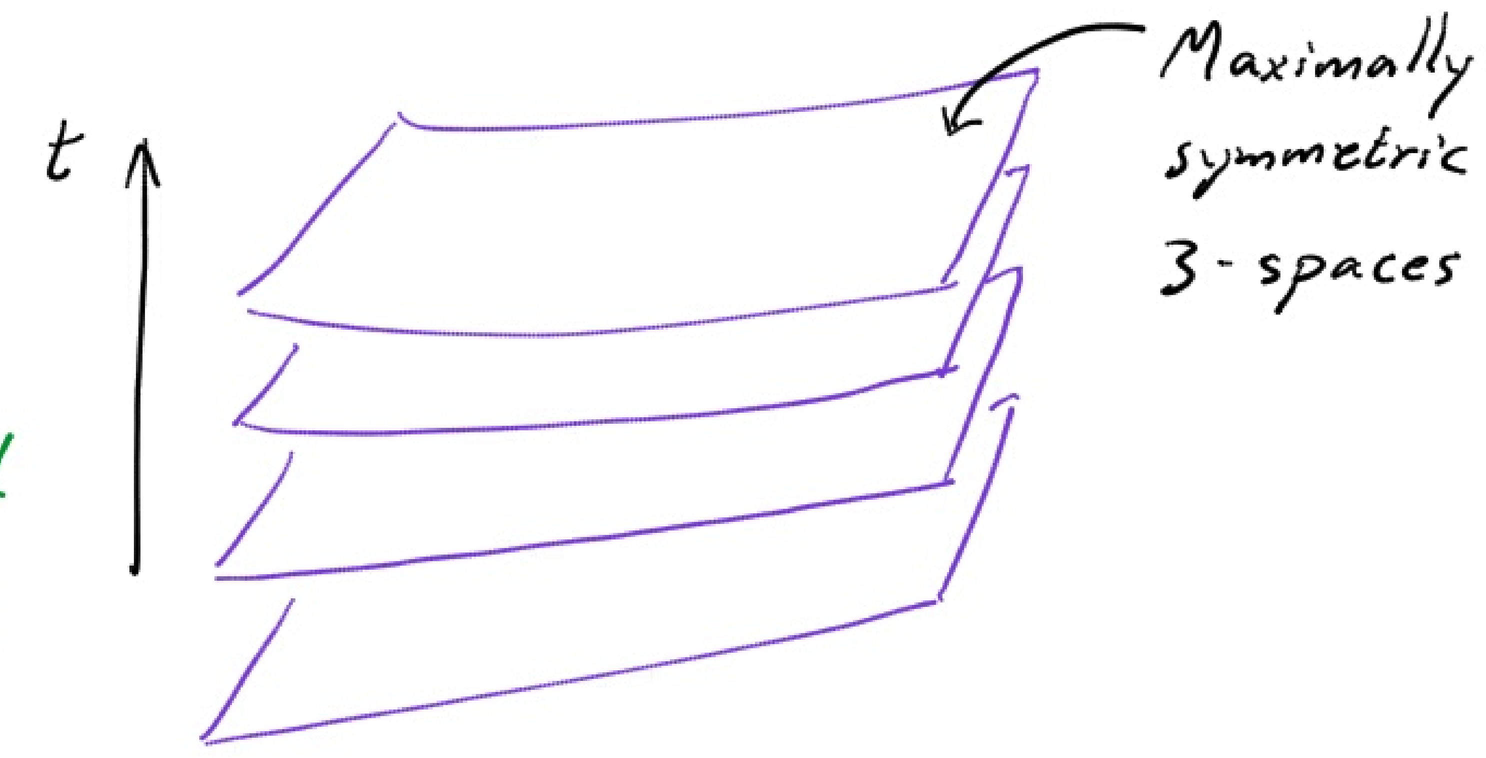
One decade later, 1929, Hubble established that the universe actually is expanding. If Einstein would have felt more confident in his original equations, he could have predicted this.

In fact, this is today counted as one of the most remarkable predictions of general relativity — that our universe must either be expanding or contracting, and that, if it is expanding it must have had some "beginning": our universe has finite age.

In modern cosmology only the first of Einstein's assumptions is kept: homogeneity and isotropy. Let us see what this means in terms of the line element.

First, it is important to realize that these assumptions are assumptions about space, not spacetime. Thus they presume a foliation of spacetime into a set of spacelike 3-hypersurfaces.

The assumption about homogeneity and isotropy means that each of these 3-spaces are homogeneous and isotropic, or in other words, that they are maximally symmetric.



There are only three possibilities:

Maximally symmetric 3-spaces:

The sphere — constant positive curvature (closed)

$$d\ell^2 = d\chi^2 + \sin^2\chi \, d\Omega^2$$

$0 \leq \chi \leq \pi$

Flat space — zero curvature (flat)

$$d\ell^2 = d\chi^2 + \chi^2 d\Omega^2$$

?

Hyperbolic space — constant negative curvature (open)

$$d\ell^2 = d\chi^2 + \sinh^2\chi \, d\Omega^2$$

By means of coord. changes we can write these line elements in unified form:

$$\begin{bmatrix} r = \sin\chi & \text{(closed)} \\ r = \chi & \text{(flat)} \\ r = \sinh\chi & \text{(open)} \end{bmatrix} \Rightarrow d\ell^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \quad \text{where} \quad \begin{bmatrix} k = +1 \\ k = 0 \\ k = -1 \end{bmatrix}$$

A spacetime foliated by one of these three kinds of 3-spaces, has to have this form:

$$ds^2 = -dt^2 + a^2(t)\, d\ell^2$$ — Robertson-Walker metrics

where $d\ell^2$ is one of these three line elements.

Why? Because

(1) we cannot have any more spatial dependence, since that would ruin the homogeneity and isotropy.

(2) a function of t in front of dt^2 is possible, but unnecessary, since it could always be absorbed into a new time coordinate.

Since this is our model for the universe, we shall imagine that it is filled with galaxies, evenly spread out. But each galaxy (at least on average) must stay on the same coord. position:

$$\left.\frac{dx^i}{dt}\right|_{\text{galaxy}} = 0 \iff d\ell\Big|_{\text{galaxy}} = 0$$ — comoving coordinates

Otherwise there would be a preferred direction, and the assumption of isotropy would not hold. Therefore these coordinates are called comoving. Thus the lines $d\ell = 0$ are geodesics, and the coordinate time t is just the proper time for the galaxies.

Clearly, the whole cosmology is determined by one single function: $a(t)$. This function gives the distance between the galaxies at a given time, when multiplied by the coord. distance:

$$d(t) = a(t)\, d_{\text{coord.}}$$

If $a(t)$ is increasing, the distance between galaxies increases.

The function $a(t)$ of course is determined by Einstein's equation, as soon as we know what kind of matter to put into our cosmological model.

We will soon return to that, but first we will see how $a(t)$ also is related to the redshift of light from distant galaxies.

Redshift

Light travelling to us from a distant source is moving on a lightlike trajectory.

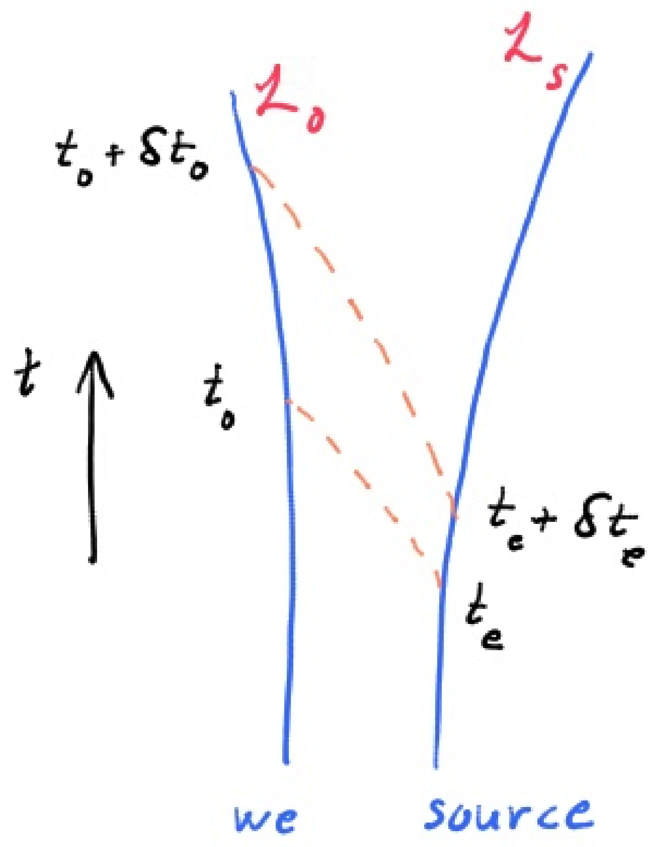
light path: $ds = 0 \implies dt = a(t)\, d\ell$

coordinate distance to source: $\ell = \int d\ell = \int_{t_e}^{t_0} \frac{dt}{a(t)}$

But since both the source and we ourselves stays at the same coordinate values, this coordinate distance is independent of time.

Thus:

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_{t_e}^{t_0} \frac{dt}{a(t)} + \int_{t_0}^{\delta t_0} \frac{dt}{a(t)} - \int_{t_e}^{\delta t_e} \frac{dt}{a(t)}$$



$$\implies \int_{t_0}^{\delta t_0} \frac{dt}{a(t)} = \int_{t_e}^{\delta t_e} \frac{dt}{a(t)} \quad \text{or} \quad \frac{\delta t_0}{a(t_0)} = \frac{\delta t_e}{a(t_e)}$$

Or in terms of frequency $\omega = \frac{2\pi}{\delta t}$:

$$\frac{\omega_0}{\omega_e} = \frac{a(t_e)}{a(t_0)} < 1 \quad \text{if universe is expanding}$$

— cosmological redshift!

Light is streched out by the same factor that universe expands during its passage from the source to us.

Conventionally one defines the redshift by

$$\text{Redshift } z = \frac{\omega_e - \omega_0}{\omega_0} = \frac{a(t_0) - a(t_e)}{a(t_e)}$$

For small distances d, we can then relate z to the distance to the source:

d small $\implies t_0 \approx t_e + d$

$$z \approx \frac{a(t_0) - a(t_0 - d)}{a(t_0 - d)} = \underbrace{\frac{a(t_0) - a(t_0-d)}{d}}_{\approx\, \dot a(t_0)} \frac{d}{a(t_0 - d)} \approx \underbrace{\frac{\dot a(t_0)}{a(t_0)}}_{H_0} d$$

H_0 — the Hubble constant

value today

$$\boxed{\text{Hubble's law: } z \approx H_0 d}$$

Note that this "law" only holds for small d, and that the Hubble "constant" is not a constant. But it is something, at least, that can be measured.

Mass / energy content of the universe

The energy that we have to use in our cosmological model of the universe is of three kinds:

- dust (= galaxies)
- radiation (= light, relativistic particles)
- vacuum energy

perfect fluids

All of these are special cases of a perfect fluid, that is, their stress-energy is of the form

$$T_{\alpha\beta} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$$

They differ in their relation between density and pressure, that is, they have different equations of state: Dust has $p = 0$, radiation has $p = \frac{\rho}{3}$ and vacuum energy has $p = -\rho$.

However, it turns out that the only thing we have to know about them is how their energy density scales when the universe expands. And that is easy.

Dust: As the universe expands, the comoving galaxes just get less and less dense, that is:

$$\rho_m(t) = \rho_m(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3$$

"matter"

Radiation: Think of the radiation as a soup of photons. Because of the expansion they are diluted by the same power of 3 as for dust. But each photon also looses energy because it is redshifted – streched out. Therefore:

$$\rho_r(t) = \rho_r(t_0)\left(\frac{a(t_0)}{a(t)}\right)^4$$

Vacuum: The energy density of the vacuum is <u>not</u> diluted as the expansion proceeds. Instead we get more and more energy as there is more space.

$$\rho_v = \text{const.}$$

This is what is called "dark energy".

That we have to add such a strange form of energy to our cosmological model to get a reasonable fit to the data, is seen by some people as a sign that there is some serious error in general relativity. Others hope that it will somehow be explained by a proper theory of quantum gravity. We don't know, but we have to include it.

Dynamics of the universe

Now we have all ingredients we need in order to see what Einstein's equation can tell us about the evolution of the universe.

$$\left.\begin{array}{l}\text{RW-line element}\\ \text{Perfect fluid } T_{\alpha\beta}\end{array}\right\} \xrightarrow[\text{put into}]{} \text{Einstein's equation}$$

$$\underset{tt\text{-comp.}}{\Longrightarrow} \quad \boxed{\dot{a}^2 - \frac{8\pi\rho}{3} a^2 = -k} \qquad \text{Friedman equation}$$

Before we solve it in some special cases, let us divide by a^2 and evaluate it today, that is, for $t = t_0$:

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 - \frac{8\pi\rho_0}{3} = -\frac{k}{a_0^2}$$

or, remembering that $H_0 = \dfrac{\dot{a}_0}{a_0}$:

$$H_0^2 - \frac{8\pi\rho_0}{3} = -\frac{k}{a_0^2}$$

H_0 can be measured, and ρ_0 can at least in principle be measured. Their relative values clearly says something about the kind of geometry:

$$\rho_0 > \frac{3H_0^2}{8\pi} \quad \Longrightarrow \quad k = +1 \quad \text{— closed}$$

$$\rho_0 = \frac{3H_0^2}{8\pi} \quad \Longrightarrow \quad k = 0 \quad \text{— flat}$$

$$\rho_0 < \frac{3H_0^2}{8\pi} \quad \Longrightarrow \quad k = -1 \quad \text{— open}$$

Hence, if we knew the total energy density (all kinds included) we could deduce the geometry of the universe!

Now, in practice it is easier to use these relations the other way around: by measuring the spatial geometry we get an estimate of the total energy density. (This is one of many ways in which we have evidence both for dark matter and dark energy.)

Let us return to the Friedman equation, and for simplicity let us consider only the flat case, $k=0$ (this is also the empirically relevant case):

$$\dot{a}^2 - \frac{8\pi\rho}{3} a^2 = 0$$

In order to solve this we just have to insert how ρ scales with a. If we first consider a universe with galaxies only, then

$$\text{matter}: \quad \rho(t) = \frac{c_1}{a^3} \quad \Rightarrow \quad \dot{a}^2 - \frac{8\pi c_1}{3a} = 0$$

This can be integrated:

$$\frac{da}{dt} = \left(\frac{8\pi c_1}{3a}\right)^{1/2} \qquad \int dt = \left(\frac{3}{8\pi c_1}\right)^{1/2} \int a^{1/2}\, da$$

$$t = \left(\frac{3}{8\pi c_1}\right)^{1/2} \frac{a^{3/2}}{3/2} \quad \Rightarrow \quad a_m \sim t^{2/3}$$

Similarly, for radiation only and vacuum energy only, we get

$$\text{radiation}: \quad a_r \sim t^{1/2}$$

$$\text{vacuum}: \quad a_v \sim e^{Ht}$$

For matter and radiation $a(0)=0$ — that is, we have a Big Bang singularity at $t=0$! Then the universe continues to expand for ever, but the speed of the expansion goes to zero. Not so in the vacuum energy case! Here the expansion is speeding up exponentially!

In practice we have a mixture of these behaviours: at early times radiation dominates the energy content, then matter becomes the dominant part, but in the end the vacuum energy completely takes over.