Lecture 7

The effective potential for the Schwarzschild spacetime

- circular orbits
- ISCO
- radial plunge orbits
- escape speed

Light ray orbits

- The effective potential for light
- The closed null orbit at $r = 3M$

Last time, we started out from the Schwarzschild geometry

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

and derived the relativistic counterpart to the effective potential in Newtonian gravity.

We made use of the two conserved quantities

energy $\quad e = -\bar{\xi} \cdot \bar{u} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \qquad (1)$

angular momentum $\quad \ell = \bar{\eta} \cdot \bar{u} = r^2 \sin^2\theta \, \frac{d\varphi}{d\tau} \overset{\theta = \frac{\pi}{2}}{=} r^2 \frac{d\varphi}{d\tau} \qquad (2)$

together with the requirement

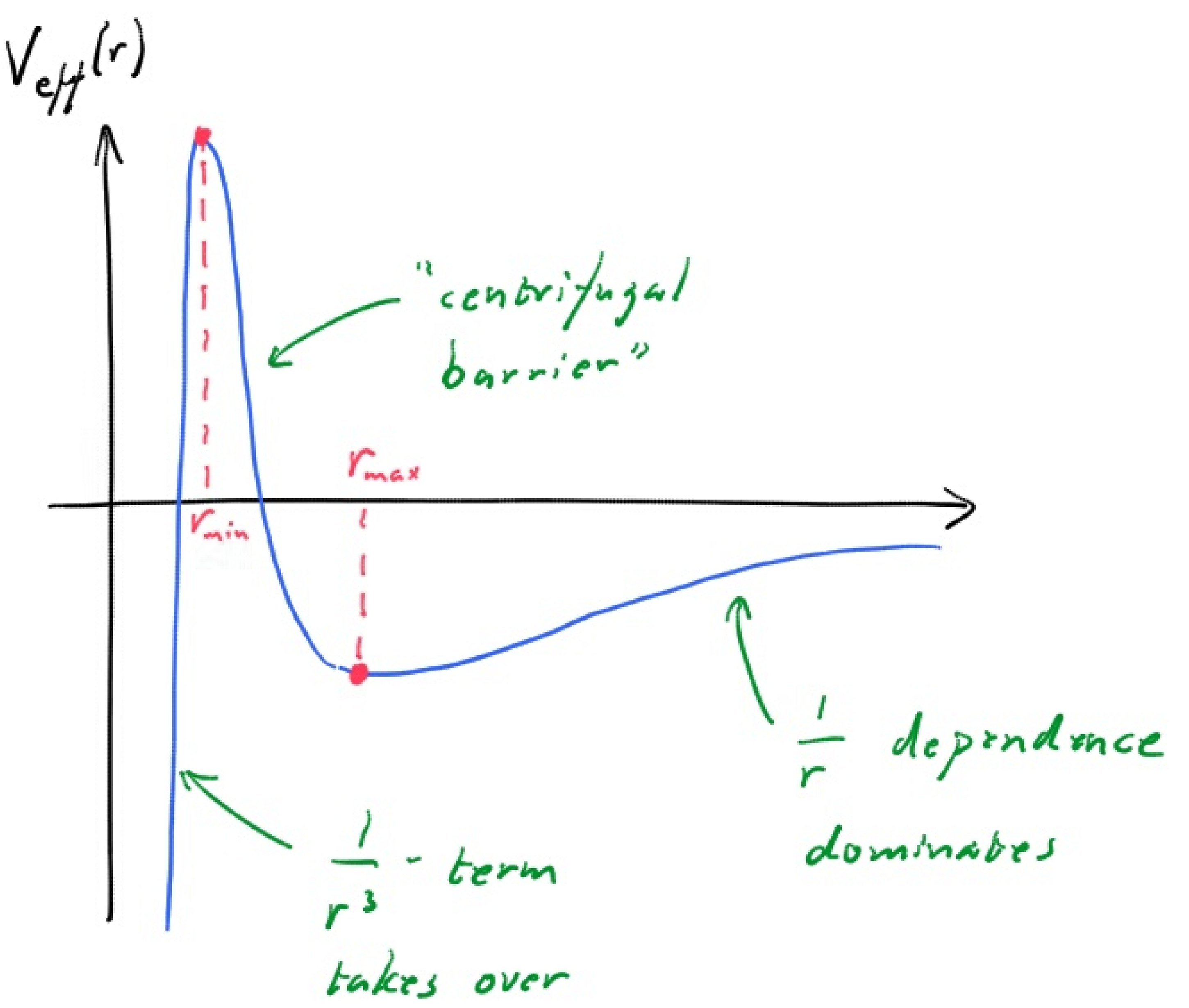
$\bar{u} \cdot \bar{u} = -1 \qquad (3)$

Then we could derive

$$\frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad (4)$$

where

$$\boxed{V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}}$$



Circular orbits

There are two circular orbits, one stable at r_{max} and one unstable at r_{min}. Where are they, and do they always exist?

$$\frac{dV_{eff}(r)}{dr} = \frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3M\ell^2}{r^4} \overset{!}{=} 0$$

$$\Rightarrow \quad Mr^2 - \ell^2 r + 3M\ell^2 = 0$$

$$r^2 - \frac{\ell^2}{M} r + 3\ell^2 = 0$$

$$r_{\substack{max \\ min}} = \frac{\ell^2}{2M} \pm \sqrt{\left(\frac{\ell^2}{2M}\right)^2 - 3\ell^2} =$$

$$= \frac{\ell^2}{2M}\left[1 \pm \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\right]$$

? So the circular orbits only exist if $\frac{\ell}{M} > \sqrt{12}$. That is, the angular momentum has to be large enough — otherwise the particle will just plunge into the black hole.

The Innermost Stable Circular Orbit (ISCO)

The stable orbit (r_{max}) gets smaller as ℓ decreases (as in the Newtonian case). The smallest value of r_{max} is thus obtained for the smallest value of $\frac{\ell}{M}$, which is

$$\frac{\ell}{M} = \sqrt{12} \implies r_{max} = r_{min} = 6M$$

For ordinary stars this is well inside the surface of the star, but for understanding accretion discs around neutron stars and black holes this ISCO is important.

Now, let us consider another kind of orbit:

Radial plunge orbits

$\ell = 0$

starting at rest at ∞ : $e = 1$ $\boxed{?}$

$$(*) \implies 0 = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r}$$

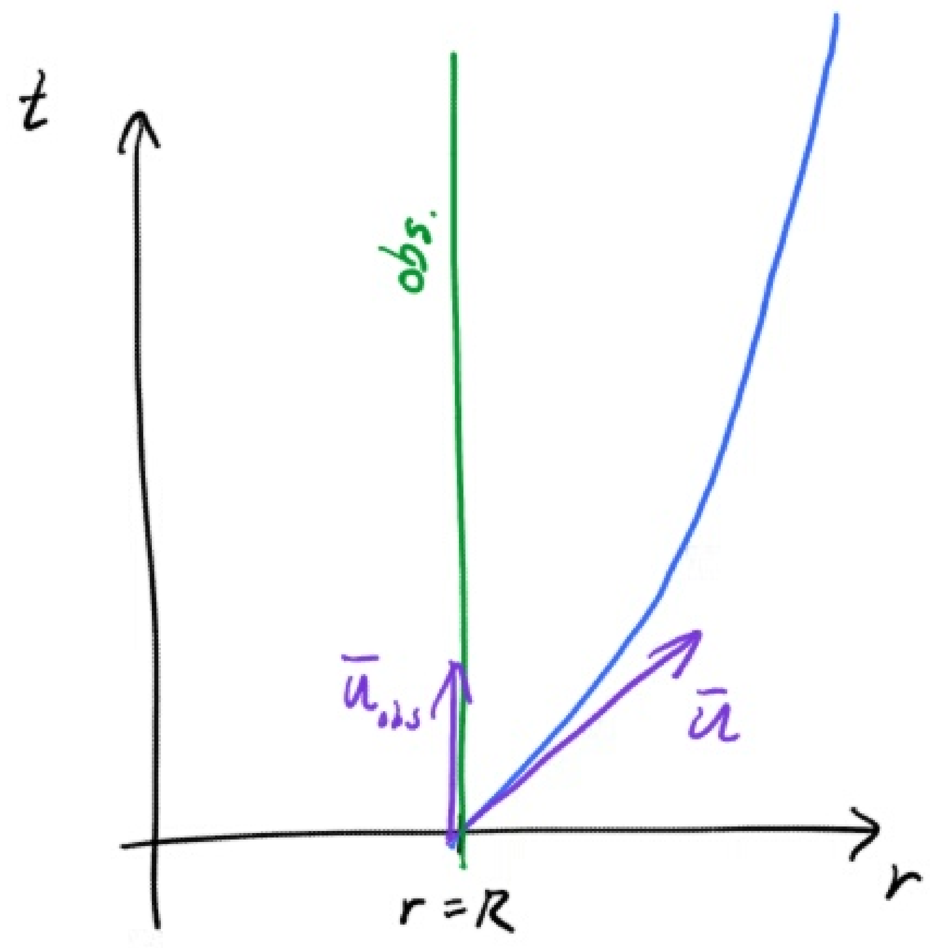
So $$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}} \qquad (5)$$

$$\frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad (6)$$

$$\implies u^\alpha = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\sqrt{\frac{2M}{r}}, 0, 0\right)$$

From (5) and (6) we could easily get the general shape of the radial plunge orbit, or $r(\tau)$. But let us do something else: let us find the escape speed from a radius R.

The escape speed from radius R



A stationary observer at radius R launches an object radially outwards. If the object's speed should be such that it just is able to leave "the gravitational field" — that is, come to rest at infinity — the 4-velocity at R must be

$$u^\alpha = \left(\left(1 - \frac{2M}{R}\right)^{-1}, \sqrt{\frac{2M}{R}}, 0, 0 \right)$$

which means 4-momentum

$$p^\alpha = m u^\alpha$$

What energy does this correspond to for the stationary observer at R?

$$E = -\bar{u}_{obs} \cdot \bar{p} = -g_{\alpha\beta} u^\alpha_{obs} p^\beta = -g_{tt} u^t_{obs} p^t = +\left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-1/2} m \left(1 - \frac{2M}{R}\right)^{-1} =$$

u^α_{obs} has only a time-component

$$= \frac{m}{\sqrt{1 - \frac{2M}{R}}}$$

What speed has an object with this energy?

$$v = \sqrt{\frac{2M}{R}}$$ or, in ordinary units, $v = \sqrt{\frac{2MG}{R}}$

This, as you may remember, is the same expression for the escape speed as in Newtonian gravity. But note that its meaning is not exactly the same, as R is the Schwarzschild coordinate, not the true distance.

Note also that as R approaches $2M$ the escape speed becomes 1.

Light ray orbits

The effective potential in (4) does not hold for lightlike trajectories. The reason is that the proper time along any lightlike line is zero. This means that we cannot parametrize lightlike geodesics in terms of proper time. This has two consequenses for our derivation of (4).

1) The tangent vector cannot be defined in terms of proper time, but must be defined in terms of an affine parameter λ:

$$u^\alpha = \frac{dx^\alpha}{d\lambda}$$

Defined so that a null geodesic will satisfy the geodesic equation. Only defined up to a constant scale factor.

(1) and (2) thus must be written

$$e = -\bar{\xi}\cdot\bar{u} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \qquad (1')$$

$$\ell = \bar{\eta}\cdot\bar{u} = r^2\frac{d\varphi}{d\lambda} \qquad (2')$$

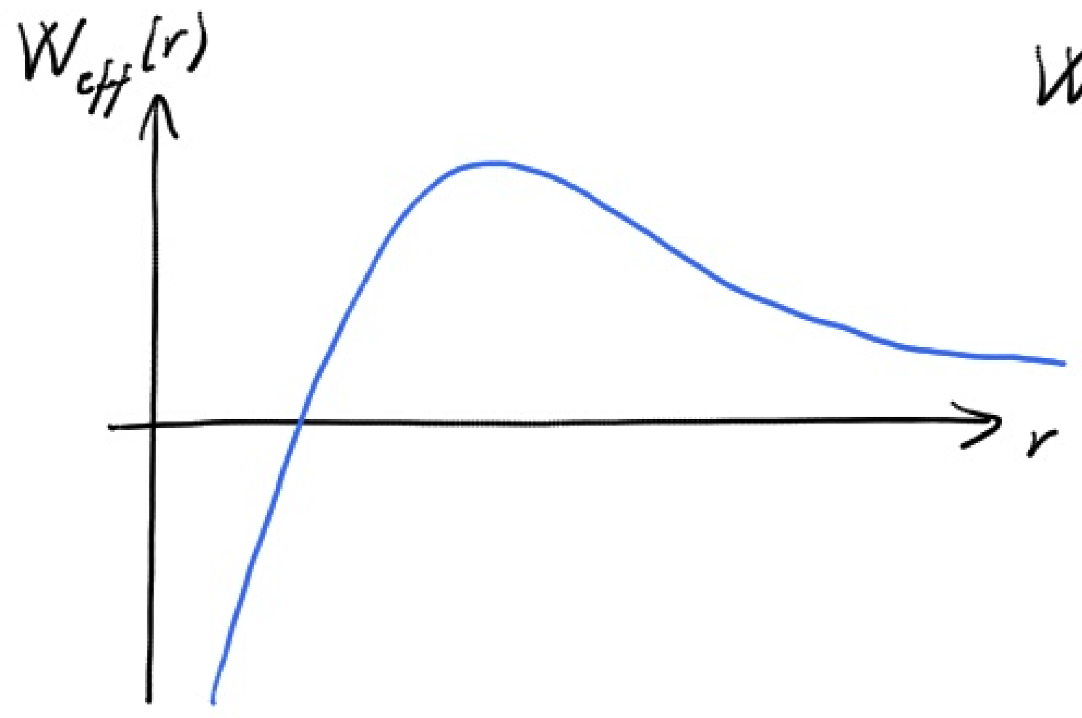
$\boxed{?}$ 2) The normalization condition (3) must be replaced by

$$\bar{u}\cdot\bar{u} = 0 \qquad (3')$$

If (3') is written out in the components of $\bar{u}$, and if (1') and (2') is used to eliminate the t and φ dependece, one obtains the result (replacing (4)):

$$\Rightarrow \frac{e^2}{\ell^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r) \qquad (4')$$

where $W_{eff}(r) = \frac{1}{r^2}\left(1-\frac{2M}{r}\right)$



$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

Remarks:

1) Since the ℓ in front of $\left(\frac{dr}{d\lambda}\right)^2$ in (4') can be absorbed into a rescaling of the affine parameter λ, the orbits only depend on one parameter: $\frac{e}{\ell}$ (Hartle shows that this is 1 over the impact parameter at infinity.)

2) There is a centrifugal barrier. But there are no stable orbits.

3) There is one unstable circular orbit, given by

$$\frac{dW_{eff}(r)}{dr} = 0 \quad \Rightarrow \quad r = 3M$$