FYS3120 Classical mechanics and electrodynamics
Mid-term exam – Spring term 2019

March 28, 2019

Important information:

- Your answers are to be submitted electronically as pdf-files, either generated from LaTeX or scanned, at the latest Friday 29th of March at 16.00 local time (GMT+1).
- This deadline is absolute.
- This mid-term exam counts for roughly 25% of the total grade in FYS3120, and you must receive a passing score on the mid-term in order to pass the course.
- As this is a home-exam you are free to use any sources of information you may want, and you may collaborate with other students on solving the problems. However, the text of the submitted answers must be your own, and the usual rules of plagiarism apply. (We may check answers for similarities.)
- The best possible score on this exam is 25 points. Up to one point will be given for clear, concise and well presented answers, including appropriate figures and/or diagrams.
- You may give your answers either in English or Norwegian.
- Good luck!

Question 1 Central potentials

Consider two objects of mass m_1 and m_2 affected by a time-independent central potential $V(r)$ in regular three-dimensional space, where $\vec{r}$ gives the distance between the objects $\vec{r} = \vec{r}_1 - \vec{r}_2$.

a) Find the number of degrees of freedom, and identify appropriate generalized coordinates. [1 point]

- Finding the number of D.O.F:

  The system consists of two bodies, $N = 2$, and no constraints, $M = 0$. Which means that $d = 3N - M = 6$.

  The components of $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{R} = \frac{\mu}{m_2}\vec{r}_1 + \frac{\mu}{m_1}\vec{r}_2$, where $\mu = \frac{m_1 m_2}{m_1+m_2}$, are chosen as the generalized coordinates.

b) Explain why the motion of three of the generalized coordinates can be solved trivially, and write down the solution. [1 point]

- The Lagrangian expressed in terms of $\vec{r}$ and $\vec{R}$ is:

$$L = \frac{1}{2}(m_1 + m_2)\vec{R} + \mu\vec{r} - V(r) \tag{1}$$

- Finding the equation of motions:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{R}_i} - \frac{\partial L}{\partial R_i} = (m_1 + m_2)\ddot{R}_i = 0 \tag{3}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i} = \mu\ddot{r}_i - \frac{\partial V}{\partial r_i} = 0 \implies \mu\ddot{r}_i = \frac{\partial V}{\partial r_i} \tag{4}$$

  (3) shows that the motion of the three components of $\vec{R}$ have trivial solutions: the center of mass moves at a constant speed.

c) Explain why the angular momentum of the reduced mass μ,

$$\vec{\ell} = \vec{r} \times \vec{p} = \vec{r} \times \mu\dot{\vec{r}}, \tag{5}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \tag{6}$$

is a constant of motion. [1 point]

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d) Explain why this leads to the motion being confined to the plane perpendicular to $\vec{\ell}$. [1 point]

$\vec{r} \times \mu\dot{\vec{r}} = 0$, means that either $\vec{r}$ or $\dot{\vec{r}}$ equals $\vec{0}$, or that the two vectors are parallel. As neither vectors are equal to $\vec{0}$, this means that neither vectors have

e) Choosing $\vec{\ell} = \ell\hat{k}$, *i.e.* that the angular momentum is in the z-direction, we can switch to the polar coordinates (r, ϕ), where ϕ is the angle in the xy-plane. Show that the equations of motion for these coordinates are

$$\mu\ddot{r} - \frac{\ell^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0, \tag{7}$$

and

$$\dot{\phi} = \frac{\ell}{\mu r^2}. \tag{8}$$

[3 points]

- As $\vec{R}$ has constant velocity (1.b), it may be chosen freely. Therefore, $\vec{R}$ is chosen to be $(0, 0, 0)$. Thus $L = \mu\vec{r} - V(r)$. As the motion is confined to a plane, polar coordinates may be used to be used to describe the system, with general coordinates: r, ϕ.

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$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r) \tag{9}$$

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \tag{10}$$

- $p_\phi = \frac{\partial L}{\partial\dot{\phi}} = \mu r^2\dot{\phi}$, and $p_\phi = \ell$, so $\dot{\phi} = \frac{\ell}{\mu r^2}$

- $$\frac{d}{dt}\frac{\partial L}{\partial\dot{r}} - \frac{\partial L}{\partial r} = 0 \tag{11}$$

  $$= \frac{d}{dt}(\mu\dot{r}) - (\mu r\dot{\phi}^2 - \frac{\partial V}{\partial r}) = \mu\ddot{r} - \mu r\dot{\phi}^2 + \frac{\partial V}{\partial r} \tag{12}$$

  $$= \mu\ddot{r} - \mu r\frac{\ell^2}{\mu^2 r^4} + \frac{\partial V}{\partial r} = \mu\ddot{r} - \frac{\ell^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0 \tag{13}$$

f) Find the corresponding Hamiltonian in terms of the generalized coordinates (r, ϕ) and generalized momenta (p_r, p_ϕ), and write down Hamilton's equations for this system. [3 points]

- $p_r = \frac{\partial L}{\partial\dot{r}} = \frac{\partial}{\partial t}\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) = \mu\dot{r}$

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$$H = \sum_i p_i \dot{q}_i - L = p_\phi \dot{\phi} + p_r \dot{r} - (\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)) \quad (14)$$

$$= p_\phi \dot{\phi} + p_r \dot{r} - \frac{1}{2} p_r \dot{r} - \frac{1}{2} p_\phi \dot{\phi} + V(r) \quad (15)$$

$$= \frac{1}{2} p_r \dot{r} + \frac{1}{2} p_\phi \dot{\phi} + V(r) \quad (16)$$

$$= \frac{1}{2}\frac{p_r^2}{\mu} + \frac{1}{2}\frac{p_\phi^2}{\mu r^2} + V(r) \quad (17)$$

g) Explain why the Hamiltonian H of this system is a constant of motion and what the physical interpretation of H is. [1 point]

Question 2 Orbital motion

Consider the gravitational potential between two objects of mass m_1 and m_2,

$$V(r) = -\frac{Gm_1 m_2}{r}, \quad (18)$$

where G is Newton's gravitational constant.

a) Find Hamilton's equations for this system. [1 point]

- A gravitational potential is a central potential, as such, H from question 1 may used (17), and the correct $V(r)$ applied:

$$H = \frac{1}{2}\frac{p_r^2}{\mu} + \frac{1}{2}\frac{p_\phi^2}{\mu r^2} - \frac{Gm_1 m_2}{r} \quad (19)$$

Where $\mu =$ and p_r and p_ϕ

b) Make a two-dimensional plot of the phase space for the distance r between the objects, and its generalized momentum p_r, for concreteness fixing the masses to be those of the International Space Station (ISS) orbiting the Earth, varying ℓ.

Give a qualitative description of the different types of motion that can be read out of the diagram and comment on how the situation changes with increasing ℓ. *Hint:* For those using `python` the plotting framework `pyplot` has a useful plotting command for phase space plots called `streamplot`. [4 points]

c) Use the Lagrange equations to show that the differential equation for the orbit equation $r(\phi)$, where ϕ is the polar angle of the motion, is

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - r + \frac{Gm_1 m_2 \mu}{\ell^2} r^2 = 0. \quad (20)$$

[2 points]

equation (10)

- $$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} - \frac{\partial L}{\partial \phi} = 0 \tag{21}$$
$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot\phi}\left(\frac{1}{2}\mu(\dot r^2 + r^2\dot\phi^2) + \frac{Gm_1m_2}{r}\right)\right) - \frac{\partial}{\partial \phi}\left(\frac{1}{2}\mu(\dot r^2 + r^2\dot\phi^2) + \frac{Gm_1m_2}{r}\right) = 0 \tag{22}$$
$$\frac{d}{dt}\left(\mu r^2\dot\phi\right) = 0 \tag{23}$$

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$$\frac{d\dot\phi}{dt} = \frac{d}{dt}\frac{\ell}{\mu r^2} = -2\frac{\ell}{\mu r^3}\frac{dr}{dt} = -2\frac{\ell}{\mu r^3}\frac{dr}{d\phi}\frac{d\phi}{dt} = -2\frac{\dot\phi^2}{r}\frac{dr}{d\phi} \tag{24}$$

$$\mu\ddot r - \frac{\ell^2}{\mu r^3} + \frac{\partial V}{\partial r} = 0 \tag{25}$$
$$\mu\frac{d}{dt}\frac{d}{dt}r - \frac{\ell^2}{\mu r^3} - \frac{\partial}{\partial r}\frac{Gm_1m_2}{r} = 0 \tag{26}$$
$$\mu\frac{d}{dt}\left(\frac{dr}{d\phi}\frac{d\phi}{dt}\right) - \frac{\ell^2}{\mu r} + \frac{Gm_1m_2}{r^2} = 0 \tag{27}$$
$$\mu\frac{d}{dt}\left(\frac{dr}{d\phi}\dot\phi\right) - \frac{\ell^2}{\mu r^3} + \frac{Gm_1m_2}{r^2} = 0 \tag{28}$$
$$\mu\dot\phi\frac{d}{dt}(\frac{dr}{d\phi}) + \mu\frac{dr}{d\phi}\frac{d\dot\phi}{dt} - \frac{\ell^2}{\mu r^3} + \frac{Gm_1m_2}{r^2} = 0 \tag{29}$$
$$\mu\dot\phi\frac{d\phi}{dt}\frac{d}{d\phi}(\frac{dr}{d\phi}) - 2\mu\frac{\dot\phi^2}{r}\left(\frac{dr}{d\phi}\right)^2 - \frac{\ell^2}{\mu r^3} + \frac{Gm_1m_2}{r^2} = 0 \tag{30}$$
$$\mu\dot\phi^2\frac{d^2r}{d\phi^2} - 2\mu\frac{\dot\phi^2}{r}\left(\frac{dr}{d\phi}\right)^2 - \frac{\ell^2}{\mu r^3} + \frac{Gm_1m_2}{r^2} = 0 \tag{31}$$
$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - \frac{1}{\mu}\frac{\mu^2r^4}{\ell^2}\frac{\ell^2}{\mu r^3} + \frac{1}{\mu}\frac{\mu^2r^4}{\ell^2}\frac{Gm_1m_2}{r^2} = 0 \tag{32}$$
$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - r + \frac{Gm_1m_2\mu}{\ell^2}r^2 = 0 \tag{33}$$
$$\tag{34}$$

d) Show that complete set of solutions for the orbit equation is

$$r(\phi) = \frac{r_0}{1 - \varepsilon\cos(\phi - \phi_0)}, \tag{35}$$

where ε is called the **eccentricity** and ϕ_0 the **phase**. Find r_0. *Hint:* The substitution $s = 1/r$ is very useful. [2 points]

$$\frac{ds}{d\phi} = \frac{ds}{dr}\frac{dr}{d\phi} = \frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} \implies \frac{dr}{d\phi} = -\frac{1}{s^2}\frac{ds}{d\phi} \tag{36}$$

Substituting into (20): **fjern første**

$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - r + \frac{Gm_1m_2\mu}{\ell^2}r^2 = 0 \tag{37}$$

$$\frac{d}{d\phi}\left(-\frac{1}{s^2}\frac{ds}{d\phi}\right) - 2s\left(-\frac{1}{s^2}\frac{ds}{d\phi}\right)^2 - \frac{1}{s} + \frac{Gm_1m_2\mu}{\ell^2s^2} = 0 \tag{38}$$

$$-\frac{d}{d\phi}\left(\frac{1}{s^2}\right)\frac{ds}{d\phi} - \frac{d}{d\phi}\left(\frac{ds}{d\phi}\right)\frac{1}{s^2} - \frac{2}{s^3}\left(\frac{ds}{d\phi}\right)^2 - \frac{1}{s} + \frac{Gm_1m_2\mu}{\ell^2s^2} = 0 \tag{39}$$

$$\frac{2}{s^3}\frac{ds}{d\phi}\frac{ds}{d\phi} - \frac{1}{s^2}\frac{d^2s}{d\phi} - \frac{2}{s^3}\left(\frac{ds}{d\phi}\right)^2 - \frac{1}{s} + \frac{Gm_1m_2\mu}{\ell^2s^2} = 0 \tag{40}$$

$$-\frac{1}{s^2}\frac{d^2s}{d\phi} - \frac{1}{s} + \frac{Gm_1m_2\mu}{\ell^2s^2} = 0 \tag{41}$$

$$\implies \frac{1}{s^2}\frac{d^2s}{d\phi} + \frac{1}{s} = \frac{Gm_1m_2\mu}{\ell^2s^2} \tag{42}$$

$$\implies \frac{d^2s}{d\phi} + s = \frac{Gm_1m_2\mu}{\ell^2} \tag{43}$$

which is a nonhomogeneous second order differential equation. The general solution to the homogeneous equation (harmonic oscillator equation):

$$\frac{d^2s}{d\phi} + s = 0 \tag{44}$$

is $s_1 = A\cos(\phi - \phi_0)$, where A is a constant, and ϕ_0 the phase.

A solution to the nonhomegenous equation is the constant solution: $s_0 = \frac{Gm_1m_2\mu}{\ell^2}$. Thus the solution of the differential equation may be written as:

$$s(\phi) = \frac{Gm_1m_2\mu}{\ell^2} + A\cos(\phi - \phi_0) = \frac{Gm_1m_2\mu}{\ell^2}\left(1 + \varepsilon\cos(\phi - \phi_0)\right) \tag{45}$$

The choice of A and ϕ_0 correspond to particular instances of the problem, such that A may be written as $-A$, thus:

$$s(\phi) = \frac{Gm_1m_2\mu}{\ell^2}\left(1 - \varepsilon\cos(\phi - \phi_0)\right) \tag{46}$$

Where $\epsilon = \frac{Al^2}{Gm_1m_2\mu}$

Solving for $r(\phi)$ by substituting in $r = \frac{1}{s}$ and rearranging:

$$r(\phi) = \frac{\frac{\ell^2}{Gm_1m_2\mu}}{1-\varepsilon\cos(\phi-\phi_0)} = \frac{r_0}{1-\varepsilon\cos(\phi-\phi_0)}$$

As shown above, $r_0 = \frac{\ell^2}{Gm_1m_2\mu}$, where ℓ is a constant of motion corresponding to a specific orbit.

e) Find an expression for the total energy in terms of the eccentricity ε. [2 points]

$$\begin{aligned}\dot{r}(\phi) &= \frac{d}{dt}\frac{r_0}{(1-\varepsilon\cos(\phi-\phi_0)} = -\frac{r_0(\varepsilon sin(\phi-\phi_0)}{(1-\varepsilon\cos(\phi-\phi_0)^2}\frac{d\phi}{dt} = -\frac{r_0(\varepsilon sin(\phi-\phi_0)}{(1-\varepsilon\cos(\phi-\phi_0)^2}\frac{\ell}{\mu(r(\phi))^2} \\ &= -\frac{r_0(\varepsilon sin(\phi-\phi_0)}{(1-\varepsilon\cos(\phi-\phi_0)^2}\frac{\ell}{\mu r_0^2}(1-\varepsilon\cos(\phi-\phi_0)^2 \\ &= -\frac{\varepsilon\ell}{\mu r_0}\sin(\phi-\phi_0)\end{aligned}$$

As $\frac{dH}{dt} = \frac{dE_{tot}}{dt} = 0$, finding the total energy, E_{tot}, may be done at an arbitrary angle ϕ. Using (19) and substituting $p_r = \mu\dot{r}(\phi)$ and $p_\phi = \ell$, yields $H = \frac{1}{2}\mu(\dot{r}(\phi))^2 + \frac{1}{2}\frac{\ell^2}{\mu(r(\phi))^2} - \frac{Gm_1m_2}{r} = E_{tot}$, may therefore be calculated at $\phi = \phi_0$, where $r(\phi_0) = \frac{\ell^2}{\eta\mu(1-\varepsilon)}$, where $\nu = Gm_1m_2$, and $\dot{r}(\phi_0) = -\frac{\varepsilon\ell}{\mu r_0}\sin(\phi_0-\phi_0) = 0$. So;

$$\begin{aligned}H &= \frac{1}{2}\frac{\ell^2}{\mu(r(\phi_0))^2} - \frac{Gm_1m_2}{r(\phi)} \\ &= \frac{1}{2}\frac{\eta^2\mu(1-\varepsilon)^2}{\ell^2} - \frac{\eta^2\mu(1-\varepsilon)}{\ell^2} = \frac{\eta^2\mu}{\ell^2}\left(\frac{(1-\varepsilon)^2}{2} - 1 + \varepsilon\right) = \frac{\eta^2\mu}{\ell^2}(\frac{\varepsilon^2}{2} - \frac{1}{2}) \\ &= \frac{G^2m_1^2m_2^2\mu}{2\ell}(\varepsilon^2-1)\end{aligned}$$

Which means that $E_{tot}(\varepsilon) = \frac{Gm_1m_2}{2r_0}(\epsilon^2-1)$

f) If an astronaut jumped off the ISS directly towards Earth, what would happen to her orbit? Assume, for simplicity, that the ISS is in a circular orbit. [2 points]

$E_0(\epsilon = 0) + \frac{1}{2}\mu\dot{r}^2 = E(\epsilon_{new})$ eller $H(p_r, p_\phi) = E(\epsilon) =??? = \ell = \mu_A r_0 V_{ISS}.\vec{\ell} = \mu\vec{r}\times\dot{\vec{r}}$

linspace meshgrid - use solution of hamiltons equation put into stream plot. google what streamplot has as inpt9999