FYS3120 Classical Mechanics and Electrodynamics

Problem set 1

January 27, 2019

Problem 1 Figure 1 shows four different mechanical systems, in **a)** a pendulum attached to a block which in turn is attached to a spring, **b)** a pendulum which is attached to a vertical ring which rotates with a fixed frequency ω, **c)** a straight rod which can tilt without sliding on the top of cylinder, while the cylinder can roll on a horizontal plane (no slipping), and **d)** a spinning top which moves on a horizontal floor.

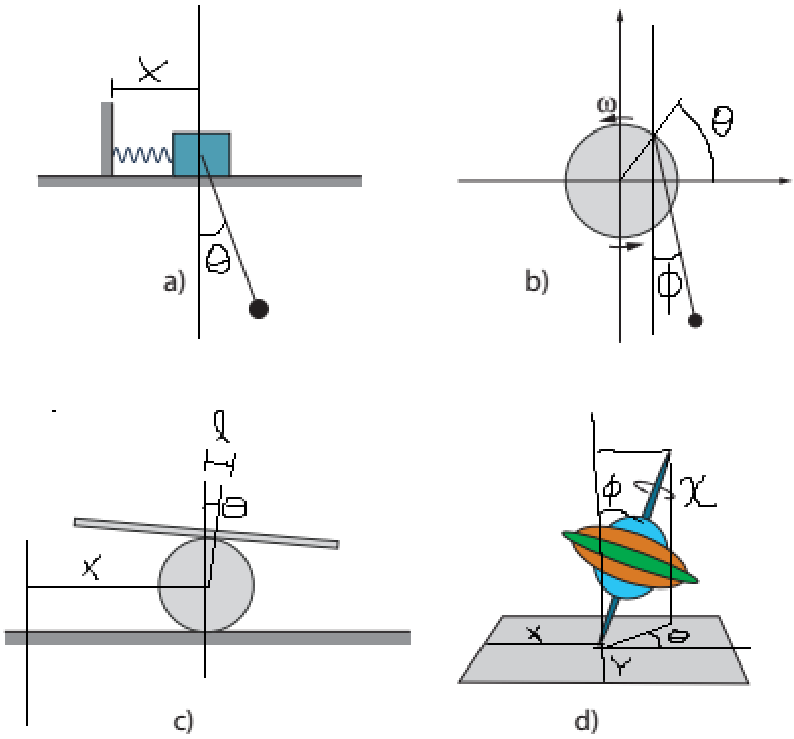

Figure 1: Four mechanical systems.

In all cases specify the number of degrees of freedom and choose an appropriate set of generalized coordinates.

Generalized coordinates are drawn onto the the system representations in figure 1.

- 1.a) $d = 2$ degrees of freedom. Generalized coordinates, q_j: $q_1 = X$ - horizontal position of block. $q_2 = \theta$ - angle from vertical axis to pendulum.
- 1.b) $d = 2$. $q_1 = \theta$ - angle from centre of wheel to string fixation point. $q_2 = \phi$ - angle between vertical axis and string.

- 1.c) $d = 3$. $q_1 = X$ - horizontal position of center of wheel, $q_2 = \theta$ - angle between vertical axis and line from center of wheel to point of contact, $q_3 = l$ - distance from point of contact to mass center of rod.

- 1.d) $d = 5$ - $q_1 = X$ - x component of point of contact with surface. $q_2 = Y$ - y component of point of ontact with sufrace. $q_3 = \theta$ - angle between projection of spinning top onto xy-plane and x-axis. $q_4 = \phi$ - angle between projection of spinning top onto xz-plane and z-axis. $q_5 = \chi$ - rotation about own mass center.

Problem 2 An Atwood machine consists of three parts, with masses $m_1 = 4m$, $m_2 = 2m$ and $m_3 = m$, that move vertically, and two rotating pulleys, which we treat as massless. The fixed lengths of the ropes, which we also consider as massless, are l_1 and l_2. We show the set-up in Fig. 2.

Explain why the number of degrees of freedom of the system is two and choose a corresponding set of generalized coordinates. Find the potential and kinetic energies of the system expressed as functions of the generalized coordinates and their time derivatives. Write down the Lagrangian of the system.

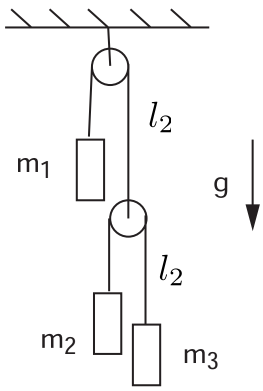

Figure 2: An Atwood machine with three weights.

Consider first a an Atwood machine with two parts; m_1 and m_2, moving vertically on a massless pulley, with a rope of fixed length, l, connecting the masses. Each mass has one degree of freedom (vertical movement along the y-axis), while the system set up imposes one constraint equation: $y_1 + y_2 = l$. Thus, a two body Atwood machine has $d = 1$ degree of freedom.

A compound Atwood machine of three masses, such as the one in figure 2, may be thought of as two separate two body Atwood machines, similar to the one described above. One Atwood machine with two parts; the part with mass m_1, and the part composed of the pulley to which m_2 and m_3 are connected, pulley 2, with mass $m_2 + m_3$. The other Atwood machine is then made up of the parts with

m_2 and m_3. Each of the two Atwood machines has one degree of freedom, which means a total of $d = 2$ degrees of freedom.

$d = 2 \implies q = 2$ general coordinates: the vertical distances Y_1 and Y_2, where Y_1 is y coordinate of pulley 2, and Y_2 is the distance from pulley 2 to the mass m_2. So;

$$\begin{aligned} y_1 &= Y_1 - l_1 \\ y_3 &= Y_2 - l_2 \\ y_2 &= Y_2 \end{aligned}$$

Thus

$$\begin{aligned} T = \frac{1}{2}m_1(\dot{Y_1})^2 + \frac{1}{2}m_2(\dot{Y_2})^2 + \frac{1}{2}m_3(\dot{Y_2})^2 &= \frac{1}{2}m_1(\dot{Y_1})^2 + \frac{1}{2}(m_2 + m_3)(\dot{Y_2})^2 \\ T &= 2m(\dot{Y_1})^2 + \frac{3}{2}m(\dot{Y_2})^2 = \frac{7}{2}m(\dot{Y_1}^2 + \dot{Y_2}^2) \\ V &= m_1 g(Y_1 - l_1) + m_2 g(Y_2 - l_2) + m_3 g Y_2 \\ V &= 4mgY_1 + 3mgY_2 - 4mgl_1 - 2mgl_2 \end{aligned}$$

Such that $L = T - V = \frac{7}{2}m(\dot{Y_1}^2 + \dot{Y_2}^2) - 4mgY_1 - 3mgY_2 + 4mgl_1 + 2mgl_2$

Problem 3 Three identicals rods of mass m and length l are connected by frictionless joints and suspended, with the distance between the points of suspension being equal to the length of the rods, as shown in Fig. 3. The rods move in the plane. Explain why the system has only one degree of freedom, and choose the angle θ as generalized coordinate.

- The movement of the system is restricted to one plane, so each system part may have up to two d.o.f. If each rod is considered to be made up of two end points, the system consists of 6 parts. Two of those parts are fixed to a surface, so that their contribution to the number of d.o.f is zero. Of the remaining 4 parts, 2 parts belong the horizontal rod, and one to each of the two vertical rods. Each of the end points of the horizontal rod is connected to one of the vertical rods, thus effectively reducing the number of parts to two, which means 4 d.o.f. Lastly, these two parts have 3 restrictions: fixed distance to each other, as well as fixed distance to the hinges fastened to the surface. Thus, the total number of d.o.f is 1.

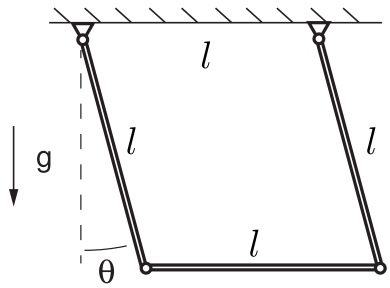

Figure 3: System of moving suspended rods.

Show that the Lagrangian, defined as the difference between the kinetic and potential energy, $L = K - V$, gets the following form as a function of θ and $\dot{\theta}$,

$$L(\theta, \dot{\theta}) = \frac{5}{6} m l^2 \dot{\theta}^2 + 2mgl\cos\theta. \tag{1}$$

We remind you that the expression for the moment of of inertia of a rod about its endpoint is $I = \frac{1}{3} m l^2$.

- Calculating $K = 2K_{\text{vertical rod}} + K_{\text{horizontal rod}}$:
  In order to find $K_{\text{vertical rod}}$, the total kinetic energy of one of the vertical rods is expressed through rotational energy about it's end point: $\frac{1}{2} I\omega^2 = \frac{1}{6} m l^2 \dot{\theta}^2$. As the horizontal rod is not rotating about any of it's end points,$K_{\text{horizontal rod}}$ is purely translational, so: $K_{\text{horizontal rod}} = \frac{1}{2} m v^2 = \frac{1}{2} m (l\dot{\theta})^2 = \frac{1}{2} m l^2 \dot{\theta}^2$. Thus $K = 2K_{\text{vertical rod}} + K_{\text{horizontal rod}} = 2\frac{1}{6} m l^2 \dot{\theta} + \frac{1}{2} m l^2 \dot{\theta}^2 = \frac{5}{6} m l^2 \dot{\theta}^2$

- $V = 2V_{\text{vertical rod}} + V_{\text{horizontal rod}}$:
  Considering that the system's constraints, $V(\theta = 0) = 0$, and $V(\theta) = V_{max}$, which means that the heigh, h, is displacement in the direction of $\vec{g}$: $\theta = 0$
  $V_{\text{vertical rod}} = mgh_{\text{center of mass}} = mgh_{\text{center of mass}} = mg(-\frac{l}{2})cos\theta$
  $V_{\text{horizontal rod}} = -mglcos\theta$, which means that $V = 2V_{\text{vertical rod}} + V_{\text{horizontal rod}} = -2mglcos\theta$

- $L = K - V = \frac{5}{6}ml^2\dot{\theta}^2 - (-2mglcos\theta) = \frac{5}{6}ml^2\dot{\theta}^2 + 2mglcos\theta$

Problem 4 A particle with mass m moves in three-dimensional space under the influence of a constraint. The constraint is expressed by the equation

$$e^{-(x^2+y^2)} + z = 0, \tag{2}$$

for the Cartesian coordinates (x, y, z) of the particle.

a) Explain why the number of degrees of freedom of the particle is two. Use x and y as generalized coordinates and find the expression for the position vector $\vec{r}$ of the particle in terms of x and y.

As the particle moves in three-dimensional space, under one constraint, the number of degrees of freedom is $d = 3N - M = 3 - 1 = 2$. Taking x, y as the generalized coordinates, the position vector of the particle may be expressed as:

$$\vec{r} = (x, y, z) = (x, y, -e^{x^2+y^2}) \tag{3}$$

b) A virtual displacement is a change in the position of the particle $\vec{r} \to \vec{r} + \delta\vec{r}$ which is caused by an inifinitesimal change in the generalized coordinates, $x \to x + \delta x$ and $y \to y + \delta y$. Find $\delta\vec{r}$ expressed in terms of δx and δy.

$$\partial\vec{r} = \sum_{j=1}^{2} \frac{\partial\vec{r}}{\partial q_j}\delta q_j = \frac{\partial\vec{r}}{\partial x}\partial x + \frac{\partial\vec{r}}{\partial y}\partial y$$
$$= (\mathbf{i} - 2xe^{x^2+y^2}\mathbf{k})\delta x + (\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k})\delta y =$$
$$\delta x\mathbf{i} + \delta y\mathbf{j} - (2xe^{x^2+y^2}\delta x + 2ye^{x^2+y^2}\delta y)\mathbf{k}$$

c) The constraint can be interpreted as a restriction for the particle to move on a two-dimensional surface in three-dimensional space. Any virtual displacement $\delta\vec{r}$ is a tangent vector to the surface while the constraint force $\vec{f}$ which acts on the particle is perpendicular to the surface. Use this to determine $\vec{f}$ (up to a normalization factor) as a function of x and y.

$$\mathbf{f} \cdot \mathbf{r} = 0$$
$$f_x \delta x + f_y \delta y - f_z(2xe^{x^2+y^2}\delta x + 2ye^{x^2+y^2}\delta y) = 0$$
$$(f_x - f_z 2xe^{x^2+y^2})\delta x + (f_y - f_z 2ye^{x^2+y^2})\delta y = 0$$

Which must hold for any value of δx and δy, thus

$$f_x - f_z 2xe^{x^2+y^2} = 0$$
$$f_y - f_z 2ye^{x^2+y^2} = 0$$
$$\vec{f}(x,y) = (2xe^{x^2+y^2}, 2ye^{x^2+y^2}, 1)f_z$$

I am not sure how to proceed from here. Any comments on this would be greatly appreciated.

d) Make a drawing of a section through the surface for $y = 0$. Indicate in the drawing the direction of the two vectors $\vec{f}$ and $\delta\vec{r}$ for a chosen point on the surface.