FYS3120 Classical Mechanics and Electrodynamics

Problem set 11

May 2, 2019

Problem 1 Figure 1 shows a rectangular current loop ABCD. In the loop's rest frame, S, the loop has length a in the x direction and width b in the y direction, the current is I and the charge density is zero. We remind you of the following general definitions of the electric dipole moment $\vec{p}$, and the magnetic dipole moment $\vec{m}$, for a given current distribution:

$$\vec{p} = \int \vec{r}\rho(\vec{r})\, d^3r \tag{1}$$

$$\vec{m} = \frac{1}{2}\int (\vec{r} \times \vec{j}(\vec{r}))\, d^3r \tag{2}$$

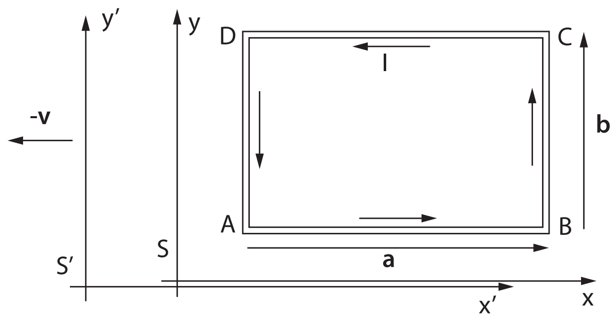

Figure 1: Illustration of current loop.

a) Show that in the rest frame the loop's electric dipole moment is zero and the magnetic moment is $\vec{m} = I\vec{a} \times \vec{b}$, where $I = j\Delta$ with j as the current density and Δ as the cross section area of the current wire.

- Δ is the cross section area of the current wire. Using this, the integral in (2) instead evaluated as a piecewize line integral:

$$\vec{m} = \Delta\frac{1}{2}[\int_A^B (\vec{x} \times \vec{j}(\vec{x}))\, dx + \int_C^D ((\vec{x} + \vec{b}) \times \vec{j}(\vec{x}))\, dx$$
$$+ \int_A^D (\vec{x} \times \vec{j}(\vec{y}))\, dy + \int_B^C ((\vec{x} + \vec{a}) \times \vec{j}(\vec{y}))\, dy]$$
$$\vec{m} = \Delta\frac{1}{2}\left[\int_A^B 0 + \, dx \int_C^D bj(\vec{x})\vec{k}\, dx + \int_A^D 0\, dy + \int_B^C aj(\vec{y})\vec{k}\, dy\right]$$

I uniform on each segment of the loop $\rightarrow \vec{j}(\vec{r}) = \vec{j}$, thus:

$$\vec{m} = \Delta\frac{1}{2}\left[\int_C^D bj\vec{k}\, dx + \int_B^C aj\vec{k}\, dy\right]$$
$$\vec{m} = \Delta\frac{1}{2}\left[jab\vec{k} + jab\vec{k}\right] = \Delta jab\vec{k} = I\vec{a} \times \vec{b}$$

Charge density $\rho(\vec{r}) = 0 \implies \vec{p} = 0$

In the following we will examine how the loop is observed in a reference frame S', where the loop is moving with velocity $\vec{v}$ to the right ($\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$). The Lorentz transformation formulas for charge and current denisities may be useful when solving the problems below.

b) What is the length and width of the loop in S'?

- Length contraction in x direction $\rightarrow a' = \frac{1}{\gamma}a$. No velocity in y direction $\rightarrow b' = b$

c) Show that the parts AB and CD of the loop have charge $\pm aIv/c^2$ in S'.

- $Q = \int \rho(\vec{r})d^3r$. Using $\Delta' = \Delta$ on loop from A to B, and the fact that $\vec{j}(\vec{r}) = \vec{j} \implies \rho'(x') = \rho$

$$Q'_{AB} = \int_A^B \Delta p'(x)dx = \Delta\rho' a'$$

Using Lorentz transformation in the x-direction, and that I is uniform:

$$\rho' = \gamma(\rho - \frac{v}{c^2}j) = -\gamma\frac{v}{c^2}j \tag{3}$$

Thus $Q'_{AB} = \Delta a' - \gamma\frac{v}{c^2}j = -\Delta a\frac{v}{c^2}j = -aIv/c^2$. Similarly, $Q'_{DC} = aIv/c^2$ as the current travels in the opposite direction on the upper part of the loop. Thus $Q = \pm aIv/c^2$ on the parts AB and CD of the loop in S'.

d) Show that in S' the loop's electric dipole moment is $\vec{p}' = -\frac{1}{c^2}\vec{m} \times \vec{v}$, and the magnetic dipole moment is $\vec{m}' = (1 - \beta^2/2)\vec{m}$.

$$\begin{aligned}
p' &= \int \vec{r}'\rho(\vec{r}')\,d^3r' \\
&= \Delta\int_A^B \vec{x}'\rho(\vec{r}')\,dx' + \Delta\int_D^C (\vec{x}' + \vec{b}')\rho(\vec{r}')\,dx' + \Delta'\int_A^D \vec{y}'\rho(\vec{r}')\,dy' + \Delta'\int_B^C (\vec{y}' + \vec{a}')\rho(\vec{r}')\,dy' \\
&= \Delta\left[\int_A^B \vec{x}'\rho(\vec{r}')\,dx' + \int_D^C (\vec{x}' + \vec{b}')\rho(\vec{r}')\,dx' + \frac{1}{\gamma}\int_A^D \vec{y}'\rho(\vec{r}')\,dy' + \frac{1}{\gamma}\int_B^C (\vec{y}' + \vec{a}')\rho(\vec{r}')\,dy'\right] \\
&= -\gamma\frac{v}{c^2}j\Delta\left[\int_A^B \vec{x}'\,dx' + \int_D^C (\vec{x}' + \vec{b}')\,dx' + \frac{1}{\gamma}\int_A^D \vec{y}'\,dy' + \frac{1}{\gamma}\int_B^C (\vec{y}' + \vec{a}')\,dy'\right] \\
&= -\gamma\frac{v}{c^2}I\left[\vec{a}' + \int_D^C (\vec{x}' + \vec{b}')\,dx' + \frac{1}{\gamma}\vec{b}' + \frac{1}{\gamma}\int_B^C (\vec{y}' + \vec{a}')\,dy'\right]
\end{aligned}$$

e) Show that the current is $I\gamma$ in the AB and CD and I/γ in BC and DA.

f) Show that the result in **e)** is consistent with charge conservation.

Problem 2 An electric point charge q is moving with constant velocity $\vec{v}$ along the x-axis of the inertial frame S, as illustrated in Fig. 2. Assume it passes the origin of S at $t = 0$.

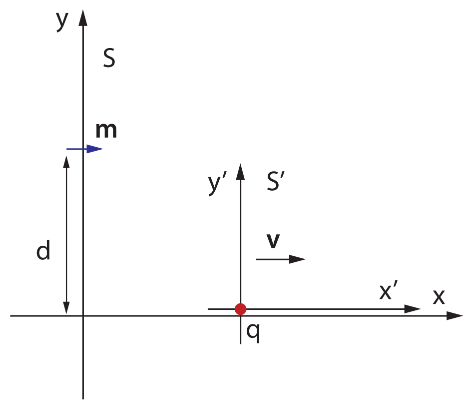

Figure 2: Charge on the move.

a) Give the expression for the scalar potential ϕ' and the vector potential $\vec{A}'$ set up by the charge in its rest frame S'. In the relativistic description the scalar and vector potentials define the four-potential A'^{μ}, with the time component related to the scalar potential as $A'^{0} = \phi'/c$. Make use of the transformation properties of the four-potential to determine its components A^{μ} in reference frame S as functions of the coordinates (ct, x, y, z) in the same frame.

- $\phi'(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}$
- $\vec{A}' = 0$, as there are no currents in the rest frame S'.
- $A'^{\mu} = (A'^{0}, \vec{A}) = (\phi'/c, \vec{0}) = (\frac{q/c}{4\pi\epsilon_0 r}, \vec{0})$
- RF S moves with velocity $-v$ with respect to RF S', thus $\phi = \gamma(\phi' + vA'_x) = \gamma\phi$.

b) Determine (the components of) the electric field $\vec{E}$ in the reference frame S, as functions of (ct, x, y, z).

c) Determine similarly the magnetic field $\vec{B}$ in reference frame S.

A magnetic dipole, with dipole moment $\vec{m}$, is at rest in S, at the position $(x, y, z) = (0, d, 0)$. The dipole vector $\vec{m}$ points in the x-direction.

d) The field from the moving charge acts with a time dependent torque on the dipole, $\vec{M} = \vec{m} \times \vec{B}$. Find the expression for the torque.

e) Assuming the magnetic dipole can be viewed as a small current loop, the force on the dipole from the field produced by the moving charge is $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$. Determine the force.