FYS3120 Classical Mechanics and Electrodynamics

Problem set 11

May 3, 2019

Problem 1 Figure 1 shows a rectangular current loop ABCD. In the loop's rest frame, S, the loop has length a in the x direction and width b in the y direction, the current is I and the charge density is zero. We remind you of the following general definitions of the electric dipole moment $\vec{p}$, and the magnetic dipole moment $\vec{m}$, for a given current distribution:

$$\vec{p} = \int \vec{r}\rho(\vec{r})\, d^3r \tag{1}$$

$$\vec{m} = \frac{1}{2}\int (\vec{r} \times \vec{j}(\vec{r}))\, d^3r \tag{2}$$

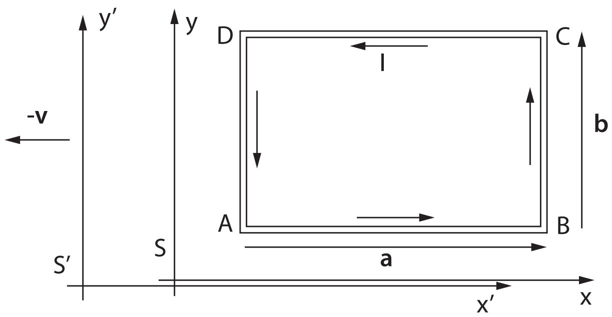

Figure 1: Illustration of current loop.

a) Show that in the rest frame the loop's electric dipole moment is zero and the magnetic moment is $\vec{m} = I\vec{a} \times \vec{b}$, where $I = j\Delta$ with j as the current density and Δ as the cross section area of the current wire.

- Δ is the cross section area of the current wire. Using this, the integral in (2) instead evaluated as a piecewize line integral:

$$\vec{m} = \Delta\frac{1}{2}[\int_A^B (\vec{x} \times \vec{j}(\vec{r}))\, dx + \int_D^C ((\vec{x} + \vec{b}) \times \vec{j}(\vec{r}))\, dx$$
$$+ \int_A^D (\vec{x} \times \vec{j}(\vec{r}))\, dy + \int_B^C ((\vec{r} + \vec{a}) \times \vec{j}(\vec{r}))\, dy]$$

I uniform on each segment of the loop $\rightarrow \vec{j} = -j\hat{x}$ on DC and $\vec{j} = j\hat{y}$ BC, thus:

$$\vec{m} = \Delta\frac{1}{2}\left[\int_A^B 0 +\, dx \int_D^C -(-bj)\vec{k}\, dx + \int_A^D 0\, dy + \int_B^C aj\vec{k}\, dy\right]$$

$$\vec{m} = \Delta\frac{1}{2}\left[\int_C^D bj\vec{k}\, dx + \int_B^C aj\vec{k}\, dy\right]$$
$$\vec{m} = \Delta\frac{1}{2}\left[jab\vec{k} + jab\vec{k}\right] = \Delta jab\vec{k} = I\vec{a} \times \vec{b}$$

- Charge density $\rho(\vec{r}) = 0 \implies \vec{p} = 0$

In the following we will examine how the loop is observed in a reference frame S', where the loop is moving with velocity $\vec{v}$ to the right ($\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$). The Lorentz transformation formulas for charge and current denisities may be useful when solving the problems below.

b) What is the length and width of the loop in S'?

- Length contraction in x direction $\rightarrow a' = \frac{1}{\gamma}a$. No velocity in y direction $\rightarrow b' = b$

c) Show that the parts AB and CD of the loop have charge $\pm aIv/c^2$ in S'.

- $Q = \int \rho(\vec{r}) d^3r$. Using $\Delta' = \Delta$ and that uniform $I \implies$ uniform ρ' on AB and DC

$$Q'_{AB} = \int_A^B \Delta p' dx = \Delta a' \rho'$$

Using Lorentz transformation in the x-direction, with velocity $-v$:

$$\rho' = \gamma(\rho + \frac{v}{c^2}j) = \gamma\frac{v}{c^2}j \tag{3}$$

Thus $Q'_{AB} = \Delta a' \gamma \frac{v}{c^2} j = \Delta a \frac{v}{c^2} j = aIv/c^2$. Similarly, $Q'_{DC} = -aIv/c^2$ as the current travels in the opposite direction on the upper part of the loop. Thus $Q = \pm aIv/c^2$ on the parts AB and CD of the loop in S'.

d) Show that in S' the loop's electric dipole moment is $\vec{p'} = -\frac{1}{c^2}\vec{m} \times \vec{v}$, and the magnetic dipole moment is $\vec{m'} = (1 - \beta^2/2)\vec{m}$.

Lorentz transformation in y-direction yields that $\rho' = 0$ on the vertical sides of the loop. Thus;

$$\begin{aligned}
p' &= \int \vec{r'}\rho(\vec{r'})\, d^3r' \\
&= \Delta \int_A^B \vec{x'}\rho'(\vec{r'})\, dx' + \Delta \int_D^C (\vec{x'} + \vec{b'})\rho'(\vec{r'})\, dx' + \Delta' \int_A^D \vec{y'}\rho'(\vec{r'})\, dy' + \Delta' \int_B^C (\vec{y'} + \vec{a'})\rho'(\vec{r'})\, dy' \\
&= \Delta \int_A^B \vec{x'}\rho(\vec{r'})\, dx' + \Delta \int_D^C (\vec{x'} + \vec{b'})\rho(\vec{r'})\, dx' + \Delta' \int_A^D \vec{y'} 0\, dy' + \Delta' \int_B^C (\vec{y'} + \vec{a'}) 0\, dy' \\
&= \gamma\frac{v}{c^2}[\Delta \int_A^B \vec{x'} j\, dx' + \Delta \int_D^C (\vec{x'} + b')(-j)\, dx'] \\
&= \gamma\frac{v}{c^2}\Delta(-\vec{b}a'j) = -\frac{1}{c^2}(vaj\Delta\vec{b}) = -\frac{1}{c^2}(vIa\vec{b}) = -\frac{1}{c^2}(\vec{v}\cdot I\vec{a})\vec{b}
\end{aligned}$$

Using that the triple product rule: $(\vec{a}\times\vec{b})\times\vec{c} = -(\vec{c}\cdot\vec{b})\vec{a} + (\vec{c}\cdot\vec{a})\vec{b}$, and the fact that $\vec{v}\cdot\vec{b} = 0$ yields $\vec{m} = \frac{1}{c^2}\vec{m}\times\vec{v}$

As the velocity of S seen from S' is constant, the magnetic dipole moment may be evaluated as the origins of the two RFs coincide. Here, the contribution from line segments AB and AD are zero, just as in S. Using that $j'_x = \gamma j_x$ on DC:

$$
\begin{aligned}
\vec{m}' &= \frac{1}{2}\int (\vec{r}'\times\vec{j}'(\vec{r}'))\, d^3r' \\
\vec{m}' &= \frac{1}{2}\left[\Delta\int_D^C ((\vec{x}'+\vec{b}')\times\vec{j}'(\vec{r})\, dx + \Delta'\int_B^C ((\vec{x}'+\vec{a}')\times\vec{j}(\vec{r}))\, dy\right] \\
\vec{m}' &= \frac{1}{2}\left[\Delta\int_D^C -b'(-j')\vec{k}dx + \Delta'\int_B^C a'j'\vec{k}\, dy\right] \\
\vec{m}' &= \frac{1}{2}\left[\Delta a'b'j'_x\vec{k} + \Delta' a'b'j'_y\vec{k}\right] = \frac{1}{2}\left[\Delta\frac{a}{\gamma}bj_x\gamma + \frac{\Delta}{\gamma}\frac{a}{\gamma}bj_y\right]\vec{k} \\
\vec{m}' &== \frac{1}{2}\left[\Delta abj + \frac{1}{\gamma^2}\Delta abj\right]\vec{k} = \frac{1}{2}\left[1+\frac{1}{\gamma^2}\right]\vec{m} \\
\vec{m}' &= \frac{1}{2}\left[1+(1+\beta^2)\right]\vec{m} \\
\vec{m}' &= (1-\beta^2/2)\vec{m}
\end{aligned}
$$

e) Show that the current is $I\gamma$ in the AB and CD and I/γ in BC and DA.

- On all line segments, $I' = \Delta' j'$. Using that $j' = \gamma j$, $\Delta' = \Delta$ on AB and CD, and that $j' = j$, $\Delta' = \frac{\Delta}{\gamma}$ on BC and DA:

$$
\begin{aligned}
I'_{AB} &= I'_{CD} = \Delta j' = \gamma\Delta j = \gamma I \\
I'_{BC} &= I'_{DA} = \Delta' j = \frac{\Delta}{\gamma} j = I/\gamma
\end{aligned}
$$

f) Show that the result in **e)** is consistent with charge conservation. $\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{j} = 0$

Problem 2 An electric point charge q is moving with constant velocity $\vec{v}$ along the x-axis of the inertial frame S, as illustrated in Fig. 2. Assume it passes the origin of S at $t = 0$.

a) Give the expression for the scalar potential ϕ' and the vector potential $\vec{A}'$ set up by the charge in its rest frame S'. In the relativistic description the scalar and vector potentials define the four-potential A'^{μ}, with the time component related to the scalar potential as $A'^{0} = \phi'/c$. Make

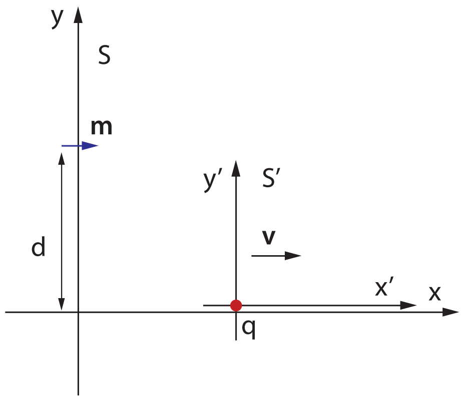

Figure 2: Charge on the move.

use of the transformation properties of the four-potential to determine its components A^μ in reference frame S as functions of the coordinates (ct, x, y, z) in the same frame.

- $\phi'(\vec{r}) = \frac{q}{4\pi\epsilon_0|\vec{r}' - \vec{r}'_q|} = \frac{q}{4\pi\epsilon_0 r'}$, where $\vec{r}'_q = 0$ is the position of the charge.
- $\vec{A}' = 0$, as there are no currents in the rest frame S'.
- $A'^\mu = (A'^0, \vec{A}) = (\phi'/c, \vec{0}) = (\frac{q/c}{4\pi\epsilon_0 r'}, \vec{0})$
- Four potential in RF S: $A^\rho = L^\rho_\mu A^\mu = (\gamma A'^0, -\gamma\beta A'^0, 0, 0) = (\gamma\phi'/c, \gamma\frac{v}{c}\phi'/c, 0, 0)$.
- $A^0 = \gamma\phi/c = \gamma/c\frac{q}{4\pi\epsilon_0|\vec{r}' - \vec{r}'_q|}$

b) Determine (the components of) the electric field $\vec{E}$ in the reference frame S, as functions of (ct, x, y, z).

c) Determine similarly the magnetic field $\vec{B}$ in reference frame S.

A magnetic dipole, with dipole moment $\vec{m}$, is at rest in S, at the position $(x, y, z) = (0, d, 0)$. The dipole vector $\vec{m}$ points in the x-direction.

d) The field from the moving charge acts with a time dependent torque on the dipole, $\vec{M} = \vec{m} \times \vec{B}$. Find the expression for the torque.

e) Assuming the magnetic dipole can be viewed as a small current loop, the force on the dipole from the field produced by the moving charge is $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$. Determine the force.