Ultrafast Optimal Sideband Cooling under Non-Markovian Evolution

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A sideband cooling strategy that incorporates (i) the dynamics induced by structured (non-Markovian) environments in the target and auxiliary systems and (ii) the optimally time-modulated interaction between them is developed. For the context of cavity optomechanics, when non-Markovian dynamics are considered in the target system, ground state cooling is reached at much faster rates and at a much lower phonon occupation number than previously reported. In contrast to similar current strategies, ground state cooling is reached here for coupling-strength rates that are experimentally accessible for the state-of-the-art implementations. After the ultrafast optimal-ground-state-cooling protocol is accomplished, an additional optimal control strategy is considered to maintain the phonon number as close as possible to the one obtained in the cooling procedure. Contrary to the conventional expectation, when non-Markovian dynamics are considered in the auxiliary system, the efficiency of the cooling protocol is undermined.

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Introduction.—Fabricating and controlling micro- and nanodevices is of primary importance to develop quantum technologies. In this effort, optomechanical devices play a central role because they may become a building block for hybrid systems [1–4]. To benefit from quantum effects, potential quantum technologies that utilize optomechanical systems need to prepare the mechanical component in the ground state. Thus, searching alternative schemes and refining present strategies to improve cooling protocols are of great interest. However, the highly nontrivial interplay between time-dependent external fields and structured environments poses constraints on the coherent control of these systems [5] that have been barely explored.

Despite the copious recent literature on optomechanical cooling [1–3,6–12], a key point of the physics at low temperature remains unexplored, namely, low temperature induces non-Markovian dynamics [13–16]. This very fact and the recent experimental evidence that the dynamics of microresonators are non-Markovian [17], certainly, suggest that the present understanding of cooling processes is incomplete. Moreover, because these approaches work on very short time scales, of the order of the period of the resonator, it is expected that non-Markovian effects, either from low-temperature fluctuations or structured environments, dominate the energy and entropy transfer.

The exploration of non-Markovian dynamics has already led to the enhancement of, e.g., quantum speed limits [18], survival of entanglement [16,19,20], improvements in quantum metrology [21], and corrections to thermal equilibrium states [15,22]. In the context of cooling, it was even shown that only under non-Markovian dynamics can the entropy of a parametrically driven resonator decrease [5]. The non-Markovian aspects of the dynamics discussed above are introduced here to one of the most promising techniques to reach the minimum phonon number, namely, finding an optimal coupling function via optimal control theory [5,8]. Based on analytic exact results derived in the context of the Feynman-Vernon influence theory, it is found that when the non-Markovian character is considered in the target resonator, the phonon number is lower than that predicted by Markovian processes; however, in contrast to the conventional expectation, the non-Markovian dynamics in the auxiliary system deteriorates the cooling protocol.

Model and theory.—For mechanical systems, sideband cooling consists in coupling a resonator to a microwave or optical resonator whose frequency is sufficiently high that it effectively sits in its ground state at ambient temperature. The coupling between the mechanical and electromagnetic modes is mediated by radiation pressure and is nonlinear. However, under realistic experimental conditions [9,23–25], the coupling can be assumed to be linear [2,4]. This common model with linear coupling is considered below [26]. Specifically, in the linear approximation, the system consists in two harmonic modes that describe the mechanical mode and the optical mode with the annihilation operators $\hat{a}$ and $\hat{b}$, respectively. The Hamiltonian reads

$$\hat{H} = \hbar\omega_{\mathrm{mm}}\hat{a}^\dagger\hat{a} + \hbar\omega_{\mathrm{om}}\hat{b}^\dagger\hat{b} + \hbar g(t)(\hat{a}^\dagger + \hat{a})(\hat{b}^\dagger + \hat{b}) \quad (1)$$

with ω_{mm} and ω_{om} the frequency of the mechanical and electromagnetic modes, respectively, and $g(t)$ the arbitrary time-dependent optomechanical coupling function to be found via optimal control theory [28]. For later convenience, the period of the mechanical mode is labeled by $\tau_{\mathrm{mm}} = 2\pi/\omega_{\mathrm{mm}}$. To describe the interaction with the environment of the resonator and the losses in the cavity, each mode is coupled to an independent thermal bath with an arbitrary spectrum and is described in the context of the

Ullersma-Caldeira-Leggett model [29,30] (see the Supplemental Material [31] for details).

The particular functional form of the Hamiltonian (1), the environment model, and the thermal initial states allow for the complete description of the dynamics in terms of the variances of the modes' coordinates (see, e.g., Ref. [16]). This was utilized to combine sideband cooling and optimal control theory [8]. Specifically, the dynamics were formulated in terms of an adjoint master equation for the quadratures of the modes derived from the Markovian version of the Brownian-motion master equation [32,33]. However, this master equation may violate the density-operator positivity (see Sec. 3.6 in Ref. [34]). Moreover, for time dependent Hamiltonians, an adjoint Lindblad master equation can be derived only if the Liouvillian of the nonunitary dynamics commutes with its associated-time-ordered propagator (see Sec. 3.2 in Ref. [34]); this is not the case for the Hamiltonian (1).

To circumvent the issues raised above, the Feynman-Vernon influence functional theory [35] is employed here. It allows for performing a description of the dynamics without any approximation (see the Supplemental Material [31]) and it further obeys the positivity of the density operator and straightforwardly incorporates driving fields. By combining the exact analytic results provided by the influence functional approach and an efficient numerical algorithm to solve integrodifferential equations, the dynamics are solved here for an arbitrary bath spectrum and driving field.

Non-Markovian optimal sideband cooling.—The potential benefits from cooling optomechanical systems have attracted a great deal of attention [1,2,7,10,23,36,37] and, recently, this has been powered by optimal control theory [3,8]. However, a key dynamical feature has been left out of the discussion, namely, the non-Markovian dynamics induced by low temperature and structured environments [13–16]. Therefore, the goal here is to calculate and investigate the dependence of the minimum phonon number in the mechanical resonator $\langle\hat{n}\rangle(t_{\mathrm{cool}})$, at the shortest possible time t_{cool}, on the non-Markovian character of the dynamics. In terms of the second moments of the position $\langle\hat{q}^2\rangle$ and momentum $\langle\hat{p}^2\rangle$, the phonon number at time t is given by

$$\langle\hat{n}\rangle(t) = \frac{1}{2\hbar\omega_{\mathrm{mm}}}\left[\frac{\langle\hat{p}^2\rangle(t)}{m} + m\omega_{\mathrm{mm}}{}^2\langle\hat{q}^2\rangle(t)\right] - \frac{1}{2}. \quad (2)$$

The optimization procedure under non-Markovian dynamics, described in the Supplemental Material [31], leads to the minimum number phonon occupation at time t_{cool}. The cooling time t_{cool} is chosen as short as possible by hand. The spectrum of the mechanical-resonator's environment is described by the spectral density $J_{\mathrm{mm}}(\omega) = \gamma\omega_{\mathrm{D}}^2/(\omega_{\mathrm{D}}^2+\omega^2)$ whereas the cavity environment is chosen, for convenience, as $J_{\mathrm{om}}(\omega) = \kappa\Omega_{\mathrm{D}}^2/(\Omega_{\mathrm{D}}^2+\omega^2)$. This spectral density relates to temperature fluctuations [38]. As in Ref. [8], it is assumed that the frequency conversion is exact and set $\omega_{\mathrm{mm}} = \omega_{\mathrm{om}}$. The corrections to this approximation are of the order of $(\omega_{\mathrm{mm}}/\omega_{\mathrm{om}})^2$.

For typical values of optomechanical setups, Fig. 1(a) shows the optimal dynamics of the phonon number in the Markovian (dashed lines) and the non-Markovian (continuous lines) regime. Figure 1 shows two alternatives to decrease the minimum phonon number, namely, (i) extending the cooling time, or (ii) considering non-Markovian effects in the dynamics. By lengthening the cooling process from $t_{\mathrm{cool}} = 0.55\tau_{\mathrm{mm}}$ (black lines) to $t_{\mathrm{cool}} = 1.25\tau_{\mathrm{mm}}$ (green and magenta lines) and for the initial phonon numbers in the mechanical mode $n_T = 10^2$ and $n_T = 10^3$, it is possible to decrease the minimum phonon number by an order of magnitude, from $\sim\langle n(t_{\mathrm{cool}})\rangle = 10^{-2}$ to $\langle n(t_{\mathrm{cool}})\rangle = 10^{-3}$. The Markovian and non-Markovian character of the dynamics does not change considerably the functional form of the optomechancal coupling; however, the small difference is enough to decrease the minimum

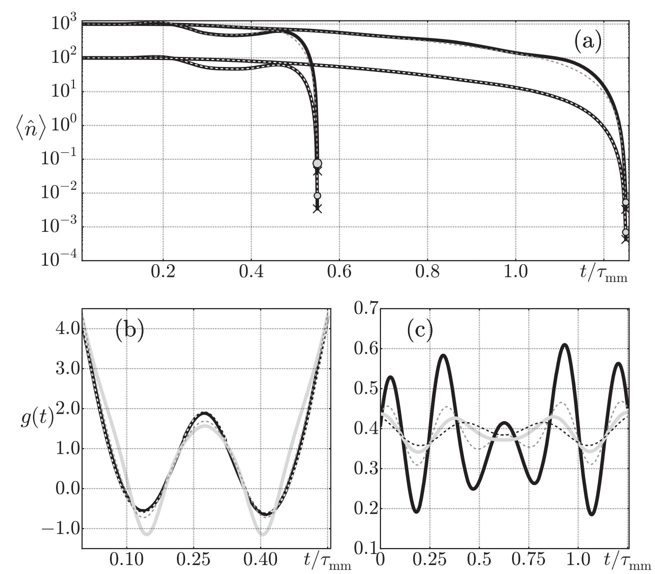

FIG. 1. (a) Time evolution of the phonon number $\hat{n}(t)$ during the optimal modulation of the interaction with $n_{\mathrm{c}} = 0$, $\gamma = 10^{-6}\omega_{\mathrm{mm}}$ and $\kappa = 2.15\times10^{-4}\omega_{\mathrm{mm}}$. In particular, for $n_T = 10^2$ ($n_T = 10^3$) and final times $t_{\mathrm{cool}} = 0.55\tau_{\mathrm{mm}}$ and $t_{\mathrm{cool}} = 1.25\tau_{\mathrm{mm}}$, the dashed white (gray) curve depicts the dynamics of the phonon number under Markovian dynamics ($\omega_{\mathrm{D}} = \Omega_{\mathrm{D}} = 100\omega_{\mathrm{mm}}$) while the continuos black curves depicts the dynamics under non-Markovian dynamics ($\omega_{\mathrm{D}} = \Omega_{\mathrm{D}} = \omega_{\mathrm{mm}}$). The y-axis is in logarithmic scale. (b) Optimal coupling function with $t_{\mathrm{cool}} = 0.55\tau_{\mathrm{mm}}$ under non-Markovian (continuous lines) and Markovian dynamics (dashed lines). (c) Optimal coupling function with $t_{\mathrm{cool}} = 1.25\tau_{\mathrm{mm}}$ under non-Markovian (continuous lines) and Markovian dynamics (dashed lines). For (b) and (c) the continuos black curve corresponds to $n_T = 10^2$ and the continuos lightgray curve correponds to $n_T = 10^3$.

TABLE I. Minimum phonon number in the mechanical resonator at a time $t_{\text{cool}} = 0.55\tau_{\text{mm}}$ for different values of the dissipation rate $\gamma/\omega_{\text{mm}}$ and for different cooling scenarios (see text). The cavity dissipation rate $\kappa = 10^{-4}\omega_{\text{mm}}$.

	$\langle\hat{n}(t_{\text{cool}})\rangle$				
$\gamma/\omega_{\text{mm}}$	$\text{opt.om}_{\text{M}} + \text{mm}_{\text{M}}$	$\text{om}_{\text{nM}} + \text{mm}_{\text{nM}}$	$\text{opt.om}_{\text{nM}} + \text{mm}_{\text{nM}}$	$\text{om}_{\text{M}} + \text{mm}_{\text{nM}}$	$\text{opt.om}_{\text{M}} + \text{mm}_{\text{nM}}$
10^{-6}	9.03×10^{-3}	8.86×10^{-3}	3.43×10^{-3}	1.91×10^{-3}	9.99×10^{-5}
10^{-5}	1.04×10^{-2}	1.02×10^{-2}	4.79×10^{-3}	2.60×10^{-3}	7.91×10^{-4}
10^{-4}	3.28×10^{-2}	2.40×10^{-2}	1.99×10^{-2}	1.63×10^{-2}	1.52×10^{-2}
10^{-3}	2.61×10^{-1}	1.61×10^{-1}	1.53×10^{-1}	1.53×10^{-1}	1.51×10^{-1}
10^{-2}	2.45	1.52	1.50	1.50	1.50
10^{-1}	21.12	14.21	13.52	13.52	13.52

phonon number by around 1 order of magnitude (see also Table I).

The coupling strength for the ultrafast cooling scenario in Fig. 1(b) is beyond the present experimental capabilities; however, the results for the fast cooling case in Fig. 1(c) are encouraging [23]. Moreover, in the regime of Fig. 1(c) cooling from room temperature is possible [12].

The nontrivial interplay between non-Markovian dynamics and driving fields is explored in Table I; it displays the predicted phonon number for a variety of scenarios, namely, (i) the minimum phonon number obtained from optimization under Markovian dynamics ($\text{opt.om}_{\text{M}}+ \text{mm}_{\text{M}}$), (ii) the phonon number obtained under non-Markovian dynamics with the optimal coupling found under Markovian dynamics ($\text{om}_{\text{nM}} + \text{mm}_{\text{nM}}$), (iii) the phonon number obtained from optimization under non-Markovian dynamics ($\text{opt.om}_{\text{nM}} + \text{mm}_{\text{nM}}$), (iv) the phonon number obtained under Markovian dynamics in the optical mode and non-Markovian dynamics in the mechanical mode with the optimal coupling found under Markovian dynamics ($\text{om}_{\text{M}} + \text{mm}_{\text{nM}}$), and (v) the phonon number obtained from optimization under Markovian dynamics in the optical mode and non-Markovian dynamics in the mechanical mode ($\text{opt.om}_{\text{M}} + \text{mm}_{\text{nM}}$).

Comparison of scenarios (i) and (ii) in Table I shows that, for the same coupling function, the presence of non-Markovian dynamics reduces the minimum phonon number and that this decrease is more noticeable when the dissipation rate of the mechanical resonator increases. The third scenario depicts the influence of the optimization process. Even though there is a reduction in the phonon number with respect to the second scenario, in absolute terms, the decrease is tiny. Besides, if the optical-mode dynamics are considered as Markovian and the dynamics of the mechanical mode as non-Markovian, the fourth scenario, there is a decrease in the number of phonons for low values of the decay factor $\gamma/\omega_{\text{mm}}$. Surprisingly, only when Markovian and non-Markovian dynamics and optimal control theory are combined, the fifth scenario, there is a substantial reduction in the phonon number reached for low mechanical decay rates. For these low-decay-factor cases, which are relevant under experimental conditions, the strategy in the fifth scenario is able to bring the number of phonons more than 1 order of magnitude below the number of phonons reached under the conditions of the first scenario.

This nontrivial and unexpected interplay between Markovian and non-Markovian dynamics and optimal control theory is also observed for a variety of values of the cavity decay rate in Table II. The advantage of a Markovian-dynamics cavity is more transparent for large values of the cavity loss rate κ; e.g., for a non-Markovian cavity ($\Omega_D = \omega_{\text{om}}$) with $\kappa = 2.15 \times 10^{-1}\omega_{\text{om}}$, there is no ground-state cooling [$\langle\hat{n}_{\text{nM}}(t_{\text{cool}}) = 2.34$].

Because the mechanical-mode initial state is highly thermally populated, the non-Markovian character of the dynamics originates mainly from the structure of the thermal bath and not from quantum fluctuations at low temperature. This explains why, for nonstructured environments, Markovian descriptions of the dynamics may have provided sensible results.

Maintaining the minimum phonon number over time.—Reaching a very low phonon number in a short period of time is a desirable goal. However, because the resonator is continuously coupled to its environment, keeping that phonon number is a must. In doing so, we introduce a second optimal control scenario where the objective is to maintain, over a long period of time, the minimum number

TABLE II. Minimum phonon number for scenarios (i) ($\text{opt.om}_{\text{M}} + \text{mm}_{\text{M}}$) $\langle\hat{n}_{\text{M}}(t_{\text{cool}})\rangle$, (iii) ($\text{opt.om}_{\text{nM}} + \text{mm}_{\text{nM}}$) $\langle\hat{n}_{\text{nM}}(t_{\text{cool}})\rangle$, and (v) ($\text{opt.om}_{\text{M}} + \text{mm}_{\text{nM}}$) $\langle\hat{n}_{\text{om}_{\text{M}}}(t_{\text{cool}})\rangle$ for different cooling times and for different parameters of cavity dissipation. The initial parameters are $n_T = 100$, $\gamma = 10^{-4}\omega_{\text{mm}}$.

$t_{\text{cool}}/\tau_{\text{mm}}$	$\kappa/\omega_{\text{om}}$	$\langle\hat{n}_{\text{M}}(t_{\text{cool}})\rangle$	$\langle\hat{n}_{\text{nM}}(t_{\text{cool}})\rangle$	$\langle\hat{n}_{\text{om}_{\text{M}}}(t_{\text{cool}})\rangle$
0.55	1×10^{-3}	0.016	0.015	0.014
0.6	1.5×10^{-2}	0.025	0.019	0.019
0.8	2.5×10^{-2}	0.029	0.030	0.021
0.8	4.5×10^{-2}	0.035	0.060	0.026
0.8	5.5×10^{-2}	0.037	0.086	0.032
1.0	1.25×10^{-1}	0.048	0.356	0.040
1.6	2.15×10^{-1}	0.056	2.34	0.044

of phonons in the resonator reached in the ultrafast cooling in Fig. 1. Define then a composite optimal optomechanical coupling function as

$$c(t) = \begin{cases} g_c(t), & 0 \leq t \leq t_{\mathrm{cool}}, \\ g_m(t), & t > t_{\mathrm{cool}}, \end{cases} \tag{3}$$

which encompasses the entire cooling process, i.e., reaching a minimum phonon number by applying $g_c(t)$ and maintaining this number once it is obtained with $g_m(t)$. Because performing the second optimization process under non-Markovian dynamics over long times requires an enormous amount of computing time, the second optimization process is performed under the assumption that for long times the Markovian approximation holds so that the optimal coupling function is calculated for Markovian dynamics.

Figure 2(a) depicts the time evolution of the phonon number under the action of the optimal coupling function $c(t)$ for a variety of experimentally relevant values of the decay rate γ. To obtain an experimentally accessible coupling-strength scenario, $|c(t)| < 10^{-1}$, the cooling time is set to $t_{\mathrm{cool}} = 6\tau_{\mathrm{mm}}$ and the initial phonon number to $n_T = 10^2$ [see Fig. 2(b)]. For the second control phase, the initial number of phonons is set to be the minimum phonon number reached at t_{cool}. Maintaining the minimum phonon number requires moving from the strong coupling regime to a weak coupling regime between the two modes, see Figs. 2(b) and 2(c). Likewise, with the optimal optomechanical coupling function $c(t)$ the phonon number is approximately maintained for 50 periods of the mechanical resonator.

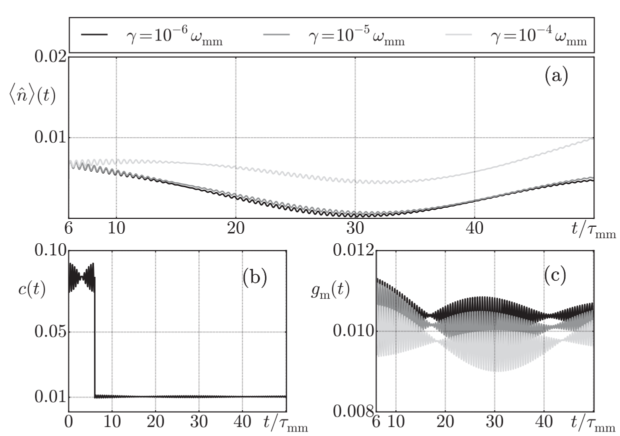

FIG. 2. (a) Phonon number as a function of time during the optimal control protocol aimed at maintaining the minimum phonon number for different values of the dissipation rate. The initial parameters are $n_T = \langle n(t_{\mathrm{cool}})\rangle$, $n_c = 0$, $\kappa = 2.15 \times 10^{-4}\omega_{\mathrm{om}}$. (b) The optimal optomechanical coupling function $c(t)$ for $\gamma = 10^{-6}\omega_{\mathrm{mm}}$. (c) The optimal optomechanical coupling function $g_m(t)$.

Discussion.—Analytical solutions and an efficient optimal control protocol showed that the presence of non-Markovian dynamics allows for lower phonon numbers than the predicted by Markovian and Rotating-Wave-Approximation based previous works. Surprisingly, significant enhancements are found as the interplay between non-Markovian dynamics in the mechanical mode, Markovian dynamics in the optical mode, and optimally designed coupling functions. To understand this effect, note that for well-behaved $J(\omega)$ non-Markovian dynamics define an effective coupling to the thermal bath [15,16] that, in general, is weaker than the Markovian one. Thus, when non-Markovian dynamics are considered in the cavity dynamics, the rate at which the electromagnetic mode releases the entropy, which is absorbed from the mechanical mode, into its environment diminishes, and therefore the number of phonons in the mechanical mode does not largely decrease compared to the case of the Markovian-dynamics cavity. Furthermore, because the mechanical mode is constantly coupled to its environment, the optimal cooling protocol was extended [see Eq. (3)] to preserve the phonon number as low as possible after cooling is reached. For this second protocol, the coupling amplitude is within reach of present technology and can be combined with already experimentally implemented cooling protocols [9,23–25].

The optimization protocol developed here relies on the optimization over the Green functions of the trajectories that minimize the influence functional (see the Supplemental Material [31]) and can be readily implemented in semiclassical formulations of quantum mechanics in phase space [39,40]. Thus, nonlinear systems can be addressed and the influence of quantum fluctuations in the design of optimal pulses of coupling functions can be analyzed. This enables the present proposal, e.g., in the context of optimal control theory of molecular processes [41].

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[1] S. Mancini, D. Vitali, and P. Tombesi, Phys. Rev. Lett. **80**, 688 (1998).
[2] F. Marquardt, J. P. Chen, A. A. Clerk, and S. M. Girvin, Phys. Rev. Lett. **99**, 093902 (2007).
[3] S. Machnes, J. Cerrillo, M. Aspelmeyer, W. Wieczorek, M. B. Plenio, and A. Retzker, Phys. Rev. Lett. **108**, 153601 (2012).
[4] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).

[5] R. Schmidt, A. Negretti, J. Ankerhold, T. Calarco, and J. T. Stockburger, Phys. Rev. Lett. **107**, 130404 (2011).
[6] F. Marquardt and S. M. Girvin, Physics **2**, 40 (2009).
[7] I. Wilson-Rae, N. Nooshi, W. Zwerger, and T. J. Kippenberg, Phys. Rev. Lett. **99**, 093901 (2007).
[8] X. Wang, S. Vinjanampathy, F. W. Strauch, and K. Jacobs, Phys. Rev. Lett. **107**, 177204 (2011).
[9] J. Chan, T. P. M. Alegre, A. H. Safavi-Naeini, J. T. Hill, A. Krause, S. Gröblacher, M. Aspelmeyer, and O. Painter, Nature (London) **478**, 89 (2011).
[10] A. H. Safavi-Naeini, J. Chan, J. T. Hill, S. Grblacher, H. Miao, Y. Chen, M. Aspelmeyer, and O. Painter, New J. Phys. **15**, 035007 (2013).
[11] Y.-C. Liu, Y.-F. Xiao, X. Luan, and C. W. Wong, Phys. Rev. Lett. **110**, 153606 (2013).
[12] Y.-C. Liu, R.-S. Liu, C.-H. Dong, Y. Li, Q. Gong, and Y.-F. Xiao, Phys. Rev. A **91**, 013824 (2015).
[13] U. Weiss, *Quantum Dissipative Systems*, Series in Modern Condensed Matter Physics (World Scientific, Singapore, 2012).
[14] L. A. Pachón and P. Brumer, J. Math. Phys. (N.Y.) **55**, 012103 (2014).
[15] L. A. Pachón, J. F. Triana, D. Zueco, and P. Brumer, arXiv:1401.1418.
[16] A. F. Estrada and L. A. Pachon, New J. Phys. **17**, 033038 (2015).
[17] S. Groblacher, A. Trubarov, N. Prigge, G. D. Cole, M. Aspelmeyer, and J. Eisert, Nat. Commun. **6**, 7606 (2015).
[18] S. Deffner and E. Lutz, Phys. Rev. Lett. **111**, 010402 (2013).
[19] M. Thorwart, J. Eckel, J. Reina, P. Nalbach, and S. Weiss, Chem. Phys. Lett. **478**, 234 (2009).
[20] S. F. Huelga, A. Rivas, and M. B. Plenio, Phys. Rev. Lett. **108**, 160402 (2012).
[21] A. W. Chin, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **109**, 233601 (2012).
[22] C.-J. Yang, J.-H. An, H.-G. Luo, Y. Li, and C. H. Oh, Phys. Rev. E **90**, 022122 (2014).
[23] J. D. Teufel, T. Donner, D. Li, J. W. Harlow, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker, K. W. Lehnert, and R. W. Simmonds, Nature (London) **475**, 359 (2011).
[24] S. Gigan, H. R. Bohm, M. Paternostro, F. Blaser, G. Langer, J. B. Hertzberg, K. C. Schwab, D. Bauerle, M. Aspelmeyer, and A. Zeilinger, Nature (London) **444**, 67 (2006).
[25] D. Kleckner and D. Bouwmeester, Nature (London) **444**, 75 (2006).
[26] Note that in standard cavity optomechanics the linearization is realized using a steady state value for the cavity field and depends on the damping rate of the cavity field. Thus, to proceed with no further refinement, it is necessary to verify that the spectral density used allows for time-independent steady states. This occurs for the spectral density used below and, in general, for spectral densities with no gaps [27].
[27] H.-N. Xiong, P.-Y. Lo, W.-M. Zhang, D. H. Feng, F. Nori, Sci. Rep. **5**, 13353 (2015).
[28] D. Kirk, Optimal Control Theory: An Introduction, *Dover Books on Electrical Engineering* (Dover Publications, New York, 2012).
[29] P. Ullersma, Physica (Utrecht) **32**, 27 (1966).
[30] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981).
[31] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.116.183602 for [brief description].
[32] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, and R. J. Schoelkopf, Rev. Mod. Phys. **82**, 1155 (2010).
[33] C. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics*, Springer Series in Synergetics (Springer, New York, 2004).
[34] H. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford Uuiversity Press, Oxford, 2007).
[35] R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals: Emended Edition* (Dover Publications, Incorporated, New York, 2012).
[36] C. Genes, D. Vitali, P. Tombesi, S. Gigan, and M. Aspelmeyer, Phys. Rev. A **77**, 033804 (2008).
[37] R. Rivière, S. Deléglise, S. Weis, E. Gavartin, O. Arcizet, A. Schliesser, and T. J. Kippenberg, Phys. Rev. A **83**, 063835 (2011).
[38] A. Cleland and M. Roukes, J. Appl. Phys. **92**, 2758 (2002).
[39] L. Pachón, G.-L. Ingold, and T. Dittrich, Chem. Phys. **375**, 209 (2010).
[40] T. Dittrich, E. A. Gómez, and L. A. Pachón, J. Chem. Phys. **132**, 214102 (2010).
[41] M. Shapiro and P. Brumer, *Quantum Control of Molecular Processes*, 2nd ed. (Wiley-VCH, Weinheim, 2012).

Supplementary Material for Ultrafast Optimal Sideband Cooling under Non-Markovian Evolution

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INFLUENCE FUNCTIONAL

The state of the mechanical mode and the optical mode $\tilde{\rho}(q''_{1+}, q''_{2+}, q''_{1-}, q''_{2}, t)$ can be determined from

$$\tilde{\rho}(q''_{1+}, q''_{2+}, q''_{1-}, q''_{2}, t) = \int_{-\infty}^{\infty} \mathrm{d}q'_{1+}\mathrm{d}q'_{2+}\mathrm{d}q'_{1-}\mathrm{d}q'_{2-}\, J(q''_{1+}, q''_{2+}, q''_{1-}, q''_{2-}, t; q'_{1+}, q'_{2+}, q'_{1-}, q'_{2-}, 0) \times \rho_{\mathrm{S}}(q'_{1+}, q'_{2+}, q'_{1-}, q'_{2-}, 0) \quad (1)$$

where $\rho_{\mathrm{S}}(q'_{1+}, q'_{2+}, q'_{1-}, q'_{2-}, 0)$ accounts for the initial state. $J(q''_{1+}, q''_{2+}, q''_{1-}, q''_{2-}, t; q'_{1+}, q'_{2+}, q'_{1-}, q'_{2-}, 0)$ is the propagating function to be obtained from the Feynman-Vernon influence functional approach. After assuming that each mode is coupled to its own Ullersma-Caldeira-Leggett-type bath at temperature $T_{1,2}$, the propagating function reads

$$\begin{aligned} &J(q''_{1+}, q''_{2+}, q''_{1-}, q''_{2-}, t; q'_{1+}, q'_{2+}, q'_{1-}, q'_{2-}, 0) = \\ &\frac{1}{N(t)} \exp\left\{ \frac{i}{\hbar} \int_0^t \mathrm{d}s \left[\sum_{\alpha=1}^{2} \left(\frac{1}{2} m_\alpha \dot{q}_{\alpha+}^2(s) - \frac{1}{2} m_\alpha \omega_\alpha^2 q_{\alpha+}^2(s) \right) - c(s) q_{1+}(s) q_{2+}(s) \right] \right\} \\ &\times \exp\left\{ -\frac{i}{\hbar} \int_0^t \mathrm{d}s \left[\sum_{\alpha=1}^{2} \left(\frac{1}{2} m_\alpha \dot{q}_{\alpha-}^2(s) - \frac{1}{2} m_\alpha \omega_\alpha^2 q_{\alpha-}^2(s) \right) - c(s) q_{1-}(s) q_{2-}(s) \right] \right\} \\ &\times \mathcal{F}[q_{1+}, q_{2+}, q_{1-}, q_{2-}] \end{aligned} \quad (2)$$

with

$$\begin{aligned} \mathcal{F}[q_{1+}, q_{2+}, q_{1-}, q_{2-}] = \prod_{\alpha=1}^{2} &\exp\left(-\frac{i m_\alpha}{2\hbar} \left\{ (q'_{\alpha+} + q'_{\alpha-}) \int_0^t \mathrm{d}s\, \gamma_\alpha(s) [q_{\alpha+}(s) - q_{\alpha-}(s)] \right.\right. \\ &\left.\left. + \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, \gamma_\alpha(s-u) [\dot{q}_{\alpha+}(u) + \dot{q}_{\alpha-}(u)][q_{\alpha+}(s) - q_{\alpha-}(s)] \right\} \right) \\ &\times \exp\left\{ -\frac{1}{\hbar} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, K_\alpha(u-s) [q_{\alpha+}(s) - q_{\alpha-}(s)][q_{\alpha+}(u) - q_{\alpha-}(u)] \right\}, \end{aligned} \quad (3)$$

where

$$\gamma_\alpha(s) = \frac{2}{m_\alpha}\int_0^\infty \frac{\mathrm{d}\omega_\alpha}{\pi}\frac{J_\alpha(\omega_\alpha)}{\omega_\alpha}\cos(\omega_\alpha s), \tag{4}$$

$$K_\alpha(s) = \int_0^\infty \frac{\mathrm{d}\omega_\alpha}{\pi} J_\alpha(\omega_\alpha)\coth\left(\frac{\hbar\beta_\alpha\omega_\alpha}{2}\right)\cos(\omega_\alpha s), \tag{5}$$

being $J_\alpha(\omega_\alpha)$ the spectral density. In the main text, $J_\alpha(\omega_\alpha)$ was chosen such that is accounts for Ohmic dissipation with a cutoff frequency $\omega_{\alpha\mathrm{D}}$. The functions $q_{\alpha\pm}(s)$ obey the following relations (1 for mechanical mode and 2 for optical mode)

$$\begin{aligned}
&\ddot{q}_{1+}(s) + \omega_1^2 q_{1+}(s) + \frac{c(s)}{m_1}q_{2+}(s) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_0^t \mathrm{d}u\,\gamma_1(s-u)(q_{1+}+q_{1-}) - \int_s^t \mathrm{d}u\,\gamma_1(u-s)(q_{1+}-q_{1-})\right] = 0,\\
&\ddot{q}_{1-}(s) + \omega_1^2 q_{1-}(s) + \frac{c(s)}{m_1}q_{2-}(s) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_0^t \mathrm{d}u\,\gamma_1(s-u)(q_{1+}+q_{1-}) + \int_s^t \mathrm{d}u\,\gamma_1(u-s)(q_{1+}-q_{1-})\right] = 0,\\
&\ddot{q}_{2+}(s) + \omega_1^2 q_{2+}(s) + \frac{c(s)}{m_2}q_{1+}(s) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_0^t \mathrm{d}u\,\gamma_2(s-u)(q_{2+}+q_{2-}) - \int_s^t \mathrm{d}u\,\gamma_2(u-s)(q_{2+}-q_{2-})\right] = 0,\\
&\ddot{q}_{2-}(s) + \omega_1^2 q_{2-}(s) + \frac{c(s)}{m_2}q_{1-}(s) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_0^t \mathrm{d}u\,\gamma_2(s-u)(q_{2+}+q_{2-}) + \int_s^t \mathrm{d}u\,\gamma_2(u-s)(q_{2+}-q_{2-})\right] = 0.
\end{aligned} \tag{6}$$

To simplify the above expressions and subsequent calculations, introduce the half sum and difference coordinates as

$$Q_\alpha = \frac{1}{2}(q_{\alpha+} + q_{\alpha-}), \qquad q_\alpha = q_{\alpha+} - q_{\alpha-} \tag{7}$$

where the Jacobian of the coordinates transformation is equal to one.

The interest is to find the minimum phonon number in the mechanical mode, which in terms of the second moments is given by

$$\langle\hat{n}\rangle(t) = \frac{1}{2\hbar\omega}\left[\frac{\langle\hat{p}^2\rangle(t)}{m} + m\omega^2\langle\hat{q}^2\rangle(t)\right] - \frac{1}{2}. \tag{8}$$

Following the technical details in Ref. [1], the second moments read

$$\left\langle q_1^2\right\rangle(t) = \int_{-\infty}^{\infty}\mathrm{d}Q_1''\,Q_1''\tilde{\rho}_1(Q_1'', q_1''=0, t), \tag{9}$$

$$\left\langle p_1^2\right\rangle(t) = -\hbar^2\int_{-\infty}^{\infty}\mathrm{d}Q_1''\,\frac{\mathrm{d}^2}{\mathrm{d}q_1''^2}\tilde{\rho}_1(Q_1'', q_1'', t)\bigg|_{q_1''=0}, \tag{10}$$

with

$$\tilde{\rho}_1(q''_{1+}, q''_{1-}, t) = \langle q''_{1+}|\mathrm{tr}_2\tilde{\rho}(t)|q''_{1-}\rangle = \int_{-\infty}^{\infty} \mathrm{d}Q''_2 \tilde{\rho}(q''_{1+}, Q''_2, q''_{1-}, q''_2 = 0, t). \tag{11}$$

EQUATIONS OF MOTION AND ITS SOLUTION

The differential equations of motion for the coordinates Q_i and q_i take the form

$$\begin{aligned} \ddot{Q}_{1,2}(s) + \omega_{1,2}^2 Q_{1,2}(s) + \frac{c(s)}{m_{1,2}} Q_{2,1}(s) + \frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \mathrm{d}u\, \gamma_{1,2}(s-u) Q_{1,2}(u) = 0, \\ \ddot{q}_{1,2}(s) + \omega_{1,2}^2 q_{1,2}(s) + \frac{c(s)}{m_{1,2}} q_{2,1}(s) - \frac{\mathrm{d}}{\mathrm{d}s}\int_s^t \mathrm{d}u\, \gamma_{1,2}(u-s) q_{1,2}(u) = 0, \end{aligned} \tag{12}$$

and have the following boundary conditions

$$Q_i(s) = \begin{cases} Q'_i, & s = 0 \\ Q''_i, & s = t \end{cases}, \qquad q_i(s) = \begin{cases} q'_i, & s = 0 \\ q''_i, & s = t \end{cases}. \tag{13}$$

The solution to Eqs. (12) can be expresed as

$$\begin{aligned} Q_1(t,s) &= U_1(t,s)Q'_1 + U_2(t,s)Q''_1 + U_3(t,s)Q'_2 + U_4(t,s)Q''_2, \\ Q_2(t,s) &= V_1(t,s)Q'_2 + V_2(t,s)Q''_2 + V_3(t,s)Q'_1 + V_4(t,s)Q''_1, \\ q_1(t,s) &= u_1(t,s)q'_1 + u_2(t,s)q''_1 + u_3(t,s)q'_2 + u_4(t,s)q''_2, \\ q_2(t,s) &= v_1(t,s)q'_2 + v_2(t,s)q''_2 + v_3(t,s)q'_1 + v_4(t,s)q''_1, \end{aligned} \tag{14}$$

where the boundary conditions that satisfied the functions U_i, V_i, u_i and v_i in Eqs. (14) are given by

$$U_1(t,0) = 1, \quad U_1(t,t) = 0, \quad U_2(t,0) = 0, \quad U_2(t,t) = 1, \quad U_i(t,0) = U_i(t,t) = 0, \tag{15}$$

$$V_1(t,0) = 1, \quad V_1(t,t) = 0, \quad V_2(t,0) = 0, \quad V_2(t,t) = 1, \quad V_i(t,0) = V_i(t,t) = 0, \tag{16}$$

$$u_1(t,0) = 1, \quad u_1(t,t) = 0, \quad u_2(t,0) = 0, \quad u_2(t,t) = 1, \quad u_i(t,0) = u_i(t,t) = 0, \tag{17}$$

$$v_1(t,0) = 1, \quad v_1(t,t) = 0, \quad v_2(t,0) = 0, \quad v_2(t,t) = 1, \quad v_i(t,0) = v_i(t,t) = 0, \tag{18}$$

for $i = 3, 4$.

One form to solve the problem is solving in terms of the initial conditions

$$Q_i(0) = q_i(0) = 1, \quad \dot{Q}_i(0) = \dot{q}_i(0) = 1, \quad Q_i(0) = q_i(0) = 0, \quad \text{and} \quad \dot{Q}_i(0) = \dot{q}_i(0) = 0,$$

where $U_i(t,s)$, $V_i(t,s)$, $u_i(t,s)$, $v_i(t,s)$ in terms of the solutions $\varphi_i(s)$, $\phi_i(s)$, $\nu_i(s)$ and $\vartheta_i(s)$ (solution functions of the problem initial condition) are given by

$$\begin{aligned} U_1(t,s) = \varphi_1(s) &- \frac{\varphi_1(t)\varphi_2(s)\phi_2(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} - \frac{\varphi_2(t)\phi_3(t)\varphi_4(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} \\ &+ \frac{\varphi_2(s)\phi_3(t)\varphi_4(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} + \frac{\varphi_1(t)\varphi_4(s)\phi_4(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)}, \end{aligned} \tag{19}$$

$$U_2(t,s) = \frac{\varphi_2(s)\phi_2(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} - \frac{\varphi_4(s)\phi_4(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)}, \tag{20}$$

$$\begin{aligned} U_3(t,s) = \varphi_3(s) &- \frac{\varphi_2(s)\phi_2(t)\varphi_3(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} - \frac{\phi_1(t)\varphi_2(t)\varphi_4(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} \\ &+ \frac{\phi_1(t)\varphi_2(s)\varphi_4(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} + \frac{\varphi_3(t)\varphi_4(s)\phi_4(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)}, \end{aligned} \tag{21}$$

$$U_4(t,s) = \frac{\varphi_2(t)\varphi_4(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} - \frac{\varphi_2(s)\phi_4(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)}, \tag{22}$$

$$\begin{aligned} V_1(t,s) = \phi_1(s) &- \frac{\phi_1(t)\varphi_2(t)\phi_2(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} - \frac{\phi_2(t)\varphi_3(t)\phi_4(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} \\ &+ \frac{\phi_1(t)\varphi_4(t)\phi_4(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} + \frac{\phi_2(s)\varphi_3(t)\phi_4(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)}, \end{aligned} \tag{23}$$

$$V_2(t,s) = \frac{\varphi_2(t)\phi_2(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} - \frac{\varphi_4(t)\phi_4(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)}, \tag{24}$$

$$V_3(t,s) = \phi_3(s) - \frac{\varphi_2(t)\phi_2(s)\varphi_3(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} - \frac{\varphi_1(t)\phi_2(t)\phi_4(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} + \frac{\phi_3(t)\varphi_4(t)\phi_4(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} + \frac{\varphi_1(t)\phi_2(s)\phi_4(t)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)}, \tag{25}$$

$$V_4(t,s) = \frac{\varphi_2(t)\phi_2(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)} - \frac{\varphi_4(t)\phi_4(s)}{\varphi_2(t)\phi_2(t) - \varphi_4(t)\phi_4(t)}, \tag{26}$$

$$u_1(t,s) = \nu_1(s) - \frac{\nu_1(t)\nu_2(s)\vartheta_2(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} - \frac{\nu_2(t)\vartheta_3(t)\nu_4(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} + \frac{\nu_2(s)\vartheta_3(t)\nu_4(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} + \frac{\nu_1(t)\nu_4(s)\vartheta_4(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)}, \tag{27}$$

$$u_2(t,s) = \frac{\nu_2(s)\vartheta_2(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} - \frac{\nu_4(s)\vartheta_4(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)}, \tag{28}$$

$$u_3(t,s) = \nu_3(s) - \frac{\nu_2(s)\vartheta_2(t)\nu_3(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} - \frac{\vartheta_1(t)\nu_2(t)\nu_4(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} + \frac{\vartheta_1(t)\nu_2(s)\nu_4(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} + \frac{\nu_3(t)\nu_4(s)\vartheta_4(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)}, \tag{29}$$

$$u_4(t,s) = \frac{\nu_2(t)\nu_4(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} - \frac{\nu_2(s)\vartheta_4(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)}, \tag{30}$$

$$v_1(t,s) = \vartheta_1(s) - \frac{\vartheta_1(t)\nu_2(t)\vartheta_2(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} - \frac{\vartheta_2(t)\nu_3(t)\vartheta_4(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} + \frac{\vartheta_1(t)\nu_4(t)\vartheta_4(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} + \frac{\vartheta_2(s)\nu_3(t)\vartheta_4(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)}, \tag{31}$$

$$v_2(t,s) = \frac{\nu_2(t)\vartheta_2(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} - \frac{\nu_4(t)\vartheta_4(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)}, \tag{32}$$

$$
\begin{aligned}
v_3(t,s) = \vartheta_3(s) &- \frac{\nu_2(t)\vartheta_2(s)\nu_3(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} - \frac{\nu_1(t)\vartheta_2(t)\vartheta_4(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} \\
&+ \frac{\vartheta_3(t)\nu_4(t)\vartheta_4(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} + \frac{\nu_1(t)\vartheta_2(s)\vartheta_4(t)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)},
\end{aligned} \tag{33}
$$

$$
v_4(t,s) = \frac{\nu_2(t)\vartheta_2(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)} - \frac{\nu_4(t)\vartheta_4(s)}{\nu_2(t)\vartheta_2(t) - \nu_4(t)\vartheta_4(t)}. \tag{34}
$$

Therefore, in terms of the solutions of the initial condition problem with φ_i, ϕ_i, ν_i and ϑ_i, the equations of motion are transformed as

$$
\begin{aligned}
&\ddot{\varphi}_{1,3}(s) + \omega_1^2\varphi_{1,3}(s) + \frac{c(s)}{m_1}\phi_{3,1}(s) + \frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \mathrm{d}u\, \gamma_1(s-u)\varphi_{1,3}(u) = 0, \\
&\ddot{\varphi}_{2,4}(s) + \omega_1^2\varphi_{2,4}(s) + \frac{c(s)}{m_1}\phi_{4,2}(s) + \frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \mathrm{d}u\, \gamma_1(s-u)\varphi_{2,4}(u) = 0, \\
&\ddot{\phi}_{1,3}(s) + \omega_2^2\phi_{1,3}(s) + \frac{c(s)}{m_2}\varphi_{3,1}(s) + \frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \mathrm{d}u\, \gamma_2(s-u)\phi_{1,3}(u) = 0, \\
&\ddot{\phi}_{2,4}(s) + \omega_2^2\phi_{2,4}(s) + \frac{c(s)}{m_2}\varphi_{4,2}(s) + \frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \mathrm{d}u\, \gamma_2(s-u)\phi_{2,4}(u) = 0,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&\ddot{\nu}_{1,3}(s) + \omega_1^2\nu_{1,3}(s) + \frac{c(s)}{m_1}\vartheta_{3,1}(s) - \frac{\mathrm{d}}{\mathrm{d}s}\int_s^t \mathrm{d}u\, \gamma_1(u-s)\nu_{1,3}(u) = 0, \\
&\ddot{\nu}_{2,4}(s) + \omega_1^2\nu_{2,4}(s) + \frac{c(s)}{m_1}\vartheta_{4,2}(s) - \frac{\mathrm{d}}{\mathrm{d}s}\int_s^t \mathrm{d}u\, \gamma_1(u-s)\nu_{2,4}(u) = 0, \\
&\ddot{\vartheta}_{1,3}(s) + \omega_2^2\vartheta_{1,3}(s) + \frac{c(s)}{m_2}\nu_{3,1}(s) - \frac{\mathrm{d}}{\mathrm{d}s}\int_s^t \mathrm{d}u\, \gamma_2(u-s)\vartheta_{1,3}(u) = 0, \\
&\ddot{\vartheta}_{2,4}(s) + \omega_2^2\vartheta_{2,4}(s) + \frac{c(s)}{m_2}\nu_{4,2}(s) - \frac{\mathrm{d}}{\mathrm{d}s}\int_s^t \mathrm{d}u\, \gamma_2(u-s)\vartheta_{2,4}(u) = 0.
\end{aligned} \tag{36}
$$

In this work, the dissipation kernels in Eqs. (35-36) for the mechanical and electromagnetic mode are given by

$$
\gamma_1(s) = \gamma_1\omega_{1\mathrm{D}}e^{-\omega_{1\mathrm{D}}s}, \qquad \gamma_2(s) = \gamma_2\omega_{2\mathrm{D}}e^{-\omega_{2\mathrm{D}}s}. \tag{37}
$$

The equations of motion to solve and optimize are obtained after replacing these dissipation kernels in Eqs. (35-36) and are given by

$$
\begin{aligned}
\ddot{\varphi}_{1,3}(s) + \omega_1^2 \varphi_{1,3}(s) + \frac{c(s)}{m_1}\phi_{3,1}(s) + \dot{J}_{1,2}(s) &= 0,\\
\ddot{\phi}_{1,3}(s) + \omega_2^2 \phi_{1,3}(s) + \frac{c(s)}{m_2}\varphi_{3,1}(s) + \dot{J}_{3,4}(s) &= 0,\\
\dot{J}_1(s) + \omega_{1\mathrm{D}} J_1(s) - \gamma_1 \omega_{1\mathrm{D}} \varphi_1(s) &= 0,\\
\dot{J}_2(s) + \omega_{1\mathrm{D}} J_2(s) - \gamma_1 \omega_{1\mathrm{D}} \varphi_3(s) &= 0,\\
\dot{J}_3(s) + \omega_{2\mathrm{D}} J_3(s) - \gamma_2 \omega_{2\mathrm{D}} \phi_1(s) &= 0,\\
\dot{J}_4(s) + \omega_{2\mathrm{D}} J_4(s) - \gamma_2 \omega_{2\mathrm{D}} \phi_3(s) &= 0,
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
\ddot{\varphi}_{2,4}(s) + \omega_1^2 \varphi_{2,4}(s) + \frac{c(s)}{m_1}\phi_{4,2}(s) + \dot{J}_{5,6}(s) &= 0,\\
\ddot{\phi}_{2,4}(s) + \omega_2^2 \phi_{2,4}(s) + \frac{c(s)}{m_2}\varphi_{4,2}(s) + \dot{J}_{7,8}(s) &= 0,\\
\dot{J}_5(s) + \omega_{1\mathrm{D}} J_5(s) - \gamma_1 \omega_{1\mathrm{D}} \varphi_2(s) &= 0,\\
\dot{J}_6(s) + \omega_{1\mathrm{D}} J_6(s) - \gamma_1 \omega_{1\mathrm{D}} \varphi_4(s) &= 0,\\
\dot{J}_7(s) + \omega_{2\mathrm{D}} J_7(s) - \gamma_2 \omega_{2\mathrm{D}} \phi_2(s) &= 0,\\
\dot{J}_8(s) + \omega_{2\mathrm{D}} J_8(s) - \gamma_2 \omega_{2\mathrm{D}} \phi_4(s) &= 0,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
\ddot{\nu}_{1,3}(s) + \omega_1^2 \nu_{1,3}(s) + \frac{c(s)}{m_1}\vartheta_{3,1}(s) - \dot{j}_{1,2}(s) &= 0,\\
\ddot{\vartheta}_{1,3}(s) + \omega_2^2 \vartheta_{1,3}(s) + \frac{c(s)}{m_2}\nu_{3,1}(s) - \dot{j}_{3,4}(s) &= 0,\\
\dot{j}_1(s) - \omega_{1\mathrm{D}} j_1(s) + \gamma_1 \omega_{1\mathrm{D}} \nu_1(s) &= 0,\\
\dot{j}_2(s) - \omega_{1\mathrm{D}} j_2(s) + \gamma_1 \omega_{1\mathrm{D}} \nu_3(s) &= 0,\\
\dot{j}_3(s) - \omega_{2\mathrm{D}} j_3(s) + \gamma_2 \omega_{2\mathrm{D}} \vartheta_1(s) &= 0,\\
\dot{j}_4(s) - \omega_{2\mathrm{D}} j_4(s) + \gamma_2 \omega_{2\mathrm{D}} \vartheta_3(s) &= 0,
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
\ddot{\nu}_{2,4}(s) + \omega_1^2 \nu_{2,4}(s) + \frac{c(s)}{m_1}\vartheta_{4,2}(s) - \dot{j}_{5,6}(s) &= 0, \\
\ddot{\vartheta}_{2,4}(s) + \omega_2^2 \vartheta_{2,4}(s) + \frac{c(s)}{m_2}\nu_{4,2}(s) - \dot{j}_{7.8}(s) &= 0, \\
\dot{j}_5(s) - \omega_{1\mathrm{D}} j_5(s) + \gamma_1 \omega_{1\mathrm{D}} \nu_2(s) &= 0, \\
\dot{j}_6(s) - \omega_{1\mathrm{D}} j_6(s) + \gamma_1 \omega_{1\mathrm{D}} \nu_4(s) &= 0, \\
\dot{j}_7(s) - \omega_{2\mathrm{D}} j_7(s) + \gamma_2 \omega_{2\mathrm{D}} \vartheta_2(s) &= 0, \\
\dot{j}_8(s) - \omega_{2\mathrm{D}} j_8(s) + \gamma_2 \omega_{2\mathrm{D}} \vartheta_4(s) &= 0,
\end{aligned} \tag{41}
$$

with

$$
\begin{aligned}
J_{1,2}(s) &= \int_s^t \mathrm{d}u\, \gamma_1 \omega_{1\mathrm{D}} e^{-\omega_{1\mathrm{D}}(u-s)} \varphi_{1,3}(u), \\
J_{3,4}(s) &= \int_s^t \mathrm{d}u\, \gamma_2 \omega_{2\mathrm{D}} e^{-\omega_{2\mathrm{D}}(u-s)} \phi_{1,3}(u), \\
J_{5,6}(s) &= \int_s^t \mathrm{d}u\, \gamma_1 \omega_{1\mathrm{D}} e^{-\omega_{1\mathrm{D}}(u-s)} \varphi_{2,4}(u), \\
J_{7,8}(s) &= \int_s^t \mathrm{d}u\, \gamma_2 \omega_{2\mathrm{D}} e^{-\omega_{2\mathrm{D}}(u-s)} \phi_{2,4}(u), \\
j_{1,2}(s) &= \int_s^t \mathrm{d}u\, \gamma_1 \omega_{1\mathrm{D}} e^{-\omega_{1\mathrm{D}}(u-s)} \nu_{1,3}(u), \\
j_{3,4}(s) &= \int_s^t \mathrm{d}u\, \gamma_2 \omega_{2\mathrm{D}} e^{-\omega_{2\mathrm{D}}(u-s)} \vartheta_{1,3}(u), \\
j_{5,6}(s) &= \int_s^t \mathrm{d}u\, \gamma_1 \omega_{1\mathrm{D}} e^{-\omega_{1\mathrm{D}}(u-s)} \nu_{2,4}(u), \\
j_{7,8}(s) &= \int_s^t \mathrm{d}u\, \gamma_2 \omega_{2\mathrm{D}} e^{-\omega_{2\mathrm{D}}(u-s)} \vartheta_{2,4}(u),
\end{aligned} \tag{42}
$$

where the Eqs. (38-41) are solved by a Runge-Kutta algorithm. A similar process can be followed for arbitrary spectral densities, where the Eqs. (35-36) have to be solve by a numerical method for integro-differential equations. In order to give an idea, the final equations to solve for any spectral density are

$$
\begin{aligned}
&\ddot{\varphi}_{1,3}(s) + [\omega_1^2 + \gamma_1(0)]\varphi_{1,3}(s) + \frac{c(s)}{m_1}\phi_{3,1}(s) + \int_0^s \mathrm{d}u\, \gamma_1'(s-u)\varphi_{1,3}(u) = 0,\\
&\ddot{\varphi}_{2,4}(s) + [\omega_1^2 + \gamma_1(0)]\varphi_{2,4}(s) + \frac{c(s)}{m_1}\phi_{4,2}(s) + \int_0^s \mathrm{d}u\, \gamma_1'(s-u)\varphi_{2,4}(u) = 0,\\
&\ddot{\phi}_{1,3}(s) + [\omega_2^2 + \gamma_2(0)]\phi_{1,3}(s) + \frac{c(s)}{m_2}\varphi_{3,1}(s) + \int_0^s \mathrm{d}u\, \gamma_2'(s-u)\phi_{1,3}(u) = 0,\\
&\ddot{\phi}_{2,4}(s) + [\omega_2^2 + \gamma_2(0)]\phi_{2,4}(s) + \frac{c(s)}{m_2}\varphi_{4,2}(s) + \int_0^s \mathrm{d}u\, \gamma_2'(s-u)\phi_{2,4}(u) = 0,
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&\ddot{\nu}_{1,3}(s) + [\omega_1^2 + \gamma_1(0)]\nu_{1,3}(s) + \frac{c(s)}{m_1}\vartheta_{3,1}(s) + \int_s^t \mathrm{d}u\, \gamma_1'(u-s)\nu_{1,3}(u) = 0,\\
&\ddot{\nu}_{2,4}(s) + [\omega_1^2 + \gamma_1(0)]\nu_{2,4}(s) + \frac{c(s)}{m_1}\vartheta_{4,2}(s) + \int_s^t \mathrm{d}u\, \gamma_1'(u-s)\nu_{2,4}(u) = 0,\\
&\ddot{\vartheta}_{1,3}(s) + [\omega_2^2 + \gamma_2(0)]\vartheta_{1,3}(s) + \frac{c(s)}{m_2}\nu_{3,1}(s) + \int_s^t \mathrm{d}u\, \gamma_2'(u-s)\vartheta_{1,3}(u) = 0,\\
&\ddot{\vartheta}_{2,4}(s) + [\omega_2^2 + \gamma_2(0)]\vartheta_{2,4}(s) + \frac{c(s)}{m_2}\nu_{4,2}(s) + \int_s^t \mathrm{d}u\, \gamma_2'(u-s)\vartheta_{2,4}(u) = 0.
\end{aligned}
\tag{44}
$$

SQUEEZING GENERATION BY NON-MARKOVIAN DYNAMICS

In general, for non-Markovian dynamics the system does not always thermalize and part of its quantum coherence could be preserved in the steady state'. In our Ref. [15] arXiv:1401.1418, we showed that even at thermal equilibrium, non-Markovian dynamics allow for (i) generating squeezing in a single harmonic mode that induces deviations from the canonical thermal state and (ii) the presence of entanglement between two harmonic modes. For measuring cooling based on the mean phonon number this may be problematic because the final state may not be thermal. However, the presence of quantum correlations assisted by non-Markovian dynamics is possible at equilibrium only when $k_{\mathrm{B}}T/\hbar\omega_{\mathrm{m}} \ll 1$.

For the case at hand, interest is in the the mechanical mode that is assumed to be in a thermal state at $t = 0$ with $k_{\mathrm{B}}T/\hbar\omega_{\mathrm{m}} \gg 1$; therefore, there is no non-Markovianally-induced squeezing initially. During the subsequent ultrafast dynamics, of the order of the mechanical period oscillation, the time-dependent character of the coupling may squeeze the normal modes of the optomechanical systems and, certainly, take the system into a non-thermal state. Note that this time-depedent-coupling-induced squeezing is also present in the Markovian case considered in Ref. [8] Phys. Rev. Lett. 107, 177204 (2011) and in one of the original proposal of sideband cooling [see Refs., 7 and 22 in the manuscript].

A possible way to characterize the deviations from a thermal state is the $g^2(t)$ correlation function; however, due to the Gaussian character of the state of the mechanical and optical states, and for the present purposes, an equivalent calculation to the $g^{(2)}$ function is the direct calculation of the squeezing parameter $r(t)$ defined as

$$r = \frac{1}{2} \log \left(\frac{\langle \Delta q \rangle^2}{\langle \Delta p \rangle^2} \right), \tag{45}$$

being $\langle \Delta q \rangle^2$ and $\langle \Delta p \rangle^2$ the dispersion of the position and momentum, respectively.

For the parameters used to generate the mean phonon dynamics depicted by the black curve in Fig. 1 of the manuscript, Fig. 1 depicts the time dependence of the squeezing parameter of the mechanical mode state $r(t)$. Because the initial state is thermal at high temperature and no initial system-bath correlations are considered, we have that $r(0) = 0$. The subsequent time-modulation of the coupling induces squeezing that goes very close to zero at the end of the cooling protocol. Specifically, $r_{\rm NM}(t_{\rm cool}) = 7.193 \times 10^{-2}$ and $r_{\rm M}(t_{\rm cool}) = 3.572 \times 10^{-2}$.

Therefore, non-Markovian dynamics generate more squeezing than the Markovian one; however, due to the parameter regime, the excess of squeezing is very small $\Delta r(t_{\rm cool}) = 3.621 \times 10^{-2}$. Thus, it is safe to consider the mean phonon number as an measure of cooling.

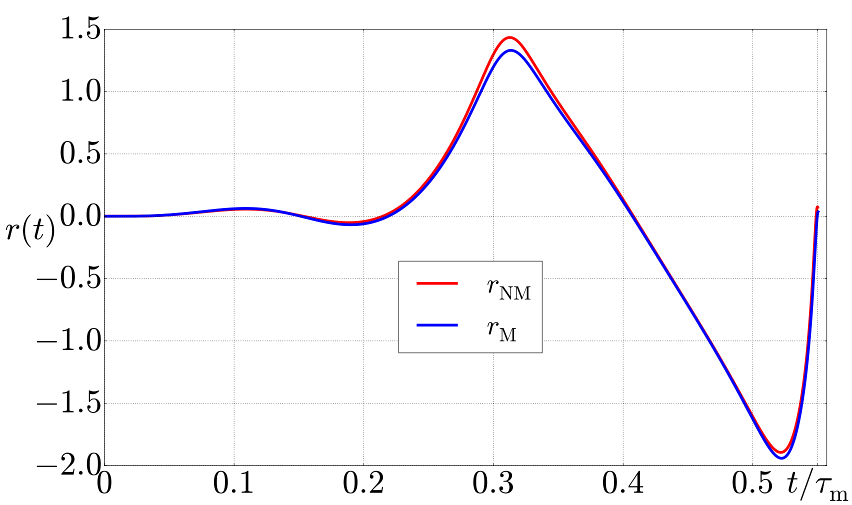

FIG. 1. Squeezing parameter $r(t)$ for the cooling protocol depicted by the black curve in Fig. 1 of the manuscript under Markovian (blue curve) and non-Markovian (red curve) dynamics. The parameters are $\gamma = 10^{-6}\omega_{\rm m}$, $\kappa = 2.15 \times 10^{-4}\omega_{\rm c}$, $n_{\rm T} = 100$ and $n_{\rm cav} = 0$.

ENTROPY TRANSFER

Interest concerns how to reduce the phonon number in the mechanical mode to achieve, as soon as possible, a state near to the ground state. An alternative cooling witness is the entropy at the mechanical mode. If the entropy of the mechanical mode decreases as the optical mode entropy increases, the effective temperature of the mechanical mode decreases, i.e., we are cooling the mechanical mode although we cannot give an absolute value for the temperature. Figure 2 depicts the von Newman entropy [2]

$$S(x) = \left(x+\frac{1}{2}\right)\log\left(x+\frac{1}{2}\right) - \left(x-\frac{1}{2}\right)\log\left(x-\frac{1}{2}\right), \tag{46}$$

$$x = \sqrt{\langle q^2\rangle\langle p^2\rangle - \langle pq+qp\rangle^2/4}, \tag{47}$$

for the mechanical and the optical mode as a function of time.

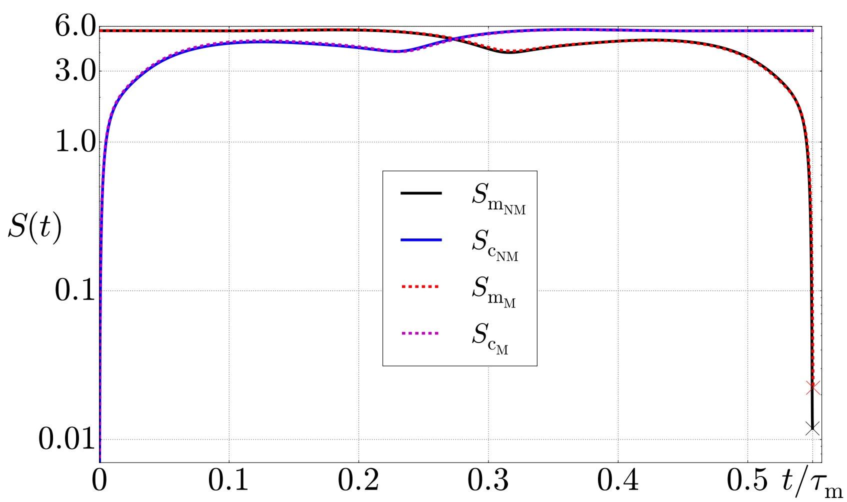

FIG. 2. von Neumann entropy of the both mechanical (S_m) and optical mode (S_c). Parameters as in Fig. 1. Continuous lines for non-Markovian dynamics and dashed lines for Markovian dynamics.

Cooling at this level is guaranteed by the fact that the entropy in the mechanical mode decreases; specifically, it decreases from $S_m(0) = 5.6101$ to $S_m(t_{cool}) = 2.23 \times 10^{-2}$ in the Markovian case and to $S_m(t_{cool}) = 2.19 \times 10^{-2}$ in the non-Markovian case. Thus, the systems sits very close to its ground state (zero entropy). For the cavity mode, it varies from $S_c(0) = 0$ to $S_c(t_{cool}) = 5.601$ in the Markovian case and to $S_c(t_{cool}) = 5.607$ in the non-Markovian. Although for this set of parameters the entropy difference is not substantial, from the analysis in the manuscript, it is clear that if the dissipative rate γ increases, then the entropy difference does so. Note that non-Markovian dynamics favours entropy flow as discussed in Ref. [5].

OPTIMIZATION ALGORITHM

To find the minimum phonon number in the mechanical mode, Eqs. (38-41) are optimized to minimize the phonon number in the mechanical mode. In this algorithm the condition to minimize is then

$$J = h\left[\mathbf{x}(t), t\right] + \int_{t_0}^{t_f} V\left[\mathbf{x}(t), \mathbf{g}(t), t\right] \, \mathrm{d}t = \langle \hat{n}\left[\mathbf{x}(t), t\right] \rangle \tag{48}$$

where $h\left[\mathbf{x}(t), t\right]$ and $V\left[\mathbf{x}(t), \mathbf{g}(t), t\right]$ are arbitrary functions, $\mathbf{x}(t)$ and $\mathbf{g}(t)$ are the vector of the equations of motion and the vector of the optimal functions, respectively. In this work, from Eq. (48), $V\left[\mathbf{x}(t), \mathbf{g}(t), t\right] = 0$, $h\left[\mathbf{x}(t), t\right] = \langle \hat{n}\left[\mathbf{x}(t), t\right] \rangle$, $\mathbf{g}(t) = c(t)$ being $c(t)$ the coupling function and

$$H = V\left[\mathbf{x}(t), \mathbf{g}(t), t\right] + \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x}, \tag{49}$$

where $\mathbf{p}$ is the vector of the costate equations.

However, the condition to minimize, the phonon number in the mechanical mode, is given in terms of the second moments [see Eq. (8)] which in turn are given in terms of the functions φ_i, ϕ_i, ν_i and ϑ_i. This makes quite tedious the calculation of the "initial" conditions for the costate equations (see below) .

The algorithm used to solve the optimization problem was "*the method of steepest descent*" [3] whose algorithm consists of 4 steps:

1. Subdivide the interval $[t_0, t_f]$ into N equal subintervals and assume an initial piecewise-constant control $g^{(0)}(t) = g(0)(t_k)$, $t \in [t_k, t_{k+1}]$ $k = 0, 1, ..., N-1$. In this case, the control function is the coupling between the two resonators $c(s)$ [see Eqs. (38-41)], the mechanical and the electromagnetic mode and the vector of the optimal functions is composed by one function.

2. Apply the assumed control $g^{(i)}$ to integrate the state equations from t_0 to t_f with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and store the state trajectory $\mathbf{x}^{(i)}$. In this case, Eqs. (38-41) are solved and the state trajectory is stored to modify the control function later.

3. Apply $g^{(i)}$ and $\mathbf{x}^{(i)}$ to integrate costate equations backward, i.e., from $[t_f, t_0]$. The "initial value" $\mathbf{p}^{(i)}(t_f)$ is obtained by:

$$\mathbf{p}^{(i)}(t_f) = \frac{\partial h\left[\mathbf{x}^{(i)}(t_f)\right]}{\partial \mathbf{x}} = \frac{\partial \left\langle \hat{n}\left[\mathbf{x}^{(i)}(t_f)\right]\right\rangle}{\partial \mathbf{x}}. \tag{50}$$

Costate equations and their "initial conditions" are given below in the section "Costate equations and their initial conditions in the optimization algorithm (Third Step)".

Evaluate $\partial H^{(i)}(t)/\partial g$, $t \in [t_0, t_f]$ and store this vector.

4. If

$$\left|\left|\frac{\partial H^{(i)}}{\partial g}\right|\right| \leq \epsilon, \tag{51}$$

$$\left|\left|\frac{\partial H^{(i)}}{\partial g}\right|\right|^2 \equiv \int_{t_0}^{t_f} \left[\left|\left|\frac{\partial H^{(i)}}{\partial g}\right|\right|\right]^T \left[\left|\left|\frac{\partial H^{(i)}}{\partial g}\right|\right|\right] \mathrm{d}t \tag{52}$$

then stop the iterative procedure. Here ϵ is a preselected small positive constant used as a tolerance. If Eq. (51) is not satisfied, adjust the piecewise-constant control function by:

$$g^{(i+1)}(t_k) = g^{(i)} - \tau \frac{\partial H^{(i)}}{\partial g}(t_k), \qquad k = 0, 1, ..., N-1. \tag{53}$$

Replace $g^{(i)}$ by $g^{(i+1)}$ and return to step 2. Here, τ is the step size.

COSTATE EQUATIONS AND THEIR INITIAL CONDITIONS IN THE OPTIMIZATION ALGORITHM (THIRD STEP)

The costate equations are calculated by

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}. \tag{54}$$

The "initial" (final) conditions for the costate equations in Eq. (50), in terms of the seconds moments, are given by

$$\begin{aligned}
p_{\varphi_1}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\varphi_1}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\varphi_1}\right), & p_{\varphi_2}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\varphi_2}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\varphi_2}\right),\\
p_{\varphi_3}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\varphi_3}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\varphi_3}\right), & p_{\varphi_4}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\varphi_4}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\varphi_4}\right),\\
p_{\phi_1}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\phi_1}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\phi_1}\right), & p_{\phi_2}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\phi_2}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\phi_2}\right),\\
p_{\phi_3}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\phi_3}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\phi_3}\right), & p_{\phi_4}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\phi_4}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\phi_4}\right),\\
p_{\varphi_1 p}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\varphi_1 p}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\varphi_1 p}\right), & p_{\varphi_2 p}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\varphi_2 p}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\varphi_2 p}\right),\\
p_{\varphi_3 p}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\varphi_3 p}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\varphi_3 p}\right), & p_{\varphi_4 p}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\varphi_4 p}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\varphi_4 p}\right),\\
p_{\phi_1 p}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\phi_1 p}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\phi_1 p}\right), & p_{\phi_2 p}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\phi_2 p}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\phi_2 p}\right),\\
p_{\phi_3 p}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\phi_3 p}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\phi_3 p}\right), & p_{\phi_4 p}(t_f) &= \frac{1}{2}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\phi_4 p}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\phi_4 p}\right),
\end{aligned} \tag{55}$$

$$p_{\nu_1}(t_f) = \frac{1}{2}\left[\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu_1}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu_1}+\sum_{i=1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu_1 i}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu_1 i}\right)\nu i_l+\sum_{i=1,i\neq 1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu i1}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu i1}\right)\nu i_r\right], \tag{56}$$

$$p_{\nu_2}(t_f) = \frac{1}{2}\left[\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu_2}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu_2}+\sum_{i=1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu_2 i}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu_2 i}\right)\nu i_l+\sum_{i=1,i\neq 2}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu i2}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu i2}\right)\nu i_r\right], \tag{57}$$

$$p_{\nu_3}(t_f) = \frac{1}{2}\left[\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu_3}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu_3}+\sum_{i=1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu_3 i}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu_3 i}\right)\nu i_l+\sum_{i=1,i\neq 3}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu i3}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu i3}\right)\nu i_r\right], \tag{58}$$

$$p_{\nu_4}(t_f) = \frac{1}{2}\left[\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu_4}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu_4}+\sum_{i=1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu_4 i}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu_4 i}\right)\nu i_l+\sum_{i=1,i\neq 4}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\nu i4}+\frac{\partial\langle\hat{q}^2\rangle}{\partial\nu i4}\right)\nu i_r\right], \tag{59}$$

$$p_{\vartheta_1}(t_f) = \frac{1}{2}\left[\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta_1} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta_1} + \sum_{i=1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta_1 i} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta_1 i}\right)\vartheta i_l + \sum_{i=1,i\neq 1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta i1} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta i1}\right)\vartheta i_r\right], \tag{60}$$

$$p_{\vartheta_2}(t_f) = \frac{1}{2}\left[\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta_2} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta_2} + \sum_{i=1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta_2 i} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta_2 i}\right)\vartheta i_l + \sum_{i=1,i\neq 2}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta i2} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta i2}\right)\vartheta i_r\right], \tag{61}$$

$$p_{\vartheta_3}(t_f) = \frac{1}{2}\left[\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta_3} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta_3} + \sum_{i=1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta_3 i} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta_3 i}\right)\vartheta i_l + \sum_{i=1,i\neq 3}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta i3} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta i3}\right)\vartheta i_r\right], \tag{62}$$

$$p_{\vartheta_4}(t_f) = \frac{1}{2}\left[\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta_4} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta_4} + \sum_{i=1}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta_4 i} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta_4 i}\right)\vartheta i_l + \sum_{i=1,i\neq 4}^{4}\left(\frac{\partial\langle\hat{p}^2\rangle}{\partial\vartheta i4} + \frac{\partial\langle\hat{q}^2\rangle}{\partial\vartheta i4}\right)\vartheta i_r\right], \tag{63}$$

with $\nu i_{l,r}$ and $\vartheta i_{l,r}$ given by

$$\begin{aligned}
\nu i_r &= \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, K_1(u-s)u_i(s), \\
\vartheta i_r &= \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, K_2(u-s)v_i(s), \\
\nu i_l &= \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, K_1(u-s)u_i(u), \\
\vartheta i_l &= \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u\, K_2(u-s)v_i(u),
\end{aligned} \tag{64}$$

where $K_i(s)$ is the noise kernel generated by the dissipation kernels in Eqs. (37) [1].

Variances

The initial conditions for the thermal states of the mechanical (1) and optical (2) modes are

$$
\begin{aligned}
\langle q_1\rangle(0) &= 0, & \langle q_2\rangle(0) &= 0,\\
\langle p_1\rangle(0) &= 0, & \langle p_2\rangle(0) &= 0,\\
\langle q_1q_1\rangle(0) &= \frac{1}{2m_1\omega_1}\coth\left(\frac{1}{2}\beta_1\omega_1\right), & \langle q_2q_2\rangle(0) &= \frac{1}{2m_2\omega_2}\coth\left(\frac{1}{2}\beta_2\omega_2\right),\\
\langle p_1p_1\rangle(0) &= \frac{m_1\omega_1}{2}\coth\left(\frac{1}{2}\beta_1\omega_1\right), & \langle p_2p_2\rangle(0) &= \frac{m_2\omega_2}{2}\coth\left(\frac{1}{2}\beta_2\omega_2\right),\\
\langle q_1p_1\rangle(0) &= 0, & \langle q_2p_2\rangle(0) &= 0,
\end{aligned} \tag{65}
$$

where β_i are the inverse temperatures of the baths, ω_i are the frequencies of the mechanical and electromagnetic modes respectively, and m_i the masses of the resonators.

The second moments of the mechanical resonator $\langle q_1q_1\rangle(t) =$ `sq1q1[t]` and $\langle p_1p_1\rangle(t) =$ `sp1p1[t]` in terms of the functions $\varphi_i =$ `Ui`, $\phi_i =$ `Vi`, $\nu_i =$ `ui` and $\vartheta_i =$ `vi` are given by

```
sq1q1[t_]:=(sp2p2i (U2p[0] U4[t]-U2[t] U4p[0])^2)/(4 (2 U2[t] V2[t]-2 U4[t]
V4[t])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)+(vep2i^2 (U2p[0] U4[t]-U2[t]
U4p[0])^2)/(4 (2 U2[t] V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)
+ ((U2p[0] U4[t]-U2[t] U4p[0])^2 (u22[t] (u4[t] v1[t]-u3[t] v2[t])^2+u2[t]^2
(u44[t] v1[t]^2-v2[t] (-(u33[t]+v11[t]) v2[t]+v1[t] (u34[t]+u43[t]+v12[t]+v21[t]))
+ v1[t]^2 v22[t])+u24[t] u3[t] u4[t] v1[t] v4[t]-u23[t] u4[t]^2 v1[t] v4[t]-u32[t]
u4[t]^2 v1[t] v4[t]+u3[t] u4[t] u42[t] v1[t] v4[t]-u4[t]^2 v1[t] v14[t] v4[t]-u24[t]
u3[t]^2 v2[t] v4[t]+u23[t] u3[t] u4[t] v2[t] v4[t]+u3[t] u32[t] u4[t] v2[t]
v4[t]-u3[t]^2 u42[t] v2[t] v4[t]+u3[t] u4[t] v14[t] v2[t] v4[t]+u3[t] u4[t]
v1[t] v24[t] v4[t]-u3[t]^2 v2[t] v24[t] v4[t]-u3[t] u34[t] u4[t] v4[t]^2+u33[t]
u4[t]^2 v4[t]^2-u3[t] u4[t] u43[t] v4[t]^2+u3[t]^2 u44[t] v4[t]^2+u4[t]^2
v11[t] v4[t]^2-u3[t] u4[t] v12[t] v4[t]^2-u3[t] u4[t] v21[t] v4[t]^2+u3[t]^2
v22[t] v4[t]^2-u4[t]^2 v1[t] v4[t] v41[t]+u3[t] u4[t] v2[t] v4[t] v41[t]+u3[t]
u4[t] v1[t] v4[t] v42[t]-u3[t]^2 v2[t] v4[t] v42[t]+u2[t] (u24[t] v1[t] (-u4[t]
v1[t]+u3[t] v2[t])+u4[t] (-u42[t] v1[t]^2+u23[t] v1[t] v2[t]+u32[t] v1[t]
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v2[t]+v1[t] v14[t] v2[t]-v1[t]^2 v24[t]+u34[t] v1[t] v4[t]+u43[t] v1[t] v4[t]+v1[t]
v12[t] v4[t]-2 u33[t] v2[t] v4[t]-2 v11[t] v2[t] v4[t]+v1[t] v21[t] v4[t]+v1[t]
v2[t] v41[t]-v1[t]^2 v42[t])+u3[t] (u42[t] v1[t] v2[t]-u23[t] v2[t]^2-u32[t]
v2[t]^2-v14[t] v2[t]^2+v1[t] v2[t] v24[t]-2 u44[t] v1[t] v4[t]+u34[t] v2[t]
v4[t]+u43[t] v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t] v21[t] v4[t]-2 v1[t] v22[t]
v4[t]-v2[t]^2 v41[t]+v1[t] v2[t] v42[t]))+(u4[t] v1[t]-u3[t] v2[t])^2 v44[t]))/(2
(u2[t] v2[t]-u4[t] v4[t])^2 (2 U2[t] V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)^2) - (vep1i vep2i (U2p[0] U4[t]-U2[t] U4p[0]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 (2 U2[t] V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)^2)-(vep2i veq1i (U2p[0] U4[t]-U2[t] U4p[0]) (U2[t] (-U1p[0]
V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0]
V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)-(sq2p2i (U2p[0] U4[t]-U2[t]
U4p[0]) (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)-(vep2i veq2i
(U2p[0] U4[t]-U2[t] U4p[0]) (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0]
V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)
- ((U2p[0] U4[t]-U2[t] U4p[0]) (u12[t] u2[t] u4[t] v1[t] v2[t]+u2[t] u21[t]
u4[t] v1[t] v2[t]-u2[t]^2 u41[t] v1[t] v2[t]+u13[t] u2[t]^2 v2[t]^2-u12[t]
u2[t] u3[t] v2[t]^2-u2[t] u21[t] u3[t] v2[t]^2+u2[t]^2 u31[t] v2[t]^2+u2[t]^2
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v13[t] v2[t]^2-u2[t]^2 v1[t] v2[t] v23[t]-2 u2[t] u24[t] u4[t] v1[t] v3[t]+2
u22[t] u4[t]^2 v1[t] v3[t]-2 u2[t] u4[t] u42[t] v1[t] v3[t]+2 u2[t]^2 u44[t]
v1[t] v3[t]+u2[t] u24[t] u3[t] v2[t] v3[t]-u2[t]^2 u34[t] v2[t] v3[t]+u2[t]
u23[t] u4[t] v2[t] v3[t]-2 u22[t] u3[t] u4[t] v2[t] v3[t]+u2[t] u32[t] u4[t]
v2[t] v3[t]+u2[t] u3[t] u42[t] v2[t] v3[t]-u2[t]^2 u43[t] v2[t] v3[t]-u2[t]^2
v12[t] v2[t] v3[t]+u2[t] u4[t] v14[t] v2[t] v3[t]-u2[t]^2 v2[t] v21[t] v3[t]+2
u2[t]^2 v1[t] v22[t] v3[t]-2 u2[t] u4[t] v1[t] v24[t] v3[t]+u2[t] u3[t] v2[t]
v24[t] v3[t]+u2[t]^2 v2[t]^2 v31[t]-u2[t]^2 v1[t] v2[t] v32[t]+u2[t] u4[t]
v1[t] v2[t] v34[t]-u2[t] u3[t] v2[t]^2 v34[t]-u12[t] u4[t]^2 v1[t] v4[t]-u21[t]
u4[t]^2 v1[t] v4[t]+u2[t] u4[t] u41[t] v1[t] v4[t]-2 u13[t] u2[t] u4[t] v2[t]
v4[t]+u12[t] u3[t] u4[t] v2[t] v4[t]+u21[t] u3[t] u4[t] v2[t] v4[t]-2 u2[t]
u31[t] u4[t] v2[t] v4[t]+u2[t] u3[t] u41[t] v2[t] v4[t]-2 u2[t] u4[t] v13[t]
v2[t] v4[t]+u2[t] u4[t] v1[t] v23[t] v4[t]+u2[t] u3[t] v2[t] v23[t] v4[t]+u24[t]
u3[t] u4[t] v3[t] v4[t]+u2[t] u34[t] u4[t] v3[t] v4[t]-u23[t] u4[t]^2 v3[t]
v4[t]-u32[t] u4[t]^2 v3[t] v4[t]+u3[t] u4[t] u42[t] v3[t] v4[t]+u2[t] u4[t]
u43[t] v3[t] v4[t]-2 u2[t] u3[t] u44[t] v3[t] v4[t]+u2[t] u4[t] v12[t] v3[t]
v4[t]-u4[t]^2 v14[t] v3[t] v4[t]+u2[t] u4[t] v21[t] v3[t] v4[t]-2 u2[t] u3[t]
v22[t] v3[t] v4[t]+u3[t] u4[t] v24[t] v3[t] v4[t]-2 u2[t] u4[t] v2[t] v31[t]
v4[t]+u2[t] u4[t] v1[t] v32[t] v4[t]+u2[t] u3[t] v2[t] v32[t] v4[t]-u4[t]^2
v1[t] v34[t] v4[t]+u3[t] u4[t] v2[t] v34[t] v4[t]+u13[t] u4[t]^2 v4[t]^2+u31[t]
u4[t]^2 v4[t]^2-u3[t] u4[t] u41[t] v4[t]^2+u4[t]^2 v13[t] v4[t]^2-u3[t] u4[t]
v23[t] v4[t]^2+u4[t]^2 v31[t] v4[t]^2-u3[t] u4[t] v32[t] v4[t]^2-u14[t] (u2[t]
v1[t]-u3[t] v4[t]) (u2[t] v2[t]-u4[t] v4[t])+u2[t] u4[t] v2[t] v3[t] v41[t]-u4[t]^2
v3[t] v4[t] v41[t]-2 u2[t] u4[t] v1[t] v3[t] v42[t]+u2[t] u3[t] v2[t] v3[t]
v42[t]+u3[t] u4[t] v3[t] v4[t] v42[t]+u2[t] u4[t] v1[t] v2[t] v43[t]-u2[t]
u3[t] v2[t]^2 v43[t]-u4[t]^2 v1[t] v4[t] v43[t]+u3[t] u4[t] v2[t] v4[t] v43[t]+2
u4[t] (u4[t] v1[t]-u3[t] v2[t]) v3[t] v44[t]+u1[t] (2 u22[t] v2[t] (-u4[t]
v1[t]+u3[t] v2[t])+u2[t] (u24[t] v1[t] v2[t]+u42[t] v1[t] v2[t]-u23[t] v2[t]^2-u32[t]
v2[t]^2-v14[t] v2[t]^2+v1[t] v2[t] v24[t]-2 u44[t] v1[t] v4[t]+u34[t] v2[t]
v4[t]+u43[t] v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t] v21[t] v4[t]-2 v1[t] v22[t]
v4[t]-v2[t]^2 v41[t]+v1[t] v2[t] v42[t])+v4[t] (u24[t] (u4[t] v1[t]-2 u3[t]
v2[t])+u4[t] (-(u34[t]+u43[t] +v12[t]+v21[t]) v4[t]+v2[t] (u23[t]+u32[t] +
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v14[t] + v41[t]) + v1[t] (u42[t] + v24[t] + v42[t]))-2 u3[t] (-(u44[t]+v22[t])
v4[t]+v2[t] (u42[t] + v24[t] + v42[t])))+2 v2[t] (-u4[t] v1[t]+u3[t] v2[t])
v44[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 (u2[t] v2[t]-u4[t] v4[t])^2
(2 U2[t] V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)+(sp1p1i
(-m U2[t] V2p[0]+m U4[t] V4p[0])^2)/(4 (2 U2[t] V2[t]-2 U4[t] V4[t])^2 (((-U2p[0]
V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2)^2)+(vep1i^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)/(4
(2 U2[t] V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)+(sq1p1i
(U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)/((2 U2[t]
V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)+(vep1i veq1i
(U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)/((2 U2[t]
V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)+(sq1q1i
(U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t]
(U2p[0] V2[t]-U4p[0] V4[t]))^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)/((2 U2[t]
V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)+(veq1i^2
(U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t]
(U2p[0] V2[t]-U4p[0] V4[t]))^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)/((2 U2[t]
V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
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(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)+(vep1i veq2i
(U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)/((2 U2[t]
V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)+(2 veq1i
veq2i (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0]
V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m
U4[t] V4p[0])^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)^2)+(sq2q2i (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0]
V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t]))^2 (-m U2[t] V2p[0]+m
U4[t] V4p[0])^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)^2)+(veq2i^2 (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0]
V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t]))^2 (-m U2[t] V2p[0]+m
U4[t] V4p[0])^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)^2)+((-u14[t] u2[t]^2 v2[t] v3[t]+u12[t] u2[t] u4[t] v2[t]
v3[t]+u2[t] u21[t] u4[t] v2[t] v3[t]-u2[t]^2 u41[t] v2[t] v3[t]-u2[t]^2 v2[t]
v23[t] v3[t]-u2[t] u24[t] u4[t] v3[t]^2+u22[t] u4[t]^2 v3[t]^2-u2[t] u4[t]
u42[t] v3[t]^2+u2[t]^2 u44[t] v3[t]^2+u2[t]^2 v22[t] v3[t]^2-u2[t] u4[t]
v24[t] v3[t]^2-u2[t]^2 v2[t] v3[t] v32[t]+u2[t]^2 v2[t]^2 v33[t]+u2[t] u4[t]
v2[t] v3[t] v34[t]+u14[t] u2[t] u4[t] v3[t] v4[t]-u12[t] u4[t]^2 v3[t] v4[t]-u21[t]
u4[t]^2 v3[t] v4[t]+u2[t] u4[t] u41[t] v3[t] v4[t]+u2[t] u4[t] v23[t] v3[t]
v4[t]+u2[t] u4[t] v3[t] v32[t] v4[t]-2 u2[t] u4[t] v2[t] v33[t] v4[t]-u4[t]^2
v3[t] v34[t] v4[t]+u4[t]^2 v33[t] v4[t]^2+u11[t] (u2[t] v2[t]-u4[t] v4[t])^2-u2[t]
u4[t] v3[t]^2 v42[t]+u2[t] u4[t] v2[t] v3[t] v43[t]-u4[t]^2 v3[t] v4[t] v43[t]+u4[t]^2
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v3[t]^2 v44[t]+u1[t]^2 (u22[t] v2[t]^2-v4[t] (-(u44[t]+v22[t]) v4[t]+v2[t]
(u24[t]+u42[t]+v24[t]+v42[t])) +v2[t]^2 v44[t])+u1[t] (u12[t] v2[t] (-u2[t]
v2[t]+u4[t] v4[t])+u2[t] (-u21[t] v2[t]^2+u24[t] v2[t] v3[t]+u42[t] v2[t]
v3[t]+v2[t] v24[t] v3[t]-v2[t]^2 v34[t]+u14[t] v2[t] v4[t]+u41[t] v2[t] v4[t]+v2[t]
v23[t] v4[t]-2 u44[t] v3[t] v4[t]-2 v22[t] v3[t] v4[t]+v2[t] v32[t] v4[t]+v2[t]
v3[t] v42[t]-v2[t]^2 v43[t])+u4[t] (-2 u22[t] v2[t] v3[t]+v4[t] (-(u14[t]
+ u41[t] + v23[t] +v32[t]) v4[t]+v3[t] (u24[t]+u42[t]+v24[t]+v42[t])+v2[t]
(u21[t]+v34[t]+v43[t]))-2 v2[t] v3[t] v44[t]))) (-m U2[t] V2p[0]+m U4[t]
V4p[0])^2)/(2 (u2[t] v2[t]-u4[t] v4[t])^2 (2 U2[t] V2[t]-2 U4[t] V4[t])^2
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2) +(m sq2p2i (U2p[0] U4[t]-U2[t]
U4p[0])^2 (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0]
V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2)^2) +(m vep2i veq2i (U2p[0] U4[t]-U2[t] U4p[0])^2
(U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0]
V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2)^2)-(m vep1i veq2i (U2p[0] U4[t]-U2[t] U4p[0]) (-m
U2[t] V2p[0]+m U4[t] V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0]
V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2
U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2) - (2 m veq1i veq2i (U2p[0] U4[t]-U2[t]
U4p[0]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]) (U2[t] (-V1p[0]
V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
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U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)
- (2 m sq2q2i (U2p[0] U4[t]-U2[t] U4p[0]) (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t]
(-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t]
V2p[0]+m U4[t] V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0]
V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2
U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2) - (2 m veq2i^2 (U2p[0] U4[t]-U2[t]
U4p[0]) (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]) (U2[t] (-V1p[0]
V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)
+(m^2 sq2q2i (U2p[0] U4[t]-U2[t] U4p[0])^2 (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0]))^2)/((2 U2[t]
V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2) +(m^2 veq2i^2
(U2p[0] U4[t]-U2[t] U4p[0])^2 (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0]
V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0]))^2)/((2 U2[t] V2[t]-2
U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2) +(m vep2i veq1i (U2p[0] U4[t]-U2[t]
U4p[0])^2 (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t]
(V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0]
V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2)^2)-(m sq1p1i (U2p[0] U4[t]-U2[t] U4p[0]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0]
V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2
U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2

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U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)-(m vep1i veq1i (U2p[0] U4[t]-U2[t]
U4p[0]) (-m U2[t] V2p[0]+m U4[t] V4p[0]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t]
(-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2) - (2 m sq1q1i
(U2p[0] U4[t]-U2[t] U4p[0]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0]
V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m
U4[t] V4p[0]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t]
V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2) - (2 m veq1i^2 (U2p[0] U4[t]-U2[t]
U4p[0]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]) (U2[t] (V2p[0]
V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)
- (2 m veq1i veq2i (U2p[0] U4[t]-U2[t] U4p[0]) (U2[t] (U4p[0] V1[t]-U3p[0]
V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t]))
(-m U2[t] V2p[0]+m U4[t] V4p[0]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t]
(-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)+(2 m^2 veq1i
veq2i (U2p[0] U4[t]-U2[t] U4p[0])^2 (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])) (U2[t] (V2p[0]
V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
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U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2)
+(m^2 sq1q1i (U2p[0] U4[t]-U2[t] U4p[0])^2 (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t]
(-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0]))^2)/((2 U2[t]
V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2) +(m^2 veq1i^2
(U2p[0] U4[t]-U2[t] U4p[0])^2 (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0]
V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0]))^2)/((2 U2[t] V2[t]-2
U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)^2);

sp1p1[t_]:=(4 sq1q1i (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t]
V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t])^2)/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2+(4 veq1i^2 (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t]
U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t])^2)/(2
U2[t] V2[t]-2 U4[t] V4[t])^2+(8 veq1i veq2i (U1p[t] U2[t] V2[t]-U1[t] U2p[t]
V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t]
V4[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t]
V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2+(4
sq2q2i (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t]
V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t])^2)/(2 U2[t] V2[t]-2 U4[t] V4[t])^2+(4
veq2i^2 (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t]
V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t])^2)/(2 U2[t] V2[t]-2 U4[t] V4[t])^2+(2
(u22[t] v2[t]^2-v4[t] (-(u44[t]+v22[t]) v4[t]+v2[t] (u24[t]+u42[t]+v24[t]+v42[t]))
+v2[t]^2 v44[t]))/(u2[t] v2[t]-u4[t] v4[t])^2+(sp2p2i (-U2p[t] U4[t]+U2[t]
U4p[t])^2)/(-m U2[t] V2p[0]+m U4[t] V4p[0])^2+(vep2i^2 (-U2p[t] U4[t]+U2[t]
U4p[t])^2)/(-m U2[t] V2p[0]+m U4[t] V4p[0])^2+(2 (-U2p[t] U4[t]+U2[t] U4p[t])^2
(u22[t] (u4[t] v1[t]-u3[t] v2[t])^2+u2[t]^2 (u44[t] v1[t]^2-v2[t] (-(u33[t]+v11[t])
v2[t]+v1[t] (u34[t]+u43[t]+v12[t]+v21[t]))+v1[t]^2 v22[t])+u24[t] u3[t] u4[t]
v1[t] v4[t]-u23[t] u4[t]^2 v1[t] v4[t]-u32[t] u4[t]^2 v1[t] v4[t]+u3[t] u4[t]
u42[t] v1[t] v4[t]-u4[t]^2 v1[t] v14[t] v4[t]-u24[t] u3[t]^2 v2[t] v4[t]+u23[t]
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u3[t] u4[t] v2[t] v4[t]+u3[t] u32[t] u4[t] v2[t] v4[t]-u3[t]^2 u42[t] v2[t]
v4[t]+u3[t] u4[t] v14[t] v2[t] v4[t]+u3[t] u4[t] v1[t] v24[t] v4[t]-u3[t]^2
v2[t] v24[t] v4[t]-u3[t] u34[t] u4[t] v4[t]^2+u33[t] u4[t]^2 v4[t]^2-u3[t]
u4[t] u43[t] v4[t]^2+u3[t]^2 u44[t] v4[t]^2+u4[t]^2 v11[t] v4[t]^2-u3[t]
u4[t] v12[t] v4[t]^2-u3[t] u4[t] v21[t] v4[t]^2+u3[t]^2 v22[t] v4[t]^2-u4[t]^2
v1[t] v4[t] v41[t]+u3[t] u4[t] v2[t] v4[t] v41[t]+u3[t] u4[t] v1[t] v4[t]
v42[t]-u3[t]^2 v2[t] v4[t] v42[t]+u2[t] (u24[t] v1[t] (-u4[t] v1[t]+u3[t]
v2[t])+u4[t] (-u42[t] v1[t]^2+u23[t] v1[t] v2[t]+u32[t] v1[t] v2[t]+v1[t]
v14[t] v2[t]-v1[t]^2 v24[t]+u34[t] v1[t] v4[t]+u43[t] v1[t] v4[t]+v1[t] v12[t]
v4[t]-2 u33[t] v2[t] v4[t]-2 v11[t] v2[t] v4[t]+v1[t] v21[t] v4[t]+v1[t]
v2[t] v41[t]-v1[t]^2 v42[t])+u3[t] (u42[t] v1[t] v2[t]-u23[t] v2[t]^2-u32[t]
v2[t]^2-v14[t] v2[t]^2+v1[t] v2[t] v24[t]-2 u44[t] v1[t] v4[t]+u34[t] v2[t]
v4[t]+u43[t] v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t] v21[t] v4[t]-2 v1[t] v22[t]
v4[t]-v2[t]^2 v41[t]+v1[t] v2[t] v42[t]))+(u4[t] v1[t]-u3[t] v2[t])^2 v44[t]))/((u2[t]
v2[t]-u4[t] v4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)-(4 vep2i veq1i (-U2p[t]
U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t]
U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]))/((2 U2[t] V2[t]-2 U4[t]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))-(4 sq2p2i (-U2p[t] U4[t]+U2[t] U4p[t])
(U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t]
U4[t] V4[t]+U3[t] U4p[t] V4[t]))/((2 U2[t] V2[t]-2 U4[t] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))-(4 vep2i veq2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t]
U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t]
U4[t] V4[t]+U3[t] U4p[t] V4[t]))/((2 U2[t] V2[t]-2 U4[t] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))+(2 (-U2p[t] U4[t]+U2[t] U4p[t]) (2 u22[t] v2[t] (-u4[t]
v1[t]+u3[t] v2[t])+u2[t] (u24[t] v1[t] v2[t]+u42[t] v1[t] v2[t]-u23[t] v2[t]^2-u32[t]
v2[t]^2-v14[t] v2[t]^2+v1[t] v2[t] v24[t]-2 u44[t] v1[t] v4[t]+u34[t] v2[t]
v4[t]+u43[t] v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t] v21[t] v4[t]-2 v1[t] v22[t]
v4[t]-v2[t]^2 v41[t]+v1[t] v2[t] v42[t])+v4[t] (u24[t] (u4[t] v1[t]-2 u3[t]
v2[t])+u4[t] (-(u34[t]+u43[t] +v12[t]+v21[t]) v4[t]+v2[t] (u23[t]+u32[t] +
v14[t] + v41[t]) +v1[t] (u42[t] + v24[t] + v42[t]))-2 u3[t] (-(u44[t]+v22[t])
v4[t]+v2[t] (u42[t] + v24[t] + v42[t])))+2 v2[t] (-u4[t] v1[t]+u3[t] v2[t])
v44[t]))/((u2[t] v2[t]-u4[t] v4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0]))+(2
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vep1i vep2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]))/((2
U2[t] V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(4 vep2i
veq1i (U2p[0] U4[t]-U2[t] U4p[0]) (U2p[t] V2[t]-U4p[t] V4[t]) (U1p[t] U2[t]
V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t]
V4[t]+U1[t] U4p[t] V4[t]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) +(4 sq2p2i (U2p[0] U4[t]-U2[t] U4p[0]) (U2p[t] V2[t]-U4p[t]
V4[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t]
V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(4 vep2i veq2i (U2p[0] U4[t]-U2[t]
U4p[0]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t]
U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]))/((2
U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(4 vep2i
veq1i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t] (-U1p[0]
V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0]
V4[t])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
+(4 sq2p2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t]
(U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0]
V2[t]-U4p[0] V4[t])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) +(4 vep2i veq2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t]
V4[t]) (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t]
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(U2p[0] V2[t]-U4p[0] V4[t])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0]
V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2)) -(2 (U2p[0] U4[t]-U2[t] U4p[0]) (U2p[t] V2[t]-U4p[t]
V4[t]) (2 u22[t] v2[t] (-u4[t] v1[t]+u3[t] v2[t])+u2[t] (u24[t] v1[t] v2[t]+u42[t]
v1[t] v2[t]-u23[t] v2[t]^2-u32[t] v2[t]^2-v14[t] v2[t]^2+v1[t] v2[t] v24[t]-2
u44[t] v1[t] v4[t]+u34[t] v2[t] v4[t]+u43[t] v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t]
v21[t] v4[t]-2 v1[t] v22[t] v4[t]-v2[t]^2 v41[t]+v1[t] v2[t] v42[t])+v4[t]
(u24[t] (u4[t] v1[t]-2 u3[t] v2[t])+u4[t] (-(u34[t]+u43[t] +v12[t]+v21[t])
v4[t]+v2[t] (u23[t]+u32[t] + v14[t] + v41[t]) +v1[t] (u42[t] + v24[t] + v42[t]))-2
u3[t] (-(u44[t]+v22[t]) v4[t]+v2[t] (u42[t] + v24[t] + v42[t])))+2 v2[t]
(-u4[t] v1[t]+u3[t] v2[t]) v44[t]))/((u2[t] v2[t]-u4[t] v4[t])^2 (2 U2[t]
V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))+(2 (-U2p[t]
U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (u12[t] u2[t] u4[t] v1[t]
v2[t]+u2[t] u21[t] u4[t] v1[t] v2[t]-u2[t]^2 u41[t] v1[t] v2[t]+u13[t] u2[t]^2
v2[t]^2-u12[t] u2[t] u3[t] v2[t]^2-u2[t] u21[t] u3[t] v2[t]^2+u2[t]^2 u31[t]
v2[t]^2+u2[t]^2 v13[t] v2[t]^2-u2[t]^2 v1[t] v2[t] v23[t]-2 u2[t] u24[t]
u4[t] v1[t] v3[t]+2 u22[t] u4[t]^2 v1[t] v3[t]-2 u2[t] u4[t] u42[t] v1[t]
v3[t]+2 u2[t]^2 u44[t] v1[t] v3[t]+u2[t] u24[t] u3[t] v2[t] v3[t]-u2[t]^2
u34[t] v2[t] v3[t]+u2[t] u23[t] u4[t] v2[t] v3[t]-2 u22[t] u3[t] u4[t] v2[t]
v3[t]+u2[t] u32[t] u4[t] v2[t] v3[t]+u2[t] u3[t] u42[t] v2[t] v3[t]-u2[t]^2
u43[t] v2[t] v3[t]-u2[t]^2 v12[t] v2[t] v3[t]+u2[t] u4[t] v14[t] v2[t] v3[t]-u2[t]^2
v2[t] v21[t] v3[t]+2 u2[t]^2 v1[t] v22[t] v3[t]-2 u2[t] u4[t] v1[t] v24[t]
v3[t]+u2[t] u3[t] v2[t] v24[t] v3[t]+u2[t]^2 v2[t]^2 v31[t]-u2[t]^2 v1[t]
v2[t] v32[t]+u2[t] u4[t] v1[t] v2[t] v34[t]-u2[t] u3[t] v2[t]^2 v34[t]-u12[t]
u4[t]^2 v1[t] v4[t]-u21[t] u4[t]^2 v1[t] v4[t]+u2[t] u4[t] u41[t] v1[t] v4[t]-2
u13[t] u2[t] u4[t] v2[t] v4[t]+u12[t] u3[t] u4[t] v2[t] v4[t]+u21[t] u3[t]
u4[t] v2[t] v4[t]-2 u2[t] u31[t] u4[t] v2[t] v4[t]+u2[t] u3[t] u41[t] v2[t]
v4[t]-2 u2[t] u4[t] v13[t] v2[t] v4[t]+u2[t] u4[t] v1[t] v23[t] v4[t]+u2[t]
u3[t] v2[t] v23[t] v4[t]+u24[t] u3[t] u4[t] v3[t] v4[t]+u2[t] u34[t] u4[t]
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v3[t] v4[t]-u23[t] u4[t]^2 v3[t] v4[t]-u32[t] u4[t]^2 v3[t] v4[t]+u3[t] u4[t]
u42[t] v3[t] v4[t]+u2[t] u4[t] u43[t] v3[t] v4[t]-2 u2[t] u3[t] u44[t] v3[t]
v4[t]+u2[t] u4[t] v12[t] v3[t] v4[t]-u4[t]^2 v14[t] v3[t] v4[t]+u2[t] u4[t]
v21[t] v3[t] v4[t]-2 u2[t] u3[t] v22[t] v3[t] v4[t]+u3[t] u4[t] v24[t] v3[t]
v4[t]-2 u2[t] u4[t] v2[t] v31[t] v4[t]+u2[t] u4[t] v1[t] v32[t] v4[t]+u2[t]
u3[t] v2[t] v32[t] v4[t]-u4[t]^2 v1[t] v34[t] v4[t]+u3[t] u4[t] v2[t] v34[t]
v4[t]+u13[t] u4[t]^2 v4[t]^2+u31[t] u4[t]^2 v4[t]^2-u3[t] u4[t] u41[t] v4[t]^2+u4[t]^2
v13[t] v4[t]^2-u3[t] u4[t] v23[t] v4[t]^2+u4[t]^2 v31[t] v4[t]^2-u3[t] u4[t]
v32[t] v4[t]^2-u14[t] (u2[t] v1[t]-u3[t] v4[t]) (u2[t] v2[t]-u4[t] v4[t])+u2[t]
u4[t] v2[t] v3[t] v41[t]-u4[t]^2 v3[t] v4[t] v41[t]-2 u2[t] u4[t] v1[t] v3[t]
v42[t]+u2[t] u3[t] v2[t] v3[t] v42[t]+u3[t] u4[t] v3[t] v4[t] v42[t]+u2[t]
u4[t] v1[t] v2[t] v43[t]-u2[t] u3[t] v2[t]^2 v43[t]-u4[t]^2 v1[t] v4[t] v43[t]+u3[t]
u4[t] v2[t] v4[t] v43[t]+2 u4[t] (u4[t] v1[t]-u3[t] v2[t]) v3[t] v44[t]+u1[t]
(2 u22[t] v2[t] (-u4[t] v1[t]+u3[t] v2[t])+u2[t] (u24[t] v1[t] v2[t]+u42[t]
v1[t] v2[t]-u23[t] v2[t]^2-u32[t] v2[t]^2-v14[t] v2[t]^2+v1[t] v2[t] v24[t]-2
u44[t] v1[t] v4[t]+u34[t] v2[t] v4[t]+u43[t] v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t]
v21[t] v4[t]-2 v1[t] v22[t] v4[t]-v2[t]^2 v41[t]+v1[t] v2[t] v42[t])+v4[t]
(u24[t] (u4[t] v1[t]-2 u3[t] v2[t])+u4[t] (-(u34[t]+u43[t] +v12[t]+v21[t])
v4[t]+v2[t] (u23[t]+u32[t] + v14[t] + v41[t]) +v1[t] (u42[t] + v24[t] + v42[t]))-2
u3[t] (-(u44[t]+v22[t]) v4[t]+v2[t] (u42[t] + v24[t] + v42[t])))+2 v2[t]
(-u4[t] v1[t]+u3[t] v2[t]) v44[t])))/((u2[t] v2[t]-u4[t] v4[t])^2 (2 U2[t]
V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(2 sp2p2i
(U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t]
V4[t]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(2 vep2i^2 (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
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(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) -(4 (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])
(U2p[t] V2[t]-U4p[t] V4[t]) (u22[t] (u4[t] v1[t]-u3[t] v2[t])^2+u2[t]^2 (u44[t]
v1[t]^2-v2[t] (-(u33[t]+v11[t]) v2[t]+v1[t] (u34[t]+u43[t] +v12[t]+v21[t]))+v1[t]^2
v22[t])+u24[t] u3[t] u4[t] v1[t] v4[t]-u23[t] u4[t]^2 v1[t] v4[t]-u32[t]
u4[t]^2 v1[t] v4[t]+u3[t] u4[t] u42[t] v1[t] v4[t]-u4[t]^2 v1[t] v14[t] v4[t]-u24[t]
u3[t]^2 v2[t] v4[t]+u23[t] u3[t] u4[t] v2[t] v4[t]+u3[t] u32[t] u4[t] v2[t]
v4[t]-u3[t]^2 u42[t] v2[t] v4[t]+u3[t] u4[t] v14[t] v2[t] v4[t]+u3[t] u4[t]
v1[t] v24[t] v4[t]-u3[t]^2 v2[t] v24[t] v4[t]-u3[t] u34[t] u4[t] v4[t]^2+u33[t]
u4[t]^2 v4[t]^2-u3[t] u4[t] u43[t] v4[t]^2+u3[t]^2 u44[t] v4[t]^2+u4[t]^2
v11[t] v4[t]^2-u3[t] u4[t] v12[t] v4[t]^2-u3[t] u4[t] v21[t] v4[t]^2+u3[t]^2
v22[t] v4[t]^2-u4[t]^2 v1[t] v4[t] v41[t]+u3[t] u4[t] v2[t] v4[t] v41[t]+u3[t]
u4[t] v1[t] v4[t] v42[t]-u3[t]^2 v2[t] v4[t] v42[t]+u2[t] (u24[t] v1[t] (-u4[t]
v1[t]+u3[t] v2[t])+u4[t] (-u42[t] v1[t]^2+u23[t] v1[t] v2[t]+u32[t] v1[t]
v2[t]+v1[t] v14[t] v2[t]-v1[t]^2 v24[t]+u34[t] v1[t] v4[t]+u43[t] v1[t] v4[t]+v1[t]
v12[t] v4[t]-2 u33[t] v2[t] v4[t]-2 v11[t] v2[t] v4[t]+v1[t] v21[t] v4[t]+v1[t]
v2[t] v41[t]-v1[t]^2 v42[t])+u3[t] (u42[t] v1[t] v2[t]-u23[t] v2[t]^2-u32[t]
v2[t]^2-v14[t] v2[t]^2+v1[t] v2[t] v24[t]-2 u44[t] v1[t] v4[t]+u34[t] v2[t]
v4[t]+u43[t] v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t] v21[t] v4[t]-2 v1[t] v22[t]
v4[t]-v2[t]^2 v41[t]+v1[t] v2[t] v42[t]))+(u4[t] v1[t]-u3[t] v2[t])^2 v44[t]))/((u2[t]
v2[t]-u4[t] v4[t])^2 (2 U2[t] V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t]
V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(4 sq1p1i (U2p[t] V2[t]-U4p[t]
V4[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t]
V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/((2
U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(4 vep1i
veq1i (U2p[t] V2[t]-U4p[t] V4[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t]
U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (-m
U2[t] V2p[0]+m U4[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0]
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V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(4 vep1i veq2i (U2p[t] V2[t]-U4p[t] V4[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 sq1q1i (U2p[t] V2[t]-U4p[t] V4[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 veq1i^2 (U2p[t] V2[t]-U4p[t] V4[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 veq1i veq2i (U2p[t] V2[t]-U4p[t] V4[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 veq1i veq2i (U2p[t] V2[t]-U4p[t] V4[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m

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U4[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
-(8 sq2q2i (U2p[t] V2[t]-U4p[t] V4[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t]
U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (U2[t]
(U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0]
V2[t]-U4p[0] V4[t])) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/((2 U2[t] V2[t]-2
U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 veq2i^2 (U2p[t] V2[t]-U4p[t]
V4[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t]
V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (U2[t] (U4p[0] V1[t]-U3p[0]
V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t]))
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0]
V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2)) -(2 (U2p[t] V2[t]-U4p[t] V4[t]) (-2 u1[t] u22[t]
v2[t]^2+2 u22[t] u4[t] v2[t] v3[t]+2 u1[t] u24[t] v2[t] v4[t]-u21[t] u4[t]
v2[t] v4[t]+2 u1[t] u42[t] v2[t] v4[t]+2 u1[t] v2[t] v24[t] v4[t]-u24[t]
u4[t] v3[t] v4[t]-u4[t] u42[t] v3[t] v4[t]-u4[t] v24[t] v3[t] v4[t]-u4[t]
v2[t] v34[t] v4[t]+u14[t] u4[t] v4[t]^2+u4[t] u41[t] v4[t]^2-2 u1[t] u44[t]
v4[t]^2-2 u1[t] v22[t] v4[t]^2+u4[t] v23[t] v4[t]^2+u4[t] v32[t] v4[t]^2+u12[t]
v2[t] (u2[t] v2[t]-u4[t] v4[t])+2 u1[t] v2[t] v4[t] v42[t]-u4[t] v3[t] v4[t]
v42[t]-u4[t] v2[t] v4[t] v43[t]+u2[t] (u21[t] v2[t]^2-u24[t] v2[t] v3[t]-u42[t]
v2[t] v3[t]-v2[t] v24[t] v3[t]+v2[t]^2 v34[t]-u14[t] v2[t] v4[t]-u41[t] v2[t]
v4[t]-v2[t] v23[t] v4[t]+2 u44[t] v3[t] v4[t]+2 v22[t] v3[t] v4[t]-v2[t]
v32[t] v4[t]-v2[t] v3[t] v42[t]+v2[t]^2 v43[t])+2 v2[t] (-u1[t] v2[t]+u4[t]
v3[t]) v44[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/((u2[t] v2[t]-u4[t] v4[t])^2
(2 U2[t] V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(4
m sq1p1i (-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t]
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U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(4 m vep1i veq1i (-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(4 m vep1i veq2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m sq1q1i (-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m veq1i^2 (-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m veq1i veq2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t]

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V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
+(8 m veq1i veq2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t]
U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t]
U4p[t] V4[t]) (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0]
V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t] V4p[0]))/((2
U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m sq2q2i
(-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t]
U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (U2[t]
(U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0]
V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t]
V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m veq2i^2 (-U2p[t] U4[t]+U2[t]
U4p[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t]
V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (U2[t] (U4p[0] V1[t]-U3p[0]
V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t]))
(V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0]
V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2))+(2 m (-U2p[t] U4[t]+U2[t] U4p[t]) (-2 u1[t] u22[t]
v2[t]^2+2 u22[t] u4[t] v2[t] v3[t]+2 u1[t] u24[t] v2[t] v4[t]-u21[t] u4[t]
v2[t] v4[t]+2 u1[t] u42[t] v2[t] v4[t]+2 u1[t] v2[t] v24[t] v4[t]-u24[t]
u4[t] v3[t] v4[t]-u4[t] u42[t] v3[t] v4[t]-u4[t] v24[t] v3[t] v4[t]-u4[t]
v2[t] v34[t] v4[t]+u14[t] u4[t] v4[t]^2+u4[t] u41[t] v4[t]^2-2 u1[t] u44[t]
v4[t]^2-2 u1[t] v22[t] v4[t]^2+u4[t] v23[t] v4[t]^2+u4[t] v32[t] v4[t]^2+u12[t]
v2[t] (u2[t] v2[t]-u4[t] v4[t])+2 u1[t] v2[t] v4[t] v42[t]-u4[t] v3[t] v4[t]
v42[t]-u4[t] v2[t] v4[t] v43[t]+u2[t] (u21[t] v2[t]^2-u24[t] v2[t] v3[t]-u42[t]
v2[t] v3[t]-v2[t] v24[t] v3[t]+v2[t]^2 v34[t]-u14[t] v2[t] v4[t]-u41[t] v2[t]
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v4[t]-v2[t] v23[t] v4[t]+2 u44[t] v3[t] v4[t]+2 v22[t] v3[t] v4[t]-v2[t]
v32[t] v4[t]-v2[t] v3[t] v42[t]+v2[t]^2 v43[t])+2 v2[t] (-u1[t] v2[t]+u4[t]
v3[t]) v44[t]) (V2p[0] V4[t]-V2[t] V4p[0]))/((u2[t] v2[t]-u4[t] v4[t])^2
(2 U2[t] V2[t]-2 U4[t] V4[t])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))+(2
m sp2p2i (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0]
V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m
U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))+(2 m vep2i^2 (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) +(4 m (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2
(u22[t] (u4[t] v1[t]-u3[t] v2[t])^2+u2[t]^2 (u44[t] v1[t]^2-v2[t] (-(u33[t]+v11[t])
v2[t]+v1[t] (u34[t]+u43[t] +v12[t]+v21[t]))+v1[t]^2 v22[t])+u24[t] u3[t]
u4[t] v1[t] v4[t]-u23[t] u4[t]^2 v1[t] v4[t]-u32[t] u4[t]^2 v1[t] v4[t]+u3[t]
u4[t] u42[t] v1[t] v4[t]-u4[t]^2 v1[t] v14[t] v4[t]-u24[t] u3[t]^2 v2[t]
v4[t]+u23[t] u3[t] u4[t] v2[t] v4[t]+u3[t] u32[t] u4[t] v2[t] v4[t]-u3[t]^2
u42[t] v2[t] v4[t]+u3[t] u4[t] v14[t] v2[t] v4[t]+u3[t] u4[t] v1[t] v24[t]
v4[t]-u3[t]^2 v2[t] v24[t] v4[t]-u3[t] u34[t] u4[t] v4[t]^2+u33[t] u4[t]^2
v4[t]^2-u3[t] u4[t] u43[t] v4[t]^2+u3[t]^2 u44[t] v4[t]^2+u4[t]^2 v11[t]
v4[t]^2-u3[t] u4[t] v12[t] v4[t]^2-u3[t] u4[t] v21[t] v4[t]^2+u3[t]^2 v22[t]
v4[t]^2-u4[t]^2 v1[t] v4[t] v41[t]+u3[t] u4[t] v2[t] v4[t] v41[t]+u3[t] u4[t]
v1[t] v4[t] v42[t]-u3[t]^2 v2[t] v4[t] v42[t]+u2[t] (u24[t] v1[t] (-u4[t]
v1[t]+u3[t] v2[t])+u4[t] (-u42[t] v1[t]^2+u23[t] v1[t] v2[t]+u32[t] v1[t]
v2[t]+v1[t] v14[t] v2[t]-v1[t]^2 v24[t]+u34[t] v1[t] v4[t]+u43[t] v1[t] v4[t]+v1[t]
v12[t] v4[t]-2 u33[t] v2[t] v4[t]-2 v11[t] v2[t] v4[t]+v1[t] v21[t] v4[t]+v1[t]
v2[t] v41[t]-v1[t]^2 v42[t])+u3[t] (u42[t] v1[t] v2[t]-u23[t] v2[t]^2-u32[t]
v2[t]^2-v14[t] v2[t]^2+v1[t] v2[t] v24[t]-2 u44[t] v1[t] v4[t]+u34[t] v2[t]

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v4[t]+u43[t] v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t] v21[t] v4[t]-2 v1[t] v22[t]
v4[t]-v2[t]^2 v41[t]+v1[t] v2[t] v42[t]))+(u4[t] v1[t]-u3[t] v2[t])^2 v44[t])
(V2p[0] V4[t]-V2[t] V4p[0]))/((u2[t] v2[t]-u4[t] v4[t])^2 (2 U2[t] V2[t]-2
U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) -(2 m vep1i vep2i (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0]
V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m
U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(4 m vep2i veq1i (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t]
U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (V2p[0]
V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m
U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(4 m sq2p2i (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t]
U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (V2p[0]
V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m
U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(4 m vep2i veq2i (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t]
U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (V2p[0]
V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m
U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(4 m vep2i veq1i (-U2p[t] U4[t]+U2[t]
U4p[t])^2 (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0]
V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t] V4p[0]))/((2
U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0]
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V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2)) -(4 m sq2p2i (-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t]
(U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0]
V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t]
V4[t])^3 (-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
-(4 m vep2i veq2i (-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (U4p[0] V1[t]-U3p[0]
V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t]))
(V2p[0] V4[t]-V2[t] V4p[0]))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m
U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))+(2 m (U2p[0] U4[t]-U2[t] U4p[0])
(-U2p[t] U4[t]+U2[t] U4p[t]) (2 u22[t] v2[t] (-u4[t] v1[t]+u3[t] v2[t])+u2[t]
(u24[t] v1[t] v2[t]+u42[t] v1[t] v2[t]-u23[t] v2[t]^2-u32[t] v2[t]^2-v14[t]
v2[t]^2+v1[t] v2[t] v24[t]-2 u44[t] v1[t] v4[t]+u34[t] v2[t] v4[t]+u43[t]
v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t] v21[t] v4[t]-2 v1[t] v22[t] v4[t]-v2[t]^2
v41[t]+v1[t] v2[t] v42[t])+v4[t] (u24[t] (u4[t] v1[t]-2 u3[t] v2[t])+u4[t]
(-(u34[t]+u43[t] +v12[t]+v21[t]) v4[t]+v2[t] (u23[t]+u32[t] + v14[t] + v41[t])
+v1[t] (u42[t] + v24[t] + v42[t]))-2 u3[t] (-(u44[t]+v22[t]) v4[t]+v2[t]
(u42[t] + v24[t] + v42[t])))+2 v2[t] (-u4[t] v1[t]+u3[t] v2[t]) v44[t]) (V2p[0]
V4[t]-V2[t] V4p[0]))/((u2[t] v2[t]-u4[t] v4[t])^2 (2 U2[t] V2[t]-2 U4[t]
V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
-(2 m (-U2p[t] U4[t]+U2[t] U4p[t])^2 (u12[t] u2[t] u4[t] v1[t] v2[t]+u2[t]
u21[t] u4[t] v1[t] v2[t]-u2[t]^2 u41[t] v1[t] v2[t]+u13[t] u2[t]^2 v2[t]^2-u12[t]
u2[t] u3[t] v2[t]^2-u2[t] u21[t] u3[t] v2[t]^2+u2[t]^2 u31[t] v2[t]^2+u2[t]^2
v13[t] v2[t]^2-u2[t]^2 v1[t] v2[t] v23[t]-2 u2[t] u24[t] u4[t] v1[t] v3[t]+2
u22[t] u4[t]^2 v1[t] v3[t]-2 u2[t] u4[t] u42[t] v1[t] v3[t]+2 u2[t]^2 u44[t]
v1[t] v3[t]+u2[t] u24[t] u3[t] v2[t] v3[t]-u2[t]^2 u34[t] v2[t] v3[t]+u2[t]
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u23[t] u4[t] v2[t] v3[t]-2 u22[t] u3[t] u4[t] v2[t] v3[t]+u2[t] u32[t] u4[t]
v2[t] v3[t]+u2[t] u3[t] u42[t] v2[t] v3[t]-u2[t]^2 u43[t] v2[t] v3[t]-u2[t]^2
v12[t] v2[t] v3[t]+u2[t] u4[t] v14[t] v2[t] v3[t]-u2[t]^2 v2[t] v21[t] v3[t]+2
u2[t]^2 v1[t] v22[t] v3[t]-2 u2[t] u4[t] v1[t] v24[t] v3[t]+u2[t] u3[t] v2[t]
v24[t] v3[t]+u2[t]^2 v2[t]^2 v31[t]-u2[t]^2 v1[t] v2[t] v32[t]+u2[t] u4[t]
v1[t] v2[t] v34[t]-u2[t] u3[t] v2[t]^2 v34[t]-u12[t] u4[t]^2 v1[t] v4[t]-u21[t]
u4[t]^2 v1[t] v4[t]+u2[t] u4[t] u41[t] v1[t] v4[t]-2 u13[t] u2[t] u4[t] v2[t]
v4[t]+u12[t] u3[t] u4[t] v2[t] v4[t]+u21[t] u3[t] u4[t] v2[t] v4[t]-2 u2[t]
u31[t] u4[t] v2[t] v4[t]+u2[t] u3[t] u41[t] v2[t] v4[t]-2 u2[t] u4[t] v13[t]
v2[t] v4[t]+u2[t] u4[t] v1[t] v23[t] v4[t]+u2[t] u3[t] v2[t] v23[t] v4[t]+u24[t]
u3[t] u4[t] v3[t] v4[t]+u2[t] u34[t] u4[t] v3[t] v4[t]-u23[t] u4[t]^2 v3[t]
v4[t]-u32[t] u4[t]^2 v3[t] v4[t]+u3[t] u4[t] u42[t] v3[t] v4[t]+u2[t] u4[t]
u43[t] v3[t] v4[t]-2 u2[t] u3[t] u44[t] v3[t] v4[t]+u2[t] u4[t] v12[t] v3[t]
v4[t]-u4[t]^2 v14[t] v3[t] v4[t]+u2[t] u4[t] v21[t] v3[t] v4[t]-2 u2[t] u3[t]
v22[t] v3[t] v4[t]+u3[t] u4[t] v24[t] v3[t] v4[t]-2 u2[t] u4[t] v2[t] v31[t]
v4[t]+u2[t] u4[t] v1[t] v32[t] v4[t]+u2[t] u3[t] v2[t] v32[t] v4[t]-u4[t]^2
v1[t] v34[t] v4[t]+u3[t] u4[t] v2[t] v34[t] v4[t]+u13[t] u4[t]^2 v4[t]^2+u31[t]
u4[t]^2 v4[t]^2-u3[t] u4[t] u41[t] v4[t]^2+u4[t]^2 v13[t] v4[t]^2-u3[t] u4[t]
v23[t] v4[t]^2+u4[t]^2 v31[t] v4[t]^2-u3[t] u4[t] v32[t] v4[t]^2-u14[t] (u2[t]
v1[t]-u3[t] v4[t]) (u2[t] v2[t]-u4[t] v4[t])+u2[t] u4[t] v2[t] v3[t] v41[t]-u4[t]^2
v3[t] v4[t] v41[t]-2 u2[t] u4[t] v1[t] v3[t] v42[t]+u2[t] u3[t] v2[t] v3[t]
v42[t]+u3[t] u4[t] v3[t] v4[t] v42[t]+u2[t] u4[t] v1[t] v2[t] v43[t]-u2[t]
u3[t] v2[t]^2 v43[t]-u4[t]^2 v1[t] v4[t] v43[t]+u3[t] u4[t] v2[t] v4[t] v43[t]+2
u4[t] (u4[t] v1[t]-u3[t] v2[t]) v3[t] v44[t]+u1[t] (2 u22[t] v2[t] (-u4[t]
v1[t]+u3[t] v2[t])+u2[t] (u24[t] v1[t] v2[t]+u42[t] v1[t] v2[t]-u23[t] v2[t]^2-u32[t]
v2[t]^2-v14[t] v2[t]^2+v1[t] v2[t] v24[t]-2 u44[t] v1[t] v4[t]+u34[t] v2[t]
v4[t]+u43[t] v2[t] v4[t]+v12[t] v2[t] v4[t]+v2[t] v21[t] v4[t]-2 v1[t] v22[t]
v4[t]-v2[t]^2 v41[t]+v1[t] v2[t] v42[t])+v4[t] (u24[t] (u4[t] v1[t]-2 u3[t]
v2[t])+u4[t] (-(u34[t]+u43[t] +v12[t]+v21[t]) v4[t]+v2[t] (u23[t]+u32[t] +
v14[t] + v41[t]) +v1[t] (u42[t] + v24[t] + v42[t]))-2 u3[t] (-(u44[t]+v22[t])
v4[t]+v2[t] (u42[t] + v24[t] + v42[t])))+2 v2[t] (-u4[t] v1[t]+u3[t] v2[t])
v44[t])) (V2p[0] V4[t]-V2[t] V4p[0]))/((u2[t] v2[t]-u4[t] v4[t])^2 (2 U2[t]
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V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) +(4 sq1q1[t] (2 U2[t] V2[t]-2 U4[t] V4[t])^2 (-(((U2p[t]
V2[t]-U4p[t] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2)+(m (-U2p[t] U4[t]+U2[t] U4p[t]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2)^2) /(-m U2[t] V2p[0]+m U4[t] V4p[0])^2+(4 m
sq2p2i (-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t]
V2[t]-2 U4[t] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)+(4 m vep2i veq2i
(-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t]
V2[t]-2 U4[t] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)-(8 m veq1i veq2i
(-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t]
U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t]
(-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0]
V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m
U4[t] V4p[0]))-(8 m sq2q2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] U4[t] V1[t]-U2[t]
U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t]
U4p[t] V4[t]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t]
V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^2
(-m U2[t] V2p[0]+m U4[t] V4p[0]))-(8 m veq2i^2 (-U2p[t] U4[t]+U2[t] U4p[t])
(U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t]
U4[t] V4[t]+U3[t] U4p[t] V4[t]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0]))+(4 m vep1i veq2i
(-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t] (-V1p[0]
V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
+(8 m veq1i veq2i (U2p[0] U4[t]-U2[t] U4p[0]) (U2p[t] V2[t]-U4p[t] V4[t])
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(U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t]
U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0]))))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m sq2q2i
(U2p[0] U4[t]-U2[t] U4p[0]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2p[t] U4[t] V1[t]-U2[t]
U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t]
U4p[t] V4[t]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t]
V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0]))))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m veq2i^2 (U2p[0] U4[t]-U2[t]
U4p[0]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t]
U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (U2[t]
(-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0]
V4[t]+V2[t] V4p[0]))))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) +(8 m veq1i veq2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t]
V4[t]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0]
V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0]))))/((2 U2[t] V2[t]-2
U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m sq2q2i (-U2p[t] U4[t]+U2[t]
U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t]
(-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t])) (U2[t] (-V1p[0]
V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t]
V4p[0]))))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
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+(8 m veq2i^2 (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t]
(U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t] (U2p[0]
V2[t]-U4p[0] V4[t])) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t]
V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m sq2p2i (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t] (-V1p[0]
V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m U4[t] V4p[0])
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m vep2i veq2i (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t] (-V1p[0]
V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m U4[t] V4p[0])
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m^2 sq2p2i (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t]
(-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0]
V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m
U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m^2 vep2i veq2i (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t]
(-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0]
V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m
U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(4 m^2 vep1i veq2i (-U2p[t] U4[t]+U2[t]
U4p[t])^2 (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
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(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0]))))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) -(8 m^2 veq1i veq2i (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t]
U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t]
U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (V2p[0] V4[t]-V2[t] V4p[0])
(U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t]
V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m^2 sq2q2i
(U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] U4[t] V1[t]-U2[t]
U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t]
U4p[t] V4[t]) (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^4 (-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) -(8 m^2 veq2i^2 (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t]
U4p[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t]
V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (V2p[0] V4[t]-V2[t] V4p[0])
(U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t]
V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m^2 veq1i
veq2i (-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t]
(-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t]
V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t]
V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
```

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U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m^2 sq2q2i
(-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0]
V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t]
V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0]))))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t]
V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m^2 veq2i^2
(-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0]
V1[t]+U3p[0] V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t]
V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0]))))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t]
V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(4 m^2 sq2q2i
(-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0]))^2)/((2 U2[t]
V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)+(4 m^2 veq2i^2
(-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t]
(V1p[0] V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0]))^2)/((2 U2[t]
V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)-(8 m^2 sq2q2i
(U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t]
V4[t]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t]
V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m^2 veq2i^2
(U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t]
V4[t]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t]
V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
```

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U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m^3 sq2q2i
(U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0] V4[t]-V2[t]
V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t]
V2p[0]+m U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m^3 veq2i^2
(U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0] V4[t]-V2[t]
V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t]
V2p[0]+m U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(4 m vep2i
veq1i (-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t]
(-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t]
V2[t]-2 U4[t] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)-(8 m sq1q1i (-U2p[t]
U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t]
U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t] (V2p[0] V3[t]-V2[t]
V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2
U2[t] V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0]))-(8 m veq1i^2
(-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t]
U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t]
(V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0]
V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m
U4[t] V4p[0]))-(8 m veq1i veq2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] U4[t]
V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t]
V4[t]+U3[t] U4p[t] V4[t]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0]
V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2
U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0]))+(4 m sq1p1i (-U2p[t] U4[t]+U2[t]
U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t]
(-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t]
```

```
V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(4 m vep1i
veq1i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t] (V2p[0]
V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
+(8 m sq1q1i (U2p[0] U4[t]-U2[t] U4p[0]) (U2p[t] V2[t]-U4p[t] V4[t]) (U1p[t]
U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t]
U4[t] V4[t]+U1[t] U4p[t] V4[t]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t]
(-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m
U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0])
(V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m veq1i^2
(U2p[0] U4[t]-U2[t] U4p[0]) (U2p[t] V2[t]-U4p[t] V4[t]) (U1p[t] U2[t] V2[t]-U1[t]
U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t]
U4p[t] V4[t]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t]
V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m veq1i veq2i (U2p[0] U4[t]-U2[t]
U4p[0]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t]
U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t] U4[t] V4[t]+U3[t] U4p[t] V4[t]) (U2[t]
(V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0]
V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) +(8 m sq1q1i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t]
V4[t]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0] V4[t])+U1[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0]
V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2
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U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m veq1i^2 (-U2p[t] U4[t]+U2[t]
U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t]
(-U2p[0] V3[t]+U1p[0] V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (U2[t] (V2p[0]
V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
+(8 m veq1i veq2i (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t])
(U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0] V4[t])+U3[t]
(U2p[0] V2[t]-U4p[0] V4[t])) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0]
V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2
U4[t] V4[t])^4 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m vep2i veq1i (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t]) (U2[t] (V2p[0]
V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m U4[t] V4p[0])
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m^2 vep2i veq1i (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t]
(V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0]
V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m
U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(4 m^2 sq1p1i (-U2p[t] U4[t]+U2[t]
U4p[t])^2 (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t]
(-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t]
V2[t]-2 U4[t] V4[t])^3 (-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0]
V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m
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(U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2
U4[t] V4[t])^2)) -(4 m^2 vep1i veq1i (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0]
V4[t]-V2[t] V4p[0]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t]
V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^3
(-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8
m^2 sq1q1i (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t]
U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t] U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t]
U4[t] V4[t]+U1[t] U4p[t] V4[t]) (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t] (V2p[0]
V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t] V2p[0]+m U4[t] V4p[0])
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m^2 veq1i^2 (U2p[0] U4[t]-U2[t]
U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U1p[t] U2[t] V2[t]-U1[t] U2p[t] V2[t]+U2p[t]
U4[t] V3[t]-U2[t] U4p[t] V3[t]-U1p[t] U4[t] V4[t]+U1[t] U4p[t] V4[t]) (V2p[0]
V4[t]-V2[t] V4p[0]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t]
V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4
(-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8
m^2 veq1i veq2i (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])
(U2p[t] U4[t] V1[t]-U2[t] U4p[t] V1[t]-U2p[t] U3[t] V2[t]+U2[t] U3p[t] V2[t]-U3p[t]
U4[t] V4[t]+U3[t] U4p[t] V4[t]) (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t] (V2p[0]
V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t]
V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t] V2p[0]+m U4[t] V4p[0])
(((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m^2 sq1q1i (-U2p[t] U4[t]+U2[t]
U4p[t])^2 (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0]
V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t]
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(V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0]
V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t] V2p[0]+m
U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m^2 veq1i^2 (-U2p[t] U4[t]+U2[t]
U4p[t])^2 (U2[t] (-U1p[0] V2[t]+U4p[0] V3[t])+U4[t] (-U2p[0] V3[t]+U1p[0]
V4[t])+U1[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t]
(V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0]
V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t] V2p[0]+m
U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8 m^2 veq1i veq2i (-U2p[t] U4[t]+U2[t]
U4p[t])^2 (U2[t] (U4p[0] V1[t]-U3p[0] V2[t])+U4[t] (-U2p[0] V1[t]+U3p[0]
V4[t])+U3[t] (U2p[0] V2[t]-U4p[0] V4[t])) (V2p[0] V4[t]-V2[t] V4p[0]) (U2[t]
(V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0]
V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t] V2p[0]+m
U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2
U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t]
V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8 m^2 veq1i veq2i (-U2p[t] U4[t]+U2[t]
U4p[t])^2 (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0])) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0]
V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2
U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)-(16 m^2 veq1i veq2i (U2p[0]
U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t] V2[t]-U4p[t] V4[t])
(U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0] V4[t]-V1[t] V4p[0])+U3[t]
(-V2p[0] V4[t]+V2[t] V4p[0])) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0]
V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0])))/((2 U2[t] V2[t]-2
U4[t] V4[t])^4 (-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t])
(-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0]
U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2))
+(16 m^3 veq1i veq2i (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2
(V2p[0] V4[t]-V2[t] V4p[0]) (U2[t] (-V1p[0] V2[t]+V1[t] V2p[0])+U4[t] (V1p[0]
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V4[t]-V1[t] V4p[0])+U3[t] (-V2p[0] V4[t]+V2[t] V4p[0])) (U2[t] (V2p[0] V3[t]-V2[t]
V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0]))))/((2
U2[t] V2[t]-2 U4[t] V4[t])^4 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2 (((-U2p[0]
V2[t]+U4p[0] V4[t]) (-m U2[t] V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t]
V4[t])^2-(m (U2p[0] U4[t]-U2[t] U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t]
V2[t]-2 U4[t] V4[t])^2)) +(4 m^2 sq1q1i (-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t]
(V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0]
V4[t]-V3[t] V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m
U4[t] V4p[0])^2)+(4 m^2 veq1i^2 (-U2p[t] U4[t]+U2[t] U4p[t])^2 (U2[t] (V2p[0]
V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t] V4p[0])+U4[t] (V3p[0] V4[t]-V3[t]
V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^2 (-m U2[t] V2p[0]+m U4[t] V4p[0])^2)-(8
m^2 sq1q1i (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t]
V2[t]-U4p[t] V4[t]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t]
V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4
(-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) -(8
m^2 veq1i^2 (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t]) (U2p[t]
V2[t]-U4p[t] V4[t]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t]
V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4
(-m U2[t] V2p[0]+m U4[t] V4p[0]) (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8
m^3 sq1q1i (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0]
V4[t]-V2[t] V4p[0]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t]
V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4
(-m U2[t] V2p[0]+m U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2)) +(8
m^3 veq1i^2 (U2p[0] U4[t]-U2[t] U4p[0]) (-U2p[t] U4[t]+U2[t] U4p[t])^2 (V2p[0]
V4[t]-V2[t] V4p[0]) (U2[t] (V2p[0] V3[t]-V2[t] V3p[0])+U1[t] (-V2p[0] V4[t]+V2[t]
V4p[0])+U4[t] (V3p[0] V4[t]-V3[t] V4p[0]))^2)/((2 U2[t] V2[t]-2 U4[t] V4[t])^4
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(-m U2[t] V2p[0]+m U4[t] V4p[0])^2 (((-U2p[0] V2[t]+U4p[0] V4[t]) (-m U2[t]
V2p[0]+m U4[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2-(m (U2p[0] U4[t]-U2[t]
U4p[0]) (V2p[0] V4[t]-V2[t] V4p[0]))/(2 U2[t] V2[t]-2 U4[t] V4[t])^2));
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[1] A. F. Estrada and L. A. Pachon, New J. Phys. **17**, 033038 (2015).

[2] R. Schmidt, A. Negretti, J. Ankerhold, T. Calarco, and J. T. Stockburger, Phys. Rev. Lett. **107**, 130404 (2011).

[3] D. Kirk, *Optimal Control Theory: An Introduction*, Dover Books on Electrical Engineering (Dover Publications, 2012).