Uncertainty and Expectations

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1. How do people make intertemporal decisions under uncertainty?

2. How do people form expectations?

Road Map

Discounted Expected Utility

Risky Dynamic Economy

Expectations

Discounted Expected Utility

- Expected utility models preferences over risky outcome

states	ω_1	ω_2	$\dots$	ω_S
probability	$\pi(\omega_1)$	$\pi(\omega_2)$		$\pi(\omega_S)$
consumption	$c(\omega_1)$	$c(\omega_2)$		$c(\omega_S)$

$$U = \sum_{s=1}^{S} \pi(\omega_s) u(c(\omega_s)) = E[u(\tilde{c})]$$

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- Discounted utility models preferences over intertemporal outcome

date	0	1	...	T
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$$U = \sum_{t=0}^{T} \beta^t u(c_t)$$

- Discounted expected utility models preferences with both dimensions

$$U = \sum_{t=0}^{T} \beta^t E[u(\tilde{c}_t)]$$

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Risky Dynamic Economy

- Dynamic: two periods $t = 0, 1$
- Risky: state of nature ω_s realized at $t = 1$, proba $\pi(\omega_s)$, s=1, ..., S
- Agents $i = 1, ..., I$
  - Discounted expected utility $EU_i = u(c_{i0}) + \beta E[u(c_{i1}(\omega_s))]$

    $\beta < 1$, $u' > 0$, $u'' < 0$
  - Endowment at $t = 0$: e_{i0}
  - Endowment at $t = 1$ in state ω_s: $e_{i1}(\omega_s)$

Idiosyncratic Risk

- Example 1: Two agents $i \in \{A, B\}$, two states of nature $\{w_A, w_B\}$ with equal probabilities
  - A earns more in state ω_A whereas B earns more in state ω_B

    $e_{A1}(\omega_A) = e_{B1}(\omega_B) = \bar{e}_1 + \Delta$ with $\Delta > 0$

    $e_{A1}(\omega_B) = e_{B1}(\omega_A) = \bar{e}_1 - \Delta$
  - $e_{A0} = e_{B0} = \bar{e}_0$
  - Example: A and B are entrepreneurs in two different industries
- Consider the autarky allocation (no financial market)
  - $c_{i0} = e_{i0}$ and $c_{i1}(\omega_s) = e_{i1}(\omega_s)$

Q1. Can you find a Pareto improvement over autarky?

Idiosyncratic Risk

- Pareto improvement over autarky: agent hit by positive shock at time 1 transfers resources to agent hit by negative shock

Q2. What market instruments can implement this Pareto improvement?

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Q2. What market instruments can implement this Pareto improvement?

- Insurance contracts with payment contingent on state of nature
- Stocks that pay dividend $e_{i1}(\omega_s)$

  Each agent sells the shares of their business to a mutual fund, and buys shares in the mutual fund

- Cooperative

Idiosyncratic Risk

Q3. What is the equilibrium allocation if markets are complete?

$$c_{i0} =$$

$$c_{i1}(\omega_s) =$$

Idiosyncratic Risk

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$c_{i0} = \overline{e}_0$

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i.e. perfect insurance (formal proofs below)

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Definition of risk-free interest rate r_f: a financial instrument paying off 1 for sure at time 1 has price $\frac{1}{1+r_f}$ at time 0

Q4. What is the value at time 0 of firm A paying off $\overline{e}_1 + \Delta$ in state ω_A and $\overline{e}_1 - \Delta$ in state ω_B?

less than / equal to / greater than $\frac{\overline{e}_1}{1+r_f}$

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equal to $\frac{\overline{e}_1}{1+r_f}$ because the risk can be diversified away

Aggregate Risk

- Example 2: Two agents, two states of nature ω_{Good} and ω_{Bad} with equal probabilities
  - Both agents earn more in good state than in bad state

    $e_{i1}(\omega_{Good}) = \bar{e}_1 + \Delta$

    $e_{i1}(\omega_{Bad}) = \bar{e}_1 - \Delta$
  - $e_{i0} = \bar{e}_0$
- Consider the autarky allocation
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Q1. Can you find a Pareto improvement over autarky? no

Aggregate Risk

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less than $\frac{\overline{e}_1}{1+r_f}$ because the risk cannot be diversified

Solving the Model with Complete Markets

- Complete markets: any payoff at time 1 can be generated by a portfolio of instruments

- Requires as many non-redundant instruments as number of states (mathematically: payoff matrix of securities has rank S)

- For example: for all $s = 1, ..., S$, the Arrow security paying off 1 in state ω_s exists. We denote its price at date 0 by $\phi(\omega_s)$

- Any other securities (stocks, bonds, derivatives, etc.) is a portfolio, i.e., a linear combination of Arrow securities

Agent's Problem

- Agent i maximizes $u(c_{i0}) + \sum_{s=1}^{S} \beta\pi(\omega_s)u(c_{i1}(\omega_s))$ by choice of

  consumption at $t = 0$: c_{i0}

  consumption at $t = 1$ in state ω_s: $c_{i1}(\omega_s)$

  position in each state ω_s-Arrow security: $a_i(\omega_s) \lesseqgtr 0$

- Subject to budget constraints

  - at $t = 0$: $c_{i0} + \sum_{s=1}^{S} a_i(\omega_s)\phi(\omega_s) \leq e_{i0}$
  - at $t = 1$ in state ω_s: $c_{i1}(\omega_s) + a_i(\omega_s)? \leq e_{i1}(\omega_s) + a_i(\omega_s)?$

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  $\Rightarrow$ intertemporal BC: $c_{i0} + \sum_{s=1}^{S} \phi(\omega_s)c_{i1}(\omega_s) \leq e_{i0} + \sum_{s=1}^{S} \phi(\omega_s)e_{i1}(\omega_s)$

Solving For Equilibrium

Step 1: Determine agents' choices given prices $\phi(\omega_s)$

Step 2: Clear markets to determine $\phi(\omega_s)$ and equilibrium allocation

Step 1: First Order Conditions

- First order conditions
  - w.r.t. c_{i0}: $u'(c_{i0}) - \lambda_i = 0$

    where λ_i = multiplier of intertemporal budget constraint
  - w.r.t. $c_i(\omega_s)$: $\beta\pi(\omega_s)u'(c_{i1}(\omega_s)) - \lambda_i\phi(\omega_s) = 0$

$\Rightarrow$ $\phi(\omega_s)u'(c_{i0}) = \pi(\omega_s)\beta u'(c_{i1}(\omega_s))$ for all i and ω_s

- Intuition: move 1 unit of consumption from time 0 to time 1 state ω_s
- Left: MU loss at time 0
- Right: expected MU gain at time 1

Step 2: Market Clearing

- Market clearing for

  (1) time 0 good: $\sum_i c_{i0} = ?$

  (2) time 1 state ω_s good: $\sum_i c_{i1}(\omega_s) = ?$

  (3) state ω_s-Arrow security: $\sum_i a_i(\omega_s) = ?$

- Walras's Law: (3) is redundant, it is implied by (1) + (2) + budget constraints

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Back to the Examples: Aggregate Risk

- Example 2: $$\begin{aligned} e_{i0} &= \overline{e}_0 \\ e_{i1}(\omega_{Good}) &= \overline{e}_1 + \Delta \\ e_{i1}(\omega_{Bad}) &= \overline{e}_1 - \Delta \end{aligned}$$

- Equilibrium: $$\begin{aligned} c_{i0} &= \overline{e}_0 \\ c_{i1}(\omega_{Good}) &= \overline{e}_1 + \Delta \\ c_{i1}(\omega_{Bad}) &= \overline{e}_1 - \Delta \end{aligned}$$ satisfy

  - Market clearing
  - First order condition for $\phi(\omega_s) = \dfrac{0.5\beta u'(\overline{e}_1(\omega_s))}{u'(\overline{e}_0)}$

- Instrument paying off in bad state has higher or lower price than instrument paying off in good state

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Asset Pricing with Aggregate Risk

- The price of the risk-free asset paying off 1 in both states is

  a. 1 b. $1+1$ c. $\phi(\omega_{Good}) + \phi(\omega_{Bad})$

-

Asset Pricing with Aggregate Risk

- The price of the risk-free asset paying off 1 in both states is

$$\phi(\omega_{Good}) + \phi(\omega_{Bad}) \equiv \frac{1}{1 + r_f}$$

- The price of an asset paying off $\overline{e}_1 + \Delta$ in good state and $\overline{e}_1 - \Delta$ in bad state is

  a. $\frac{1}{2}(\overline{e}_1 + \Delta) + \frac{1}{2}(\overline{e}_1 - \Delta)$

  b. $(\overline{e}_1 + \Delta) \times \phi(\omega_{Good}) + (\overline{e}_1 - \Delta) \times \phi(\omega_{Bad})$

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$$(\overline{e}_1 + \Delta) \times \phi(\omega_{Good}) + (\overline{e}_1 - \Delta) \times \phi(\omega_{Bad})$$

$$= \frac{\overline{e}_1}{1+r_f} + \Delta \times (\phi(\omega_{Good}) - \phi(\omega_{Bad})) < \frac{\overline{e}_1}{1+r_f}$$

Idiosyncratic Risk

- Example 1: $$\begin{aligned} e_{i0} &= \bar{e}_0 \\ e_{A1}(\omega_A) = e_{B1}(\omega_B) &= \bar{e}_1 + \Delta \\ e_{A1}(\omega_B) = e_{B1}(\omega_A) &= \bar{e}_1 - \Delta \end{aligned}$$

- Equilibrium: $$\begin{aligned} c_{i0} &= \bar{e}_0 \\ c_{i1}(\omega_s) &= \bar{e}_1 \end{aligned}$$ satisfy

  - Market clearing

  - First order condition for $\phi(\omega_s) = \dfrac{0.5\beta u'(\bar{e}_1)}{u'(\bar{e}_0)}$

- Arrow securities for each state have the same price

Asset Pricing with Idiosyncratic Risk

- Firm A paying off $\overline{e}_1 + \Delta$ in state ω_A and $\overline{e}_1 - \Delta$ in state ω_B has price

$$(\overline{e}_1 + \Delta) \times \phi(\omega_A) + ?$$

Asset Pricing with Idiosyncratic Risk

- Firm A paying off $\overline{e}_1 + \Delta$ in state ω_A and $\overline{e}_1 - \Delta$ in state ω_B has price

$$(\overline{e}_1 + \Delta) \times \phi(\omega_A) + (\overline{e}_1 - \Delta) \times \phi(\omega_B)$$

$$= \overline{e}_1 \times (\phi(\omega_A) + \phi(\omega_B)) \equiv \frac{\overline{e}_1}{1+r_f}$$

Pop quiz

Q. By this logic, which real-world financial instruments should be expensive?

a. Out-of-the-money puts on the stock market

b. Insurance against global cyber-attack

c. Car insurance

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(In)Complete Markets

- Different degrees of market completeness
  - No financial markets (autarky): not realistic in modern economies, a benchmark to evaluate gains from financial development
  - Risk-free asset only (possibly with credit constraints): often used in macro with heterogeneous agents (see Macro 2)
  - Risk-free asset + some risky assets only (e.g., public companies' stocks are traded but not income from labor or private businesses)
  - Complete markets: all possible assets and contracts can be traded

Market (In)Completeness

Q. In practice, which shocks can/cannot be insured? Why?

Market (In)Completeness

Q. In practice, which shocks can/cannot be insured? Why?

- Easier to insure against shocks whose probability of occurrence
  1. is outside of the control of agents (else, moral hazard)
  2. agents don't have private information about (else, adverse selection)
- See Micro 2

Road Map

Discounted Expected Utility

Risky Dynamic Economy

Expectations

Rational Expectations

- Implicit assumption so far: agents have **rational expectations**, i.e., they know the distribution of random variables
- How can they?
- Learning: infer distribution from past realizations
- But:
  - Requires stationarity
  - Estimating probability of rare events requires long time-series
  - E.g., how to learn about climate risk?
- If being perfectly informed is costly (time, data, etc.), agents may rationally choose to be imperfectly informed: "rational inattention" (Sims, 2003)

Testing Rational Expectations

- Testing whether agents have rational expectations (RE) is challenging because it requires to observe and compare:

  1. People's expectations

  2. The true distribution

A Test of Rational Expectations

Coibion and Gorodnichenko 2015. Information Rigidity and the Expectations Formation Process: A Simple Framework and New Facts. *American Economic Review* [pdf]

- Idea: while RE do not imply that (ex-post) forecast errors are uniformly zero because of uncertainty,

  RE imply that forecast errors are <u>unpredictable</u>

  otherwise the forecast could have been improved, i.e., it was not rational

A Test of Rational Expectations

- Denote

  x_{t+h}: variable being forecast

  $E_t[x_{t+h}]$: RE forecast at time t

  $F_t[x_{t+h}]$: people's forecast at time t

- Do people have RE: $F_t[x_{t+h}] = E_t[x_{t+h}]$?

- Assume $F_t[x_{t+h}] = (1-\lambda)E_t[x_{t+h}] + \lambda F_{t-1}[x_{t+h}]$

  $\lambda = 0$: RE

  $\lambda \in (0,1)$: under-react to new information

  $\lambda < 0$: over-react to new information

A Test of Rational Expectations

$$\Leftrightarrow \underbrace{E_t[x_{t+h}] - F_t[x_{t+h}]}_{\text{deviation from RE}} = \frac{\lambda}{1-\lambda} \underbrace{\Big(F_t[x_{t+h}] - F_{t-1}[x_{t+h}]\Big)}_{\text{forecast revision}}$$

- In words: upon positive news, an under-reacting person ($\lambda \in (0,1)$) revises their forecast up but not enough, so their new forecast is below the RE forecast

- Can be estimated by OLS

$$\Leftrightarrow E_t\Big[x_{t+h} - F_t[x_{t+h}]\Big] = \frac{\lambda}{1-\lambda}\Big(F_t[x_{t+h}] - F_{t-1}[x_{t+h}]\Big)$$

$$\Leftrightarrow x_{t+h} - F_t[x_{t+h}] = a + b\Big(F_t[x_{t+h}] - F_{t-1}[x_{t+h}]\Big) + \epsilon_{t+h}$$

with $a = 0$, $b = \frac{\lambda}{1-\lambda}$, and $\epsilon_{t+h} = x_{t+h} - E_t[x_{t+h}]$ is unpredictable by definition of RE hence uncorrelated with the past forecast revision

A Test of Rational Expectations

- Consensus inflation forecast by professional forecasters

  $F_t[x_{t+3}]$: quarter t consensus forecast of quarter $t+3$ inflation

  x_{t+3}: realized quarter $t+3$ inflation

- Time-series regression

$$x_{t+3} - F_t[x_{t+3}] = a + b\left(F_t[x_{t+3}] - F_{t-1}[x_{t+3}]\right) + \epsilon_{t+3}$$

- Result: $b \simeq 1 \Rightarrow \lambda \simeq 0.5$ per quarter $\Rightarrow$ under-reaction

More Evidence

- Evidence of over-reaction in other settings, e.g., forecasts of stock market returns[1]
  - A potential mechanism for stock market bubbles
- Deviation from RE depends on
  - Who: households vs. firm managers vs. professional forecasters
  - What: macro vs. local variables
- Robust evidence of large dispersion in individual forecasts
  - More so for households than firms, and more so for firms than professional forecasters
  - Many determinants: personal experience, political affiliation, IQ, incentives, etc.[2]

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[1] Bordalo, Gennaioli, La Porta and Shleifer 2019. Diagnostic Expectations and Stock Returns. *Journal of Finance* [pdf]

[2] Coibion, D'Acunto, Gorodnichenko and Weber 2022. The Subjective Inflation Expectations of Households and Firms: Measurement, Determinants and Implications. *Journal of Economic Perspectives* [pdf]