Problem Set for Friday 24 November: Solution

Solution to Problem 1

a. False. Alice is indifferent between 100 apples today and $100/\beta_A = 102$ apples next period, so she prefers 105 apples next period over 100 apples today.

b. False. A smaller time discount factor means higher impatience to consume.

c. Alice is more patient than Bob, so a profitable trade is that Alice lends to Bob. For example, Alice lends fraction $x \in (0, 1)$ of an apple to Bob at $t = 0$ with an interest rate of r, and Bob gives back $(1 + r)x$ apples to Alice at $t = 1$. Let us check that we can find values of r such that this trade increases utility of both agents relative to autarky.

   Under autarky:

$$U^{aut}_{Alice} = 1 + \beta_A$$
$$U^{aut}_{Bob} = 1 + \beta_B$$

   When Alice lends to Bob:

$$U_{Alice} = 1 - x + \beta_A(1 + (1 + r)x)$$
$$U_{Bob} = 1 + x + \beta_B(1 - (1 + r)x)$$

   The trade increases Alice's utility if and only if

$$\beta_A(1 + r) > 1$$

   and it increases Bob's utility if and only if

$$\beta_B(1 + r) < 1$$

   Therefore, any interest rate in the range

$$\frac{1 - \beta_A}{\beta_A} < r < \frac{1 - \beta_B}{\beta_B}$$

   generates gains from trade.

   This transaction is a debt contract.

d. Agent i maximizes discounted utility

$$\log(c_{i0}) + \beta_i \log(c_{i1})$$

   subject to the budget constraint at each date

$$c_{i0} + b_i = 1$$
$$c_{i1} = 1 + (1 + r)b_i$$

   where b_i denotes savings in the bond instrument between $t = 0$ and $t = 1$. We eliminate b_i to obtain the intertemporal budget constraint

$$c_{i0} + \frac{c_{i1}}{1 + r} = 1 + \frac{1}{1 + r}$$

   We use the intertemporal budget constraint to substitute c_{i1} in the objective function:

$$\log(c_{i0}) + \beta_i \log\big(1 + (1 + r)(1 - c_{i0})\big).$$

The objective function is now written as a convex function of c_{i0}, so the optimum is determined by the first order condition:

$$\frac{1}{c_{i0}} - \beta_i \frac{1+r}{1+(1+r)(1-c_{i0})},$$

which gives

$$c_{i0} = \frac{1}{1+\beta_i}\frac{2+r}{1+r}$$

Substituting c_{i0} in the intertemporal budget constraint, we obtain

$$c_{i1} = \frac{\beta_i(2+r)}{1+\beta_i}$$

e. Agent i's net lending is

$$b_i = 1 - c_{i0} = 1 - \frac{1}{1+\beta_i}\frac{2+r}{1+r}.$$

Net lending is increasing in r. Intuitively, agents lend more/borrow less when the interest rate is high.

f. The equilibrium interest is determined by market clearing in the bond market: $b_A + b_B = 0$, that is, the amount borrowed by one agent must be equal to the amount lent by the other agent. We obtain

$$r = \frac{2(1-\beta_A\beta_B)}{\beta_A + \beta_B + 2\beta_A\beta_B}.$$

When agents are more patient i.e. have higher time discount factors, they are more willing to save and lend and less willing to borrow, therefore the equilibrium interest rate that clears the credit market is lower.

Solution to Problem 2

a. $40 = \beta \times 50$, that is, $\beta = 0.8$ per week.

b. Yes.

c. Harry's time preferences do not satisfy *exponential discounting*, that is, his discounted utility at time t cannot be represented as $U_t = \sum_{\tau \geq 0} \beta^\tau u(c_{t+\tau})$ with a constant β. Instead, Harry has *hyperbolic discounting*, that is, he uses a lower discount factor for the short term than for the long term. One way to model such preferences is $U_t = u(c_t) + \delta \sum_{\tau \geq 1} \beta^\tau u(c_{t+\tau})$ with $\delta < 1$. Note that Harry's preferences are *time inconsistent*. If asked today, he prefers a 60 euro dinner in one year and one week than a 40 euro dinner in one year. But if faced with the same choice in one year time, he will prefer having the 40 euro dinner immediately than wait for a week to have the 60 euro dinner.

Solution to Problem 3

a. The budget constraint at date t is

$$c_t \leq y_t + b_{0t}.$$

Using this equation to substitute b_{0t} in the date 0 budget constraint, we obtain the intertemporal budget constraint

$$\sum_{t=0}^{T} q_{0t} c_t \leq \sum_{t=0}^{T} q_{0t} y_t$$

Note that, as in the two-period case studied in class, the LHS of the intertemporal budget constraint is the date 0 value of intertemporal consumption and the RHS is the date 0 value of intertemporal income.

b. We can derive the first order condition heuristically. Suppose the agent marginally reduces consumption at date 0 to buy one unit of the t-period bond at price q_{0t} and increases consumption at date t. Doing so reduces date 0 utility by $q_{0t}u'(c_0)$ and increases date t utility by $\beta^t u(c_t)$. Therefore, the chosen consumption profile is optimal only if

$$q_{0t}u'(c_0) = \beta^t u'(c_t).$$

We can also derive this first order condition by writing the Lagrangian, taking derivatives w.r.t. c_0 and c_t and eliminating the Lagrangian multiplier from the intertemporal budget constraint. As usual, that the first order condition is sufficient for optimality follows from the concavity of the agent's problem.

Market clearing in the good market implies $c_t = y_t$ at all t. Therefore,

$$q_{0t} = \frac{\beta^t u'(y_t)}{u'(y_0)}$$

c. Constant relative risk aversion. $\gamma = -u''c/u'$ is the parameter of relative risk aversion.

d. Substitute $u'(c) = c^{-\gamma}$ in the bond price and calculate the interest rate.

e. If we denote by g_{0t} the annualized growth rate of GDP between date 0 and date t in our economy, the t-year interest rate is equal to

$$r_{0t} = \beta^{-1}(1 + g_{0t})^{\gamma} - 1.$$

Therefore, a low long-term GDP growth rate leads to a low long-term interest rate.

Solution to Problem 4

a. We denote by b_i the amount lent (or borrowed if negative) by agent i at $t = 0$. Agent i maximizes $u(c_{i1})$ subject to the budget constraints

$$\begin{aligned} t = 0: &\qquad k_i + b_i \leq 1 \\ t = 1: &\qquad c_{i1} \leq z_i k_i^{\alpha} + (1 + r)b_i \end{aligned}$$

The sequential budget constraints can be consolidated into the intertemporal budget constraint

$$\frac{c_{i1}}{1 + r} + k_i \leq 1 + \frac{z_i k_i^{\alpha}}{1 + r}$$

The first order condition with respect to k_i implies

$$1 + r = z_i \alpha k_i^{\alpha - 1}$$

Intuitively, the agent invests up to the point where the marginal return on investment (RHS) is equal to the cost of capital (LHS). Therefore

$$k_i = \left(\frac{z_i \alpha}{1 + r}\right)^{\frac{1}{1-\alpha}}$$

b. The market clearing condition for goods at $t = 0$ writes

$$\int_i k_i \, di = 1$$

Using the expression for k_i from the previous question, we obtain

$$1 + r^* = \alpha \overline{z}$$

c. Substituting the expression for r^* back into k_i, the equilibrium investment is

$$k_i^* = \left(\frac{z_i}{\overline{z}}\right)^{\frac{1}{1-\alpha}}$$

d. At a Pareto optimum, investment levels k_i must maximize total date 1 production $\int_i z_i k_i^\alpha \, di$ subject to the resource constraint at date 0. By contradiction, if it was not the case, the planner could reshuffle investment across agents at date 0 to achieve higher aggregate production at date 1, and therefore increase the consumption of every agents, leading to a Pareto improvement.

The problem of maximizing aggregate production is

$$\begin{aligned} \max_{\{k_i\}} \quad & \int_i z_i k_i^\alpha \, di \\ s.t. \quad & \int_i k_i \, di \leq 1 \end{aligned}$$

Denoting by λ the Lagrange multiplier of the resource constraint, the first order condition w.r.t. k_i is

$$z_i \alpha k_i^{\alpha-1} = \lambda$$

for all i. Therefore

$$k_i = \left(\frac{z_i \alpha}{\lambda}\right)^{\frac{1}{1-\alpha}}$$

We find λ by substituting k_i into the resource constraint:

$$\lambda = \alpha \left(\int_i z_i^{\frac{1}{1-\alpha}} \, di\right)^{1-\alpha}$$

Substituting back into k_i, we obtain that $k_i = k_i^*$.

Capital is optimally allocated in the competitive equilibrium. This is an incarnation of the First Welfare Theorem: the competitive equilibrium is a Pareto optimum.

e. Agent i's intertemporal budget constraint is now

$$\frac{c_{i1}}{1+r} + k_i \leq 1 + \frac{(1-\tau_i) z_i k_i^\alpha}{1+r}$$

Taking the first order condition with respect to k_i, we obtain

$$k_i = \left(\frac{(1-\tau_i) z_i \alpha}{1+r}\right)^{\frac{1}{1-\alpha}}$$

Using the market clearing condition at $t = 0$, we obtain

$$1 + r = \alpha \left(\int_i [(1-\tau_i) z_i]^{\frac{1}{1-\alpha}} \, di\right)^{1-\alpha}$$

Equilibrium investment is

$$k_i = \frac{[(1-\tau_i) z_i]^{\frac{1}{1-\alpha}}}{\int_j [(1-\tau_j) z_j]^{\frac{1}{1-\alpha}} \, dj}$$

f. $k_i = k_i^*$ if and only if τ_i is constant across agents. A necessary and sufficient condition for taxes not to distort investment is that all agents face the same tax rate: $\tau_i = \tau$ for all i.

If the tax rate differs across agents, agents facing a higher tax rate do not invest enough (relative to a Pareto-efficient allocation) and agents facing a lower tax rate invest too much, that is, capital is misallocated in equilibrium.