Introduction to Social Influence Models

Auto-Logistic Actor Attribute Models (ALAAMs)

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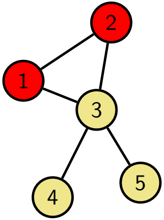
Preamble

- All material is on the workshop repository https://github.com/johankoskinen/CHDH-SNA
  - Download the RMarkdown file CHDH-SNA-2.Rmd
- In order to run the Markdown you need
  - The R-package
  - The RStudio interface
- We will predominantly use the packages
  - `sna`
  - `network`
- as well as `MultivarALAAMalt.R` from GitHub

The issue



Nodes: $V = \{1, 2, \ldots, n\}$

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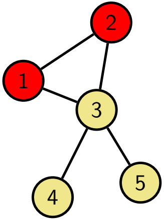
Tie-variables:

$$X_{ij} = \begin{cases} 1, & \text{if tie from } i \text{ to } j \\ 0, & \text{else} \end{cases}$$



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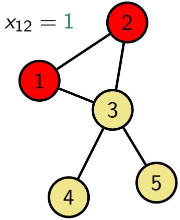

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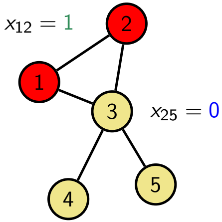

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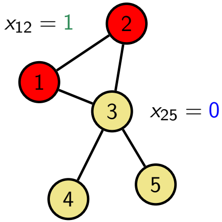

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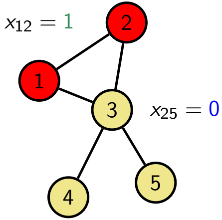

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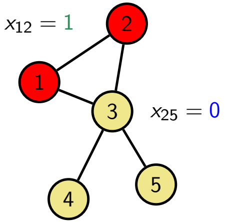

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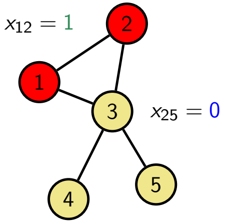

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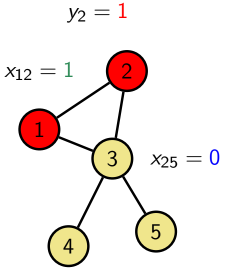

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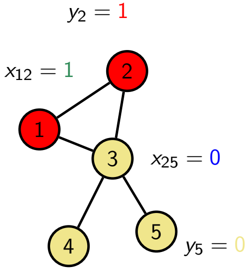

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Attribute vector

$$\boldsymbol{y} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^\top$$

How model binary outcomes?

i	y_i	x_i
1	0	2
2	1	2
3	1	3
4	1	3
5	0	3
6	1	4
7	1	4
8	1	5
9	1	5

Table: Nine binary observations on a binary dependent variable and covariate

Regressing raw values of $\boldsymbol{y}$ on $\mathbf{X}$?

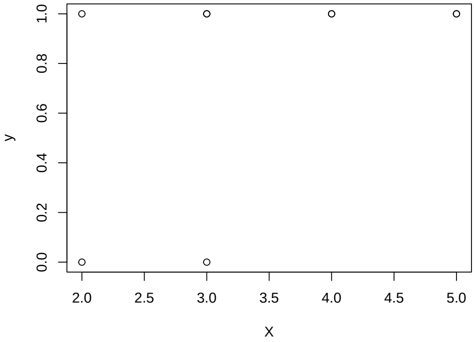

Figure: Scatter plot of data in Table 1

Regressing proportions?

Conditional means in regression:

$$\hat{p}_x = \alpha + \beta x + \epsilon_x$$

	$\hat{p}_x$	x
1	0.5000000	2
2	0.6666667	3
3	1.0000000	4
4	1.0000000	5

Table: The conditional proportions of the nine binary observations in Table 1 grouped by outcomes on x

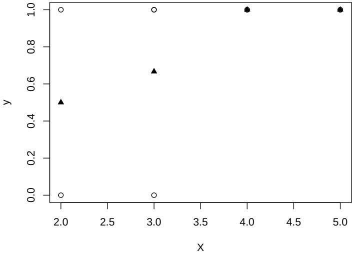

Figure: Scatter plot of data in Table ?? with conditional means (triangles)

Modelling binary dependent variable

Log odds and the logit function

For a probability p, we define the odds as

$$\frac{p}{1-p}.$$

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For logistic regression it is convenient to work with the logarithm of the odds, the log odds, or *logit*

$$\mathrm{logit}(p) = \log\left(\frac{p}{1-p}\right)$$

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$$\mathrm{logit}(p) = \log\left(\frac{p}{1-p}\right)$$

We shall see that $\mathrm{logit}(p)$ takes values in all of $\mathbb{R}$

Logit is a function that maps probabilities ℝ

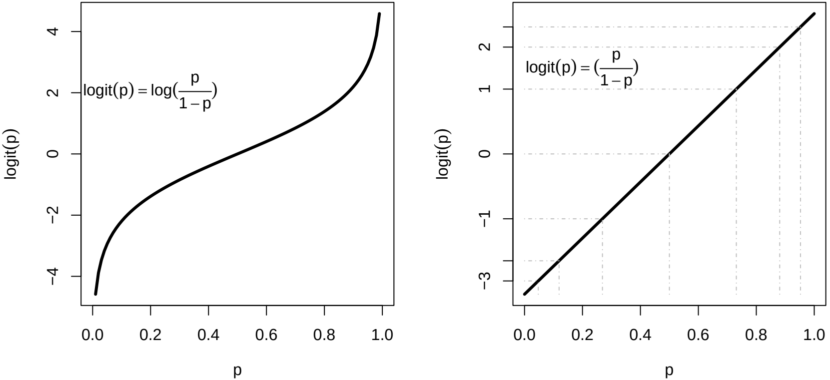

Figure: Probabilities v logit function. Logit scale on vertical axis (right)

Logistic regression

We can regress the logit on any combination of variables

$$\operatorname{logit}(p_i) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_1 x_{ik}$$

We only have a linear predictor for the logit and *NO* random errors.

- increasing function of linear predictor - larger linear comb, large prob
- takes all values on real line

Logistic regression - how probabilities?

For the equation

$$\operatorname{logit}(p_i) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_1 x_{ik}$$

we can easily get the predicted probabilities $\Pr(Y_i = 1 | \mathbf{x}_i)$

$$p_i = \frac{e^{\beta_0 + \beta_1 x_{i1} + \cdots + \beta_1 x_{ik}}}{\underbrace{1 + e^{\beta_0 + \beta_1 x_{i1} + \cdots + \beta_1 x_{ik}}}_{\text{to ensure } p+q=1}}$$

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A probability in $[0, 1]$

$$p_i + (1 - p_i) = 1$$

Logistic regression

When logit is the **link function**, calling the inverse logit

$$\mathrm{logit}^{-1}(\eta) = G(\eta)$$

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G is the **inverse link function**
Logistic regression, is regression of the linear predictor

$$\eta_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_1 x_{ik}$$

So that

$$E(Y_i \mid \mathbf{x}) = G(\eta_i)$$

Logistic regression - how probabilities?

Note that we can write

$$\Pr(Y_i = y | \mathbf{x}_i) = \frac{e^{y\eta_i}}{1 + e^{\eta_i}}$$

Logistic regression - how probabilities?

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$$\Pr(Y_i = y|\mathbf{x}_i) = \frac{e^{y\eta_i}}{1 + e^{\eta_i}}$$

A probability in $[0, 1]$

$$\Pr(Y_i = 0|\mathbf{x}_i) + \Pr(Y_i = 1|\mathbf{x}_i) = \frac{e^{0\times\eta_i}}{1 + e^{\eta_i}} + \frac{e^{1\times\eta_i}}{1 + e^{\eta_i}} = \frac{1 + e^{\eta_i}}{1 + e^{\eta_i}} = 1$$

Example (cont)

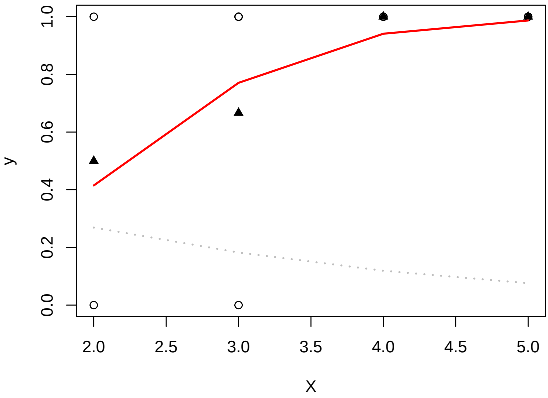


Figure: Original data points of Table 1, with grouped proportions (triangles), the fitted probabilities using β_{ML} (red, solid)

Using network statistics - The Markov model

Auto-Logistic Actor Attribute Model (ALAAM)

What if we let $\Pr(Y_i = 1|\mathbf{X})$ depend on i's position in the network?

Auto-Logistic Actor Attribute Model (ALAAM)

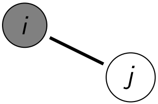
What if we let $\Pr(Y_i = 1|\mathbf{X})$ depend on i's position in the network? For example

$$\eta_i = \beta_0 + \beta_{\mathrm{deg}} \sum_j x_{ij} + \beta_{\mathrm{var}} \sum_{j,k} x_{ij}x_{ik} + \beta_{\mathrm{tri}} \sum_{j,k} x_{ij}x_{ik}x_{jk}$$

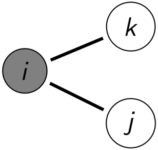
which gives us a model

$$p(\boldsymbol{y} \mid \mathbf{X}) = \exp\left\{\boldsymbol{\beta}^\top z(\boldsymbol{y}, \mathbf{X}) - \psi(\boldsymbol{\beta})\right\}$$

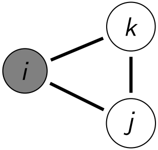
where $z(\boldsymbol{y}, \mathbf{X}) = (z_1, \ldots, z_p)^\top$, $z_1 = \sum y_i$, and



$z_2 = \sum_i y_i x_{i+}$



$z_3 = \sum_i y_i \sum_{j,k} x_{ij}x_{ik}$



$z_4 = \sum_i y_i \sum_{j,k} x_{ij}x_{ik}x_{jk}$

Auto-Logistic Actor Attribute Model (ALAAM)

If $\beta_{\mathrm{deg}} > 0$ then nodes with high degree centrality are more likely to have $y_i = 1$ than nodes with low degree

The network activity ALAAM

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Markov dependence assumption (Robins et al., 2001)

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The outcomes Y_i and X_{ij} are conditionally dependent as $\{i\} \cap \{i, j\} = \{i\}$

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Deriving model from dependence (as in ERGM)

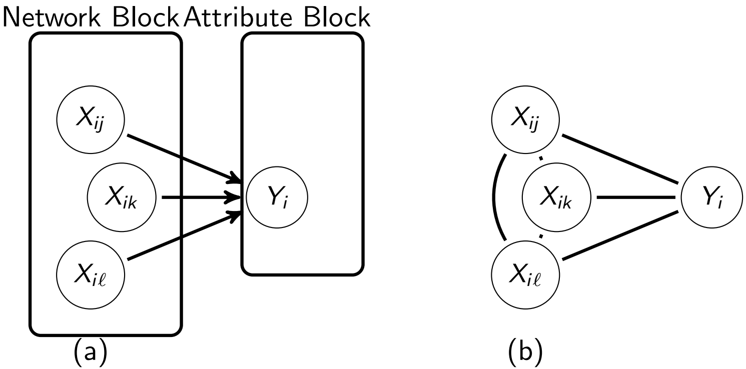

Figure: Dependence graph (a) and Moral graph (b) of network activity dependence model (Robins et al., 2001)

The network activity ALAAM

The statistics z_r correspond to cliques in the Moral graph, and includes

- intercept: $\sum y_i$
- degree: $\sum y_i \sum_j x_{ij}$
- stars: $\sum y_i \sum x_{ij_1} \cdots x_{ij_k}$

But crucially, **no** statistics of the type

$$y_i y_j x_{ij}$$

and thus Y_i and Y_j are independent given $\mathbf{X}$

$\Pr(Y_i = y_i, Y_j = y_j \mid \mathbf{X}, \boldsymbol{y}_{-i,j})$

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$$\Pr(Y_i = y_i, Y_j = y_j \mid \mathbf{X}, \boldsymbol{y}_{-i,j}) = \Pr(Y_i = y_i \mid \mathbf{X}, \boldsymbol{y}_{-i,j}) \Pr(Y_j = y_j \mid \mathbf{X}, \boldsymbol{y}_{-i,j})$$

The network activity ALAAM - logistic regression

The network activity ALAAM is equivalent to logistic regression with

$$\mathrm{logit}(p_i) = \beta_0 + \beta_1 z_{i1} + \cdots + \beta_1 z_{ip}$$

where the statistics z_{ih} are summaries of i's network position

The network activity ALAAM - logistic regression

Example (Modern contraceptive use in rural Kenya)

	Mean	Description
mcUse	0.35	Do you use modern contraceptive techniques?
Age	34.41	Age (sd:16.04)
Female	0.60	Female (1) or Male (0)
HasChildren	0.68	Have one child or more
relevanOthersApprove	0.45	Other people's approval is important
relevanOthersUse	0.67	I care if other people use modern contraceptives
mcUseConflict	0.68	The use of modern contraceptives is contentious and causes conflict
numFriends	0.88	Tallied: the number of names of people they spend their free time with

Table: Variables in Kenya study on Modern contraception usage (Not exact question wordings)(NSF-CMMI-2005661)

The network activity ALAAM - logistic regression

Example (Modern contraceptive use in rural Kenya (cont.) $n = 1303$)

Estimated logistic regression

	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	-0.6340	0.2601	-2.44	0.0148
Age	-0.0554	0.0067	-8.24	0.0000
Female	-1.0232	0.1538	-6.65	0.0000
HasChildren	1.9622	0.2068	9.49	0.0000
relevanOthersApprove	1.4696	0.1514	9.70	0.0000
relevanOthersUse	0.3415	0.1720	1.99	0.0471
mcUseConflict	-0.3835	0.1474	-2.60	0.0093
numFriends	0.3349	0.0828	4.04	0.0001

The network activity ALAAM - logistic regression

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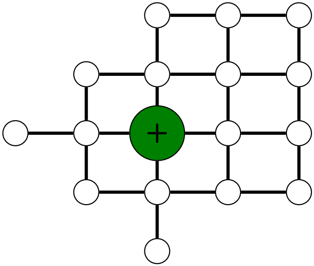
relevanOthersApprove	1.4696	0.1514	9.70	0.0000
relevanOthersUse	0.3415	0.1720	1.99	0.0471
mcUseConflict	-0.3835	0.1474	-2.60	0.0093
numFriends	0.3349	0.0828	4.04	0.0001

How much is the increase in the probability of `mcUse` if you acquire another friend?

A network contagion ALAAM

Ising model (Besag, 1972)

Probability spin $+ \approx \sharp$ neighbours $j \in N(i)$ with spin $+$



$$\Pr(Y_i = 1 | Y_{N(i)} = y_{N(i)}) \propto \exp\{\theta_1 + \theta_2 \sum_{j \in N(i)} y_j\}$$

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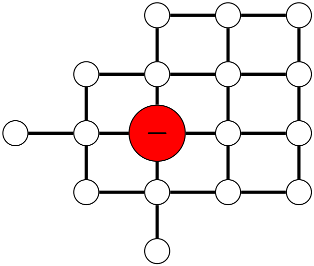

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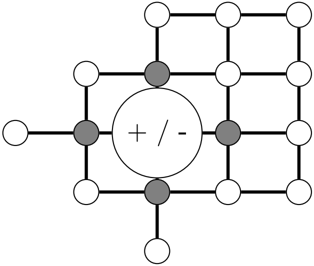

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Markov random fields for Social Networks

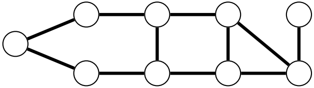
- ppls' networks are not regular lattices
- ppls' attitudes/behaviours also depend on SES, SEX, Education, etc

Social dependence is messy

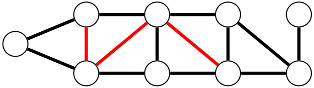
In Graphical models
Conditional independence graph: $i \sim j$ unless

$$X_i \perp X_j | X_{V \setminus \{i,j\}}$$

each node represents one variable (with many observations)
some dependence structures are easier than others



not decomposable



decomposable

Adding dependence between outcomes

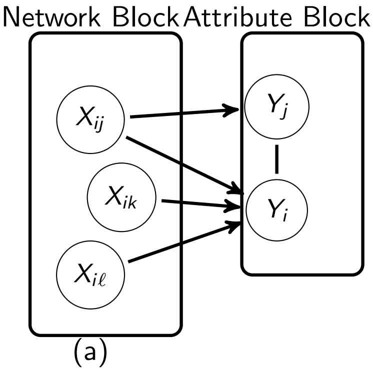


Figure: Dependence graph (a) and Moral graph (b) of model with dependence between attributes that share tie-variables

Deriving contagion statistics is non-trivial

To derive a non-trivial set of statistics use *realization-dependence* (Baddeley & Möller, 1989).

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- Partial dependence graph $\mathcal{Q}_{\mathcal{B}}$, is a graph on $\mathcal{V}_{-\mathcal{B}}$
- where $\{i, j\} \in \mathcal{Q}_{\mathcal{B}}$ if
  - ✓ variables i and j are not conditionally independent conditional on variables $\mathcal{V}_{-\mathcal{B},i,j}$,
  - ✓ and all variables corresponding to the index set $\mathcal{B}$ are zero.

In the model, the parameter for the statistic $A \subset \mathcal{V}$ is non-zero only if A is a clique of $\mathcal{M}$ and A is a clique of $\mathcal{Q}_{\mathcal{B}}$ for all $\mathcal{B}$.
Daraganova (2009) - derived statistics

Standard ALAAM

From this, and

- Making some Homogeneity assumptions and
- setting some higher-order statistics to zero,

we arrive at the following contagion model

$$p_\theta(\boldsymbol{y}|\mathbf{X}) = \exp\left\{\theta_0 \sum_{i=1}^{n} y_i + \theta_{out} \sum_{i=1}^{n} y_i \sum_{j \neq i} x_{ij} + \theta_{in} \sum_{i=1}^{n} y_i \sum_{j \neq i} x_{ji} + \theta_{con} \sum_{i,j:i \neq yj} y_i y_j (x_{ij} + x_{ji}) - \psi(\theta)\right\}$$

This includes an interaction term similar to that of Besag's (1972) classic auto-logistic model but it is subtly different in the definition of the neighbourhood.

Auto-Logistic Actor Attribute Model (ALAAM)

ALAAM defines a distribution on attributes $\boldsymbol{y} \in \mathcal{Y} = \{0, 1\}^V$

ALAAM pmf

$$p_\theta(\boldsymbol{y}|\mathbf{X}) = \exp\{\theta^\top z(\boldsymbol{y}; \mathbf{X}) - \psi(\theta)\}$$

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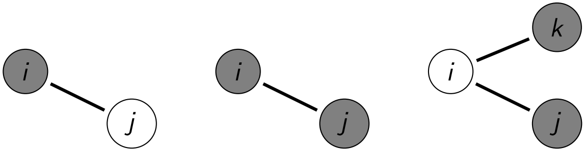
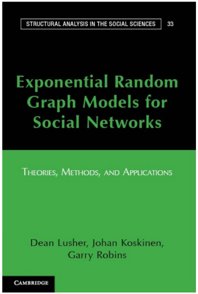


$\sum_i y_i x_{i+}$ (degree) $\sum_{i,j} y_i y_j x_{ij}$ (contagion) $\sum_{i,j} y_i y_k x_{ij} x_{jk}$ (indirect contagion)

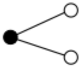
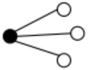
. . . not new

In book

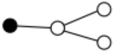
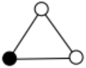


And Robins, Elliot, Pattison (2001); Daraganova (2009)

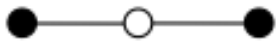
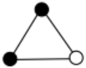
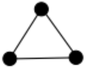
CUP ERGM book - available in MPNet (1)

Configuration	Statistic	Parameter
	$\sum_i y_i$	attribute-density
	$\sum_i y_i \sum_j x_{ij}$	actor activity
	$\sum_i y_i \sum_{j,k} x_{ij} x_{ik}$	actor 2-star
	$\sum_i y_i \sum_{j,k,l} x_{ij} x_{ik} x_{il}$	actor 3-star

CUP ERGM book - available in MPNet (2)

	$\sum y_i \binom{x_{i+}}{k}$	actor *k*-star
	$\sum_i y_i \sum_{j,k} x_{ij} x_{jk}$	partner activity actor 2-path
	$\sum_i y_i \sum_j x_{ij} \binom{x_{j+}}{2}$	partner 2-star
	$\sum y_i \sum_j x_{ij} \binom{x_{j+}}{m}$	partner *m*-star
	$\sum_i y_i \sum_{j,k} x_{ij} x_{ik} x_{jk}$	actor triangle

CUP ERGM book - available in MPNet (3)

Configuration	Statistic	Parameter
	$\sum_{i<j} y_i y_j x_{ij}$	partner attribute
	$\sum_{i<k} y_i y_k \sum_{j \neq i,k} x_{ij} x_{jk}$	indirect partner attribute
	$\sum_{i,j} y_i y_j x_{ij} x_{j+}$	partner attribute activity
	$\sum_{i,j,k} y_i y_j y_k x_{ij} x_{jk}$	partner-partner attribute
	$\sum_{i,j,k} y_i y_j x_{ij} x_{ik} x_{jk}$	partner attribute triangle
	$\sum_{i,j,k} y_i y_j y_k x_{ij} x_{ik} x_{jk}$	partner-partner attribute triangle

The network activity ALAAM - social influence

Example (Modern contraceptive use in rural Kenya (cont.))

Estimated ALAAM

	Posterior		95% CI	
	Estimate	sd	0.025	0.975
intercept	−0.762	0.291	−1.273	−0.188
contagion	0.457	0.076	0.303	0.592
Age	−0.049	0.007	−0.063	−0.035
Female	−1.091	0.178	−1.461	−0.747
HasChildren	1.710	0.233	1.240	2.154
relevanOthersApprove	1.473	0.165	1.140	1.802
relevanOthersUse	0.353	0.179	−0.005	0.697
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How much is the increase in the probability of `mcUse` if your friend uses?

The network activity ALAAM - social influence

A closer look at the pmf

$$p(\boldsymbol{y} \mid \mathbf{X}) =$$

The network activity ALAAM - social influence

A closer look at the pmf

$$p(\boldsymbol{y} \mid \mathbf{X}) = \exp\{\theta^\top z(\boldsymbol{y}; \mathbf{X}) - \underbrace{\psi(\theta)}_{\text{norm. const.}}\}$$

The network activity ALAAM - social influence

A closer look at the pmf

$$p(\boldsymbol{y} \mid \mathbf{X}) = \exp\{\theta^\top z(\boldsymbol{y}; \mathbf{X}) - \underbrace{\psi(\theta)}_{\text{norm. const.}}\} = \frac{e^{\theta^\top z(\boldsymbol{y}, \mathbf{X})}}{\underbrace{\sum_{\boldsymbol{y} \in \mathcal{X}} e^{\theta^\top z(\boldsymbol{y}, \mathbf{X})}}_{2^n \text{ terms}}}$$

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We can **only** evaluate *conditional* probabilities

$$\Pr(Y_i = 1 \mid \mathbf{X}, \boldsymbol{y}_{-i}) = \frac{e^{\theta^\top z(\boldsymbol{y}^{i+},\mathbf{X})}}{e^{\theta^\top z(\boldsymbol{y}^{i+},\mathbf{X})} + e^{\theta^\top z(\boldsymbol{y}^{i-},\mathbf{X})}}$$

where $\boldsymbol{y}^{i+}$ is $\boldsymbol{y}$ with $y_i = 1$, and $\boldsymbol{y}^{i-}$ is $\boldsymbol{y}$ with $y_i = 0$

How did we get estimates - Estimation

Simulating from likelihood

We cannot evaluate likelihood for any θ, but for any θ we can simulate Y_i given $y_1, \ldots, y_{i-1}, y_{i+1}, \cdots, y_n$ using probabilities

$$\mathrm{logit}\left\{ \Pr_\theta(Y_i = 1 | \boldsymbol{y}_{-i}, \mathbf{X}) \right\} = \theta^\top \{ z(\boldsymbol{y}^{i+}, \mathbf{X}) - z(\boldsymbol{y}^{i-}, \mathbf{X}) \}$$

giving us samples from

$$\boldsymbol{y} \mid \mathbf{X}, \theta$$

We will use this for

- estimation, and
- goodness-of-fit (GOF)

`MPNet` uses samples in stochastic approximation for MLE

MCMC for un-normalized distributions

MCMC: Sample $\theta^{(0)}, \theta^{(1)}, \ldots$ from $\pi(\theta)$ by

- propose update $\theta^{(t)}$ to $\theta^* \; q(\theta^* | \theta^{(t)})$
- set $\theta^{(t+1)} := \theta^*$ w.p. $\min\{1, H\}$

$$H = \frac{\pi(\theta^*)}{\pi(\theta^{(t)})} \frac{q(\theta^{(t)} | \theta^*)}{q(\theta^* | \theta^{(t)})}$$

(Works when $\pi(\theta) = f(\theta)/c(\theta)$ and $c(\theta)$ intractable)

For our target distribtuion $\pi(\theta|z)$

$$H = \frac{\exp\{\theta^{*\top} z(\boldsymbol{y}; \mathbf{X}) - \psi(\theta^*)\}\pi(\theta^*)}{\exp\{\theta^{(t)\top} z(\boldsymbol{y}; \mathbf{X}) - \psi(\theta^{(t)})\}\pi(\theta^{(t)})} \frac{q(\theta^{(t)}|\theta^*)}{q(\theta^*|\theta^{(t)})}$$

normalising constant $\psi(\cdot)$ of *likelihood* cannot be evaluated (*model is doubly intractable*)

Solution to double intractability

Approximate $\hat{\lambda}(\theta, \theta^*) \approx \exp\{\psi(\theta) - \psi(\theta^*)\}$

- off-line importance sample (Koskinen, 2004)
- 'exact' auxiliary variable-based online importance sample with sample size of 1 - (Møller et al., 2006)
- 'exact' online (linked) path sampler auxiliary variable (Koskinen, 2008; Koskinen, 2009)
- online self-tuning auxiliary variable (Murray et al., 2006) [Approximate Exchange Algorithm]

ERGO: we can obtain posterior for θ when y is observed

Monitoring performance of MCMC

Ideally, in our MCMC sample

$$\theta^{(0)}, \theta^{(1)}, \ldots, \theta^{(M)}$$

the samples points are independent draws

$$\theta^{(m)} \overset{iid}{\sim} \pi(\theta|\boldsymbol{y}, \mathbf{X})$$

so that we use Monte Carlo estimators

$$\hat{E}(\theta|\boldsymbol{y}, \mathbf{X}) = \bar{\theta} = \tfrac{1}{M} \sum_{m=1}^{M} \theta^{(m)}$$

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$$\hat{E}(\theta|\boldsymbol{y}, \mathbf{X}) = \bar{\theta} = \frac{1}{M}\sum_{m=1}^{M}\theta^{(m)} \text{ , and } \widehat{Cov}(\theta|\boldsymbol{y}, \mathbf{X}) = \frac{1}{M}\sum_{m=1}^{M}(\theta^{(m)}-\bar{\theta})(\theta^{(m)}-\bar{\theta})^{\top}$$

as well as approximate probabilities $\Pr(\theta \in C)$

Monitoring performance of MCMC - trace plots

In plots, *trace plots*, of

$$\theta^{(0)}, \theta^{(1)}, \dots, \theta^{(M)}$$

we should **not** see any

- trend/drift (independence of starting point)
  - select the number of initial iterations to discard - burnin
- serial correlation (good mixing)
  - space out sample points $\theta^{(k)}, \theta^{(2k)}, \theta^{(3k)}, \dots$ - thinning of sample

Monitoring performance of MCMC - SACF & ESS

The *sample autocorrelation function* (**SACF**) measures serial correlation between sample points

$$\theta^{(m-k)}, \theta^{(m)}$$

at different lags k

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The *effective sample size* (**ESS**) tells us roughly how many independent sample points we have

Improving mixing

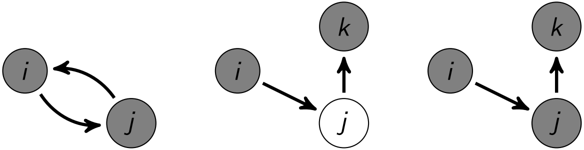
In our implementation the proposal distribution in each iteration

$$\theta^* \mid \theta^{(t)} \sim N(\theta^{(t)}, \Sigma_p)$$

SACF can be lowered and mixing improved through improved Σ_p.

More elaborate effects

A number of more elaborate forms of contagion/influence are admissible



reciprocal indirect reinforced indir

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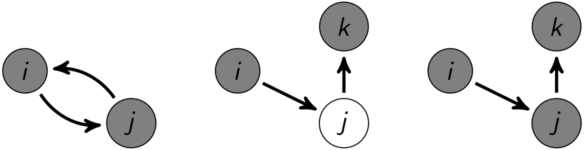
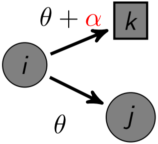


reciprocal indirect reinforced indir

Interaction:

influence from some nodes can be θ and for others $\theta + {\color{red}\alpha}$



SBC (Koskinen and Daraganova, 2022)

Stockholm Birth Cohort (SBC) cohort study, Stockholm Metropolitan area (Stenberg et al., 2006; Stenberg et al. 2007).

- best-friend network with a cap of three nominations (May 1966)
- Let y be indicators $y_i = 1$ of whether pupils i said that they intended to proceed to higher secondary school, and $y_i = 0$ otherwise (see Koskinen and Stenberg, 2012)
- Here: 19 school classes, six of which are from a school in a suburb in the south of Stockholm and the rest are from three inner-city schools
- The proportion of missing entries range from 0 to 0.286

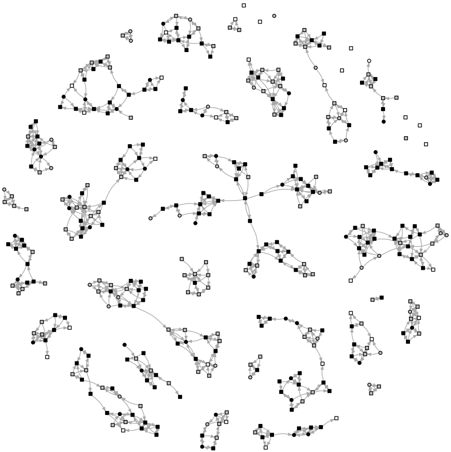


Figure: Bffs in 4 schools. Squares (girl) and circles (boys), and outcome black ($y_i = 1$), grey ($y_i = 0$), and white for missing.

More elaborate effects - interaction example

Example (Simple contagion of intention to go to higher secondary school)

	mean	sd	ESS	SACF 10	SACF 30	2.5 perc	97.5 perc
intercept	-9.67	1.11	178.03	0.68	0.32	-11.83	-7.51
contagion	0.16	0.10	183.10	0.68	0.32	-0.04	0.35
indegree	-0.07	0.11	183.55	0.67	0.32	-0.29	0.13
sex	-0.09	0.29	134.35	0.70	0.39	-0.66	0.47
family attitude	0.48	0.09	164.22	0.70	0.32	0.33	0.65
marks	0.99	0.15	168.66	0.68	0.32	0.69	1.28
social class 1	0.59	0.32	198.40	0.66	0.24	-0.06	1.19

Table: Posterior summaries for model with controls estimated for contagion-model for progression to upper-secondary school in SBC (thinned sample of 10,000 iterations, taking every 20th iteration, with burnin of 1000)

More elaborate effects - interaction example

Example (Contextual contagion of intention to go to higher secondary school)

	mean	sd	ESS	SACF 10	SACF 30	2.5 perc	97.5 perc
intercept	-10.13	1.19	168.32	0.76	0.44	-12.81	-8.04
contagion	0.24	0.12	143.31	0.72	0.39	0.02	0.48
indegree	-0.08	0.12	122.80	0.75	0.41	-0.33	0.13
sex	-0.09	0.28	126.04	0.76	0.45	-0.69	0.47
family attitude	0.48	0.08	140.26	0.72	0.38	0.34	0.65
marks	1.01	0.14	265.08	0.72	0.40	0.76	1.31
composition	0.91	0.55	137.33	0.74	0.39	-0.25	1.97
social class 1	0.57	0.34	143.59	0.73	0.37	-0.07	1.21
contagion int	-0.21	0.16	152.15	0.72	0.37	-0.51	0.11

Table: Posterior summaries for model with controls estimated for contagion-model for progression to upper-secondary school in SBC (thinned sample of 10,000 iterations, taking every 20th iteration, with burnin of 1000) with social class interacted with contagion

Missing data (cp Bayesian data augmentation for ERGM)

Under assumption of Missing at Random (MAR)

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Define the missing data mechanism $f(I|y,\phi)$, where

$$I_i = \begin{cases} 1, & \text{if response } y_i \text{ is unobserved for } i \\ 0, & \text{else} \end{cases}$$

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$$\min\left[1, \exp\{\theta^\top(z(\Delta_i y, x) - z(y,x))\}\frac{f(I|\Delta_i y,\phi)}{f(I|y,\phi)}\right]$$

where $\Delta_i y$ is y with element i toggled and set to $1 - y_i$.

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where $\Delta_i y$ is y with element i toggled and set to $1 - y_i$.
Update ϕ, with MH-updating and Hastings ratio

$$\min\left\{1, \frac{f(I|y,\phi^*)\pi(\phi^*)}{f(I|y,\phi)\pi(\phi)}\right\}.$$

Missing data (cp Bayesian data augmentation for ERGM)

In the actual estimation, simply define

$$y_i = \begin{cases} 1, & \text{if response } y_i = 1 \text{ is unobserved for } i \\ 0, & \text{if response } y_i = 0 \text{ is unobserved for } i \\ NA, & \text{if response is missing for } i \end{cases}$$

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Sampling will return draws

$$(\theta^{(0)}, \boldsymbol{y}_{miss}^{(0)}), (\theta^{(1)}, \boldsymbol{y}_{miss}^{(1)}), \dots, (\theta^{(M)}, \boldsymbol{y}_{miss}^{(M)})$$

Other issues

Model selection: technical issue

BMI: basic marginal likelihood identity
Evidence (= probability of data marginalising with respect to parameters)

$$m(\boldsymbol{y}) = \frac{p_\theta(\boldsymbol{y}|\mathbf{X})\pi(\theta)}{\pi(\theta|\boldsymbol{y})}$$

requires that we evaluate p and π in a point $\theta^* \in \Theta$.
(technically possible but computationally costly)

Model selection: philosophical issue

BMI: basic marginal likelihood identity
Evidence (= probability of data marginalising with respect to parameters)

$$m(\boldsymbol{y}) = \frac{p_\theta(\boldsymbol{y}|\mathbf{X})\pi(\theta)}{\pi(\theta|\boldsymbol{y})}$$

requires that we have a subjective (and interpretable) prior $\pi(\theta)$.
(... if your mother told you this was easy she lied to you)

Goodness-of-fit (GOF)

Once we have a draw

$$\theta^{(0)}, \theta^{(1)}, \dots, \theta^{(M)}$$

from $\pi(\theta|\boldsymbol{y})$, we can generate draws

$$\boldsymbol{y}^{(0)}, \boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(M)}$$

each from

$$p_{\theta^{(m)}}(\boldsymbol{y}^{(m)}|\mathbf{X})$$

GOF evaluation

If

$$\boldsymbol{y}^{(0)}, \boldsymbol{y}^{(1)}, \dots, \boldsymbol{y}^{(M)}$$

are 'similar' to $\boldsymbol{y}$, then model has good fit