Econometrics of asset pricing

The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging

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1 Introduction

Commodities are not like other assets as they play a central role in the economy. There are at the origin of everything produced and transformed. They have been traded for centuries and the way we exchange them was transformed through the ages. Today, with the development of electronic markets, the exchanges are easier and much more quicker, resulting in more transparency in the price of the resources. For a few decades, the fast development of derivatives markets have resulted in an expansion of the commodities markets, for hedging or speculation purposes. Although, as most of the traders on commodity derivatives are not physical traders, they have no use of the commodity itself but just the derivative contract. In times of extreme events, as it happens today, this can lead to very strange phenomenon. Indeed, for a few hours, a future contract on american oil was priced negatively and reached -37 dollars. Considering the total value of exchanges every day on commodity markets, a model to describe the movements of prices is needed.
As we said, commodities are not like other assets. On the market, this results in parameters that are not always observable. The spot price and the instantaneous convenience yield of a commodity is not available. We can only observe the prices of futures contracts that are traded on most trading platforms. Most of the time, the price of the closest to maturity future is used as proxy for spot price. In his paper, Schwartz proposes three models to replicate the spot price of commodity and the instantaneous convenience yield. With these models he tries to replicate the mean reversion effect observed on commodity markets. The models are built to provide a closed form solution for the spot price and the convenience yield. Then, it is easy to compare models with one another but it require assumptions that can be discussed. The difficulties of non-observable variables are dealt with the Kalman filter method, which is known to deal with such issues.

2 State of the art

Commodity prices has been studied in many articles before Schwartz, 1995. The first articles assumed that all the risk could be included in one factor: the spot price. This was in the following of Black & Scholes article (1973). It included articles like: Schwartz (1982), Gibson & Schwartz (1991) for commodities related contracts, Brennan & Schwartz (1985), Paddock, Siguel & Smith (1988), Cortazar & Schwartz (1993) for real assets. After that, seeing that the convenience yield was not constant, were developped two-factor models where the spot price and the convenience yield followed correlated stochastic movements. Jamshidian and Fein (1990) obtained a closed-form solution for oil futures and European options, in the framework of the two-factor model developed by Gibson and Schwartz (1990). Others methods were computed in order to estimate prices from a two-factor model. Brennan (1991), in "The price of convenience and the valuation of commodity contingent claims", used as input of a one-factor model the prices of all futures contracts on the market and their maturities, as well as the estimated parameters of a two-factor model. Here we have the simplicity of computing a one-factor model with the results of a two-factor model.
Advanced studies were made in order to estimate the mean reversion characteristic of commodities. Bessembinder, Coughenour, Seguin, and Smoller in "Mean reversion in equilibrium asset prices: evidence from the futures term structure", 1995, estimated this parameter for several commodities. They showed that for agricultural commodities and for oil this parameter was very large. On the contrary it was much smaller for metals, but still significant. We will see that Schwartz obtains the same results for oil and copper. They also showed that this characteristic was inherent to commodities and found only weak evidence of it in financial asset prices. This can explains a part of the results on gold, that is, as we will see, a very particular commodity.
Cortazar and Schwartz (1994), in "The evaluation of commodity contingent claims", applied a principal component analysis on daily returns of futures contracts on copper, from 1978 to 1990. They showed that a three-factor model described the dynamic of copper futures prices.
Regarding consequences of commodity price models on valuating projects such as mines, refineries or oil wells, Bhappu and Guzman stated in Engineering and Mining Journal (1995) : "The use of discounted cash flow (DCF) techniques to project valuations appears to be the industry standard. One disturbing result of the study, however, is the inability to explain the high premium that market values command over DCF valuations. The well known and used DCF analysis does not allow for placing premium values on projects under consideration. Perhaps the newer techniques such as option pricing methods of valuation may provide more accurate market value results.". It shows here the need to have a better approach to price such projects,

taking into account the specific models mentioned above.

3 Main results

3.1 Methodology

As we said in the introduction, commodity parameters such as spot price or convenience yield are not observable. The only data easily reachable are future prices. The methodology used is the application of a Kalman filter. The observable variables are future prices for different maturities and the associated unobservable state variables is the spot price and the instantaneous convenience yield. The latter are generated via the transition equation and is linked with the observable via the measurement equation. The Kalman procedure allow us to compute the optimal estimator of state variables.

3.2 Data used

The data used in this paper are weekly observations of five futures contracts price for oil, copper and gold. With F_i the i^{th} contract closest to maturity, we have the following data set:

- for oil: F1, F3, F5, F7 and F9 on two periods (1/2/85 to 2/17/95 and 1/2/90 to 2/17/95); F1, F5, F9, F13 and F17 on one period (1/2/90 to 2/17/95); ten futures contracts from Enron data (1/15/93 to 5/16/96)
- for copper: F1, F3, F5, F7 and F9 on one period (7/29/88 to 6/13/95)
- for gold: F1, F3, F6, F9 and F11 on two periods (1/2/85 to 6/13/95 and 11/21/90 to 6/13/95); F1, F5, F9, F13 and F18 on one period (11/21/90 to 6/13/95)

3.3 First model: one-factor model

In this model we assume that the spot price of the commodity has the following dynamic:

$$dS = \kappa(\mu - ln(S))Sdt + \sigma Sdz$$

Then, by applying Itô's Lemma, the log-price of the commodity is following an Ornstein-Uhlenbeck process:
$dX = d(ln(S)) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 = 0dt + \frac{1}{S}dS - \frac{\sigma^2}{2}dt = \kappa(\mu - \frac{\sigma^2}{2\kappa} - ln(S))dt + \sigma dz = \kappa(\alpha - X)dt + \sigma dz$

With $\alpha = \mu - \frac{\sigma^2}{2}$, κ the mean-reversion speed towards the long run log price α and z being a standard Brownian motion.
This equation can be rewritten under the risk-neutral probability $dX = \kappa(\alpha^* - X)dt + \sigma dz^*$ with $\alpha^* = \alpha - \lambda$, λ the market price of risk and $Z*$ is a Brownian motion under the risk neutral probability.
By applying Itô's Lemma to $Y_t = X_t e^{\kappa t}$ we can easily have the analytical form of X:

$$X_t = X_0 e^{-\kappa t} + \alpha^*(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-(t-s)} dz_s^*$$

Then, we can find that $\mathbb{E}_0(X_T) = X_0 e^{-\kappa T} + \alpha^*(1 - e^{-\kappa T})$ and that $\mathbb{V}_0(X_T) = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa T})$.

Since the model have to match the future prices on the market, we can compute the future of maturity T, under the assumption of constant interest rate, with the formula:

$$F(S,T) = \mathbb{E}[S(T)] = \mathbb{E}[e^{X(T)}] = e^{\mathbb{E}_0(X_T) + \frac{1}{2}\mathbb{V}_0(X_T)} = e^{ln(S)e^{-\kappa T} + \alpha^*(1-e^{-\kappa T}) + \frac{\sigma^2}{4\kappa}(1-e^{-2\kappa T})}$$

Or equivalently:

$$lnF(S,T) = ln(S)e^{-\kappa T} + \alpha^*(1 - e^{-\kappa T}) + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa T})$$

From this equation we derive the measurement equation needed for the Kalman procedure:

$$y_t = d_t + Z_t X_t + \epsilon_t$$

Here, $y_t = [ln(F(T_i))]$ the vector of the futures prices for the maturities observed. Similarly we have $d_t = [\alpha^*(1 - e^{-\kappa T_i}) + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa T_i})]$ and $Z_t = [e^{-\kappa T_i}]$. ϵ_t is the observation noise which we assume uncorrelated, centered and with a variance matrix H.
The transition equation is derived from the dynamic of X and we have:

$$X_t = c_t + Q_t X_{t-1} + \eta_t$$

Where $c_t = \kappa\alpha\Delta t$, $Q_t = (1 - \kappa\Delta t)$ and η is an uncorrelated disturbance, centered and of variance $V(\eta_t) = \sigma^2\Delta t$.

Below are presented and discussed the empirical results of this model for oil and copper. This model could not be fitted to the gold data, giving a first indication there is no mean reversion for this commodity.

One Factor Model: Oil

Period	1/2/85 to 2/17/95	1/2/90 to 2/17/95	1/2/90 to 2/17/95	1/15/93 to 5/16/96
Contracts	F1, F3, F5, F7, F9	F1, F3, F5, F7, F9	F1, F5, F9, F13, F17	Enron Data
NOBS	510	259	259	163
κ	0.301 (0.005)	0.694 (0.010)	0.428 (0.008)	0.099 (0.003)
μ	3.093 (0.346)	3.037 (0.228)	2.991 (0.280)	2.857 (0.635)
σ	0.334 (0.005)	0.326 (0.008)	0.257 (0.007)	0.129 (0.007)
λ	−0.242 (0.346)	−0.072 (0.228)	0.002 (0.279)	−0.320 (0.636)
$\xi 1$	0.049 (0.003)	0.045 (0.005)	0.080 (0.006)	0.079 (0.012)
$\xi 2$	0.018 (0.001)	0.017 (0.002)	0.031 (0.004)	0.046 (0.033)
$\xi 3$	0	0	0.010 (0.001)	0.029 (0.025)
$\xi 4$	0.012 (0.002)	0.009 (0.002)	0	0.014 (0.005)
$\xi 5$	0.022 (0.003)	0.015 (0.003)	0.007 (0.001)	0
$\xi 6$				0.007 (0.001)
$\xi 7$				0.018 (0.003)
$\xi 8$				0.031 (0.015)
$\xi 9$				0.035 (0.029)
$\xi 10$				0.035 (0.019)
Log-likelihood function	8130	4369	4345	5146

* (Standard errors in parentheses)
NOBS = number of observations.

Figure 1: Estimation of the first model to oil data.

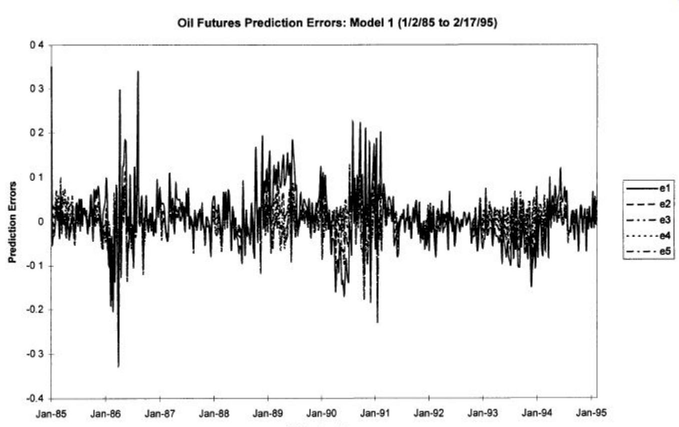


Figure 2: For each week, Model 1 oil futures prediction errors for the five futures contracts used in the estimation, starting from 1/2/85 to 2/17/95.

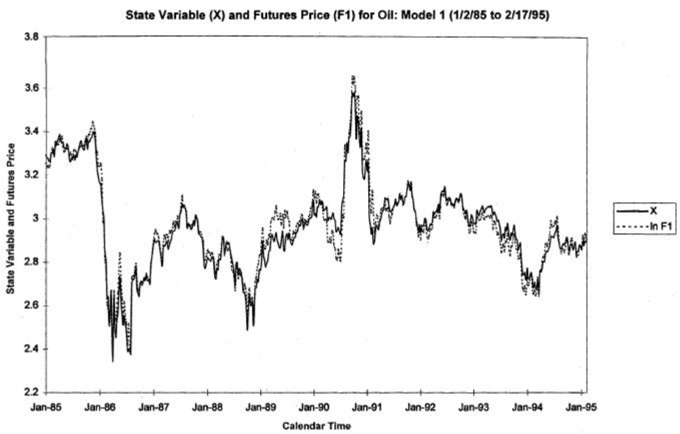


Figure 3: For each week, Model 1 estimated state variable (the logarithm of the spot price) and the logarithm of the oil futures price for the contract closest to maturity, starting from 1/2/85 to 2/17/95.

We can see from Figure 1 that there is a significant and strong mean reversion speed factor, which strongly depends on the data considered. The first and second column use the same future contracts but on two different periods, we can see that on the latter period the mean reversion speed is as much as twice higher. The second and the third column represent the same period but longer maturities for the third column. We can see that this has a negative impact on mean reversion speed, has it shifts from 0.69 to 0.43. This will have implication on hedging or pricing strategies discussed later. All the other parameters are quite similar, regardless of the periods and/or contracts considered. The risk price λ is not significant in any case. In each case we have a standard deviation of the measurement error going to zero. It seems that this is common in this kind of analysis. We can assume that the model seeks to fit very well on a particular contract (first and second column: contract F5; third column: contract F9) and leaving errors to others contracts.

Figure 2 represents the prediction errors for the five futures contracts. The average absolute error is around one percent of the log price of the contract closest to maturity. On figure 3 we can observe the log of the spot price (X) and the log price of the future with the smallest maturity in the data set. The future price seems to accurately follow the spot price with some divergences. In particular, it seems a bit more volatile, with higher pics and deeper shortfalls.

One Factor Model: Copper

Period	7/29/88 to 6/13/95
Contracts	F1, F3, F5, F7, F9
NOBS	347
κ	0.369 (0.009)
μ	4.854 (0.230)
σ	0.233 (0.007)
λ	−0.339 (0.230)
$\xi 1$	0.064 (0.002)
$\xi 2$	0.023 (0.001)
$\xi 3$	0
$\xi 4$	0.015 (0.001)
$\xi 5$	0.021 (0.002)
Log-likelihood function	5482

* (Standard errors in parentheses)
NOBS = number of observations.

Figure 4: Estimation of the first model to copper data.

On Figure 4 we have the estimation of the model for copper data. We can see that, as for oil, there is a strong mean reversion effect. Risk price is still not significant.

3.4 Second model: two-factor model

The second model is based on Gibson and Schwartz (1990) and introduce a stochastic convenience yield:

$$dS = (\mu - \delta)Sdt + \sigma_1 Sdz_1$$

$$d\delta = \kappa(\alpha - \delta)dt + \sigma_2 dz_2$$

The spot price S is following a standard process and the convenience yield δ is following an Ornstein-Uhlenbeck process. Note that if δ were a deterministic function of the spot price, ie $\delta = \kappa ln(S)$, the model 2 would be equivalent to model 1. It is therefore an extension of the first model. The two Brownian motions z_1 and z_2 are correlated and $dz_1 dz_2 = \rho dt$

As before, with $X = ln(S)$ we have the following equation by applying Itô's Lemma:

$$dX = (\mu - \delta - \frac{\sigma_1^2}{2})dt + \sigma_1 dz_1$$

Finally, we can rewrite this equation under the risk neutral probability:

$$dS = (r - \delta)Sdt + \sigma_1 Sdz_1^*$$

$$d\delta = [\kappa(\alpha - \delta) - \lambda]dt + \sigma_2 dz_2^*$$

$$dz_1^* dz_2^* = \rho dt$$

with λ being the market risk price of the convenience yield, assumed constant.
The partial differential equation that futures prices must satisfy is available in the article, equation (17) on page 6, we chose not to put it for clarity purpose. Jamshidian and Fein (1990) and Bjerksund (1991) have shown that this equation is solved with:

$$lnF(S, \delta, T) = ln(S) - \delta\frac{1 - e^{-\kappa T}}{\kappa} + A(T)$$

$$A(T) = (r - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa})T + \sigma_2^2\frac{1 - e^{-2\kappa T}}{4\kappa^3} + (\hat{\alpha}\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa})\frac{1 - e^{-\kappa T}}{\kappa^2}$$

where $\hat{\alpha} = \alpha - \frac{\lambda}{\kappa}$.

From the equations above we can deduce the measurement and the transition equations needed for the Kalman procedure:

$$y_t = d_t + Z_t[X_t, \delta_t]' + \epsilon_t$$

$$[X_t, \delta_t]' = c_t + Q_t[X_{t-1}, \delta_{t-1}]' + \eta_t$$

where $y_t = [ln(F(T_i))]$, $d_t = [A(T_i)]$, $Z_t = [1, -\frac{1-e^{-\kappa T_i}}{\kappa}]$ and ϵ is an uncorrelated disturbance, assumed centered and of variance matrix H.
We also have $c_t = [(\mu - \frac{\sigma_1^2}{2})\Delta t\ ,\ \kappa\alpha\Delta t]'$, $Q_t = \begin{vmatrix} 1 & -\Delta t \\ 0 & 1 - \kappa\Delta t \end{vmatrix}$, and η is an uncorrelated disturbance, assumed centered and of variance matrix $V(\eta_t) = \begin{vmatrix} \sigma_1\Delta t & \rho\sigma_1\sigma_2\Delta t \\ \rho\sigma_1\sigma_2\Delta t & \sigma_1\Delta t \end{vmatrix}$.
The interest rate r was chosen constant and equal to 6%, which was the average on the period considered. The fact that r is constant does not represent a strong assumption as its real deviation from 6% are absorbed by the stochastic convenience yield.

Below are presented and discussed the empirical results of this model for oil, copper and gold.

Two Factor Model: Oil

Period	1/2/85 to 2/17/95	1/2/90 to 2/17/95	1/2/90 to 2/17/95	1/15/93 to 5/16/96
Contracts	F1, F3, F5, F7, F9	F1, F3, F5, F7, F9	F1, F5, F9, F13, F17	Enron Data
NOBS	510	259	259	163
μ	0.142 (0.125)	0.244 (0.150)	0.238 (0.160)	0.082 (0.120)
κ	1.876 (0.024)	1.829 (0.033)	1.488 (0.027)	1.187 (0.026)
α	0.106 (0.025)	0.184 (0.110)	0.180 (0.126)	0.090 (0.086)
$\sigma1$	0.393 (0.007)	0.374 (0.011)	0.358 (0.010)	0.212 (0.011)
$\sigma2$	0.527 (0.015)	0.556 (0.024)	0.426 (0.017)	0.187 (0.012)
ρ	0.766 (0.013)	0.882 (0.013)	0.922 (0.006)	0.845 (0.024)
λ	0.198 (0.166)	0.316 (0.203)	0.291 (0.190)	0.093 (0.101)
$\xi1$	0.022 (0.001)	0.020 (0.001)	0.043 (0.002)	0.027 (0.001)
$\xi2$	0.001 (0.001)	0	0.006 (0.001)	0.006 (0.001)
$\xi3$	0.003 (0.001)	0.004 (0.000)	0.003 (0.000)	0
$\xi4$	0	0	0	0.002 (0.000)
$\xi5$	0.005 (0.000)	0.006 (0.000)	0.004 (0.000)	0
$\xi6$				0.004 (0.000)
$\xi7$				0.014 (0.003)
$\xi8$				0.032 (0.015)
$\xi9$				0.043 (0.036)
$\xi10$				0.055 (0.039)
Log-likelihood function	10267	5256	5139	6182

* (Standard errors in parentheses)
NOBS = number of observations.

Figure 5: Estimation of the second model to oil data.

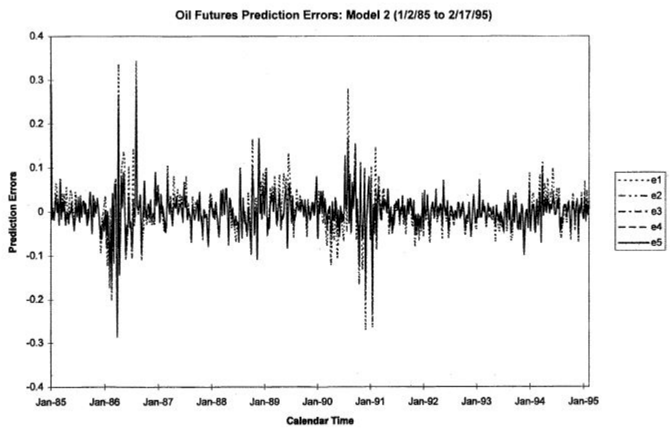


Figure 6: For each week, Model 2 oil futures prediction errors for the five futures contracts used in the estimation, starting from 1/2/85 to 2/17/95.

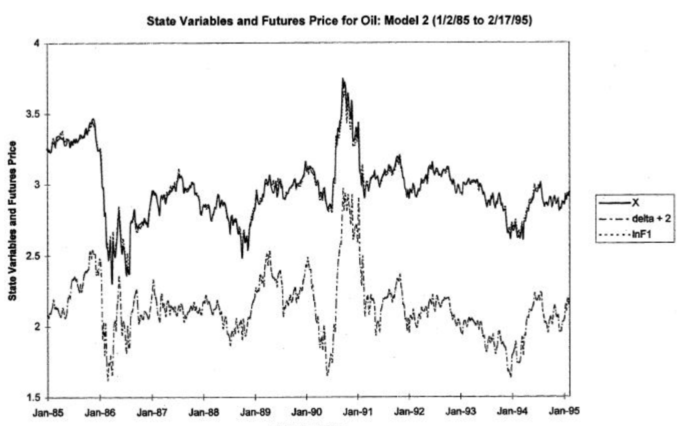


Figure 7: For each week, Model 2 estimated state variables (the logarithm of the spot price and the instantaneous convenience yield) and the logarithm of the oil futures price for the contract closest to maturity, starting from 1/2/85 to 2/17/95.

Similarly as before, we can see on Figure 5 that the mean reversion parameter of the convenience yield is highly significant and so is the correlation factor ρ. The average convenience yield α and its market price risk λ are not significant in all cases. The average spot price μ is not significant as well. As in the first model, the mean reversion speed is decreasing as the maturity of the contract is increasing. This is quite intuitive as we expect the long term futures to be less volatile than short term ones, as they do not follow precisely every moves of the spot price.
Figure 6 shows prediction errors and they are lower than in model 1. This can be seen on Figure 7, where the log price of the closest to maturity future and the log of the spot price are much more similar than in the first model. We can also observe the evolution of the second state variable, the instantaneous convenience yield, and see that it is very correlated to the other state variable (spot price). Figure 5 tells us that the correlation is of 0.77.

Two Factor Model: Copper

Period	7/29/88 to 6/13/95
Contracts	F1, F3, F5, F7, F9
NOBS	347
μ	0.326 (0.110)
κ	1.156 (0.041)
α	0.248 (0.098)
$\sigma 1$	0.274 (0.012)
$\sigma 2$	0.280 (0.017)
ρ	0.818 (0.020)
λ	0.256 (0.114)
$\xi 1$	0.033 (0.001)
$\xi 2$	0.003 (0.001)
$\xi 3$	0.006 (0.000)
$\xi 4$	0.005 (0.000)
$\xi 5$	0.009 (0.001)
Log-likelihood function	6473

* (Standard errors in parentheses)
NOBS = number of observations.

Figure 8: Estimation of the second model to copper data.

Figure 8 shows similar results for copper data. We have strong and highly significant mean reversion speed factor and correlation factor between the two state variables. The average spot price μ and the market risk price of the convenience yield λ are not significant. The average convenience yield α is significant and quite high as it represents the convenience yield for the very moment (ie the capacity to have storage instantaneously).

Two Factor Model: Gold

Period	1/2/85 to 6/13/95	11/21/90 to 6/13/95	11/21/90 to 6/13/95
Contracts	F1, F3, F6, F9, F11	F1, F3, F6, F9, F11	F1, F5, F9, F13, F18
NOBS	527	230	230
μ	0.039 (0.044)	0.033 (0.054)	0.030 (0.054)
κ	0.011 (0.008)	0.114 (0.015)	0.298 (0.018)
α	−0.002 (0.322)	0.018 (0.052)	0.019 (0.023)
$\sigma 1$	0.135 (0.003)	0.106 (0.004)	0.107 (0.004)
$\sigma 2$	0.016 (0.001)	0.0124 (0.0007)	0.015 (0.001)
ρ	0.056 (0.034)	0.113 (0.066)	0.250 (0.068)
λ	0.0067 (0.0036)	0.0076 (0.0060)	0.008 (0.007)
$\xi 1$	0.003 (0.000)	0.002 (0.000)	0.004 (0.000)
$\xi 2$	0	0	0
$\xi 3$	0.001 (0.000)	0.001 (0.000)	0.001 (0.001)
$\xi 4$	0	0	0.001 (0.000)
$\xi 5$	0.001 (0.000)	0.001 (0.000)	0.012 (0.001)
Log-likelihood function	14437	6662	5660

* (Standard errors in parentheses)
NOBS = number of observations.

Figure 9: Estimation of the second model to gold data.

On Figure 9 we see that results for gold are quite different from what we have seen until now. The main difference is that the mean reversion speed parameter is very and even insignificant for the first data set. Secondly, the correlation between the log of the spot price and the instantaneous convenience yield is significant only for the longer maturities and is low (25%). As in the oil data, the average spot price μ, the average convenience yield α and its market risk price λ are not significant. This may come from the fact that gold plays a very specific role in the economy and is highly correlated with interest rates or currencies. We cannot consider gold as a standard commodity like oil or copper.

3.5 Third model: three-factor model

In this model we introduce a new stochastic parameters: the instantaneous interest rate. We have an extension of the model 2 with the risk free rate assumed to follow an Ornstein-Uhlenbeck process as in Vasicek (1977). We thus have the following equations:

$$dS = (r - \delta)Sdt + \sigma_1 S dz_1^*$$
$$d\delta = \kappa(\hat{\alpha} - \delta)dt + \sigma_2 dz_2^*$$
$$dr = a(m^* - r)dt + \sigma_3 dz_3^*$$
$$dz_1^* dz_2^* = \rho_1 dt,\ dz_2^* dz_3^* = \rho_2 dt,\ dz_1^* dz_3^* = \rho_3 dt$$

The futures prices have to satisfy the partial differential equation (25) on page 7 of the paper. It can be shown that the log form solution is:

$$ln(F(S, \delta, r, T)) = ln(S) - \delta \frac{1 - e^{-\kappa T}}{\kappa} + r \frac{1 - e^{-aT}}{a} + C(T)$$

where C(T) is given by equation (28) on page 7 of the paper, that we will not give here for the sake of clarity.

It is assumed that the interest rate process is not affected by the commodity future prices. Therefore, the parameters of the interest rate are first estimated. Once done, we estimate the spot price and the convenience yield parameters using model 3 and Kalman procedure. The measurement equation is given by:

$$y_t = d_t + Z_t[X_t, \delta_t]' + \epsilon_t$$

where $y_t = [ln(F(T_i))]$, $d_t = [r_t \frac{1-e^{-aT_i}}{a} + C(T_i)]$, $Z_t = [1\ ,\ -\frac{1-e^{-\kappa T_i}}{\kappa}]$ and ϵ is an uncorrelated disturbance, assumed centered and of variance matrix H.
As we estimate the same state variables, the transition equation is the same as in model 2.

Below are presented and discussed the empirical results of this model for oil, copper and gold.

Three Factor Model: Oil

Period	1/2/90 to 2/17/95	1/15/93 to 5/16/96
Contracts	F1, F5, F9, F13, F17	Enron Data
NOBS	259	163
μ	0.315 (0.125)	0.008 (0.109)
κ	1.314 (0.027)	0.976 (0.022)
α	0.249 (0.093)	0.038 (0.077)
$\sigma 1$	0.344 (0.009)	0.196 (0.009)
$\sigma 2$	0.372 (0.014)	0.145 (0.008)
$\rho 1$	0.915 (0.007)	0.809 (0.027)
λ	0.353 (0.123)	0.013 (0.075)
$\xi 1$	0.045 (0.002)	0.028 (0.001)
$\xi 2$	0.007 (0.001)	0.006 (0.001)
$\xi 3$	0.003 (0.000)	0
$\xi 4$	0	0.002 (0.000)
$\xi 5$	0.004 (0.000)	0.000 (0.001)
$\xi 6$		0.005 (0.000)
$\xi 7$		0.013 (0.002)
$\xi 8$		0.024 (0.008)
$\xi 9$		0.032 (0.014)
$\xi 10$		0.053 (0.023)
Log-likelihood function	5128	6287
$\sigma 3$	0.0081	0.0073
a	0.2	0.2
$R(\infty)$	0.07	0.07
$\rho 2$	−0.0039	0.0399
$\rho 3$	−0.0293	−0.0057

* (Standard errors in parentheses)
NOBS = number of observations.

Figure 10: Estimation of the third model to oil data.

Three Factor Model: Copper

Period	7/29/88 to 6/13/95
Contracts	F1, F3, F5, F7, F9
NOBS	347
μ	0.332 (0.094)
κ	1.045 (0.030)
α	0.255 (0.078)
$\sigma1$	0.266 (0.011)
$\sigma2$	0.249 (0.014)
$\rho1$	0.805 (0.022)
λ	0.243 (0.082)
$\xi1$	0.032 (0.001)
$\xi2$	0.004 (0.001)
$\xi3$	0.005 (0.000)
$\xi4$	0.005 (0.000)
$\xi5$	0.007 (0.000)
Log-likelihood function	6520
$\sigma3$	0.0096
a	0.2
$R(\infty)$	0.07
$\rho2$	0.1243
$\rho3$	0.0964

* (Standard errors in parentheses)
NOBS = number of observations.

Figure 11: Estimation of the third model to copper data.

Three Factor Model: Gold

Period	11/21/90 to 6/13/95
Contracts	F1, F5, F9, F13, F18
NOBS	230
μ	0.023 (0.054)
κ	0.023 (0.023)
α	0.021 (0.189)
$\sigma1$	0.106 (0.004)
$\sigma2$	0.009 (0.001)
$\rho1$	0.208 (0.069)
λ	0.002 (0.004)
$\xi1$	0.003 (0.000)
$\xi2$	0
$\xi3$	0.001 (0.000)
$\xi4$	0.001 (0.000)
$\xi5$	0.015 (0.001)
Log-likelihood function	5688
$\sigma3$	0.0082
a	0.2
$R(\infty)$	0.07
$\rho2$	−0.4005
$\rho3$	−0.0260

* (Standard errors in parentheses)
NOBS = number of observations.

Figure 12: Estimation of the third model to gold data.

Parameters for the interest rate are estimated using 3-month T-bill yields. The infinite maturity yield is fixed to 7% and is described by $R(\infty) = m^* - \frac{\sigma_3^2}{2a^2}$. ρ_2 and ρ_3 were computed using weekly data from T-bills and values of the state variables estimated with model 2. We can see on figures 10, 11 and 12 that these correlations are close to zero except for the convenience yield of gold. So the assumption taken on the interest rate was not wrong for oil and copper. However, the fact that the mean reversion speed parameter of the convenience yield for gold is now insignificant shows that our assumptions was wrong. Furthermore, it shows that the model is not correct for gold when included stochastic interest rates with mean reversion. As we said before, the particular role of gold in the economy and its strong negative correlation with interest rates are not fitted for this kind of models.
For oil and copper, we observe on figures 10 and 11 similar results as in model 2. We still have a strong and highly significant mean reversion speed of convenience yield parameter and correlation ρ_1 between the two state variables (log spot price and the instantaneous convenience yield).

3.6 Models comparison

In this section we will compare the performance of the three models. We chose to focus on results on futures term structure and on long maturity futures contracts. The paper presents results for three data set: long term oil futures, copper futures and Enron futures. Gold is obviously not included as the models are not fitted to this commodity (see previous section). We will focus only on long term oil futures and on copper futures as it does not change the interpretation of the results.
The analysis will be organised as follow: discussion around models performance for three specific dates of the data set, comparison of models performance for 50 out of sample dates that were not included to estimate the models parameters (prediction capacities).

3.6.1 Futures term structure

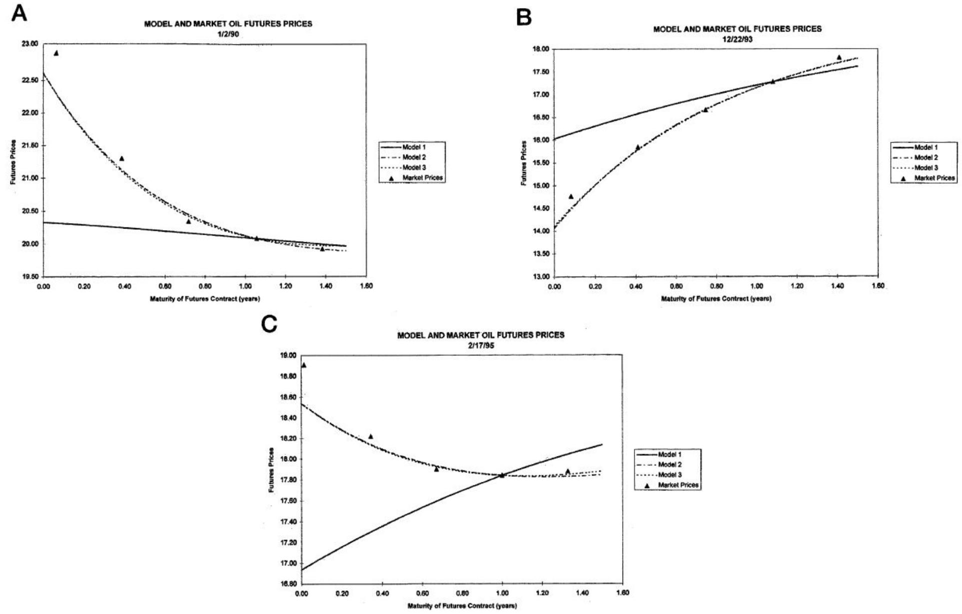


Figure 13: (A) Term structure of oil futures prices for 1/2/90, the starting date for the long term oil data, and the term structure implied by the three models. (B) Term structure of oil futures prices for 12/22/93, a date on which oil futures prices were in contango, and the term structure implied by the three models. (C) Term structure of oil futures prices for 2/17/95, the last date for the oil data, and the term structure implied by the three models.

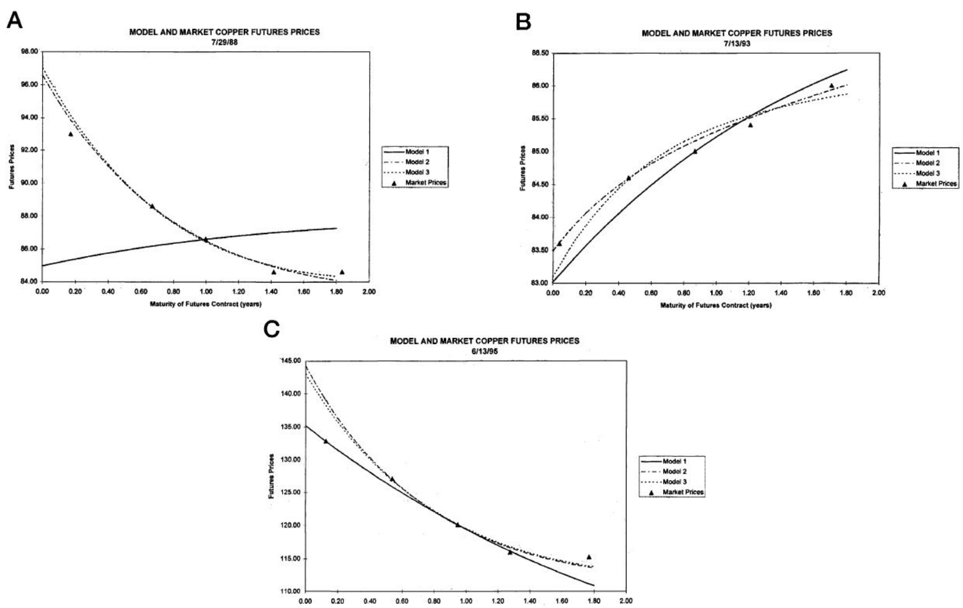


Figure 14: (A) Term structure of copper futures prices for 7/29/88, the starting date for the copper data, and the term structure implied by the three models. (B) Term structure of copper futures prices for 7/13/93, a date on which copper futures prices were in contango, and the term structure implied by the three models. (C) Term structure of copper futures prices for 6/13/95, the last date for the copper data, and the term structure implied by the three models.

On figures 13 and 14 we can see that the first model is incapable of fit the data in many cases. We also see that model 2 and 3 are often not distinguishable and very close. In most situations, backwardation and contango, they can fit the market prices well. We note that they have more difficulties to price short maturities futures. In order to have a more quantitative comparison, out-of-sample errors are computed and analyzed below.

To have an out-of-sample error, we must compute the parameters up to information available at time $t-1$ and then estimates the futures prices at time t. The author adopts the following methodology: construction of a lower bound of the error by estimating prices with models fitted on all the data, construction of an upper bound of the error by estimating prices with models fitted on all the data except the last 50 observations. The results are presented on the two figures below.

Time Series Comparison Between Models 1, 2, and 3: Last 50 Observations of Oil Data

	RMSE (of Log Prices)			Mean Error (of Log Prices)		
Model	1	2	3	1	2	3
			Panel A: In-Sample Parameter Estimation			
F1	0.0830	0.0538	0.0540	0.0628	0.0291	0.0279
F5	0.0390	0.0240	0.0248	0.0260	0.0049	0.0040
F9	0.0230	0.0195	0.0194	0.0096	0.0003	−0.0007
F13	0.0171	0.0170	0.0170	0.0012	0.0012	0.0001
F17	0.0157	0.0161	0.0160	−0.0033	0.0034	0.0016
All	0.0435	0.0300	0.0299	0.0193	0.0077	0.0066
			Panel B: Out-of-Sample Parameter Estimation			
F1	0.0919	0.0551	0.0541	0.0770	0.0315	0.0289
F5	0.0378	0.0249	0.0246	0.0273	0.0045	0.0030
F9	0.0198	0.0195	0.0195	0.0020	−0.0004	−0.0017
F13	0.0210	0.0170	0.0170	−0.0125	0.0014	0.0002
F17	0.0262	0.0166	0.0162	−0.0212	0.0050	0.0028
All	0.0477	0.0303	0.0299	0.0145	0.0084	0.0066

Figure 15: (Panel A) Lower bound (Panel B) Upper bound.

Time Series Comparison Between Models 1, 2, and 3: Last 50 Observations of Copper Data

	RMSE (of Log Prices)			Mean Error (of Log Prices)		
Model	1	2	3	1	2	3
			Panel A: In-Sample Parameter Estimation			
F1	0.0453	0.0430	0.0412	0.0046	−0.0253	−0.0232
F3	0.0294	0.0216	0.0217	0.0061	0.0029	0.0024
F5	0.0192	0.0194	0.0190	0.0027	0.0070	0.0062
F7	0.0207	0.0180	0.0180	−0.0000	−0.00126	−0.0016
F9	0.0255	0.0187	0.0186	0.0140	−0.00021	−0.0003
All	0.0296	0.0260	0.0253	0.0055	−0.0034	−0.0033
			Panel B: Out-of-Sample Parameter Estimation			
F1	0.0455	0.0467	0.0434	0.0058	−0.0301	−0.0264
F3	0.0295	0.0218	0.0216	0.0068	0.0030	0.0023
F5	0.0193	0.0199	0.0193	0.0031	0.0080	0.0068
F7	0.0207	0.0182	0.0181	0.0001	−0.0013	−0.0015
F9	0.0256	0.0188	0.0188	0.0141	−0.0018	−0.0012
All	0.0297	0.0273	0.0261	0.0060	−0.0044	−0.0040

Figure 16: (Panel A) Lower bound (Panel B) Upper bound.

Here it is clear that model 2 and 3 outperform model 1, both on oil and copper data. We can also see that model 3 is a bit better than model 2, both on RMSE and Mean error, and both on copper and oil data.

3.6.2 Long maturity futures contracts

The data available on futures prices (except for Enron data) are up to a maturity of two years. We will then look at the implications on long maturities futures prices computed by the models fitted with short maturities futures prices.

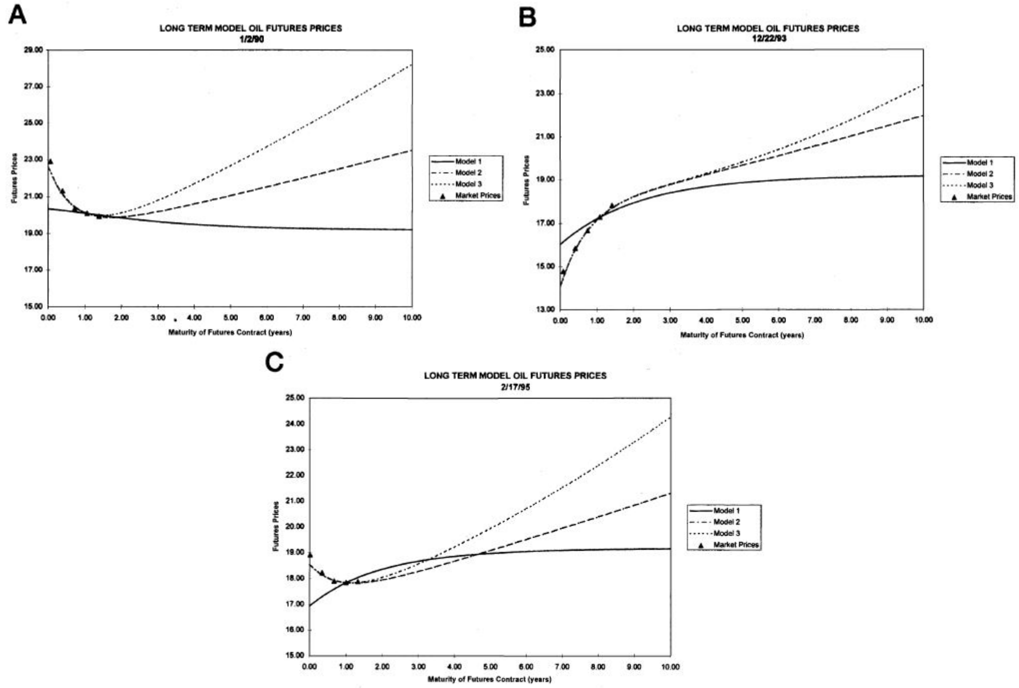


Figure 17: (A) Term structures of oil futures prices for 1/2/90 implied by the three models for maturities up to ten years. (B) Term structure of oil futures prices for 12/22/93 implied by the three models for maturities up to ten years. (C) Term structure of oil futures prices for 2/17/95 implied by the three models for maturities up to ten years.

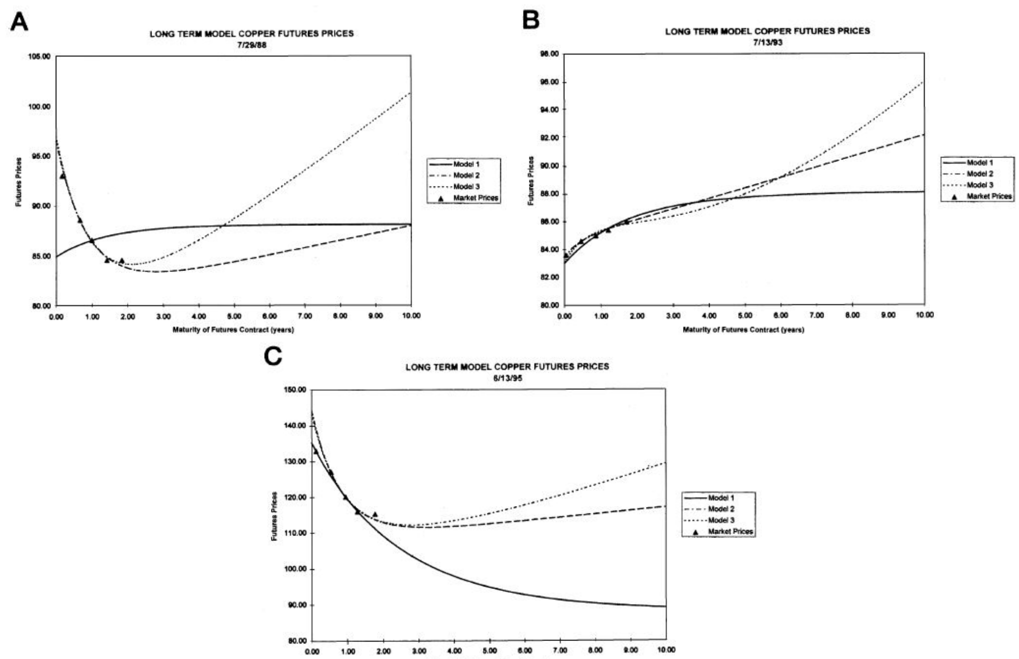


Figure 18: (A) Term structure of copper futures prices for 7/29/88 implied by the three models for maturities up to ten years. (B) Term structure of copper futures prices for 7/13/93 implied by the three models for maturities up to ten years. (C) Term structure of copper futures prices for 6/13/95 implied by the three models for maturities up to ten years.

We see on figures 17 and 18 that even though the second and the third models are very similar on short maturities contracts, they can diverge on long maturities contracts. For both models, the rate at which the futures prices is changing with maturity can be easily computed and is independent of the initial values of the state variables. We have, respectively for model 2 and 3:

$$\frac{1}{F}\frac{\partial F}{\partial T}(T \longrightarrow \infty) = r - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\rho\sigma_1\sigma_2}{\kappa}$$

$$\frac{1}{F}\frac{\partial F}{\partial T}(T \longrightarrow \infty) = m^* - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\rho_1\sigma_1\sigma_2}{\kappa} + \frac{\sigma_3^2}{2a^2} + \frac{\rho_3\sigma_1\sigma_3}{a} - \frac{\rho_2\sigma_2\sigma_3}{\kappa a}$$

Even if the future price term structure is decreasing at the beginning, it will go upward at T increases. We have, respectively for model 2 and 3, rates of 2.71% for oil and 0.85% for copper, 4.19% for oil and 2.70% for copper. These differences between the models can lead to significant differences in futures prices for longer maturities. We note that the infinity rate $R(\infty)$ fixed at 7% in model 3 as a big impact on the rates above and are positively correlated:

$$\frac{1}{F}\frac{\partial F}{\partial T}(T \longrightarrow \infty) = R(\infty) - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\rho_1\sigma_1\sigma_2}{\kappa} + \frac{\sigma_3^2}{a^2} + \frac{\rho_3\sigma_1\sigma_3}{a} - \frac{\rho_2\sigma_2\sigma_3}{\kappa a}$$

To compare the long term performances of the models more quantitatively, they are fitted using the first contracts of Enron data (maturities of 2 months to 1.5 years). Then, they estimate the price of the last five forward contracts of Enron data (maturities of 2 to 9 years). Results are shown below.

Cross-Section Comparison Between Models 1, 2, and 3 Out-of-Sample Enron Oil Data 1/15/93 to 5/16/96

	RMSE in Dollars				RMSE in Percentage			
Model	1	2	3 (7%)	3 (6%)	1	2	3 (7%)	3 (6%)
2 Years	0.35	0.09	0.16	0.20	1.91	0.49	0.86	1.12
3 Years	0.65	0.29	0.37	0.38	3.42	1.45	1.91	1.98
5 Years	1.29	0.80	0.96	0.65	6.42	3.94	4.89	3.24
7 Years	1.79	1.40	1.93	0.87	8.59	6.84	9.53	4.24
9 Years	2.24	2.14	3.27	1.33	10.53	10.22	15.68	6.38
All	1.44	1.20	1.76	0.79	6.95	5.81	8.54	3.85

Figure 19

We can see that model 2 always outperform model 1 and model 3 with a 7% infinite maturity discount yield. However, model 3 with a 6% infinite maturity discount yield has the better results with RMSE of 3.85% in average. This shows that the rates parameters in model 3 have to be carefully considered in order to have good performance on the long term.

3.7 Hedging contracts for future delivery

As we said in the introduction, on of the advantage of commodity derivatives market is to be able to hedge against long time forward and unknown commitments. The models presented above allow to compute such an hedging strategy. The difference between model 1, 2 and 3 is that in the latter, interest rates are stochastic.

Generally, forwards are hedged using futures contracts. The difference is that future contracts are settled all along the way with margin calls, eliminating credit risk, while forwards are settled once at the end. We can see now that in model 1 and 2, forwards and futures contracts will have the same price, as interest rate is constant. However in the third model this is no longer the case.

In order to hedge a forward commitment, its present value sensitivity to the underlying factors must be equal to the sensitivity of the hedge portfolio with respect to the same factors. Therefore, the number of future contracts used for hedging must be equal to the number of factors used in the models.

In model 1 and 2, we have respectively one and two factors. As forwards prices are equal to futures prices, the hedging strategy is simply obtained by discounting the future prices. In model 3 however, the present value of a forward commitment is given by equation (30) of the paper. The hedging strategy, consisting of long position w_i of future contracts with maturity t_i, of a forward commitment to deliver one unit of commodity at time T is given by solving these equations:

For model 1:

$$w_1 F_S(S, t_1) = e^{-rT} F_S(S, T)$$

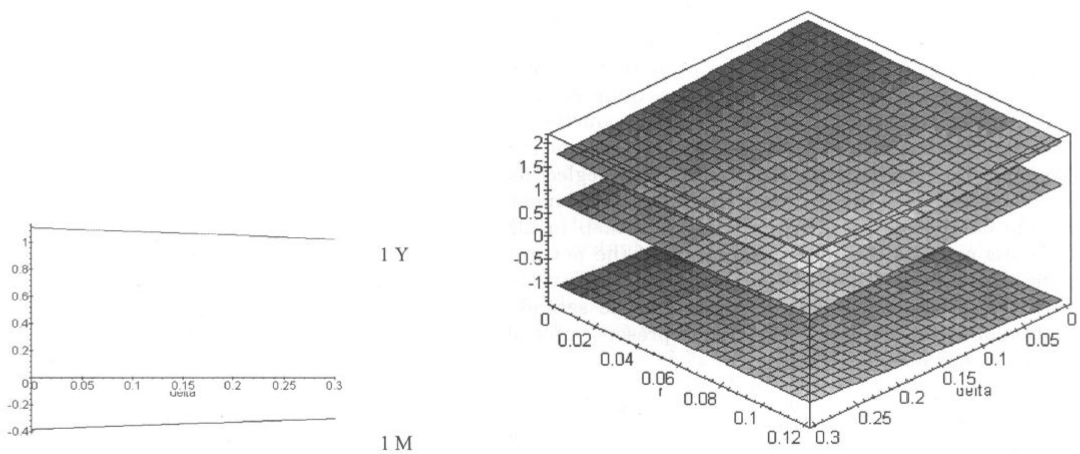
For model 2:

$$\begin{aligned} w_1 F_S(S, \delta, t_1) + w_2 F_S(S, \delta, t_2) &= e^{-rT} F_S(S, \delta, T) \\ w_1 F_\delta(S, \delta, t_1) + w_2 F_\delta(S, \delta, t_2) &= e^{-rT} F_\delta(S, \delta, T) \end{aligned}$$

For model 3:

$$\begin{aligned} w_1 F_S(S, \delta, r, t_1) + w_2 F_S(S, \delta, r, t_2) + w_3 F_S(S, \delta, r, t_3) &= P_S(S, \delta, r, T) \\ w_1 F_\delta(S, \delta, r, t_1) + w_2 F_\delta(S, \delta, r, t_2) + w_3 F_\delta(S, \delta, r, t_3) &= P_\delta(S, \delta, r, T) \\ w_1 F_r(S, \delta, r, t_1) + w_2 F_r(S, \delta, r, t_2) + w_3 F_r(S, \delta, r, t_3) &= P_r(S, \delta, r, T) \end{aligned}$$

Below are represented illustrations of such hedging strategies for a ten years forward commitment on oil, for model 2 and 3. We do not represent model 1 as it is not able to price futures correctly.



B. Model 2: 1 month and one year futures

C. Model 3: 1 and 6 month futures and 1 year discount bond

Figure 20: Hedging ten and five year oil forward commitments with future contracts (B) The figure shows the number of one-month and one-year oil futures contracts required in Model 2 to hedge a ten-year forward commitment to deliver one barrel of oil, as a function of the current instantaneous convenience yield. (C) The f igure shows the number of one-month and six-month oil futures contracts and one-year unit discount bonds required in Model 3 to hedge a five-year forward commitment to deliver one barrel of oil, as a function of the current instantaneous interest rate and convenience yield.

On panel B we can see the number of 1-month and 1-years future contracts needed to hedge the 10-year commitment, as a function of the convenience yield. On panel C, the 1-month (bottom surface), 1-year (middle surface) and 6-month (top surface) quantities are represented as a function of the convenience yield and the interest rate. In model 2 and 3, the quantities are independent of the spot level S, we can see here that they are also not very sensitive to δ and r.

4 Opening and conclusion

We have seen in the state of the art that several models have been developed to implement commodity spot price dynamic. First, one-factor model in continuation of Black Scholes, taking into account the spot price alone. Secondly, two-factor models emerged with the necessity to model spot price jointly with a stochastic convenience yield, which absorbs the mean-reversion effect. Then, the assumptions of constant interest rate being obviously too strong, three-factor models were developed with stochastic interest rate. This paper models the stochastic interest rate as in Vasicek (1977), the rate following an Ornstein-Uhlenbeck process and its parameters are fit on three-month T-Bills data.
This paper gather the three types of models and allow a comparison between them. The first and second models were implemented the same way as it was already done in previous research. However, the innovation on the third model comes from the fact that it specifies a dynamics for all the three factors and fits its parameters to market data. Each model provides an analytical form of the solution which is very convenient to compare them with one another. Therefore, we see this paper as a grouping of existing methods to model spot price of commodities, and adding a few innovations mentioned above.

An interesting point of this paper is the gathering of several applications of such models, that are discussed in several papers as well: hedging strategies and investment under uncertainty. The first point has been discussed through the dedicated section. The latter however is a true innovation and a very interesting point, that we did not discussed above for conciseness reasons. As mentioned in the state of the art, the DCF method for valuation is very limited for commodity projects. In this paper, Schwartz proposes an application of the three models to project valuation and compares them to two industry benchmarks (DCF and real option approach assuming that commodity prices follow a random walk, neglecting the mean reversion effect). He therefore provides interesting results. In the example of a copper mine, he shows that copper price at which the project is profitable ($NVP > 0$) can vary from 0.26 to 0.90 dollar an ounce depending on which model we chose, that is a factor of 346% between prices. Also, the spot price at which it is optimal to invest in the project varies from 0.73 to 1.36 dollar an ounce, ie a factor of 186%. This shows that model selection have huge impact on the prevision of the profitability state of the project and on the optimal time to invest.

A point that is unclear in this paper is how Schwartz precisely applied the Kalman procedure to the data. As this article gather several models and methods, more details on this point would have been appreciated for clarity purpose. For example, a point that explains the procedure and specifically shows how we obtain the optimal state variables. Also, only for clarity reasons, the layout of the figures and sections is messy and could be better organized.

There are several point to develop from this article. First, we have seen that gold does not fit the models. A we said, it plays a very specific role in our economy and cannot be considered as a regular commodity. Its very condensed state and its high value compared to volume occupied have clearly a role to play in the difference between its convenience yield dynamic and the one of commercial metals. Also, as it is used for currency reserves by central banks around the world, we must think of a model that is half way between a commodity and a currency. We have seen that mean reversion has no effect on its convenience yield and that it is very correlated to interest rate dynamic. Then, we have to think of a model that is able to compute the parameters without assuming that convenience yield (or spot price) is uncorrelated to interest rate, as Schwartz did in model 3.

As the recent events showed us, future prices can get negative in extreme conditions. Such shocks in market conditions can be included in model 3 by having a convenience yield that could explode at once. We could add very infrequent positive jumps into its dynamic using Poisson process. However by doing this, it would be much more difficult to have an analytical solution. We could still compute numerical solutions and see what happens. We implemented simulations of model 3 on oil data using the parameters given in figure 10. We computed data for two years with a time step of one day. We added an exogenous shock on the market resulting in a instantaneous decrease of 10% on the spot price and a convenience yield increase of 150%. This shock happens at a time T_{shock} generated using an exponential law of parameter $1/500$: the shock happens on average every 500 (business) days, that is every 2 years. Obviously it would be better to chose these parameters more carefully, but this is approximately what to expect from shock as we have lived in the recent days. Below are represented the evolution of the spot price and the instantaneous convenience yield.

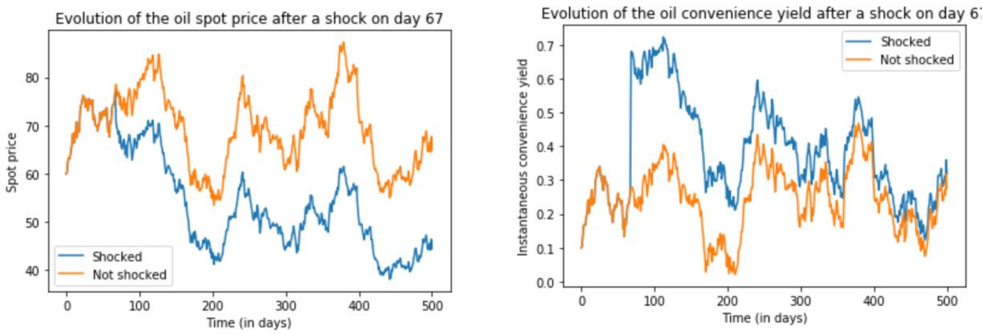


Figure 21: Evolution of the spot price and the instantaneous convenience yield on a two year period with an exogenous shock

We can see on these simulations that the convenience yield takes a long time to recover from such a shock. Also, the spot price shocked remains clearly under its "clean" counterpart and never recovers, even when convenience yield has recovered. This shows that an instantaneous exogenous shock can have a long term effect on state variables. We assumed here that the shock does not modify the other parameters of the model, such as mean-reversion speed or long term average. The interest rate dynamic follows an Ornstein-Uhlenbeck process and is not impacted by the shock, therefore we do not represent it. Python codes used for this implementation are available in the appendix.

Below are presented the results of a shock on another simulation where we computed also the one-month future price. We used the closed form solution provided for model 3 and made the assumptions that despite the shock, future prices still follow this relation. We can see that right after the shock, the difference between the spot price and the future price is much larger than before. This goes in the direction of what we saw on the market a few days ago, with a smaller spot price. However, the fact that the future price goes under zero requires a numerical solution because the analytical form is always positive.

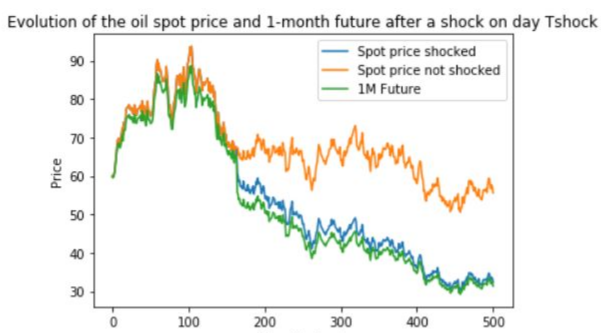


Figure 22: Evolution of the spot price and the 1-month future price on a two year period with and exogenous shock

To develop the models proposed by Schwartz, we could also introduce correlated stochastic volatilities in model 3, for the spot price and the convenience yield. On the same time, we could have defined a stochastic interest rate following a CIR process, which is more advanced than Vasicek. This would result in more parameters to estimate and of course we would lose the analytical form for futures prices.

5 Appendix

E. S. Schwartz; The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging, The Journal of Finance VOL.LII, NO. 3 . July 1997

Bessembinder, H., J. F. Coughenour, P. J. Seguin, and M. M. Smoller, 1995, Mean reversion in equilibrium asset prices: evidence from the futures term structure, Journal of Finance 50:1, 361-375.

Jamshidian, F. and M. Fein, 1990, Closed-form solutions for oil futures and European options in the Gibson-Schwartz model: A note, Working paper, Merrill Lynch Capital Markets.

Bhappu, R. R., and J. Guzman, 1995, Mineral investment decision making: A study of mining company practices, Engineering and Mining Journal 70, 36-38.

Brennan, M. J., 1991, The price of convenience and the valuation of commodity contingent claims, in D. Lund and B. Oksendal, Eds.: Stochastic Models and Option Values, North Holland.

Cortazar, G., and E. S. Schwartz, 1994, The evaluation of commodity contingent claims, Journal of Derivatives 1, 27-39.

Python code used for the simulation of a shock on oil spot price, convenience yield and future price

```
import numpy as np
from numpy import random
import matplotlib.pyplot as plt

T = 2.0 #two years
dt = 1.0/250.0 #one business day
timestep = int(T/dt)

#Parameters from the paper, third model, oil data
sigma1 = 0.344
kappa = 1.314
alpha = 0.249
sigma2 = 0.372
a = 0.2
Rinf = 0.07
sigma3 = 0.0081
rho1 = 0.915
rho2 = -0.0039
rho3 = -0.0293

m = Rinf + sigma3*sigma3/(2*a*a)
mean = [0.0,0.0,0.0]
cov = [[1.0,rho1,rho3],[rho1,1.0,rho2],[rho3,rho2,1.0]]

#shock time generated with an exponential law, shock happens on average every 500 days = 2
T_shock = int(-np.log(np.random.uniform(0,1))*500)

#Initial state
S = [60.0]
Sc= [60.0]
delta = [0.1]
```

```
deltac = [0.1]
r = [0.05]
shockd = 0.0 #shock on convenience yield
shocks = 0.0 #shock on spot price

#Generating paths
for i in range(0,timestep):

#Define shock instant
    if i==T_shock:
        shockd = 1.5
        shocks = 0.1
        #print('s')
    else:
        shockd = 0.0
        shocks = 0.0

    #Generate three correlated Brownian motion increments
    dz = np.random.multivariate_normal(mean, cov, 1).T
    dz1, dz2, dz3 = dz[0,0], dz[1,0], dz[2,0]

    #Compute increments
    ds = (r[-1]-delta[-1])*S[-1]*dt + sigma1*S[-1]*dz1*np.sqrt(dt) - shocks*S[-1]
    ddelta = kappa*(alpha-delta[-1])*dt + sigma2*dz2*np.sqrt(dt) + shockd*abs(delta[-1])
    dr = a*(m-r[-1])*dt + sigma3*dz3*np.sqrt(dt)

#Generate the paths not shocked
    if i>=T_shock:
        dsc = (r[-1]-deltac[-1])*Sc[-1]*dt + sigma1*Sc[-1]*dz1*np.sqrt(dt)
        ddeltac = kappa*(alpha-deltac[-1])*dt + sigma2*dz2*np.sqrt(dt)
    else:
        dsc = ds
        ddeltac = ddelta

    #Updating variables
    S.append(S[-1] + ds)
    Sc.append(Sc[-1] + dsc)
    delta.append(delta[-1] + ddelta)
    deltac.append(deltac[-1] + ddeltac)
    r.append(r[-1] + dr)

plt.plot(S)
plt.plot(Sc)
plt.xlabel('Time (in days)')
plt.ylabel('Spot price')
plt.title('Evolution of the oil spot price after a shock on day 67')
plt.legend(['Shocked','Not shocked'])
plt.show()

plt.plot(delta)
plt.plot(deltac)
plt.xlabel('Time (in days)')
plt.ylabel('Instantaneous convenience yield')
plt.title('Evolution of the oil convenience yield after a shock on day 67')
```

```
plt.legend(['Shocked','Not shocked'])
plt.show()

maturity = 20.0/250 #arround 1 month in business day

#Computing terms for calculating future value
first = (kappa*alpha+sigma1*sigma2*rho1)*((1-np.exp(-kappa*maturity))-kappa*maturity)/(kapp
second = -sigma2**2*(4*(1-np.exp(-kappa*maturity)) - (1-np.exp(-2*kappa*maturity)) -2*kappa
third = -1*(a*m + sigma1*sigma3*rho3)*((1-np.exp(-a*maturity)) - a*maturity)/(a**2)
fourth = -sigma3**2*(4*(1-np.exp(-a*maturity)) - (1-np.exp(-2*a*maturity)) - 2*a*maturity),
fifth = sigma2*sigma3*rho2*((1-np.exp(-kappa*maturity)) + (1-np.exp(-a*maturity)) - (1-np.e
sixth = sigma2*sigma3*rho2*(kappa**2*(1-np.exp(-a*maturity)) + a**2*(1-np.exp(-kappa*matur

C_T = first +second + third + fourth + fifth + sixth

# Using the analytical form provided in the article
F = np.array(S) * np.exp(-np.array(delta)*(1-np.exp(-kappa*maturity))/kappa + np.array(r)*(

plt.plot(S)
plt.plot(Sc)
plt.plot(F)
plt.xlabel('Time (in days)')
plt.ylabel('Price')
plt.title('Evolution of the oil spot price and 1-month future after a shock on day Tshock')
plt.legend(['Spot price shocked','Spot price not shocked','1M Future'])
plt.show()
```