A Objective Tensor Fields of Arbitrary Order

The definition of objective vector fields given in Eq. 10 directly generalizes to general $\binom{r}{s}$ tensor fields $\mathbf{T}\left(\omega^1,\ldots,\omega^r,\mathbf{x}_1,\ldots,\mathbf{x}_s\right)$,[3] with r covector (1-form) arguments ω^j, and s vector arguments $\mathbf{x}_i$, because $\mathbf{T}$ is a multi-linear function of these arguments, and we can apply the definition for vectors argument-wise. We refer to App. M for an overview of the necessary tensor notation. We define, in abbreviated notation, that $\mathbf{T}$ is objective if, under the group action Φ with g, it transforms as

$$\begin{aligned}\mathbf{T}^*\big((\omega^1)^*,\ldots,(\omega^r)^*,(\mathbf{x}_1)^*,\ldots,(\mathbf{x}_s)^*\big) &= \\ \mathbf{T}\big(g\cdot(\omega^1)^*,\ldots,g\cdot(\omega^r)^*,g^{-1}\cdot(\mathbf{x}_1)^*,&\ldots,g^{-1}\cdot(\mathbf{x}_s)^*\big).\end{aligned} \tag{A.1}$$

Giving the group actions precisely requires the pushforward and pullback of the diffeomorphism ϕ_g, corresponding to $g\in G$, and its inverse $\phi_{g^{-1}}$. Specifically, a tensor field $\mathbf{T}$ is objective if it transforms as

$$\begin{aligned}\mathbf{T}^*\big((\omega^1)^*,\ldots,(\omega^r)^*,(\mathbf{x}_1)^*,\ldots,(\mathbf{x}_s)^*\big) &= \\ \mathbf{T}\big(\phi_g^*(\omega^1)^*,\ldots,\phi_g^*(\omega^r)^*,\mathrm{d}\phi_{g^{-1}}(\mathbf{x}_1)^*,&\ldots,\mathrm{d}\phi_{g^{-1}}(\mathbf{x}_s)^*\big).\end{aligned} \tag{A.2}$$

We refer to App. S for details on pushforwards and pullbacks. In fact, Eq. A.2 simply states that a tensor field $\mathbf{T}$ is objective, if it transforms as $\mathbf{T}^* = \phi_{g^{-1}}^*\mathbf{T}$, meaning as the *pullback* of $\mathbf{T}$.[4] That is, Eq. A.2 is identical to the *definition* of the pullback of a tensor $\mathbf{T}$. We also note that the pullback $\phi_g^*\omega$ of a 1-form ω is *defined* via the pushforward $\mathrm{d}\phi_g$, i.e.,

$$\big(\phi_g^*\omega\big)\big(\mathbf{v}\big) := \omega\big(\mathrm{d}\phi_g(\mathbf{v})\big). \tag{A.3}$$

For example, a $\binom{1}{1}$ tensor field $\mathbf{S}$, i.e., a multi-linear map that maps one covector and one vector to a scalar, is objective if it transforms as

$$\begin{aligned}\mathbf{S}^*\big((\omega^1)^*,(\mathbf{x}_1)^*\big) &= \mathbf{S}\big(g\cdot(\omega^1)^*,g^{-1}\cdot(\mathbf{x}_1)^*\big),\\ &= \mathbf{S}\big(\phi_g^*(\omega^1)^*,\mathrm{d}\phi_{g^{-1}}(\mathbf{x}_1)^*\big).\end{aligned} \tag{A.4}$$

This is equivalent to Eq. 12, where the tensor $\mathbf{S}$ is equivalently interpreted as a linear transformation of vectors (or a vector-valued 1-form).

B Guide to the Remaining Appendixes

We give a brief overview of the appendixes, to provide a guide to their usage. We provide three different kinds of appendixes: Appendixes that (1) are related to the core of the paper, covering additional details such as derivations and proofs; (2) describe explicit computations to help with implementing our approach; and (3) give summaries of important concepts from differential geometry that we build upon in the paper.

Regarding (1), the following appendixes correspond to the core parts of the paper: We first rewrite the standard *Euclidean observer transformations* from continuum mechanics, as defined and described, for example, by Truesdell and Noll [64], Holzapfel [32], and Ogden [46], in a more general form in App. C, and then describe the accordingly generalized *Observer transformations on general manifolds* in App. D. It is not straightforward to see that an approach such as the one presented here is in fact objective. We therefore provide detailed *Objectivity proofs* for our method in App. E. Regarding (2) and (3), see App. B.1 and B.2, respectively. Finally, App. T provides a table summarizing the mathematical *Notation* used throughout the paper and the appendixes.

B.1 Guide to Reimplementation

In order to implement our approach for a curved manifold, such as the sphere, one needs to define coordinate charts. In each chart, our intrinsic approach requires the corresponding metric tensor field (comprising a 2×2 matrix for each mesh vertex; due to symmetry, only three unique values need to be stored), as well as the corresponding Christoffel symbols (comprising a $2\times 2\times 2$ set of values for each mesh vertex; due to symmetry, only six unique values need to be stored). Using a very simple setup of six coordinate charts orthogonally projected onto the (hemi)sphere, we give the explicit computation of the *Metric and Christoffel symbols of the 2-sphere* in App. P. We note that all of these quantities are computed *analytically*, and therefore do not contain any numerical error. We also note that any other choice of charts for the sphere (e.g., with less distortions than orthogonal projection) can easily be used instead, and can be derived in the same way as done in App. P. We have used orthogonally projected charts solely for simplicity.

In addition to the above analytically computed intrinsic properties of the manifold under consideration, we also require numerically approximated partial derivatives of vector components that are given numerically at mesh vertices. (Of the vector fields that we denote by $\mathbf{v}$ and $\mathbf{u}$, respectively.) We use an icosahedral mesh that triangulates the sphere. Vector field components are given at the mesh vertices; via two components for a vector. App. Q describes the corresponding *Numerical computation of partial derivatives*, using a simple, precomputed least-squares approach. This results in filter stencil weights for the 1-ring neighborhood of each vertex, i.e., weights for all vertices connected to the center vertex via an edge, and a weight for the center.

App. G describes how we compute *Lie derivatives in curved spaces*, which is necessary in order to compute *observed time derivatives*. As derived in that appendix, they can be computed solely from partial derivatives, which we estimate numerically as described in App. Q.

B.2 Guide to Basic Material

We provide summaries, written using the notation that we use in the paper, of basic differential geometry concepts. These concepts are standard in differential geometry and mathematical physics, but not all of them are well-known or often used in visualization. We describe *Tensors as multi-linear maps, and their bases* in App. M, and describe *Metric tensor fields* in App. O. App. F describes *Lie derivatives*, and App. N describes *Intrinsic covariant derivatives*. App. R describes the mathematical concept of *The flow of a vector field*, and App. S describes the basic concepts of *Pushforwards and pullbacks* of a given map (often a diffeomorphism) ϕ between manifolds.

Symmetry groups and Lie group actions are described in App. H, the corresponding *Lie algebra actions and induced vector fields* in App. I, and *Lie algebras and the exponential map* in App. L. More specifically, we describe *The Lie algebra of all Killing vector fields* in App. J, and the *Isometry Group and Killing Fields of the 2-Sphere* in App. K.

C Euclidean Observer Transformations

We repeat the spatial part of the reference frame transformation of Truesdell and Noll [64, p.41], from our Eq. 4, which is

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\,\mathbf{x}. \tag{C.1}$$

However, this transformation is specific to Euclidean space and hard to generalize directly. The reason for this is that a general notion of time-dependent diffeomorphisms corresponding to the transformation between the frames denoted by $\mathbf{x}^*$ and $\mathbf{x}$, respectively, is missing. In order to prepare for—and be able to compare to—transformations on arbitrary manifolds, we introduce a one-parameter (i.e., time-dependent) family of diffeomorphisms ϕ_t, mapping between the two frames, from a manifold $M=\mathbb{R}^n$ for the first frame (points labeled $\mathbf{x}$), to a manifold $N=\mathbb{R}^n$ for the second frame (points labeled $\mathbf{x}^*$). That is, we define

$$\begin{aligned}\phi_t : M &\to N,\\ \mathbf{x} &\mapsto \phi_t(\mathbf{x}) =: \mathbf{x}^*.\end{aligned} \tag{C.2}$$

We equivalently also write $\phi(\mathbf{x},t) := \phi_t(\mathbf{x})$. This gives simply

$$\mathbf{x}^* = \phi(\mathbf{x},t). \tag{C.3}$$

[3] For brevity, we list the arguments in this order, but the analogous applies for an argument list of contravariant and covariant arguments in any order.

[4] In fact, this is only possible because ϕ_g is a diffeomorphism, which guarantees that it is invertible. For non-invertible maps, the pullback of tensors of *mixed variance* is *not defined*. For diffeomorphisms, it is well-defined [39, p.321].

Below, we will obtain the special case of Eq. C.1 by simply defining ϕ_t as the one-parameter family of Euclidean isometries (with $M = N = \mathbb{R}^n$)

$$\phi_t(\mathbf{x}) := \mathbf{c}(t) + \mathbf{Q}(t)\,\mathbf{x}. \tag{C.4}$$

The map ϕ_t applies at any point $\mathbf{x} \in M$, so in order to transform a curve $t \mapsto \mathbf{x}(t) \in M$ into the corresponding curve $t \mapsto \mathbf{x}^*(t) \in N$, we have

$$\mathbf{x}^*(t) = \phi(\mathbf{x}(t), t). \tag{C.5}$$

Taking the derivative with respect to time t on both sides, we derive that a velocity field $\mathbf{v} := \mathrm{d}\mathbf{x}(t)/\,\mathrm{d}t$ transforms to $\mathbf{v}^* := \mathrm{d}\mathbf{x}^*(t)/\,\mathrm{d}t$ by

$$\begin{aligned}
\mathbf{x}^*(t) &= \phi(\mathbf{x}(t), t), \\
\frac{\mathrm{d}}{\mathrm{d}t} t \mapsto \mathbf{x}^*(t) &= \frac{\mathrm{d}}{\mathrm{d}t} t \mapsto \phi(\mathbf{x}(t), t), \\
\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^*(t)\right)_{\mathbf{x}^*(t)} &= \frac{\partial}{\partial \mathbf{x}}\bigg|_{\mathbf{x}=\mathbf{x}(t)} \phi(\mathbf{x}, t) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\right) + \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=t} \phi\big(\mathbf{x}(t), \tau\big), \\
\left(\frac{\mathrm{d}\mathbf{x}^*(t)}{\mathrm{d}t}\right)_{\mathbf{x}^*(t)} &= (\mathrm{d}\phi_t)_{\mathbf{x}(t)} \left(\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}\right) + \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=t} \phi\big(\mathbf{x}(t), \tau\big).
\end{aligned} \tag{C.6}$$

The map $\mathrm{d}\phi_t$ is the *pushforward* of ϕ_t, i.e., $(\mathrm{d}\phi_t)_{\mathbf{x}} \colon T_{\mathbf{x}}M \to T_{\mathbf{x}^*}N$, with $\mathbf{x}^* = \phi_t(\mathbf{x})$. The transformation of $\mathbf{v}$ to $\mathbf{v}^*$, for any fixed t, is therefore

$$\mathbf{v}^*\big(\mathbf{x}^*(t), t\big) = (\mathrm{d}\phi_t)_{\mathbf{x}(t)}\big(\mathbf{v}\big(\mathbf{x}(t), t\big)\big) + \frac{\mathrm{d}}{\mathrm{d}\tau}\phi\big(\mathbf{x}(t), \tau\big). \tag{C.7}$$

But because Eq. C.7 is correct for any pair of positions $\big(\mathbf{x}^*(t), \mathbf{x}(t)\big)$, $\mathbf{x}^*(t) = \phi_t(\mathbf{x}(t))$, we can drop the position-dependence on t again and write, for any corresponding pair of positions $(\mathbf{x}^*, \mathbf{x})$ with $\mathbf{x}^* = \phi_t(\mathbf{x})$,

$$\mathbf{v}^*\big(\mathbf{x}^*, t\big) = (\mathrm{d}\phi_t)_{\mathbf{x}}\big(\mathbf{v}(\mathbf{x}, t)\big) + \frac{\mathrm{d}}{\mathrm{d}t}\phi\big(\mathbf{x}, t\big). \tag{C.8}$$

We can further introduce a vector field $\mathbf{w}$ on N, i.e., in the frame $\mathbf{x}^*$, as

$$\mathbf{w}(\mathbf{x}^*, t) := \frac{\mathrm{d}}{\mathrm{d}t}\phi\big(\mathbf{x}, t\big), \quad \text{with} \quad \mathbf{x} = \phi_t^{-1}(\mathbf{x}^*). \tag{C.9}$$

This allows us to write the general transformation rule as

$$\mathbf{v}^*\big(\mathbf{x}^*, t\big) = (\mathrm{d}\phi_t)_{\mathbf{x}}\big(\mathbf{v}(\mathbf{x}, t)\big) + \mathbf{w}\big(\mathbf{x}^*, t\big). \tag{C.10}$$

C.1 Euclidean Observer Motions as Killing Fields

For Euclidean spaces, we can write the general pushforward $\mathrm{d}\phi_t$ as

$$\begin{aligned}
(\mathrm{d}\phi_t)_{\mathbf{x}} \colon T_{\mathbf{x}}M &\to T_{\mathbf{x}^*}N, \\
\mathbf{v} &\mapsto (\mathrm{d}\phi_t)_{\mathbf{x}}(\mathbf{v}) = \left(\frac{\partial}{\partial \mathbf{x}}\phi(\mathbf{x}, t)\right) \cdot \mathbf{v}.
\end{aligned} \tag{C.11}$$

The specific map ϕ_t of Eq. C.4, for each time t, corresponds to a general isometry from Euclidean $\mathbb{R}^n$ to itself. The derivatives of this ϕ_t are then

$$\begin{aligned}
(\mathrm{d}\phi_t)_{\mathbf{x}} &= \frac{\partial}{\partial \mathbf{x}}\phi(\mathbf{x}, t) = \mathbf{Q}(t), \\
\left(\frac{\partial}{\partial t}\phi(\mathbf{x}, t)\right)_{\mathbf{x}^*} &= \dot{\mathbf{c}}(t) + \dot{\mathbf{Q}}(t)\,\mathbf{x}, \\
&= \dot{\mathbf{c}}(t) + \dot{\mathbf{Q}}(t)\,\phi_t^{-1}(\mathbf{x}^*), \\
&= \dot{\mathbf{c}}(t) + \dot{\mathbf{Q}}(t)\,\mathbf{Q}^T(t)\,(\mathbf{x}^* - \mathbf{c}(t)), \\
&= \dot{\mathbf{c}}(t) + \mathbf{\Omega}(t)\,(\mathbf{x}^* - \mathbf{c}(t)), \\
&= \dot{\mathbf{c}}(t) - \mathbf{\Omega}(t)\,\mathbf{c}(t) + \mathbf{\Omega}(t)\,\mathbf{x}^*,
\end{aligned} \tag{C.12}$$

where $\mathbf{\Omega} := \dot{\mathbf{Q}}\mathbf{Q}^T$ constitutes an anti-symmetric spin tensor. For brevity, we now define $\mathbf{t}(t) := \dot{\mathbf{c}}(t) - \mathbf{\Omega}(t)\,\mathbf{c}(t)$. We can then give Eq. C.8 as

$$\mathbf{v}^*\big(\mathbf{x}^*, t\big) = \mathbf{Q}(t)\,\mathbf{v}\big(\mathbf{x}, t\big) + \mathbf{t}(t) + \mathbf{\Omega}(t)\,\mathbf{x}^*. \tag{C.13}$$

The expression $\mathbf{t} + \mathbf{\Omega}\,\mathbf{x}^*$ is the derivative of a translation and rotation, i.e., an infinitesimal isometry of Euclidean space. We now see that here the vector field $\mathbf{w}$ (Eq. C.9) is $\mathbf{w}(\mathbf{x}^*, t) = \mathbf{t}(t) + \mathbf{\Omega}(t)\,\mathbf{x}^*$, and finally get

$$\mathbf{v}^*\big(\mathbf{x}^*, t\big) = \mathbf{Q}(t)\,\mathbf{v}\big(\mathbf{x}, t\big) + \mathbf{w}\big(\mathbf{x}^*, t\big). \tag{C.14}$$

Or, more briefly and analogously to Eq. C.1, but probably less clearly,

$$\mathbf{v}^* = \mathbf{Q}(t)\,\mathbf{v} + \mathbf{w}(t). \tag{C.15}$$

Here, $\mathbf{w}(\mathbf{x}^*, t)$ is a Killing vector field, because the velocity gradient tensor $\nabla\mathbf{w}$ is $\nabla\mathbf{w} \equiv \mathbf{\Omega}$, and thus it is anti-symmetric at all $\mathbf{x}^* \in N$.

In contrast to Eq. C.1, the notion of Killing fields is well-defined for arbitrary (Riemannian) manifolds, not just for $\mathbb{R}^n$. This inspires building on Killing vector fields, determining infinitesimal isometries, for a generalization of objectivity to isometric observer transformations.

C.2 Correspondence to Our General Approach

Our general approach is built on the *group action* Φ of a Lie group G on a manifold M (see App. H for more details), given by

$$\begin{aligned}
\Phi \colon G \times M &\to M, \\
(g, x) &\mapsto \Phi(g, x).
\end{aligned} \tag{C.16}$$

For brevity, we denote the specific diffeomorphism corresponding to the action Φ with an element $g \in G$ by the map $\phi_g(x) := \Phi(g, x)$.

We can now consider the symmetry groups of Euclidean space, comprising all isometries of $\mathbb{R}^n$, which are often denoted as $\mathrm{ISO}(n)$ or $\mathrm{E}(n)$. However, these groups also include reflections. For our purposes of observer transformations, we want to restrict the allowed isometries to translations and rotations only. We therefore choose the subgroup $G = \mathrm{SE}(n)$, the special Euclidean group not including reflections. Each ϕ_g generated by a group element $g \in \mathrm{SE}(n)$ is then given by

$$\begin{aligned}
\phi_g \colon \mathbb{R}^n &\to \mathbb{R}^n, \\
\mathbf{x} &\mapsto \phi_g(\mathbf{x}) = \mathbf{c} + \mathbf{Q}\,\mathbf{x},
\end{aligned} \tag{C.17}$$

where $\mathbf{c}$ is a translation vector, and $\mathbf{Q}$ is a proper orthogonal tensor.

In our framework, an observer transformation is a smooth path $t \mapsto g(t) \in G$, i.e., a one-parameter group of transformations. For $G = \mathrm{SE}(n)$, this gives each individual transformation, for a fixed t, as

$$\begin{aligned}
\phi_{g(t)} \colon \mathbb{R}^n &\to \mathbb{R}^n, \\
\mathbf{x} &\mapsto \phi_{g(t)}(\mathbf{x}) = \mathbf{c}(t) + \mathbf{Q}(t)\,\mathbf{x}.
\end{aligned} \tag{C.18}$$

Given any Lie group G, we can consider its Lie algebra $\mathfrak{g}$, comprising the derivatives of all paths $t \mapsto h(t) \in G$ through the identity element e of G, at the identity element e, i.e., the tangent space T_eG. As described in detail in App. I, the vector field $\mathbf{w}$ generated on M by a Lie algebra element $W \in \mathfrak{g}$, from the curve $t \mapsto W(t) \in \mathfrak{g}$, at a certain t, is given by

$$\mathbf{w}(\mathbf{x}^*, t) := \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \phi_{h(\tau)}(\mathbf{x}^*), \quad \text{with} \quad h(\tau) := e^{\tau W(t)}. \tag{C.19}$$

To understand the relationship of $t \mapsto W(t)$ with $t \mapsto g(t)$, we note that the derivative $W(t) \in T_eG$ maps to $g'(t) \in T_{g(t)}G$ via *left-translation* ($L_g = g\cdot$) in G [20, p.399], i.e., $W(t) = \mathrm{d}L_{g^{-1}(t)}\big(g'(t)\big)$. With this $W(t)$,

$$\begin{aligned}
\mathbf{w}(\mathbf{x}^*, t) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \phi_{h(\tau)}(\mathbf{x}^*) = \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \phi_{h(\tau)}\big(\phi_{g(t)}(\mathbf{x})\big), \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \phi_{h(\tau)g(t)}(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=t} \phi_{g(\tau)}(\mathbf{x}).
\end{aligned} \tag{C.20}$$

When G is the isometry group of M, $\mathfrak{g}$ gives all infinitesimal isometries of M, and every vector field $\mathbf{w}$ is a *Killing vector field* on M, for each t, as already derived above specifically for the Euclidean case $M = \mathbb{R}^n$, i.e., compare Eq. C.20 with the second part of Eq. C.12 and below.

App. I describes Lie algebra actions and their induced vector fields in detail, and App. K describes our approach, described here for Euclidean space, instead applied to the isometries of the sphere $\mathbb{S}^2$.

D Observer Transformations on General Manifolds

Similarly to above, but now for general manifolds M and N, we define

$$\begin{aligned} \phi_t : M &\to N, \\ x &\mapsto \phi_t(x) =: x^*. \end{aligned} \tag{D.1}$$

To consider a curve $t \mapsto x(t) \in M$, transforming into the corresponding curve $t \mapsto x^*(t) \in N$, we define $\phi(x,t) := \phi_t(x)$ and write

$$x^*(t) = \phi(x(t),t). \tag{D.2}$$

Now, taking the derivative with respect to the time t on both sides, gives that a velocity field $\mathbf{v} := \mathrm{d}x(t)/\,\mathrm{d}t$ transforms to $\mathbf{v}^* := \mathrm{d}x^*(t)/\,\mathrm{d}t$ by

$$\begin{aligned} x^*(t) &= \phi(x(t),t), \\ \frac{\mathrm{d}}{\mathrm{d}t} t \mapsto x^*(t) &= \frac{\mathrm{d}}{\mathrm{d}t} t \mapsto \phi(x(t),t), \\ \left(\frac{\mathrm{d}x^*(t)}{\mathrm{d}t}\right)_{x^*(t)} &= (\mathrm{d}\phi_t)_{x(t)} \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right) + \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=t} \phi\big(x(t),\tau\big), \\ \left(\frac{\mathrm{d}x^*(t)}{\mathrm{d}t}\right)_{x^*(t)} &= (\mathrm{d}\phi_t)_{x(t)} \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right) + \mathbf{w}\big(x^*(t),t\big). \end{aligned} \tag{D.3}$$

Here, we have introduced the vector field $\mathbf{w}$, on the manifold N, as

$$\mathbf{w}(x^*,t) := \frac{\mathrm{d}}{\mathrm{d}t} \phi\big(x,t\big), \quad \text{with} \quad x = \phi_t^{-1}(x^*). \tag{D.4}$$

This gives us the general transformation rule

$$\big(\mathbf{v}^*(t)\big)_{\phi_t(x)} = (\mathrm{d}\phi_t)_x\big(\mathbf{v}(t)\big) + \big(\mathbf{w}(t)\big)_{\phi_t(x)}. \tag{D.5}$$

E Objectivity Proofs

According to Eq. D.5, a velocity field $\mathbf{v}$ transforms under the group action Φ with $g(t) \in G$, and corresponding diffeomorphism $\phi_{g(t)}$, as

$$\big(\mathbf{v}^*(t)\big)_{\phi_{g(t)}(x)} = (\mathrm{d}\phi_{g(t)})_x\big(\mathbf{v}(t)\big) + \big(\mathbf{w}(t)\big)_{\phi_{g(t)}(x)}. \tag{E.1}$$

If, as above, we again define $x^* := \phi_{g(t)}(x)$, we can also write this as

$$\big(\mathbf{v}^*(t)\big)_{x^*} = (\mathrm{d}\phi_{g(t)})_{\phi^{-1}_{g(t)}(x^*)}\big(\mathbf{v}(t)\big) + \big(\mathbf{w}(t)\big)_{x^*}. \tag{E.2}$$

The vector field $\mathbf{w}$ is determined by the $\phi_{g(t)}$, and is (see also Eq. I.2),

$$\big(\mathbf{w}(t)\big)_{\phi_{g(t)}(x)} = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=t} \tau \mapsto \phi_{g(\tau)}(x). \tag{E.3}$$

The vector field $\mathbf{w}$ is a Killing field if and only if $\phi_{g(t)}$ is an isometry.

Euclidean space. As above, when $\phi_{g(t)}$ is an isometry of Euclidean space, the pushforward $\mathrm{d}\phi_{g(t)}$ is the same proper orthogonal (rotation) tensor $\mathbf{Q}$, i.e., $(\mathrm{d}\phi_{g(t)})_x = \mathbf{Q}(t)$, at all $x \in M$. This is well-known, e.g., O'Neill [47, p.107]. In non-Euclidean spaces, however, this is not true.

E.1 Objectivity of Killing Operator

We first determine the transformation behavior of the tensor field $K\mathbf{u}$ under the group action Φ with $g \in G$, with ϕ_g an isometry. We evaluate for (arbitrary) arguments related by $\mathbf{x}^* = \mathrm{d}\phi_g(\mathbf{x})$, and $\mathbf{y}^* = \mathrm{d}\phi_g(\mathbf{y})$. For a velocity field $\mathbf{u}$ transforming as $\mathbf{u}^* = \mathrm{d}\phi_g(\mathbf{u}) + \mathbf{w}$ (Eq. E.1), we get

$$\begin{aligned} (K\mathbf{u}^*)_{\phi_g(x)}(\mathbf{x}^*,\mathbf{y}^*) &= K\big((\mathrm{d}\phi_g)_x(\mathbf{u}) + \mathbf{w}\big)(\mathbf{x}^*,\mathbf{y}^*), \\ &= K\big((\mathrm{d}\phi_g)_x(\mathbf{u})\big)(\mathbf{x}^*,\mathbf{y}^*) + K\mathbf{w}(\mathbf{x}^*,\mathbf{y}^*), \\ &= K\big((\mathrm{d}\phi_g)_x(\mathbf{u})\big)(\mathbf{x}^*,\mathbf{y}^*), \\ &= K\big((\mathrm{d}\phi_g)_x(\mathbf{u})\big)\big((\mathrm{d}\phi_g)_x(\mathbf{x}),(\mathrm{d}\phi_g)_x(\mathbf{y})\big), \\ &= (K\mathbf{u})_x(\mathbf{x},\mathbf{y}). \end{aligned} \tag{E.4}$$

In the first step, we have used the linearity of the operator K, and in the second step $K\mathbf{w} = 0$, because the Killing operator vanishes for a Killing field $\mathbf{w}$. Covariant derivatives (with the Levi-Civita connection) are invariant under isometries [40, p.125], thus, from Eq. 21, the Killing operator is also invariant, giving the last step above. According to Eq. A.2, a tensor field $\mathbf{T}$ of type $\binom{0}{2}$ is objective, if it transforms as

$$\mathbf{T}^*(\mathbf{x}^*,\mathbf{y}^*) = \mathbf{T}\big(\mathrm{d}\phi_{g^{-1}}(\mathbf{x}^*),\mathrm{d}\phi_{g^{-1}}(\mathbf{y}^*)\big). \tag{E.5}$$

From the transformation behavior given by Eq. E.4, we confirm that

$$\begin{aligned} K\mathbf{u}^*(\mathbf{x}^*,\mathbf{y}^*) &= K\mathbf{u}\big(\mathbf{x},\mathbf{y}\big), \\ &= K\mathbf{u}\big(\mathrm{d}\phi_{g^{-1}}(\mathrm{d}\phi_g(\mathbf{x})),\mathrm{d}\phi_{g^{-1}}(\mathrm{d}\phi_g(\mathbf{y}))\big), \\ &= K\mathbf{u}\big(\mathrm{d}\phi_{g^{-1}}(\mathbf{x}^*),\mathrm{d}\phi_{g^{-1}}(\mathbf{y}^*)\big). \end{aligned} \tag{E.6}$$

Thus, the tensor $K\mathbf{u}$ ($\mathbf{u}$ arbitrary) transforms under the group action Φ with $g \in G$ according to Eq. E.5, and is therefore objective.

E.2 Objectivity of Relative Velocity Fields

We first determine the transformation behavior of the (arbitrary) relative velocity vector field $\mathbf{v} - \mathbf{u}$ under the group action Φ with $g \in G$. Since both velocity vector fields $\mathbf{v},\mathbf{u}$ transform as $\mathbf{v}^* = \mathrm{d}\phi_g(\mathbf{v}) + \mathbf{w}$, and $\mathbf{u}^* = \mathrm{d}\phi_g(\mathbf{u}) + \mathbf{w}$ (Eq. E.1), respectively, we get the transformation

$$\begin{aligned} (\mathbf{v}^* - \mathbf{u}^*)_{\phi_g(x)} &= (\mathbf{v}^*)_{\phi_g(x)} - (\mathbf{u}^*)_{\phi_g(x)}, \\ &= (\mathrm{d}\phi_g)_x(\mathbf{v}) + \mathbf{w} - (\mathrm{d}\phi_g)_x(\mathbf{u}) - \mathbf{w}, \\ &= (\mathrm{d}\phi_g)_x(\mathbf{v}) - (\mathrm{d}\phi_g)_x(\mathbf{u}), \\ &= (\mathrm{d}\phi_g)_x(\mathbf{v} - \mathbf{u}). \end{aligned} \tag{E.7}$$

Thus, the relative velocity $\mathbf{v} - \mathbf{u}$ ($\mathbf{v}$, $\mathbf{u}$ arbitrary) transforms under the group action Φ with $g \in G$ according to Eq. 10, and is therefore objective.

E.3 Objectivity of Lie Derivatives of Objective Tensors

The time-dependent Lie derivative $L_{\mathbf{u}}\mathbf{v}$ of an objective vector field $\mathbf{v}$ is objective, even when the vector field $\mathbf{u}$ with respect to whose flow the Lie derivative is taken is not objective. We prove below that under the group action Φ with $g \in G$, $L_{\mathbf{u}}\mathbf{v}$ transforms according to Eq. 10. We start from Eq. E.1 and Eq. 10. We then follow the general proof of Marsden and Hughes [43, p.101], applied to our specific diffeomorphism $\phi_{g(t)}$.

For the proof it is crucial to keep track of which variables (spatial positions, times) are held fixed, and which are variable arguments for derivatives to be taken. We consider a fixed point $\bar{x}$ at fixed time $\bar{t}$, such that $\bar{x} = \phi_g(x)$ with the diffeomorphism $\phi_g := \phi_{g(\bar{t})}$. That is, the corresponding fixed point $x = \phi_{g^{-1}}(\bar{x})$. Because we have to keep the point $\bar{x}$ fixed for different times t, we introduce the "moving source" $t \mapsto x(t) := \phi_{g^{-1}(t)}(\bar{x})$. For the time-dependence of $\mathbf{v}$, we write $\mathbf{v}_t := \mathbf{v}(t)$. The proof of objectivity can then be given as (with Eq. E.9 below),

$$\begin{aligned} (L_{\mathbf{u}^*}\mathbf{v}^*)_{\bar{x}} &= L_{(\mathrm{d}\phi_g)_x(\mathbf{u})+\mathbf{w}}\big((\mathrm{d}\phi_g)_x(\mathbf{v})\big), \\ &= \mathscr{L}_{(\mathrm{d}\phi_g)_x(\mathbf{u})+\mathbf{w}}\big((\mathrm{d}\phi_g)_x(\mathbf{v})\big) + \frac{\partial}{\partial t}\big((\mathrm{d}\phi_{g(t)})_{x(t)}(\mathbf{v}_t)\big), \\ &= (\mathrm{d}\phi_g)_x(\mathscr{L}_{\mathbf{u}}\,\mathbf{v}) + \mathscr{L}_{\mathbf{w}}\big((\mathrm{d}\phi_g)_x(\mathbf{v})\big) + \frac{\partial}{\partial t}\big((\mathrm{d}\phi_{g(t)})_{x(t)}(\mathbf{v}_t)\big), \\ &= (\mathrm{d}\phi_g)_x(\mathscr{L}_{\mathbf{u}}\,\mathbf{v}) + L_{\mathbf{w}}\big((\mathrm{d}\phi_g)_x(\mathbf{v})\big), \\ &= (\mathrm{d}\phi_g)_x(\mathscr{L}_{\mathbf{u}}\,\mathbf{v}) + (\mathrm{d}\phi_g)_x\Big(\frac{\partial \mathbf{v}}{\partial t}\Big), \\ &= (\mathrm{d}\phi_g)_x\Big(\mathscr{L}_{\mathbf{u}}\,\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}\Big), \\ &= (\mathrm{d}\phi_g)_x\big(L_{\mathbf{u}}\,\mathbf{v}\big). \end{aligned} \tag{E.8}$$

We have used that $\mathscr{L}_{(\mathrm{d}\phi_g)_x(\mathbf{u})}\big((\mathrm{d}\phi_g)_x(\mathbf{v})\big) = (\mathrm{d}\phi_g)_x(\mathscr{L}_{\mathbf{u}}\,\mathbf{v})$ [43, p.98], and resolved the Lie derivative $L_{\mathbf{w}}$. For the latter, in the derivation below we insert the definition of the time-dependent Lie derivative (Eq. F.3) for $L_{\mathbf{w}}$ with flow $\psi_{t,\bar{t}}$, with pullback $\psi^*_{t,\bar{t}}\colon T_{\psi_{t,\bar{t}}(\bar{x})}M \to T_{\bar{x}}M$, define $\tilde{x}(t) := \psi_{t,\bar{t}}(\bar{x})$, and define $\hat{x}(t) := \phi_{g^{-1}(t)}(\tilde{x}(t))$. We also use that $\psi_{t,\bar{t}} = \phi_{g(t)} \circ \phi_{g^{-1}(\bar{t})} = \phi_{g(t)g^{-1}(\bar{t})}$, and rewrite, by definition, the pullback $\phi^*_{g(t)g^{-1}(\bar{t})}$ as the pushforward $\mathrm{d}\phi_{g(\bar{t})g^{-1}(t)}$. This gives,

$$
\begin{aligned}
\left(L_{\mathbf{w}}\left((\mathrm{d}\phi_g)_x(\mathbf{v})\right)\right)_{\bar{x}} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=\bar{t}}\ \psi^*_{t,\bar{t}}\big((\mathrm{d}\phi_{g(t)})_{\hat{x}(t)}(\mathbf{v}_t)\big),\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=\bar{t}}\ \phi^*_{g(t)g^{-1}(\bar{t})}\big((\mathrm{d}\phi_{g(t)})_{\hat{x}(t)}(\mathbf{v}_t)\big),\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=\bar{t}}\ \big(\mathrm{d}\phi_{g(\bar{t})g^{-1}(t)}\big)_{\tilde{x}(t)}\big((\mathrm{d}\phi_{g(t)})_{\hat{x}(t)}(\mathbf{v}_t)\big),\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=\bar{t}}\ (\mathrm{d}\phi_{g(\bar{t})})_x(\mathbf{v}_t),\\
&= (\mathrm{d}\phi_{g(\bar{t})})_x\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=\bar{t}}\ \mathbf{v}_t\Big),\\
&= (\mathrm{d}\phi_g)_x\Big(\frac{\partial \mathbf{v}}{\partial t}\Big).
\end{aligned}
\tag{E.9}
$$

Eqs. E.8 and E.9 prove that the Lie derivative of the objective vector field $\mathbf{v}$ is objective, even though the field $\mathbf{u}$ is not. In fact, essentially the same derivation proves that the same is true for objective tensor fields $\mathbf{t}$ of any order and variance, not only for vector fields [43, p.101].

Observed time derivative. Because in Eq. E.7 we have proved that the velocity field $\mathbf{v}-\mathbf{u}$ is objective, the proof given above immediately implies that the *observed time derivative* (Eq. 1) of $\mathbf{v}-\mathbf{u}$, given by

$$\frac{\mathscr{D}}{\mathscr{D}t}(\mathbf{v}-\mathbf{u}) = \frac{\partial}{\partial t}(\mathbf{v}-\mathbf{u}) + \mathscr{L}_{\mathbf{u}}(\mathbf{v}-\mathbf{u}) = L_{\mathbf{u}}(\mathbf{v}-\mathbf{u}), \tag{E.10}$$

is objective.

F Lie Derivatives

The *Lie derivative* measures the rate of change of a tensor field on a manifold M with respect to the *flow* (App. R) generated by a vector field on M. For a time-independent tensor field $\mathbf{t}$, the Lie derivative $\mathscr{L}_{\mathbf{u}}\,\mathbf{t}$ with respect to a vector field $\mathbf{u}$ with flow ϕ_t, is defined, at $x \in M$, as

$$\left(\mathscr{L}_{\mathbf{u}}\,\mathbf{t}\right)_x := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{d}\phi_{-t}\big(\mathbf{t}_{\phi_t(x)}\big). \tag{F.1}$$

Here, $\mathrm{d}\phi_t$ is the differential of the flow ϕ_t, and $\phi_{-t} = \phi_t^{-1}$. When $\mathbf{t}$ is a vector field $\mathbf{v}$, the Lie derivative $\mathscr{L}_{\mathbf{u}}\mathbf{v}$ is the same as the Lie bracket [20, Ch. 4] between the two vector fields, i.e., $\mathscr{L}_{\mathbf{u}}\mathbf{v} = [\mathbf{u},\mathbf{v}]$. For any given torsion-free connection on a manifold M, such as the Levi-Civita connection, the Lie bracket, and thus the Lie derivative, is then (cf. Eq. N.7)

$$\mathscr{L}_{\mathbf{u}}\,\mathbf{v} = \nabla\mathbf{v}\,(\mathbf{u}) - \nabla\mathbf{u}\,(\mathbf{v})\,. \tag{F.2}$$

If the field $\mathbf{t}$ is time-dependent, the definition of the Lie derivative must be extended to the time-dependent Lie derivative [43, p.95], which is

$$\left(L_{\mathbf{u}}\,\mathbf{t}\right)_x := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s} \psi^*_{t,s}\left(\mathbf{t}_{\psi_{t,s}(x)}\right) = \left(\frac{\partial \mathbf{t}}{\partial t} + \mathscr{L}_{\mathbf{u}}\,\mathbf{t}\right)_x, \tag{F.3}$$

at the point $x \in M$, at time s. The pullback $\psi^*_{t,s}$ is given by $\psi^*_{t,s} = \mathrm{d}\psi_{s,t}$. We refer to Marsden and Hughes [43, Ch. 1.6], and Frankel [20, Ch. 4].

G Lie Derivatives in Curved Spaces

Lie derivatives are independent of the metric $\mathbf{g}$ defined on the manifold M [43, p.96]. For a vector field $\mathbf{v}$, this can be seen by expanding

$$
\begin{aligned}
\mathscr{L}_{\mathbf{u}}\,\mathbf{v} &= \nabla\mathbf{v}\,(\mathbf{u}) - \nabla\mathbf{u}\,(\mathbf{v})\,,\\
&= \left(\nabla_j v^i u^j - \nabla_j u^i v^j\right)\mathbf{e}_i,\\
&= \left((\partial_j v^i + \Gamma^i{}_{jk}v^k)u^j - (\partial_j u^i + \Gamma^i{}_{jk}u^k)v^j\right)\mathbf{e}_i,\\
&= \left(\partial_j v^i u^j + \Gamma^i{}_{jk}v^k u^j - \partial_j u^i v^j - \Gamma^i{}_{jk}u^k v^j\right)\mathbf{e}_i,\\
&= \left(\partial_j v^i u^j - \partial_j u^i v^j\right)\mathbf{e}_i.
\end{aligned}
\tag{G.1}
$$

That is, all terms with Christoffel symbols $\Gamma^i{}_{jk}$ cancel out. This property always holds, given that the connection is torsion-free, which means that the symmetry $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$ holds (for $\{\mathbf{e}_i\}$ a coordinate basis). This applies in our framework, because we are using the *Levi-Civita connection*, which, by definition, is both metric-compatible and torsion-free.

To make parsing the tensor expressions above easier, we note that an expression like $\partial_j v^i$ can be seen as a matrix of partial derivatives, with row index i and column index j, and $\partial_j v^i u^j$ is equivalent to matrix-vector multiplication with a column vector u^j with row index j. We also give the explicit expansion and summations for the 2D case:

$$
\begin{aligned}
\mathscr{L}_{\mathbf{u}}\,\mathbf{v} =& \nabla\mathbf{v}\,(\mathbf{u}) - \nabla\mathbf{u}\,(\mathbf{v})\,,\\
=& \left(\sum_{j=1,2}\Big((\partial_j v^1)\,u^j - (\partial_j u^1)\,v^j\Big)\right)\mathbf{e}_1 +\\
& \left(\sum_{j=1,2}\Big((\partial_j v^2)\,u^j - (\partial_j u^2)\,v^j\Big)\right)\mathbf{e}_2.
\end{aligned}
\tag{G.2}
$$

H Symmetry Groups and Lie Group Actions

The *group action* Φ [31, p.209], more specifically a smooth left action, of a Lie group G on a manifold M, is a smooth map

$$
\begin{aligned}
\Phi\colon G\times M &\to M,\\
(g,x) &\mapsto \Phi(g,x),
\end{aligned}
\tag{H.1}
$$

such that

1. $\Phi(e,x) = x$, for all $x \in M$, and
2. $\Phi(g,\Phi(h,x)) = \Phi(gh,x)$, for all $g,h \in G$ and $x \in M$,

By setting $\phi_g(x) := \Phi(g,x)$ the properties of the group action can be written in the more concise form

$$\phi_g\phi_h = \phi_{gh} \quad \text{and} \quad \phi_e = \mathrm{id}_M. \tag{H.2}$$

It follows then from $\phi_g\phi_{g^{-1}} = \phi_{gg^{-1}} = \phi_e = \mathrm{id}_M$ that $\phi_{g^{-1}} = \left(\phi_g\right)^{-1}$, and since both ϕ_g and $\phi_{g^{-1}}$ are smooth by definition that, for every $g \in G$, the map

$$
\begin{aligned}
\phi_g\colon M &\to M,\\
x &\mapsto \phi_g(x)
\end{aligned}
\tag{H.3}
$$

is a *diffeomorphism*.

I Lie Algebra Actions and Induced Vector Fields

We are interested in determining a correspondence of elements of the Lie algebra $\mathfrak{g}$ of a given Lie group G, such as a matrix Lie group, and the space of vector fields (again as a Lie algebra) on a given manifold M. For example, we want to construct a correspondence between a Lie algebra of anti-symmetric matrices and the Killing vector fields on M.

We can do this by defining a Lie algebra homomorphism (or an isomorphism) as the *action* of the Lie algebra $\mathfrak{g}$ on the manifold M, *generating vector fields* on M. We can write this as a map

$$
\begin{aligned}
\Xi\colon \mathfrak{g} &\to \mathfrak{X}(M),\\
X &\mapsto \mathbf{x}.
\end{aligned}
\tag{I.1}
$$

$\mathfrak{X}(M)$ denotes the Lie algebra of (smooth) vector fields on M. Using the Lie group action Φ with $g \in G$, and the corresponding diffeomorphism ϕ_g on M, the vector field $\mathbf{x}$ *generated* on M by $X \in \mathfrak{g}$, is given by

$$\mathbf{x}(x) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_{g(t)}(x), \quad \text{with} \quad g(t) := e^{tX}. \tag{I.2}$$

The definition of $g(t)$ on the right defines a *one-parameter group* of diffeomorphims $\phi_{g(t)}$ on M. This then means that the vector field $\mathbf{x}(x)$, at all points $x \in M$, is given by the point-wise derivative of the diffeomorphism $\phi_{g(t)}$, at each x, evaluated at the parameter value $t = 0$.

For the common, and our, case where the Lie algebra $\mathfrak{g}$ is a matrix Lie algebra, corresponding to a matrix Lie group G, the exponential map $X \mapsto e^{tX}$ is given by the standard matrix exponential. See App. L.

Roughly speaking, the exponential map *integrates* an infinitesimal transformation X (such as an infinitesimal rotation, with X an anti-symmetric matrix) from $g = e$, i.e., the identity element of G, such as the identity matrix, at time $t = 0$, to the corresponding finite transformation $g(t)$, at time t (such as a rotation, with $g(t)$ a proper orthogonal matrix).

Appendixes K and L give a detailed description of the relationship between the isometry group G, the action of its Lie algebra $\mathfrak{g}$, and the corresponding vector fields on M, for the case of the two-sphere $\mathbb{S}^2$.

J The Lie Algebra of all Killing Vector Fields

The set of all possible Killing vector fields on a given (Riemannian) manifold M has a lot of structure that can be exploited. In fact, the set of Killing fields constitutes a *Lie algebra*, which is a vector space with an additional Lie bracket operation (see App. K). Therefore, as a vector space, we can talk about the dimensionality of the space of all possible Killing fields on a manifold M, which we now denote by k.

Knowing k for a given manifold M gives a lot of insight on all possible isometries of M. For example, it is known that curved surfaces embedded in $\mathbb{R}^3$ can have at most three linearly independent Killing fields, i.e., they can only have $k \leq 3$ [7, 45]. However, a continuous isometry is a rather strong condition, and therefore many general manifolds have only the trivial intrinsic isometry ($k = 0$). In these cases, approximate Killing fields can be computed [7]. For observers and objectivity, we are interested in spaces with non-trivial isometries.

Knowing k is the same as knowing the dimensionality of the isometry group of M. The isometry group (without reflections) of Euclidean space $\mathbb{R}^n$ is $\mathrm{SO}(n) \ltimes \mathrm{T}(n)$. $\mathrm{SO}(n)$ is the rotation group, $\mathrm{T}(n)$ the group of translations, and $\ltimes$ the semi-direct product. The Euclidean plane $\mathbb{R}^2$ thus has $k = 3$; Euclidean 3-space $\mathbb{R}^3$ has $k = 6$. The (direct) isometry group of the two-sphere $\mathbb{S}^2$ is SO(3), and therefore it has $k = 3$. A cylinder has $k = 2$, and a generic surface of revolution has $k = 1$ [7].

K Isometry Group and Killing Fields of the 2-Sphere

The (direct) isometry group of the standard two-sphere $\mathbb{S}^2$, embedded in the ambient space $\mathbb{R}^3$ as $\mathbb{S}^2 := \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$, with the standard Euclidean topology and metric, is the Lie group $G = \mathrm{SO}(3)$, which is also a smooth, non-linear manifold. SO(3) is a *matrix Lie group*, where each element $g \in G$ is a proper orthogonal matrix, with $\det g = 1$, corresponding to a rigid rotation of $\mathbb{R}^3$. See also App. L. The corresponding Lie algebra $\mathfrak{g} = T_eG = \mathfrak{so}(3)$ is the tangent space (i.e., a vector space) of G at the identity group element e. As every Lie algebra, $\mathfrak{so}(3)$ is a vector space with an additional Lie bracket operation, which in this case is the matrix commutator $[X,Y] := XY - YX$. The Lie algebra $\mathfrak{so}(3)$ is the algebra of all real, anti-symmetric 3×3 matrices

$$X = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathfrak{so}(3), \text{ with } \omega_i \in \mathbb{R}. \tag{K.1}$$

An isomorphism between a Lie algebra $\mathfrak{g}$ and vector fields induced on a manifold M can be determined via a group action of G on M, and the corresponding action of $\mathfrak{g}$, described in App. I. App. L gives an isomorphism for $\mathfrak{g} = \mathfrak{so}(3)$ and the manifold $M = \mathbb{S}^2$. Using this isomorphism, each $X \in \mathfrak{so}(3)$ corresponds to a uniquely determined Killing field $\mathbf{x}$ on $\mathbb{S}^2$. For the Lie algebra of vector fields $\mathbf{x}$ induced on the manifold M, the Lie bracket is the differential geometric *Lie bracket of vector fields* $[\mathbf{x},\mathbf{y}]$, corresponding to but not the same as the matrix commutator $[X,Y]$. In fact, $[\mathbf{x},\mathbf{y}]$ is the Lie derivative $\mathscr{L}_{\mathbf{x}}\mathbf{y}$.

The Lie algebra $\mathfrak{so}(3)$ is three-dimensional as a vector space. This can be seen by giving a basis for all anti-symmetric 3×3 matrices, e.g., the three basis vectors X_1, X_2, X_3 in Eq. L.7. Because the Lie algebra of Killing fields on $\mathbb{S}^2$ and the matrix Lie algebra $\mathfrak{so}(3)$ are isomorphic, the former is therefore also three-dimensional. Any Killing field $\mathbf{x}$ on $\mathbb{S}^2$ can thus be given as a unique linear combination of three *basis Killing fields* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Fig. 5 depicts three linearly independent basis Killing fields on $\mathbb{S}^2$ (we could also choose another basis), where each field $\mathbf{x}_i$ corresponds to a basis vector X_i of $\mathfrak{so}(3)$ as a vector space.

Extrinsic computation in $\mathbb{R}^3$. The simplest way to construct the (intrinsic) Killing field $\mathbf{x}$ on $\mathbb{S}^2$, corresponding to a given Lie algebra element $X \in \mathfrak{so}(3)$, is the following. For any point $x \in M = \mathbb{S}^2$, given via a position vector $\mathbf{r}(x) \in \mathbb{R}^3$, pointing from the origin of $\mathbb{R}^3$ to the point x on $\mathbb{S}^2$, the extrinsic Killing vector $\tilde{\mathbf{x}}(x) \in \mathbb{R}^3$ at x is simply

$$\tilde{\mathbf{x}}(x) = X \cdot \mathbf{r}(x). \tag{K.2}$$

(We note that this could also be written as the cross product of the angular velocity vector corresponding to X with the position vector $\mathbf{r}$.) From the extrinsic vector $\tilde{\mathbf{x}}(x) \in \mathbb{R}^3$, we can obtain the intrinsic vector $\mathbf{x} \in T_xM$ (two components) via the dual basis (1-forms) given in App. P,

$$\mathbf{x}(x) = \tilde{\omega}^1\big(\tilde{\mathbf{x}}(x)\big)\,\mathbf{e}_1 + \tilde{\omega}^2\big(\tilde{\mathbf{x}}(x)\big)\,\mathbf{e}_2. \tag{K.3}$$

L Lie Algebras and the Exponential Map

The *exponential map* is a mapping from a Lie algebra $\mathfrak{g}$ to the corresponding Lie group G, i.e.,

$$\exp\colon \mathfrak{g} \to G, \tag{L.1}$$

$$X \mapsto \exp(X). \tag{L.2}$$

In case of a matrix Lie group it is defined for a matrix X in the Lie algebra, like the exponential map for real numbers, by the power series

$$\exp(X) = e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots, \tag{L.3}$$

where I is the identity matrix.

The one-parameter group $g(t)$ generated by X is then given by $g(t) = \exp(tX)$. We immediately get the corresponding one-parameter group of actions Φ with $g(t)$, and the corresponding diffeomorphisms $\phi_{g(t)}$.

For example, for $G = \mathrm{SO}(2)$, and its Lie algebra $\mathfrak{g} = \mathfrak{so}(2)$, if we choose the basis vector (matrix) $X \in \mathfrak{so}(2)$

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{L.4}$$

we can compute

$$\begin{aligned}
(tX)^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
(tX)^1 &= \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}, \\
(tX)^2 &= \begin{pmatrix} -t^2 & 0 \\ 0 & -t^2 \end{pmatrix}, \\
(tX)^3 &= \begin{pmatrix} 0 & t^3 \\ -t^3 & 0 \end{pmatrix}, \\
(tX)^4 &= \begin{pmatrix} t^4 & 0 \\ 0 & t^4 \end{pmatrix} = t^4 (tX)^0,
\end{aligned} \tag{L.5}$$

and conclude that $(tX)^{k+4} = t^4 (tX)^k$ by induction. Thus,

$$\begin{aligned}
\exp\left(tX\right) &= e^{tX} \\
&= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots & -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \dots \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \end{pmatrix} \qquad \text{(L.6)} \\
&= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in \mathrm{SO}(2).
\end{aligned}$$

For $G = \mathrm{SO}(3)$, and its Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$, with essentially the same calculations, we get for the basis vectors (matrices) $X_i \in \mathfrak{so}(3)$

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{(L.7)}$$

the corresponding exponentials

$$\begin{aligned}
\exp\left(tX_1\right) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \in \mathrm{SO}(3), \\
\exp\left(tX_2\right) &= \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix} \in \mathrm{SO}(3), \qquad \text{(L.8)} \\
\exp\left(tX_3\right) &= \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3).
\end{aligned}$$

M Tensors as Multi-Linear Maps, and Their Bases

We view a tensor field as a multi-linear *coordinate-independent* map that, at any point $x \in M$ maps a set of vector and covector (1-form) arguments to a scalar. For a detailed description, including the general concept of tensor bundles over a manifold M, we refer to the books by Spivak [57] and Frankel [20]. Two basic concepts that are important in our context are the order of a tensor, and its corresponding type.

We say a tensor $\mathbf{T}$ is of *type* $\binom{r}{s}$, and correspondingly of *order* $(r+s)$, if it acts on r covector arguments and s vector arguments.

In order to be able to work with components referred to coordinates, we have to expand tensors using basis vectors and basis covectors (basis 1-forms). Higher-order tensors must appropriately combine the basis vectors and basis 1-forms, respectively, using the tensor product $\otimes$.

We exclusively use *coordinate bases* (and not non-coordinate frames), where all basis vectors are derivatives of the coordinate functions x^i of a given coordinate chart [20, p.25]. This is often denoted by $\mathbf{e}_i := \boldsymbol{\partial}_i = \frac{\partial}{\partial x^i}$. This implies that the Lie brackets $[\mathbf{e}_i, \mathbf{e}_j]$ of the basis vector fields $\{\mathbf{e}_i\}$ are zero, i.e., the basis vector fields commute, which simplifies equations. See Frankel [20] for details. We denote the corresponding dual bases by $\{\omega^i\}$, with $\omega^i(\mathbf{e}_j) = \delta^i_{\ j}$. Because $\{\mathbf{e}_i\}$ is a coordinate basis, the ω^i are coordinate differentials, i.e., $\omega^i = dx^i$.

We give examples for the types of tensors that we use in this paper:

- A vector is referred to a basis $\{\mathbf{e}_i\}$, and expanded as $v^i\, \mathbf{e}_i$.
- A 1-form is referred to a dual basis $\{\omega^i\}$, and expanded as $v_i\, \omega^i$. We can also write $v_i\, dx^i$ for coordinate bases $\omega^i := dx^i$.
- A $\binom{1}{1}$ tensor, as a bi-linear map of one covector and one vector argument to a scalar, is referred to a basis $\{\mathbf{e}_i \otimes \omega^j\}$, and expanded as $T^i_{\ j}\, \mathbf{e}_i \otimes \omega^j$. We can also interpret this as a linear map of vectors.
- A covariant second-order tensor (a $\binom{0}{2}$ tensor), such as the metric $\mathbf{g}$, is referred to a basis $\{\omega^i \otimes \omega^j\}$, and expanded $g_{ij}\, \omega^i \otimes \omega^j$.
- A contravariant second-order tensor, (a $\binom{2}{0}$ tensor), e.g., the inverse metric $\mathbf{g}^{-1}$, is referred to a basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ as $g^{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j$.

We can understand an expression such as $T^i_{\ j}\, \mathbf{e}_i \otimes \omega^j$ as a linear map $\mathbf{T}$, acting on a vector $\mathbf{v}$, and giving a result vector $\mathbf{T}(\mathbf{v})$, by writing

$$\begin{aligned}
\mathbf{T}(\mathbf{v}) &= (T^i_{\ j}\, \mathbf{e}_i \otimes \omega^j)\,(\mathbf{v})\,, \\
&= T^i_{\ j}\, \mathbf{e}_i\, \omega^j(\mathbf{v}) = T^i_{\ j}\, \omega^j(\mathbf{v})\, \mathbf{e}_i, \qquad \text{(M.1)} \\
&= T^i_{\ j} v^j\, \mathbf{e}_i.
\end{aligned}$$

We note that when the 1-form ω^j is applied to the vector argument $\mathbf{v}$, the tensor product $\otimes$ simply turns into a regular product. This behavior is part of the definition of the tensor product. It corresponds to the fact that the contraction $\omega^j(\mathbf{v})$ has turned the $\binom{1}{1}$ tensor $\mathbf{T}$ into a $\binom{1}{0}$ tensor $\mathbf{T}(\mathbf{v})$, i.e., a *vector*. Correspondingly, the remaining basis is solely the basis $\{\mathbf{e}_i\}$ for vectors. We note that the first-order tensor $\mathbf{T}(\mathbf{v})$ can be interpreted directly as a vector, or still be interpreted as a scalar-valued *function* acting on the argument of a *covector* (as one definition of a vector in tensor analysis). For example, we get the *scalar* that is the i'th component of the vector $\mathbf{T}(\mathbf{v})$ referred to the basis $\{\mathbf{e}_i\}$, by computing

$$\mathbf{T}(\mathbf{v})\big(\omega^i\big) := \omega^i\big(\mathbf{T}(\mathbf{v})\big) = \omega^i\big(T^i_{\ j} v^j\, \mathbf{e}_i\big) = T^i_{\ j} v^j \omega^i\big(\mathbf{e}_i\big) = T^i_{\ j} v^j. \qquad \text{(M.2)}$$

When "executed" for all "rows" i, the final expression $T^i_{\ j} v^j$ is a matrix-vector multiplication of components. However, in the entire derivation above, the notation has helped us avoid mixing components of different variance and the corresponding bases. Overall, tensor notation is a powerful way of using basis vectors and 1-forms, and tensors referred to components, in a general context, simplifying the use of arguments of different variance (covariant, contravariant) and higher-order tensors.

A concrete example is the definition of the covariant derivative $\nabla \mathbf{v}$ of a vector field $\mathbf{v}$ in Eqs. N.8 and N.9 below, which is a $\binom{1}{1}$ tensor. However, we note that in that context, "covariant" refers to "general covariance," not to covariant arguments. See Frankel [20, p.430].

N Intrinsic Covariant Derivatives

Coordinate-free definition. The *covariant derivative* (also called an *affine connection*) generalizes the directional derivative of tensor fields in Euclidean space to arbitrary manifolds [65, Chapter 6]. We define the (intrinsic) velocity gradient tensor $\nabla \mathbf{v}$ as the covariant derivative of a vector field $\mathbf{v}$ on a given manifold M. It has the following properties:

1. The map $(\mathbf{v}, \mathbf{w}) \mapsto \nabla_{\mathbf{w}} \mathbf{v}$ is $\mathbb{R}$-bilinear, that is
$$\begin{aligned} \nabla_{a\mathbf{w}_1 + b\mathbf{w}_2} \mathbf{v} &= a \nabla_{\mathbf{w}_1} \mathbf{v} + b \nabla_{\mathbf{w}_2} \mathbf{v}, \quad \text{and} \\ \nabla_{\mathbf{w}} (a\mathbf{v}_1 + b\mathbf{v}_2) &= a \nabla_{\mathbf{w}} \mathbf{v_1} + b \nabla_{\mathbf{w}} \mathbf{v_2} \end{aligned} \qquad \text{(N.1)}$$
for all $a, b \in \mathbb{R}$.
2. The map $\mathbf{w} \mapsto \nabla_{\mathbf{w}} \mathbf{v}$ (or $\nabla \mathbf{v}(\mathbf{w})$) is linear with respect to smooth functions, that is
$$\nabla_{\mathbf{w}} (f\mathbf{v}_1 + g\mathbf{v}_2) = f \nabla_{\mathbf{w}} \mathbf{v}_1 + g \nabla_{\mathbf{w}} \mathbf{v}_2 \qquad \text{(N.2)}$$
for all smooth functions f, g on M.
3. The map $\mathbf{v} \mapsto \nabla_{\mathbf{w}} \mathbf{v}$ is a *derivation*, i.e., it satisfies the *Leibniz rule*
$$\nabla_{\mathbf{w}} (f\mathbf{v}) = (\mathbf{w} f)\mathbf{v} + f \nabla_{\mathbf{w}} \mathbf{v} \qquad \text{(N.3)}$$
for all smooth functions f on M.

If we write Eq. N.1 as

$$\nabla \mathbf{v}(a\mathbf{w}_1 + b\mathbf{w}_2) = a \nabla \mathbf{v}(\mathbf{w}_1) + b \nabla \mathbf{v}(\mathbf{w}_2), \qquad \text{(N.4)}$$

and define

$$\begin{aligned} \nabla \mathbf{v} \colon\, & T_x^* M \times T_x M \to \mathbb{R}, \\ & (\omega, \mathbf{w}) \mapsto \omega\big(\nabla \mathbf{v}(\mathbf{w})\big), \end{aligned} \qquad \text{(N.5)}$$

it follows that $\nabla \mathbf{v}$ is a multi-linear map, and thus a $\binom{1}{1}$ tensor (field) (see App. M). In addition, on a (Riemannian) manifold with a metric $\mathbf{g}$, there is a *unique* covariant derivative [65, Theorem 6.6] that is

1. combatible with the metric, that is[5]

$$\nabla \mathbf{g} = 0, \qquad \text{and} \tag{N.6}$$

2. torsion-free, that is

$$\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{v} - [\mathbf{v},\mathbf{w}] = 0. \tag{N.7}$$

The notation $[\mathbf{v},\mathbf{w}]$ gives the Lie bracket of the vector fields $\mathbf{v}$ and $\mathbf{w}$. This unique connection is called the *Levi-Civita connection*.

Computation in a chart. Referred to a coordinate basis $\{\mathbf{e}_i \otimes \omega^j\}$, the (intrinsic) velocity gradient $\nabla\mathbf{v}$ as a *covariant derivative* is given by

$$\nabla\mathbf{v} = \left(\nabla_j v^i\right)\mathbf{e}_i \otimes \omega^j := \left(\partial_j v^i + \Gamma^i{}_{jk} v^k\right)\mathbf{e}_i \otimes \omega^j. \tag{N.8}$$

The tensor $\nabla\mathbf{v}$ evaluated in direction $\mathbf{x}$ is the vector (see App. M),

$$\nabla\mathbf{v}(\mathbf{x}) = \nabla_{\mathbf{x}}\mathbf{v} = \left(\partial_j v^i + \Gamma^i{}_{jk} v^k\right)\omega^j(\mathbf{x})\,\mathbf{e}_i. \tag{N.9}$$

The Christoffel symbols $\Gamma^i{}_{jk}$, corresponding to the (unique) Levi-Civita connection for a metric $\mathbf{g}$ on M, can be derived intrinsically from the components g_{ij} of $\mathbf{g}$, referred to the same basis (and its dual), via

$$\Gamma^i{}_{jk} = \frac{1}{2}\, g^{im}\left(\partial_k g_{mj} + \partial_j g_{mk} - \partial_m g_{jk}\right). \tag{N.10}$$

See [17, p.56]. g_{ij} is the metric $\mathbf{g}$ referred to the basis $\{\omega^i \otimes \omega^j\}$, and g^{ij} is its inverse $\mathbf{g}^{-1}$, i.e., $g^{ik} g_{kj} = \delta^i_j$, referred to the basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$.

Relation to Cartesian tensors. The tensor $\nabla\mathbf{v}$ *only* consists solely of partial derivatives when (1) affine or Cartesian coordinates are used; and thus (2) the manifold is intrinsically flat, such as $M = \mathbb{R}^n$. Only then do the Christoffel symbols on M vanish. The above intrinsic formulation can be used on abstract manifolds M, without any known immersion into a Euclidean ambient space. However, even when an immersion of M into a higher-dimensional ambient space $\mathbb{R}^m$ is known, such as for a two-manifold embedded as a curved surface in $\mathbb{R}^3$, it is extremely useful for intrinsic (lower-dimensional) computations.

O Metric Tensor Fields

Coordinate-free definition. A (Riemannian) metric $\mathbf{g}$ on a manifold M defines an inner product on each tangent space T_xM. This is usually written as $\langle\mathbf{x},\mathbf{y}\rangle := \mathbf{g}(\mathbf{x},\mathbf{y})$ for $\mathbf{x},\mathbf{y} \in T_xM$. Specifically, $\mathbf{g}$ is

1. *symmetric*, that is

$$\langle\mathbf{x},\mathbf{y}\rangle = \langle\mathbf{y},\mathbf{x}\rangle \tag{O.1}$$

   for all $\mathbf{x},\mathbf{y} \in T_xM$,
2. *bilinear*, that is

$$\langle a\mathbf{x}+b\mathbf{y},\mathbf{z}\rangle = a\langle\mathbf{x},\mathbf{z}\rangle + b\langle\mathbf{y}.\mathbf{z}\rangle = \langle\mathbf{z},a\mathbf{x}+b\mathbf{y}\rangle \tag{O.2}$$

   for all $\mathbf{x},\mathbf{y},\mathbf{z} \in T_xM$, and all $a,b \in \mathbb{R}$, and
3. *positive definite*, that is

$$\langle\mathbf{x},\mathbf{x}\rangle \geq 0 \tag{O.3}$$

   for all $\mathbf{x} \in T_xM$ with $\langle\mathbf{x},\mathbf{x}\rangle = 0$ if and only if $\mathbf{x} = 0$.

Furthermore, $\mathbf{g}$ is required to be smooth in the sense that in all charts the coordinate functions are smooth. Consequently, $\mathbf{g}$ is a covariant second-order tensor field (see App. M).

Computation in a chart. If we define $g_{ij} := \langle\mathbf{e}_i,\mathbf{e}_j\rangle$, we get

$$\begin{aligned}\mathbf{g}(\mathbf{x},\mathbf{y}) &= \langle\mathbf{x},\mathbf{y}\rangle = \langle x^i\mathbf{e}_i, y^j\mathbf{e}_j\rangle = x^i y^j\langle\mathbf{e}_i,\mathbf{e}_j\rangle = x^i y^j g_{ij} \\ &= \omega^i(\mathbf{x})\omega^j(\mathbf{y})g_{ij} = g_{ij}(\omega^i\otimes\omega^j)(\mathbf{x},\mathbf{y}),\end{aligned} \tag{O.4}$$

since $\{\omega^i\}$ is dual to $\{\mathbf{e}_i\}$, i.e., $\omega^i(\mathbf{e}_j) = \delta^i_j$, and the tensor product of two covectors is simply their product. That is, the g_{ij} are the components of the metric tensor $\mathbf{g}$ with respect to the basis $\omega^i \otimes \omega^j$.

We can write $x^i y^j g_{ij}$ in matrix notation as

$$\mathbf{x}^T\mathbf{g}\,\mathbf{y}. \tag{O.5}$$

Inverse metric. Given a vector $\mathbf{x} = x^i\,\mathbf{e}_i$, the map $\mathbf{y} \mapsto \mathbf{g}(\mathbf{x},\mathbf{y})$ defines a covector (or 1-form). From Eq. O.4, we get

$$\begin{aligned}\mathbf{g}(\mathbf{x},\mathbf{y}) &= g_{ij}x^i\omega^j(\mathbf{y}), \quad \text{or} \\ (\mathbf{y}\mapsto\mathbf{g}(\mathbf{x},\mathbf{y})) &= g_{ij}x^i\omega^j.\end{aligned} \tag{O.6}$$

This means that the components of the covector are simply $g_{ij}x^i$. If we set $x_j := g_{ij}x^i$, we can write the covector as $x_j\omega^j$. Thus, using the metric to convert a vector into a covector, we have effectively *lowered* the index of the components.

The matrix $\mathbf{g}$ with components g_{ij} is invertible with inverse $\mathbf{g}^{-1}$, whose components are denoted by g^{ij}. This means that

$$g_{jk}g^{ki} = g^{ik}g_{kj} = \delta^i_j, \tag{O.7}$$

or in matrix notation

$$\mathbf{g}\,\mathbf{g}^{-1} = \mathbf{g}^{-1}\,\mathbf{g} = I, \tag{O.8}$$

with I the identity matrix. Given the components x_j of a covector ω, we now obtain the components of the corresponding vector by *raising* the index: $x^i = g^{ij}x_j$. See Lee [40, p. 26] for more details.

P Metric and Christoffel Symbols of the 2-Sphere

We derive the coordinate bases and their derivatives of charts $U \subset \mathbb{R}^2$, corresponding to orthogonal projection of each chart onto a hemisphere of some arbitrary radius r. See Fig. 10. From these, we can derive the metric tensor and the Christoffel symbols referred to the chart. We emphasize that, although here we perform some derivations in the ambient space $\mathbb{R}^3$, the resulting metric (Eq. P.6) and the Christoffel symbols (Eq. P.9) are completely *intrinsic* and are given as 2D quantities. Using the Christoffel symbols, we can compute the covariant derivative of any vector field $\mathbf{v}$ (Eq. P.12, Eq. N.8) in a completely intrinsic manner.

Each chart $U \subset \mathbb{R}^2$ is defined in a disk of radius r in the (u,v) plane, i.e., $U := \{(u,v)\colon u^2+v^2 \leq r^2\}$. We describe the map from a chart in $\mathbb{R}^2$ (intrinsic view) to ambient $\mathbb{R}^3$ (extrinsic view) via an *inclusion map*

$$\begin{aligned}\iota^{x\perp y}\colon U \subset \mathbb{R}^2 &\hookrightarrow \mathbb{R}^3, \\ (u,v) &\mapsto (u,v,\bar{w}).\end{aligned} \tag{P.1}$$

The third component is $\bar{w} := \sqrt{r^2-u^2-v^2}$. This chart is defined via projection onto the hemisphere on the x,y plane, denoted by $x \perp y$. The entire sphere is covered by six analogous charts. In total, we define

$$\begin{aligned}\iota^{x\perp y}&\colon (u,v)\mapsto(u,v,\bar{w}), & \iota^{-x\perp y}&\colon (u,v)\mapsto(-u,v,-\bar{w}), \\ \iota^{z\perp y}&\colon (u,v)\mapsto(-\bar{w},v,u), & \iota^{-z\perp y}&\colon (u,v)\mapsto(\bar{w},v,-u), \\ \iota^{x\perp -z}&\colon (u,v)\mapsto(u,\bar{w},-v), & \iota^{x\perp z}&\colon (u,v)\mapsto(u,-\bar{w},v).\end{aligned} \tag{P.2}$$

To avoid too severe distortions, apart from overlaps to facilitate transitions between neighboring charts, each chart is only used where $u^2 \leq \bar{w}^2$ *and* $v^2 \leq \bar{w}^2$. Outside this region, another chart will be used.

We now consider the basis vectors $\frac{\partial}{\partial x^i} = \boldsymbol{\partial}_i = \mathbf{e}_i$, $i \in \{1,2\}$, denoting coordinate functions $x^1, x^2 := u,v$. In the chart, $\mathbf{e}_1,\mathbf{e}_2$ are by definition given by components $(1,0)$, $(0,1)$, respectively. In ambient space $\mathbb{R}^3$, for the chart $x \perp y$, they map to the partial derivatives of Eq. P.1, i.e.,

$$\tilde{\mathbf{e}}_1\Big|_{(u,v)} = \begin{pmatrix}1\\0\\-u/\bar{w}\end{pmatrix}, \quad \tilde{\mathbf{e}}_2\Big|_{(u,v)} = \begin{pmatrix}0\\1\\-v/\bar{w}\end{pmatrix}. \tag{P.3}$$

[5] This is often written in the equivalent, but less intuitive, form $\nabla_{\mathbf{u}}\langle\mathbf{v},\mathbf{w}\rangle = \langle\nabla_{\mathbf{u}}\mathbf{v},\mathbf{w}\rangle + \langle\mathbf{v},\nabla_{\mathbf{u}}\mathbf{w}\rangle$ [40, Proposition 5.5].

These components are referred to Cartesian coordinates in $\mathbb{R}^3$. We will now also use the shorthand notations $a^2 := r^2 - u^2$, $b^2 := r^2 - v^2$. The dual basis ω^i, with $\omega^i(\mathbf{e}_j) = \delta^i{}_j$, mapped to ambient space $\mathbb{R}^3$, is

$$\tilde{\omega}^1\Big|_{(u,v)} = \frac{1}{r^2}\begin{pmatrix} a^2 \\ -uv \\ -u\bar{w} \end{pmatrix}, \quad \tilde{\omega}^2\Big|_{(u,v)} = \frac{1}{r^2}\begin{pmatrix} -uv \\ b^2 \\ -v\bar{w} \end{pmatrix}. \tag{P.4}$$

In order to be able to directly use Eq. P.8 below, these two dual basis vectors $\tilde{\omega}^1$ and $\tilde{\omega}^2$ were computed such that they correspond to *orthogonal projection* from the ambient space $\mathbb{R}^3$ into the tangent plane of the immersion of M into $\mathbb{R}^3$. An easy way to do this is to compute an orthogonal third extrinsic basis vector $\tilde{\mathbf{e}}_3 := \tilde{\mathbf{e}}_1 \times \tilde{\mathbf{e}}_2$, and compute the extrinsic dual basis by inverting the 3×3 matrix with columns $\{\tilde{\mathbf{e}}_i\}$ to get $\{\tilde{\omega}^i\}$. The basis vector $\tilde{\mathbf{e}}_3$, and its corresponding dual $\tilde{\omega}^3$, are

$$\tilde{\mathbf{e}}_3\Big|_{(u,v)} = \begin{pmatrix} u/\bar{w} \\ v/\bar{w} \\ 1 \end{pmatrix}, \quad \tilde{\omega}^3\Big|_{(u,v)} = \frac{1}{r^2}\begin{pmatrix} u\bar{w} \\ v\bar{w} \\ \bar{w}^2 \end{pmatrix}. \tag{P.5}$$

The components of the *intrinsic* metric tensor $\mathbf{g}$ can then be computed as $g_{ij} = \tilde{\mathbf{e}}_i \cdot \tilde{\mathbf{e}}_j$, with $\cdot$ the usual Euclidean dot product. For our chart, the metric $\mathbf{g}$ (components g_{ij}) and its inverse $\mathbf{g}^{-1}$ (components g^{ij}), are

$$g_{ij}\Big|_{(u,v)} = \frac{1}{\bar{w}^2}\begin{bmatrix} b^2 & uv \\ uv & a^2 \end{bmatrix}, \quad g^{ij}\Big|_{(u,v)} = \frac{1}{r^2}\begin{bmatrix} a^2 & -uv \\ -uv & b^2 \end{bmatrix}. \tag{P.6}$$

The partial derivatives of the basis vectors in the ambient $\mathbb{R}^3$, in the directions $x^1, x^2 := u, v$, evaluated at the point (u,v) in the chart, are

$$\begin{aligned} \partial_1\tilde{\mathbf{e}}_1\Big|_{(u,v)} &= -\frac{1}{\bar{w}^3}\begin{pmatrix} 0 \\ 0 \\ b^2 \end{pmatrix}, \quad \partial_1\tilde{\mathbf{e}}_2\Big|_{(u,v)} = -\frac{1}{\bar{w}^3}\begin{pmatrix} 0 \\ 0 \\ uv \end{pmatrix}, \\ \partial_2\tilde{\mathbf{e}}_1\Big|_{(u,v)} &= -\frac{1}{\bar{w}^3}\begin{pmatrix} 0 \\ 0 \\ uv \end{pmatrix}, \quad \partial_2\tilde{\mathbf{e}}_2\Big|_{(u,v)} = -\frac{1}{\bar{w}^3}\begin{pmatrix} 0 \\ 0 \\ a^2 \end{pmatrix}. \end{aligned} \tag{P.7}$$

From the immersion in $\mathbb{R}^3$, we can now derive the Christoffel symbols $\Gamma^i{}_{jk}$. From the basis vector field partial derivatives $\partial_j\tilde{\mathbf{e}}_i$ just computed, reading off components in the tangent plane with the dual basis gives

$$\Gamma^i{}_{jk} = \tilde{\omega}^i(\partial_j\tilde{\mathbf{e}}_k), \qquad \text{for} \quad i,j,k \in \{1,2\}. \tag{P.8}$$

Due to the way in which we have computed the dual basis $\{\tilde{\omega}^1, \tilde{\omega}^2\}$, this is equivalent to a completely intrinsic computation from the metric using Eq. N.10, but easier to compute. We emphasize that using this extrinsic "shortcut" computation does not in any way change the fact that afterwards we can perform all computations requiring Christoffel symbols, i.e., covariant derivatives, in a fully intrinsic manner.

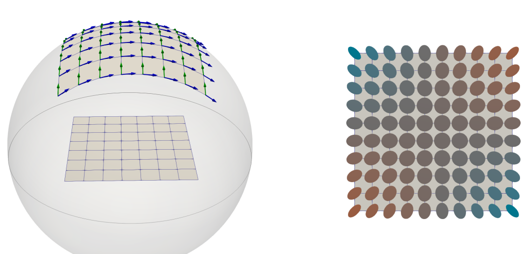

Fig. 10. **Intrinsic description of the sphere.** We describe everything intrinsically in 2D coordinate charts. At each coordinate (u,v) in a chart $U \subset \mathbb{R}^2$, we know the corresponding metric tensor (visualization on the right) in components g_{ij}, and the corresponding Christoffel symbols $\Gamma^i{}_{jk}$.

The Christoffel symbols that we need, given with respect to the chart $U \subset \mathbb{R}^2$, are (only six are unique, because $\Gamma^1{}_{12} = \Gamma^1{}_{21}$, $\Gamma^2{}_{12} = \Gamma^2{}_{21}$),

$$\begin{aligned} \Gamma^1{}_{11}\Big|_{(u,v)} &= cub^2, \quad \Gamma^1{}_{21}\Big|_{(u,v)} = cu^2v, \\ \Gamma^1{}_{12}\Big|_{(u,v)} &= cu^2v, \quad \Gamma^1{}_{22}\Big|_{(u,v)} = cua^2, \\ \Gamma^2{}_{11}\Big|_{(u,v)} &= cvb^2, \quad \Gamma^2{}_{21}\Big|_{(u,v)} = cuv^2, \\ \Gamma^2{}_{12}\Big|_{(u,v)} &= cuv^2, \quad \Gamma^2{}_{22}\Big|_{(u,v)} = cva^2. \end{aligned} \tag{P.9}$$

Here, we have used the shorthand $c := 1/(r^2\bar{w}^2)$. One can verify that with these Christoffel symbols we now have, extrinsically in $\mathbb{R}^3$,

$$\nabla_{\tilde{\mathbf{e}}_j}\tilde{\mathbf{e}}_k = \Gamma^i{}_{jk}\,\tilde{\mathbf{e}}_i, \qquad \text{for} \quad i,j,k \in \{1,2\}, \tag{P.10}$$

where $\nabla_{\tilde{\mathbf{e}}_j}\tilde{\mathbf{e}}_k$ always lies in the tangent plane at the point corresponding to (u,v). However, most importantly, we now never need to refer to the ambient space $\mathbb{R}^3$ again, and can compute everything intrinsically in the chart, with the same values for the Christoffel symbols $\Gamma^i{}_{jk}$, giving

$$\nabla_{\mathbf{e}_j}\mathbf{e}_k = \Gamma^i{}_{jk}\,\mathbf{e}_i, \qquad \text{for} \quad i,j,k \in \{1,2\}. \tag{P.11}$$

Because the covariant derivative is linear in each of its arguments, Eq. P.11 determines Eq. N.8 for the covariant derivative $\nabla\mathbf{v}$ of any vector field $\mathbf{v}$. In a 2D chart, we can thus expand Eq. N.8 as the matrix

$$\begin{bmatrix} \nabla_1 v^1 & \nabla_2 v^1 \\ \nabla_1 v^2 & \nabla_2 v^2 \end{bmatrix} = \begin{bmatrix} \partial_1 v^1 + \Gamma^1{}_{11}v^1 + \Gamma^1{}_{12}v^2 & \partial_2 v^1 + \Gamma^1{}_{21}v^1 + \Gamma^1{}_{22}v^2 \\ \partial_1 v^2 + \Gamma^2{}_{11}v^1 + \Gamma^2{}_{12}v^2 & \partial_2 v^2 + \Gamma^2{}_{21}v^1 + \Gamma^2{}_{22}v^2 \end{bmatrix}. \tag{P.12}$$

Evaluating $\nabla_{\mathbf{x}}\mathbf{v} = \nabla\mathbf{v}(\mathbf{x})$ (Eq. N.9) in the chart thus becomes a matrix-vector multiply of the matrix $\nabla_j v^i$, times the vector components x^i.

All charts. Due to the symmetry of all charts, the metric components (Eq. P.6) and the Christoffel symbols (Eq. P.9) are *the same* in all charts, although above we have derived them only for the chart $x \perp y$.

Q COMPUTING PARTIAL DERIVATIVES

As above, we work in charts $U \subset \mathbb{R}^2$. Each chart is triangulated, with mesh vertices $\{x_k\}$ at 2D coordinates $\big(u(x_k), v(x_k)\big) = (u_k, v_k) \in \mathbb{R}^2$. To compute the partial derivatives $\partial_1 v^i$ and $\partial_2 v^i$ of an $\mathbb{R}$-valued function $v^i(x)$ given at the vertices, we consider the 1-form dv^i, with basis $\{\omega^i\}$,

$$dv^i = (\partial_1 v^i)\,\omega^1 + (\partial_2 v^i)\,\omega^2. \tag{Q.1}$$

To compute dv^i for a single triangle comprising the vertices x_0, x_1, x_2, with coordinates $(u_0,v_0), (u_1,v_1), (u_2,v_2) \in \mathbb{R}^2$, and known function values $v^i(x_0), v^i(x_1), v^i(x_2) \in \mathbb{R}$, we can solve the 2×2 linear system

$$\begin{bmatrix} (u_1-u_0) & (v_1-v_0) \\ (u_2-u_0) & (v_2-v_0) \end{bmatrix}\begin{bmatrix} \partial_1 v^i \\ \partial_2 v^i \end{bmatrix} = \begin{bmatrix} v^i(x_1) - v^i(x_0) \\ v^i(x_2) - v^i(x_0) \end{bmatrix}, \tag{Q.2}$$

in order to obtain $\partial_1 v^i$ and $\partial_2 v^i$. For a 1-ring around a given vertex x_0 (see Fig. 11), labeling its vertices as $x_0, x_1, x_2, x_3, \ldots, x_{n-1}$, we can solve, in the least-squares sense, the over-determined $(n-1)\times 2$ system

$$\begin{bmatrix} (u_1-u_0) & (v_1-v_0) \\ (u_2-u_0) & (v_2-v_0) \\ \vdots & \vdots \\ (u_{n-1}-u_0) & (v_{n-1}-v_0) \end{bmatrix}\begin{bmatrix} \partial_1 v^i \\ \partial_2 v^i \end{bmatrix} = \begin{bmatrix} v^i(x_1) - v^i(x_0) \\ v^i(x_2) - v^i(x_0) \\ \vdots \\ v^i(x_{n-1}) - v^i(x_0) \end{bmatrix}. \tag{Q.3}$$

If we write the system above in the abbreviated form $\mathbf{A}\,\mathbf{d} = \mathbf{v}$, we can solve the 2×2 square system $\mathbf{A}^T\mathbf{A}\,\mathbf{d} = \mathbf{A}^T\mathbf{v}$, i.e., $\mathbf{d} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{v}$, corresponding to the normal equations of the least-squares problem.

We can simplify the structure of this computation by computing weights $\{(w^1_j, w^2_j)\}_{j=0}^{n-1}$ for each vertex x_j in the 1-ring of vertex x_0.

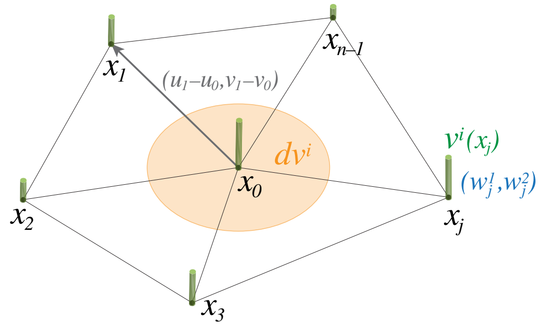

Fig. 11. **1-ring neighborhood** of a triangle vertex x_0 for approximating 1-forms $dv^i = (\partial_1 v^i)\,\omega^1 + (\partial_2 v^i)\,\omega^2$ of $\mathbb{R}$-valued functions v^i on M.

These weights form an n-tap filter stencil for computing a weighted average of the 1-ring neighborhood of vertex x_0. From them, we can compute the components $\partial_1 v^i, \partial_2 v^i$ of the 1-form dv^i at vertex x_0 as

$$\begin{aligned} \partial_1 v^i\Big|_{(u_0,v_0)} &= w_0^1\, v^i(x_0) + w_1^1\, v^i(x_1) + \ldots + w_{n-1}^1\, v^i(x_{n-1}), \\ \partial_2 v^i\Big|_{(u_0,v_0)} &= w_0^2\, v^i(x_0) + w_1^2\, v^i(x_1) + \ldots + w_{n-1}^2\, v^i(x_{n-1}). \end{aligned} \tag{Q.4}$$

In order to compute all weights $\{(w_j^1, w_j^2)\}_{j=0}^{n-1}$ in the stencil, we introduce the $2\times(n-1)$ matrix $\mathbf{W} := (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$, labeling components as $\mathbf{W}_{ij}$, with i the row and j the column index, respectively. Considering the structure of the $(n-1)\times 1$ right-hand side above, we directly obtain

$$\begin{aligned} w_0^i &= -\sum_{j=1}^{n-1} \mathbf{W}_{ij}, \qquad & i \in \{1,2\}, \\ w_j^i &= \mathbf{W}_{ij}, & i \in \{1,2\}; 1 \le j \le (n-1). \end{aligned} \tag{Q.5}$$

We pre-compute the $2n$ weights of each 1-ring neighborhood, with n vertices, storing them with the corresponding center vertex (above: x_0).

We note that all filter stencils depend solely on the geometry (the vertex positions) of the triangle mesh, but not on any specific function $v^i(x)$. We can therefore associate the filter weights with each triangle vertex, and then use them to compute the partial derivatives of arbitrary functions, e.g., the v^1 and v^2 of the previous section, see Eq. P.12. We also emphasize that these partial derivatives are the only numerically approximated quantities. The metric components g_{ij} (Eq. P.6) and the Christoffel symbols Γ^i_{jk} (Eq. P.9) are accurately computed analytically.

R The Flow of a Vector Field

We briefly summarize the standard concepts of the *flow* of a vector field as they are typically defined in differential geometry. This appendix is the same as the corresponding one in Hadwiger et al. [28, App. B].

The flow of a *time-independent* vector field $\mathbf{u}$ on a manifold M is a map $\phi\colon J\times M \to M$ for a suitable interval $J \subseteq \mathbb{R}$, such that $t \mapsto \phi(t,x)$ is the unique maximal integral curve of $\mathbf{u}$ through $x \in M$ [39, Th. 9.12]. That is, ϕ maps a point x to its image along the integral curve of $\mathbf{u}$ after time t, which we also denote by $\phi_t(x)$. Important properties of ϕ are:

- The map $\phi_t\colon M \to M$ is a (local) diffeomorphism for all $t \in J$.
- For all $t_1,t_2 \in J$, $x \in M$, $\phi_{t_2}(\phi_{t_1}(x)) = \phi_{t_1+t_2}(x)$, $\phi_0(x) = x$. The inverse of ϕ_t is ϕ_{-t}, i.e., $\phi_t^{-1}(\phi_t(x)) = \phi_{-t}(\phi_t(x)) = x$. ϕ is an *action* of the additive group $\mathbb{R}$ on M, ϕ_t is a one-parameter group.
- The *linear* map $\mathrm{d}\phi_t\colon T_xM \to T_{\phi_t(x)}M$, called the differential of ϕ_t, or the (pointwise) *push-forward*, is an isomorphism between the two tangent spaces at each $x \in M$ and $\phi_t(x) \in M$, for each $t \in J$. $\mathrm{d}\phi_t$ maps tangent vectors to all possible curves through a point $x \in M$ to the corresponding tangent vectors of the images of these curves under the diffeomorphism ϕ_t, through the point $\phi_t(x) \in M$.

When the vector field $\mathbf{u}$ is *time-dependent*, the corresponding time-dependent flow $\psi\colon J\times J\times M \to M$ maps a point $x \in M$ to its image along the integral curve from time s to time t [39, Th. 9.48], which we denote by $\psi_{t,s}(x)$. The map ψ has similar properties to the map ϕ:

- The map $\psi_{t,s}\colon M \to M$ is a (local) diffeomorphism for all $s,t \in J$.
- For all $s,t_1,t_2 \in J$, $x \in M$, $\psi_{t_2,t_1}(\psi_{t_1,s}(x)) = \psi_{t_2,s}(x)$, $\psi_{s,s}(x) = x$. The inverse of $\psi_{t,s}$ is $\psi_{s,t}$, i.e., $\psi_{t,s}^{-1}(\psi_{t,s}(x)) = \psi_{s,t}(\psi_{t,s}(x)) = x$.
- The *linear* map $\mathrm{d}\psi_{t,s}\colon T_xM \to T_{\psi_{t,s}(x)}M$, called the differential (the *push-forward*) of $\psi_{t,s}$, is an isomorphism between the tangent spaces at each $x \in M$ and $\psi_{t,s}(x) \in M$, for each $s,t \in J$. $\mathrm{d}\psi_{t,s}$ maps tangent vectors to all possible curves through a point $x \in M$ to the corresponding tangent vectors of the images of these curves under the diffeomorphism $\psi_{t,s}$, through the point $\psi_{t,s}(x) \in M$.

We note that the notation $\psi_{t,s}(x)$ can of course also be consistently used for the case of time-independent flow. In that case, $\psi_{t,s}(x) = \phi_{t-s}(x)$.

S Pushforwards and Pullbacks

A smooth map $\phi : M \to M$ induces for each $x \in M$ a *linear* map

$$\begin{aligned} (\mathrm{d}\phi)_x : T_xM &\to T_{\phi(x)}M, \\ \mathbf{x} &\mapsto (\mathrm{d}\phi)_x(\mathbf{x}), \end{aligned} \tag{S.1}$$

called the *differential* or *pushforward*, from the tangent space at x to the tangent space at $\phi(x)$.

This is illustrated geometrically in Fig. 12: Choosing a smooth curve through the point $x \in M$ defines a tangent vector $\mathbf{x} \in T_xM$. The map ϕ maps this smooth curve to another smooth curve through the point $\phi(x) \in M$, defining the tangent vector $(\mathrm{d}\phi)_x(\mathbf{x}) \in T_{\phi(x)}M$.

The differential of a smooth map allows to pushforward single tangent vectors to other points on the manifold. Our definition of objectivity requires to pushforward whole vector fields (Sec. 5.2). This is not possible with an arbitrary smooth map ϕ, as this fails to define tangent vectors at points not hit by ϕ (ϕ is not onto), or might define tangent vectors ambiguously at points hit several times (ϕ is not one-to-one).

Diffeomorphisms. However, if a smooth map ϕ happens to be both one-to-one and onto, then it uniquely defines another vector field with the tangent vectors being the pointwise pushforwards [39, Proposition 8.10]. Such a smooth map ϕ that has an inverse, if this inverse is also smooth, is called a *diffeomorphism*. That is, we can use a map ϕ to pushforward a whole vector field precisely when ϕ is a diffeomorphism.

Pullbacks. The corresponding concept to the pushforward of a vector field is the pullback of a covector (1-form) field [39, Ch. 11]. The pullback ϕ^* of a covector field is defined as

$$\begin{aligned} (\phi^*)_x : T^*_{\phi(x)}M &\to T^*_xM, \\ \omega &\mapsto (\phi^*)_x(\omega). \end{aligned} \tag{S.2}$$

Since $(\phi^*)_x(\omega)$ is a covector (1-form), in order to define it we have to specify how it acts on a vector $\mathbf{x} \in T_xM$. We use the pushforward and

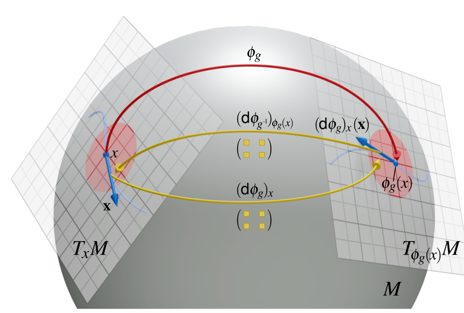

Fig. 12. **Pushforward and pullback** of a diffeomorphism ϕ_g are linear maps between the tangent spaces T_xM and $T_{\phi_g(x)}M$, and cotangent spaces $T^*_{\phi_g(x)}M$ and T^*_xM, respectively. The pushforward $(\mathrm{d}\phi_g)_x$ maps a tangent vector $\mathbf{x} \in T_xM$ to the vector $(\mathrm{d}\phi_g)_x(\mathbf{x}) \in T_{\phi_g(x)}M$. The pullback ϕ_g^* maps a covector (1-form) $\omega \in T^*_{\phi_g(x)}M$ to the covector (1-form) $\phi_g^*\omega \in T^*_xM$.

the fact that ω is a covector of the tangent space at $\phi(x)$:

$$(\phi^*)_x(\omega)(\mathbf{x}) := \omega((\mathrm{d}\phi)_x(\mathbf{x})). \tag{S.3}$$

In contrast to vector fields, covector fields always pull back to covector fields, even when the map ϕ is not a diffeomorphism.

T NOTATION TABLE

M,N	manifolds
$(M,\mathbf{g})$	Riemannian manifold M with metric $\mathbf{g}$
T_xM	tangent space at $x \in M$
T_x^*M	cotangent space at $x \in M$
TM	tangent bundle of a manifold M
$\mathbf{e}_i$	basis vector fields/bases in each tangent space T_xM
ω^i	basis covectors/1-forms (dual to $\{\mathbf{e}_i\}$)
$\mathbf{e}_i \otimes \omega^j$	basis of a $\binom{1}{1}$ tensor
$\mathbf{e}_i \otimes \mathbf{e}_j$	basis of a $\binom{2}{0}$ tensor
$\omega^i \otimes \omega^j$	basis of a $\binom{0}{2}$ tensor
$\mathbf{g}, \langle\cdot,\cdot\rangle$	metric (on tangent spaces of manifold M)
g_{ij}	comps. of metric in basis $\{\omega^i \otimes \omega^j\}$; $g_{ij} := \langle \mathbf{e}_i, \mathbf{e}_j \rangle$
ϕ	flow of a vector field
ϕ_t	one-parameter group generated by the flow ϕ
Φ	Lie group action on manifold M
ϕ_g	diffeomorphism generated by Lie group element g
$\mathrm{d}\phi_g$	pushforward/differential (lin. map betw. tang. spaces)
ϕ_g^*	pullback of ϕ_g (linear map between cotangent spaces)
Ξ	Lie algebra action on manifold M
∂_i, $\partial_{\mathbf{e}_i}$	partial derivative in direction $\mathbf{e}_i$
∇_i, $\nabla_{\mathbf{e}_i}$	covariant derivative in direction $\mathbf{e}_i$
$\Gamma^i{}_{jk}$	Christoffel symbols; with respect to a basis $\{\mathbf{e}_i\}$
$\partial_j v^i$	partial derivatives of vector comps. v^i (not a tensor!)
$\nabla\mathbf{v}$	velocity gradient tensor (field) of vector field $\mathbf{v}$
$\nabla_j v^i$	components of $\nabla\mathbf{v}$ with respect to a basis $\{\mathbf{e}_i \otimes \omega^j\}$
$\nabla_j v_i$	comp. of cov. deriv. of cov. field $v_i\omega^i$; basis $\{\omega^i \otimes \omega^j\}$
$\mathbf{u}$	observer velocity field
$\mathscr{L}_{\mathbf{u}}$	(autonomous) Lie derivative with resp. to the field $\mathbf{u}$
$L_{\mathbf{u}}$	time-dependent Lie derivative with resp. to the field $\mathbf{u}$
G	Lie group
$\mathfrak{g}$	Lie algebra of Lie group G
g	Lie group element $g \in G$
X	Lie algebra Element $X \in \mathfrak{g}$
$\mathrm{Isom}(M)$	isometry group of M (a Lie group)
$\mathfrak{isom}(M)$	Lie algebra of the isometry group of M
$\mathrm{O}(n)$	orthogonal group
$\mathrm{SO}(n)$	special ($\det g = 1$) orthogonal group (rotations)
$\mathfrak{so}(n)$	Lie algebra of $\mathrm{SO}(n)$; all $X \in \mathfrak{so}(n)$ are anti-symmetric
$\mathrm{T}(n)$	translation group of $\mathbb{R}^n$
$\mathrm{E}(n)$	Euclidean group of $\mathbb{R}^n$; $\mathrm{E}(n) = \mathrm{O}(n) \ltimes \mathrm{T}(n)$
$\mathrm{SE}(n)$	special Euclidean group; $\mathrm{SE}(n) = \mathrm{SO}(n) \ltimes \mathrm{T}(n)$
$K\mathbf{u}$	Killing operator applied to $\mathbf{u}$
$E\mathbf{u}$	Killing energy (density) of $\mathbf{u}$ at point $x \in M$
$\int_M E\mathbf{u}$	Killing energy of $\mathbf{u}$ on M
$\langle \mathbf{T}, \mathbf{S} \rangle_{\mathbf{g}}$	tensor inner product of $\mathbf{T}$ and $\mathbf{S}$ with resp. to metric $\mathbf{g}$
$\|\mathbf{T}\|_{\mathbf{g}}$	tensor norm of $\mathbf{T}$ with respect to metric $\mathbf{g}$
$(\mathscr{D}/\mathscr{D}t)$	observed time derivative (wrt. an observer field $\mathbf{u}$)
(D/Dt)	material time derivative