Analysis of Environmental Data

Distributions: Notation, Functions, and Probability

Michael France Nelson

Eco 602 – University of Massachusetts, Amherst
Michael France Nelson

Probability and Distributions: Probability Theory

Probability theory concerns the likelihood of events

Distributions are tools for describing the likelihood of observing specific events from the set of all possible events.

- They map ***events*** to ***likelihoods***

There are many named *parametric* distributions with well-understood, useful, and sometimes surprising properties.

Probability theory gets complicated and difficult *very* quickly!

- I'll attempt to help you develop intuition about the most essential parts.
- This isn't a course on probability theory – we'll only cover the basics.

Probability Theory Essentials

Probabilities are non-negative	Sample space: the set of all possible events
• A probability can be any value between zero and 1.0, inclusive. • The probability of a specific event is usually less than 1.0 • **Law of total probability:** The sum of the probabilities of all possible events is 1.0	• Events: a possible outcome of a stochastic process • The definition of event is context-specific: • "What is the probability of catching a fish that weighs 405 grams?" • "What is the probability of catching a fish that weighs between 399 and 411 grams?" • "What is the probability of catching a fish that weighs less than 200 grams?" • "What is the probability that I observe 2 gray jays?"

Probability Notation Basics

Basic probability

- $\Pr(\mathrm{A}) = 0.05$
  - Read as: "The probability that event A occurs is 5%"

Joint probability

- $\Pr(\mathrm{A}\ and\ B)$ = $\Pr(\mathrm{A} \cap B) = 0.05$
  - Read as: "The probability that both events A and B occur is 5%"

Conditional Probability

- $\Pr(\mathrm{A}|B) = 0.05$
  - Read as: "The probability that event A occurs, given that B has already occurred is 5%"

Independent Events

$$\mathbf{Pr}(\mathbf{A} \cap \boldsymbol{B}) = \mathbf{Pr}(\boldsymbol{A}) \times \mathbf{Pr}(\boldsymbol{B})$$

- The probability that A and B both occur is equal to the product of the individual probabilities…
- We'll dissect this surprisingly important definition.

Independent events

Events are independent if knowing the value of one observation gives us no information about the value of another observation:

1. I measure the temperature in Neuquén, Argentina on November 23, 1823.
2. I measure the temperature in Amherst on July 4, 2020.

The Neuquén temperature in 1823 probably doesn't tell me much about Amherst in 2020

- Likewise, the temperature here today probably won't tell me much about what to expect there! [Other than knowing that it is fall here, and it was spring/summer there!]

Independent events

Non-Independent Temperatures

Compare the previous temperature example to:

1. I measure the temperature in Amherst on July 4, 2020 at 4:05PM (it is 20C)
2. I measure the temperature in Amherst on July 4, 2020 at 4:11PM (it is 21C)

The temperature at 4:05 gives me a lot of information about what the temperature will be in the same location six minutes later.

Independent events

Suppose we are equally likely to observe these temperatures:	Independent events: joint probability is product of individual probabilities
Temperatures on July 4$^{\text{th}}$: $Pr(temp = 19C) = 0.05$ $Pr(temp = 20C) = 0.05$ $Pr(temp = 21C) = 0.05$	If successive temperature measurements were independent: • $Pr(20)*Pr(21)$=0.05∗0.05=0.0025 or about 0.25% Do you think observing a temperature of 20, followed by another temperature of 20 in the same location 6 minutes later is only 0.25%??? It's probably much higher than 5% (the *unconditional* probability of observing 20C.)

Independent events

If events are independent, the probability of observing a *specific* set of events (the joint probability) is the same as the product of the events of the individual events.

- I pick up an acorn in each hand simultaneously, from a *very large* collection of acorns of several species.
- Does knowing that the acorn in my left hand is from a Bur Oak tell me anything about the acorn in my right hand?

Independence and Maximum Likelihood

- This may not seem important now, but it is *crucial* to the likelihood concepts we'll examine later.
- It's also key to understanding Bayes' Rule.

Functions and Formulae

components and intuition

Key concepts

Notation	Common Formula Chunks
• Variables and Constants • Arithmetic operators • Summation notation • Set notation • Bar notation • Capital and lower-case letters	• Means: summation and bar notation: $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ • Sums of Squares: summation and bar notation $SSE = \sum_{i=1}^{n} (x_i - \bar{x})^2$

Interpreting formulae

Strategies	This lecture focuses on three common chunks
• Identify constants and variables • Learn to recognize common 'chunks' • Try to identify long-term behavior • Try to identify the class of the function	1. Means 2. Sums of Squares 3. Variance/Covariance

Common chunks

Sums of squares

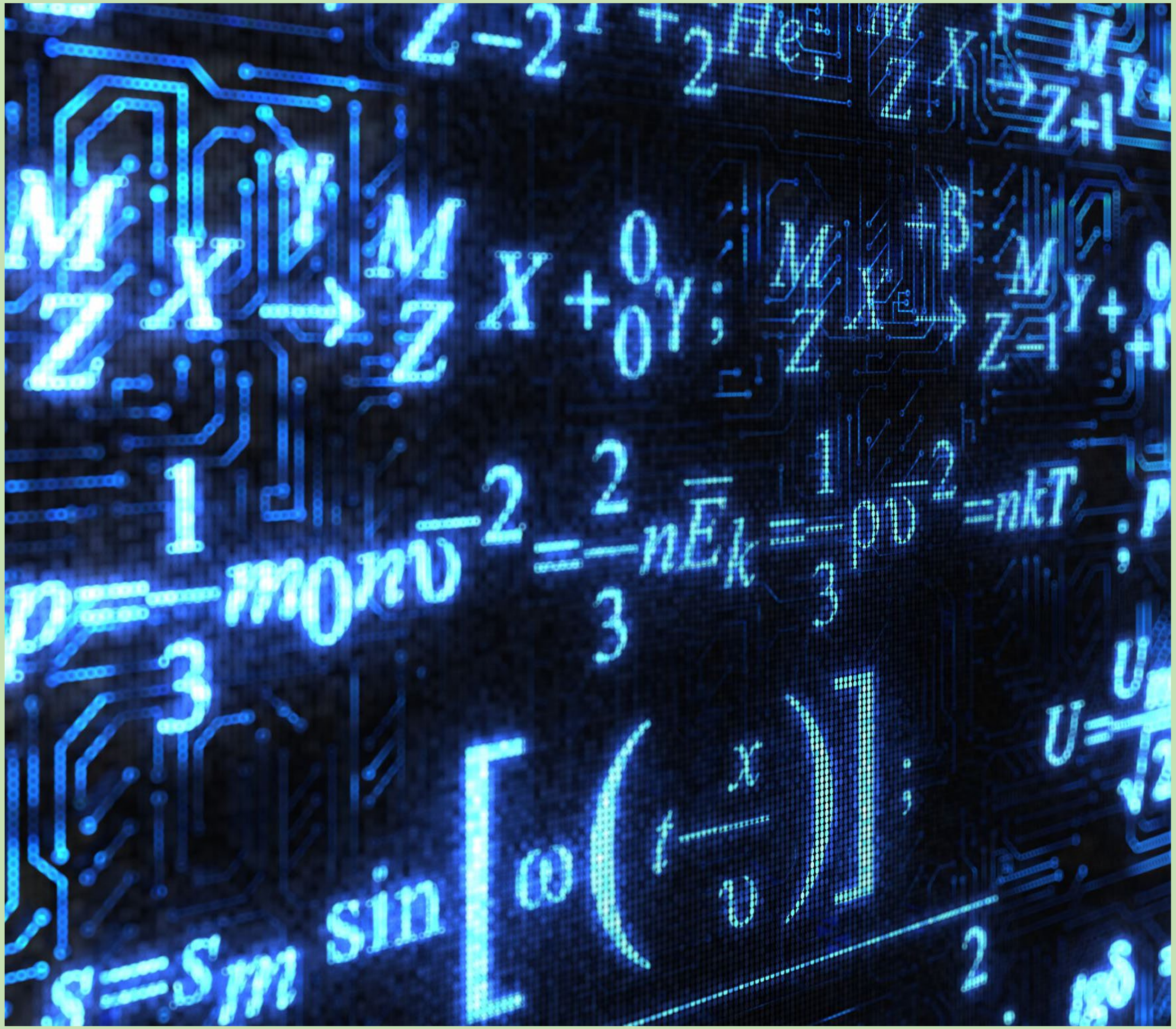
- often used to quantify some sort of 'error'
- use in variance and covariance formulas

Normalizing constants

- These are often nightmarish, but if you look closely, you can usually ignore them!

Sample size, sample size correction

- N-1, n-1



Starting simple: the mean

Arithmetic Mean

$$\frac{\sum_{i=1}^{n} x}{n}$$

The mean is a simple concept, right?

- It’s just the average value…
- It’s what we get if we add up all the numbers and divide by the count.

What do we need to know?

- Our data:
  - A vector of numbers (in R-speak)
- Our quantities:
  - The number of observations
  - The sum of all the observations

Starting simple: the mean

We can practice our notation skills:

- Set notation
- Capital/lowercase notation
- Summation notation
- Bar notation
- Normalizing and Sample size notation

Our x-values in set notation:

$$X = \{x_1, x_2, x_3, \ldots, x_n\}$$

*note the capital X for the set, and the lowercase x for the elements

The sum of values in sigma notation:

$$\sum_{i=1}^{n} x_i$$

The sample size: n

Starting simple: the mean

Putting it all together: the overall formula	The mean is also known as the Expected Value
$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$	$E(X) = \frac{\sum_{i=1}^{n} x_i}{n}$

Sum of squared errors

This is a common chunk	SSE
• *Error* is the difference between an observation and the expected value. • In words: "The sum of the squared differences between each value and the mean" • In words: "The sum of the squared errors"	$$SSE = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Sum of squared errors

We already know the mean: $\bar{x}$

The SSE chunk:

$$SSE = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

The sigma decorations are often dropped:

$$SSE = \sum (x_i - \bar{x})^2$$

It can be expanded to:

$$SSE = \sum (x_i - \bar{x})(x_i - \bar{x})$$

Not so bad. We can learn to see this as a chunk, rather than individual terms!

Sum of squared errors

Why do we want to know this anyway?

Squaring has some desirable properties

- Converts negative values to positive.
- Penalizes high values, i.e. large *errors* are very influential because the squaring function is nonlinear.
- What happens if you take the sum of the non-squared errors?

Variance and Covariance

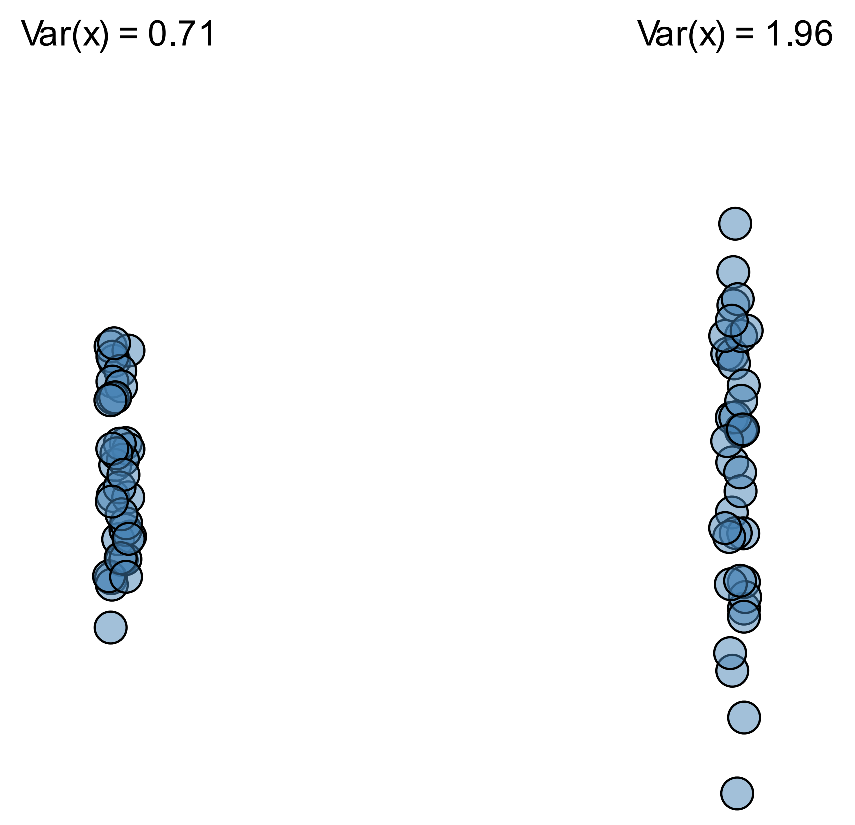
Variance and Covariance: Key Concepts

- Variance and covariance are measures of *spread* or *dispersion.*
  - What are some other measures of dispersion that we know about?
- Variance and covariance are sample statistics - we can use them to estimate population parameters.
- The formulas look intimidating… You won't need to memorize them
- My goal in this section is to build intuition about what they do and how they are notated.
- Variance is a univariate statistic.
- Covariance measures an association between two variables: this is directly analogous to correlation!

Variance

What does variance tell us?

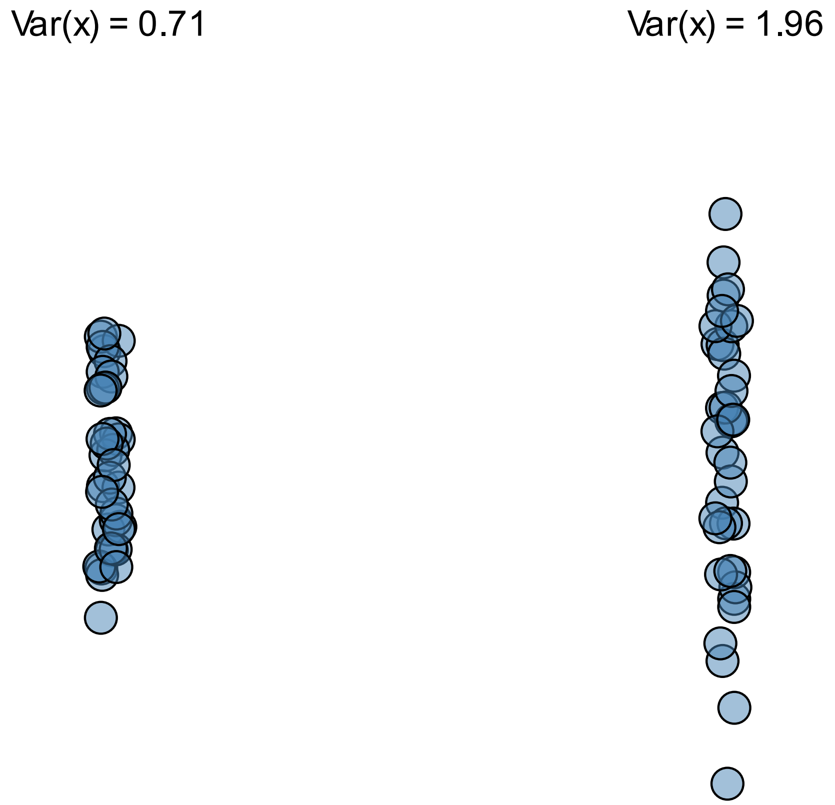
- Recall this form for the SSE:
  $SSE = \sum(x_i - \bar{x})(x_i - \bar{x})$
- It includes the $(x_i - \bar{x})$ twice.



Variance

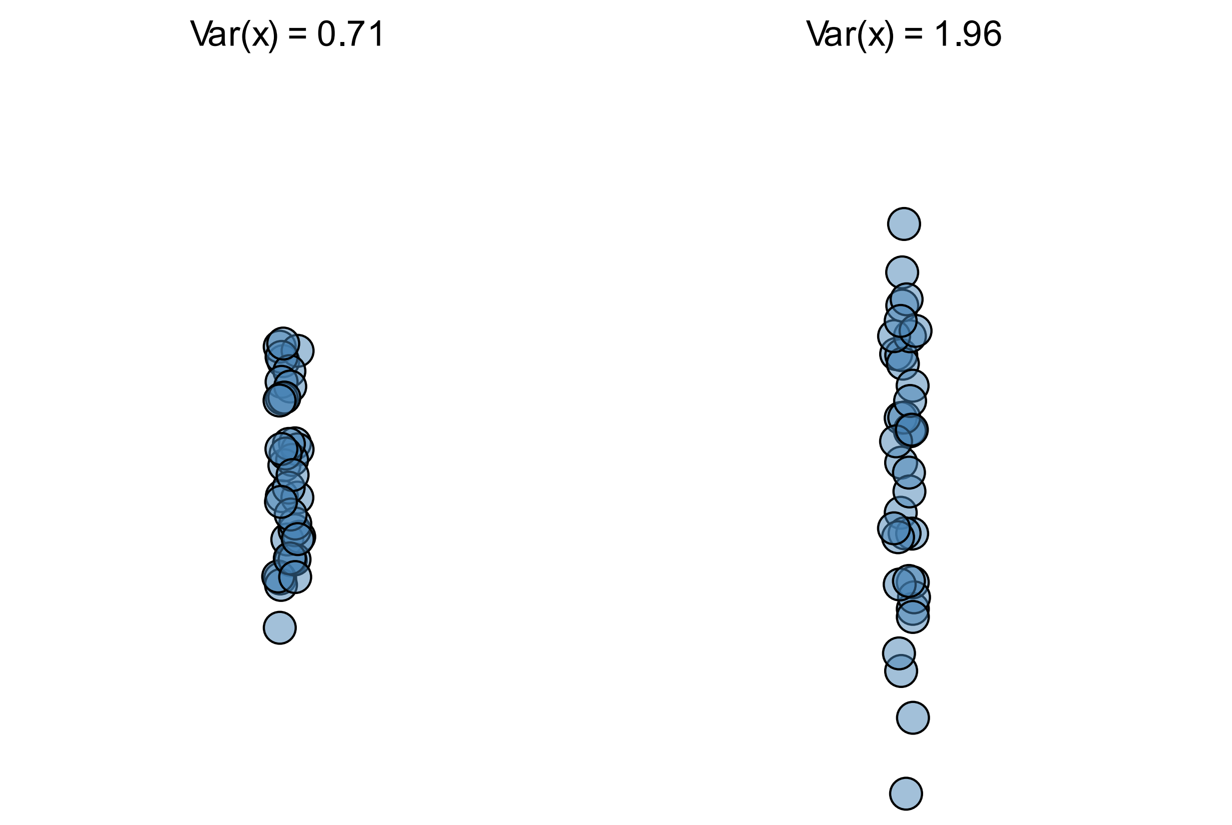
It is a *univariate* statistic!

- It characterizes how much spread there is within a variable, with reference to *itself*.
  - That's why we have the $(x_i - \bar{x})$ two times.



Variance

What if we want to know if two variables change in a *coordinated* way?



Variance

Variance is a measure of *dispersion* or *spread*	**Why do we want to use squared differences?**
• In words: "The variance is the average of the squared differences from the mean." • It's just the SSE normalized by the [adjusted] sample or population size. • For a population we use N, a sample uses n - 1	$(x_i - \bar{x})^2$ • What is the sign of this term? • What would happen to the sum if we used the unsquared differences? • Why not just use the absolute value?

Variance

Variance is a measure of *dispersion* or *spread*	**Why do we want to use squared differences?**

• In words: "The variance is the average of the squared differences from the mean."
 	
• It's just the SSE normalized by the [adjusted] sample or population size.
    • For a population we use N, a sample uses n - 1
	$(x_i - \bar{x})^2$
	
	• What is the sign of this term?
	    • It is always positive!
	• What would happen to the sum if we used the unsquared differences?
	    • They would sum to zero for both dispersed and clustered data.
	• Why not just use the absolute value?
	    • The squaring penalizes large deviations, this has desirable theoretical and practical consequences.

Variance

Formulae: Populations and Samples

- for populations

$$Var(x) = \frac{1}{N}\sum(x_i - \bar{x})^2 = \frac{\sum(x_i - \bar{x})(x_i - \bar{x})}{N}$$

- Samples require a sample size correction:

$$Var(x) = \frac{1}{\mathrm{n} - 1}\sum(x_i - \bar{x})(x_i - \bar{x}) = \frac{\sum(x_i - \bar{x})^2}{\mathrm{n} - 1}$$

Covariance

Covariance measures the *dispersion* of one variable, x, in the context of the *dispersion* of a second variable, y.	**It turns out that *variance* is a special case of *covariance*.**
$$Cov(x,y) = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{N}$$ • Covariance tells us the amount by which the changes in one variable are *coordinated* with changes in another. • $(x_i - \bar{x})(y_i - \bar{y})$ is like a *crossed* version of the squared errors… • But the term *cross product* is already taken.	$$Cov(x,x) = Var(x)$$

Variance and Covariance

Case 1: positive covariance.

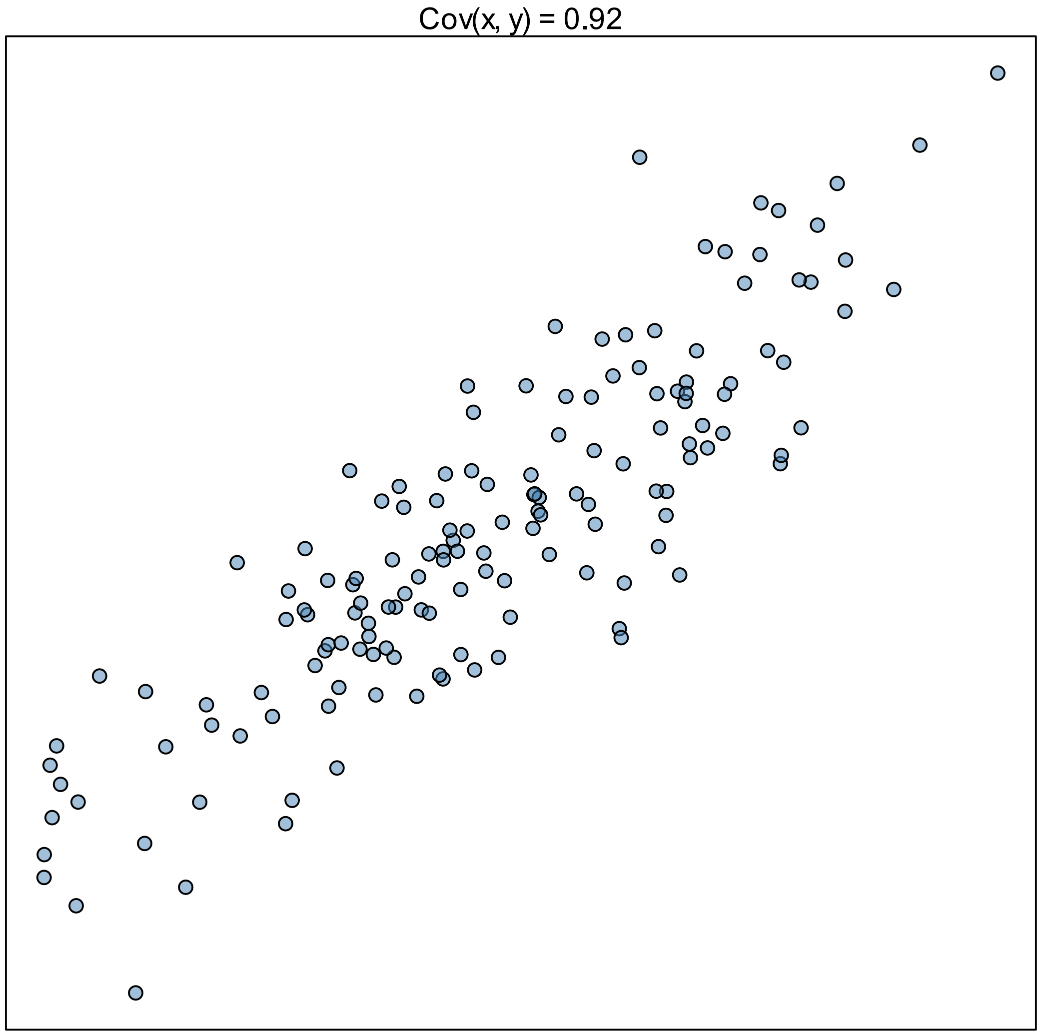
- If the values of x and y were somehow coordinated, we might expect that high values of x would tend to co-occur with high values of y.
- We use $(x_i - \bar{x})$ to symbolize the deviation of a sampling unit's x-value from the mean of x.
- Similarly $(y_i - \bar{y})$ is the deviation of a sampling unit's y-value from the average value of y.
- The (non-squared) sum of all the deviations of x is zero (by the definition of the mean).

Covariance

Case 1: Positive covariance: High x-values tend to co-occur with high y-values.	
Most terms will be positive	**Few terms will be negative**
$(x_i > \bar{x})$ **AND** $(y_i > \bar{y}) = \text{positive}$ $(x_i < \bar{x})$ **AND** $(y_i < \bar{y}) = \text{positive}$	$(x_i > \bar{x})$ **AND** $(y_i < \bar{y}) = \text{negative}$ $(x_i < \bar{x})$ **AND** $(y_i > \bar{y}) = \text{negative}$

Covariance

Positive Covariance

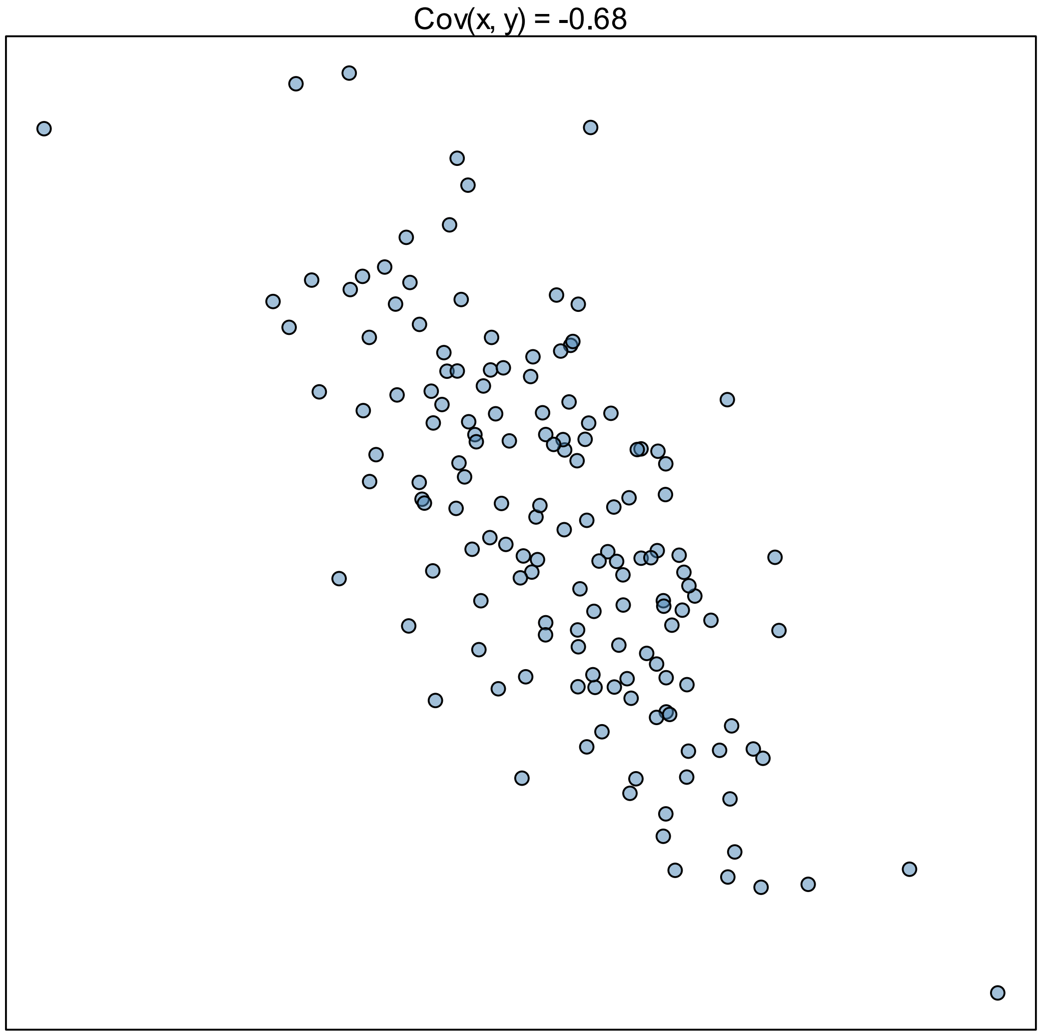


Covariance

Case 2: Negative covariance: High x-values tend to co-occur with low y-values.	
Few terms will be positive	**Most terms will be negative**
$(x_i > \bar{x})$ AND $(y_i > \bar{y}) = \text{positive}$ $(x_i < \bar{x})$ AND $(y_i < \bar{y}) = \text{positive}$	$(x_i > \bar{x})$ AND $(y_i < \bar{y}) = \text{negative}$ $(x_i < \bar{x})$ AND $(y_i > \bar{y}) = \text{negative}$

Covariance

Negative Covariance

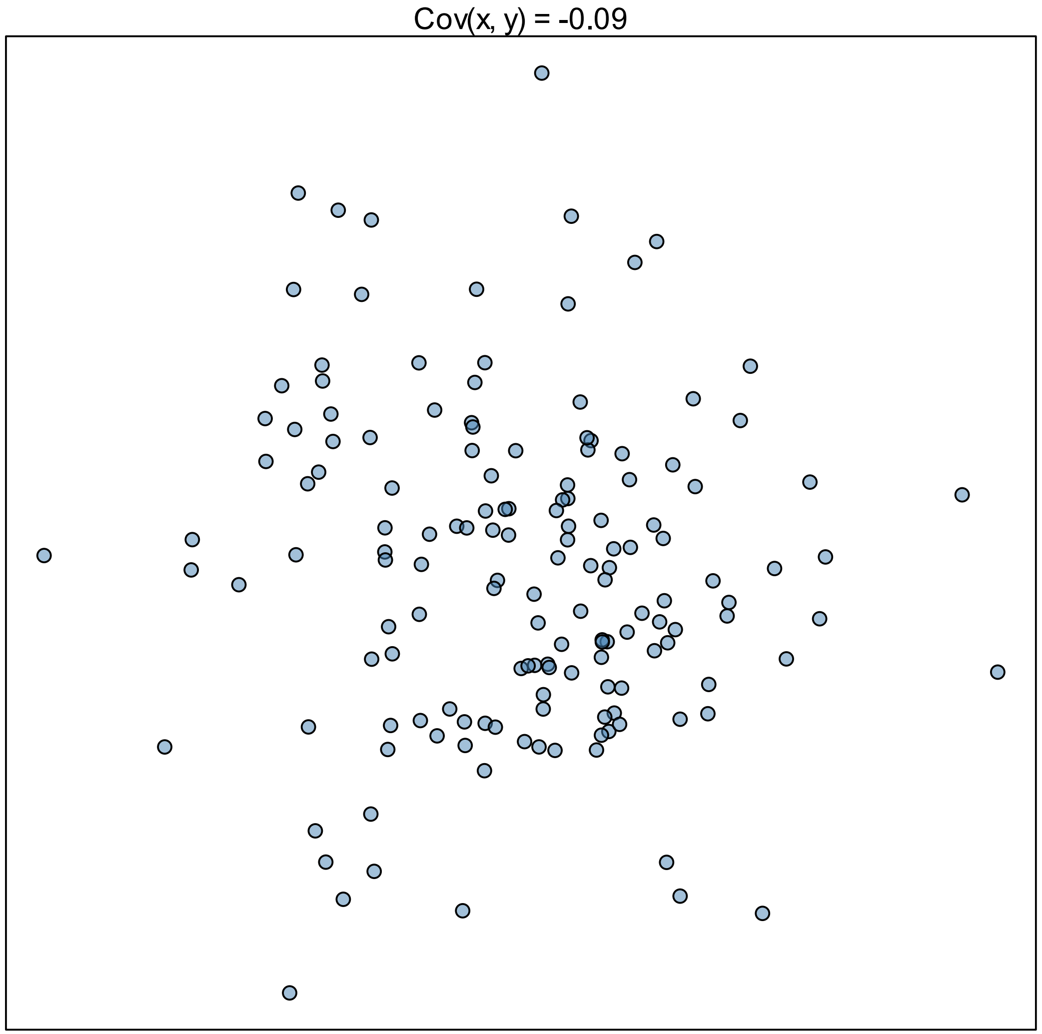


Covariance

Case 3: no covariance: No association between above average x and above average y Negative and positive values cancel – sum is near zero	
About half the terms will be positive	About half the terms will be negative
$(x_i > \bar{x})$ AND $(y_i > \bar{y}) = \text{positive}$ $(x_i < \bar{x})$ AND $(y_i < \bar{y}) = \text{positive}$	$(x_i > \bar{x})$ AND $(y_i < \bar{y}) = \text{negative}$ $(x_i < \bar{x})$ AND $(y_i > \bar{y}) = \text{negative}$

Covariance

Zero Covariance



Recap

Common chunks

- Means: $\bar{x} = \frac{\sum x}{n}$
- Sums of errors: $\sum(x_i - \bar{x})$
  - Remember this sum is zero!
- Sums of squared errors: $\sum(x_i - \bar{x})^2$
- Sums of squared errors: $\sum(x_i - \bar{x})(x_i - \bar{x})$
- Sums of crossed errors: $\sum(x_i - \bar{x})(y_i - \bar{y})$
- Normalizing by population size:
  - $\frac{1}{N}$ and $\frac{1}{N-1}$

Next time…

Pearson correlation walkthrough

Normal PDF

Discrete Distributions

A Parametric Frequentist Approach

Key concepts

- What is discrete?
- Discrete sample spaces
- Combinations and permutations
- Bernoulli, Binomial, and Poisson distributions

Discrete Distributions

In the world of theoretical distributions, discrete refers to a measurement scale that is:

- Numeric, i.e. not categorical
  - Not ordinal because the interval between adjacent points is constant:

$$4 - 3 = 5 - 4$$

but

$$medium - low \neq high - medium$$

- Cannot take on fractional values
  - They are integers

Counts, or censuses, are [usually] considered discrete data type

What’s a Sample Space?

- The set of all possible events in the domain of a distribution!

Sample Spaces

- Events in a discrete distribution can only be integer values.
- That means a discrete distribution has a finite sample space, right?
    - No! Non! Nej! Não! ¡No!
- It may seem unintuitive, but many discrete [theoretical] distributions have infinite sample spaces.

The Simplest Distribution?

One of the easiest distributions to understand is the *Bernoulli Distribution*.

- Its sample space has only two elements, which we might label as:
  - true/false, success/sailure, present/absent
- Realizations of a *Bernoulli process* produces *binary* outcome.
- It has one parameter: the probability of *success*.

It's a special case of the *binomial distribution*

- A realization of the *Bernoulli process* is called a **trial**

The Binomial distributions

A binomial process is a collection of n independent Bernoulli trials.

- Each Bernoulli trial must have the same probability of success
- Binomial has two *parameters*: n and p
  - n is the number of trials
  - p is the probability of *success* in an individual trial (just like the Bernoulli dist.)
- The *sample space* of a binomial distribution has $n + 1$ elements:
- It's the possible counts of successes, i.e. the set $\{0,1,3,\ldots,n\}$

Classic Example: Coin Tosses

A series of independent coin tosses is a lot like a binomial process…

Wait a minute, I think I remember something about *independent events* and *joint probabilities*….

- A single coin flip is not very interesting, but consider the sample space for two flips ($n = 2$):

$$\{(T, T), (T, H), (H, T), (H, H)\}$$

- The *sample space* of a binomial distribution has $n + 1$ elements, which should be 3, but there are 4 elements in the set!
  - Something seems wrong.

Independent Coin Flips

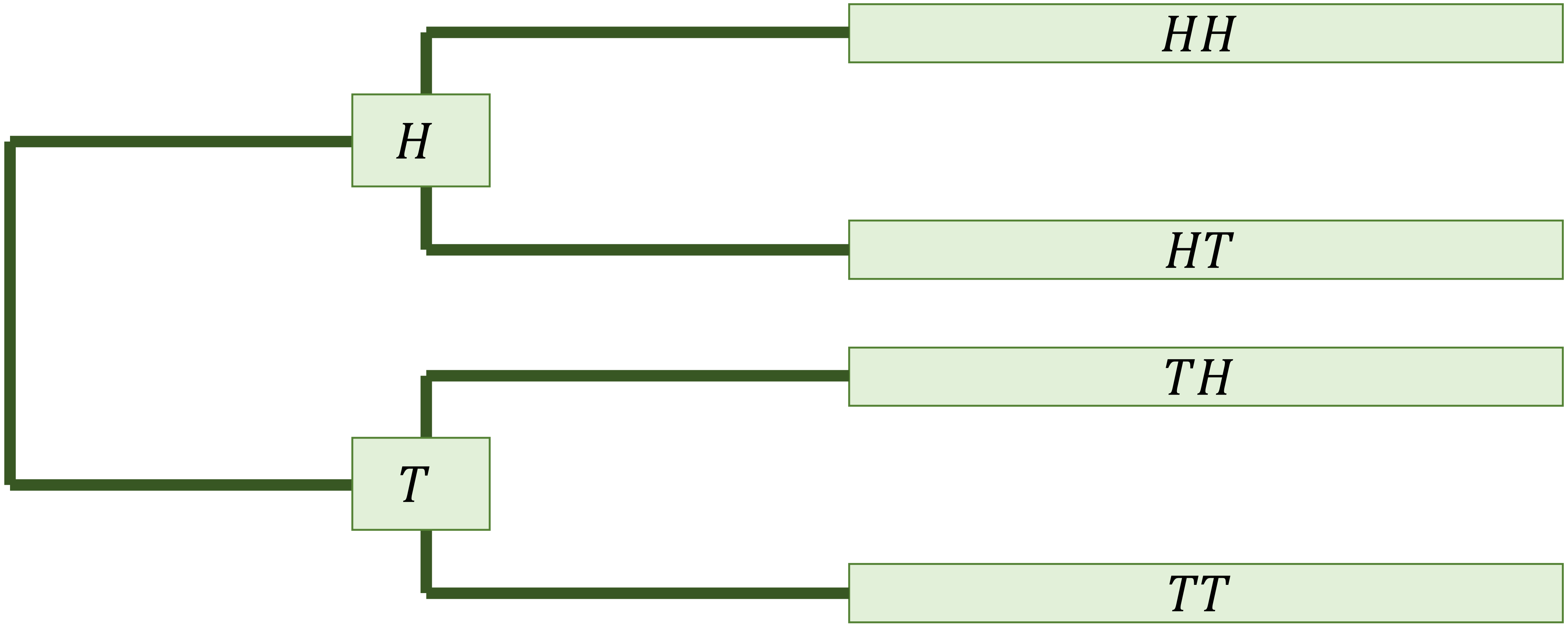
- Think of each flip as a junction in a tree.
- The first flip has two branches:

Independent Coin Flips

- If the probabilities are equal:

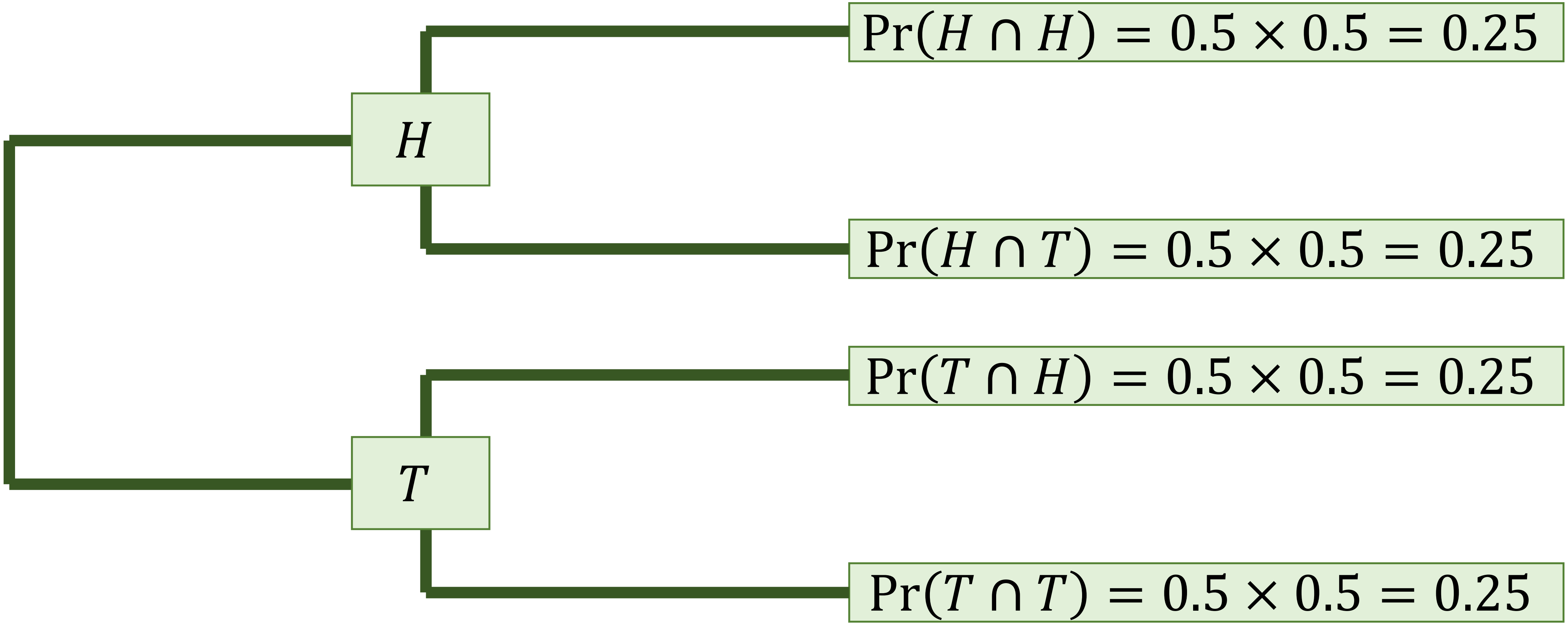
Independent Coin Flips

- Each of those branches has two branches:



Independent Coin Flips

- Probabilities of independent events are multiplied:



Binomial processes and order

Consider the sample space for two flips ($n = 2$):

$$\{(T,T),(T,H),(H,T),(H,H)\}$$

There are *four* elements… But what if we think of them as the number of heads?

$$\{0,1,1,2\}$$

- Now the size of the sample space makes more sense.
- The *sample space* is really only *three* elements because (H,T) and (T,H) are equivalent in the binomial world.

The *sample space* of a binomial is the range of possible count of successes, i.e. the set $\{0,1,3,\ldots,n\}$

Combinations and Permutations

When we consider a binomial process, we care about *combinations*…

- but we also need to know about *permutations* to characterize the sample space.
- Recall the possible outcomes for 2 coin flips: $\{(T,T),(T,H),(H,T),(H,H)\}$
    - Let's assume that $Pr(H) = Pr(T) = 0.5$.

If the flips are independent, what is $Pr(H,H)$?

- The joint probability of independent events is the produ…..
- That seems relevant, but the wording doesn't feel intuitive.
- More on combinations and permutations later

The Poisson Distribution

Another important discrete distribution is the Poisson.

It has a single parameter: λ

- The Poisson distribution describes counts:
  - A Poisson event is a count, or census.
  - The sample space is $\{0,1,2,\ldots,\infty\}$
  - It has an *infinite sample space*!

How can a discrete distribution have an infinite sample space?

- Recall that it's a *theoretical* distribution
- Compare the sample to a binomial sample space.

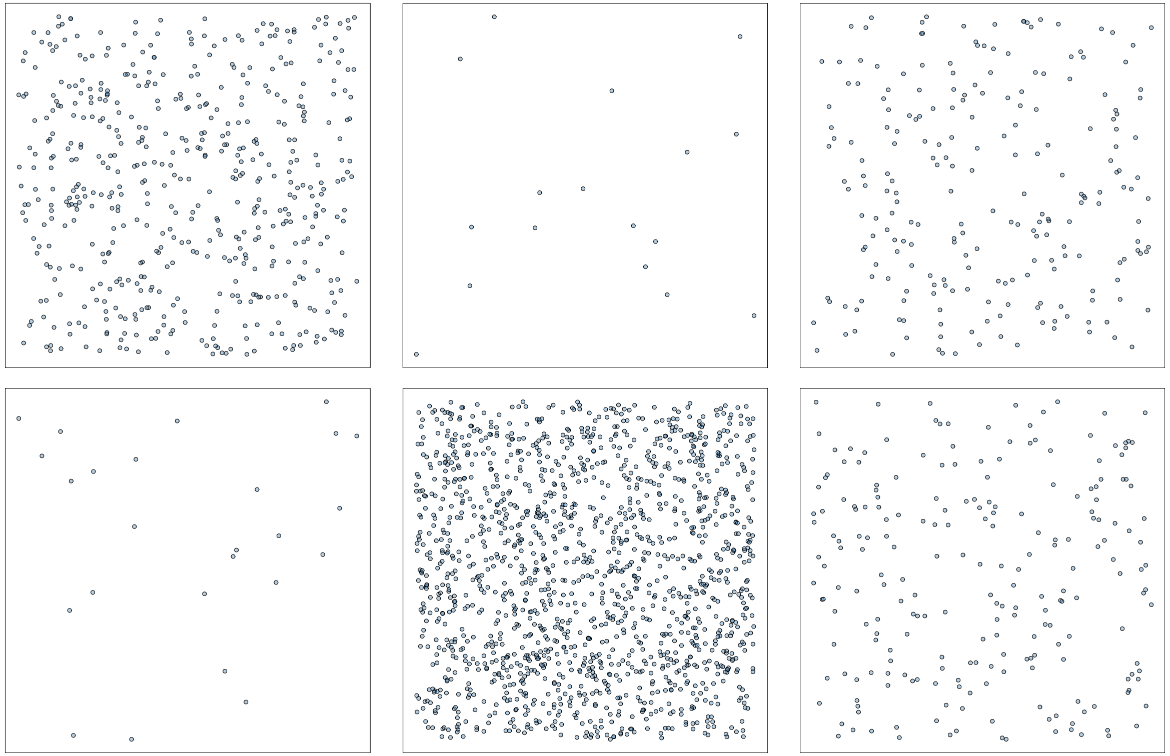
Poisson Processes

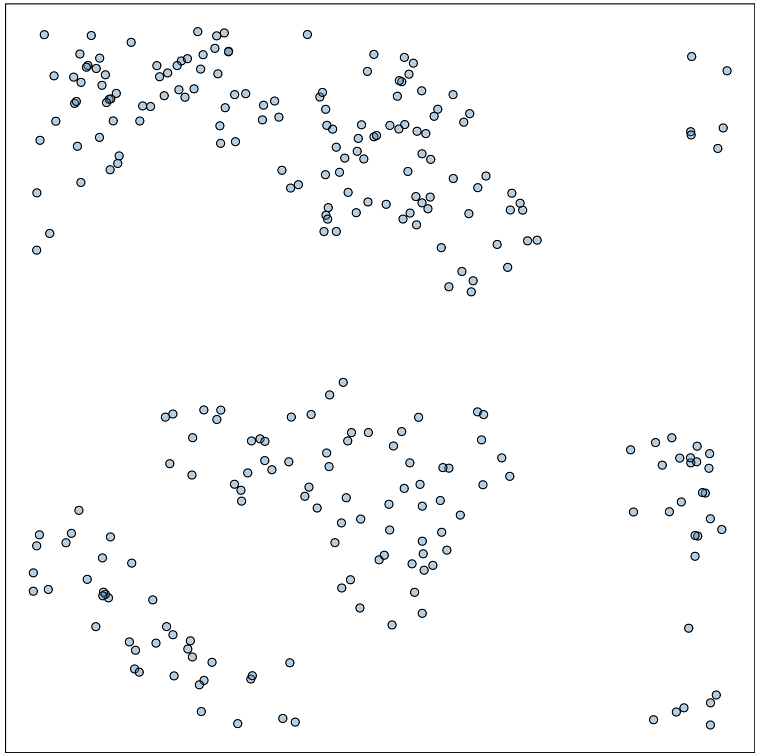
Poisson distribution is often appropriate for things that occur *randomly* but at a certain *constant rate*.

- If you could repeat a census many times (either in the same location, or simultaneously in many similar locations) you data can be modeled with a Poisson distribution.
- We'll talk more about modeling counts with the Poisson, binomial, Bernoulli, and other discrete distributions when we talk about extending the simple linear model later in the course.

The Poisson Distribution

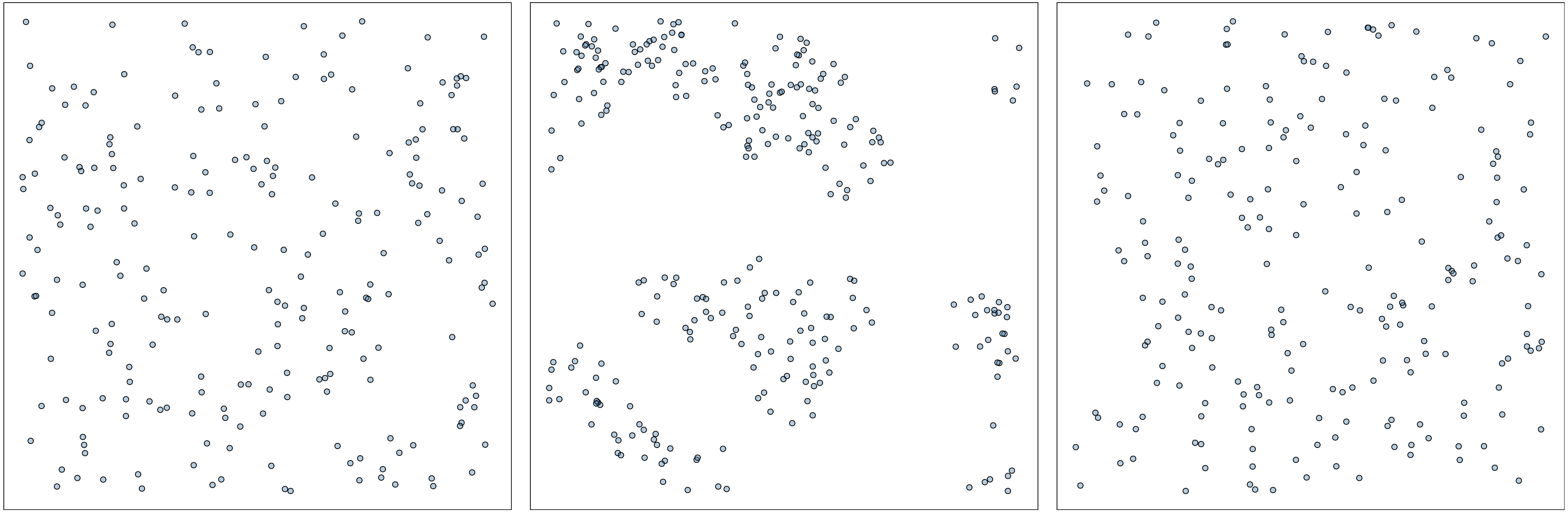
The Poisson distribution is very important *null model* in spatial statistics





The Poisson Distribution

Completely Spatially Random (CSR) point patterns follow a Poisson Distribution – It is a great model for **point processes.**



Sampling with Replacement

- The Hypergeometric Distribution models sampling with replacement:
- Binary outcomes, traditionally described using balls of 2 colors in an urn.
  - But acorns in a bag work too!
- Fixed number of trials: the total number of acorns removed from the bag.
- The number of red and brown acorns in the bag is fixed, but you might not be able to know ahead of time.
- Sampling without replacement: removal of the first red acorn changes the proportion of red and brown acorns remaining in the bag.

Sampling With and Without Replacement

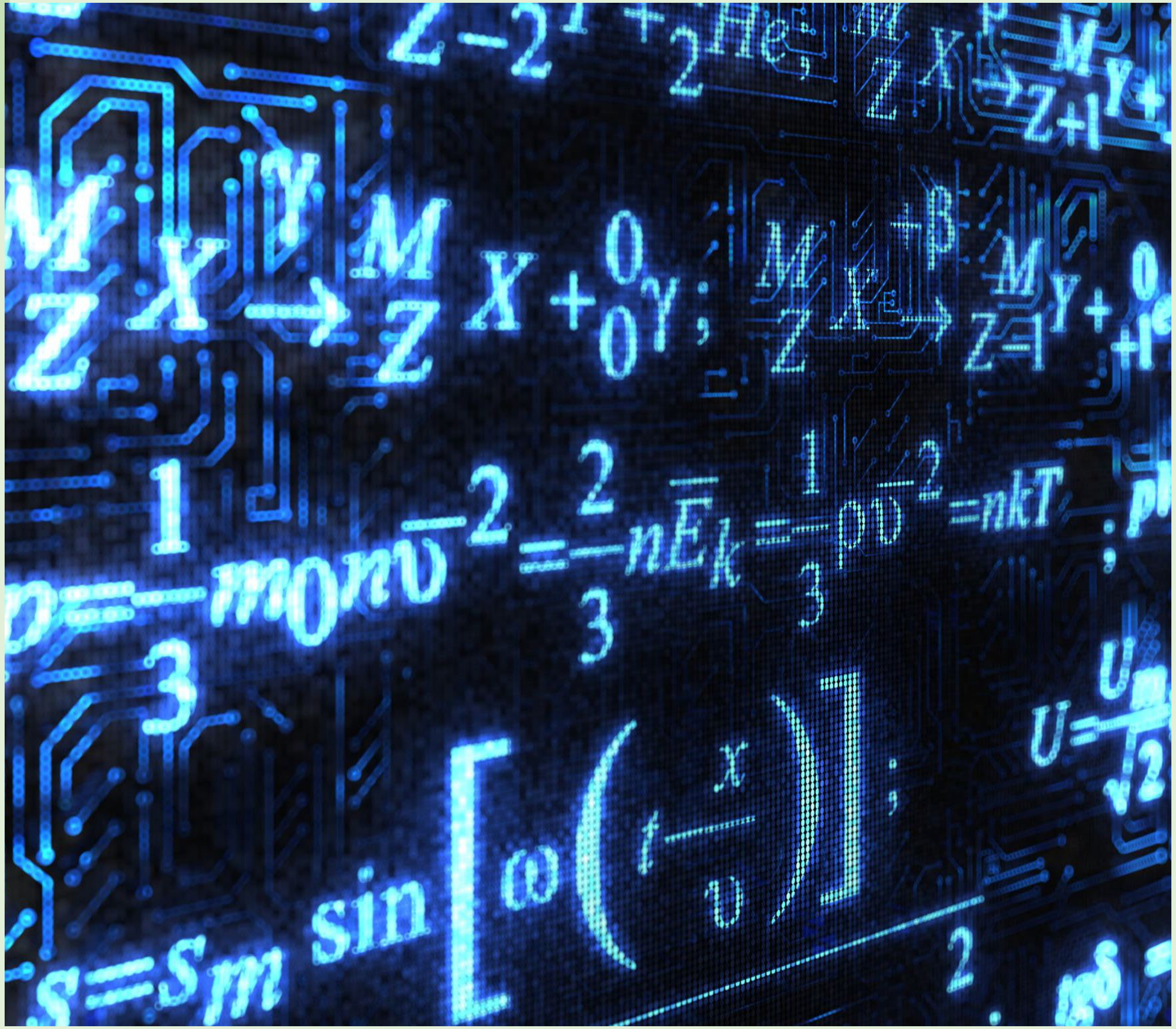
With Replacement: Binomial	Without Replacement: Hypergeometric

• Used to infer Pr(Success)
• Fixed number of trials
• Trials are independent
    • Because the acorns are replaced after each trial.
• Pr(Success) is constant for each trial
• Can be used instead of hypergeometric if N is large and Pr(success) is small.
	• Used to infer numbers of successes/failures
	• Fixed number of trials
	• Trials are non-independent
	    • You keep the acorn, changing the proportions in the remaining acorns.
	• Pr(Success) changes as you remove individuals from the sample.
	• Useful when N is small, and trials are not independent

Recap

Important concepts

- What is a discrete distribution?
- Discrete sample spaces
- Combinations and permutations
- Bernoulli, Binomial, and Poisson distributions



Combinations and Permutations

Combinatorics studies the possible ways we can *arrange* sets of objects.

- Is order important?
- do we consider (T, H) and (H, T) to be the same or different events.
- How many *categories* of objects are there?
- Sampling - to replace or not to replace?

Combinatorics is the key to understanding probability in discrete distributions.

Combinations and Permutations

Does order matter?

- If we care about order we consider (T, H) and (H, T) to be *distinct* events.
- If we do not care about order (T, H) and (H, T) are *equivalent* events.

When order is important, we work with *permutations*.

When order is unimportant, we have *combinations*.

- There are usually more permutations than combinations.

Si ektywqzuxim qldebxow*

***An interesting question. I looked for 'an interesting question' in the library, but the best I could find was three successive words with the right number of letters!**

Does the spelling of words concern *combinations* or *permutations*?

- What can we learn from the Library of Babel?

Combinations, Permutations, and Cards

- How many combinations are there for a deck of 52 cards?
  - A trick question, there is only one combination since order doesn't matter!
- How many permutations are there for a deck of 52 cards?
  - Let's calculate:
    - There are 52 possibilities for card 1
    - 51 possibilities for card 2
    - 50 for card 3
    - And so on…
    - That's 52 * 51 * 50 * … * 3 * 2 * 1
  - There are 52!, that is 52 factorial, permutations of a deck of cards. That's a huge number
- The card deck permutation question is like sampling without replacement.

Combinations, Permutations, and Four-Letter Words

- How many four-letter words are possible using the English alphabet?
  - Let's calculate:
    - There are 26 possibilities for letter 1
    - 26 for letter 2
    - 26 for letter 3, etc.
  - $26^4 = 456976$
- How many five-letter words are there?
  - By what factor does the number increase for each extra letter?
- The word question is like sampling with replacement.

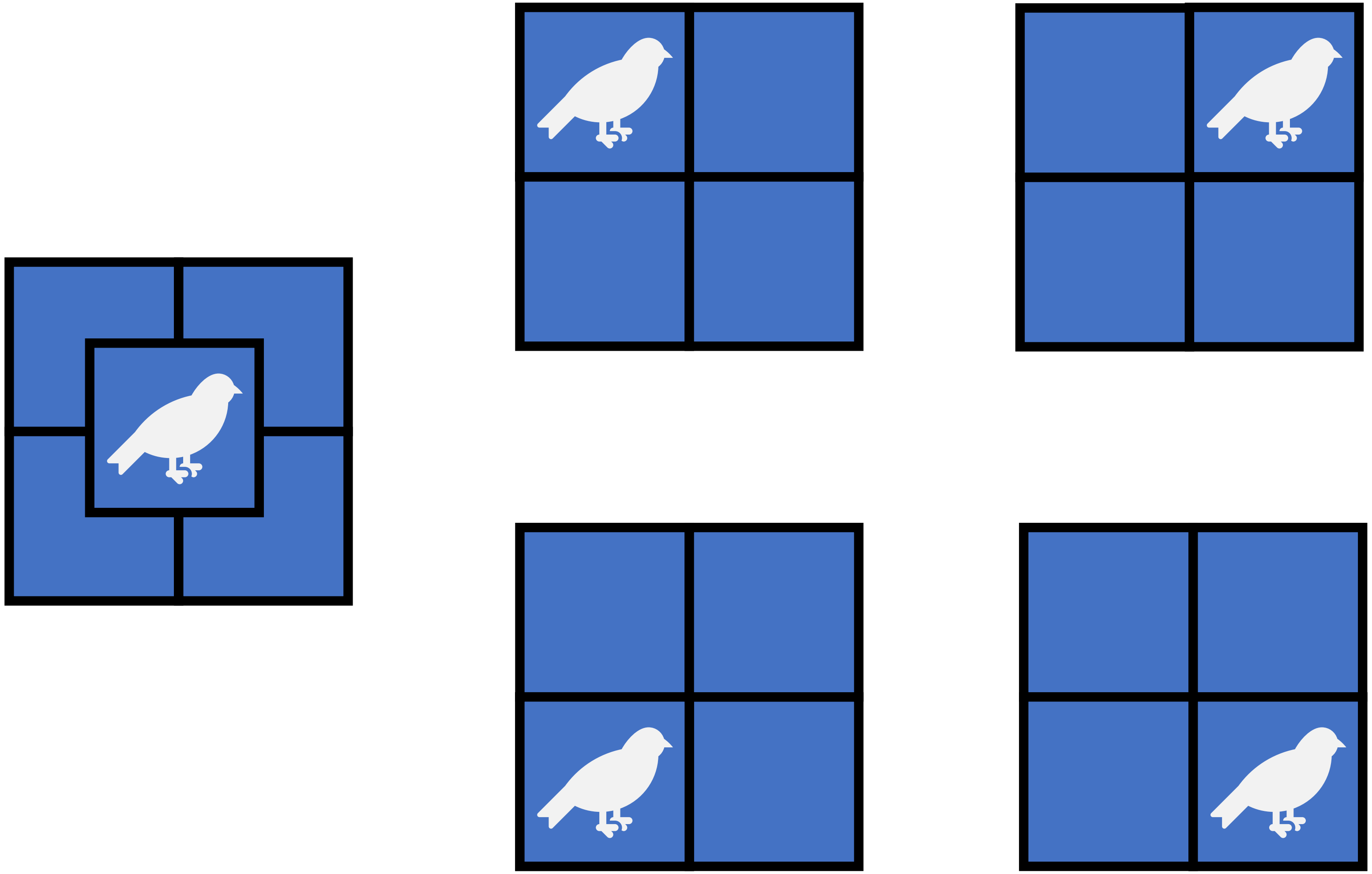
How big is 52!

- It doesn't fit in this world!

For Next Tuesday

- Make sure you've read the acorn/probability questions. We'll work on them in class using real acorns.

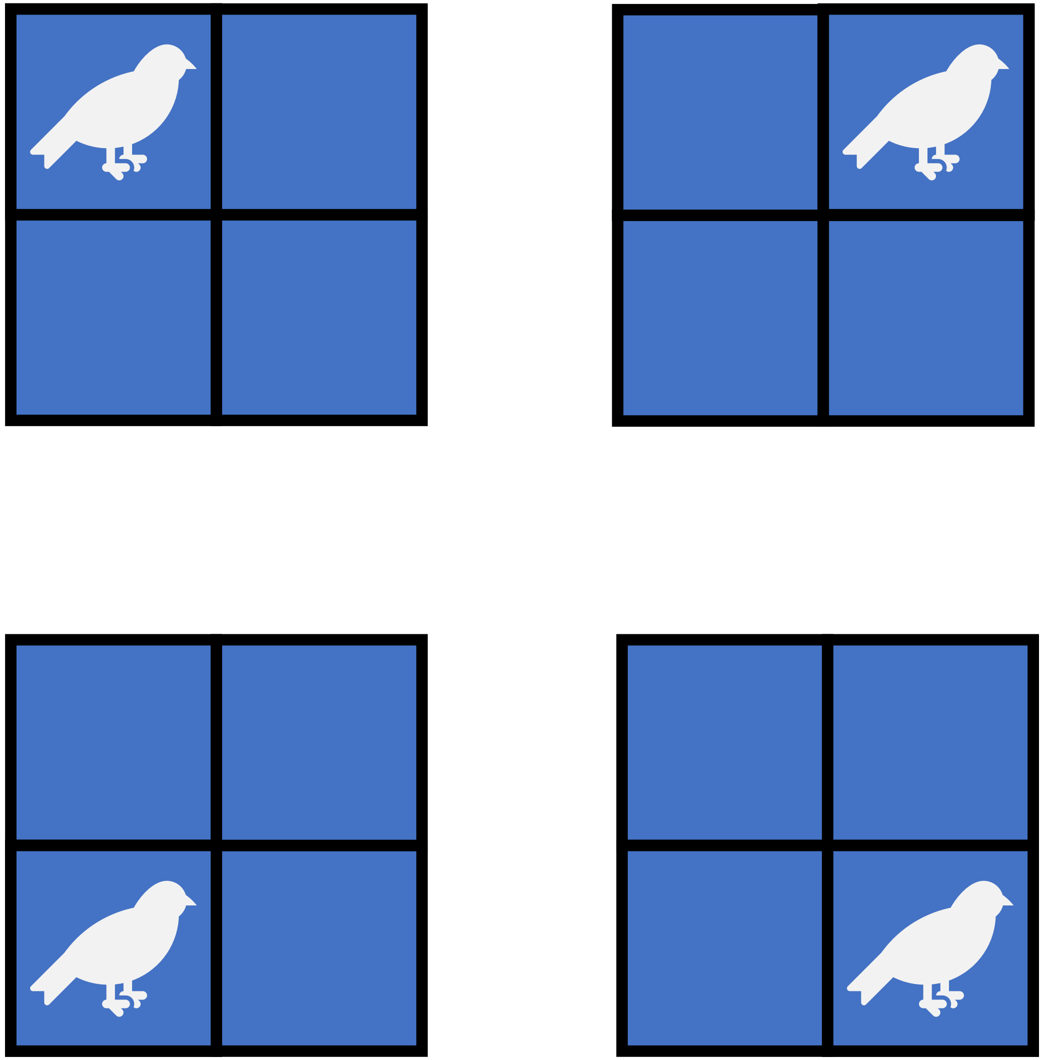
Events, Probabilities, Combinations, and Permutations



Events, Probabilities, Combinations, and Permutations

We use generally use **combinations** to enumerate **events** and populate **sample spaces** when **order** or arrangement **doesn't matter.**

We need **permutations** to figure out **probabilities.** We use permutations to figure out events and sample spaces when **order or arrangement is important.**



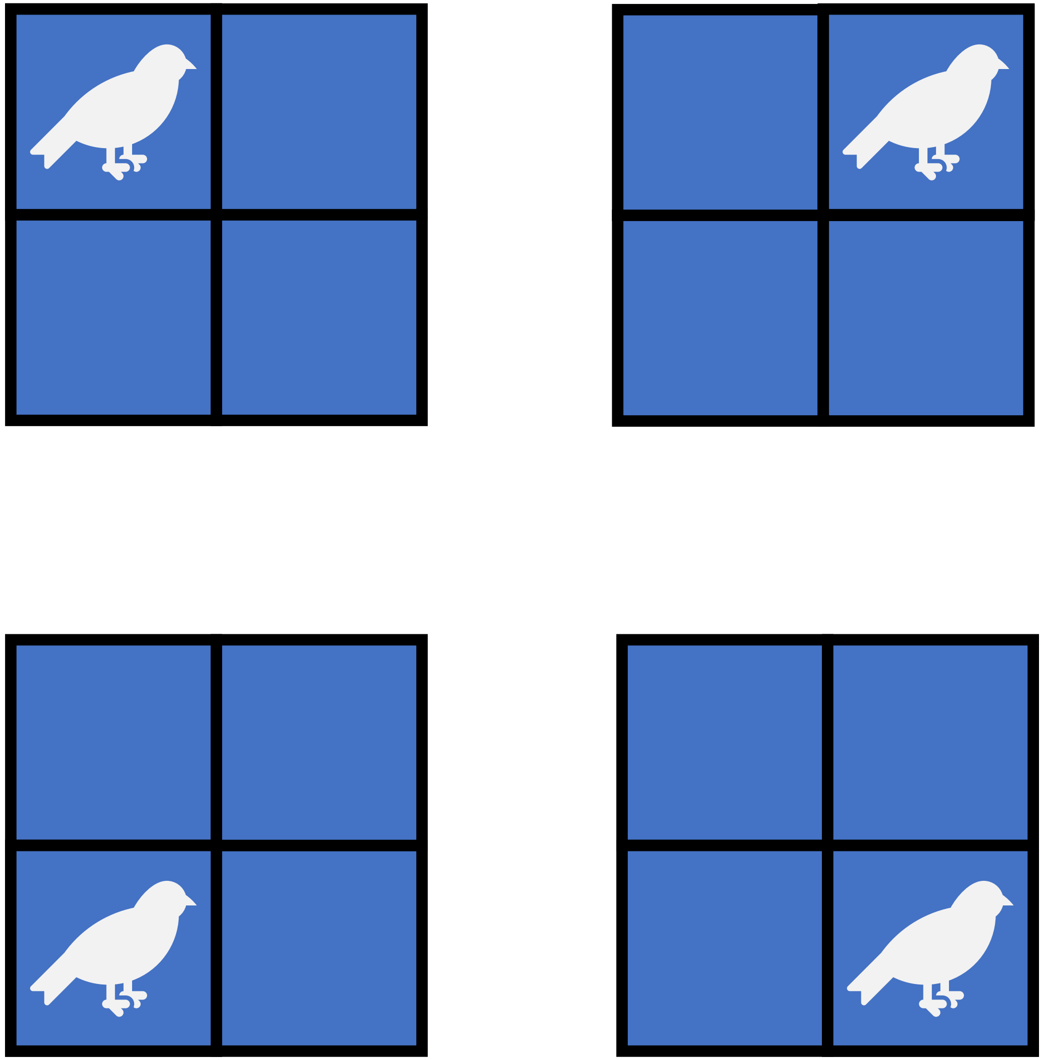
Events, Probabilities, Combinations, and Permutations

We use generally use **combinations** to enumerate **events.**

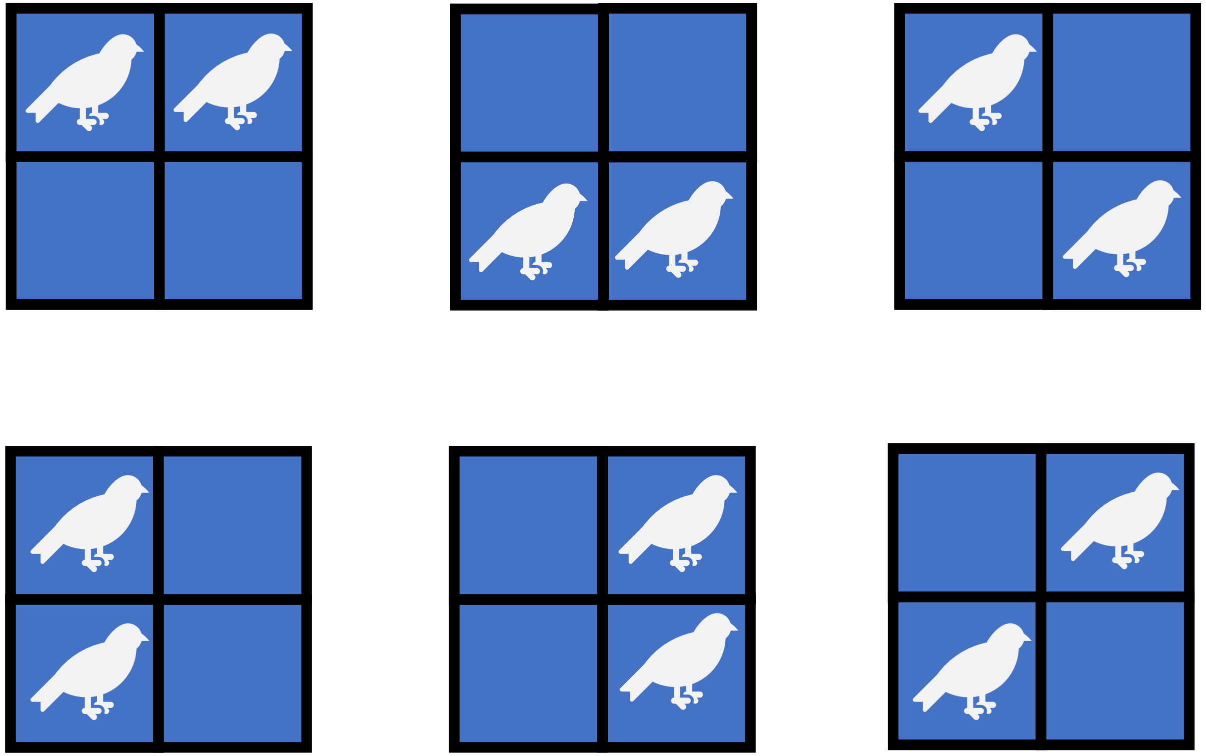
- How many sites had bird presences?

We need **permutations** to figure out **probabilities.**

- How many ways could a single site have a bird presence?

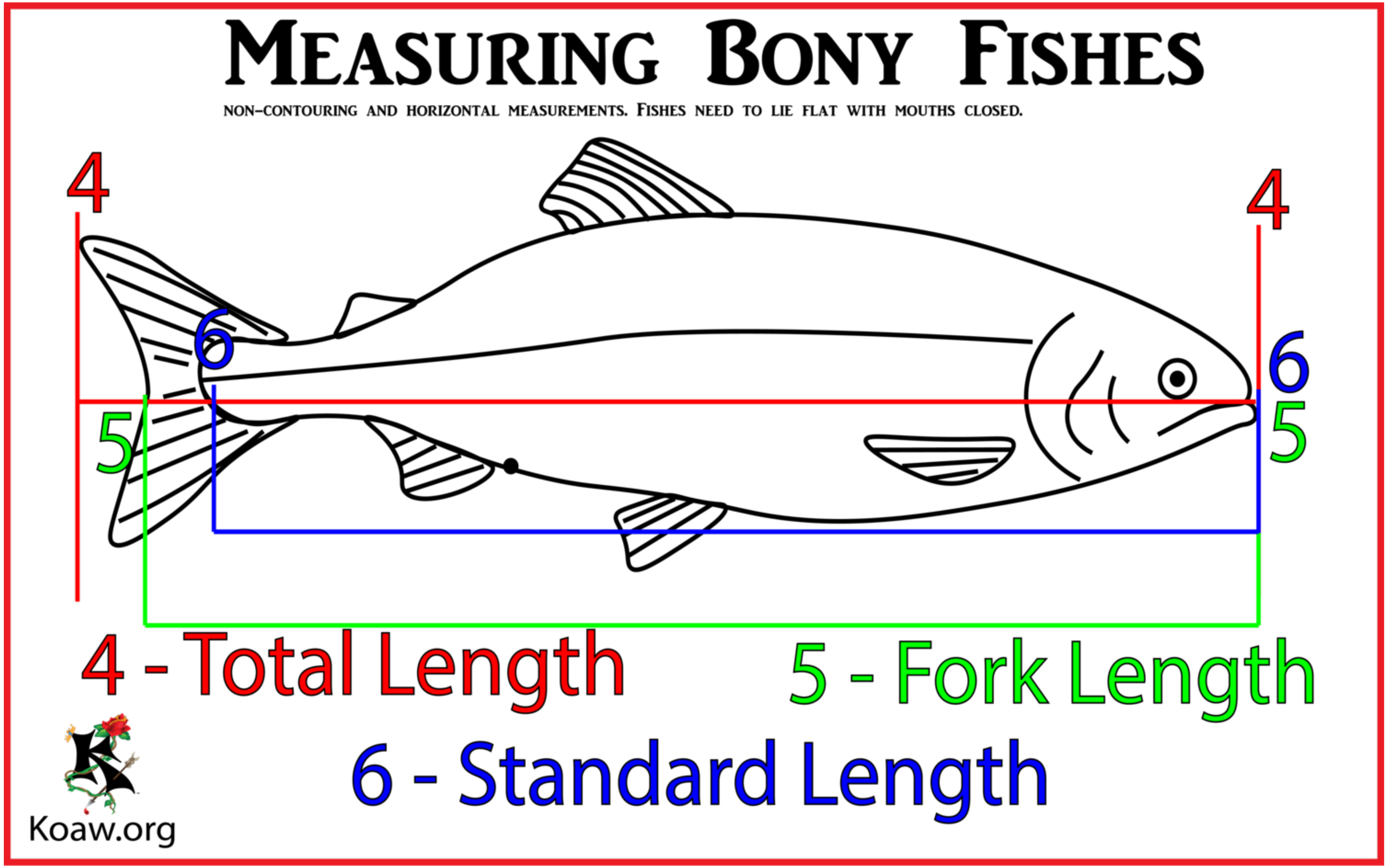


One Combination, Six Permutations

Combinations	**Permutations**
• One event: Two sites with bird presence • The event can be permuted six ways!	

Probability Distribution Functions

Probability Distribution Functions



What questions should a distribution function answer?

- Am I more likely to observe a fish that is 20cm, or a fish that is 11cm?
  - Probability Density Function: relative likelihood of x
- What is the probability that a fish is longer than 20cm?
  - Likelihood of x or smaller: cumulative Density Function
- How long is a fish in the 90th percentile?
  - Quantile Function

Probability Density/Mass Functions

Probability density or mass functions answer the questions:

- Am I more likely to catch a fish that measures 6cm or 14.5cm?
- What is the probability that I collect *exactly* two acorns of Red Oak out of a mixture of Red Oak and Bur Oak acorns?

They associate an event with a measure of likelihood

- This is the probability of the event for discrete distributions
- For continuous distributions it is more complicated, but you can think of it as relative likelihood.

Probability Density/Mass Functions

They are maps of events in the sample space to probabilities.

- **Probability Density Functions** for continuous distributions
- **Probability Mass Functions** for discrete distributions.

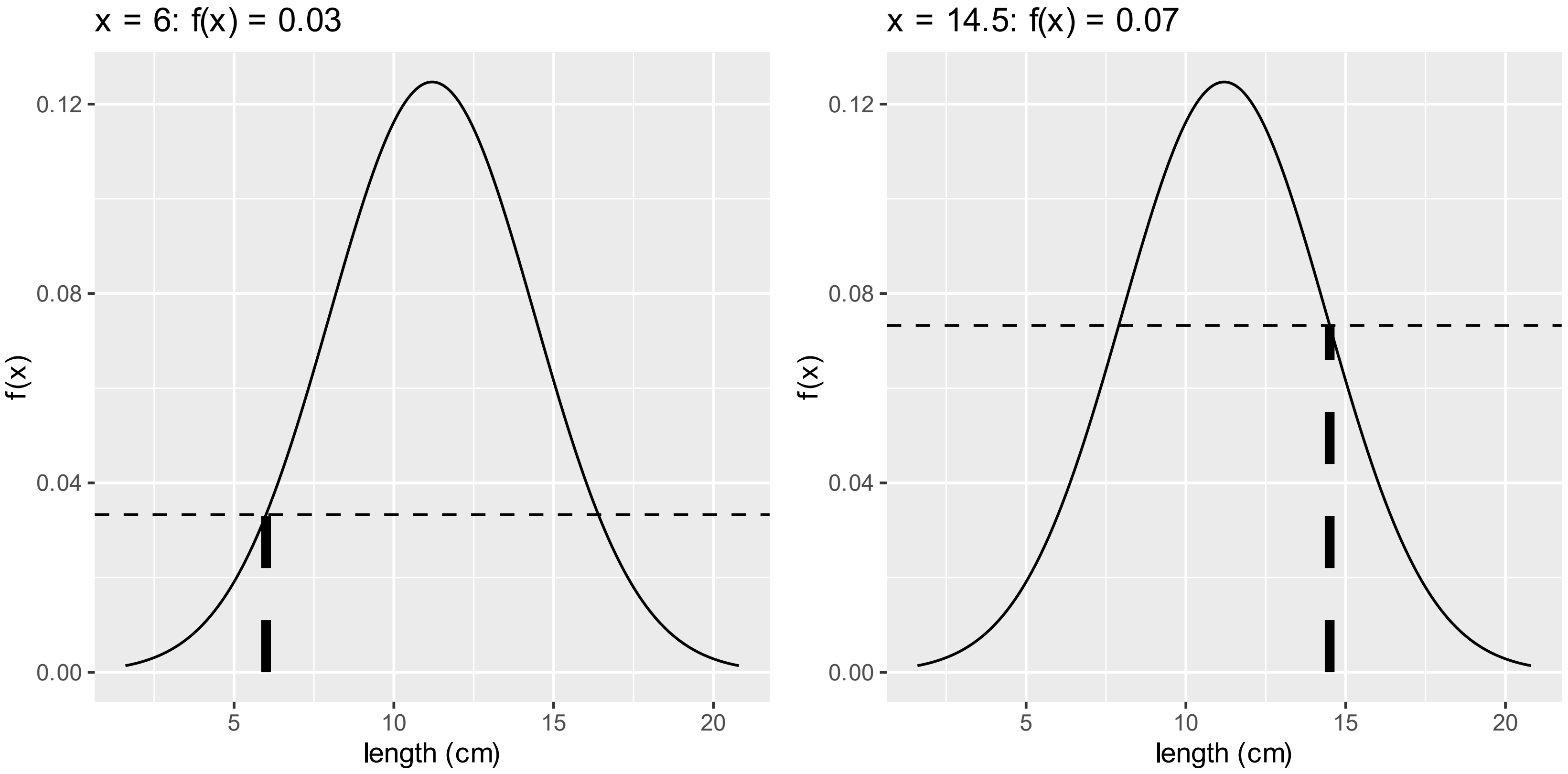
Probabilities are always between zero and 1:

- The values of PDFs and PMFs are always non-negative.

Probability Density Function

PDFs tell us about relative likelihoods

Am I more likely to catch a fish that measures 6cm or 14.5cm?



Cumulative Probability Functions: CDFs & CMFs

The CDF/CMF answers:

- What is the probability that I catch a fish that weighs 153g or less?
- What is the probability that *at least* 3 of the acorns are Bur oak?

Cumulative density is the **accumulated area under the density curve** to the left of x.

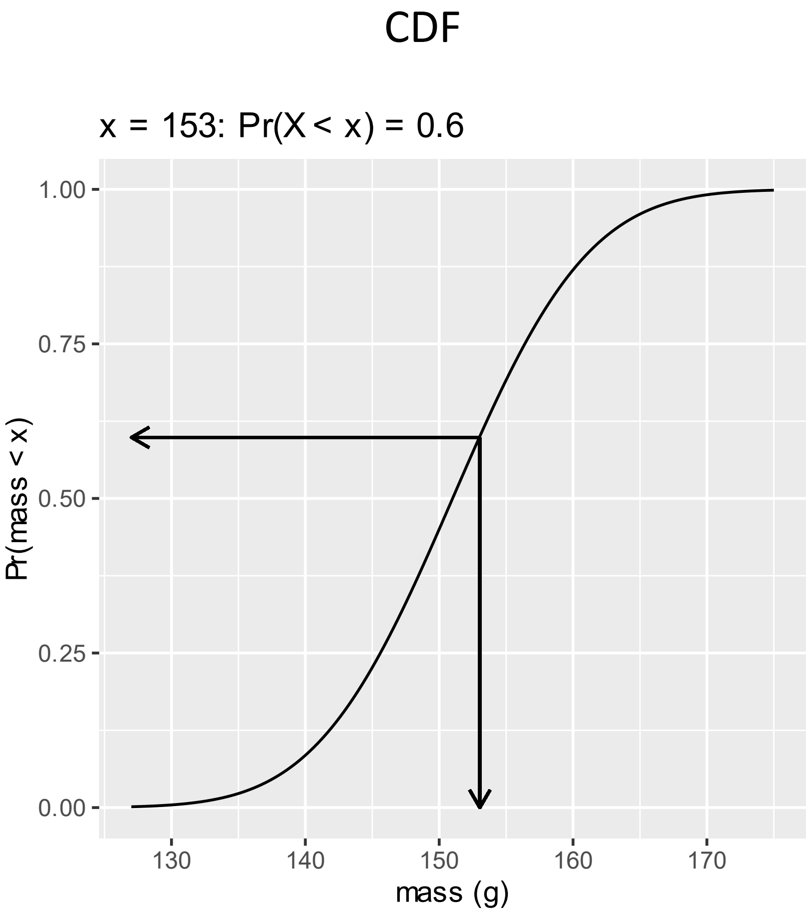
- It's an integral (or a sum for discrete distributions).
- It is the probability of observing a value equal to or less than x.
- The nth percentile (quantile).

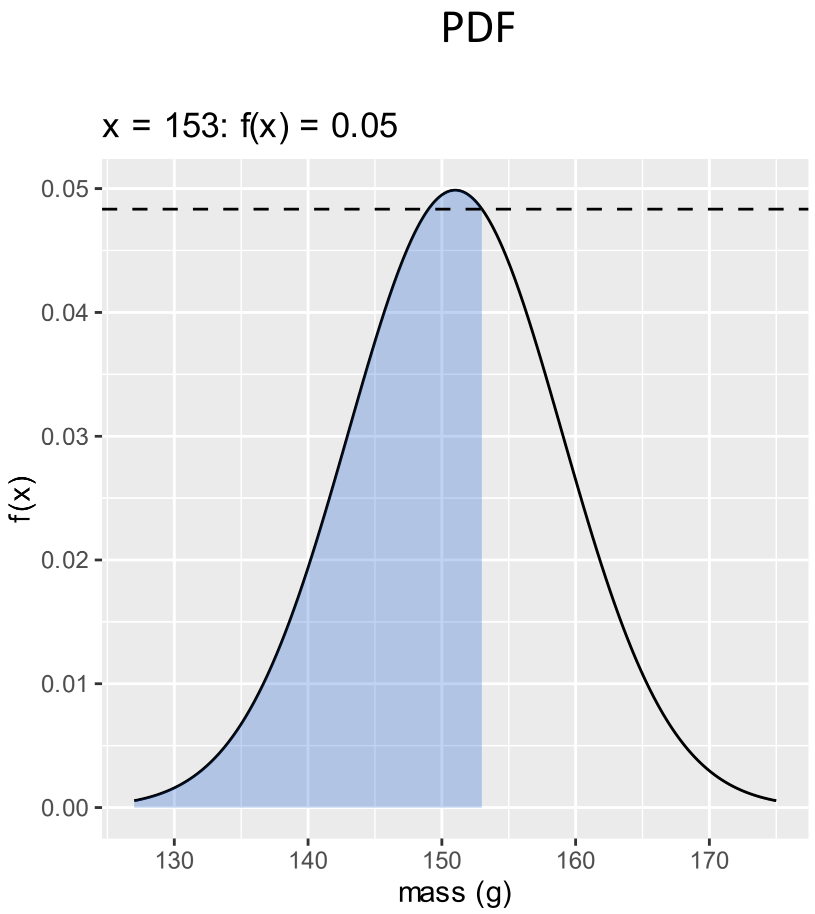
Cumulative Probability Functions: CDFs & CMFs

CDFs tell us the probability of an event:

What is the probability that I catch a fish that weighs 153g or less?

- **Read the mass on the x-axis: 153g.**
- **Read the corresponding probability on the left: 60%**



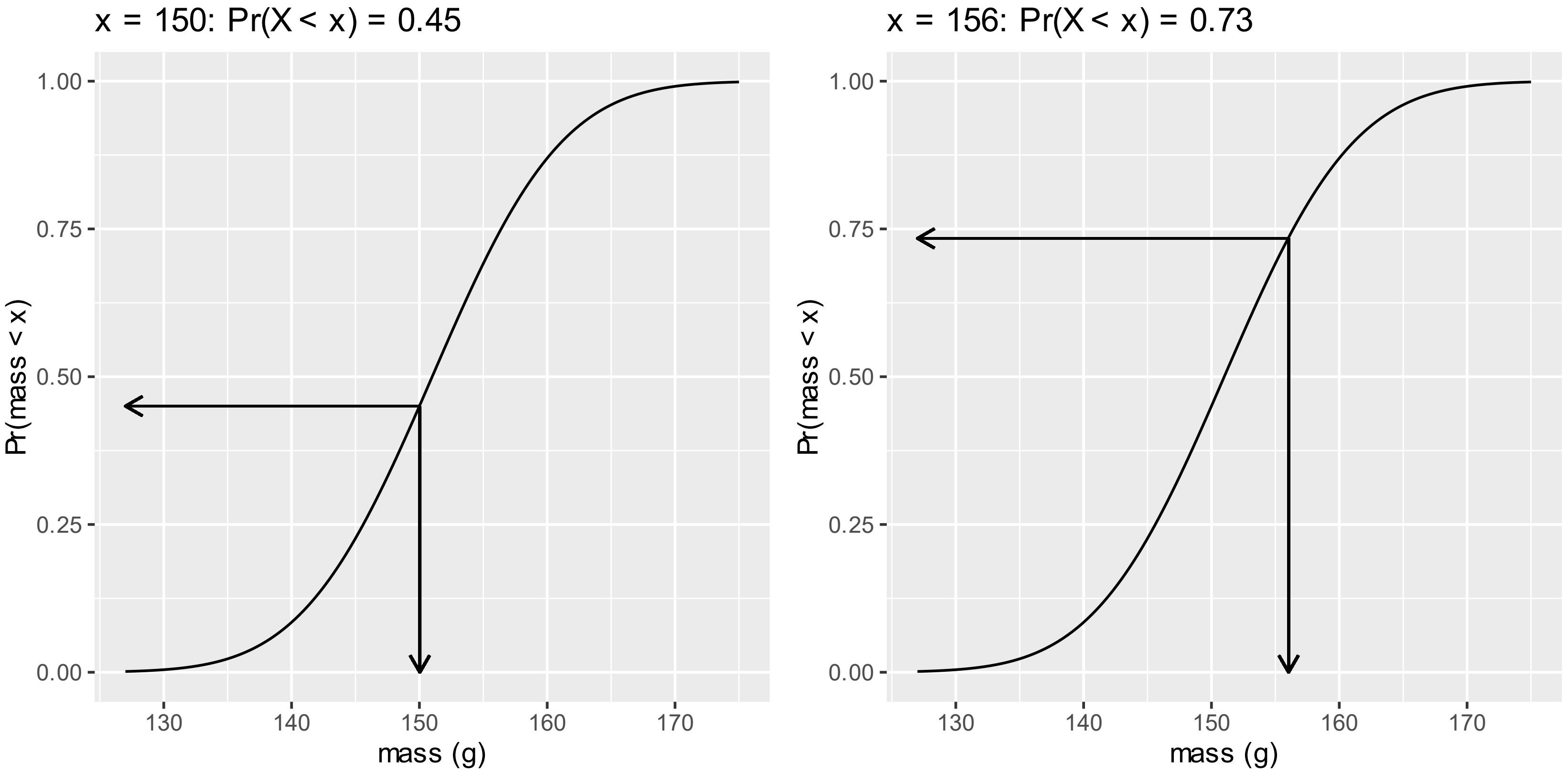


Cumulative Probability Functions: CDFs & CMFs

CDFs tell us the probability of an event:

What is the probability that I catch a fish that weighs between 150g and 156g?

- **Take the difference of probabilities from the CDF: 0.73 – 0.45 = 28%**

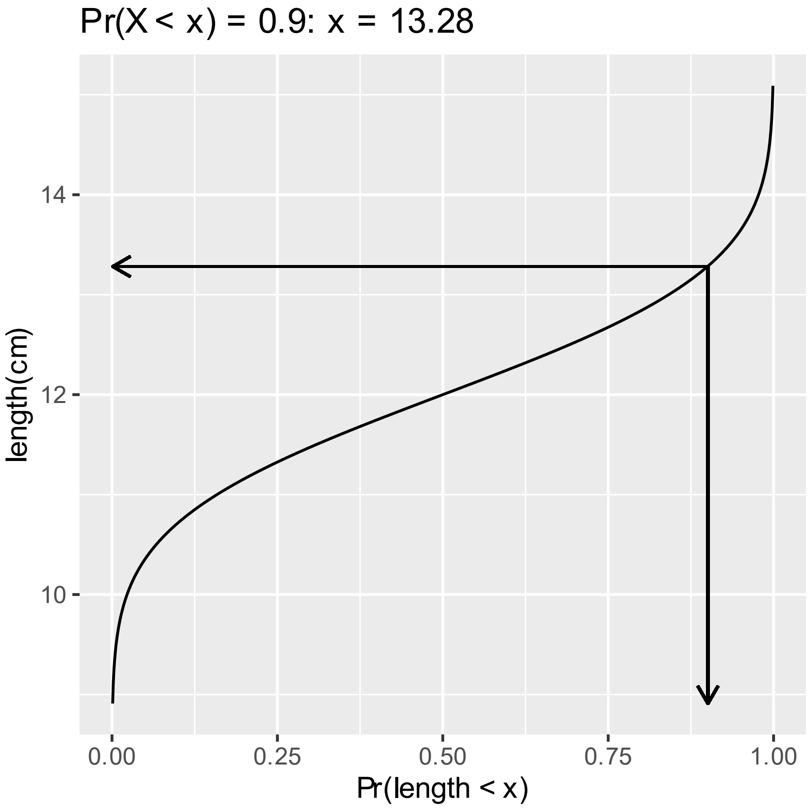


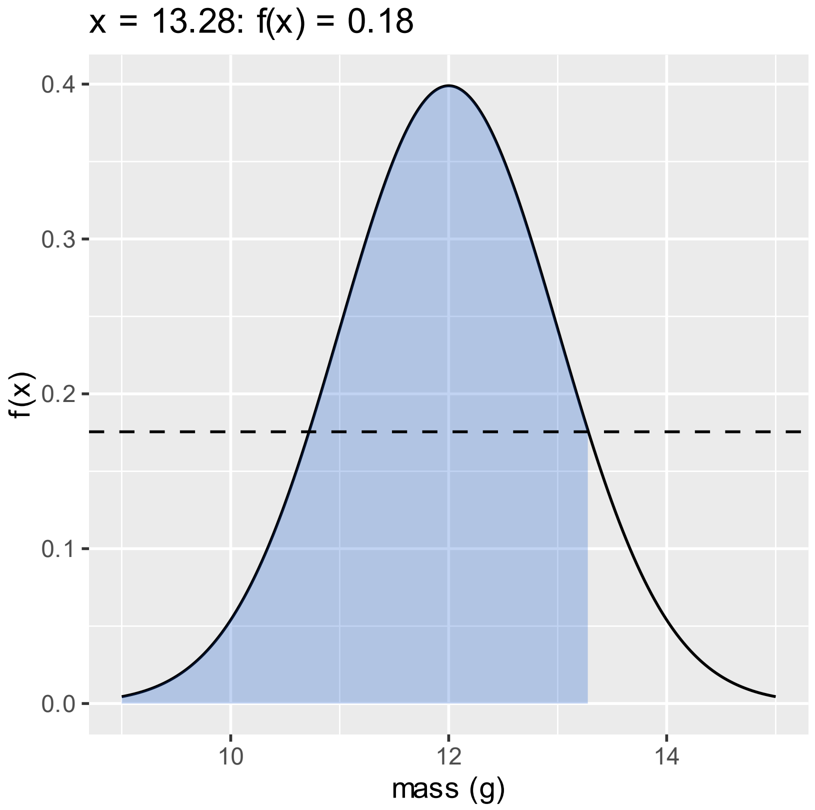
Quantile Functions

Quantile functions tell us about percentiles:

What length will 90% of all fishes will be shorter than?

- **Read the percentile on the x-axis.**
- **Read the size on the y-axis.**



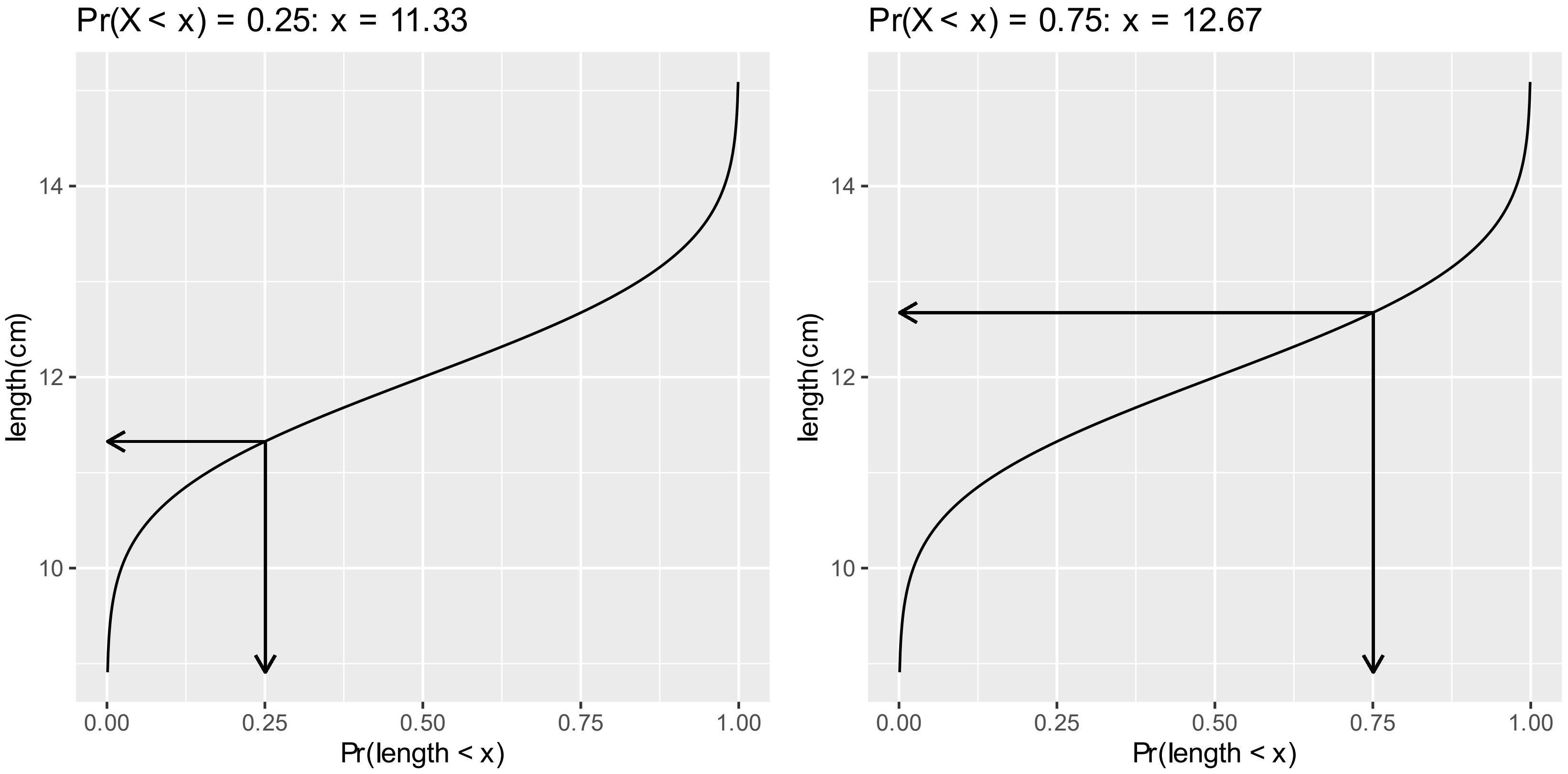


Quantile Functions

Quantile functions tell us about percentiles:

What lengths span the middle 50% of the range?

- **Read the percentiles on the x-axis.**
- **Read the sizes on the y-axis: 11.3cm – 12.7cm**



Parametric (Theoretical) Distributions

Parametric distributions are defined by mathematical *functions*

- The functions have one or more *parameters* that define how probabilities are allocated to events.
- We often want to estimate the parameters from samples.

The *binomial distribution* has two parameters: n, p.

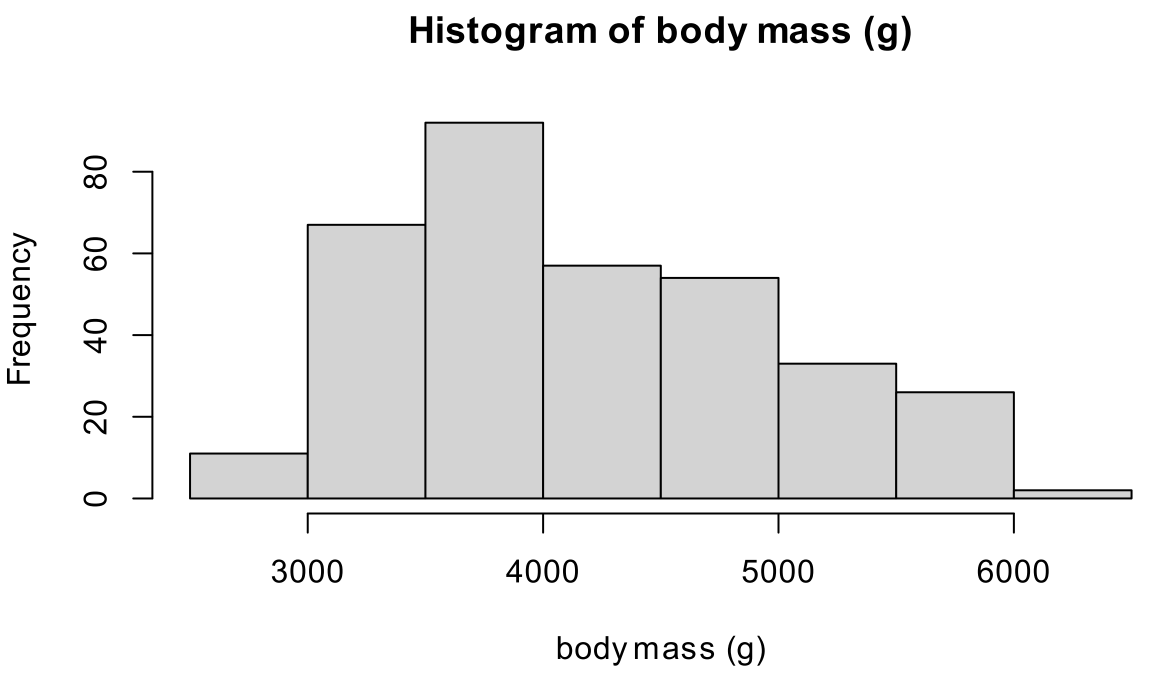
The *Poisson* distribution has only one parameter: λ

Empirical Distributions

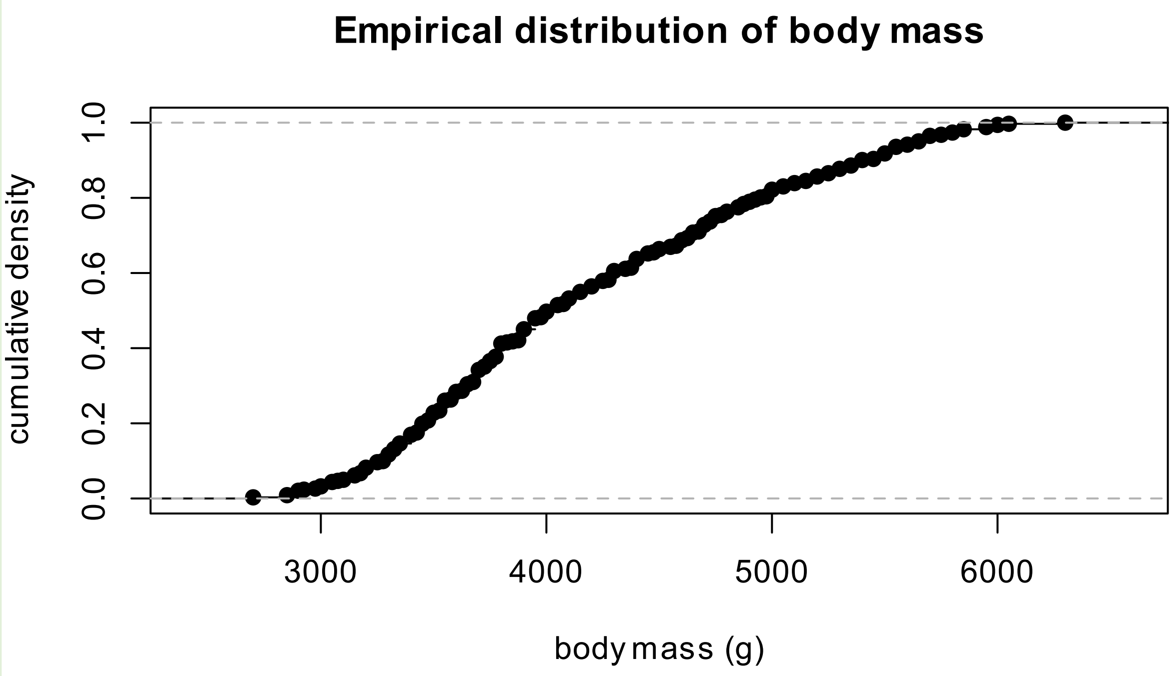
Empirical distributions are computed from *observations*.

- **There is no analytical function: the shape is computed from data.**
- **We can compare empirical distributions to parametric distributions.**

Histograms are analogous to a PDF/PMF



Empirical cumulative distribution function are analogous to the CDF/CMF



Recap

- Theoretical and empirical distributions
- Parameters
- Distribution functions
  - Probability Density/Mass
  - Cumulative Density/Mass
  - Quantile Functions

Key concepts

- Continuous sample spaces
- Normal distribution
- Exponential distribution
- T distributions

Continuous Sample Space

Continuous distributions' sample space: the *real* numbers

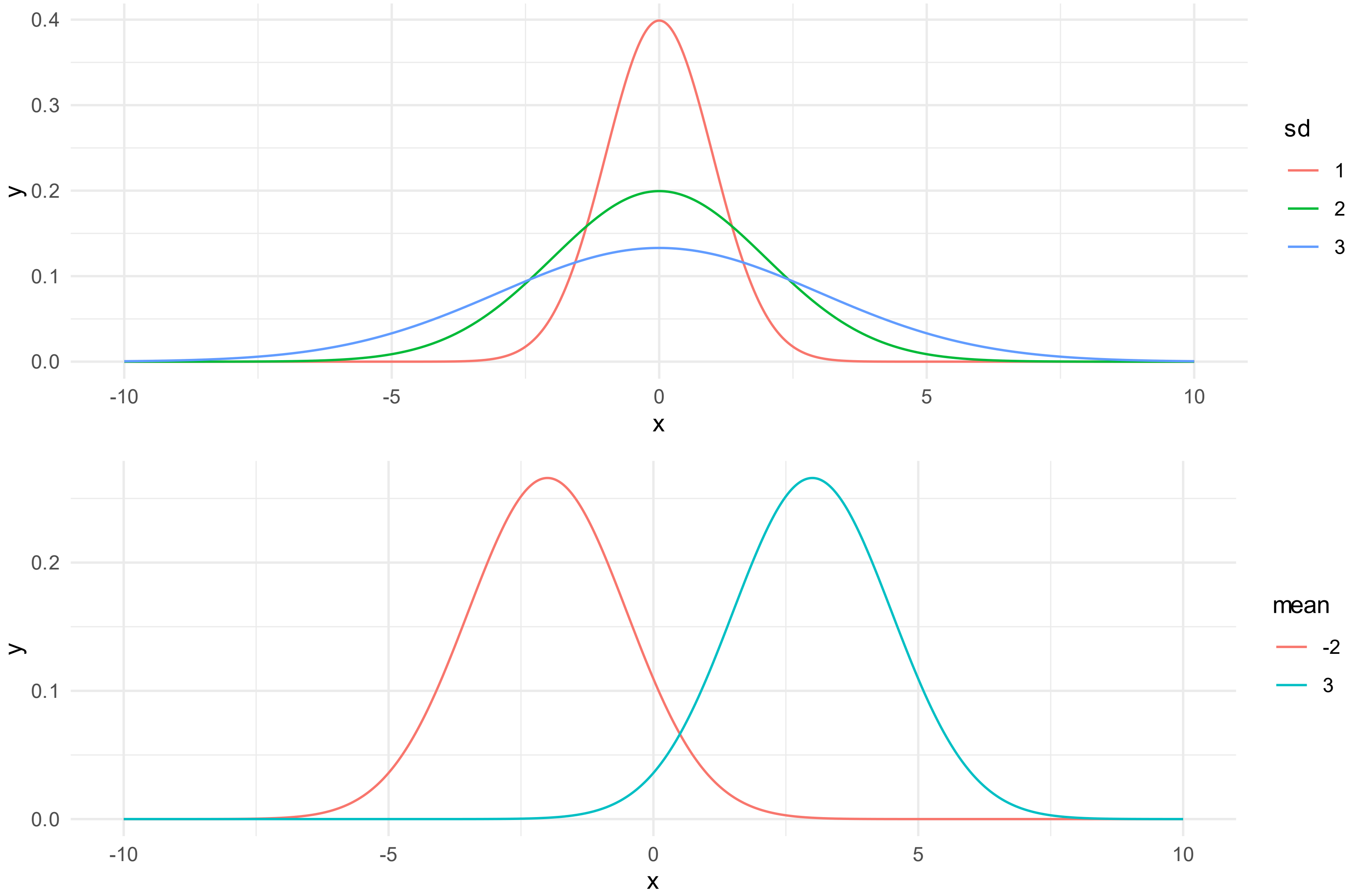
- All continuous distributions have *infinite* sample spaces.
- Continuous distributions may have *bounded* or *unbounded* sample spaces.

PDFs are *continuous functions*.

Normal Distribution

The normal distribution has 2 parameters: μ and σ

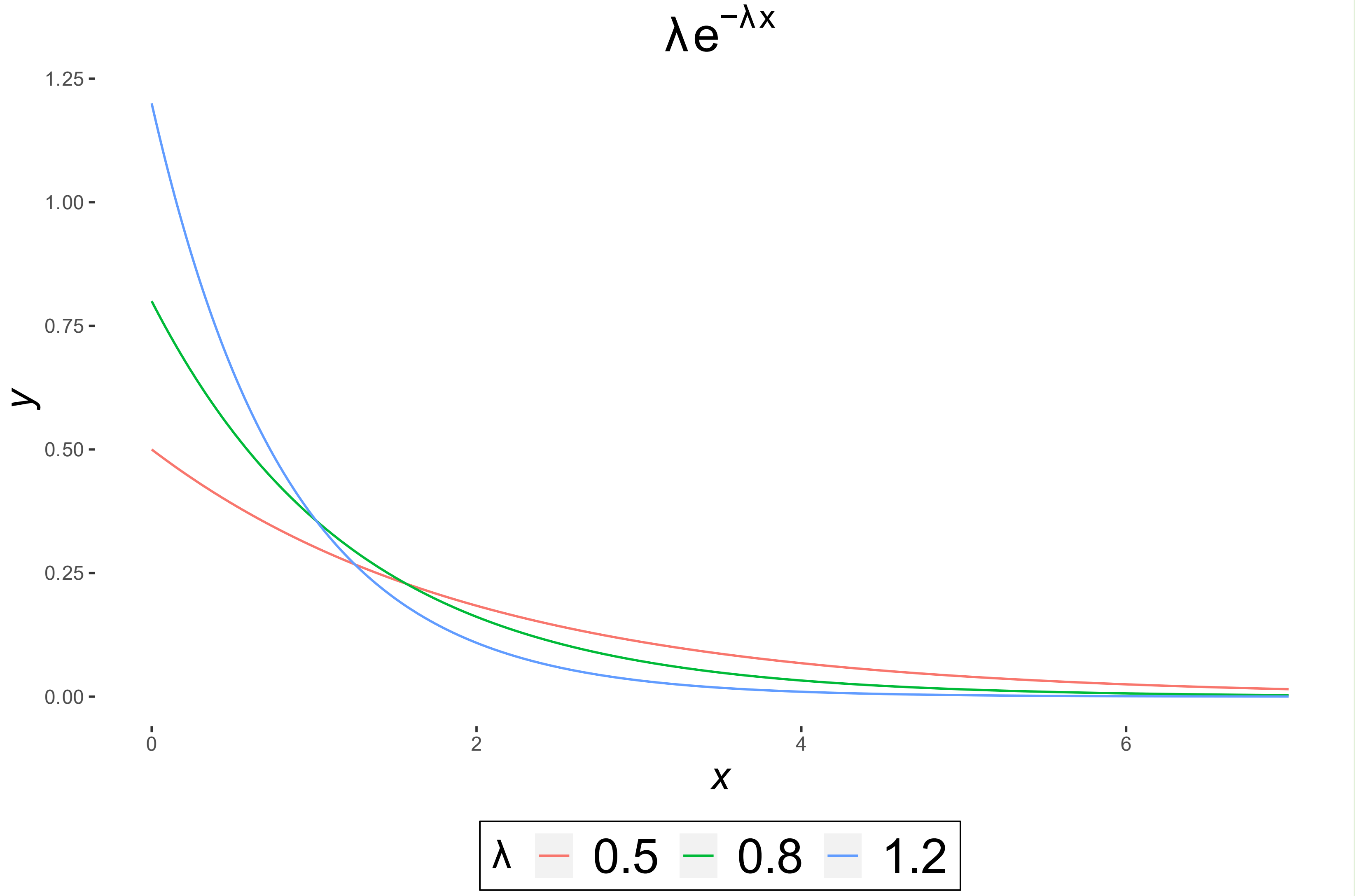
- The Standard Normal distribution has $\mu = 0$ and $\sigma = 1$
- The mean, μ moves the curve left or right.
- The standard deviation σ controls the width.



Exponential Distribution

Exponential distribution models exponential decay.

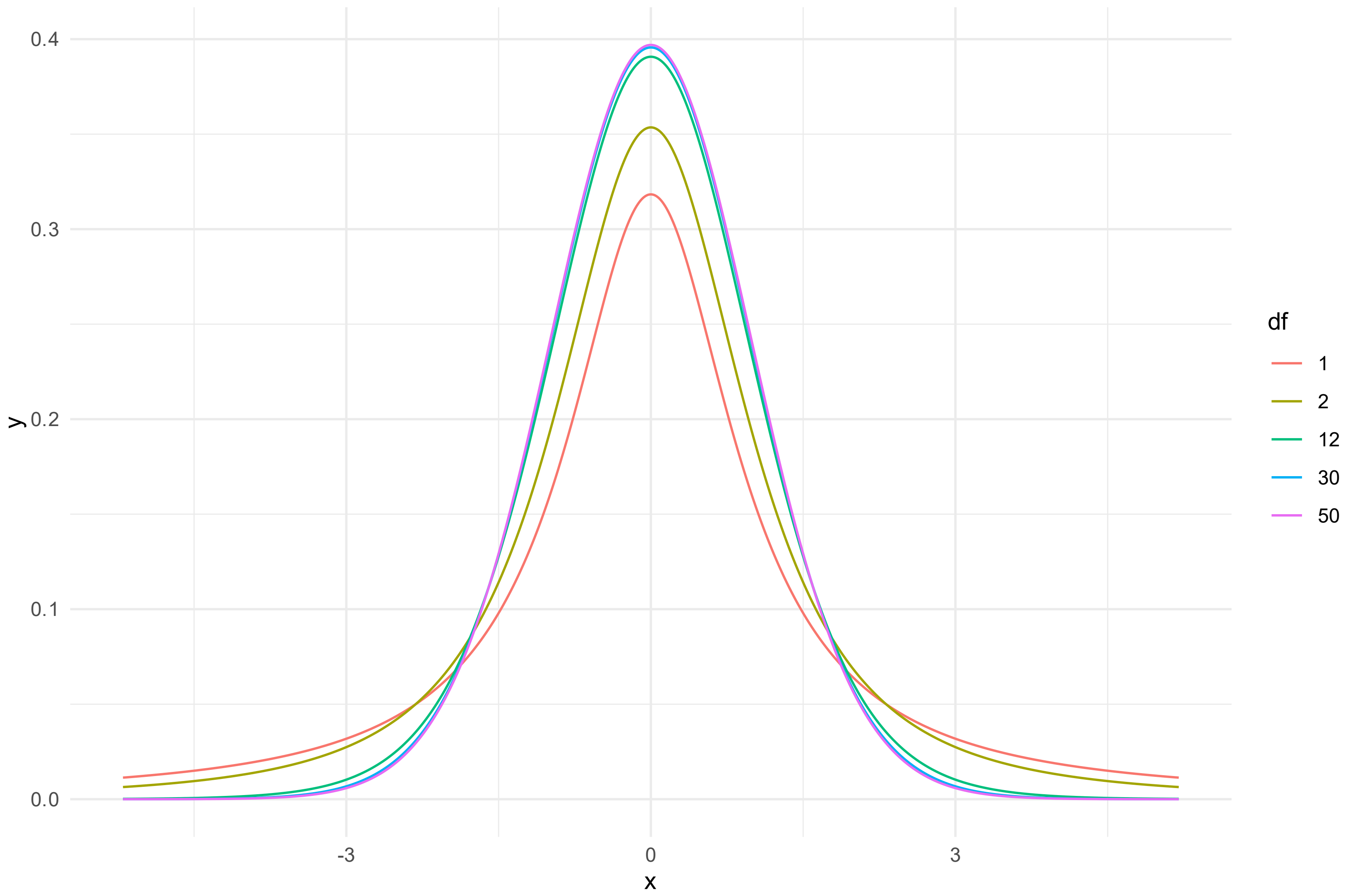
- Small observations are common, large observations are rare.



The T Distribution

The t-distributions are like a sample-size adjusted version of the standard Normal

- The adjustment is via the *degrees of freedom* parameter
- As $df \rightarrow \infty$ the t-distribution approaches the standard Normal



Skew and Kurtosis

Skew and Kurtosis are *higher-order* moments of distributions

- Mean is the 1st moment, variance is the 2nd moment
- Skew is a measure of asymmetry
- kurtosis is a measure of *pointiness*
    - Platykurtotic: flat with short tails, extreme events are less common.
    - Leptokurtotic: pointy with long tails, extreme events are more common.
- Skew and kurtosis are measured in reference to a normal distribution

Kurtosis

- **Platykurtotic = too flat**
- **Leptokurtotic = too pointy**

Key concepts

- Continuous sample spaces
- Normal distribution
- Exponential distribution
- T distributions