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Discrete Constant-Velocity-Equivalent Multirate Models for Target Tracking

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Abstract—A set of constant-velocity-equivalent multirate (MR) models are derived in this paper, which can be used in multirate signal estimation and decomposition. The connections between the conventional constant-velocity model and MR models are established, which can be used in performance comparison. An overall improvement in estimation quality using MR models is shown by Monte Carlo simulations. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Signal estimation and decomposition has found many applications in estimation theory, communication, and signal processing. In signal processing applications, for example, it is often expected that an unknown (deterministic or random) signal can be estimated from its measurement data and, meanwhile, be decomposed according to its resolution (frequency components at different levels). For this purpose, optimal estimation-decomposition techniques are needed and efficient computational algorithms are required. In random signal estimation and decomposition, a general approach is first to estimate the unknown signal using its measurement data and then to decompose the estimated signal according to the resolution requirement. Such a two-step approach is off-line in nature and is often not desirable for real-time applications. In [1], a technique for simultaneous signal estimation and decomposition was proposed, using a multiresolutional filtering approach [2–5]. The technique was developed based on the standard Kalman filtering scheme, which preserved the merits of the Kalman filter in the sense that it produced an optimal (linear, unbiased, and minimum error-variance) estimate of the unknown signal [6].

It is well known that the Kalman filter is a model-based optimal filter, which requires exact knowledge of process and measurement models as well as process and measurement noise statistics (first and second moments). However, for some applications, such as target tracking [7–9], the exact knowledge of the process model is difficult (if not impossible) to derive. For these kinds of applications, the process model is derived based on the first principle of common physical laws. Constant-velocity is a commonly-used assumption to derive a process model, which will be briefly reviewed.

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When applied in the temporal domain, the estimation and decomposition technique proposed in [1] becomes a multirate estimation and decomposition technique. This paper derives a set of multirate (MR) constant-velocity-equivalent models which can be used for multirate signal estimation and decomposition when exact models are not available and the constant-velocity assumption is valid.

The main application of the models derived in this paper will be in target tracking. This paper is related to reference [1] in the sense that both are performing multirate filtering. However, in [1] the multirate models are derived from an any given full-rate model, therefore, the dimensionality of the multirate model in [1] is higher. This paper takes advantage of the assumption on the Constant Velocity (CV) and derives lower-rate models at the *same* dimension of the CV model. When tracking nonmaneuvering targets, it is a waste of time and computational resource to update the tracks at a full-rate. Therefore, the models provided by the paper enjoy significant computational savings, yet better performance for tracking a nonmaneuvering target. The full strength of the models will be shown when they are applied to multirate multiple model target tracking, where both maneuvering and nonmaneuvering targets are tracked. The multirate CV models will be used as low-rate monmaneuvering models

The rest of the paper is organized as follows. Section 2 briefly reviews a discrete constant-velocity model. Section 3 derives a full-rate MR model and Section 4 presents a half-rate MR model. A special case is studied in Section 5, and a special case of a quarter-rate MR model is derived in Section 6. An example is given in Section 7, and Section 8 concludes the paper.

2. A DISCRETE CONSTANT-VELOCITY MODEL

For the *uniform rectilinear motion*, in which the velocity of the object is constant, we have the following process model for the object

$$\underline{x}(k+1) = \underline{x}(k) + \underline{v}T \tag{1}$$

$$= \underline{x}(k) + \underline{\dot{x}}T, \tag{2}$$

and

$$\underline{\ddot{x}} = 0, \tag{3}$$

where T is a sampling period. However, for real-world applications, the perfect constant-velocity assumption is unrealistic. A relaxation is introduced by allowing *some variation* of the velocity and this variation is described by a *piecewise constant white acceleration.* Therefore, a more realistic process model is given by

$$\underline{x}(k+1) = \underline{x}(k) + \underline{\dot{x}}(k)T + \frac{1}{2}\underline{\ddot{x}}(k)T^2, \tag{4}$$

where the piecewise constant white acceleration is described by a zero-mean Gaussian white noise

$$\underline{\ddot{x}}(k) \sim N\left(0, \sigma^2(k)\right),$$

where $\sigma(\mathbf{k}) = \mathbf{I}\sigma(k)$ and $\mathbf{I}$ is an identity matrix with an appropriate dimension. The degree of relaxation of the constant-velocity assumption is controlled by $\sigma(k)$. Putting equation (4) into a vector format, we have

$$\begin{bmatrix} \underline{x}(k+1) \\ \underline{\dot{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{T} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \underline{x}(k) \\ \underline{\dot{x}}(k) \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\mathbf{T}^2 \\ \mathbf{T} \end{bmatrix} \underline{\ddot{x}}(k), \qquad \begin{bmatrix} \frac{1}{2}\mathbf{T}^2 \\ \mathbf{T} \end{bmatrix} \underline{\ddot{x}}(k) \sim N(0, \mathbf{Q}(k)), \tag{5}$$

where $\mathbf{T} = \mathbf{I}T$ and

$$\mathbf{Q}(k) = \begin{bmatrix} \frac{1}{4}\mathbf{T}^4 & \frac{1}{2}\mathbf{T}^3 \\ \frac{1}{2}\mathbf{T}^3 & \mathbf{T}^2 \end{bmatrix} \sigma^2(k)$$

characterizes the modeling uncertainty and $\sigma(k)$ should be of the order of the maximum acceleration magnitude. Assume we only measure the positional quantities $\underline{z}(k)$. The measurement model can be given by

$$\underline{z}(k) = [\mathbf{H} \quad \mathbf{0}] \begin{bmatrix} \underline{x}(k) \\ \dot{\underline{x}}(k) \end{bmatrix} + \underline{w}(k), \qquad \underline{w}(k) \sim N(0, \mathbf{R}(k)), \tag{6}$$

where the measurement uncertainty is specified by $\mathbf{R}(k)$. Equations (5) and (6) constitute a complete set of models needed for Kalman filtering.

3. A FULL-RATE MR MODEL

In signal estimation and decomposition, it is desirable to describe the models in terms of *lowpass filtered* and *highpass filtered* quantities [1]. In this section, a full-rate MR model is derived. For simplicity, only two-tap filters are used and MR models for other filters with more than two taps can be similarly obtained. A lowpass filtering process is characterized by

$$\underline{x}_L(k) = [h_1 \quad h_2] \begin{bmatrix} \underline{x}(k-1) \\ \underline{x}(k) \end{bmatrix}, \tag{7}$$

where $[h_1 \; h_2]$ is a two-tap lowpass filter. A highpass filtered quantity can be given by

$$\underline{x}_H(k) = [g_1 \quad g_2] \begin{bmatrix} \underline{x}(k-1) \\ \underline{x}(k) \end{bmatrix}, \tag{8}$$

where $[g_1 \; g_2]$ is a two-tap highpass filter. With a set of constraints including that the lowpass filter and the highpass filter constitute a pair of Quadrature Mirror Filters (QMF) [10,11], an orthogonal transformation is formed as

$$\begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} = \mathbf{W} \begin{bmatrix} \underline{x}(k-1) \\ \underline{x}(k) \end{bmatrix}, \tag{9}$$

where

$$\mathbf{W} = \begin{bmatrix} h_1 & h_2 \\ g_1 & g_2 \end{bmatrix}$$

is an orthogonal matrix.

An equivalent constant-velocity MR model can be derived based on the common sense that highpass filtering is equivalent to taking derivative and lowpass filtering is similar to integration. An acceleration-equivalent highpass-highpass filtered quantity is derived as

$$\begin{aligned} \underline{x}_{HH}(k) &= g_1 \underline{x}_H(k-1) + g_2 \underline{x}_H(k) && (10) \\ &= g_1 \underline{x}_H(k-1) + g_2 [g_1 \underline{x}(k-1) + g_2 \underline{x}(k)] && (11) \\ &= g_2^2 \underline{x}(k) + g_1 g_2 \underline{x}(k-1) + g_1 \underline{x}_H(k-1). && (12) \end{aligned}$$

Rearranging equation (12) yields

$$\underline{x}(k) = -\frac{g_1}{g_2} \underline{x}(k-1) - \frac{g_1}{g_2^2} \underline{x}_H(k-1) + \frac{1}{g_2^2} \underline{x}_{HH}(k). \tag{13}$$

Similarly, we have

$$\underline{x}(k+1) = -\frac{g_1}{g_2} \underline{x}(k) - \frac{g_1}{g_2^2} \underline{x}_H(k) + \frac{1}{g_2^2} \underline{x}_{HH}(k+1). \tag{14}$$

A propagation equation for the lowpass filtered quantity is obtained as

$$\underline{x}_L(k+1) = h_1\underline{x}(k) + h_2\underline{x}(k+1) \tag{15}$$

$$= -\frac{g_1}{g_2}[h_1\underline{x}(k-1) + h_2\underline{x}(k)] - \frac{g_1}{g_2^2}[h_1\underline{x}_H(k-1) + h_2\underline{x}_H(k)] + \frac{1}{g_2^2}[h_1\underline{x}_{HH}(k) + h_2\underline{x}_{HH}(k+1)]. \tag{16}$$

Because of the following equalities,

$$\underline{x}_{HH}(k) = g_1\underline{x}_H(k-1) + g_2\underline{x}_H(k),$$
$$\frac{h_1}{g_2^2}\underline{x}_{HH}(k) = \frac{h_1 g_1}{g_2^2}\underline{x}_H(k-1) + \frac{h_1}{g_2}\underline{x}_H(k),$$

and

$$-\frac{g_1 h_1}{g_2^2}\underline{x}_H(k-1) + \frac{h_1}{g_2^2}\underline{x}_{HH}(k) = \frac{h_1}{g_2}\underline{x}_H(k).$$

Equation (16) can be rewritten as

$$\underline{x}_L(k+1) = -\frac{g_1}{g_2}\underline{x}_L(k) + \left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right)\underline{x}_H(k) + \frac{h_2}{g_2^2}\underline{x}_{HH}(k+1). \tag{17}$$

A propagation equation for the highpass filtered quantity could also be derived from equation (10) as

$$\underline{x}_H(k+1) = -\frac{g_1}{g_2}\underline{x}_H(k) + \frac{1}{g_2}\underline{x}_{HH}(k+1). \tag{18}$$

Putting equations (17) and (18) into a vector form, we have derived a full-rate constant-velocity-equivalent model

$$\begin{bmatrix}\underline{x}_L(k+1)\\ \underline{x}_H(k+1)\end{bmatrix} = \begin{bmatrix} -\frac{g_1}{g_2} & \left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right) \\ 0 - \frac{g_1}{g_2} & \end{bmatrix}\begin{bmatrix}\underline{x}_L(k)\\ \underline{x}_H(k)\end{bmatrix} + \begin{bmatrix}\frac{h_2}{g_2^2}\\ \frac{1}{g_2}\end{bmatrix}\underline{x}_{HH}(k+1), \tag{19}$$
$$\underline{x}_{HH}(k+1) \sim N\left(0, \sigma_{HH}^2\right),$$

where the modeling uncertainty vector $\left[\begin{smallmatrix}h_2/g_2^2\\ 1/g_2\end{smallmatrix}\right]\underline{x}_{HH}(k+1)$ is zero-mean with a covariance matrix being

$$\mathbf{Q}_{HH} = \begin{bmatrix}\frac{h_2^2}{g_2^4} & \frac{h_2}{g_2^3}\\ \frac{h_2}{g_2^3} & \frac{1}{g_2^2}\end{bmatrix}\sigma_{HH}^2. \tag{20}$$

The measurement equation accompanying equation (19) can be derived by substituting the following equation into equation (6)

$$\underline{x}(k) = [\,h_2\ \ g_2\,]\begin{bmatrix}\underline{x}_L(k)\\ \underline{x}_H(k)\end{bmatrix}, \tag{21}$$

which results in

$$\underline{z}(k) = \mathbf{H}\underline{x}(k) + \underline{w}(k) = [\mathbf{H}h_2 \quad \mathbf{H}g_2]\begin{bmatrix}\underline{x}_L(k)\\ \underline{x}_H(k)\end{bmatrix} + \underline{w}(k), \qquad \underline{w}(k) \sim N(0, \mathbf{R}(k)). \tag{22}$$

A comment on the full-rate MR model is now in order. Unlike the constant-velocity model given by equations (5) and (6), the full-rate MR model given by equations (19) and (22) is independent of the sampling period T, which is a quite attractive feature. Although, the estimates in the forms of lowpass and highpass filtered quantities ($\hat{\underline{x}}_L(k)$ and $\hat{\underline{x}}_H(k)$) can sometimes be useful [1], it is desired that the estimates of the original vectors ($\hat{\underline{x}}(k)$ and $\hat{\dot{\underline{x}}}(k)$) can be derived. To do that, we need to connect derivative and highpass filtering processes first. The first-order discrete derivative can be expressed in terms of lowpassed and highpassed quantities

$$\dot{\underline{x}}(k) = \begin{bmatrix} -\frac{1}{T} & \frac{1}{T} \end{bmatrix} \begin{bmatrix} \underline{x}(k-1) \\ \underline{x}(k) \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} -\frac{1}{T} & \frac{1}{T} \end{bmatrix} \mathbf{W}' \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} \tag{24}$$

$$= \begin{bmatrix} h & g \end{bmatrix} \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} \tag{25}$$

$$= h\underline{x}_L(k) + g\underline{x}_H(k), \tag{26}$$

where

$$h = \frac{1}{T}(-h_1 + h_2) \quad \text{and} \quad g = \frac{1}{T}(-g_1 + g_2).$$

The estimate $\hat{\underline{x}}(k)$ is obtained as equation (21)

$$\hat{\underline{x}}(k) = h_2\hat{\underline{x}}_L(k) + g_2\hat{\underline{x}}_H(k), \tag{27}$$

and $\hat{\dot{\underline{x}}}(k)$ is derived as equation (26)

$$\hat{\dot{\underline{x}}}(k) = h\hat{\underline{x}}_L(k) + g\hat{\underline{x}}_H(k). \tag{28}$$

In order to compare equation (19) with equation (5), a connection between $\ddot{\underline{x}}(k+1)$ and $\underline{x}_{HH}(k+1)$ needs to be established. From equation (4), we have the following:

$$\ddot{\underline{x}}(k+1) = -\frac{2}{T}\dot{\underline{x}}(k) + \frac{2}{T}\dot{\underline{x}}(k+1) \tag{29}$$

$$= \frac{2}{T}h[-\underline{x}_L(k) + \underline{x}_L(k+1)] + \frac{2}{T}g[-\underline{x}_H(k) + \underline{x}_H(k+1)], \tag{30}$$

and since

$$-\underline{x}_L(k) + \underline{x}_L(k+1) = \left(-1 - \frac{g_1}{g_2}\right)\underline{x}_L(k) + \left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right)\underline{x}_H(k) + \frac{h_2}{g_2^2}\underline{x}_{HH}(k+1) \tag{31}$$

and

$$-\underline{x}_H(k) + \underline{x}_H(k+1) = \left(-1 - \frac{g_1}{g_2}\right)\underline{x}_H(k) + \frac{1}{g_2}\underline{x}_{HH}(k+1). \tag{32}$$

Equation (30) can be written as

$$\begin{aligned}\ddot{\underline{x}}(k+1) &= \frac{2}{T}h\left(-1 - \frac{g_1}{g_2}\right)\underline{x}_L(k) + \frac{2}{T}h\left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right)\underline{x}_H(k) + \frac{2}{T}h\frac{h_2}{g_2^2}\underline{x}_{HH}(k+1) \\ &\quad + \frac{2}{T}g\left(-1 - \frac{g_1}{g_2}\right)\underline{x}_H(k) + \frac{2}{T}g\frac{1}{g_2}\underline{x}_{HH}(k+1) \qquad (33) \\ &= a_1\,\underline{x}_L(k) + a_2\,\underline{x}_H(k) + a_3\,\underline{x}_{HH}(k+1), \qquad (34)\end{aligned}$$

where

$$a_1 = \frac{2}{T}h\left(-1 - \frac{g_1}{g_2}\right), \qquad a_2 = \frac{2}{T}\left(h\frac{h_1}{g_2} - h\frac{g_1 h_2}{g_2^2} - g - g\frac{g_1}{g_2}\right),$$

and

$$a_3 = \frac{2}{T}\left(h\frac{h_2}{g_2^2} + \frac{g}{g_2}\right).$$

Rearranging equation (34) gives

$$\underline{x}_{HH}(k+1) = -\frac{a_1}{a_3}\underline{x}_L(k) - \frac{a_2}{a_3}\underline{x}_H(k) + \frac{1}{a_3}\underline{\ddot{x}}(k+1). \tag{35}$$

Substituting equation (35) into equations (17) and (18), we have derived a constant-velocity-equivalent equation with an identical acceleration term as in equation (4)

$$\underline{x}_L(k+1) = -\frac{g_1}{g_2}\underline{x}_L(k) + \left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right)\underline{x}_H(k) + \frac{h_2}{g_2^2}\underline{x}_{HH}(k+1) \tag{36}$$

$$= \left(-\frac{g_1}{g_2} - \frac{h_2 a_1}{g_2^2 a_3}\right)\underline{x}_L(k) + \left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2} - \frac{h_2 a_2}{g_2^2 a_3}\right)\underline{x}_H(k) + \frac{h_2}{g_2^2 a_3}\underline{\ddot{x}}(k+1) \tag{37}$$

and

$$\underline{x}_H(k+1) = -\frac{g_1}{g_2}\underline{x}_H(k) - \frac{a_1}{g_2 a_3}\underline{x}_L(k) - \frac{a_2}{g_2 a_3}\underline{x}_H(k) + \frac{1}{g_2 a_3}\underline{\ddot{x}}(k+1) \tag{38}$$

$$= -\frac{a_1}{g_2 a_3}\underline{x}_L(k) + \left(-\frac{g_1}{g_2} - \frac{a_2}{g_2 a_3}\right)\underline{x}_H(k) + \frac{1}{g_2 a_3}\underline{\ddot{x}}(k+1). \tag{39}$$

Finally, we have

$$\begin{bmatrix} \underline{x}_L(k+1) \\ \underline{x}_H(k+1) \end{bmatrix} = \begin{bmatrix} \left(-\frac{g_1}{g_2} - \frac{h_2 a_1}{g_2^2 a_3}\right) & \left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2} - \frac{h_2 a_2}{g_2^2 a_3}\right) \\ -\frac{a_1}{g_2 a_3} & \left(-\frac{g_1}{g_2} - \frac{a_2}{g_2 a_3}\right) \end{bmatrix} \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} + \begin{bmatrix} \frac{h_2}{g_2^2 a_3} \\ \frac{1}{g_2 a_3} \end{bmatrix} \underline{\ddot{x}}(k+1). \tag{40}$$

The full-rate model given by equation (19) is sufficient for multirate Kalman filtering, and equation (40) is used when a performance comparison between conventional constant-velocity model and the equivalent MR model is needed (see Section 7).

4. A HALF-RATE MR MODEL

A half-rate model is necessary when Kalman filtering is applied to half-rate data $\underline{z}_L(k)$, generated by

$$\underline{z}_L(k) = h_1 \underline{z}(k-1) + h_2 \underline{z}(k). \tag{41}$$

Repeatedly using equation (17), a half-rate propagation equation for the lowpassed quantity can be derived

$$\underline{x}_L(k+2) = -\frac{g_1}{g_2}\underline{x}_L(k+1) + \left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right)\underline{x}_H(k+1) + \frac{h_2}{g_2^2}\underline{x}_{HH}(k+2) \tag{42}$$

$$= \frac{g_1^2}{g_2^2}\underline{x}_L(k) - \frac{g_1}{g_2}\left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right)\underline{x}_H(k) - \frac{g_1 h_2}{g_2^3}\underline{x}_{HH}(k+1) \tag{43}$$

$$- \frac{g_1}{g_2}\left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right)\underline{x}_H(k) + \frac{1}{g_2}\left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right)\underline{x}_{HH}(k+1)$$

$$+ \frac{h_2}{g_2^2}\underline{x}_{HH}(k+2) \tag{43}$$

$$= b_1\, \underline{x}_L(k) + b_2\, \underline{x}_H(k) + b_3\, \underline{x}_{HH}(k+1) + b_4\, \underline{x}_{HH}(k+2), \tag{44}$$

where

$$b_1 = \frac{g_1^2}{g_2^2}, \qquad b_2 = -2\frac{g_1}{g_2}\left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right),$$

and

$$b_3 = \frac{h_1}{g_2^2} - 2\frac{g_1 h_2}{g_2^3}, \qquad b_4 = \frac{h_2}{g_2^2}.$$

Similarly, a half-rate propagation equation for the highpass filtered quantity can also be obtained by repeatedly using equation (18)

$$\underline{x}_H(k+2) = -\frac{g_1}{g_2}\underline{x}_H(k+1) + \frac{1}{g_2}\underline{x}_{HH}(k+2) \tag{45}$$

$$= b_1\underline{x}_H(k) + b_5\underline{x}_{HH}(k+1) + b_6\underline{x}_{HH}(k+2), \tag{46}$$

where

$$b_5 = -\frac{g_1}{g_2^2} \quad \text{and} \quad b_6 = \frac{1}{g_2}.$$

Putting equations (44) and (46) together results in a vector form of the half-rate propagation equation (system equation)

$$\begin{bmatrix} \underline{x}_L(k+2) \\ \underline{x}_H(k+2) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} + \begin{bmatrix} b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} \begin{bmatrix} \underline{x}_{HH}(k+1) \\ \underline{x}_{HH}(k+2) \end{bmatrix}, \tag{47}$$

where the modeling uncertainty is reflected by

$$\begin{bmatrix} \underline{x}_{HH}(k+1) \\ \underline{x}_{HH}(k+2) \end{bmatrix} \sim N(0, \mathbf{Q}_{HH}).$$

The half-rate measurement equation can be derived as

$$\underline{z}_L(k) = h_1\underline{z}(k-1) + h_2\underline{z}(k) = [\mathbf{H}\ 0] \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} + \underline{w}_L(k), \qquad \underline{w}_L(k) \sim N(0, \mathbf{R}_L(k)), \tag{48}$$

where

$$\mathbf{R}_L(k) = h_1^2\mathbf{R}(k-1) + h_2^2\mathbf{R}(k). \tag{49}$$

The transformation between the estimates of the lowpass and highpass filtered quantities and the original vectors can also be derived. For half-rate filtering, two-point vector transformations are needed. For the positional quantity, we have (equation (9))

$$\begin{bmatrix} \hat{\underline{x}}(k-1) \\ \hat{\underline{x}}(k) \end{bmatrix} = \mathbf{W}' \begin{bmatrix} \hat{\underline{x}}_L(k) \\ \hat{\underline{x}}_H(k) \end{bmatrix}. \tag{50}$$

However, due to the specifics of the half-rate operation, exact recovery of $\hat{\underline{x}}(k)$ is only available at every two points

$$\hat{\underline{x}}(k) = h\hat{\underline{x}}_L(k) + g\hat{\underline{x}}_H(k) \tag{51}$$

and

$$\hat{\underline{x}}(k-1) \cong \hat{\underline{x}}(k), \tag{52}$$

which is quite accurate due to the assumption of constant velocity. Also for the purpose of performance comparison, we can describe the modeling uncertainty in its original terms, $\ddot{\underline{x}}(k+1)$ and $\ddot{\underline{x}}(k+2)$. From equation (35), the following can be obtained:

$$\underline{x}_{HH}(k+1) = -\frac{a_1}{a_3}\underline{x}_L(k) - \frac{a_2}{a_3}\underline{x}_H(k) + \frac{1}{a_3}\ddot{\underline{x}}(k+1) \tag{53}$$

and

$$\underline{x}_{HH}(k+2) = -\frac{a_1}{a_3}\underline{x}_L(k+1) - \frac{a_2}{a_3}\underline{x}_H(k+1) + \frac{1}{a_3}\ddot{\underline{x}}(k+2) \tag{54}$$

$$= \frac{a_1 g_1}{a_3 g_2}\underline{x}_L(k) - \frac{a_1}{a_3}\left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right)\underline{x}_H(k) - \frac{a_1 h_2}{a_3 g_2^2}\underline{x}_{HH}(k+1)$$

$$+ \frac{a_2 g_1}{a_3 g_2}\underline{x}_H(k) - \frac{a_2}{a_3 g_2}\underline{x}_{HH}(k+1) + \frac{1}{a_3}\ddot{\underline{x}}(k+2) \tag{55}$$

$$= c_1\,\underline{x}_L(k) + c_2\,\underline{x}_H(k) + c_3\,\ddot{\underline{x}}(k+1) + c_4\,\ddot{\underline{x}}(k+2), \tag{56}$$

where

$$c_1 = \frac{a_1 g_1}{a_3 g_2} + \frac{a_1^2 h_2}{a_3^2 g_2^2} + \frac{a_1 a_2}{a_3^2 g_2},$$
$$c_2 = -\frac{a_1}{a_3}\left(\frac{h_1}{g_2} - \frac{g_1 h_2}{g_2^2}\right) + \frac{a_1 a_2 h_2}{a_3^2 g_2^2} + \frac{a_2 g_1}{a_3 g_2} + \frac{a_2^2}{a_3^2 g_2},$$
$$c_3 = -\frac{a_1 h_2}{a_3^2 g_2^2} - \frac{a_2}{a_3^2 g_2}, \qquad \text{and}$$
$$c_4 = \frac{1}{a_3}.$$

Therefore, a half-rate model for the lowpassed quantity containing $\ddot{\underline{x}}(k+1)$ and $\ddot{\underline{x}}(k+2)$ is obtained by substituting equations (53) and (56) into equation (44)

$$\underline{x}_L(k+2) = d_1\, \underline{x}_L(k) + d_2\, \underline{x}_H(k) + d_3\, \ddot{\underline{x}}(k+1) + d_4 \ddot{\underline{x}}(k+2), \tag{57}$$

where

$$d_1 = b_1 - b_3 \frac{a_1}{a_3} + b_4 c_1, \qquad d_2 = b_2 - b_3 \frac{a_2}{a_3} + b_4 c_2,$$
$$d_3 = \frac{b_3}{a_3} + b_4 c_3, \qquad d_4 = b_4 c_4.$$

Similarly, we have

$$\underline{x}_H(k+2) = e_1\, \underline{x}_L(k) + e_2\, \underline{x}_H(k) + e_3\, \ddot{\underline{x}}(k+1) + e_4 \ddot{\underline{x}}(k+2), \tag{58}$$

where

$$e_1 = \frac{a_1 g_1}{a_3 g_2^2} + \frac{c_1}{g_2}, \qquad e_2 = \frac{a_2 g_1}{a_3 g_2^2} + \frac{c_2}{g_2} + \frac{g_1^2}{g_2^2},$$

and

$$e_3 = -\frac{g_1}{a_3 g_2^2} + \frac{c_3}{g_2}, \qquad e_4 = \frac{c_4}{g_2}.$$

A vector form is derived by combining equations (57) and (58)

$$\begin{bmatrix} \underline{x}_L(k+2) \\ \underline{x}_H(k+2) \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ e_1 & e_2 \end{bmatrix} \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} + \begin{bmatrix} \ddot{\underline{x}}_L \\ \ddot{\underline{x}}_H \end{bmatrix}, \tag{59}$$

where

$$\begin{bmatrix} \ddot{\underline{x}}_L \\ \ddot{\underline{x}}_H \end{bmatrix} = \begin{bmatrix} d_3 \\ e_3 \end{bmatrix} \ddot{\underline{x}}(k+1) + \begin{bmatrix} d_4 \\ e_4 \end{bmatrix} \ddot{\underline{x}}(k+2)$$

is zero-mean with a covariance of

$$\mathbf{Q}(k) = \begin{bmatrix} \left(d_3^2 + d_4^2\right) & \left(d_3 e_3 + d_4 e_4\right) \\ \left(d_3 e_3 + d_4 e_4\right) & \left(e_3^2 + e_4^2\right) \end{bmatrix} \sigma^2. \tag{60}$$

5. A SPECIAL CASE

In this section, we study a special case where the two-tap filters are Haar filters

$$h_1 = \frac{\sqrt{2}}{2}, \qquad h_2 = \frac{\sqrt{2}}{2}, \qquad g_1 = -\frac{\sqrt{2}}{2}, \qquad \text{and} \qquad g_2 = \frac{\sqrt{2}}{2}. \tag{61}$$

Since for this special case,

$$h = 0, \qquad g = \frac{\sqrt{2}}{T}, \qquad a_1 = 0, \qquad a_2 = 0, \qquad a_3 = \frac{4}{T^2}$$

the full-rate MR model becomes

$$\begin{bmatrix} \underline{x}_L(k+1) \\ \underline{x}_H(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} + \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} \underline{x}_{HH}(k+1) \tag{62}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2\sqrt{2}} \\ \frac{T^2}{2\sqrt{2}} \end{bmatrix} \ddot{\underline{x}}(k+1) \tag{63}$$

with its model error covariance being 1

$$\mathbf{Q}(k) = \begin{bmatrix} \frac{T^4}{8} & \frac{T^4}{8} \\ \frac{T^4}{8} & \frac{T^4}{8} \end{bmatrix} \sigma^2. \tag{64}$$

Similarly, for the half-rate MR model, since

$$\begin{array}{llll} b_1 = 1, & b_2 = 4, & b_3 = 3\sqrt{2}, & b_4 = \sqrt{2}, \\ c_1 = 0, & c_2 = 0, & c_3 = 0, & c_4 = \frac{T^2}{4}, \\ d_1 = 1, & d_2 = 4, & d_3 = \frac{3T^2}{2\sqrt{2}}, & d_4 = \frac{T^2}{2\sqrt{2}}, \\ e_1 = 0, & e_2 = 1, & e_3 = \frac{T^2}{2\sqrt{2}}, & e_4 = \frac{T^2}{2\sqrt{2}}, \end{array}$$

we have

$$\begin{bmatrix} \underline{x}_L(k+2) \\ \underline{x}_H(k+2) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} + \begin{bmatrix} \frac{3T^2}{2\sqrt{2}} \\ \frac{T^2}{2\sqrt{2}} \end{bmatrix} \ddot{\underline{x}}(k+1) + \begin{bmatrix} \frac{T^2}{2\sqrt{2}} \\ \frac{T^2}{2\sqrt{2}} \end{bmatrix} \ddot{\underline{x}}(k+2) \tag{65}$$

with

$$\mathbf{Q}(k) = \begin{bmatrix} \frac{5T^4}{4} & \frac{T^4}{2} \\ \frac{T^4}{2} & \frac{T^4}{4} \end{bmatrix} \sigma^2. \tag{66}$$

The full-rate and half-rate measurement models for this special case can be obtained by replacing h_1 and h_2 with $\sqrt{2}/2$ in equations (22) and (48).

6. A QUARTER-RATE MR MODEL

To simplify the derivation, a quarter-rate model is derived under the assumption of Haar filters equation (61). Repeatedly using equation (65) generates

$$\begin{aligned} \underline{x}_L(k+4) = \underline{x}_L(k) + 8\underline{x}_H(k) + \left(\sqrt{2}T^2 + \frac{3T^2}{2\sqrt{2}}\right) \ddot{\underline{x}}(k+1) + \left(\sqrt{2}T^2 + \frac{T^2}{2\sqrt{2}}\right) \ddot{\underline{x}}(k+2) \\ + \frac{3T^2}{2\sqrt{2}} \ddot{\underline{x}}(k+3) + \frac{T^2}{2\sqrt{2}} \ddot{\underline{x}}(k+4) \end{aligned} \tag{67}$$

and

$$\underline{x}_H(k+4) = \underline{x}_H(k) + \frac{T^2}{2\sqrt{2}} \ddot{\underline{x}}(k+1) + \frac{T^2}{2\sqrt{2}} \ddot{\underline{x}}(k+2) + \frac{T^2}{2\sqrt{2}} \ddot{\underline{x}}(k+3) + \frac{T^2}{2\sqrt{2}} \ddot{\underline{x}}(k+4). \tag{68}$$

Putting equations (67) and (68) together in a vector form yields a quarter-rate MR model

$$\begin{bmatrix} \underline{x}_L(k+4) \\ \underline{x}_H(k+4) \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} + \begin{bmatrix} \ddot{\overline{\underline{x}}}_L \\ \ddot{\overline{\underline{x}}}_H \end{bmatrix}, \tag{69}$$

where

$$\begin{bmatrix} \ddot{\overline{\underline{x}}}_L \\ \ddot{\overline{\underline{x}}}_H \end{bmatrix} = \begin{bmatrix} \sqrt{2}T^2 + \dfrac{3T^2}{2\sqrt{2}} \\ \dfrac{T^2}{2\sqrt{2}} \end{bmatrix} \ddot{\underline{x}}(k+1) + \begin{bmatrix} \sqrt{2}T^2 + \dfrac{T^2}{2\sqrt{2}} \\ \dfrac{T^2}{2\sqrt{2}} \end{bmatrix} \ddot{\underline{x}}(k+2)$$
$$+ \begin{bmatrix} \dfrac{3T^2}{2\sqrt{2}} \\ \dfrac{T^2}{2\sqrt{2}} \end{bmatrix} \ddot{\underline{x}}(k+3) + \begin{bmatrix} \dfrac{T^2}{2\sqrt{2}} \\ \dfrac{T^2}{2\sqrt{2}} \end{bmatrix} \ddot{\underline{x}}(k+4)$$

is a zero-mean model error with a covariance matrix as

$$\mathbf{Q}(k) = \begin{bmatrix} \dfrac{21T^4}{2} & 2T^4 \\ 2T^4 & \dfrac{T^4}{2} \end{bmatrix} \sigma^2.$$

The corresponding measurement model is given by

$$\begin{aligned} \underline{z}_L(k) &= \left(\frac{\sqrt{2}}{2}\right)^2 \underline{z}(k-3) + \left(\frac{\sqrt{2}}{2}\right)^2 \underline{z}(k-2) + \left(\frac{\sqrt{2}}{2}\right)^2 \underline{z}(k-1) + \left(\frac{\sqrt{2}}{2}\right)^2 \underline{z}(k) \\ &= [\mathbf{H}\ 0] \begin{bmatrix} \underline{x}_L(k) \\ \underline{x}_H(k) \end{bmatrix} + \underline{v}_L(k), \end{aligned} \tag{70}$$

where

$$\mathbf{R}_L(k) = \left(\frac{\sqrt{2}}{2}\right)^4 \mathbf{R}(k-3) + \left(\frac{\sqrt{2}}{2}\right)^4 \mathbf{R}(k-2) + \left(\frac{\sqrt{2}}{2}\right)^4 \mathbf{R}(k-1) + \left(\frac{\sqrt{2}}{2}\right)^4 \mathbf{R}(k). \tag{71}$$

7. AN EXAMPLE AND DISCUSSIONS

A target moves on a two-dimensional surface with a constant velocity, except for a few small intervals in which the velocity is not constant (nonzero acceleration). The true trajectory of the object is shown in Figure 1(a) and the noisy measurements of the trajectory are shown in Figure 1(b), with a sampling period $T = 2$ sec.

A conventional constant-velocity model for this example is specified by equations (5) and (6) with the following statistics:

$$\mathbf{Q}(k) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad \text{and} \quad \mathbf{R}(k) = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix},$$

and initial values

$$[x_0\ y_0\ \dot{x}_0\ \dot{y}_0]' = [200\,\mathrm{m}\ 85000\,\mathrm{m}\ 0\,\mathrm{m/s} - 1\,\mathrm{m/s}]' \quad \text{and} \quad \mathbf{P}_0 = \mathbf{I}.$$

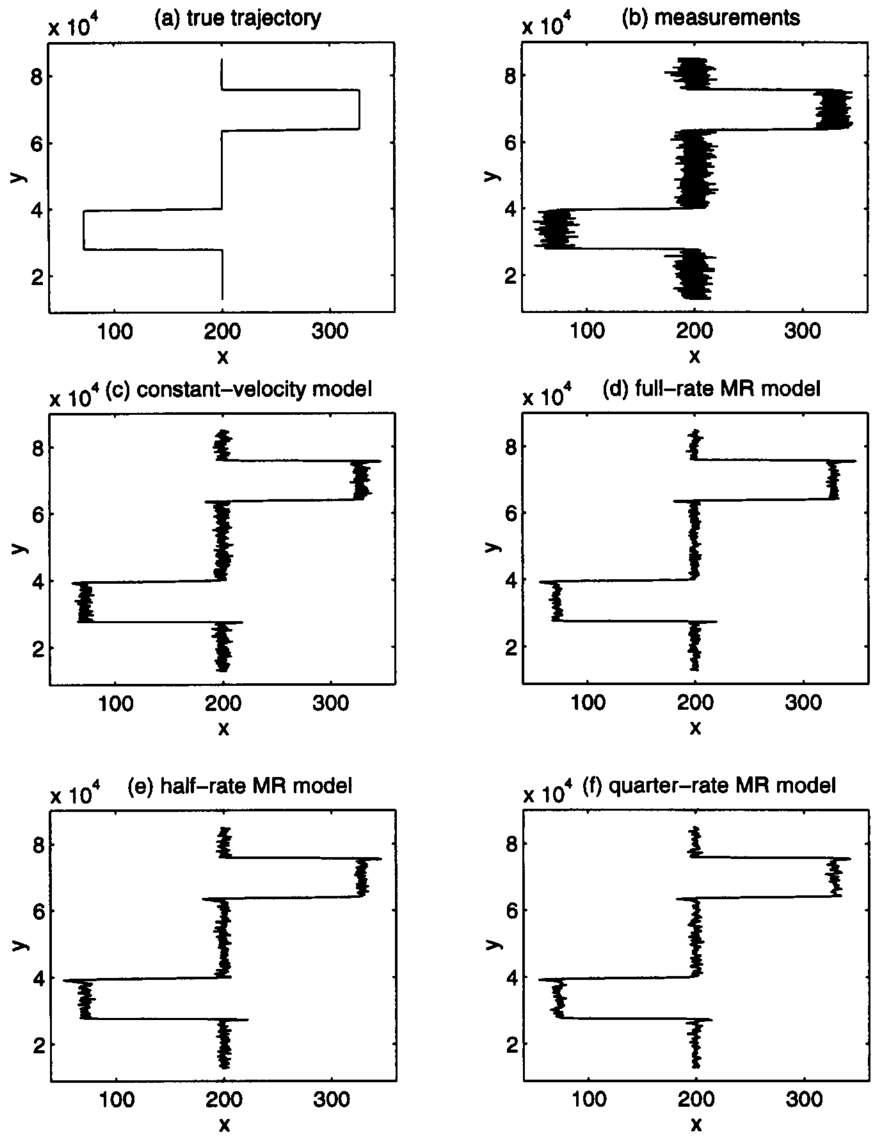

Figure 1. (a) True trajectory of the object; (b) noisy measurement; (c) estimated trajectory using a constant-velocity model; (d) estimated trajectory using a full-rate MR model; (e) estimated trajectory using a half-rate MR model; and (f) estimated trajectory using a quarter-rate MR model.

The choice of the $\mathbf{Q}(k)$ matrix is a matter of compromise between keeping constant-velocity assumption for better filtering quality at the segments where the object has a constant velocity and the ability to track the object when the velocity is not constant. The estimated trajectory using the conventional constant-velocity model is plotted in Figure 1(c). Estimated trajectories by full-rate, half-rate, and quarter-rate MR models using the same statistics of $\mathbf{Q}(k)$ and $\mathbf{R}(k)$ are shown in Figures 1(d) to 1(f), respectively. An overall improvement in estimation quality can obviously be seen by using MR models. To quantify the comparison, a set of Monte Carlo simulations are performed and the results are listed in Table 1. The improvement is due to one of the advantages of multirate filtering [1–5]: additional smoothing effects on the estimates (improving SNR). Another major advantage of multirate filtering (excluding full-rate filtering) is computational efficiency, which can be seen from Table 2 where floating-point-operations (FLOPs) needed for each simulation are recorded. Computational savings are shown when half-rate and quarter-rate filtering was performed.

8. CONCLUSIONS

This paper presents a set of constant-velocity-equivalent multirate (MR) models which can be used for multirate signal estimation and decomposition when exact models are not available and the constant-velocity assumption is valid. For comparison purpose, a connection in the modeling error statistics has also been derived. An overall improvement in estimation quality using MR models is demonstrated by Monte Carlo simulations.

Table 1. Comparison of the MR models with conventional constant-velocity models (50 Monte Carlo runs).

	RMS Error in x Coordinate	RMS Error in y Coordinate
Constant-Velocity Model	3.0855	3.1413
Full-Rate MR Model	2.3854	2.3855
Half-Rate MR Model	2.8761	3.0652
Quarter-Rate MR Model	2.5252	2.8936

Table 2. Comparison of computational complexity.

Constant-Velocity Model	1,746,472 (FLOPs)
Full-Rate MR Model	1,751,270 (FLOPs)
Half-Rate MR Model	888,000 (FLOPs)
Quarter-Rate MR Model	457,200 (FLOPs)

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