A

Library of Standard Elements

This part of the book collects the most useful standard building blocks of linear dynamic systems. Each element is described by its time domain and frequency domain equations, and its most important features are displayed (step responses, Bode diagrams, etc.).

Moreover, a continuous-time realization using operational amplifiers and a discrete-time realization suitable for a digital computer is shown. Note that the analog realization has a negative sign. The discrete-time realization assumes two hardware drivers ("analog_input" and "analog_output") to be available. The transformation from the continuous-time to the discrete-time description is made using the Tustin transformation introduced in Lecture 14. Accordingly, this description is valid only if the sampling time T_s is much smaller than the smallest time constant of the element.

A.1 Integrator Element

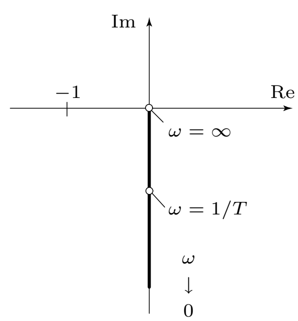
Element Acronym: $\boxed{\mathrm{I}}$

Transfer Function: $\Sigma(s) = \frac{1}{T \cdot s}$

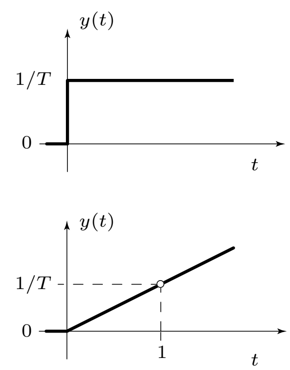
Poles/Zeros: $\pi_1 = 0$, $\zeta_1 = \infty$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{1}{T} \cdot u(t)$
$y(t) = x(t)$

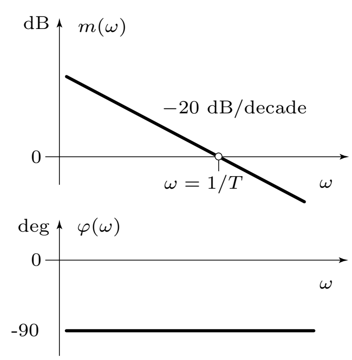
Nyquist Diagram



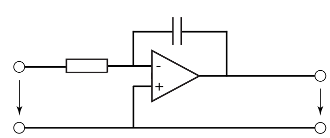
Impulse/Step Response



Bode Diagram



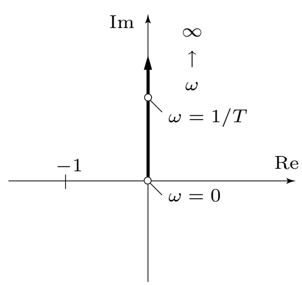
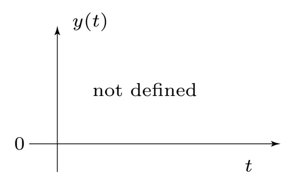
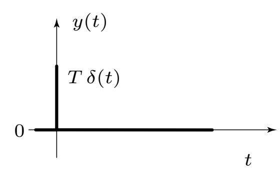
Analog/Digital Realization

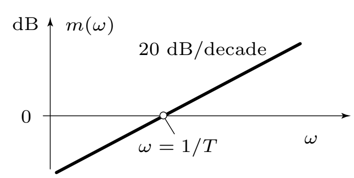
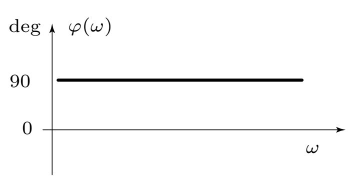
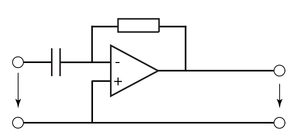



```
...
analog_input(e_k);
u_k=u_k_1+T_s/(2*T)*(e_k+e_k_1);
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.2 Differentiator Element

Element Acronym:	$\boxed{\mathrm{D}}$
Transfer Function:	$\Sigma(s) = T \cdot s$
Poles/Zeros:	$\pi_1 = \infty$, $\zeta_1 = 0$
Internal Description:	$y(t) = T \cdot \frac{\mathrm{d}}{\mathrm{d}t} u(t)$

Nyquist Diagram	Impulse/Step Response
 	 
Bode Diagram	**Analog/Digital Realization**


 	

 	
	
	...
	analog_input(e_k);
	u_k=2*T*(e_k-e_k_1)/T_s - u_k_1;
	analog_output(u_k);
	u_k_1=u_k;
	e_k_1=e_k;
	...

A.3 First-Order Element

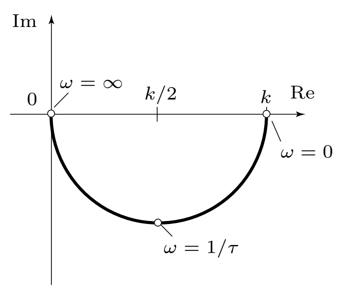
Element Acronym: $\boxed{\text{LP-1}}$

Transfer Function: $\Sigma(s) = \frac{k}{\tau \cdot s + 1}$

Poles/Zeros: $\pi_1 = -\frac{1}{\tau}$, $\zeta_1 = \infty$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\frac{1}{\tau} \cdot x(t) + \frac{1}{\tau} \cdot u(t)$
$y(t) = k \cdot x(t)$

<table>
<tr><td>Nyquist Diagram</td><td>Impulse/Step Response</td></tr>
<tr><td>



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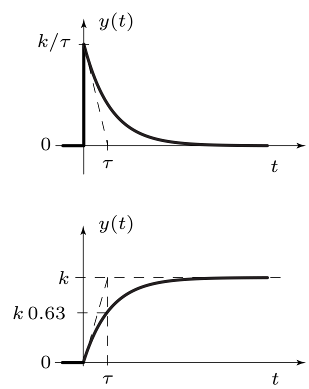
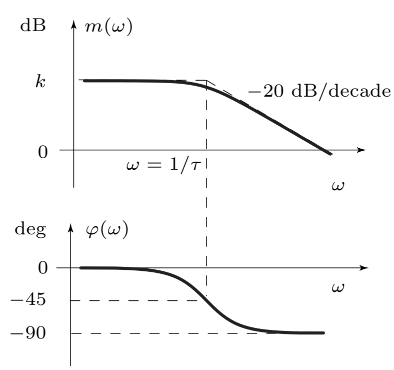


</td></tr>
<tr><td>Bode Diagram</td><td>Analog/Digital Realization</td></tr>
<tr><td>



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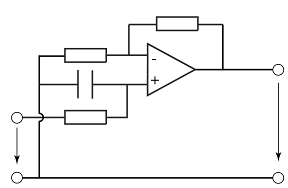


```
...
analog_input(e_k);
u_k=u_k_1*(2*tau-T_s)/(2*tau+T_s)+...
    (e_k_1+e_k)*(k*T_s)/(2*tau+T_s);
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

</td></tr>
</table>

A.4 Realizable Derivative Element

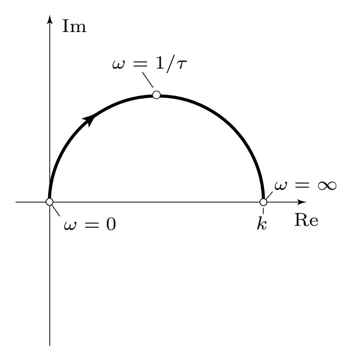
Element Acronym: $\boxed{\text{HP-1}}$

Transfer Function: $\Sigma(s) = k \cdot \frac{\tau \cdot s}{\tau \cdot s + 1} = k \cdot \left(1 - \frac{1}{\tau \cdot s + 1}\right)$

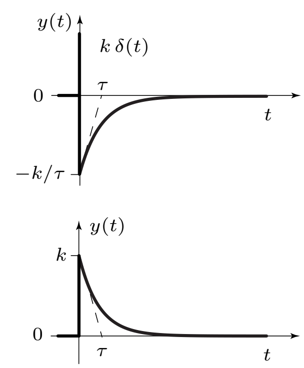
Poles/Zeros: $\pi_1 = -\frac{1}{\tau}$, $\zeta_1 = 0$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\frac{1}{\tau} \cdot x(t) + \frac{1}{\tau} \cdot u(t)$
$y(t) = -k \cdot x(t) + k \cdot u(t)$

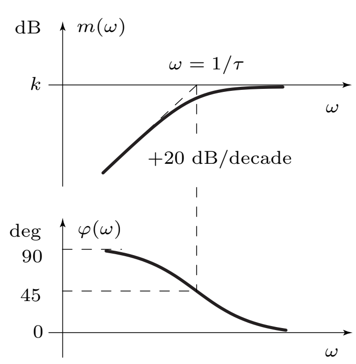
Nyquist Diagram



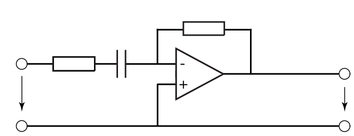
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
u_k=1/(T_s+2*tau)*(u_k_1*(2*tau-T_s)+...
                (e_k-e_k_1)*2*k*tau);
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.5 Second-Order Element

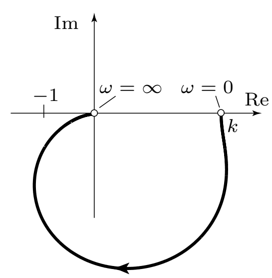
Element Acronym: $\boxed{\text{LP-2}}$

Transfer Function: $\Sigma(s) = k \cdot \frac{\omega_0^2}{s^2+2\cdot\delta\cdot\omega_0\cdot s+\omega_0^2}$

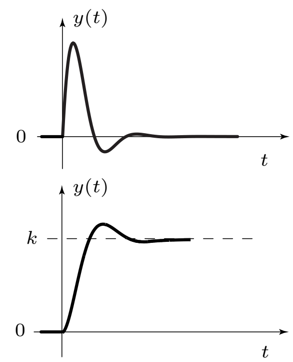
Poles/Zeros: $\pi_{1,2} = -w_0 \cdot \delta \pm w_0\sqrt{\delta^2-1}$, $\zeta_{1,2} = \infty$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = x_2(t)$,
$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = -\omega_0^2 \cdot x_1(t) - 2\cdot\delta\cdot\omega_0\cdot x_2(t) + \omega_0^2 \cdot u(t)$
$y(t) = k \cdot x_1(t)$

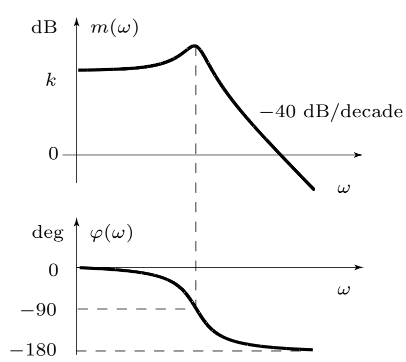
Nyquist Diagram



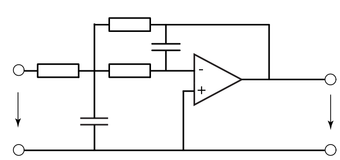
Impulse/Step Response



Bode Diagram



Analog/Digital Realization




```
...
analog_input(e_k);
...
use Matlab's c2dm
...
analog_output(u_k);
u_k_2=u_k_1;
e_k_2=e_k_1;
u_k_1=u_k;
e_k_1=e_k;
...
```

A.6 Lag Element

Element Acronym: $\boxed{\text{LG-1}}$

Transfer Function: $\Sigma(s) = k \cdot \frac{T \cdot s+1}{\alpha \cdot T \cdot s+1} = \frac{k}{\alpha} + k \cdot \frac{1-1/\alpha}{\alpha \cdot T \cdot s+1} \qquad 1 < \alpha$

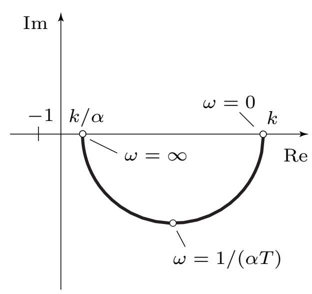
Poles/Zeros: $\pi_1 = -\frac{1}{\alpha \cdot T},\, \zeta_1 = -\frac{1}{T}$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{\alpha \cdot T} \cdot x(t) + \frac{1}{\alpha \cdot T} \cdot u(t)$

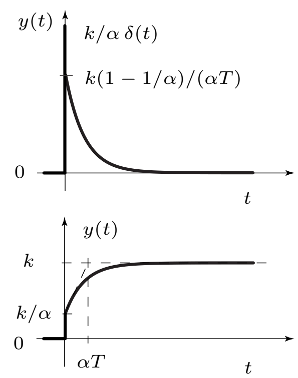
$y(t) = \frac{k \cdot (\alpha-1)}{\alpha} \cdot x(t) + \frac{k}{\alpha} \cdot u(t)$

Phase minimum: $\hat{\varphi} = \arctan(1/\sqrt{\alpha}) - \arctan(\sqrt{\alpha})$ at $\hat{\omega} = (T \cdot \sqrt{\alpha})^{-1}$

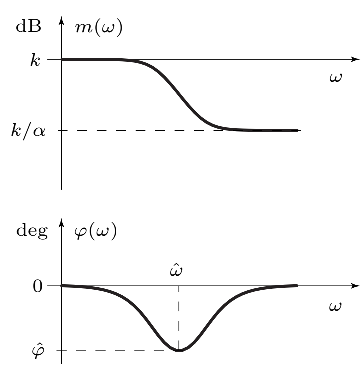
Nyquist Diagram



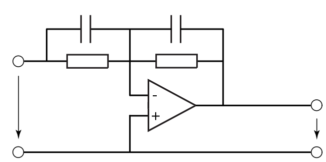
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
u_k=u_k_1*(2*T*T_s*alpha)/ ...
    (T_s+2*T*alpha)+...
    e_k*k(T_s+2*T*alpha)+...
    e_k_1*k(T_s-2*T)/(T_s+2*T*alpha);
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.7 Lead Element

Element Acronym: $\boxed{\text{LD-1}}$

Transfer Function: $\Sigma(s) = k \cdot \frac{T \cdot s+1}{\alpha \cdot T \cdot s+1} = \frac{k}{\alpha} + k \cdot \frac{1-1/\alpha}{\alpha \cdot T \cdot s+1} \qquad 0 < \alpha < 1$

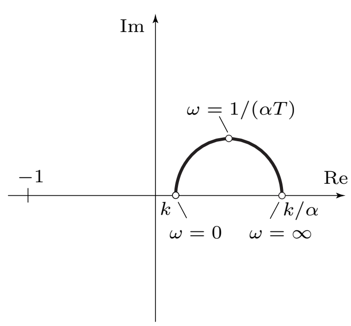
Poles/Zeros: $\pi_1 = -\frac{1}{\alpha \cdot T}, \; \zeta_1 = -\frac{1}{T}$

Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\frac{1}{\alpha \cdot T} \cdot x(t) + \frac{1}{\alpha \cdot T} \cdot u(t)$

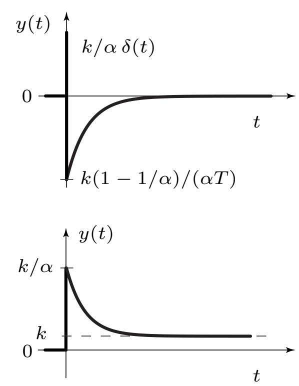
$y(t) = \frac{k \cdot (\alpha - 1)}{\alpha} \cdot x(t) + \frac{k}{\alpha} \cdot u(t)$

Phase maximum: $\hat{\varphi} = \arctan(1/\sqrt{\alpha}) - \arctan(\sqrt{\alpha})$ at $\hat{\omega} = (T \cdot \sqrt{\alpha})^{-1}$

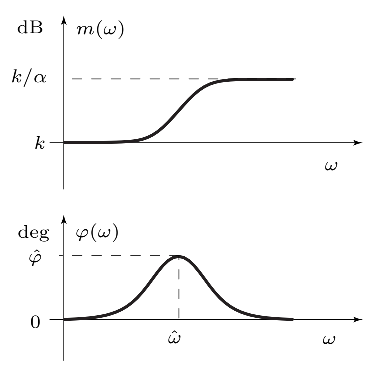
Nyquist Diagram



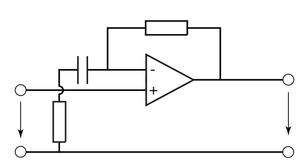
Impulse/Step Response



Bode Diagram



Analog/Digital Realization



```
...
analog_input(e_k);
u_k=u_k_1*(2*T*T_s*alpha)/ ...
    (T_s+2*T*alpha)+...
    e_k*k(T_s+2*T)/(T_s+2*T*alpha)+...
    e_k_1*k*(T_s-2*T)/(T_s+2*T*alpha);
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.8 PID Element

Element Acronym: $\boxed{\text{PID}}$

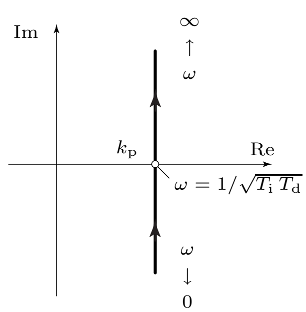
Transfer Function: $\Sigma(s) = k_\mathrm{p} \cdot \frac{T_\mathrm{d} \cdot T_\mathrm{i} \cdot s^2 + T_\mathrm{i} \cdot s + 1}{T_\mathrm{i} \cdot s} = k_\mathrm{p} \cdot (1 + \frac{1}{T_\mathrm{i} \cdot s} + T_\mathrm{d} \cdot s)$

Poles/Zeros: $\pi_1 = 0,\ \pi_2 = \infty,\ \zeta_{1,2} = -\frac{1}{2 \cdot T_\mathrm{d}} \pm \sqrt{\frac{1}{4 \cdot T_\mathrm{d}^2} - \frac{1}{T_\mathrm{i} \cdot T_\mathrm{d}}}$

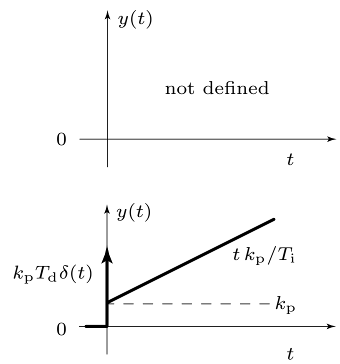
Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t} x_1(t) = \frac{1}{T_i} \cdot u(t)$

$y(t) = k_\mathrm{p} \cdot \big(u(t) + x_1(t) + T_\mathrm{d} \cdot \frac{\mathrm{d}}{\mathrm{d}t} u(t)\big)$

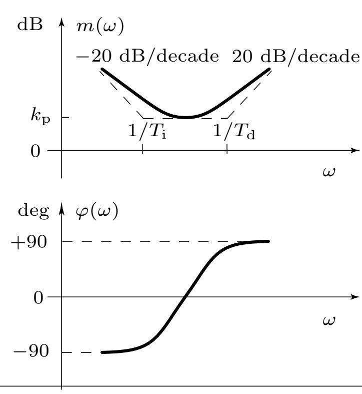
Nyquist Diagram



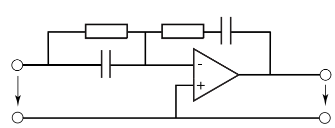
Impulse/Step Response



Bode Diagram



Analog/Digital Realization




```
...
analog_input(e_k);
u_k=e_k*kp*(1+T_s/(2*T_i)+2*T_d/T_s)+ ...
     e_k_1*kp*(T_s/T_i-4*T_d/T_s)+...
     e_k_2*kp*(-1+T_s/(2*T_i)+2*T_d/T_s)+...
     u_k_2;
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.9 First-Order All-Pass Element

Element Acronym: $\boxed{\text{AP-1}}$

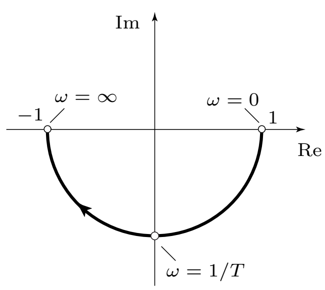
Transfer Function: $\Sigma(s) = \frac{-T\cdot s+1}{T\cdot s+1} = -1 + \frac{2}{T\cdot s+1}$

Poles/Zeros: $\pi_1 = -\frac{1}{T}$, $\zeta_1 = \frac{1}{T}$

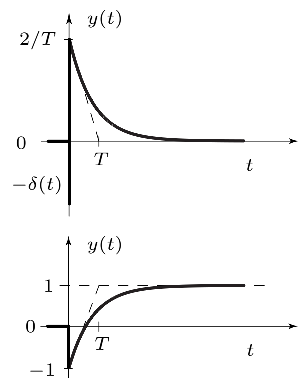
Internal Description: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\frac{1}{T}\cdot x(t) + \frac{1}{T}\cdot u(t)$

$y(t) = 2\cdot x(t) - u(t)$

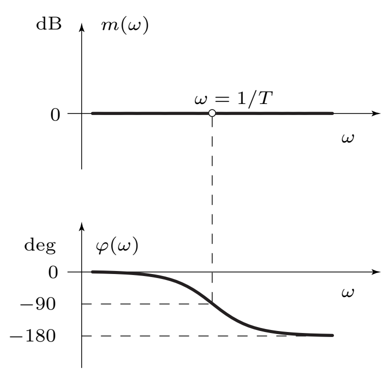
Nyquist Diagram



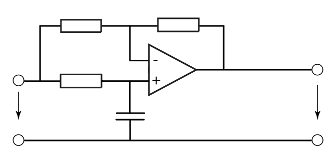
Impulse/Step Response



Bode Diagram



Analog/Digital Realization




```
...
analog_input(e_k);
u_k=(e_k-u_k_1)*(T_s-2*T)/(T_s+2*T)+e_k_1;
analog_output(u_k);
u_k_1=u_k;
e_k_1=e_k;
...
```

A.10 Delay Element

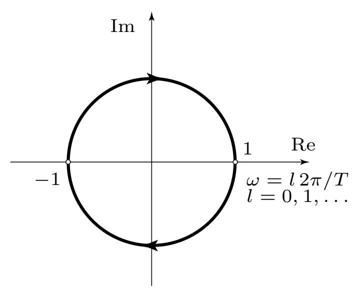
Element Acronym: $\boxed{-}$

Transfer Function: $\Sigma(s) = \mathrm{e}^{-s \cdot T}$

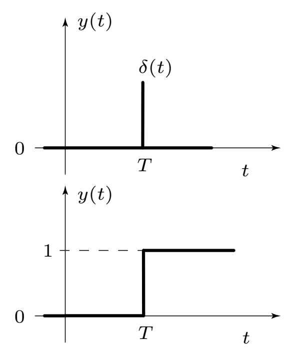
Poles/Zeros: not a real-rational element

Internal Description: $y(t) = u(t - T)$

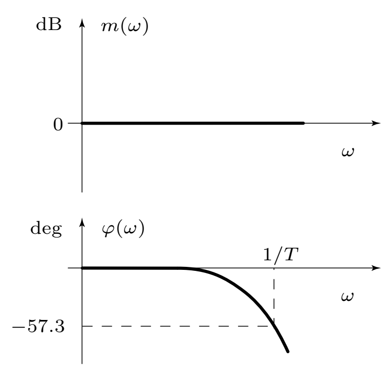
Nyquist Diagram



Impulse/Step Response



Bode Diagram



Analog/Digital Realization

Analog: use Padé elements (allpass elements) as approximation

```
KTZ=integer(T/T_s);
...
analog_input(e_k);
u_k=e_alt(KTZ);
analog_output(u_k);
for i=1:KTZ-1
        e_alt(i+1)=e_alt(i);
end;
e_alt(1)=e_k;
...
```