

stochastics and probability

Lecture 1

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indroduction

stochastic model

$$\underline{x}, t$$

stochastic simulation

$$\underline{x}_0, \underline{x}_1, \underline{x}_2, \dots, \underline{x}_T$$

$$\forall t = 0, \dots, T : \underline{x}_t \sim P(\underline{x}, t | \underline{x}_0, t_0)$$

problems

- 1) $P(.)$ not known in closed form
→ Master Equations
- 2) generating random numbers $\sim P(.)$ is not easy/ possible
→ sampling
- 3) time is continuous $\underline{x}(t)$
→ stochastic calculus
- 4) $P(.)$ changes as a function of history
→ non-Markovian processes

elementary probability

definitions:

population: a collection of objects

sample: a subset of a population

experiment: measuring something of a sample

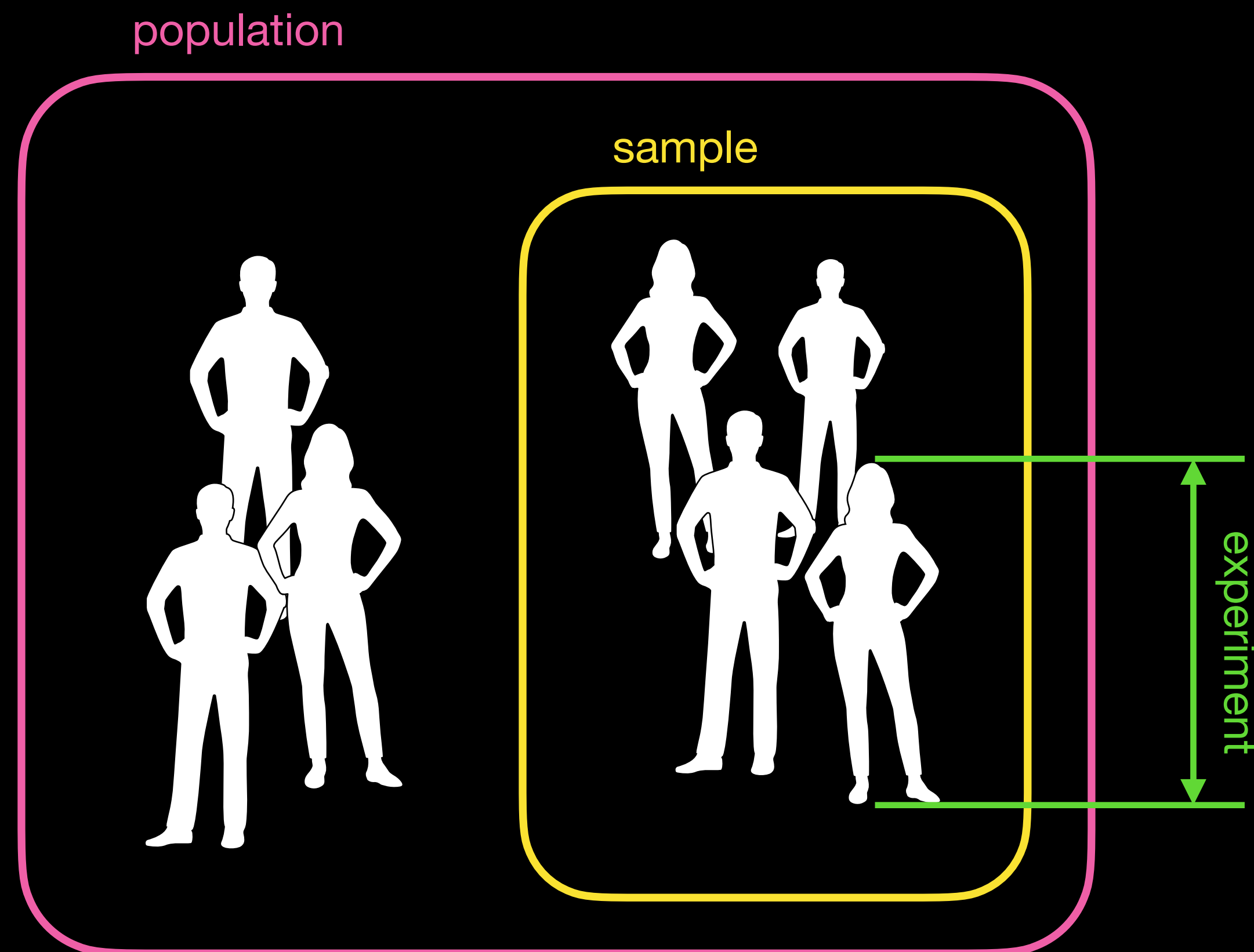
event: a concrete set of outcomes of an experiment

sample space: space of all possible events Ω

probability: likelihood of an event in a sample space

$P(A)$ for the event $A \subseteq \Omega$

example elementary probability



event

140

190

175

130

Population: a collection of objects

Sample: a subset of a population

Experiment: measuring something of a sample

Event: a concrete set of outcomes of an experiment

Sample space: space of all possible events Ω

Probability: likelihood of an event in a sample space
 $P(A)$ for the event $A \subseteq \Omega$

sample space

$$\Omega = [0, 300]$$

probability

$$P(X \in [50, 200]) = 0.98$$

Kolmogorov axioms

$$1) P(A) \geq 0 \quad \forall A \subseteq \Omega$$

$$2) P(\Omega) = 1$$

$$3) P(A_1 \cup A_2 \cup \dots) = \sum P(A_i) \quad \text{if } A_i \cap A_j = \emptyset \text{ for all } i \neq j$$

implications

$$P(\emptyset) = 0$$

$$P(\overline{A}) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

exercice

Prove: $P(\bar{A}) = 1 - P(A)$

$$\begin{aligned} 1 &= P(\Omega) \\ &= P(\Omega \setminus A \cup A) \\ &= P(\bar{A} \cup A) \\ &= P(\bar{A}) + P(A) \\ \rightarrow P(\bar{A}) &= 1 - P(A) \quad \square \end{aligned}$$

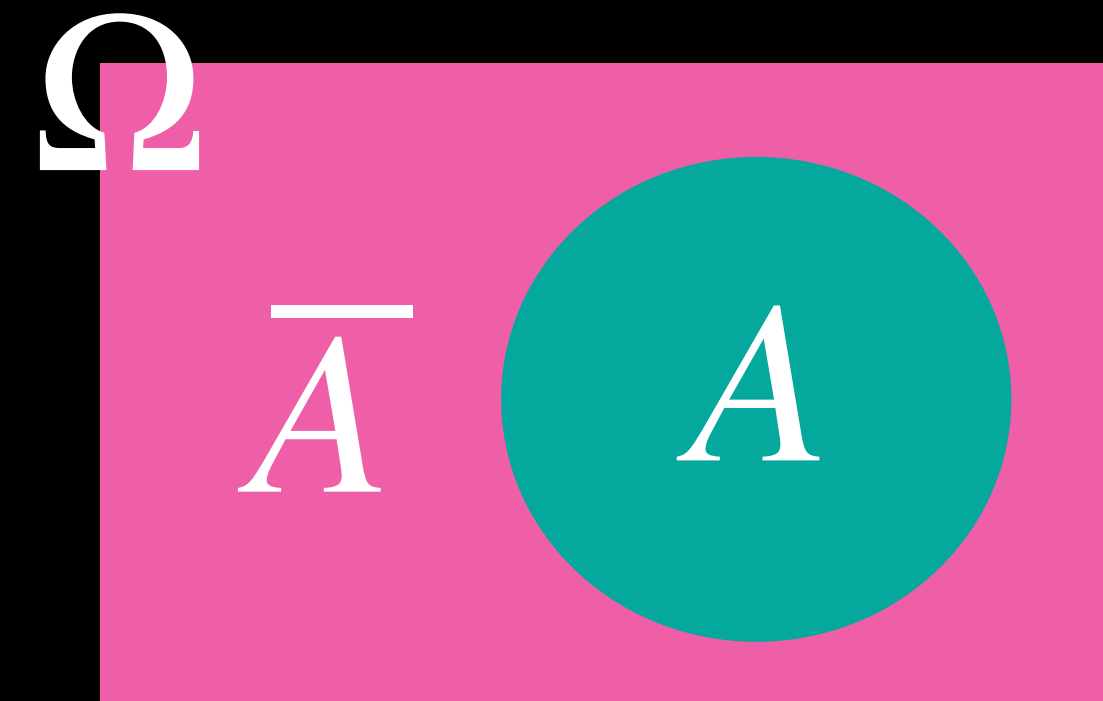
Axioms:

$$\forall A : P(A) \geq 0$$

$$P(\Omega) = 1$$

$$P(A \cup B) = P(A) + P(B) \quad \text{if } P(A \cap B) = 0$$

$$\bar{A} = A^c = A' = \neg A := \Omega \setminus A = \Omega - A$$



$$\bar{A} \cap A = \emptyset$$

conditional probabilities

definitions:

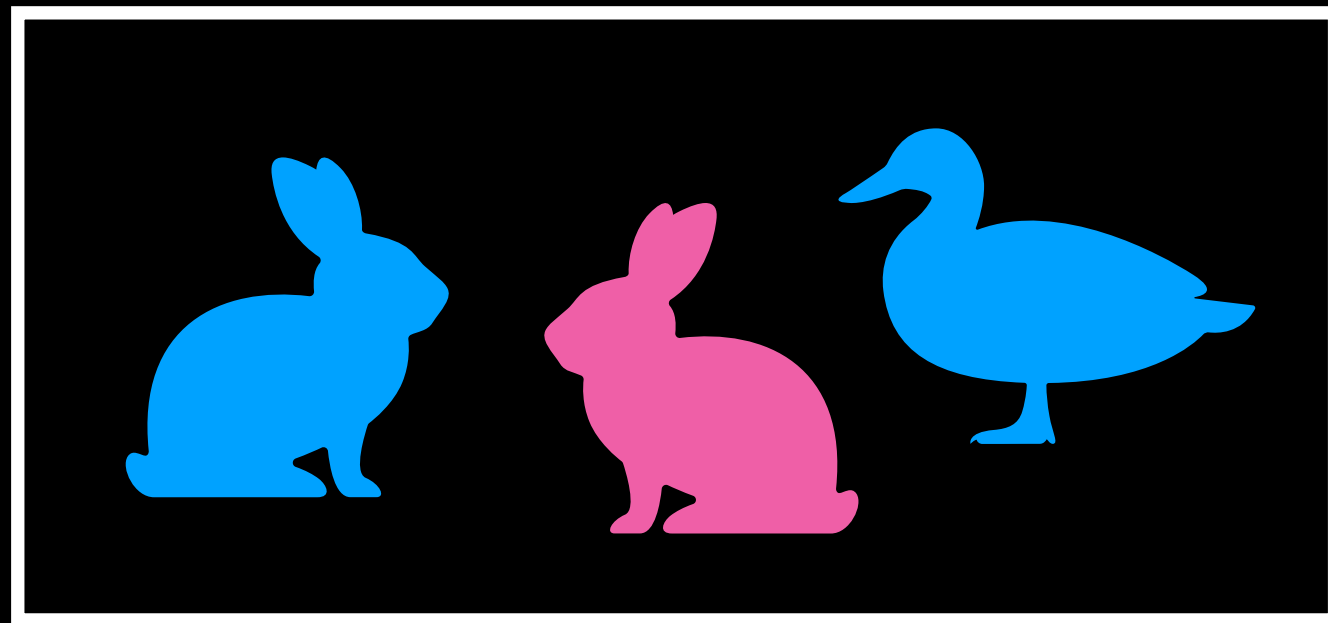
conditional probabilities

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

"A given B"

example conditional probabilities

draw



$$P(\text{blue rabbit}) = \frac{1}{3} \quad P(\text{pink rabbit}) = \frac{1}{3} \quad P(\text{blue duck}) = \frac{1}{3}$$

$$\begin{aligned} P(\text{blue rabbit} \mid \text{blue}) &= \frac{P(\text{blue rabbit} \cap \text{blue})}{P(\text{blue})} = \frac{P(\text{blue rabbit})}{P(\text{blue rabbit} \cup \text{blue duck})} = \frac{P(\text{blue rabbit})}{P(\text{blue rabbit}) + P(\text{blue duck})} \\ &= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} = \frac{1}{2} \end{aligned}$$

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

$$P(A \cup B) = P(A) + P(B) \\ \text{if } P(A \cap B) = 0$$

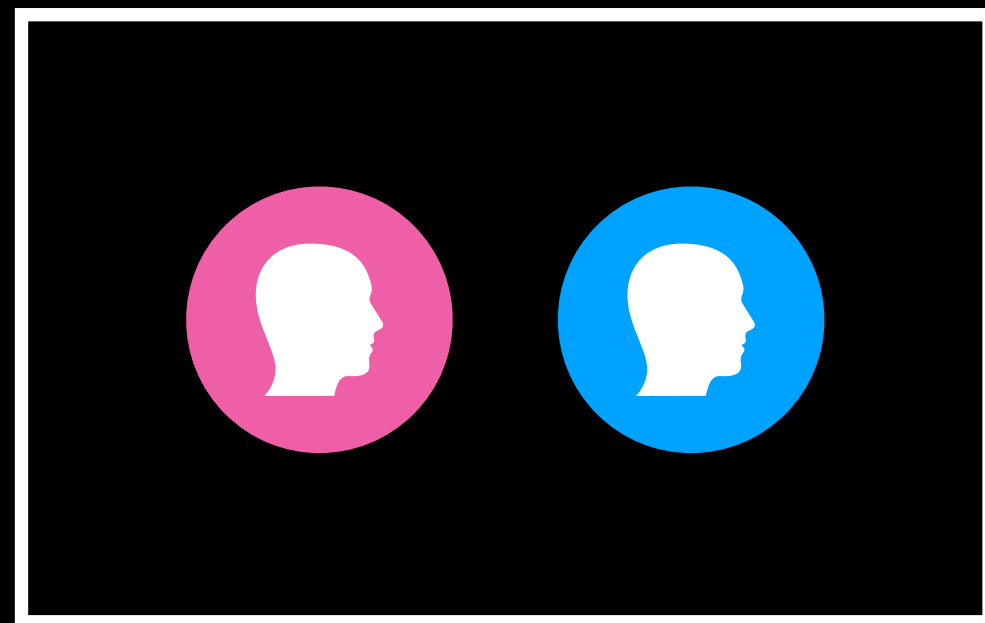
conditional probabilities

definitions:

An event A is independent of event B iff $P(A | B) = P(A)$.

$$\longrightarrow P(A \cap B) = P(A) \cdot P(B)$$

draw and flip



$$P(\text{head}) = \frac{1}{2} \quad P(2\$) = \frac{1}{2}$$

$$P(\text{head} | \text{pink}) = \frac{1}{2} = P(\text{head})$$

exercice

Prove: $P(A \cap B) = P(A) \cdot P(B)$
for A, B independent

$$\begin{aligned} P(A \cap B) &= P(A | B) \cdot P(B) \\ &= P(A) \cdot P(B) \end{aligned}$$

□

conditional probabilities:

$$P(A | B) := \frac{P(A \cap B)}{P(B)}$$

A independent of B :

$$P(A | B) = P(A)$$

Bayes' theorem

posterior likelihood prior

The diagram illustrates the components of Bayes' theorem. Three labels at the top—'posterior', 'likelihood', and 'prior'—have arrows pointing down to the corresponding parts of the formula. 'posterior' points to $P(A | B)$, 'likelihood' points to $P(B | A)$, and 'prior' points to $P(A)$. The label 'marginal' at the bottom has an arrow pointing up to $P(B)$ in the denominator.

$$P(A | B) = \frac{P(B | A) \cdot P(A)}{P(B)}$$

marginal

exercice

prove Bayes' theorem:

$$P(A | B) = \frac{P(B | A) \cdot P(A)}{P(B)}$$

$$P(A \cap B) = P(A | B) \cdot P(B)$$

$$P(B \cap A) = P(B | A) \cdot P(A)$$

$$P(A \cap B) = P(B \cap A)$$

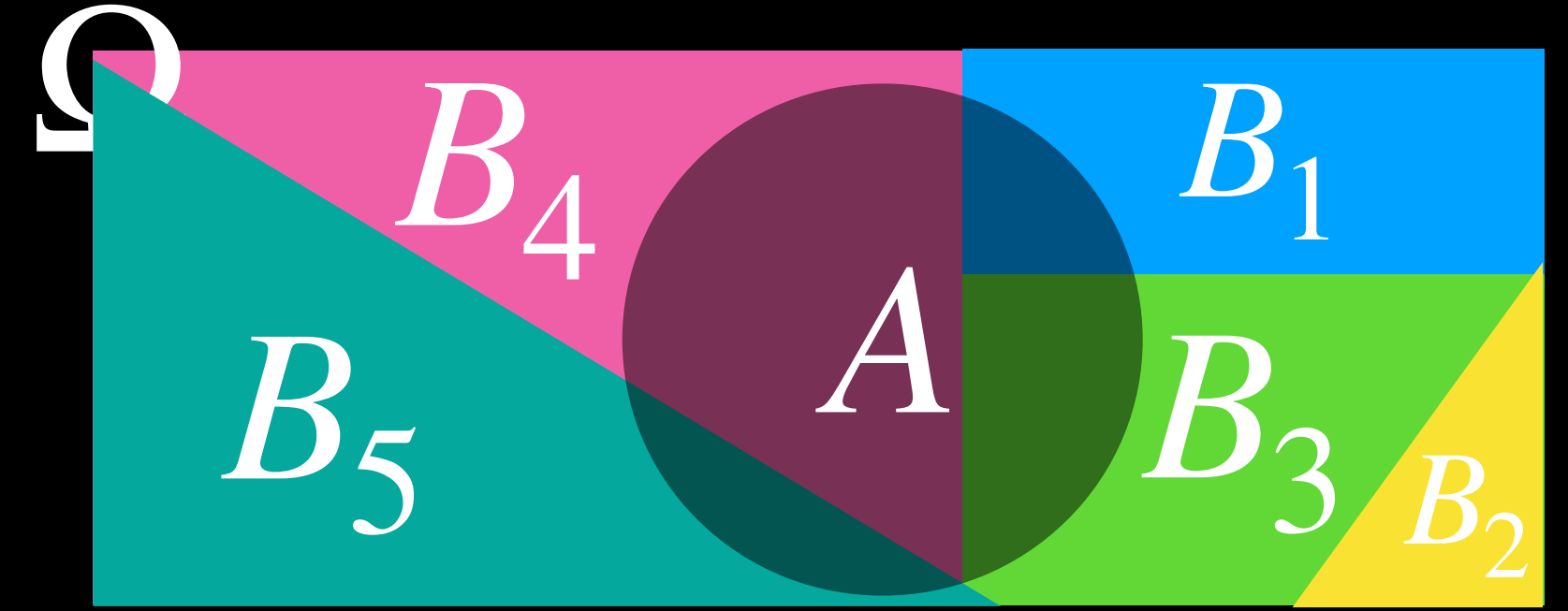
$$P(A | B) \cdot P(B) = P(B | A) \cdot P(A) \quad \longrightarrow \quad P(A | B) = \frac{P(B | A) \cdot P(A)}{P(B)}$$

conditional probabilities:

$$P(A | B) := \frac{P(A \cap B)}{P(B)}$$



Law of total probability



$$P(A) = \sum_{i=1}^n P(A | B_i) \cdot P(B_i)$$

$$\bigcup_{i=1}^n B_i = \Omega \quad B_i \cap B_j = \emptyset$$

prove

$$P(A \cup B) = P(A) + P(B) \quad \text{if } P(A \cap B) = 0$$

$$P(A \cap B) = P(A | B) \cdot P(B)$$

$$P(A) = P(A \cap \Omega) = P\left(A \cap \bigcup_{i=1}^n B_i\right) = P\left(\bigcup_{i=1}^n (A \cap B_i)\right) = \sum_{i=1}^n P(A \cap B_i) = \sum_{i=1}^n P(A | B_i) \cdot P(B_i)$$

probability expansion

$$\begin{aligned} P(A_1 \cap A_2 \cap \dots \cap A_n) \\ = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot \dots \cdot P(A_n | A_1 \cap \dots \cap A_{n-1}) \end{aligned}$$

prove idea

$$P(A \cap B) = P(A | B) \cdot P(B)$$

$$B_{n-1} := A_1 \cap \dots \cap A_{n-1}$$

$$\begin{aligned} P(A_1 \cap \dots \cap A_n) &= P(A_n \cap B_{n-1}) = P(A_n | B_{n-1}) \cdot P(B_{n-1}) \\ &= P(A_1 \cap \dots \cap A_{n-1}) \cdot P(A_n | A_1 \cap \dots \cap A_{n-1}) \\ &= \dots \end{aligned}$$

probability expansion example

draw 4 cards from 52 cards deck

What is the probability that all 4 aces are drawn?

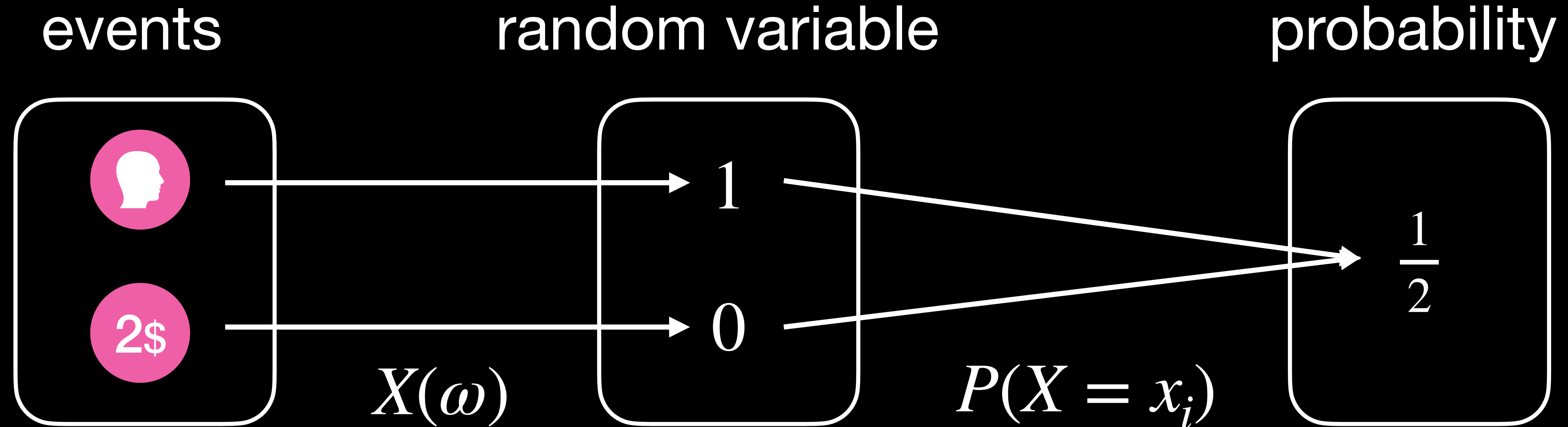
A_i : draw an aces in i -th draw

$$P(A_1) = \frac{4}{52}, \quad P(A_2 | A_1) = \frac{3}{51}, \quad P(A_3 | A_1 \cap A_2) = \frac{2}{50}, \quad P(A_4 | A_1 \cap A_2 \cap A_3) = \frac{1}{49}$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49}$$

random variable

A random variable is a number associated to an event.



Random variable

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binary random variable/ indicator variable/ Bernoulli random variables

$$\mathbb{B} = \{0,1\}$$

$$I : \Omega \rightarrow \mathbb{B}$$

$$I(\omega) = \begin{cases} 1 & \text{if } \omega = \tilde{\omega} \\ 0 & \text{else} \end{cases} \quad \tilde{\omega} \in \Omega$$

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discrete random variable

$$S := \{x_1, x_2, x_3, \dots\} \quad \text{finite or countable infinite}$$

$$X : \Omega \rightarrow S \quad X(\omega) = x_\omega \in S$$

$$P(X = x_i) = ?$$

Random variable

A random variable is a number associated to an event.

binary random variable/ indicator variable/ Bernoulli random variables

$$\mathbb{B} = \{0,1\} \quad I : \Omega \rightarrow \mathbb{B} \quad I(\omega) = \begin{cases} 1 & \text{if } \omega = \tilde{\omega} \quad \tilde{\omega} \in \Omega \\ 0 & \text{else} \end{cases}$$

discrete random variable

$$S := \{x_1, x_2, x_3, \dots\} \quad X : \Omega \rightarrow S \quad X(\omega) = x_\omega \in S$$

continuous random variables

$$J \subseteq \mathbb{R}$$

$$X : \Omega \rightarrow J$$

$$P(X = x_i) = 0$$

$$P(X > x_i) = ?$$

end