stochastics and probability

Lecture 4

Dr. Johannes Pahlke

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$\{X_n : n \in \mathbb{N}\}$ follows the strong law of large numbers iff, $P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1.$

(SLLN) (behavior at the limit)

theorem:

The strong law of large numbers implies the weak law of large numbers. (SLLN $\to$ WLLN)

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If $\{X_n : n \in \mathbb{N}\}$ is an i.i.d. process and $\mathbb{E}(\bar{X}_n)$ exists then

laws of large numbers

def.: Given an a stochastic process $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \to \mathbb{S}$.

Be $\mu = \mathbb{E}[X_0] = \mathbb{E}[X_1] = \dots$. and $\bar{X}_n := \frac{1}{n}\sum_{i=0}^{n-1} X_i$.

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The strong law of large numbers implies the weak law of large numbers. (SLLN → WLLN)

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If $\{X_n : n \in \mathbb{N}\}$ is an i.i.d. process and $\mathbb{E}(\bar{X}_n)$ exists then strong law of large numbers holds.

Monte Carlo integration (intuition)

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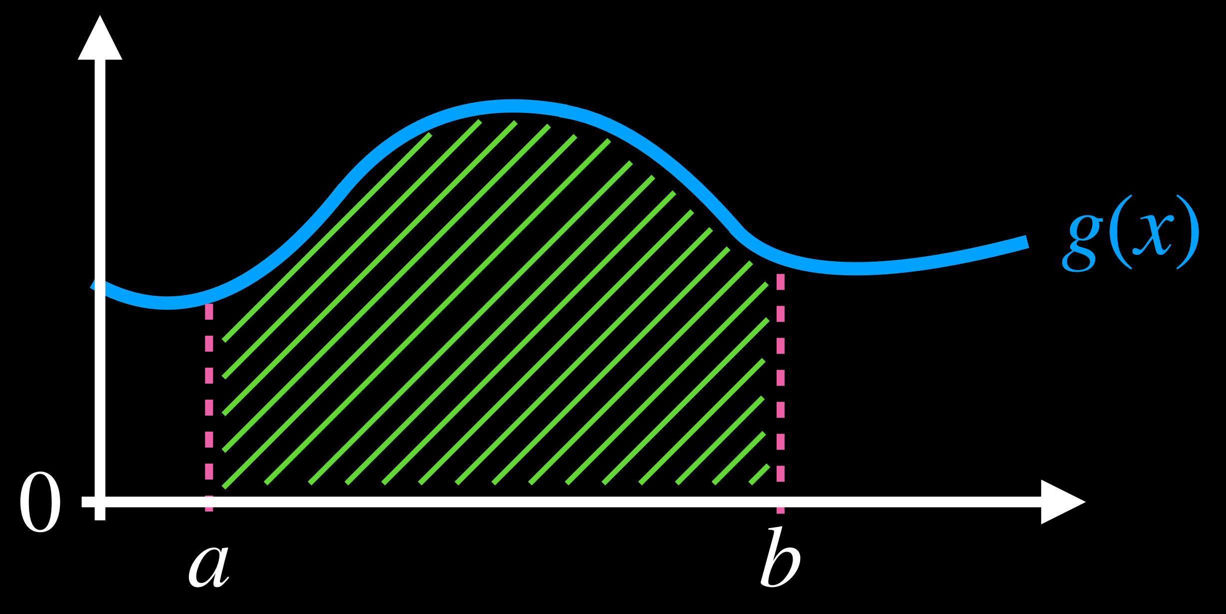
$$\int_a^b g(x)\,dx$$

Monte Carlo integration (intuition)



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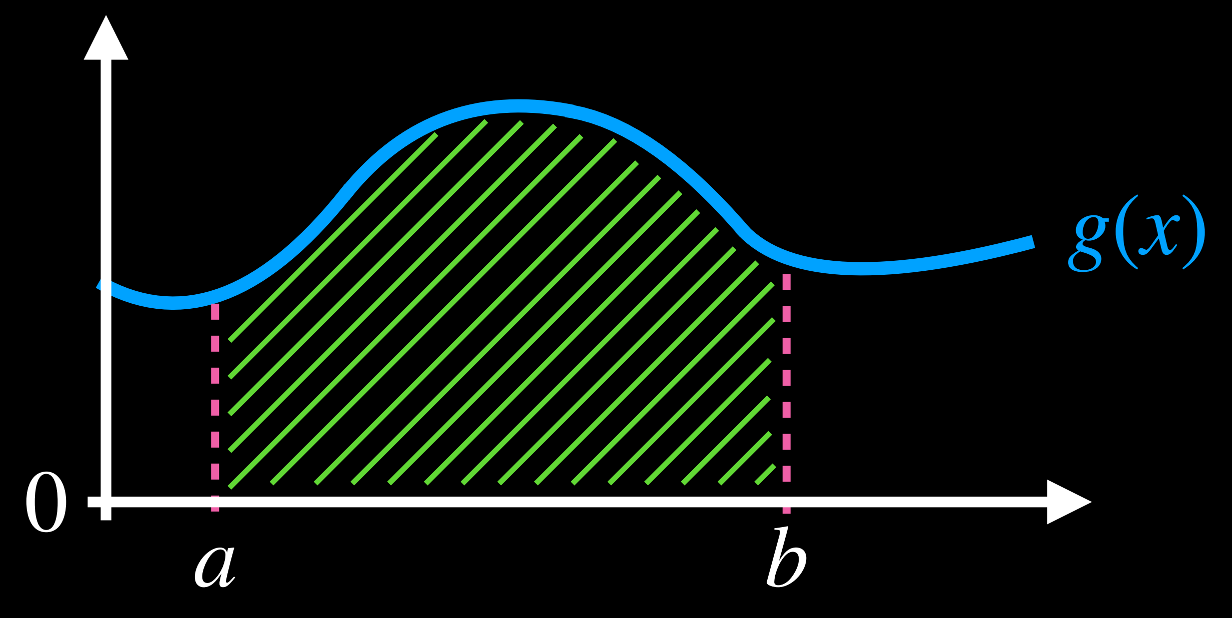

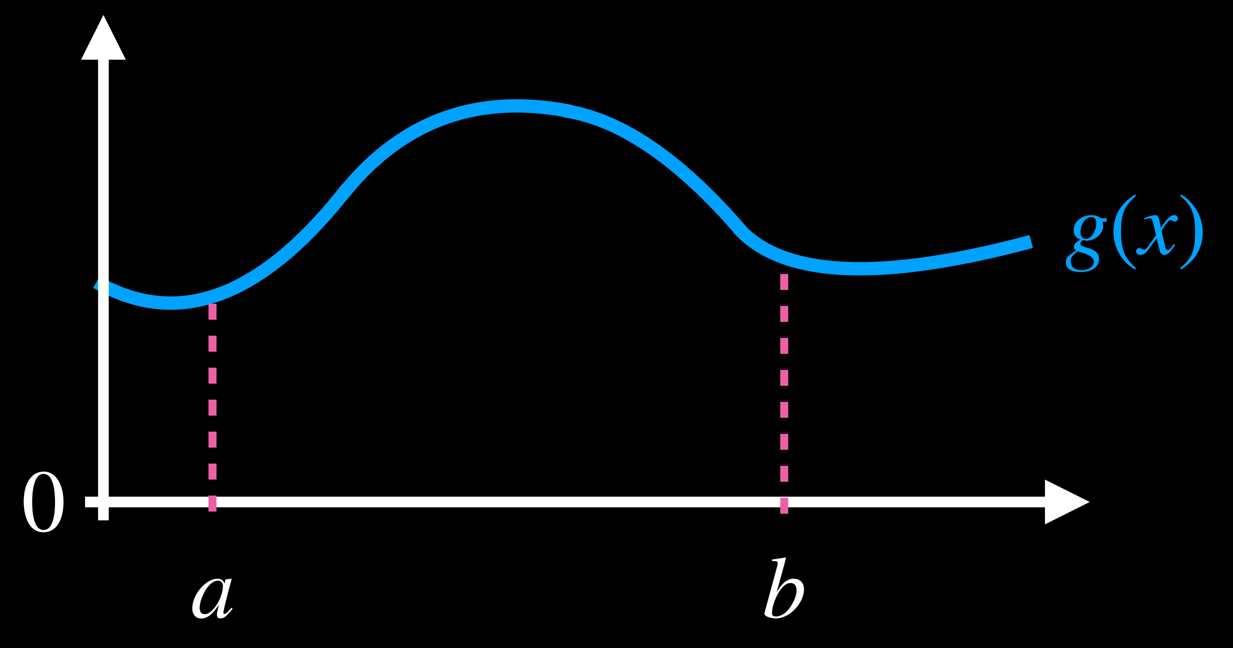

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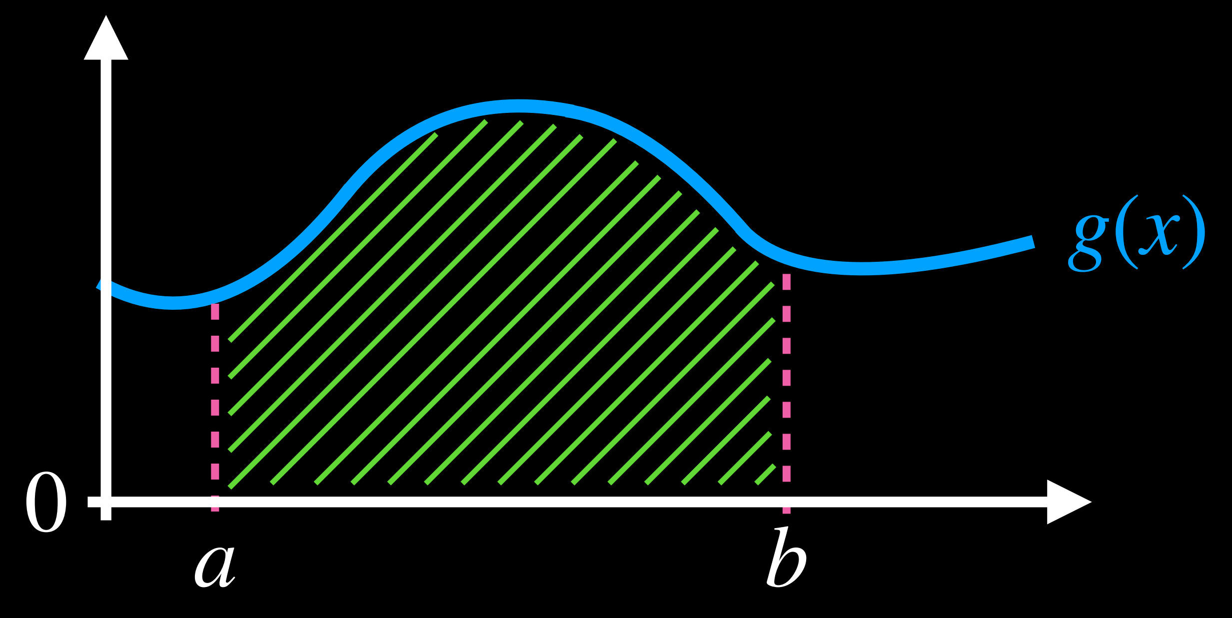

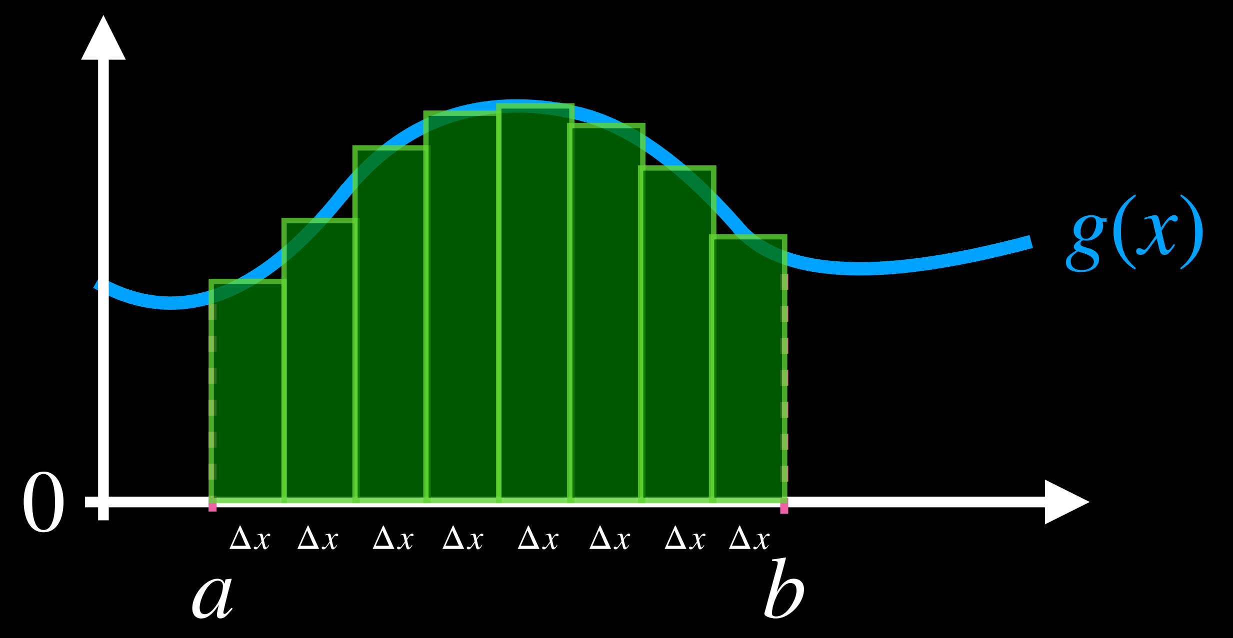

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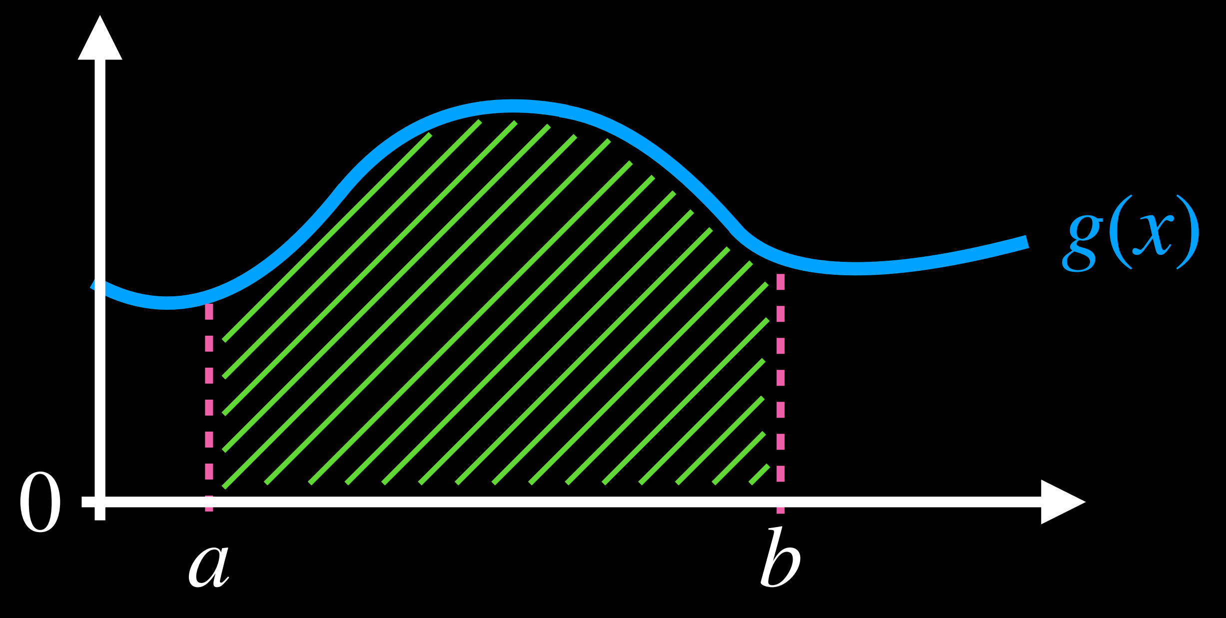

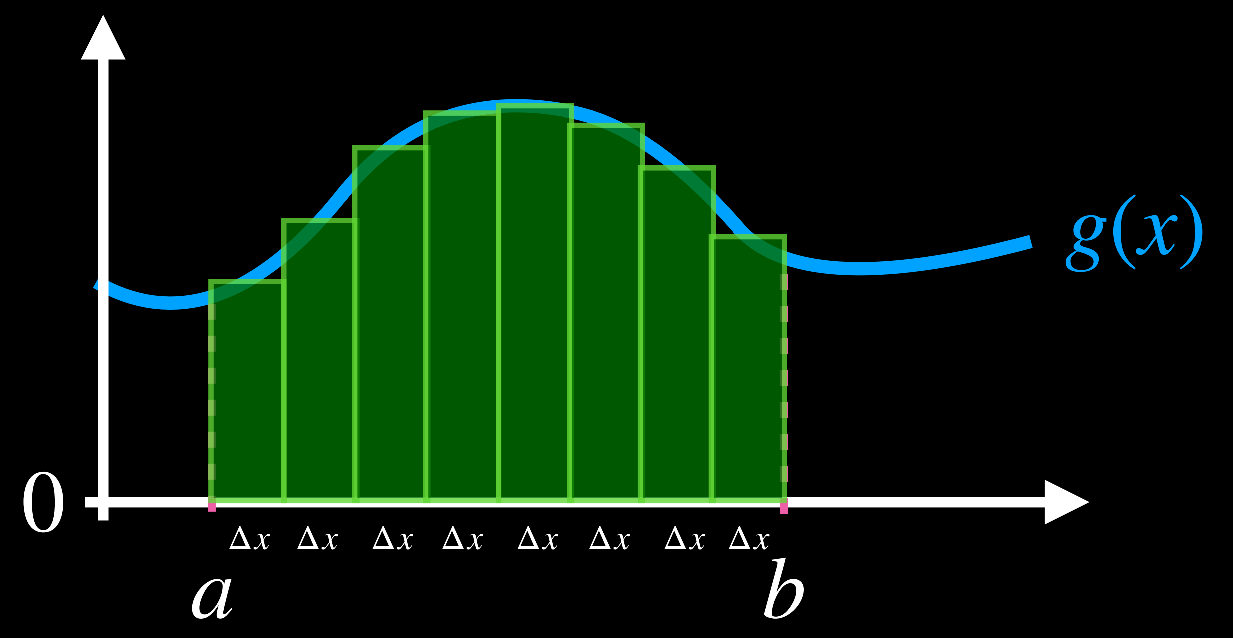

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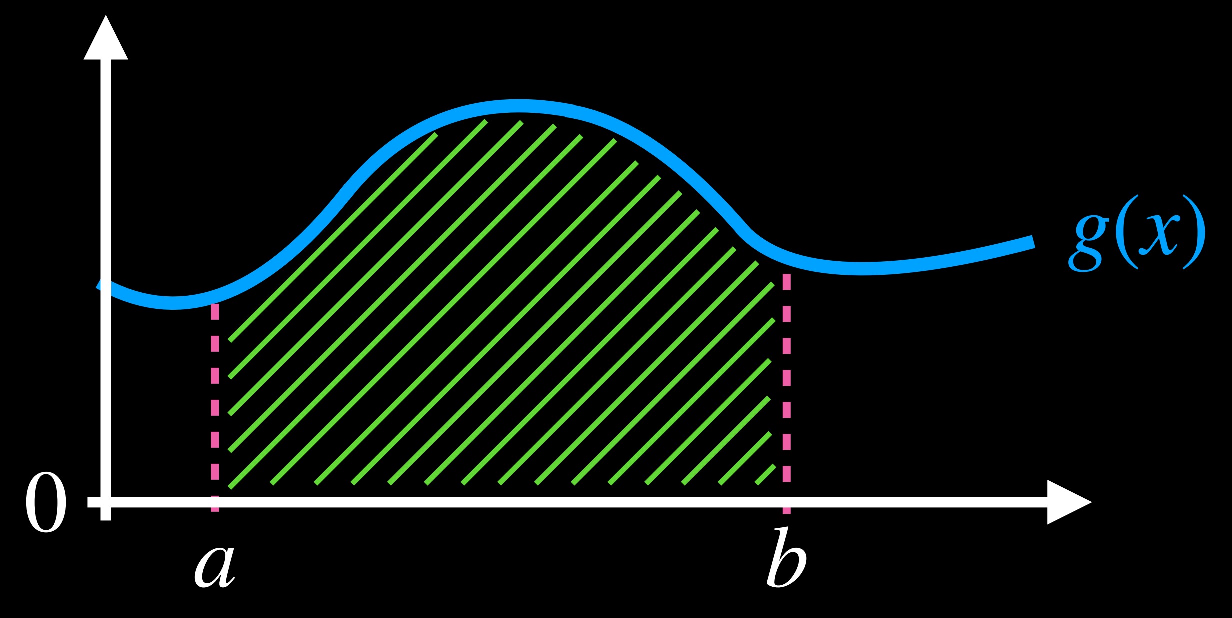

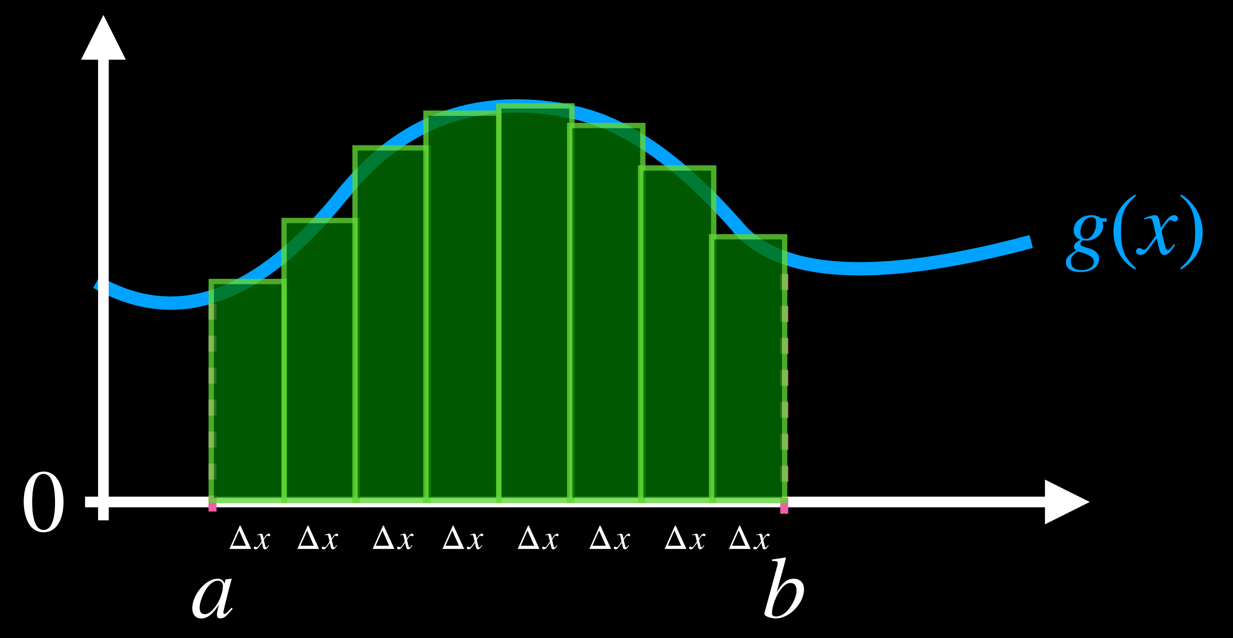

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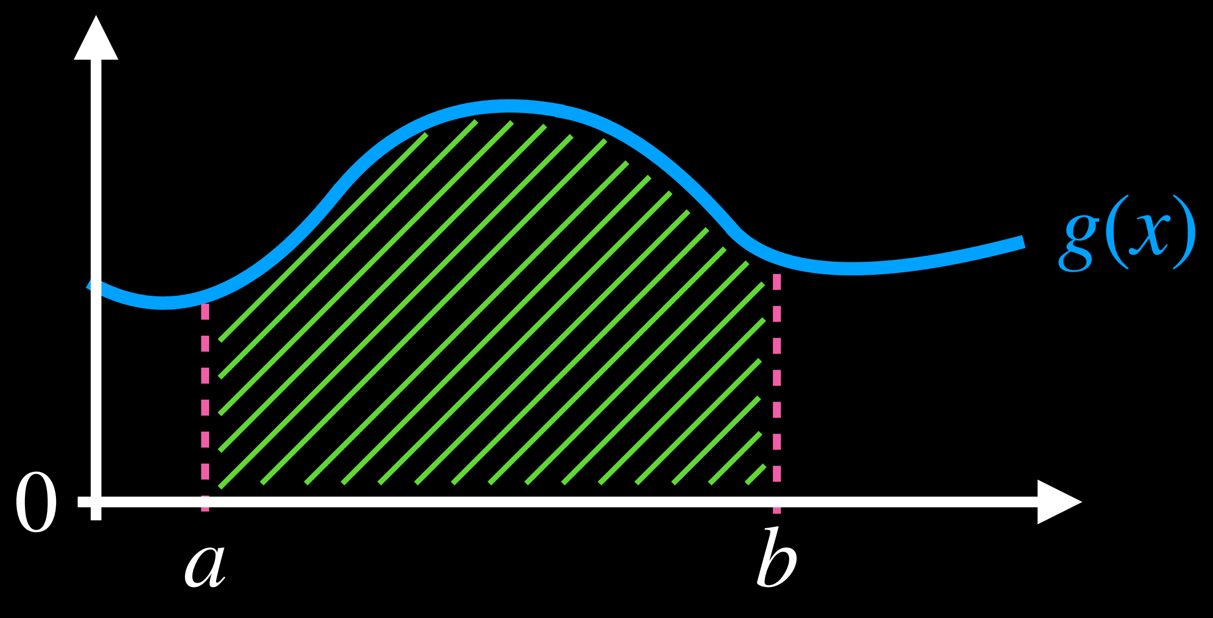

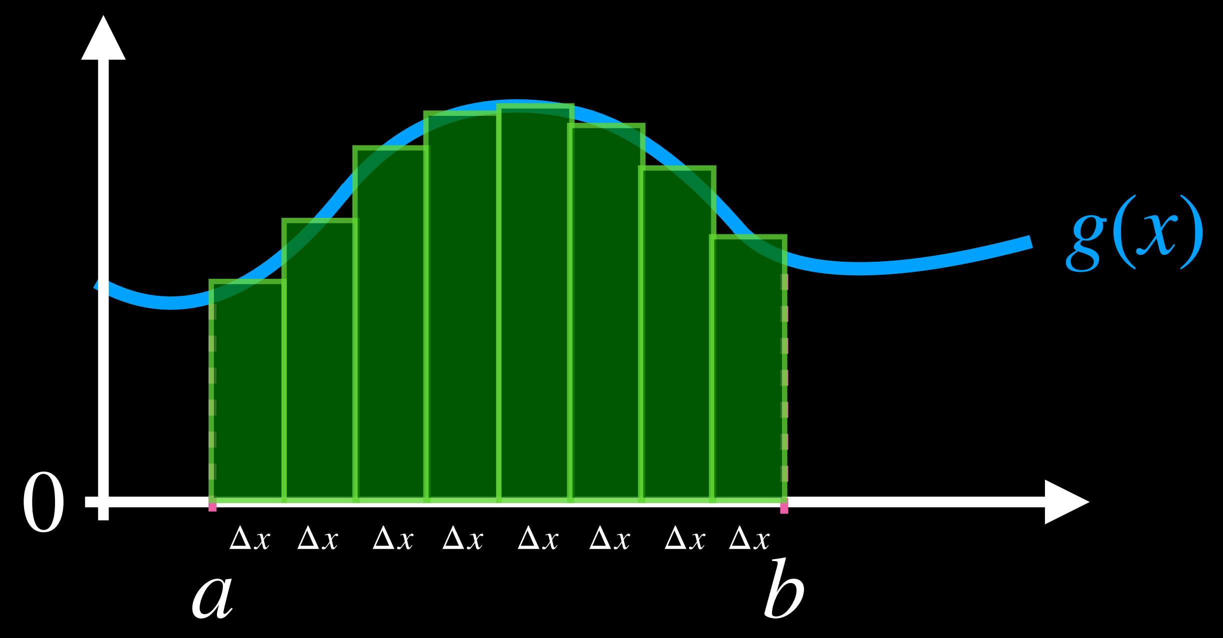

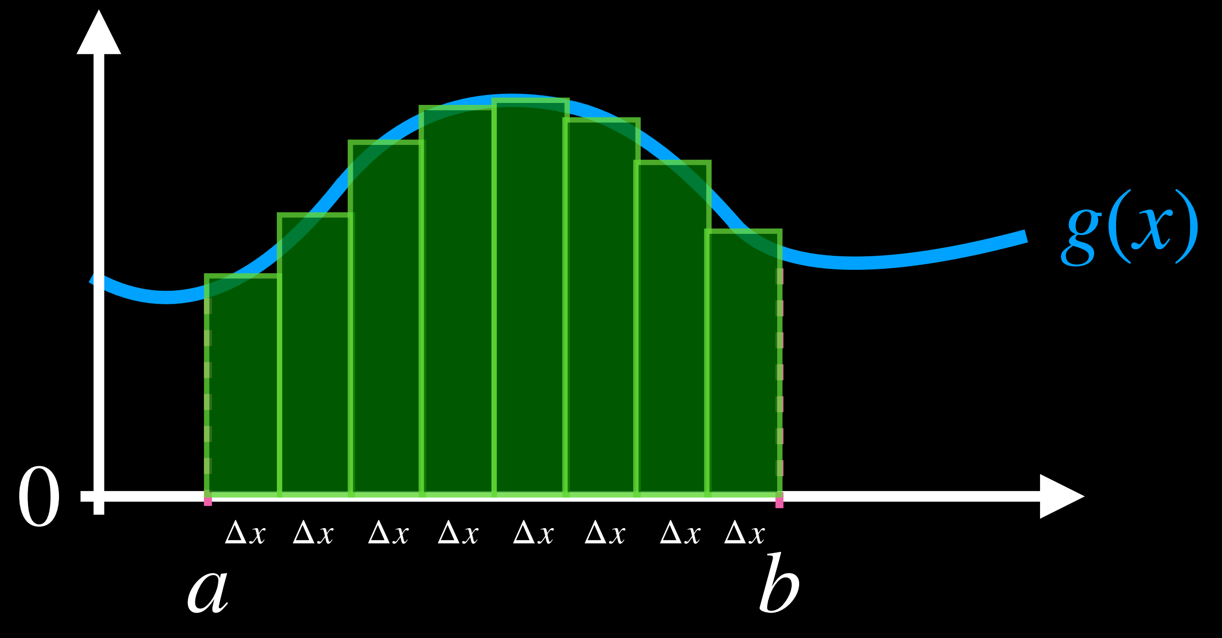

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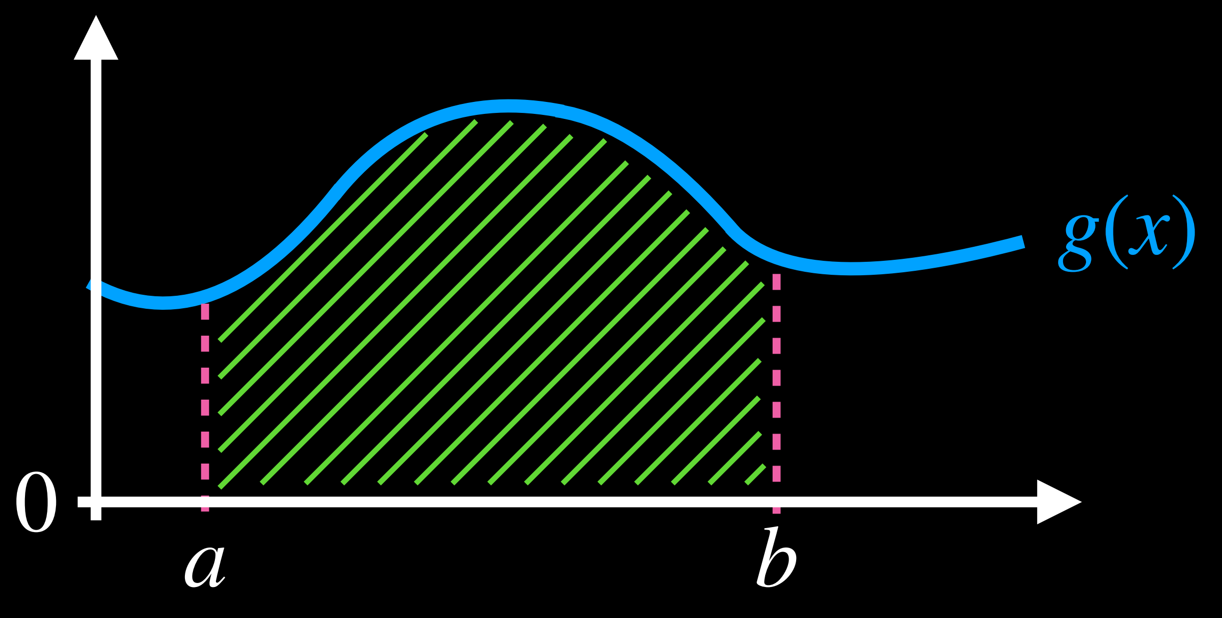

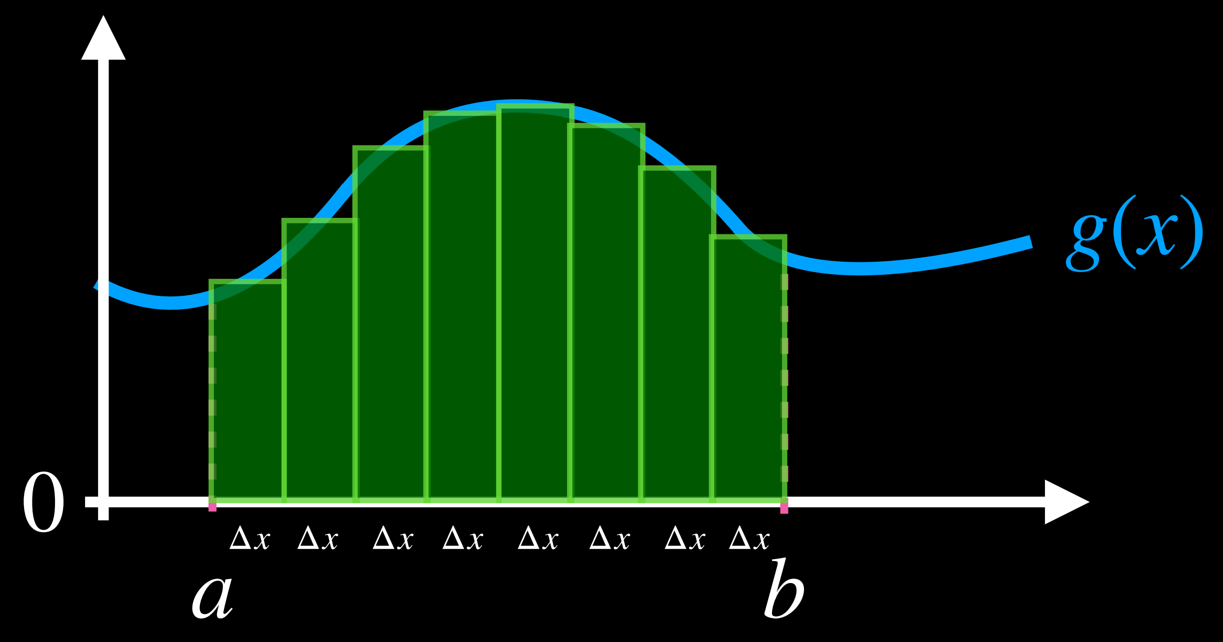

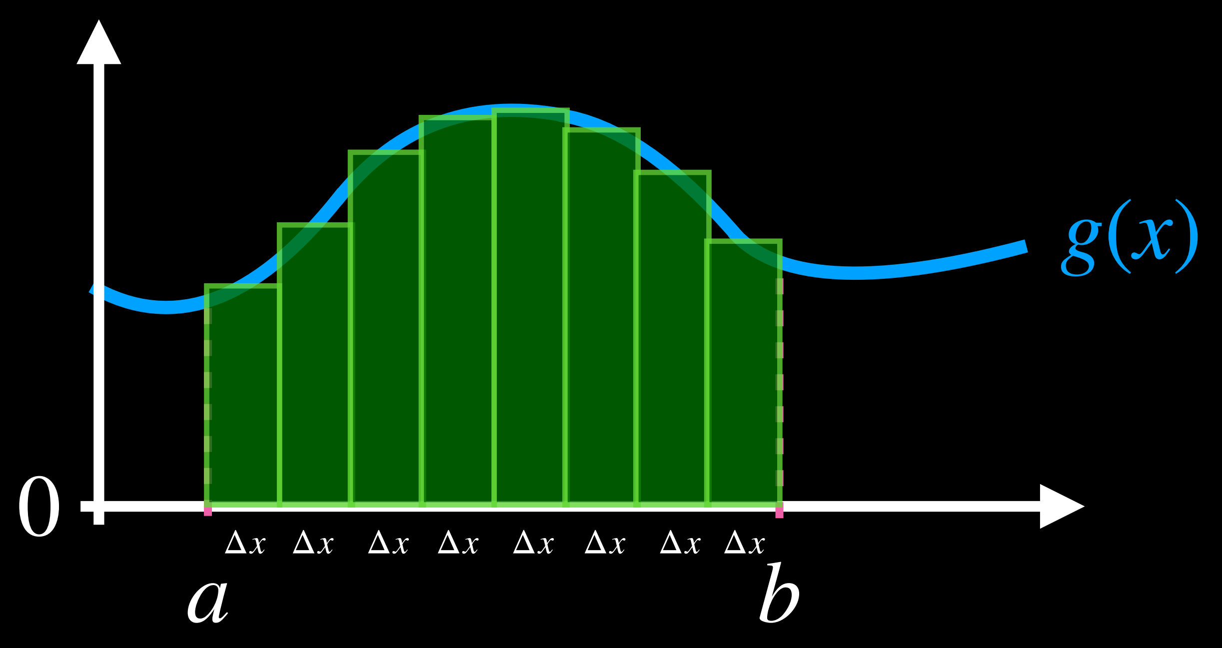

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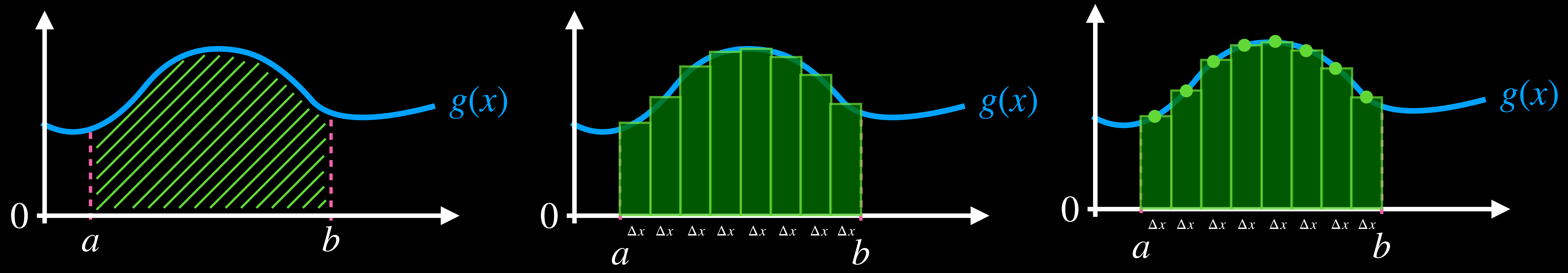

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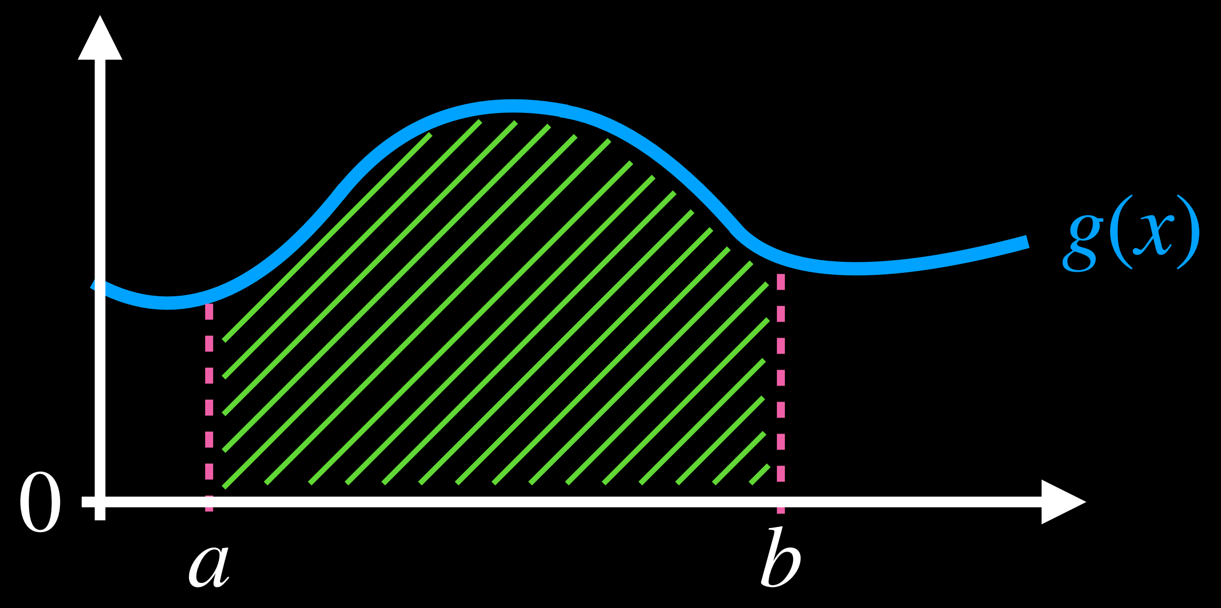

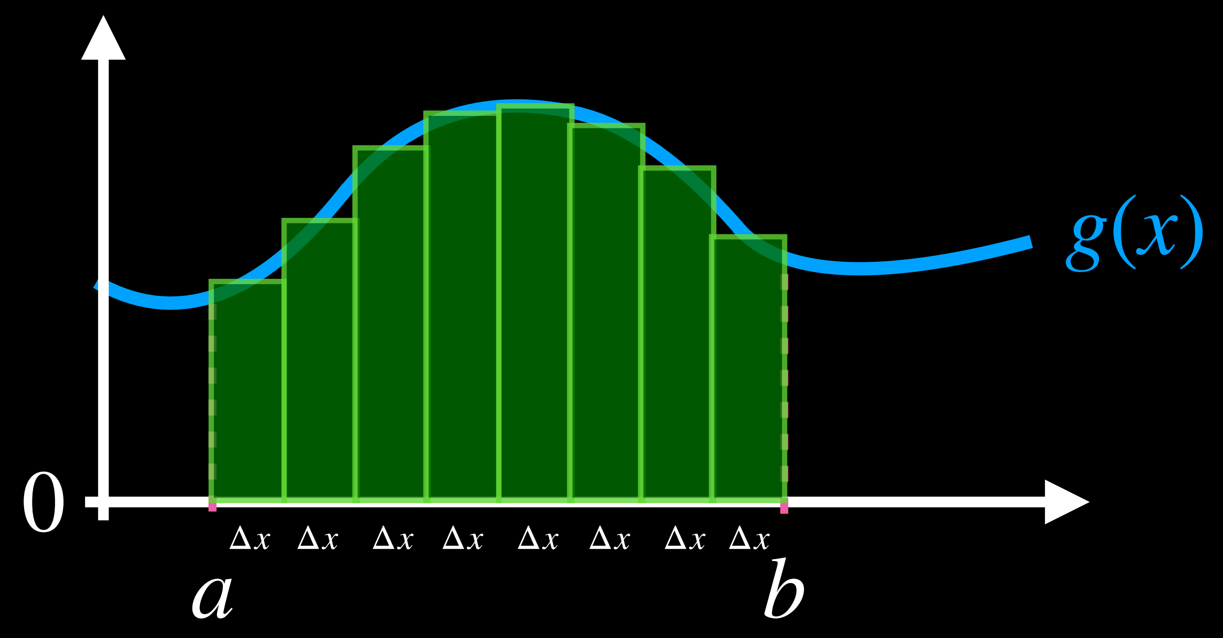

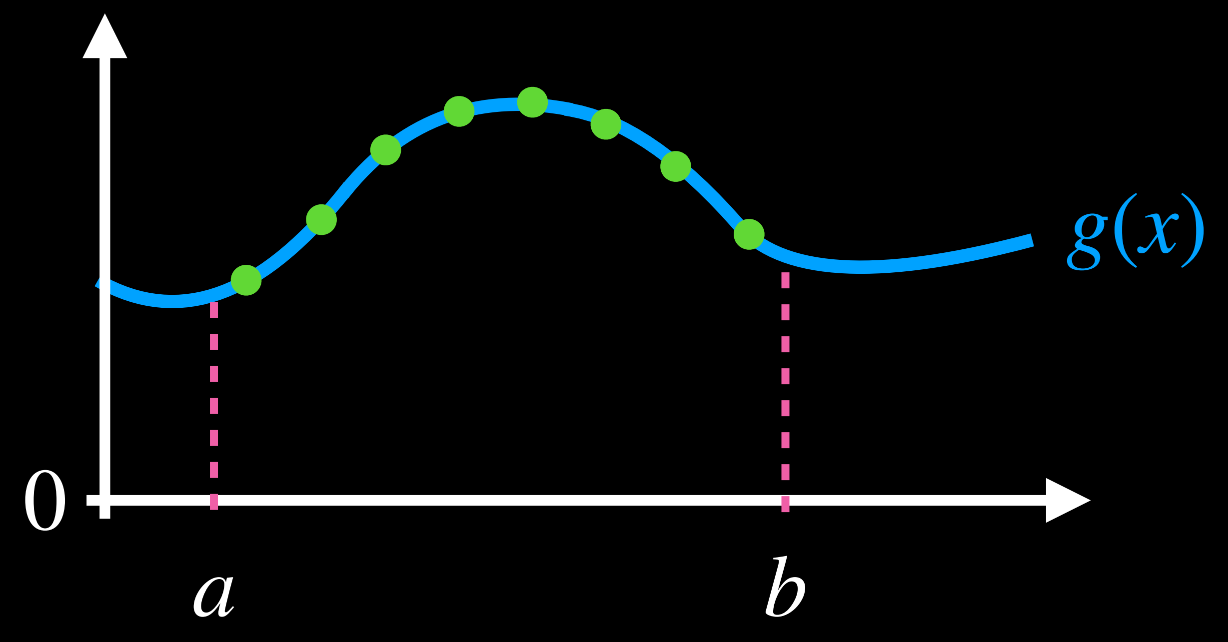

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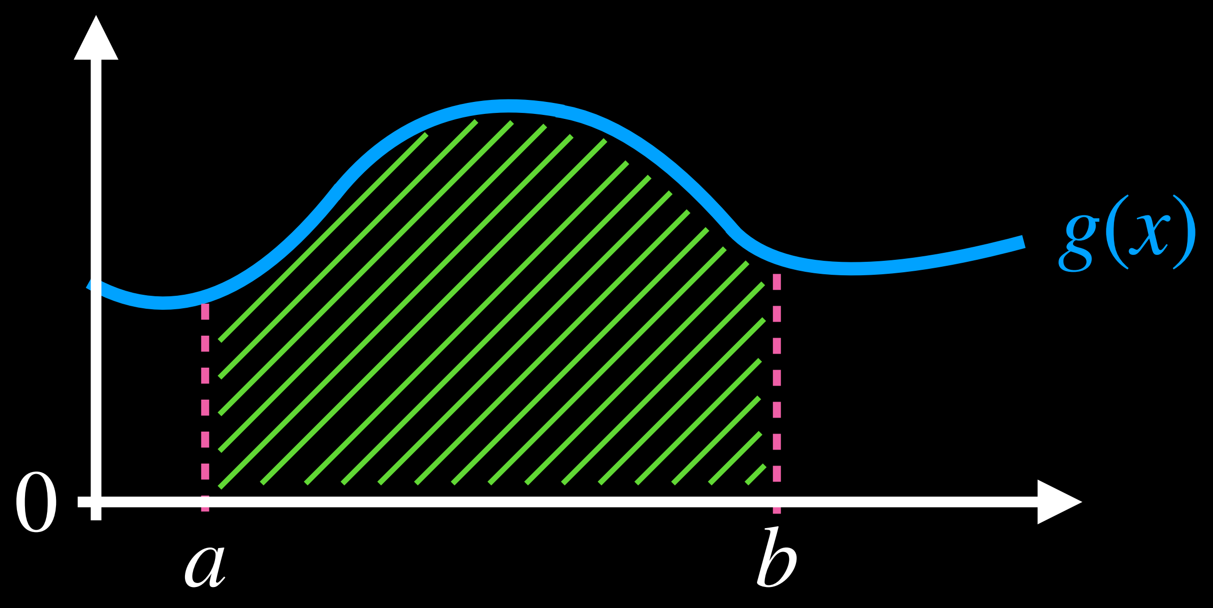

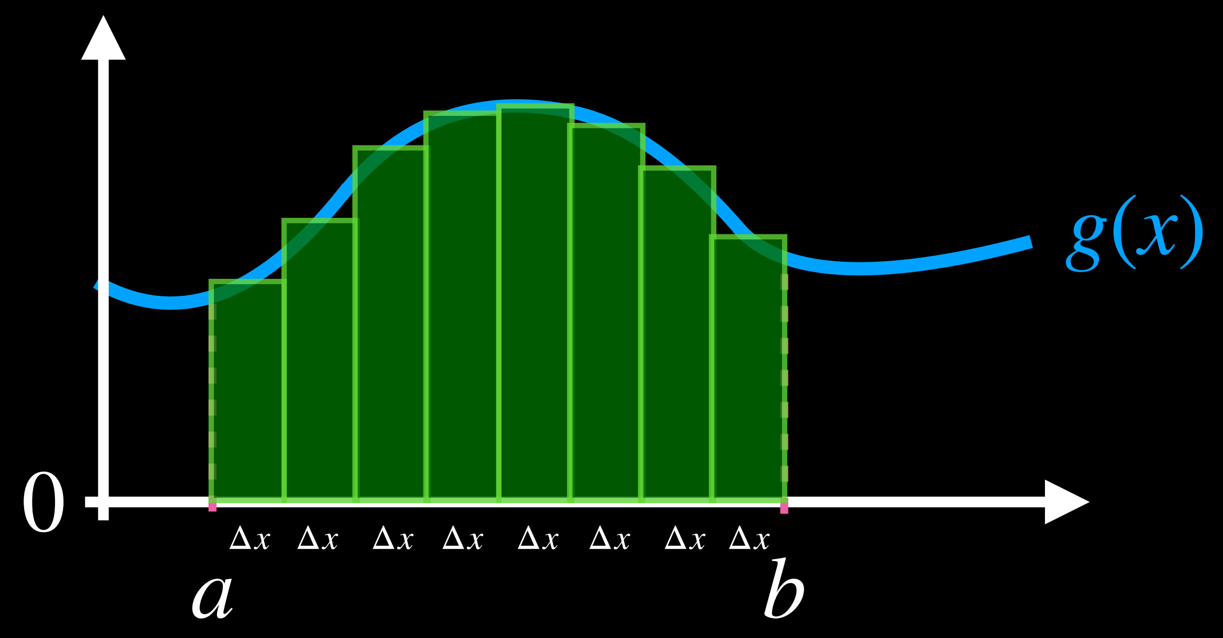

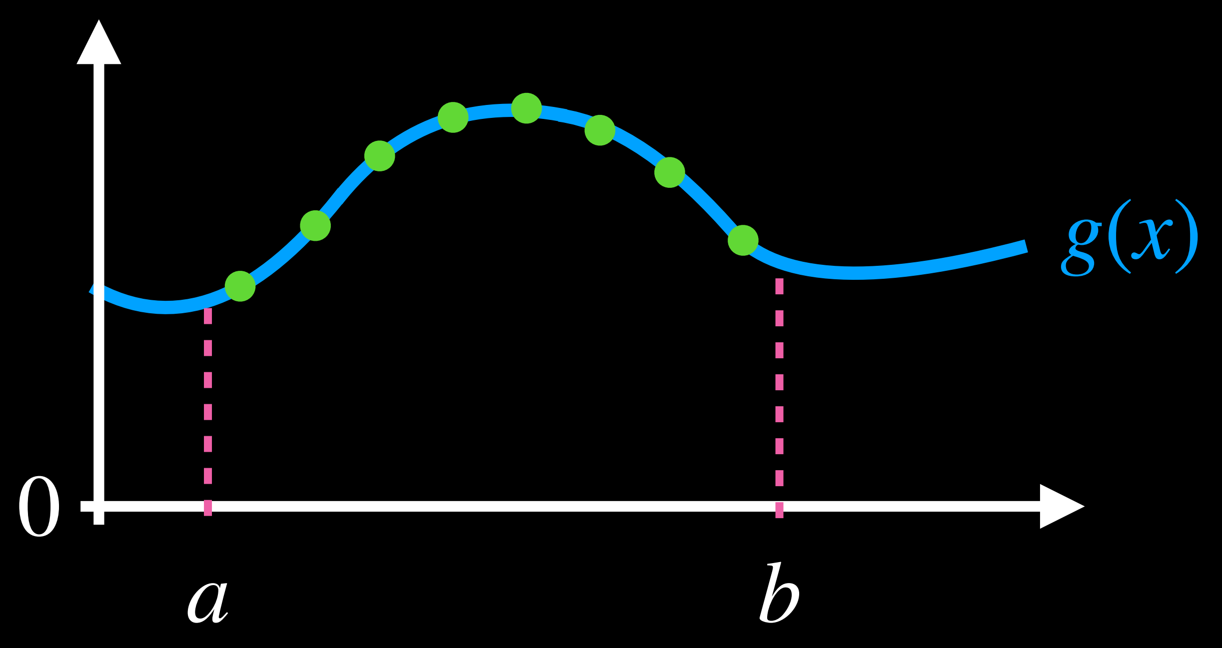

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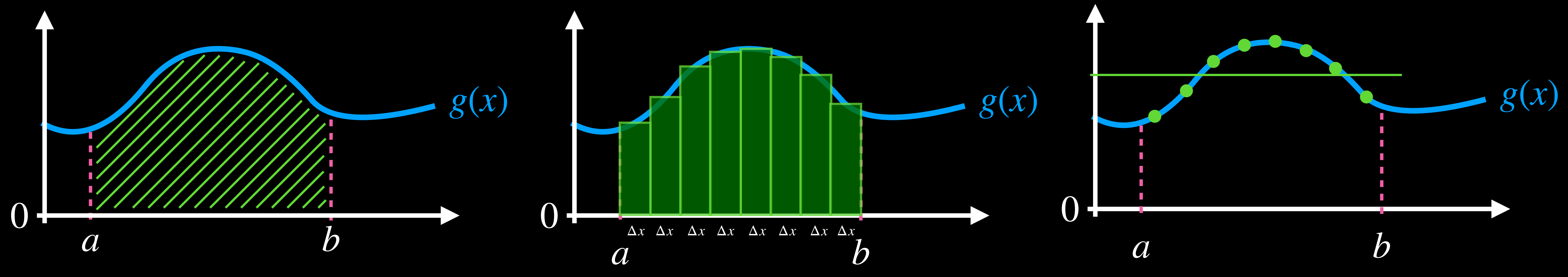

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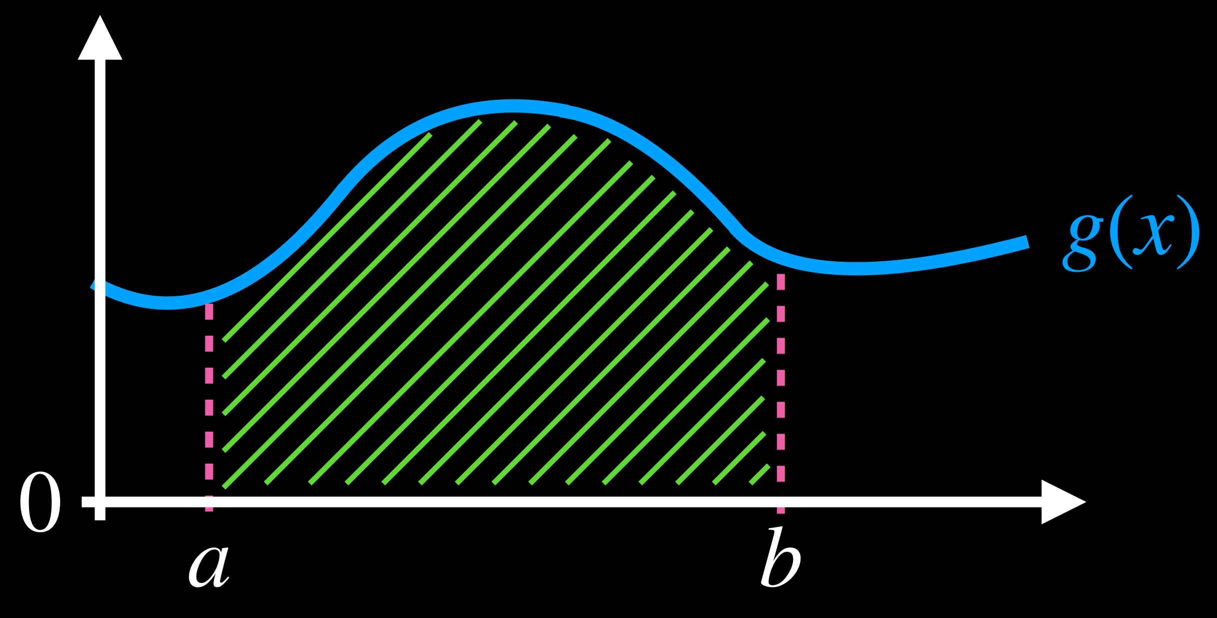

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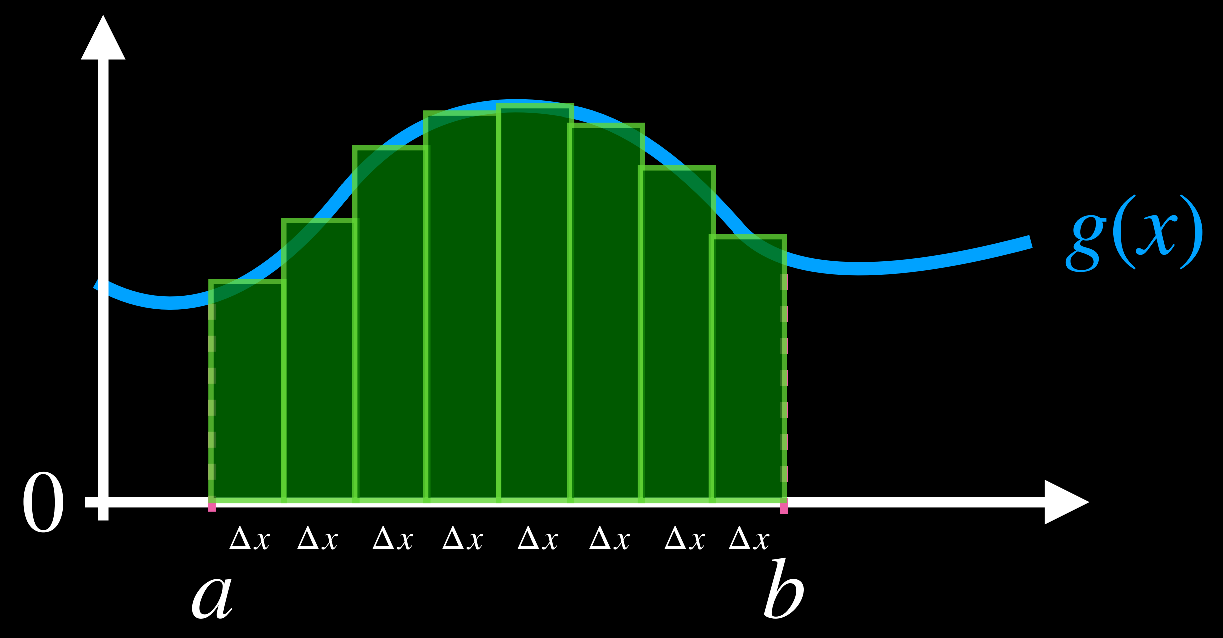

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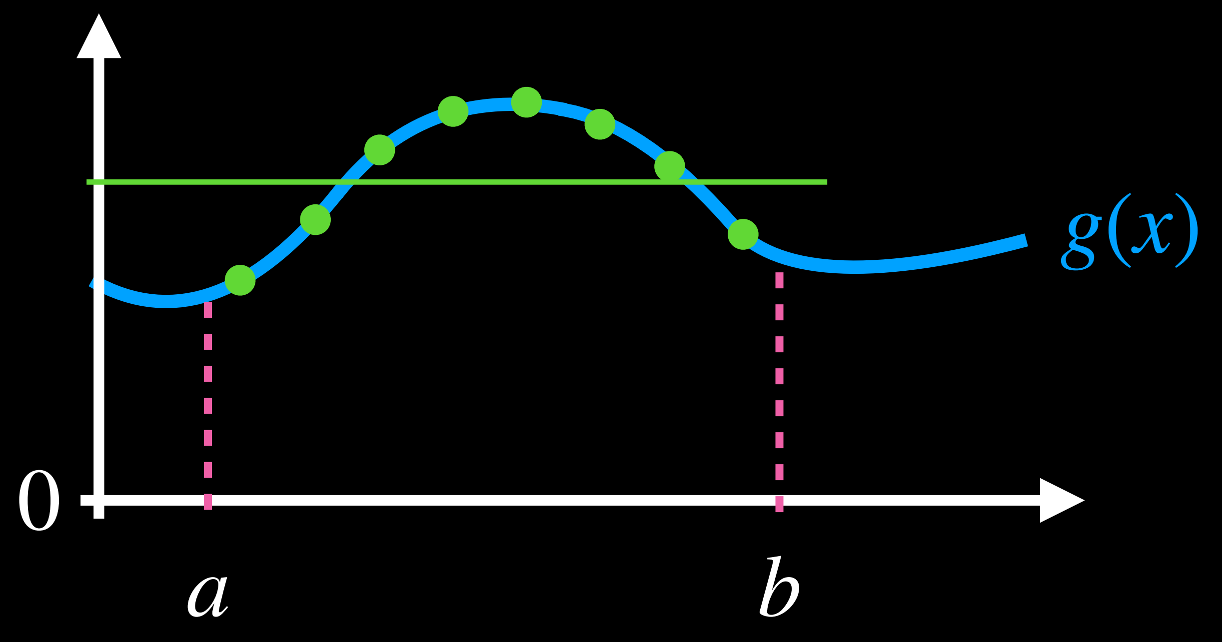

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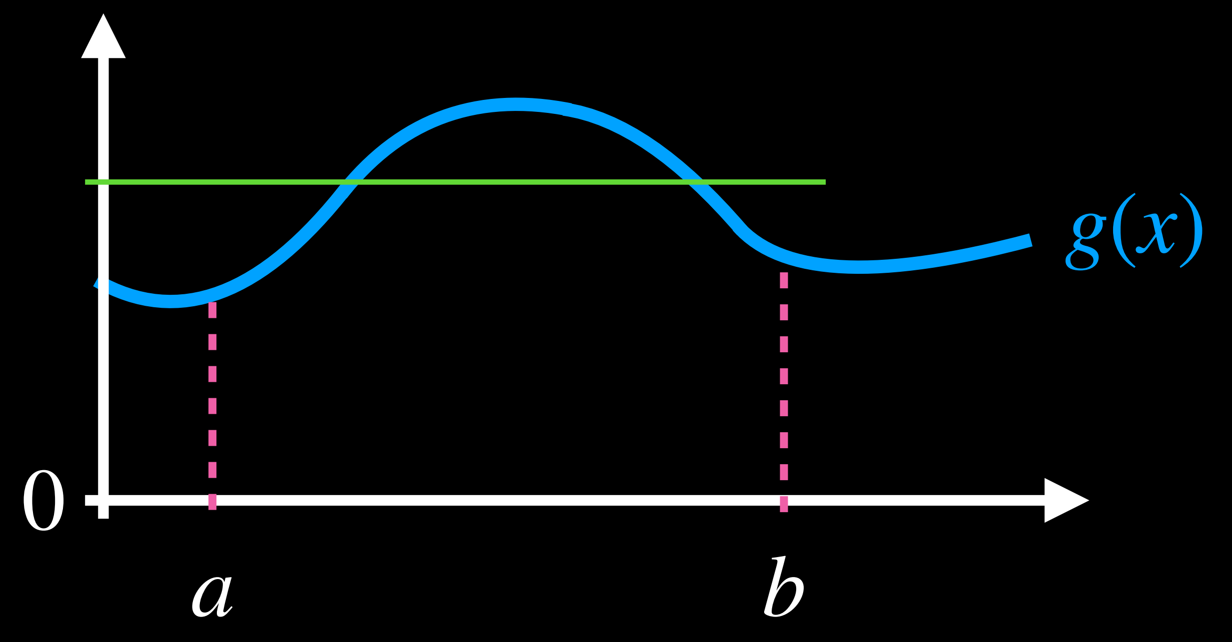

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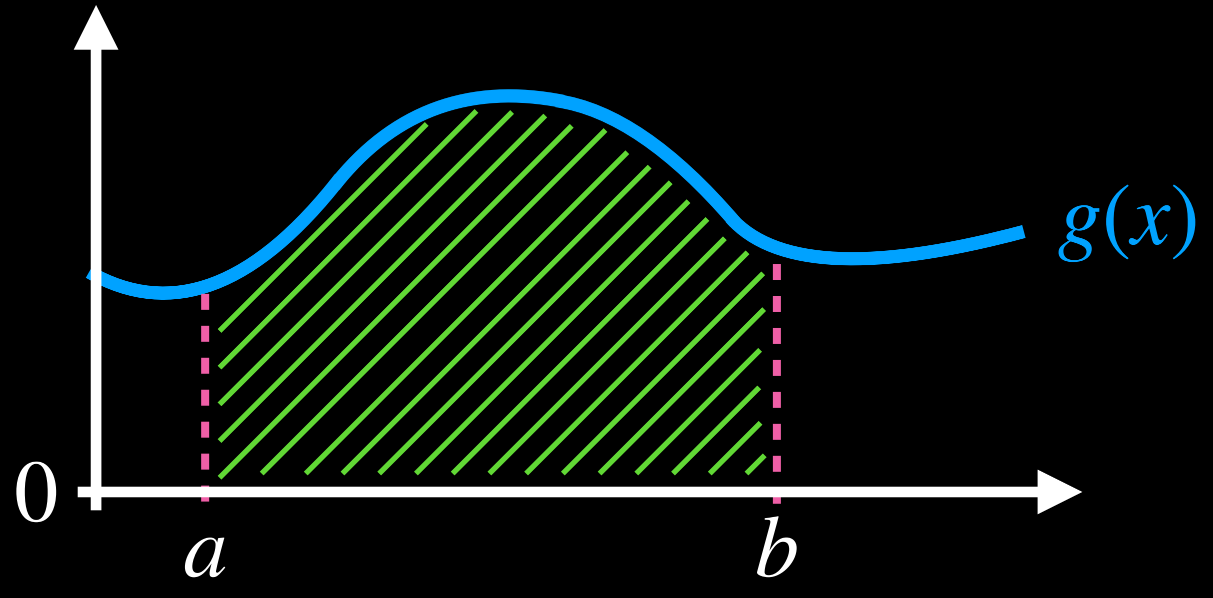

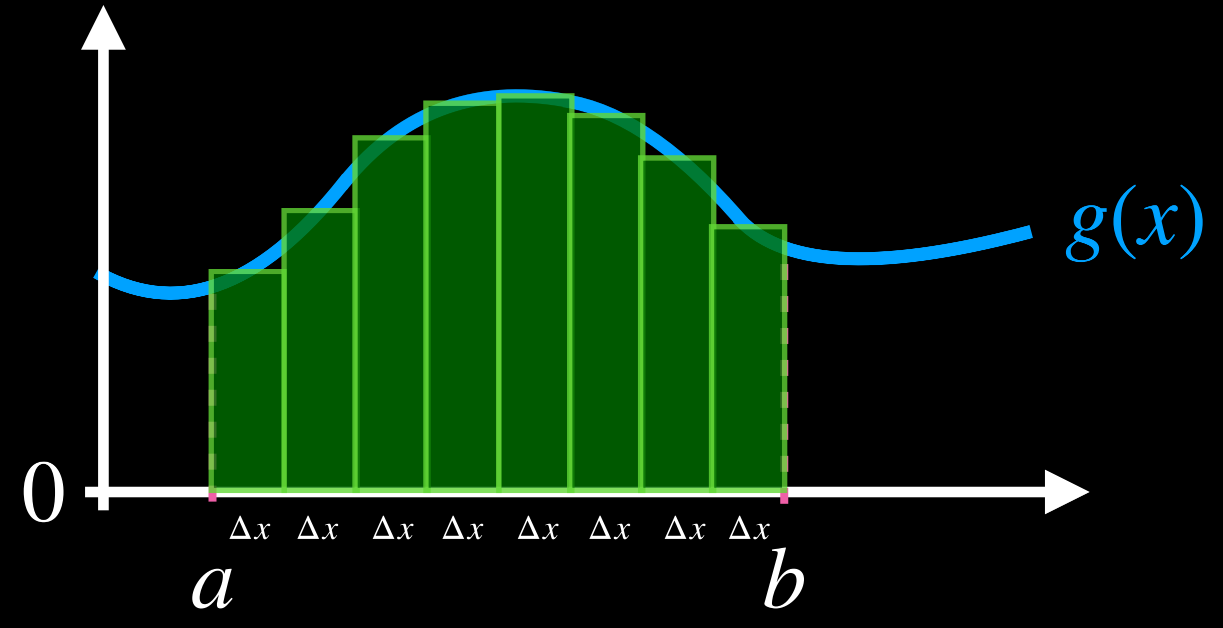

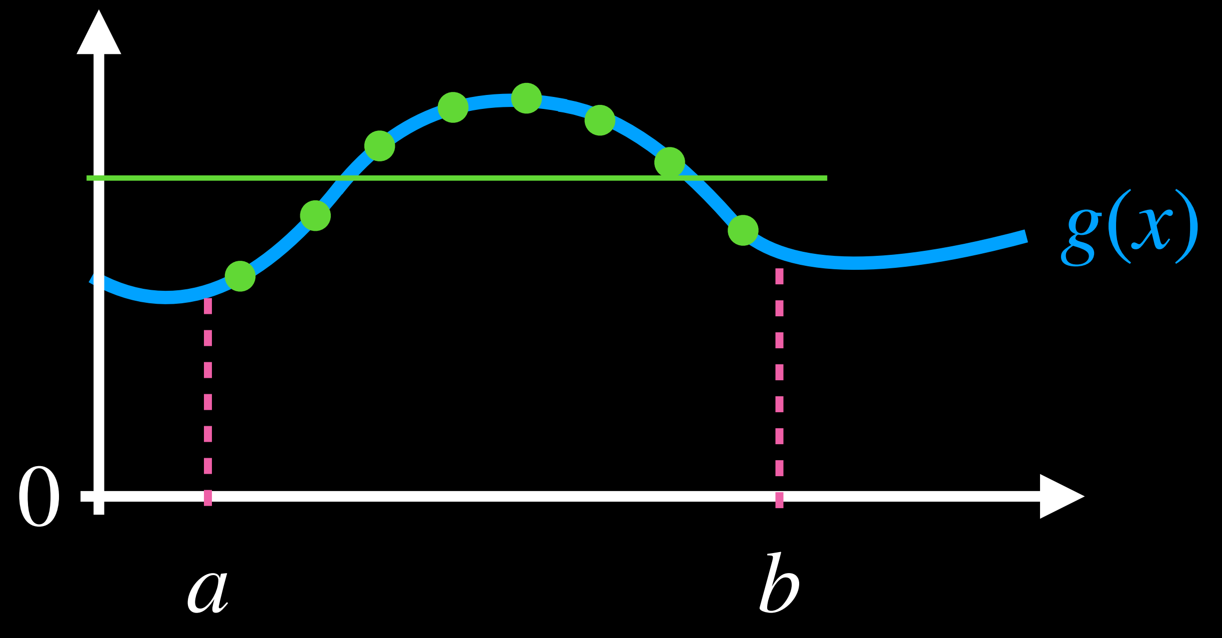

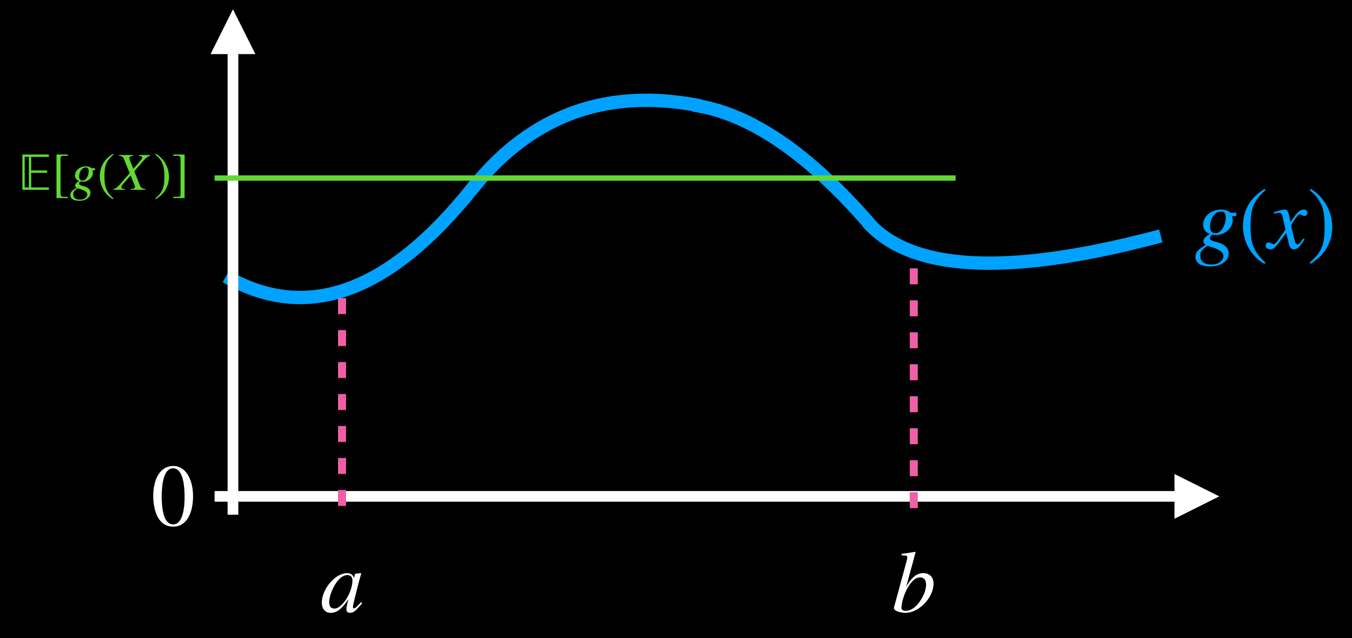

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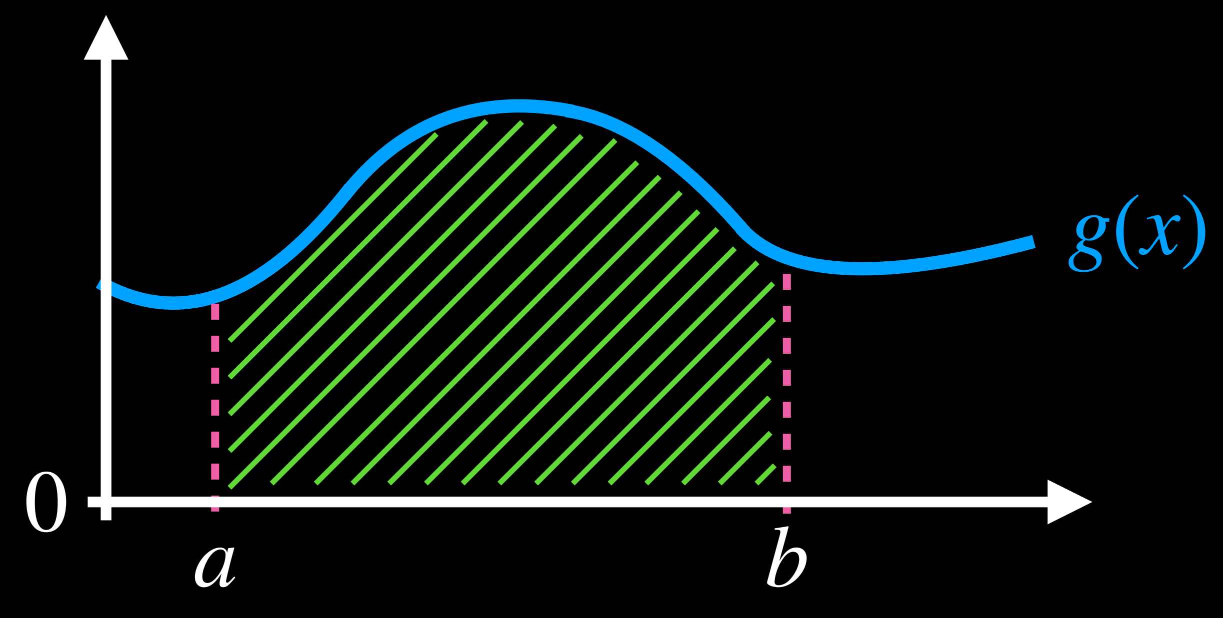

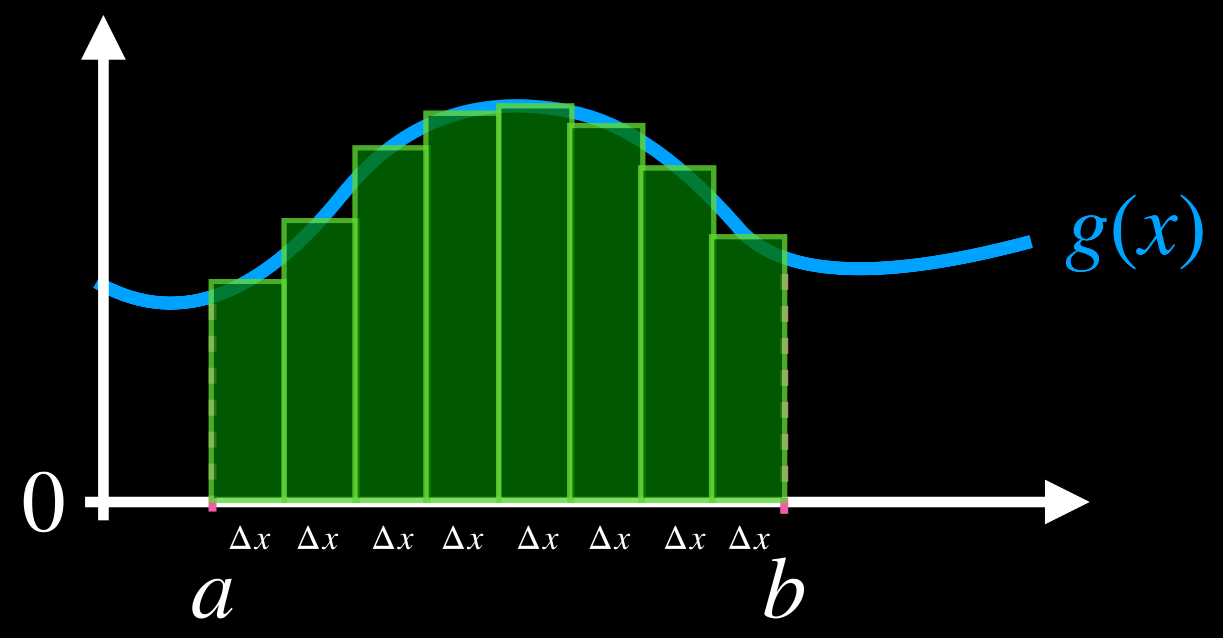

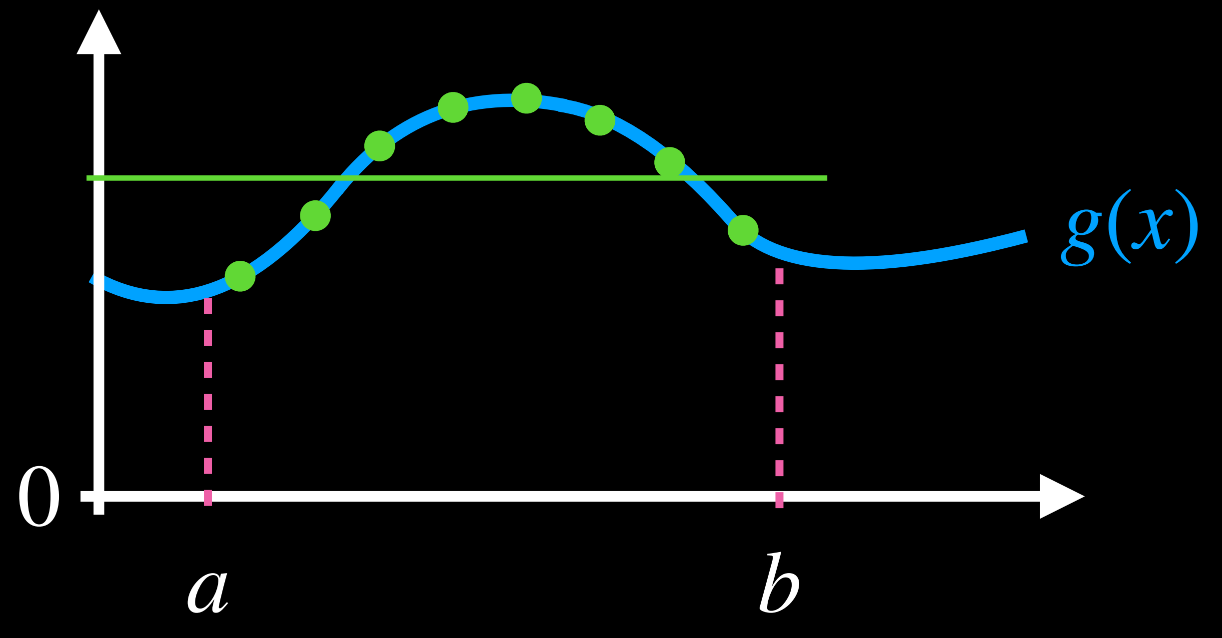

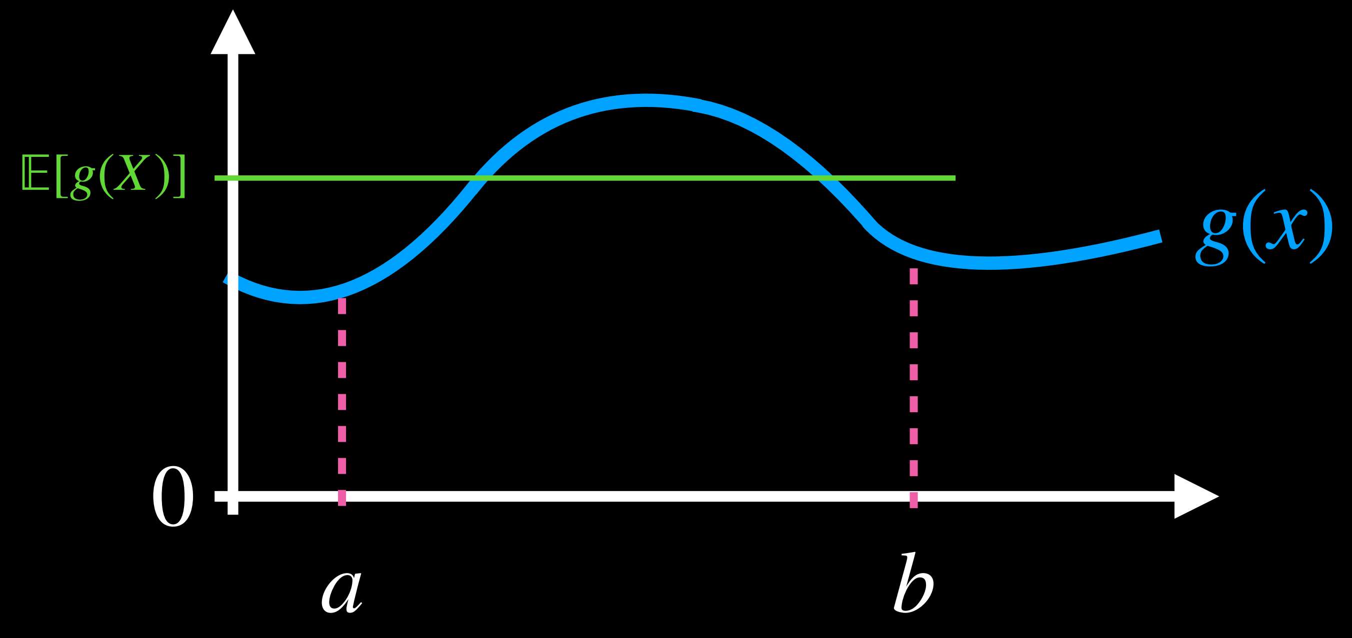

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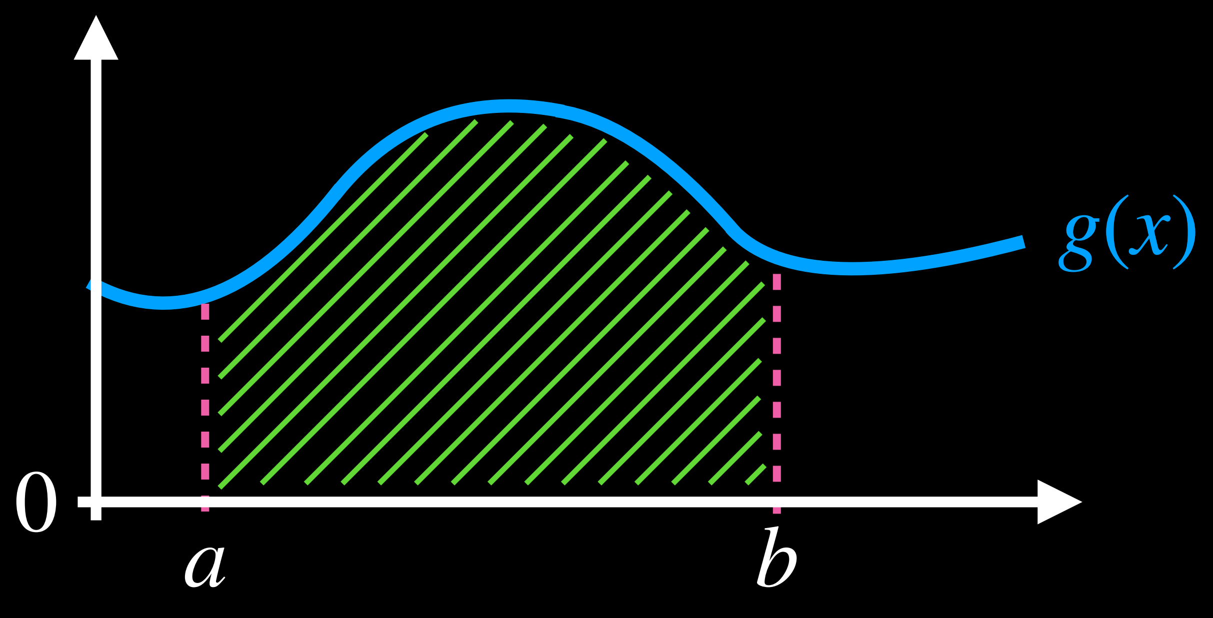

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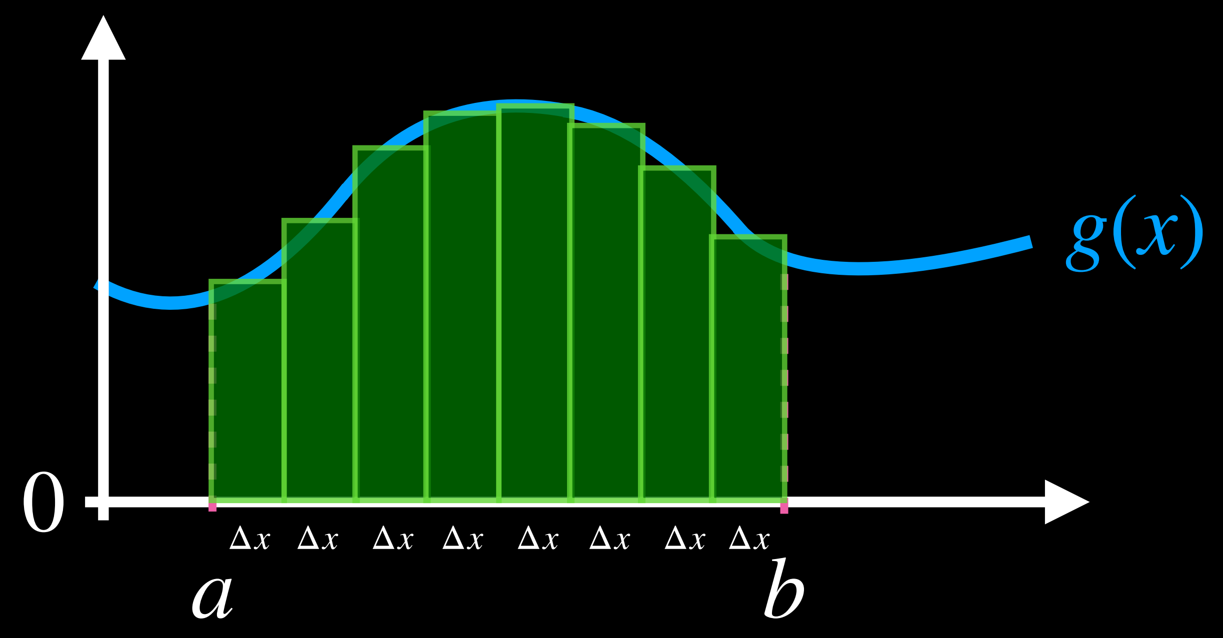

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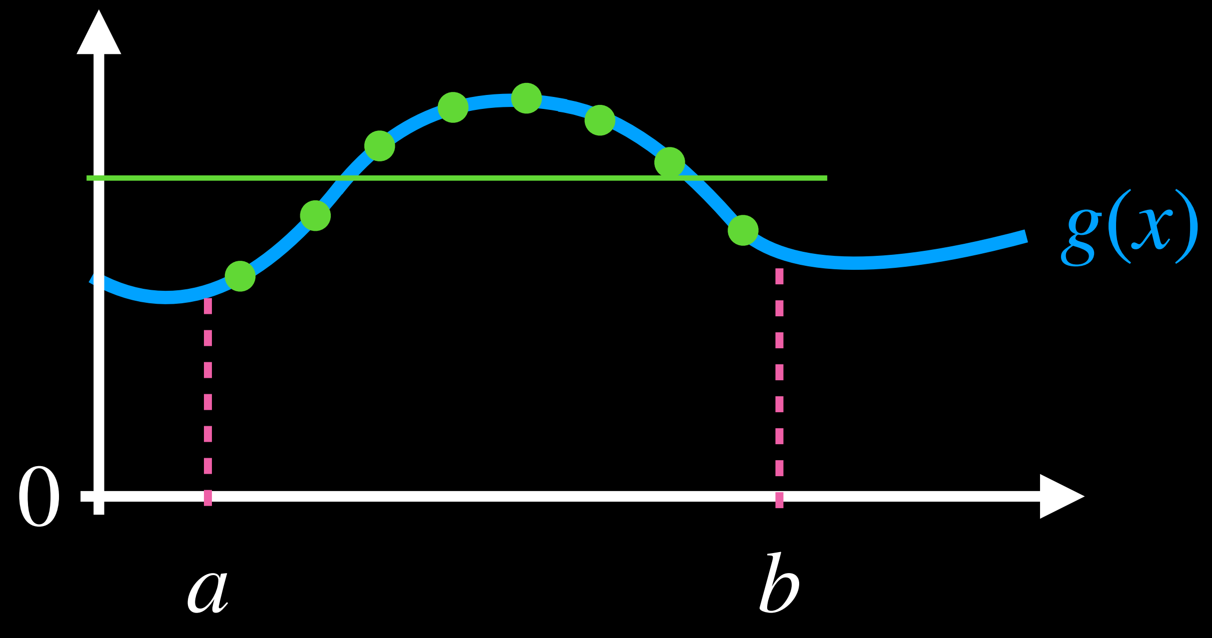

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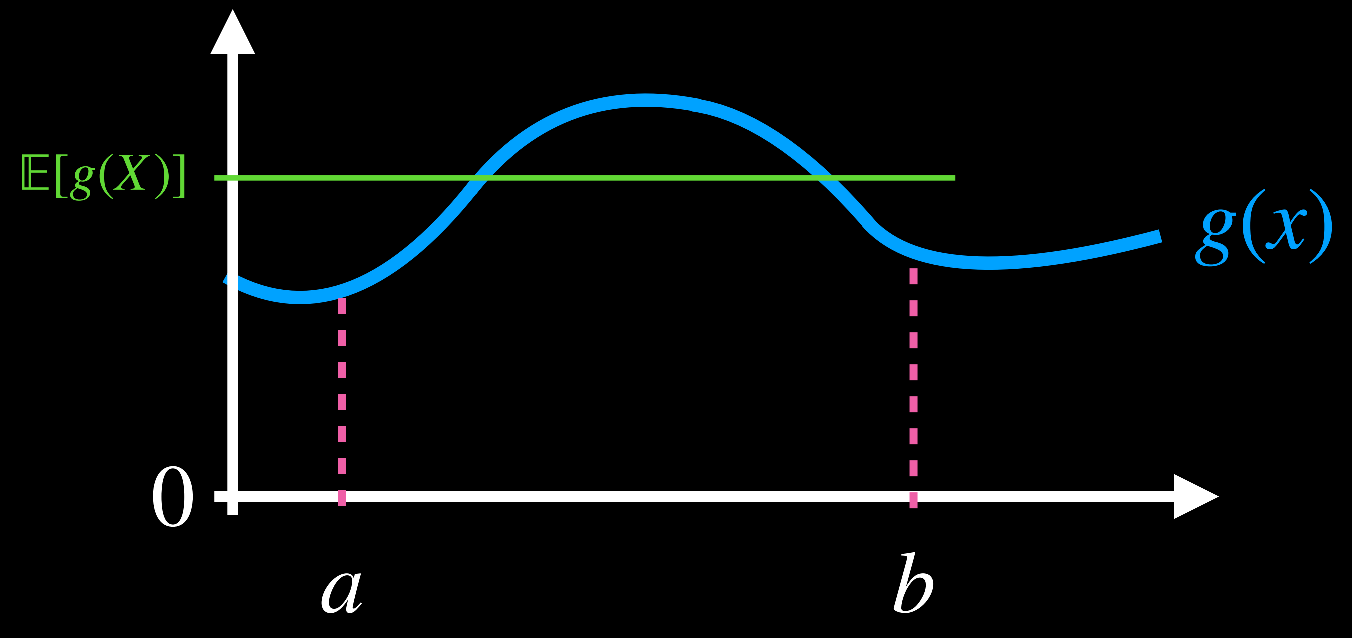

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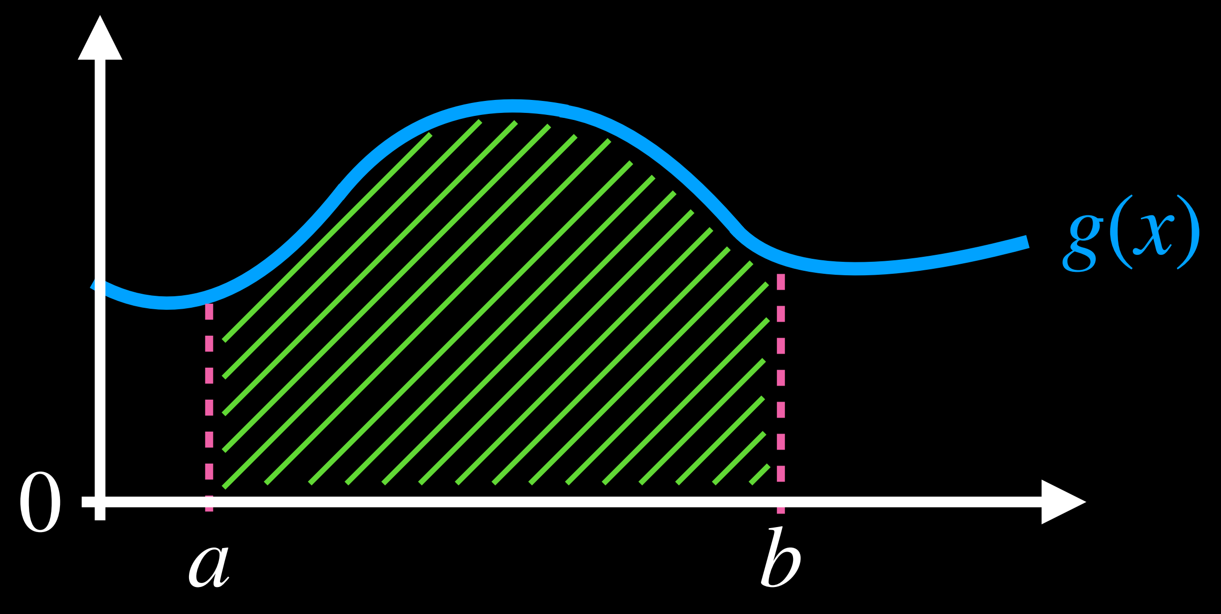
$$= \lim_{n \to \infty} \sum_{i=0}^{n-1} g(x_i)\,\frac{b-a}{n}$$

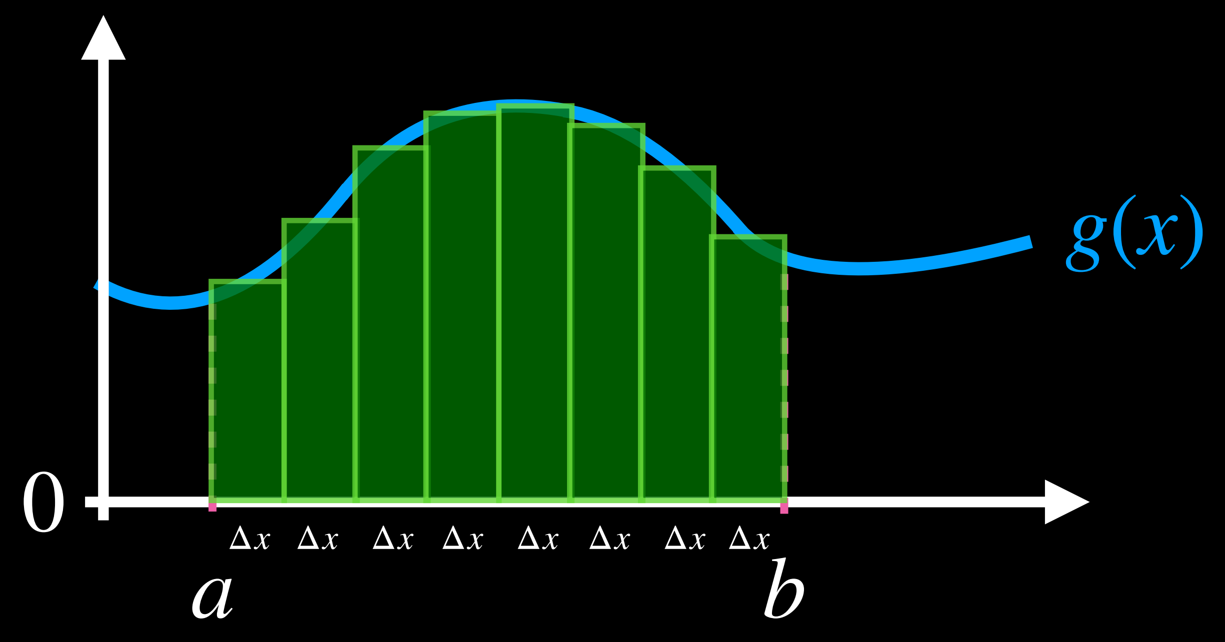
$$= (b-a)\,\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(x_i)$$

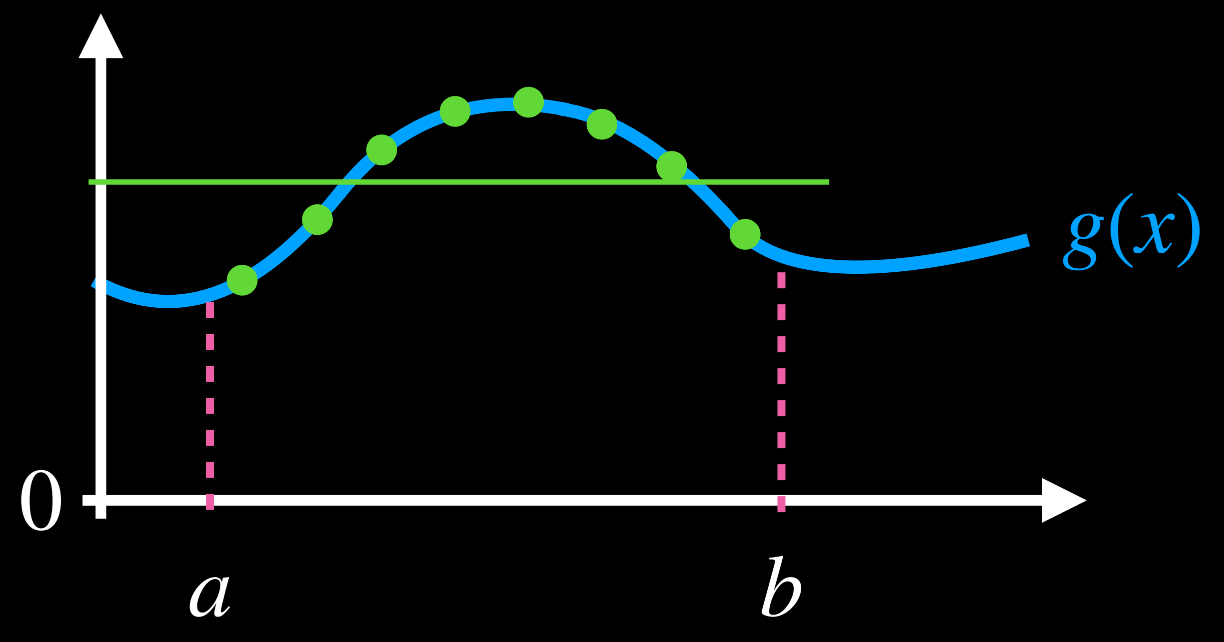
$$\bar{Y}_n := \frac{1}{n}\sum_{i=1}^{n} Y_i \qquad Y := g(X) \qquad \mu = \mathbb{E}[Y] = \mathbb{E}[g(X)] \qquad X \sim \mathcal{U}(a,b)$$

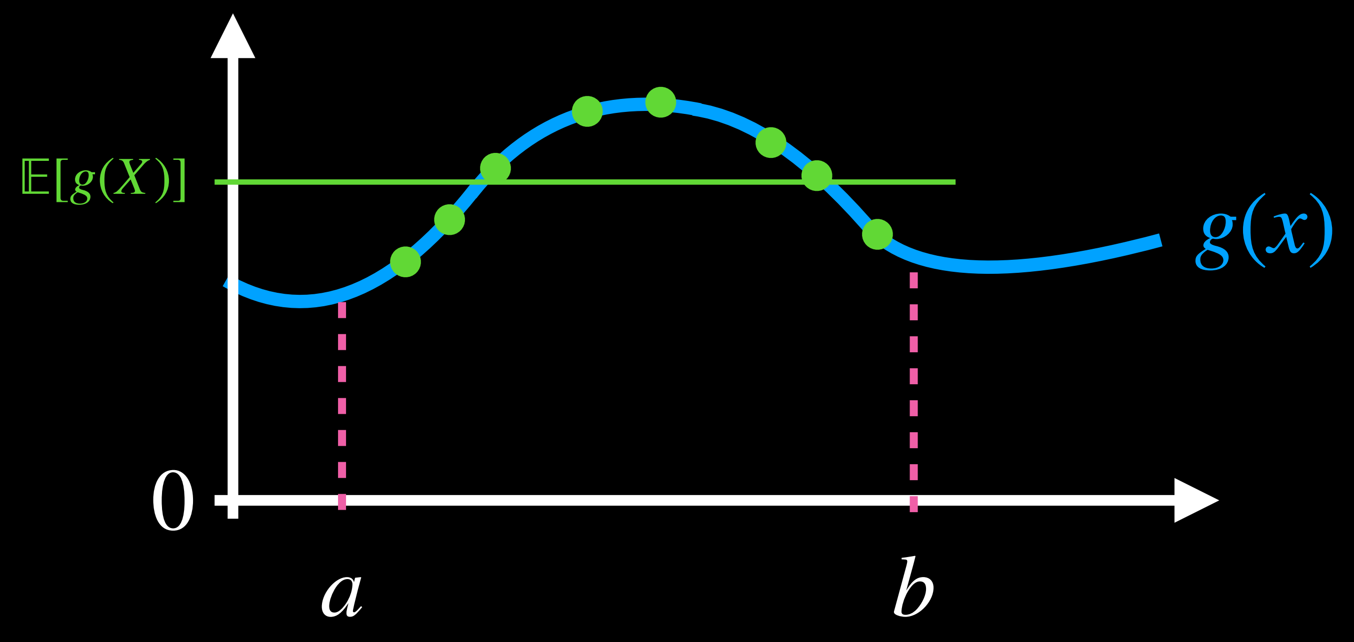
(SLLN) $$1 = P\left(\lim_{n\to\infty} \bar{Y}_n = \mu\right) = P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} Y_i = \mu\right) = P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} g(X_i) = \mathbb{E}[g(X)]\right)$$

Monte Carlo integration (intuition)









$$\int_a^b g(x)\,dx$$

$$= \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} g(x_i)\,\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} g(x_i)\,\frac{b-a}{n}$$

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Monte Carlo integration (derivation)

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx$$

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx \quad = \int_a^b g(x)\,1\,dx$$

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx \quad = \int_a^b g(x)\,1\,dx \quad = \int_a^b g(x)\,\frac{b-a}{b-a}\,dx$$

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx \quad = \int_a^b g(x)\,1\,dx \quad = \int_a^b g(x)\,\frac{b-a}{b-a}\,dx \quad = (b-a)\int_a^b g(x)\,\frac{1}{b-a}\,dx$$

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx \quad = \int_a^b g(x)\,1\,dx \quad = \int_a^b g(x)\,\frac{b-a}{b-a}\,dx \quad = (b-a)\int_a^b g(x)\,\overbrace{\frac{1}{b-a}}^{=f_X(x)}\,dx$$

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Monte Carlo integration (derivation)

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$=$
almost
always

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx \;=\; \int_a^b g(x)\,1\,dx \;=\; \int_a^b g(x)\,\frac{b-a}{b-a}\,dx \;=\; (b-a)\int_a^b g(x)\,\overbrace{\frac{1}{b-a}}^{=f_X(x)}\,dx$$

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almost always

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"Monte Carlo estimator"

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx \quad = \int_a^b g(x)\,1\,dx \quad = \int_a^b g(x)\,\frac{b-a}{b-a}\,dx \quad = (b-a)\int_a^b g(x)\,\overbrace{\frac{1}{b-a}}^{=f_X(x)}\,dx \qquad X \sim \mathcal{U}(a,b)$$

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"Monte Carlo estimator"

multi-dimensional

Monte Carlo integration (derivation)

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"Monte Carlo estimator"

multi-dimensional

$$\int_{\mathbb{S}} g(\underline{x})\,d\underline{x}$$

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx = \int_a^b g(x)\,1\,dx = \int_a^b g(x)\,\frac{b-a}{b-a}\,dx = (b-a)\int_a^b g(x)\,\overbrace{\frac{1}{b-a}}^{=f_X(x)}\,dx \qquad X \sim \mathcal{U}(a,b) \qquad \left[\ \mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x) f_X(x)\,dx\ \right]$$

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almost always

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"Monte Carlo estimator"

multi-dimensional

$$\int_{\mathbb{S}} g(\underline{x})\,d\underline{x} \approx V\,\frac{1}{n}\sum_{i=0}^{n-1} g(\underline{X}_i)$$

Monte Carlo integration (derivation)

$X \sim \mathcal{U}(a,b)$

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almost always

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"Monte Carlo estimator"

multi-dimensional

(volume)

$$\int_{\mathbb{S}} g(\underline{x})\,d\underline{x} \approx V\,\frac{1}{n}\sum_{i=0}^{n-1} g(\underline{X}_i)$$

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx \quad = \int_a^b g(x)\,1\,dx \quad = \int_a^b g(x)\,\frac{b-a}{b-a}\,dx \quad = (b-a)\int_a^b g(x)\,\overbrace{\frac{1}{b-a}}^{=f_X(x)}\,dx \qquad X \sim \mathcal{U}(a,b)$$

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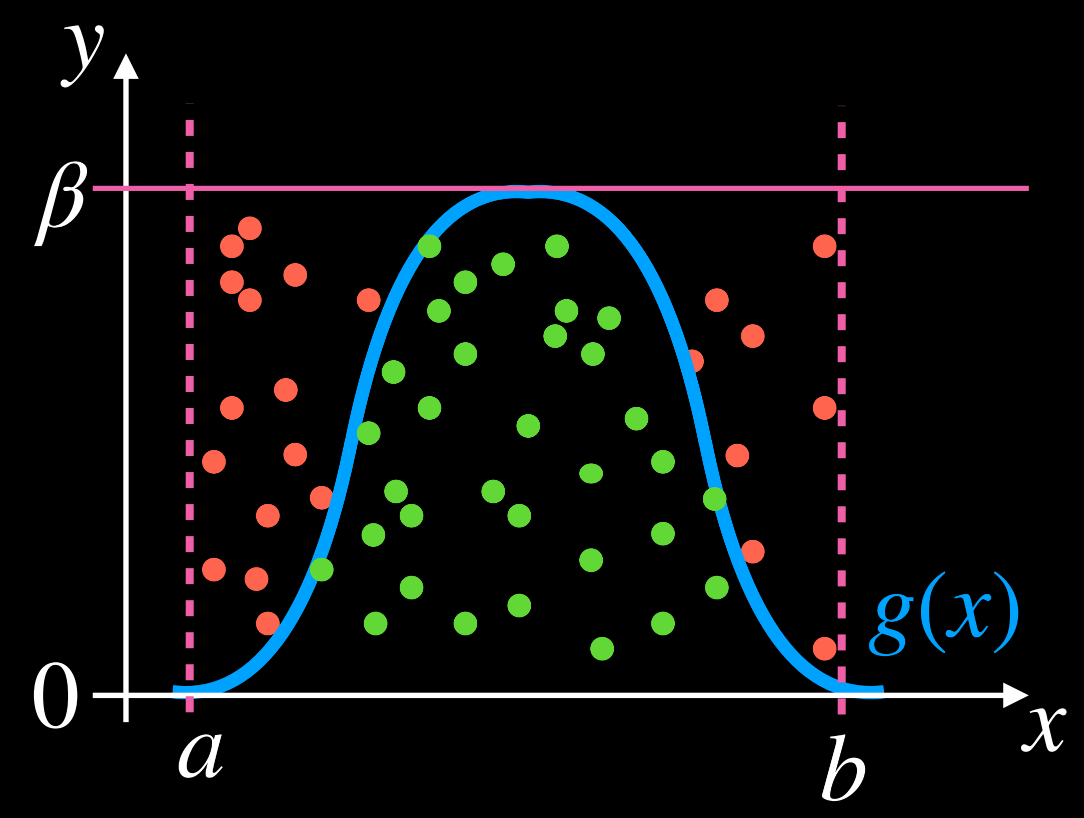
"Monte Carlo estimator"

multi-dimensional

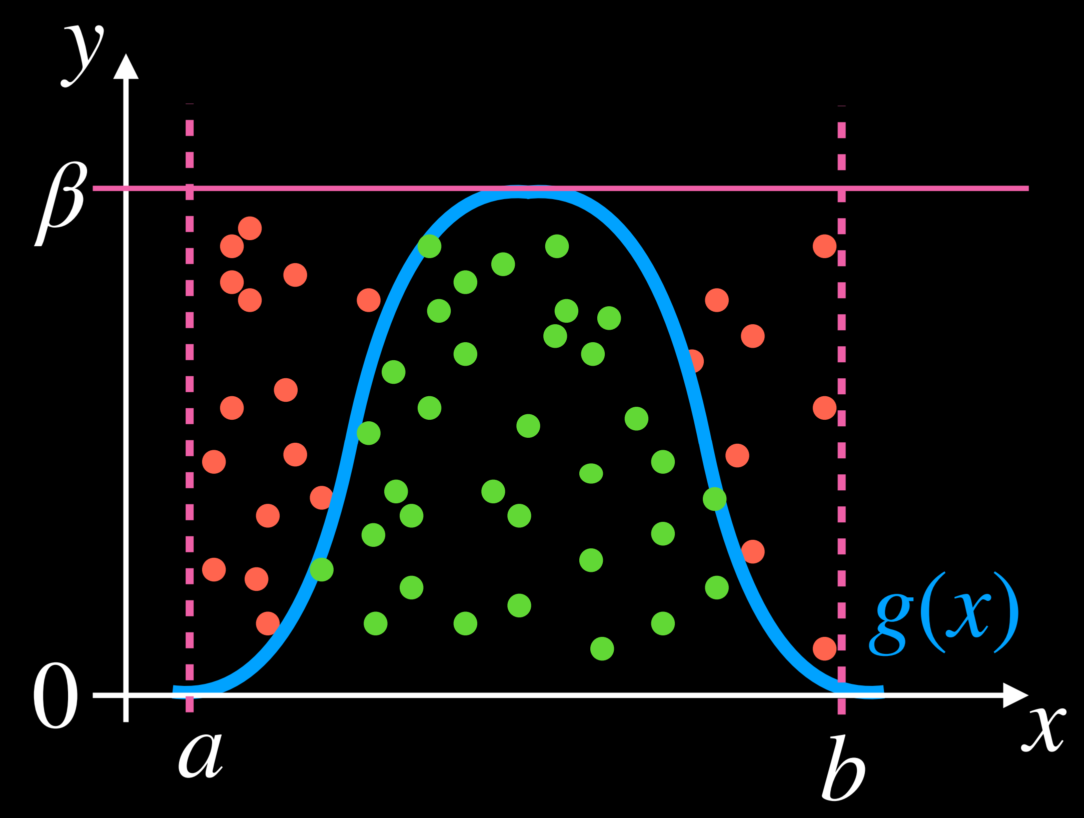
$$\int_{\mathbb{S}} g(\underline{x})\,d\underline{x} \approx \overset{\text{(volume)}}{V}\,\frac{1}{n}\sum_{i=0}^{n-1} g(\underline{X}_i) \quad =: \vartheta_n$$

Monte Carlo integration (alternative like accept-reject sampling)

Monte Carlo integration (alternative like accept-reject sampling)



Monte Carlo integration (alternative like accept-reject sampling)

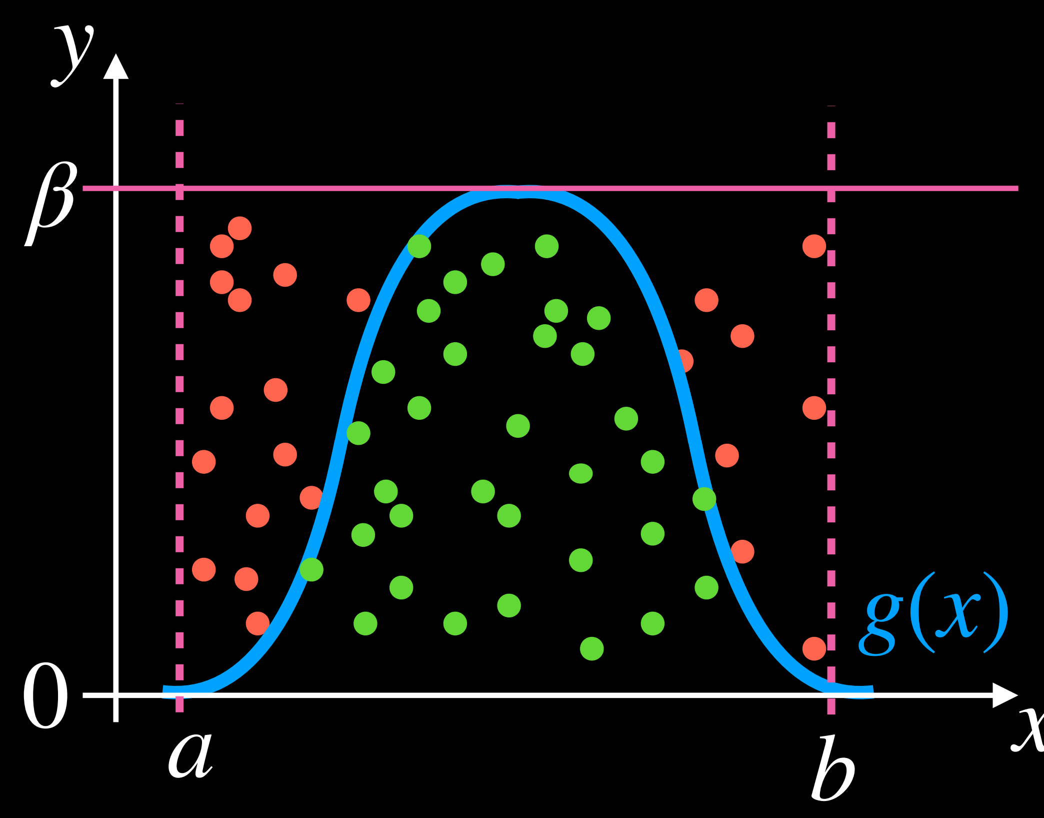


$x \in [a, b]$

Monte Carlo integration (alternative like accept-reject sampling)

$$x \in [a, b]$$

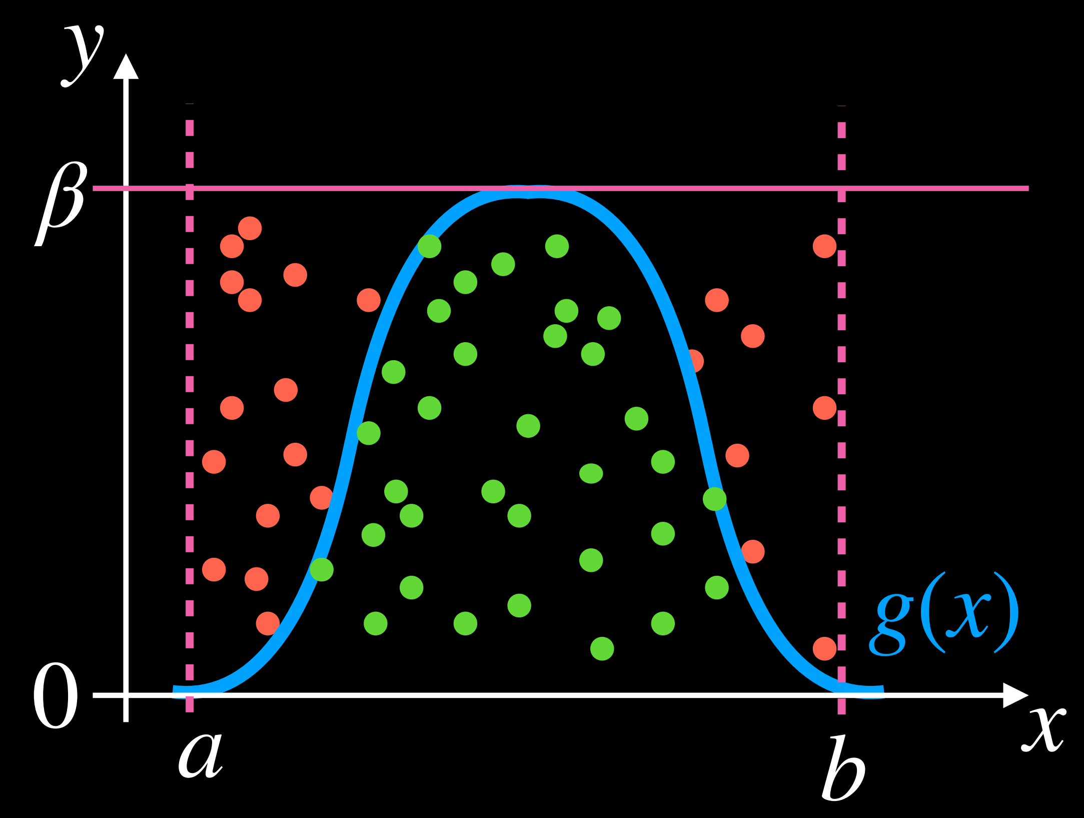
$$g(x) \in [0, \beta]$$



Monte Carlo integration (alternative like accept-reject sampling)

$$x \in [a, b]$$
$$g(x) \in [0, \beta]$$

$$\mathbb{I}_g(x, y) := \begin{cases} 1 & \text{if } y \leq g(x) \\ 0 & \text{else} \end{cases}$$



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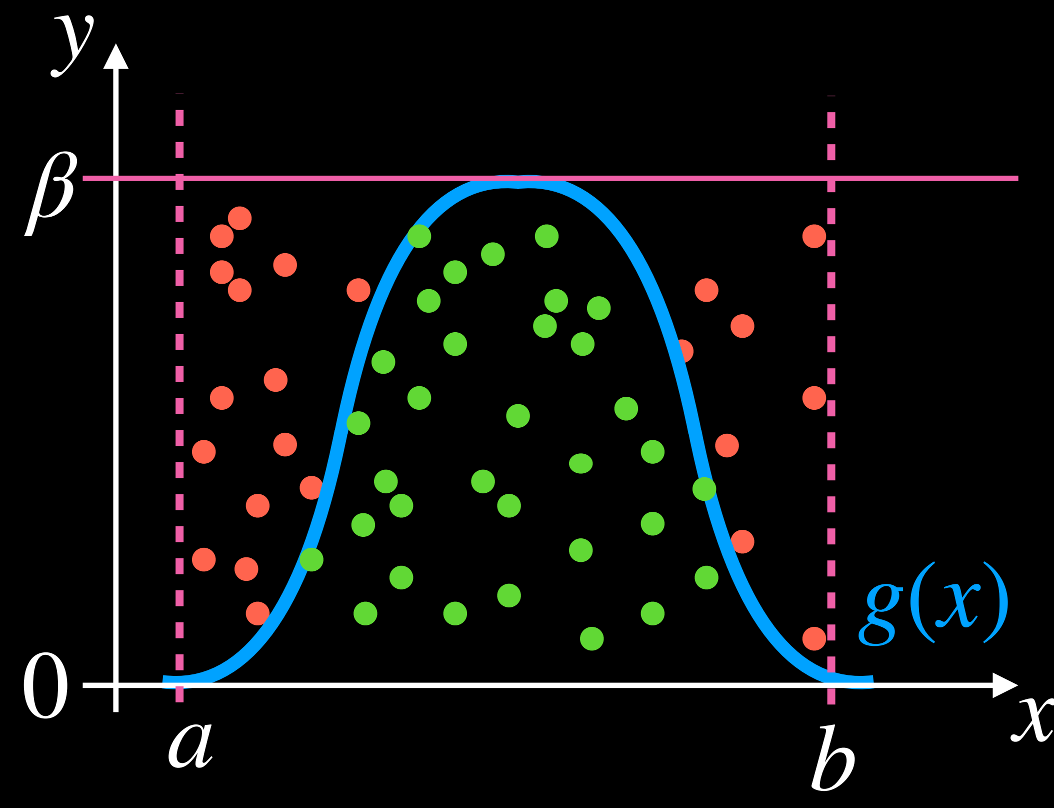

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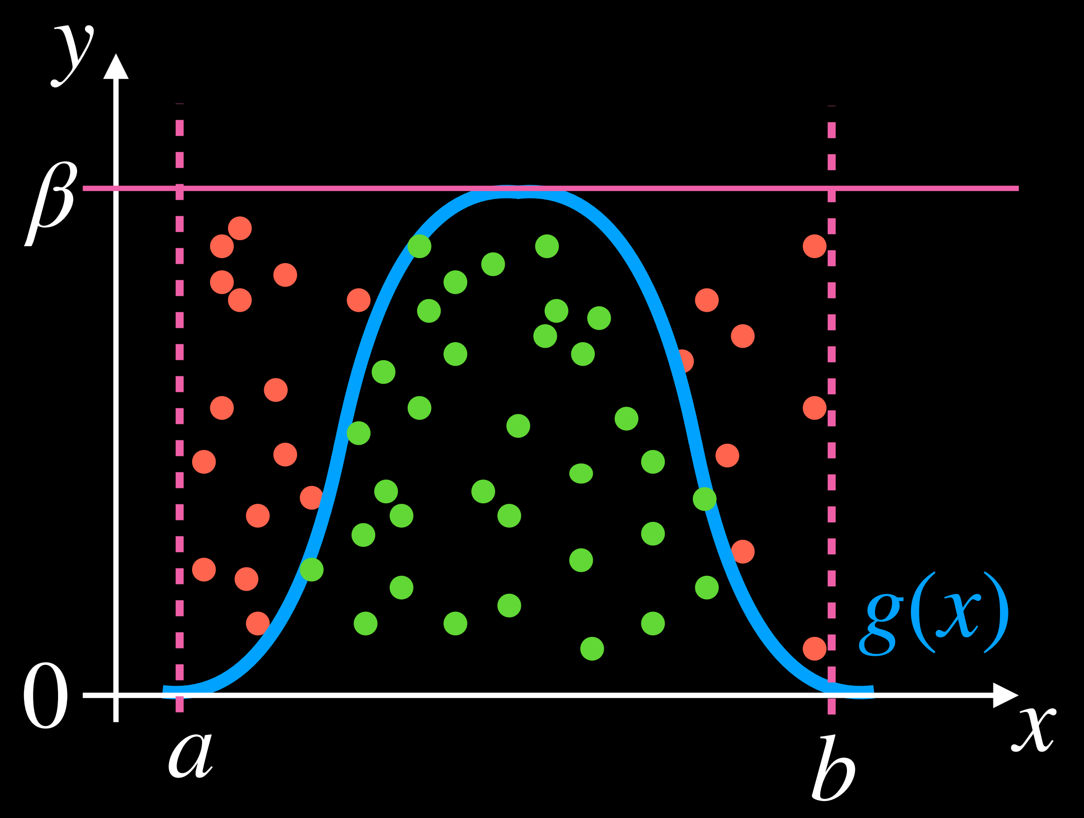
$$\int_a^b g(x)\, dx$$

Monte Carlo integration (alternative like accept-reject sampling)

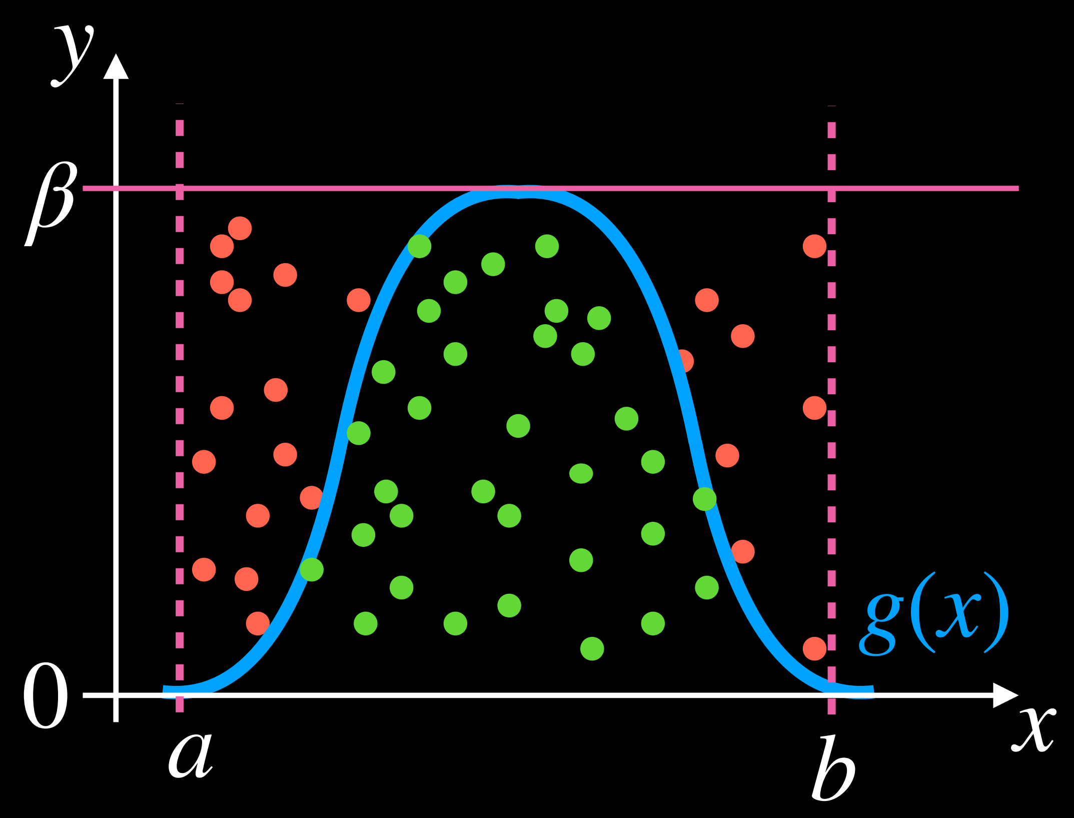


$$x \in [a, b]$$
$$g(x) \in [0, \beta]$$

$$\mathbb{1}_g(x, y) := \begin{cases} 1 & \text{if } y \leq g(x) \\ 0 & \text{else} \end{cases}$$

$$\int_a^b g(x)\, dx = \int_0^\beta \int_a^b \mathbb{1}_g(x, y)\, dx\, dy$$

Monte Carlo integration (alternative like accept-reject sampling)

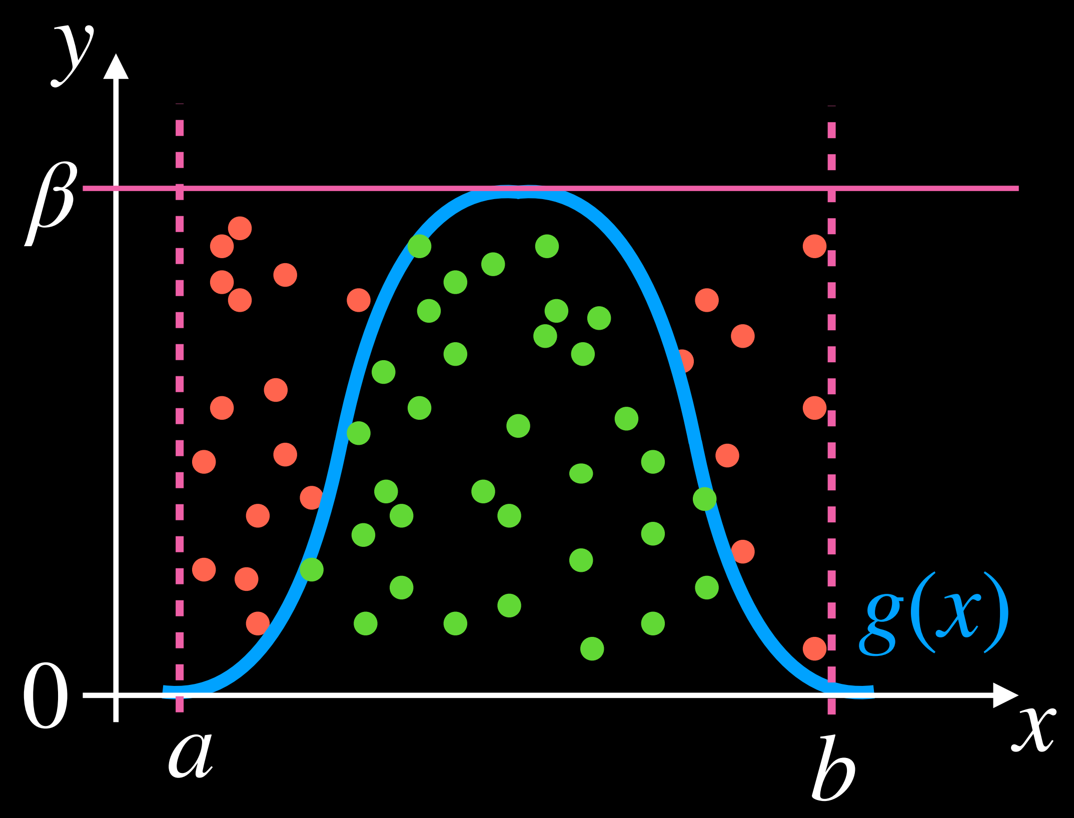


$$x \in [a, b] \qquad g(x) \in [0, \beta]$$

$$\mathbb{I}_g(x, y) := \begin{cases} 1 & \text{if } y \leq g(x) \\ 0 & \text{else} \end{cases}$$

$$\int_a^b g(x)\, dx = \int_0^\beta \int_a^b \mathbb{I}_g(x, y)\, dx\, dy = \int_0^\beta \int_a^b \mathbb{I}_g(x, y)\, 1\, dx\, dy$$

Monte Carlo integration (alternative like accept-reject sampling)



$$x \in [a, b]$$
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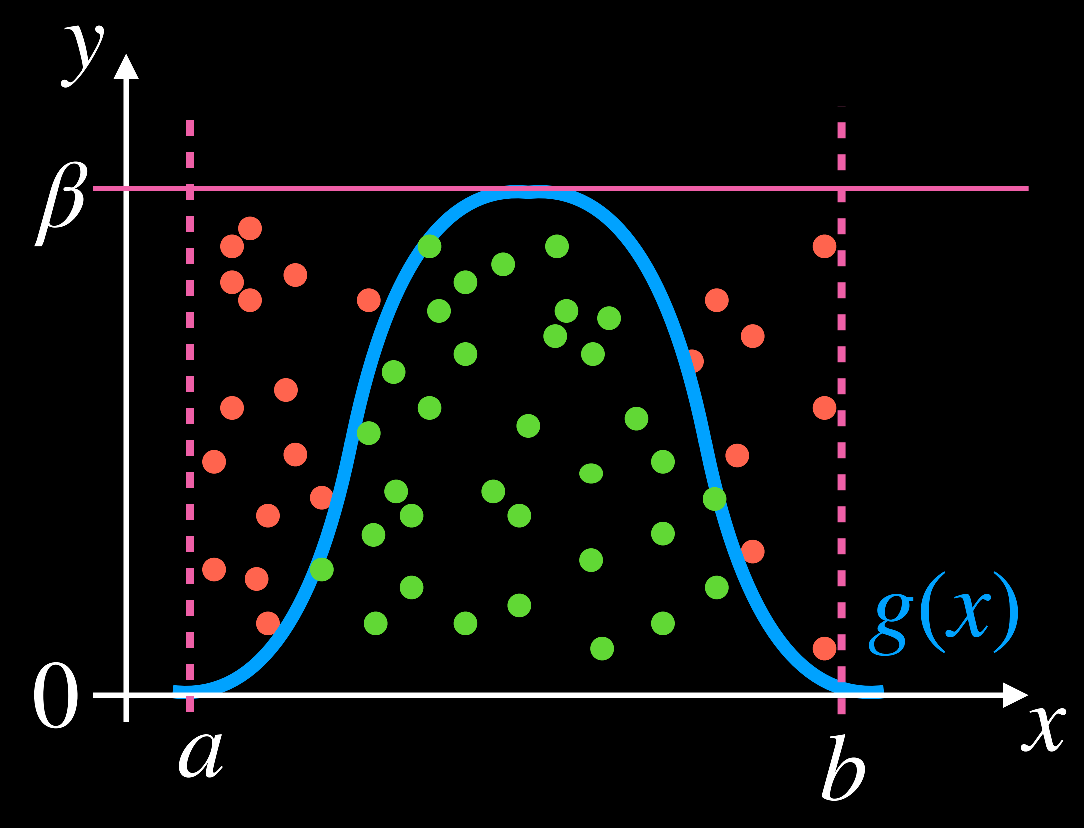
Monte Carlo integration (alternative like accept-reject sampling)



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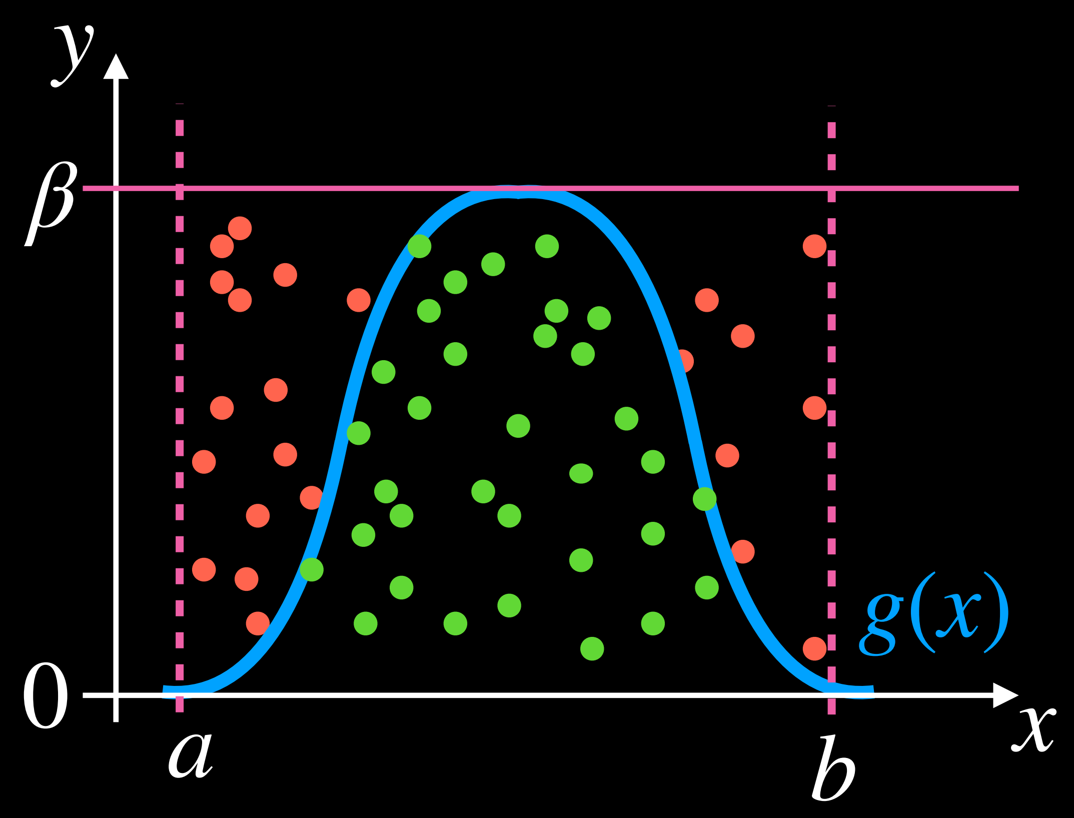
$$= \beta(b - a) \int_0^\beta \int_a^b \mathbb{1}_g(x, y)\, \frac{1}{\beta(b - a)}\, dx\, dy$$

Monte Carlo integration (alternative like accept-reject sampling)



$$x \in [a,b]$$
$$g(x) \in [0,\beta]$$

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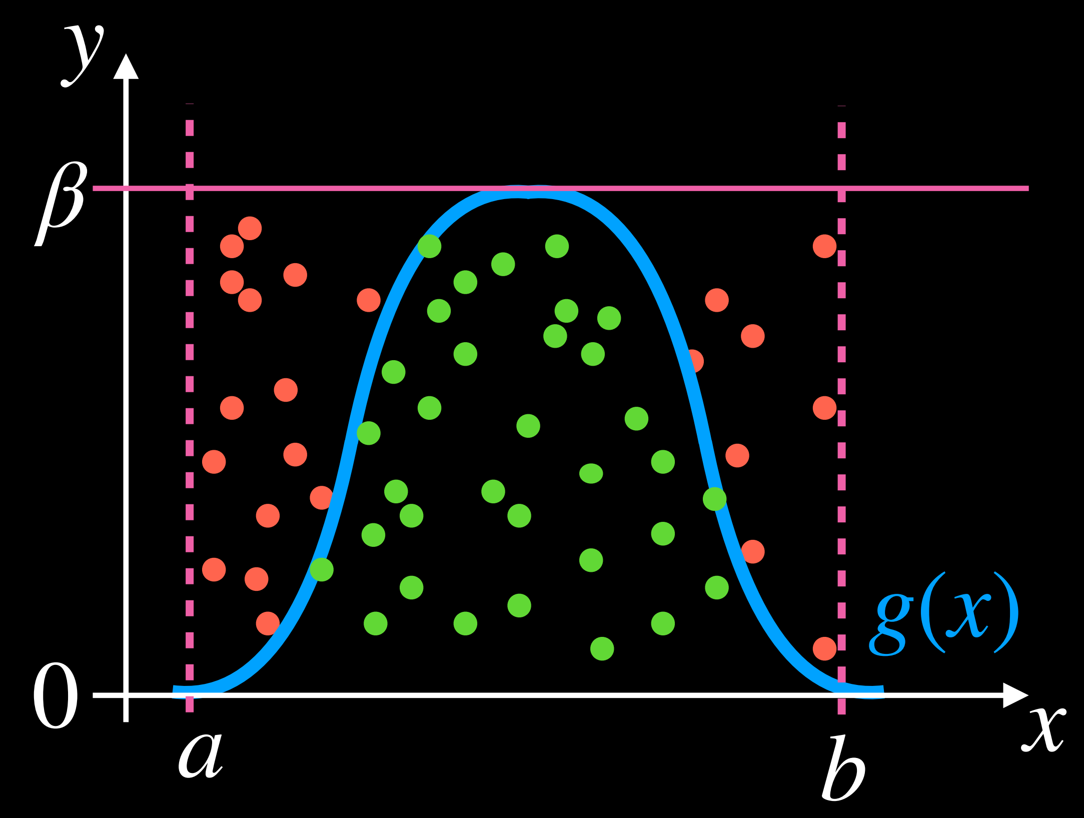
$$\int_a^b g(x)\,dx = \int_0^\beta \int_a^b \mathbb{1}_g(x,y)\,dx\,dy = \int_0^\beta \int_a^b \mathbb{1}_g(x,y)\,1\,dx\,dy = \int_0^\beta \int_a^b \mathbb{1}_g(x,y)\,\frac{(\beta-0)(b-a)}{(\beta-0)(b-a)}\,dx\,dy$$

$$= \beta(b-a)\int_0^\beta \int_a^b \mathbb{1}_g(x,y)\,\underbrace{\frac{1}{\beta(b-a)}}_{=f_{XY}(x,y)}\,dx\,dy$$

Monte Carlo integration (alternative like accept-reject sampling)



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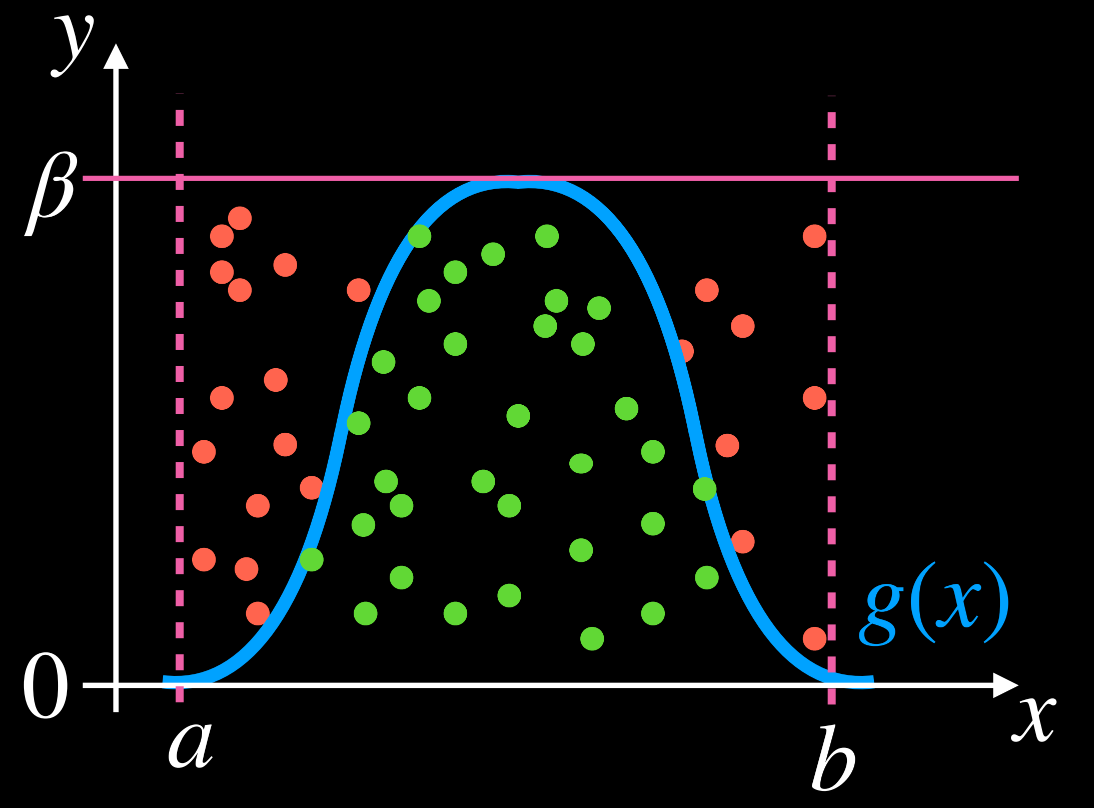
Monte Carlo integration (alternative like accept-reject sampling)



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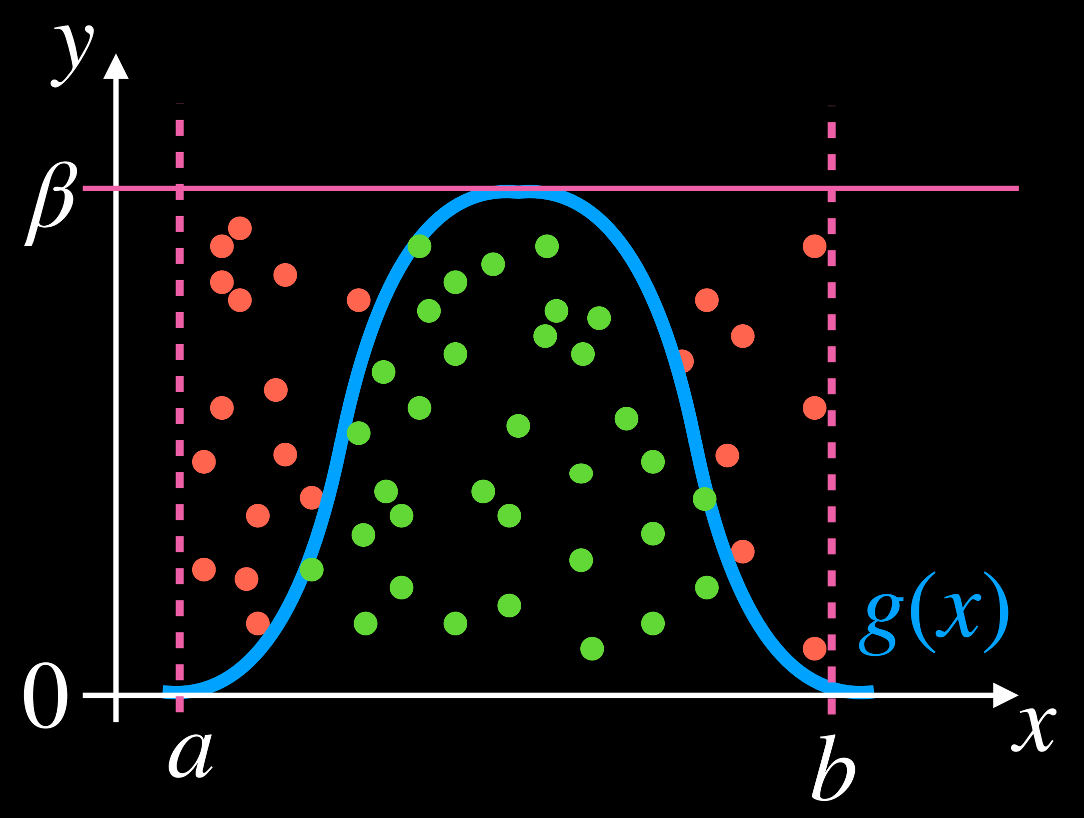

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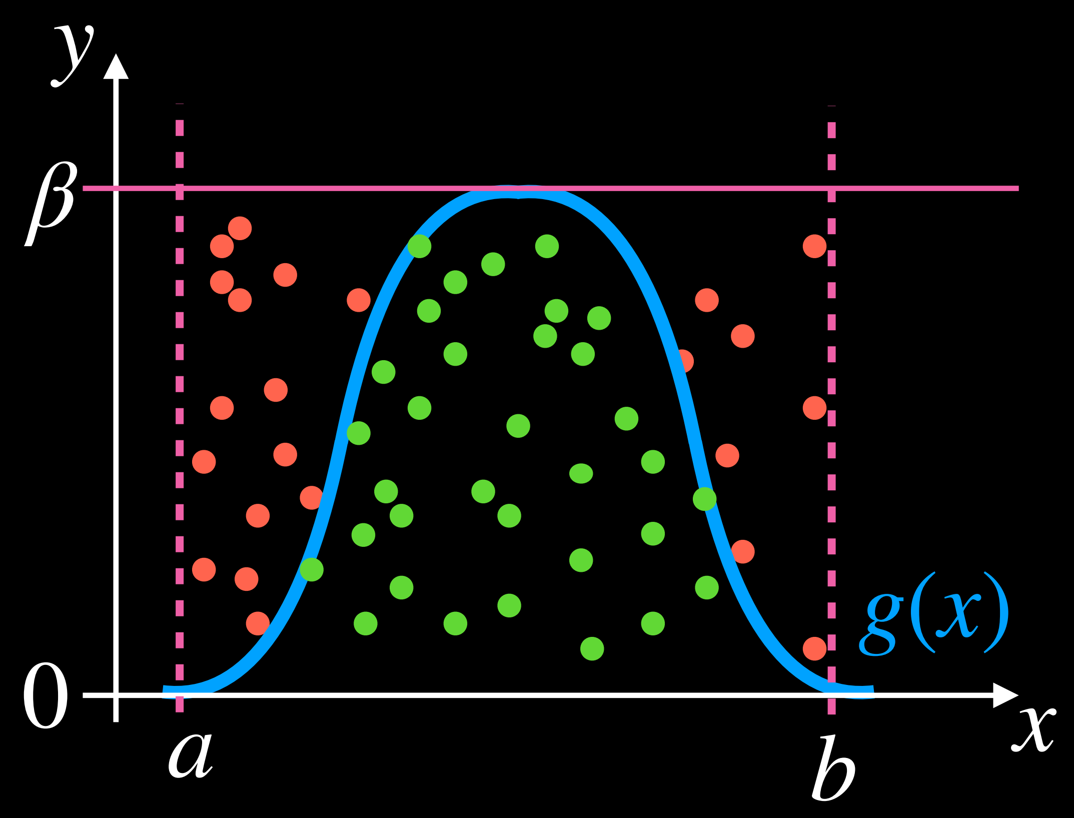

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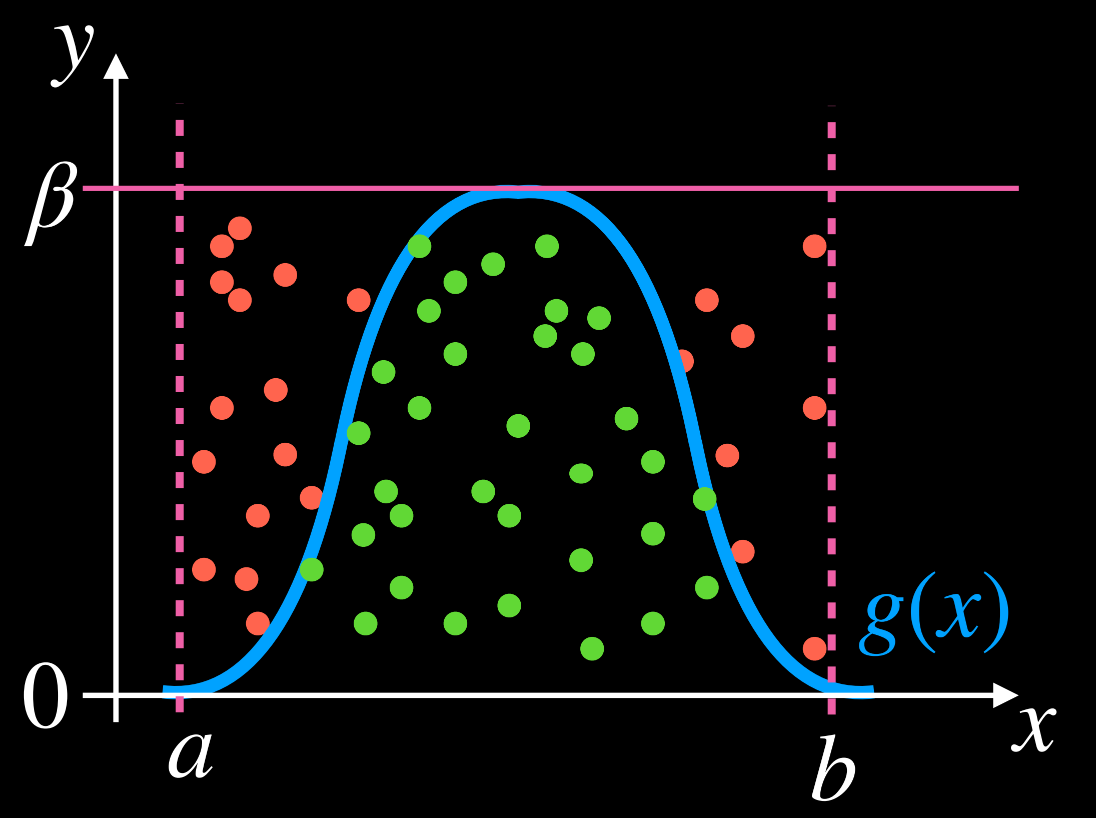

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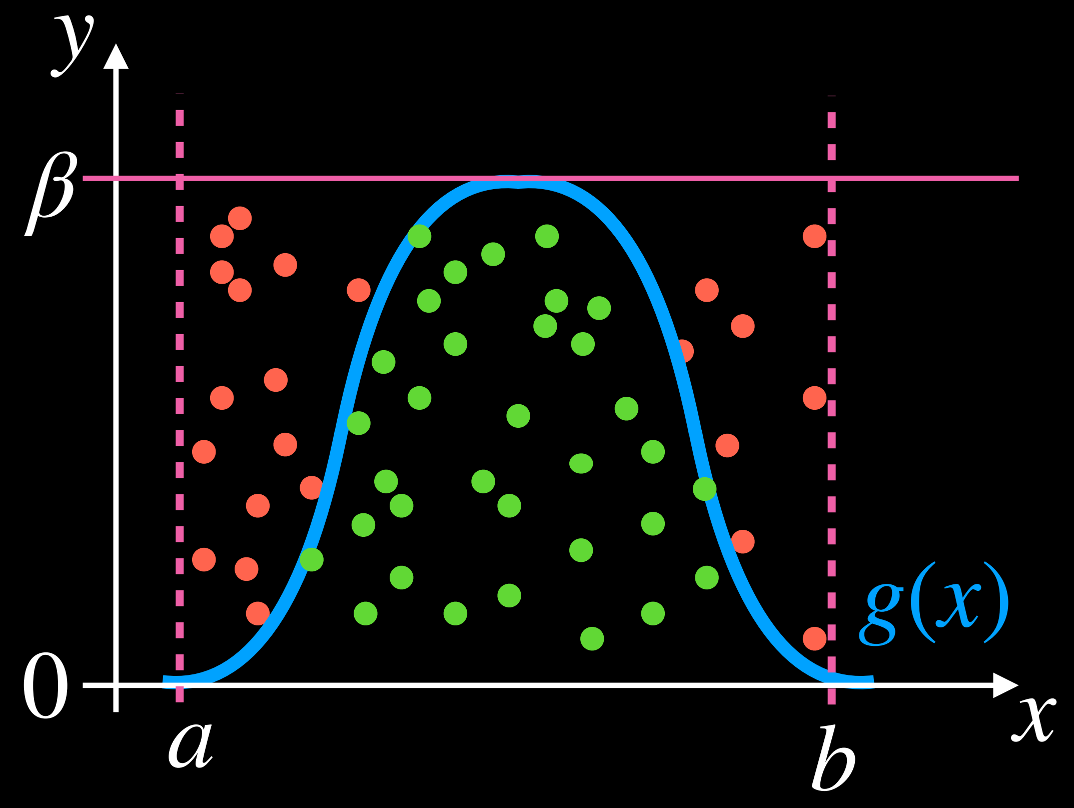
example:

Monte Carlo integration (alternative like accept-reject sampling)



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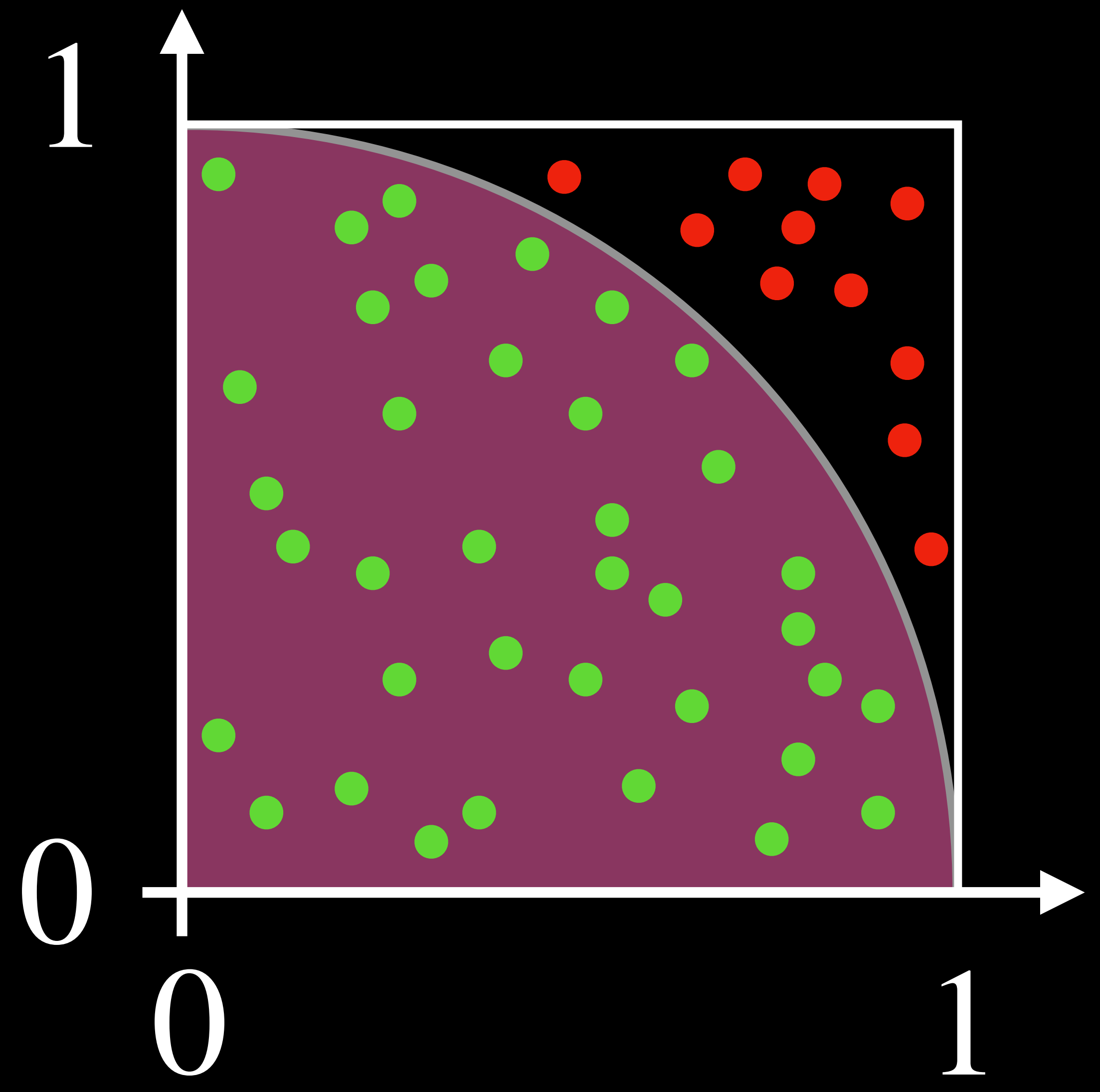
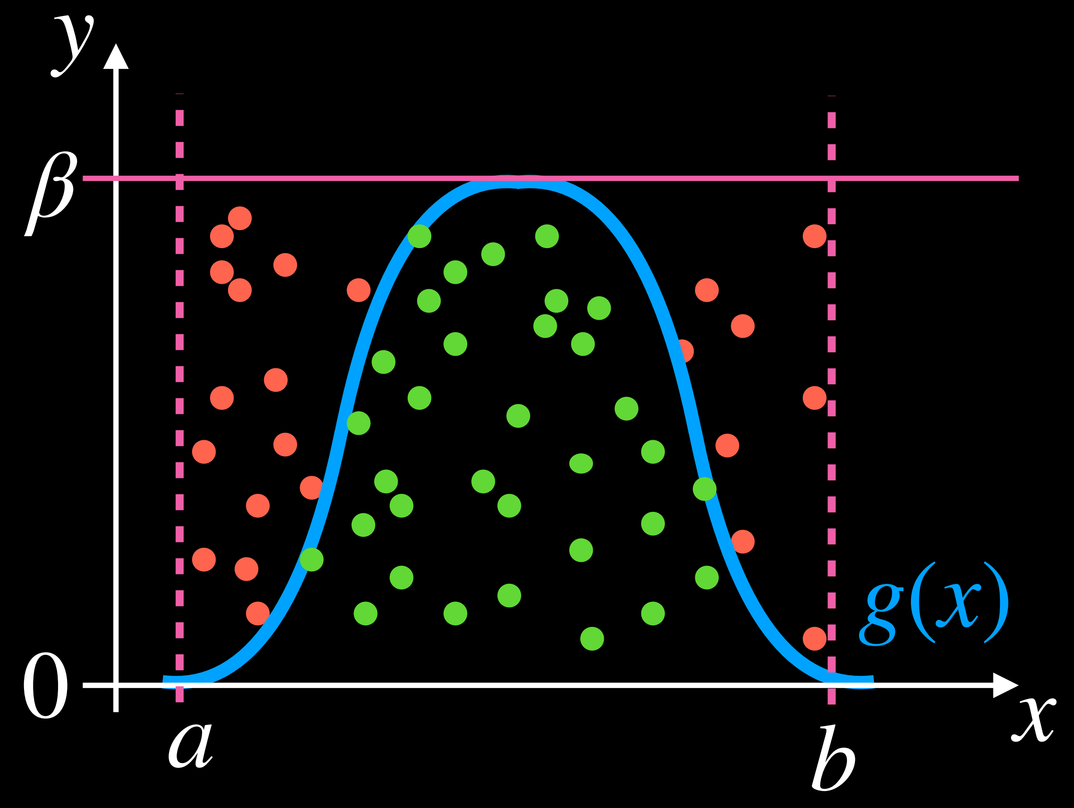


Monte Carlo integration (alternative like accept-reject sampling)



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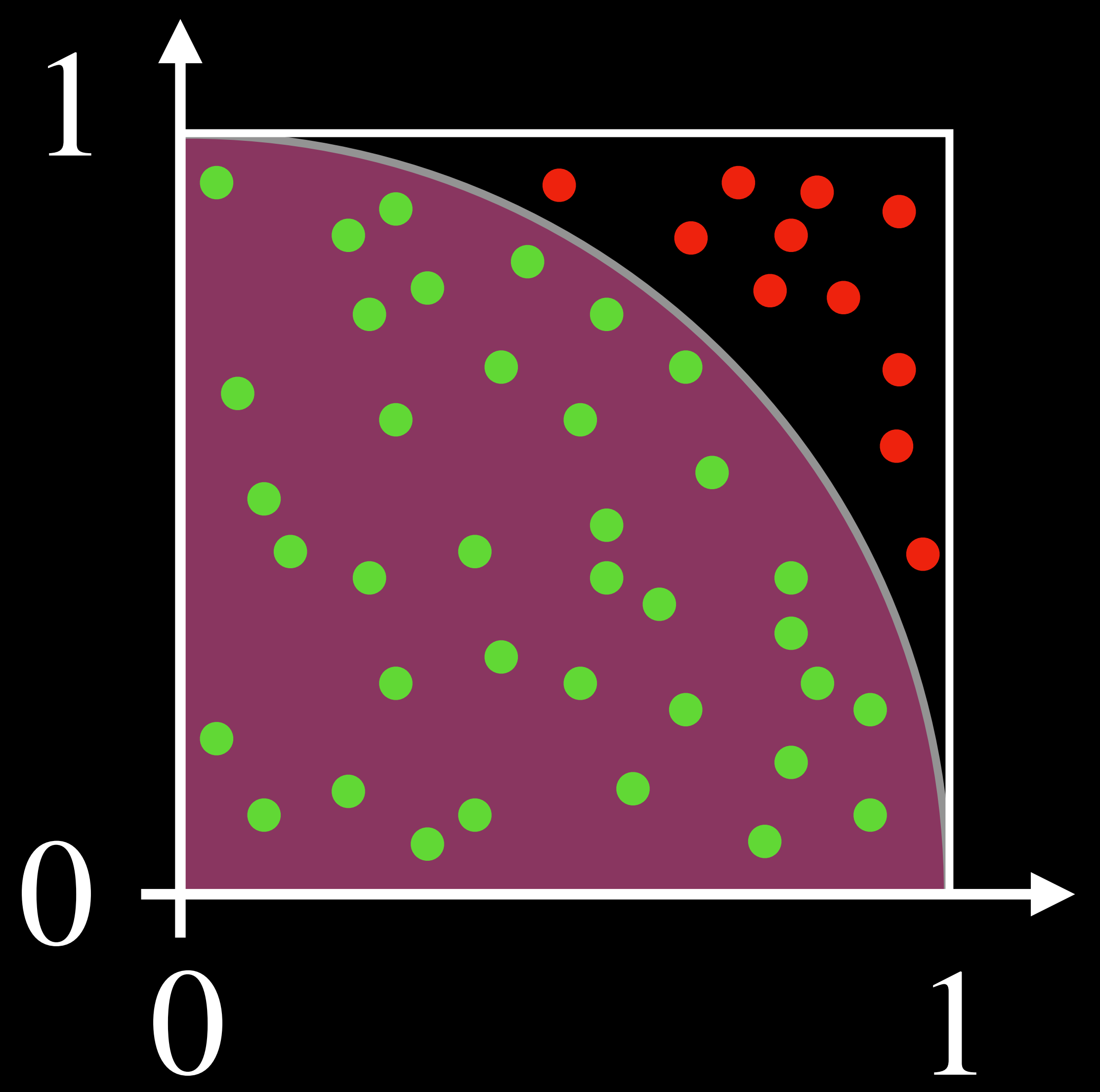
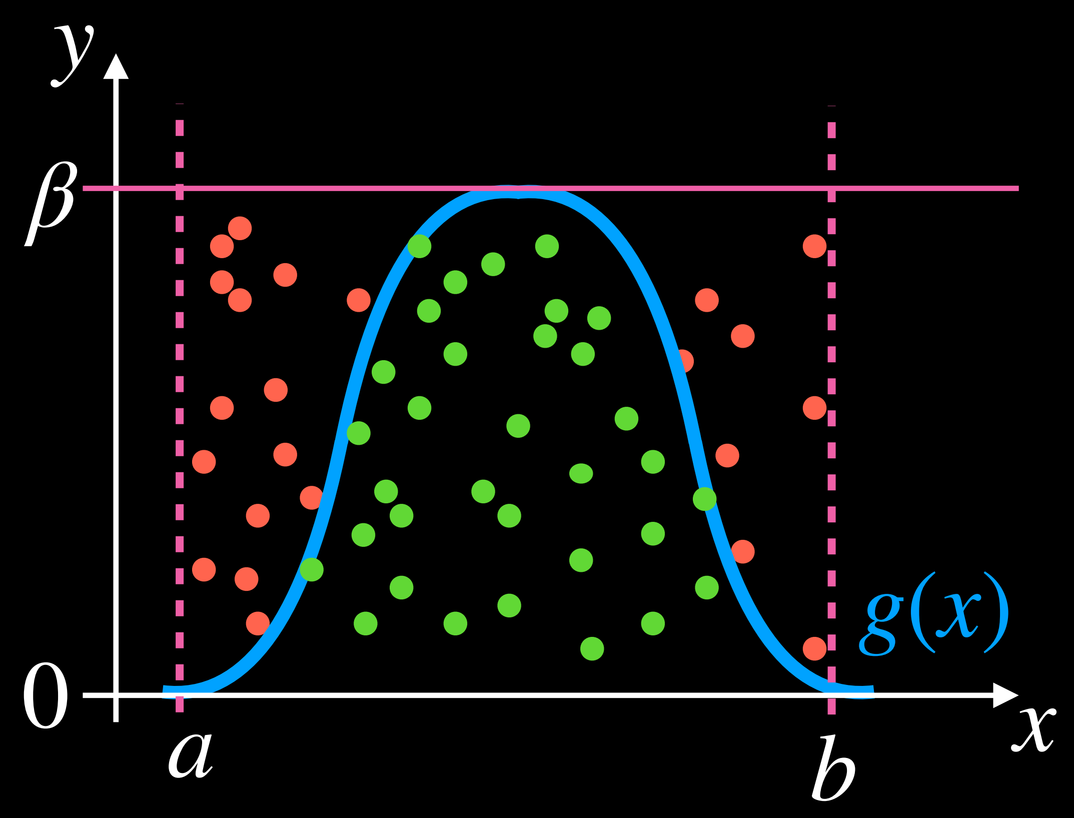


$$x \in [0,1]$$

Monte Carlo integration (alternative like accept-reject sampling)



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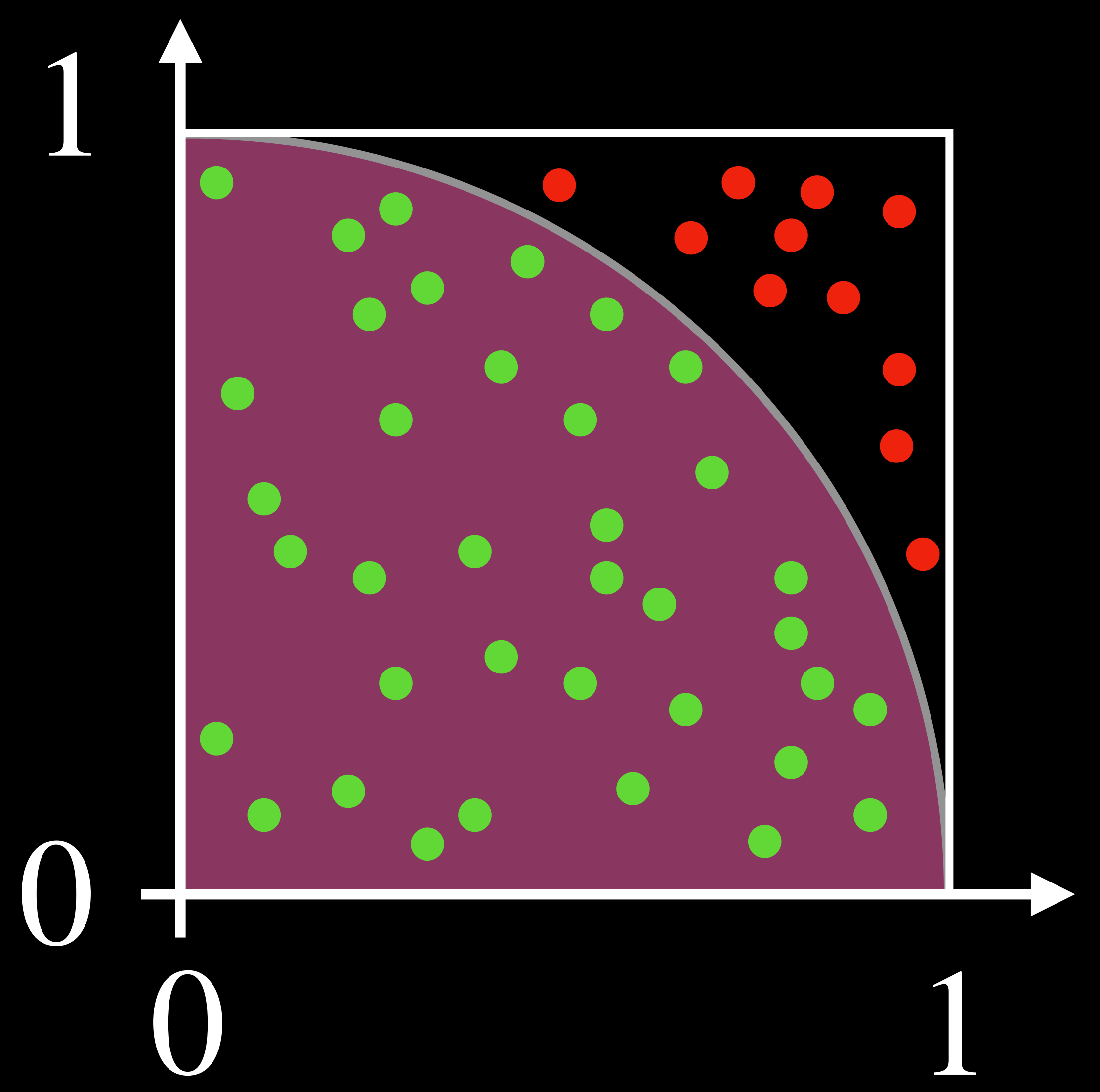

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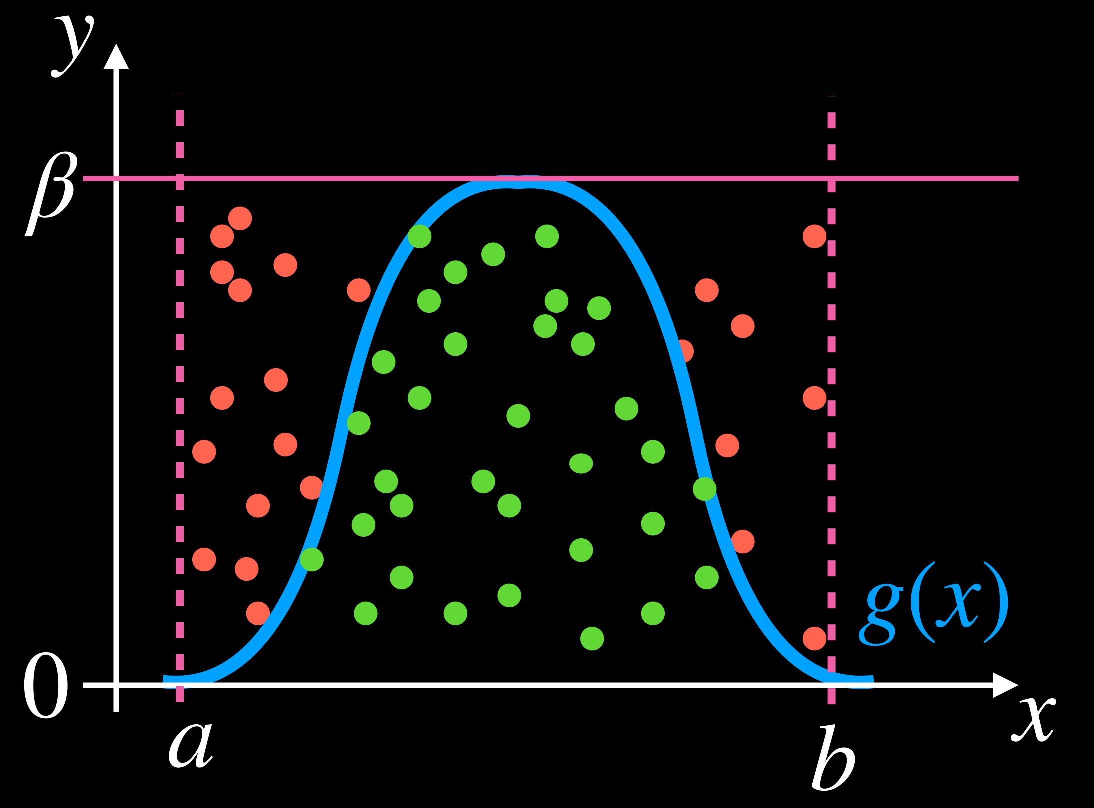
Monte Carlo integration (alternative like accept-reject sampling)



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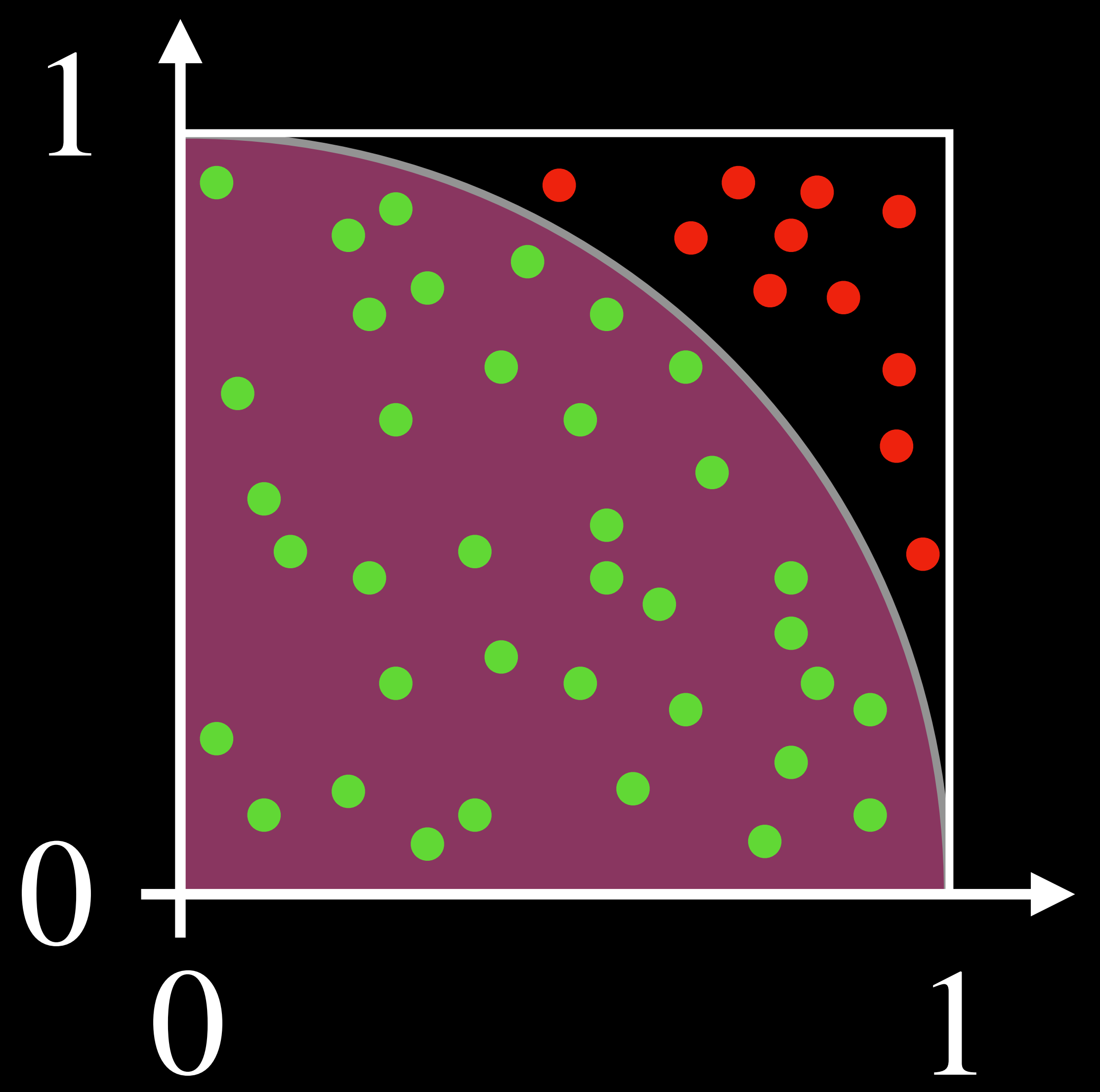

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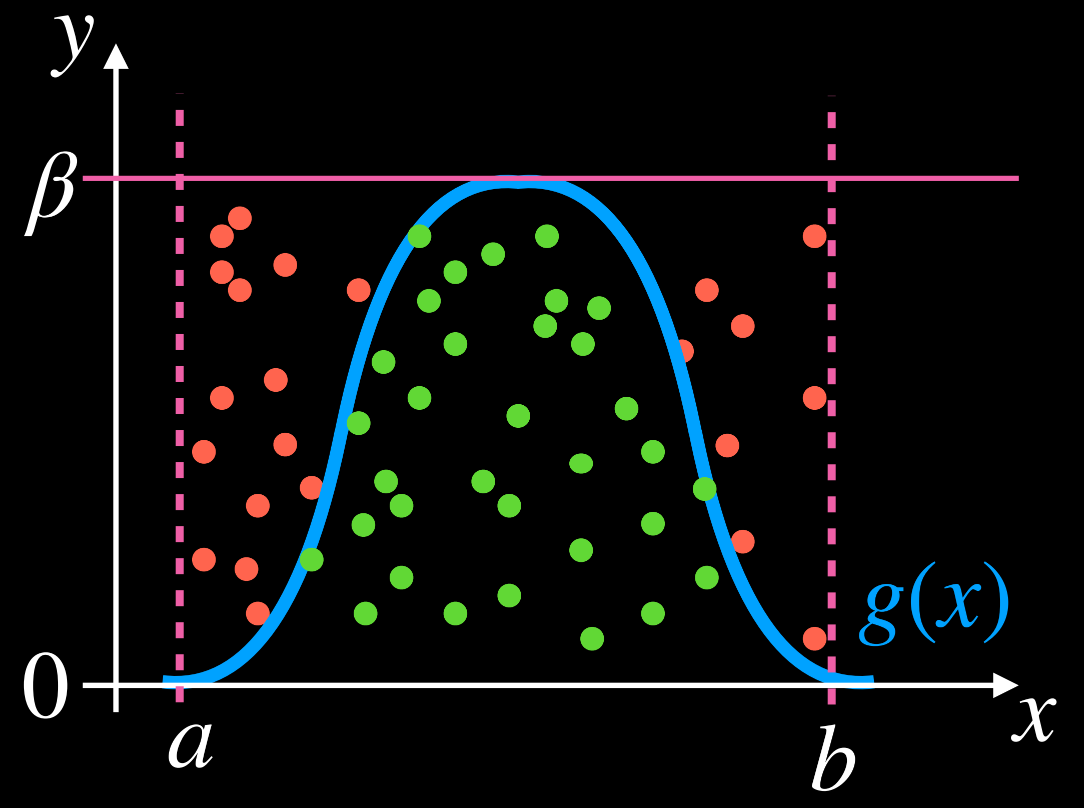

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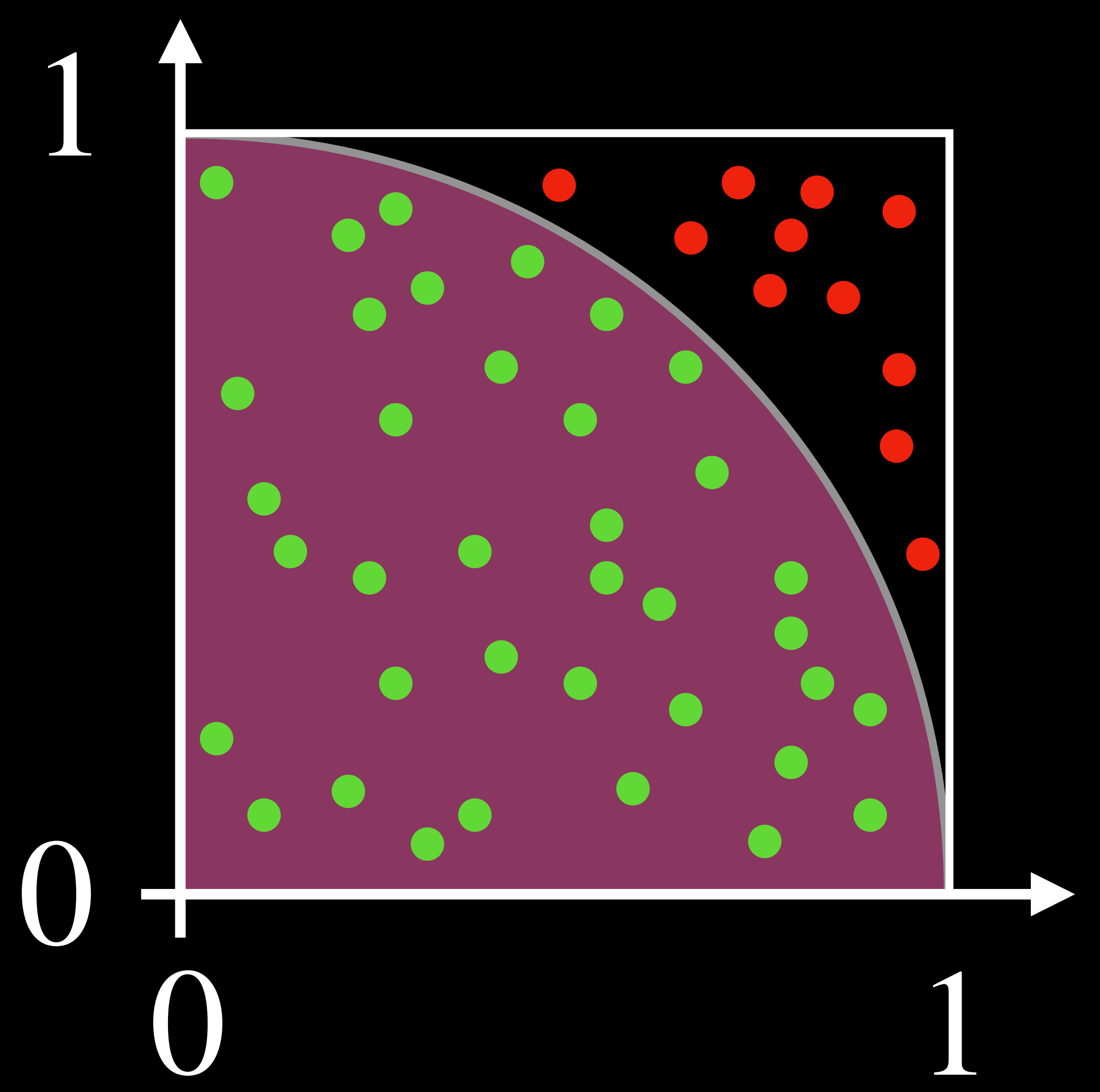

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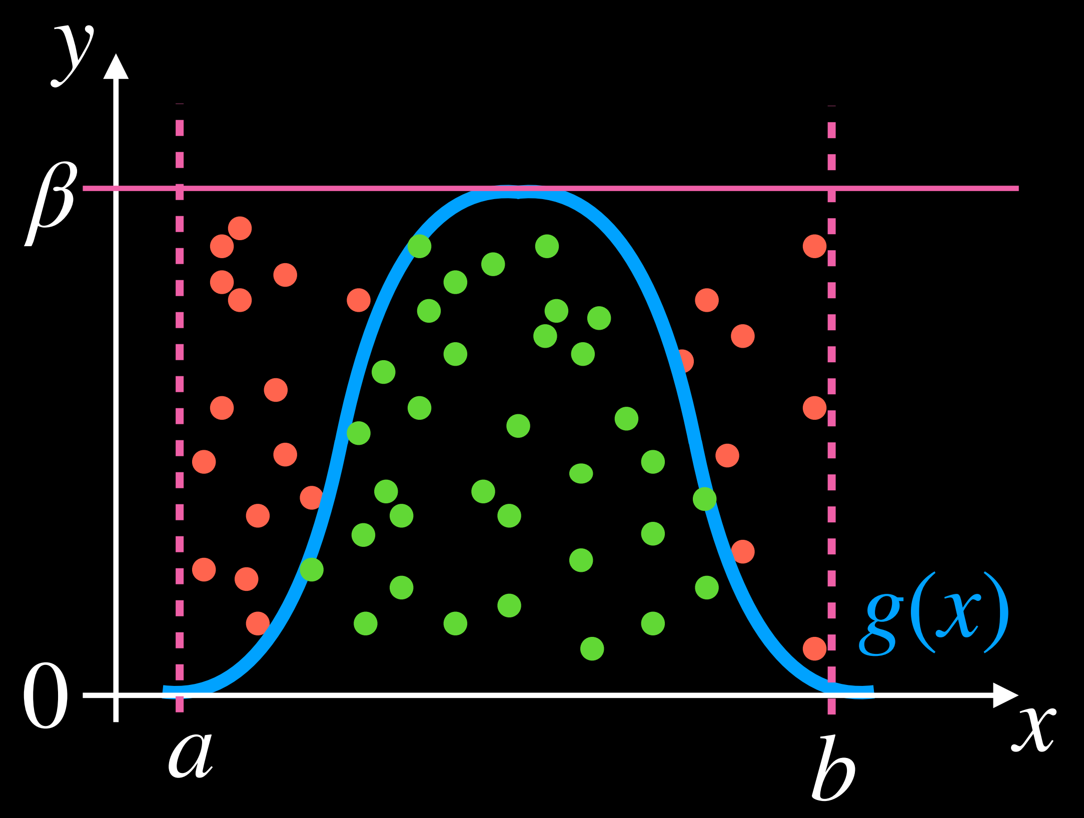
Monte Carlo integration (alternative like accept-reject sampling)



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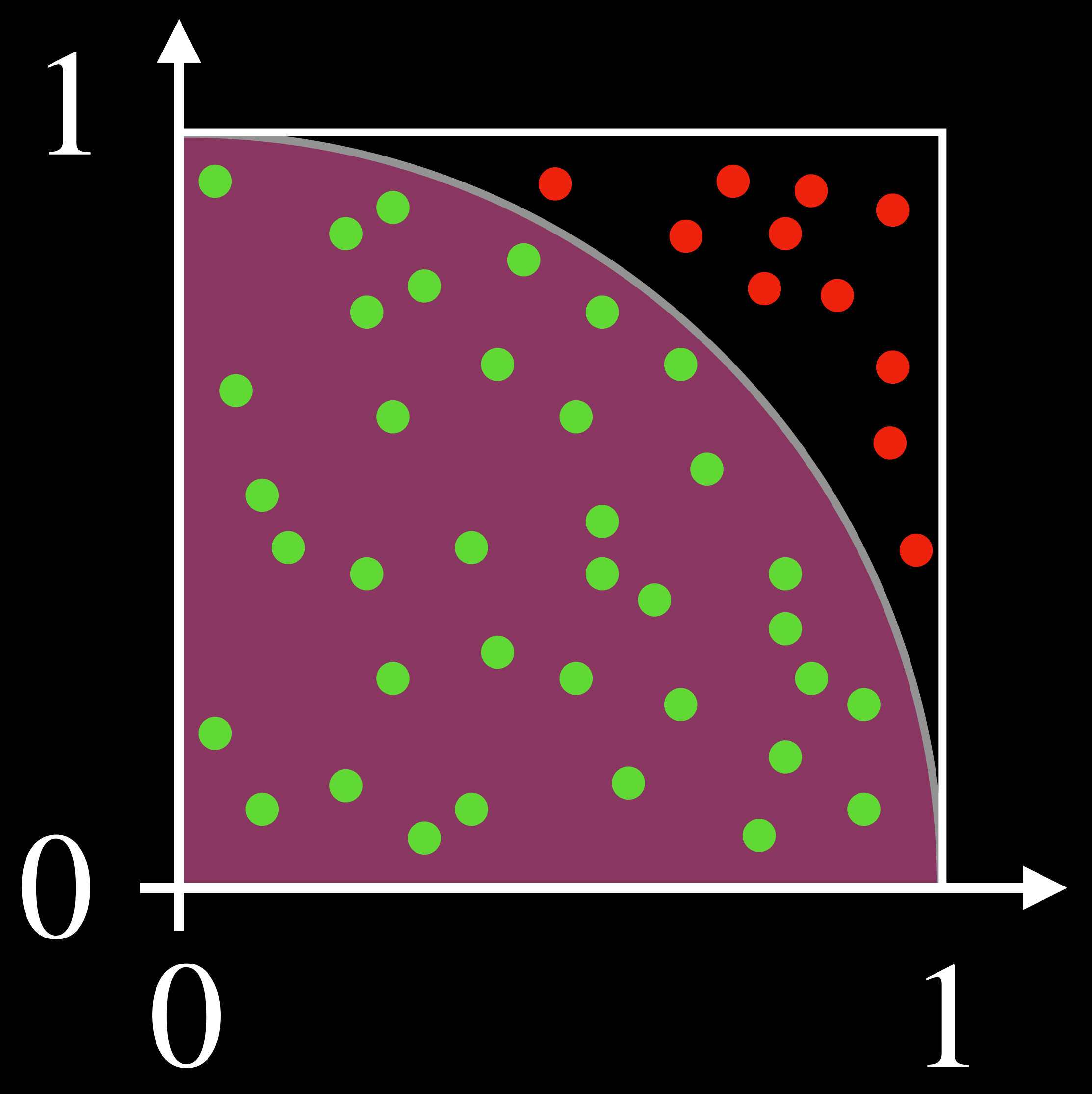

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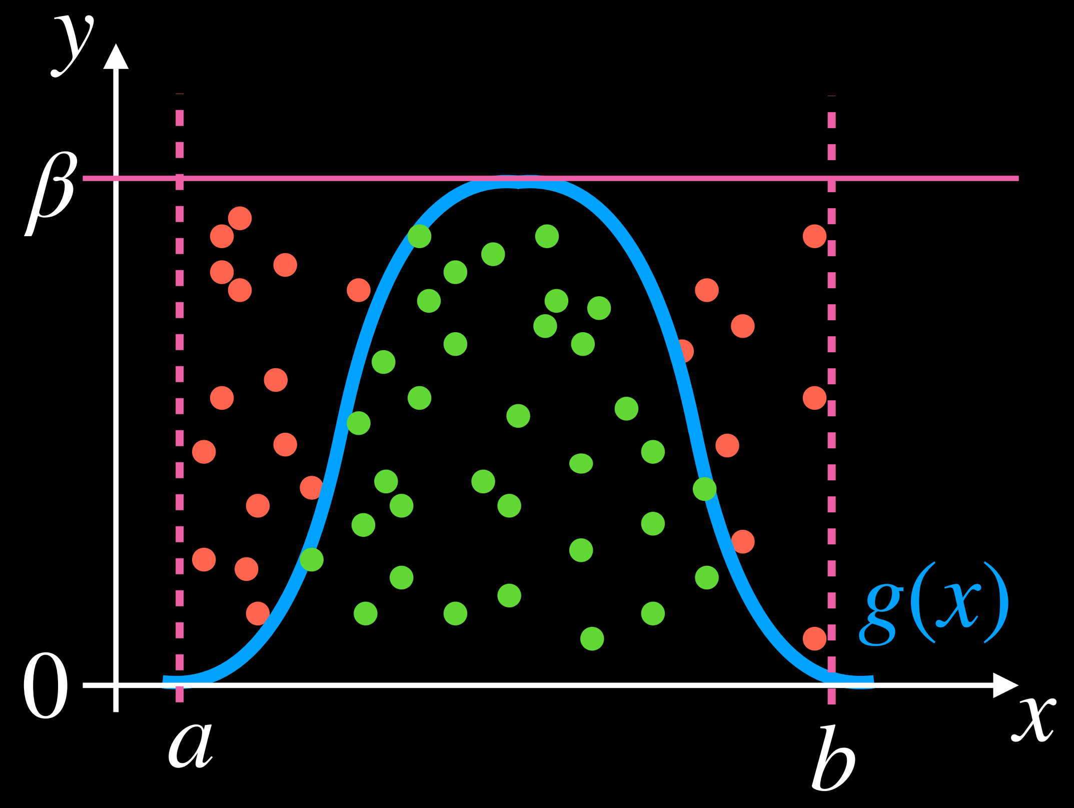
Monte Carlo integration (alternative like accept-reject sampling)



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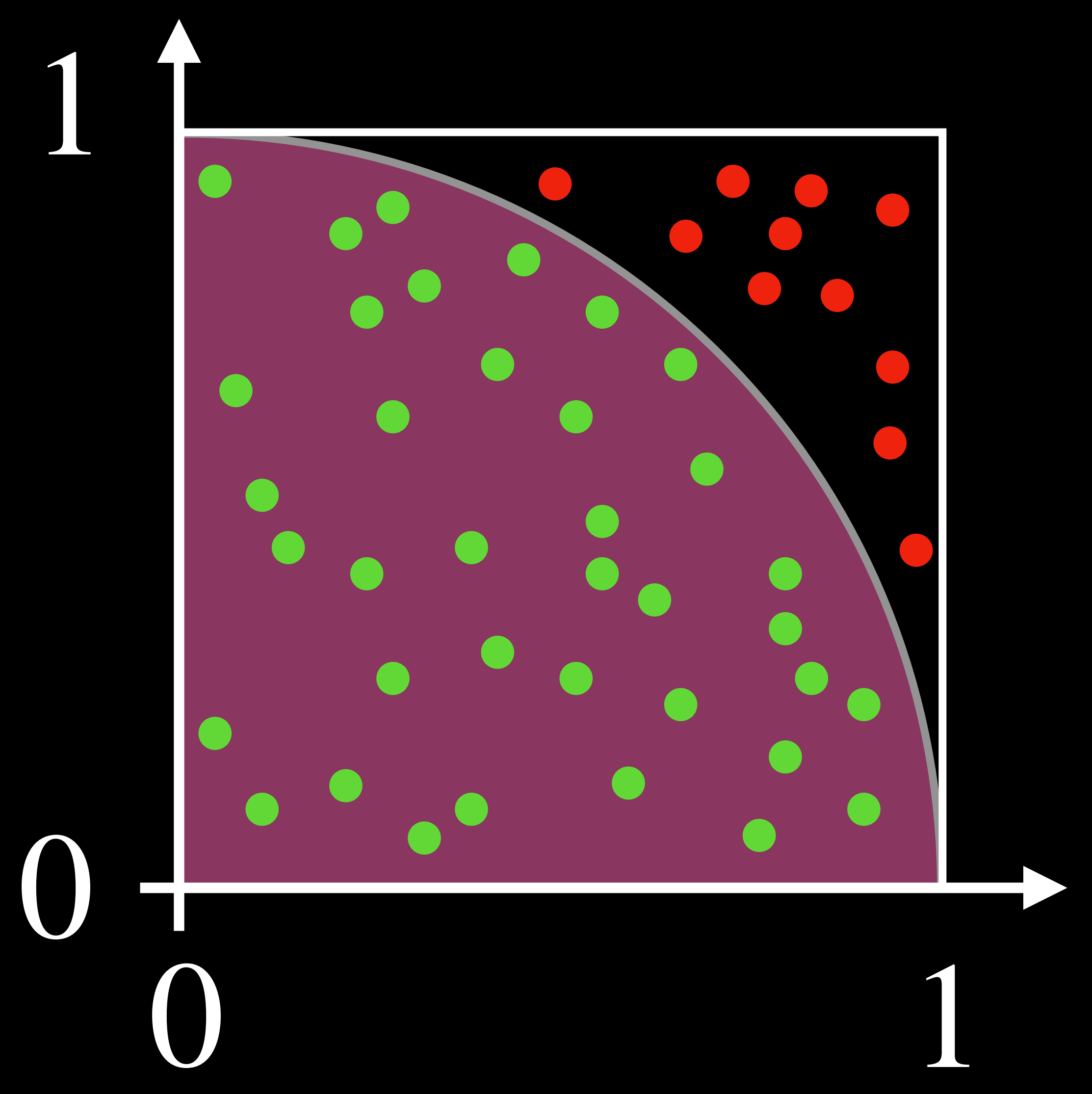

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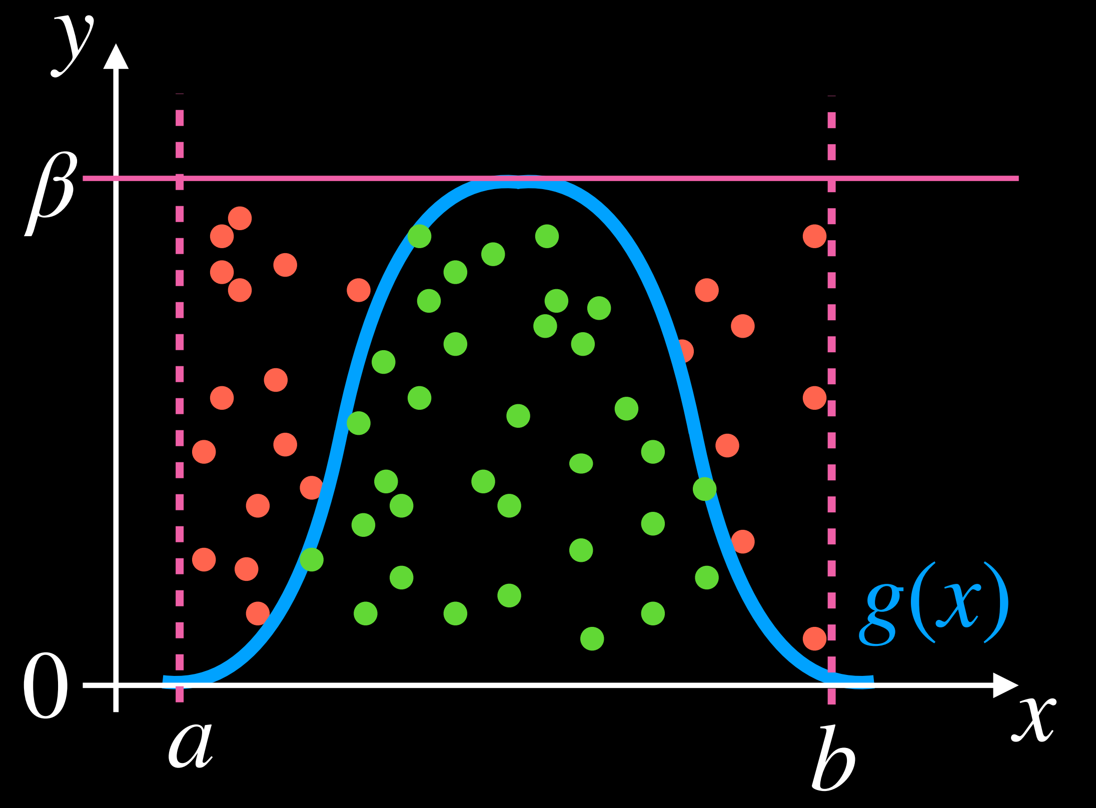

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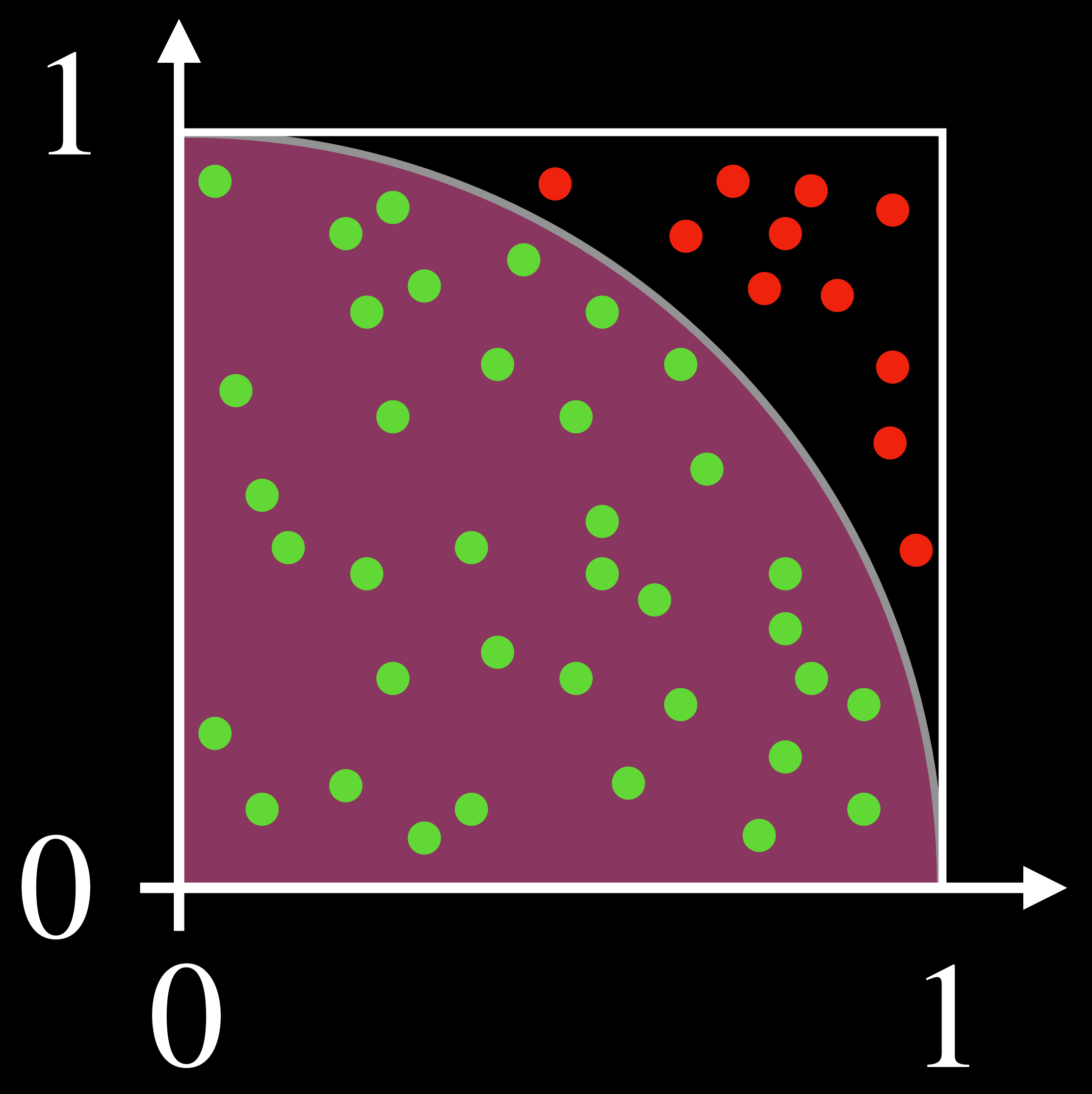

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Monte Carlo integration (error)

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$error_{\vartheta_n}$

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$$= \frac{b-a}{\sqrt{n}}\sqrt{Var\left(g(X)\right)}$$

Monte Carlo integration (error)

$$error_{\vartheta_n} \approx \sqrt{Var(\vartheta_n)} \quad = \sqrt{Var\left((b-a)\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) \right)}$$

$$= \frac{b-a}{n}\sqrt{Var\left(\sum_{i=0}^{n-1} g(X_i)\right)} \qquad X_i \text{ i.i.d.}$$

$$= \frac{b-a}{n}\sqrt{\sum_{i=0}^{n-1} Var\left(g(X_i)\right)}$$

$$= \frac{b-a}{n}\sqrt{n\ Var\left(g(X)\right)}$$

$$= \frac{b-a}{\sqrt{n}}\sqrt{Var\left(g(X)\right)} \qquad \propto \frac{1}{\sqrt{n}}$$

Monte Carlo integration (error)

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X_i i.i.d.

$$= \frac{b-a}{n}\sqrt{\sum_{i=0}^{n-1} Var\left(g(X_i)\right)}$$

$$= \frac{b-a}{n}\sqrt{n\ Var\left(g(X)\right)}$$

higher dimension error

$$= \frac{b-a}{\sqrt{n}}\sqrt{Var\left(g(X)\right)} \qquad \propto \frac{1}{\sqrt{n}}$$

Monte Carlo integration (error)

$$error_{\vartheta_n} \approx \sqrt{Var(\vartheta_n)} \quad = \sqrt{Var\left((b-a)\frac{1}{n}\sum_{i=0}^{n-1} g(X_i)\right)}$$

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$error_{\vartheta_n}$

Monte Carlo integration (error)

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higher dimension error

(volume)

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$$= \frac{b-a}{\sqrt{n}}\sqrt{Var\left(g(X)\right)} \qquad \propto \frac{1}{\sqrt{n}}$$

higher dimension error

$$error_{\vartheta_n} \approx \frac{\overset{\text{(volume)}}{V}}{\sqrt{n}}\sqrt{Var\left(g(X)\right)}$$

Monte Carlo integration (importance sampling)

Monte Carlo integration (importance sampling)

intuition:

Monte Carlo integration (importance sampling)

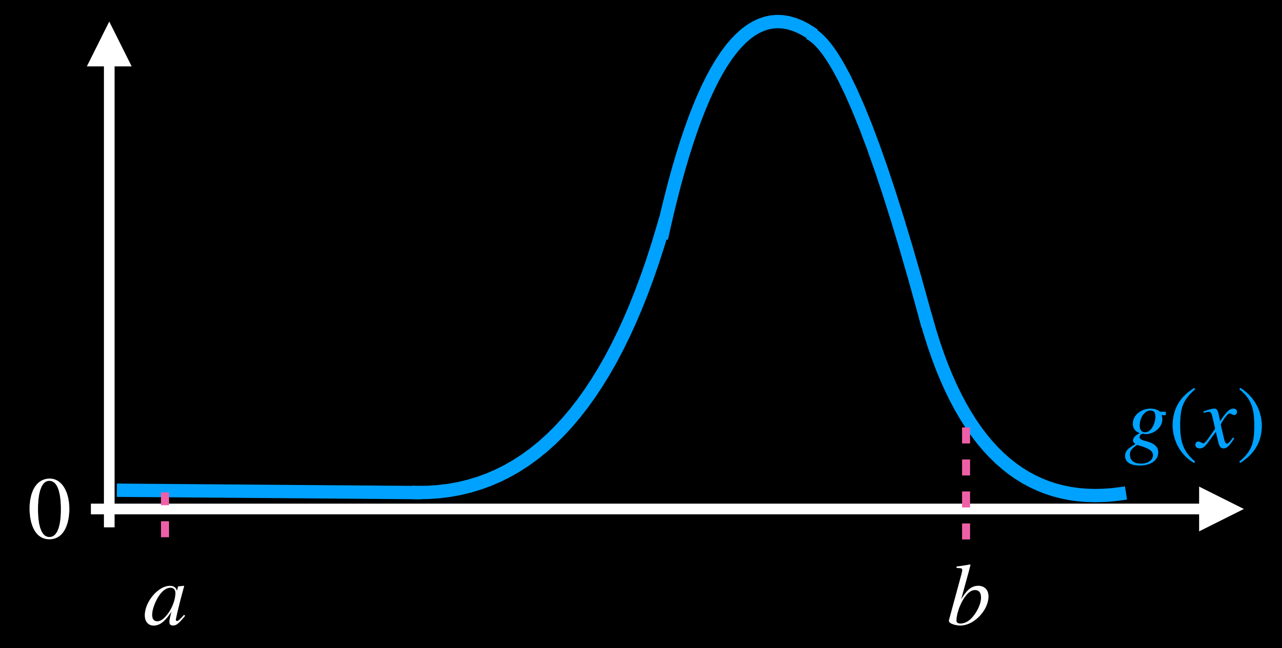
intuition:

$$\int_a^b g(x)\,dx$$

Monte Carlo integration (importance sampling)

intuition:

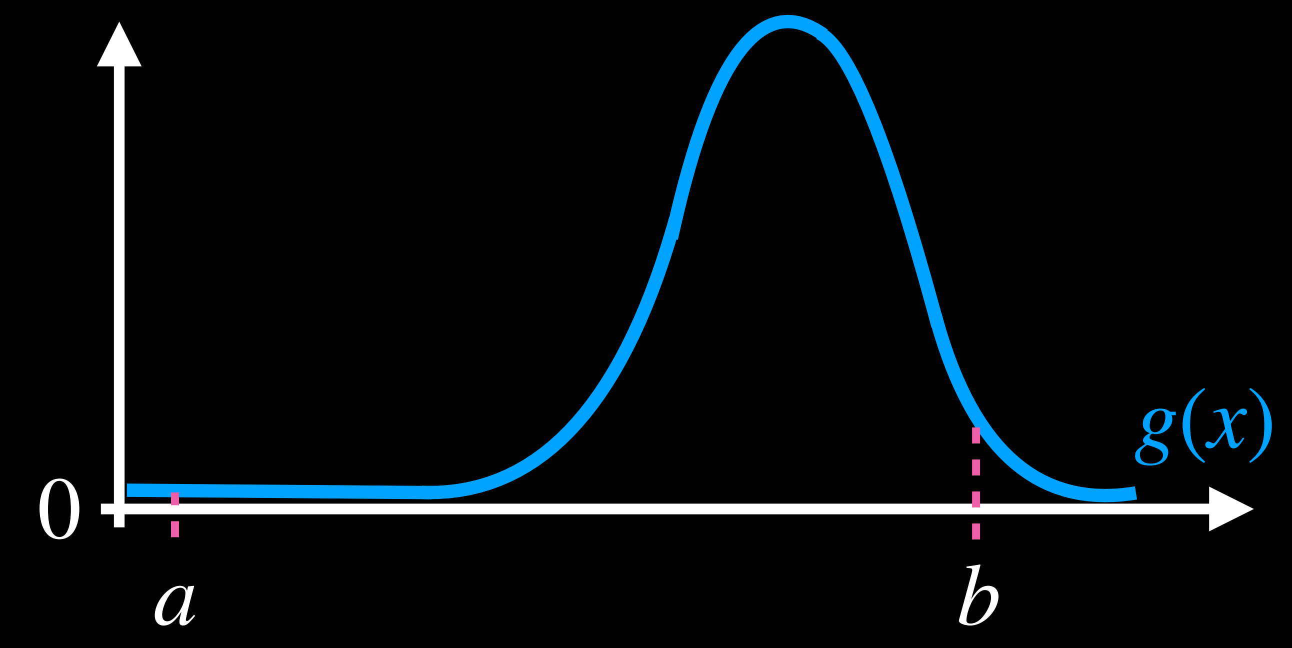
$$\int_a^b g(x)\,dx$$



Monte Carlo integration (importance sampling)

intuition:

$$\int_a^b g(x)\,dx \approx \sum_{i=0}^{n-1} g(x_i)\,\Delta x_i$$



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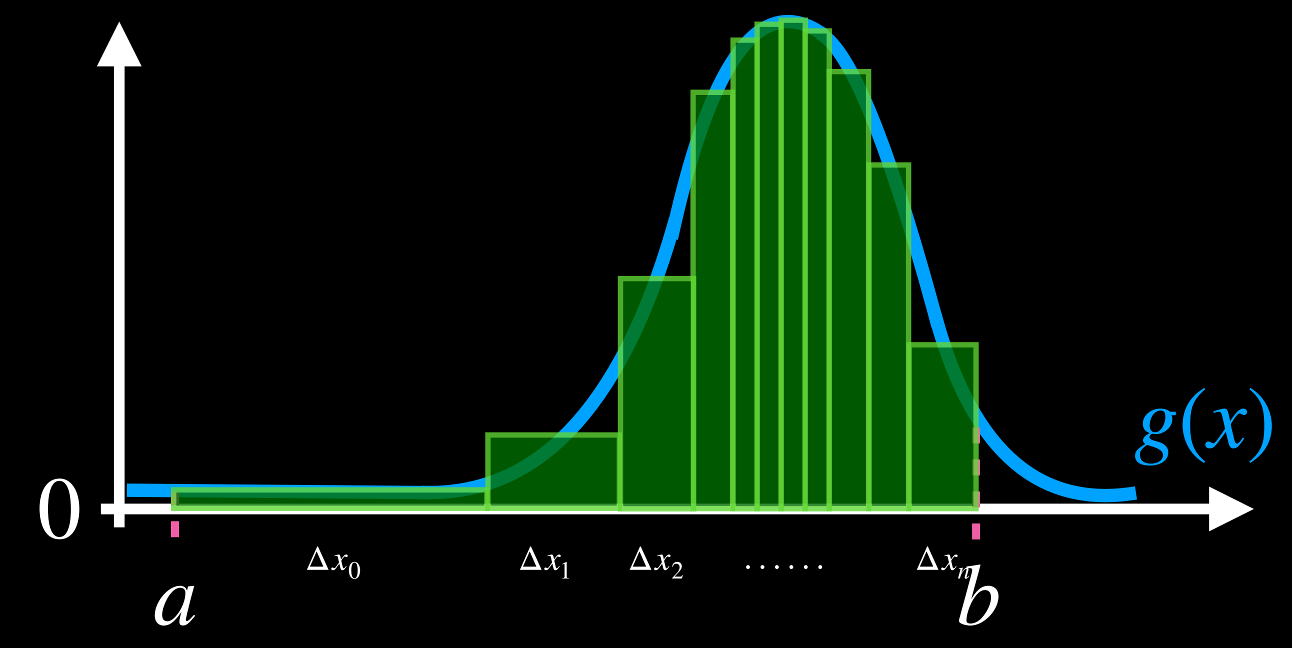

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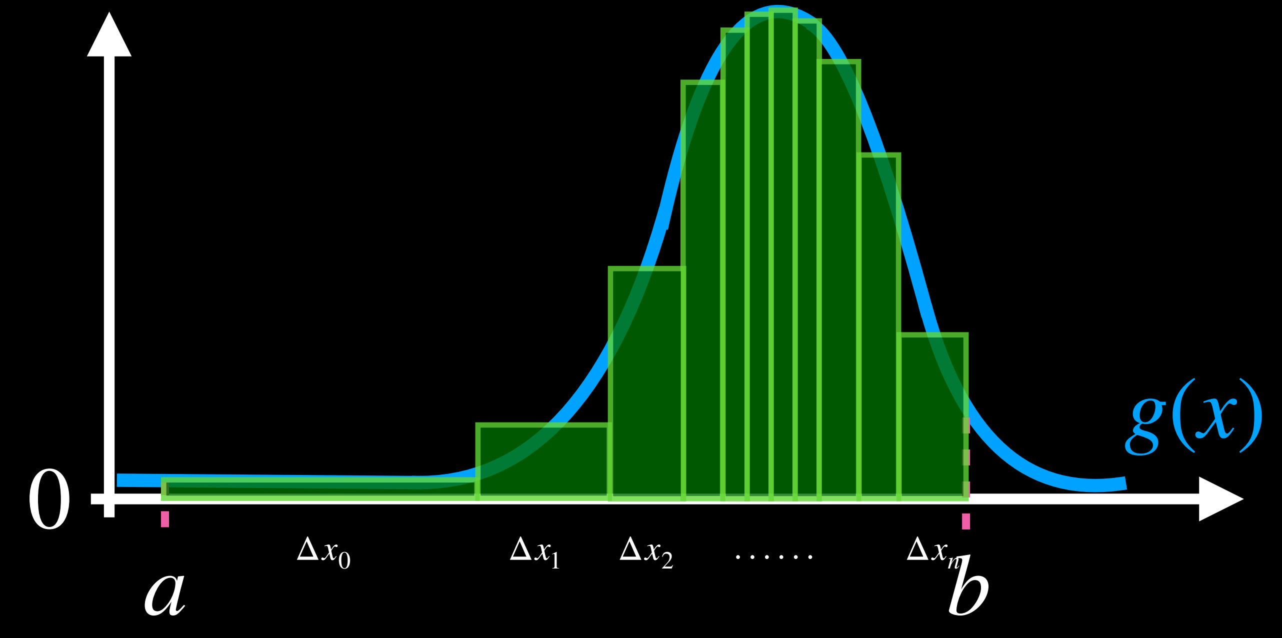
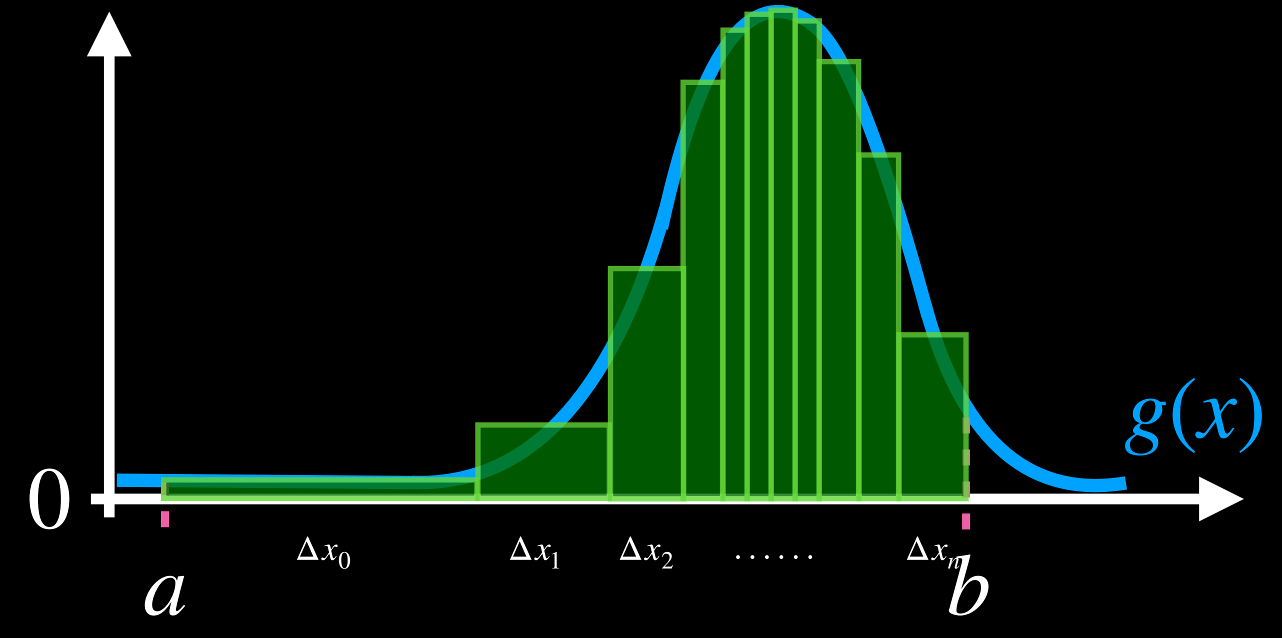


$$\Delta x = \frac{b-a}{n}$$

Monte Carlo integration (importance sampling)



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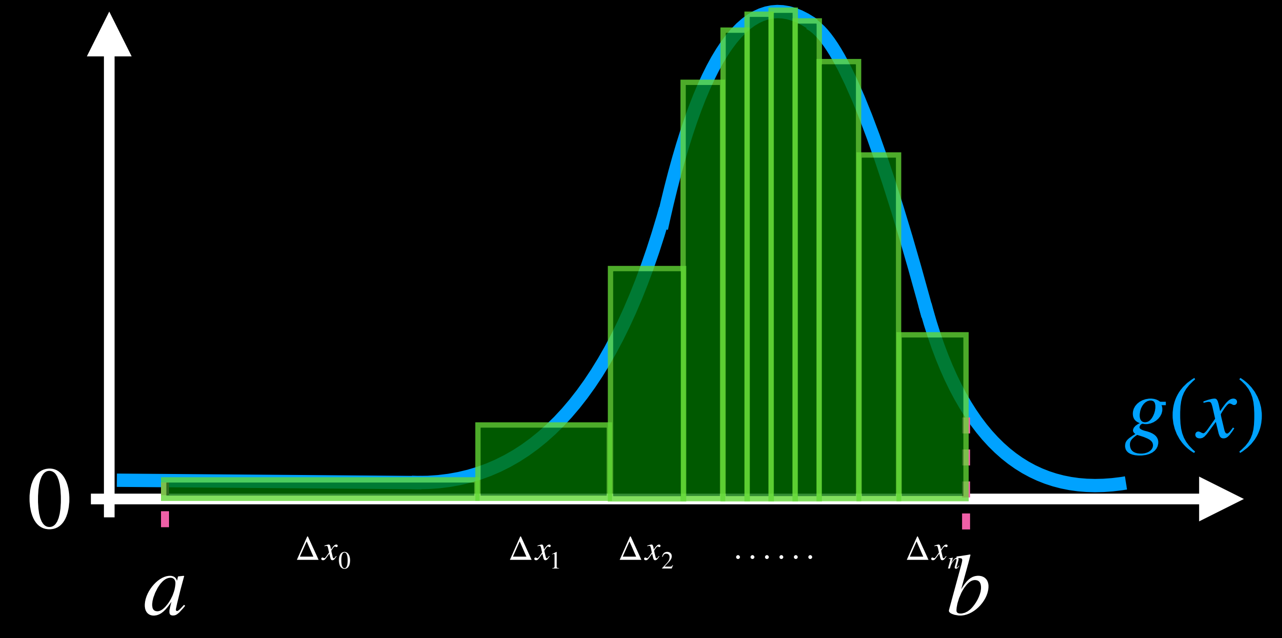

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Monte Carlo integration (importance sampling)

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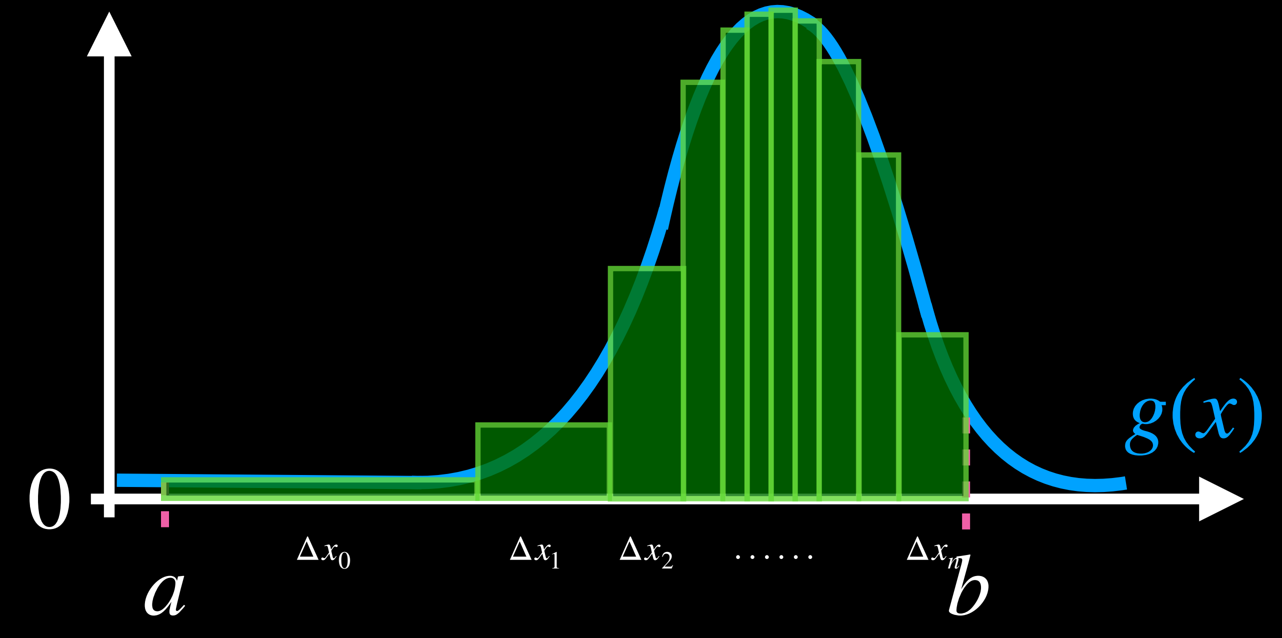

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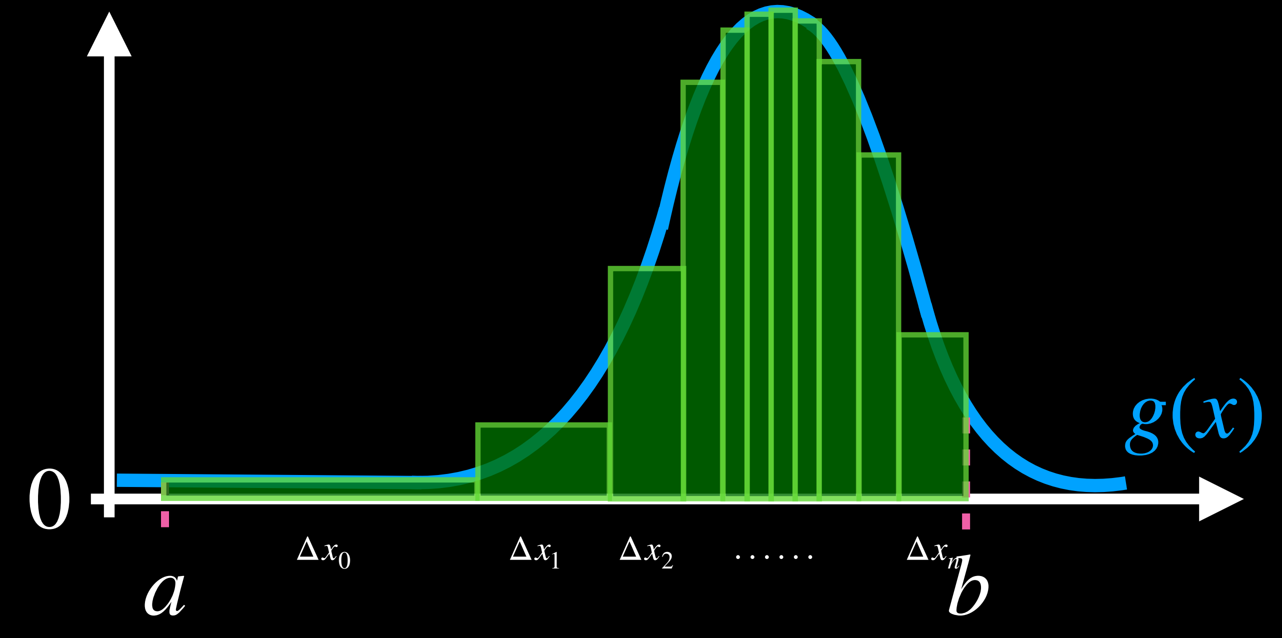

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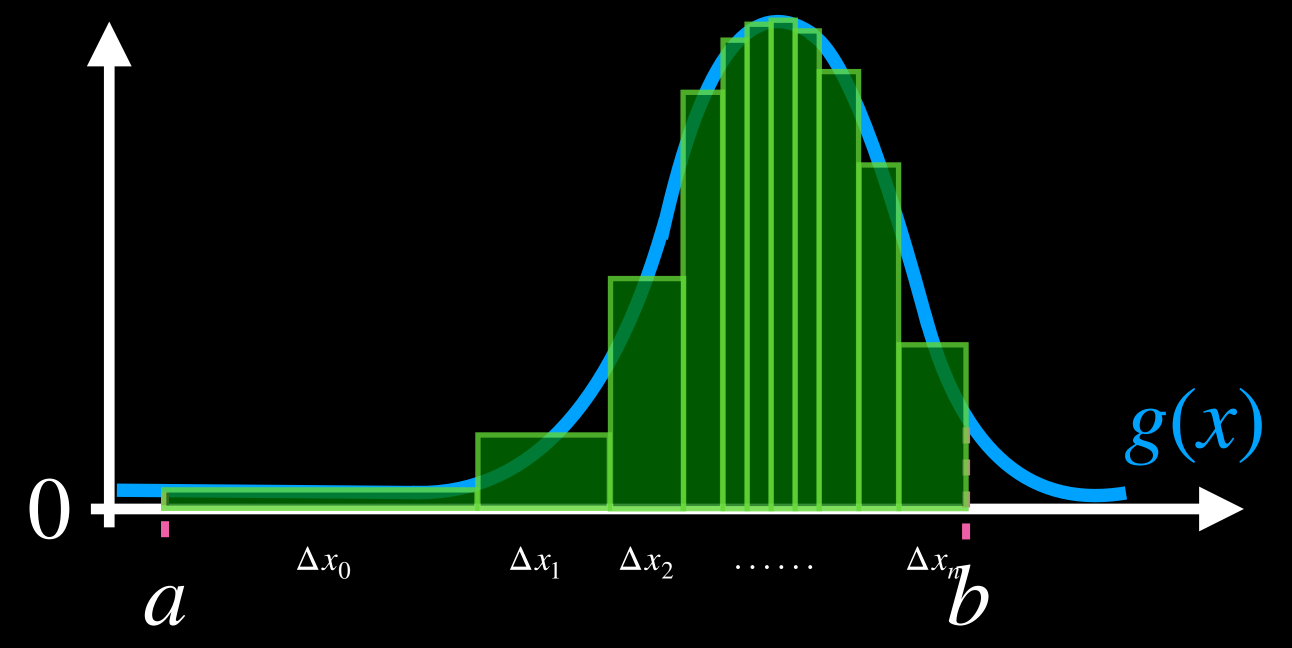
Monte Carlo integration (importance sampling)



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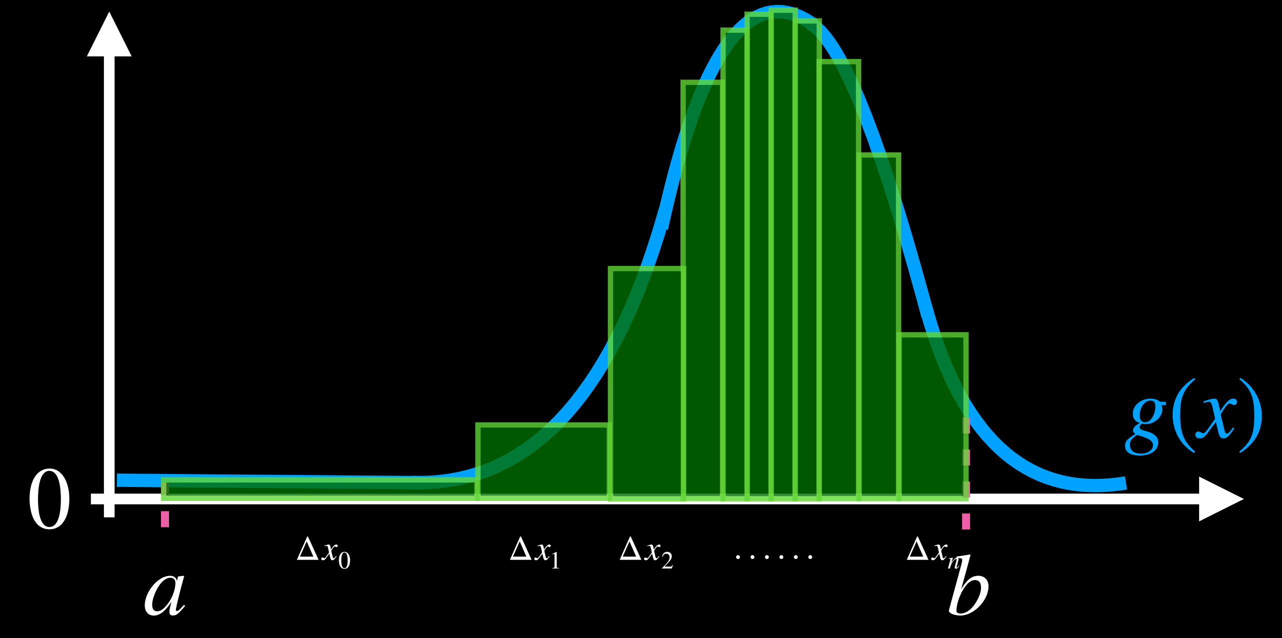

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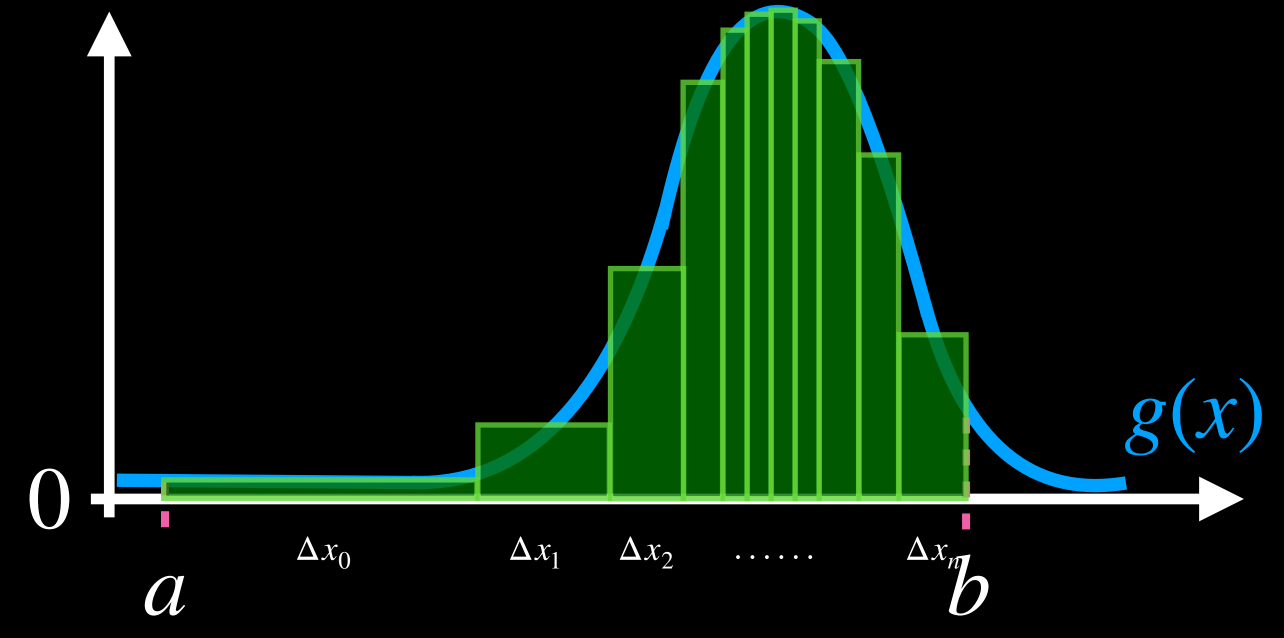

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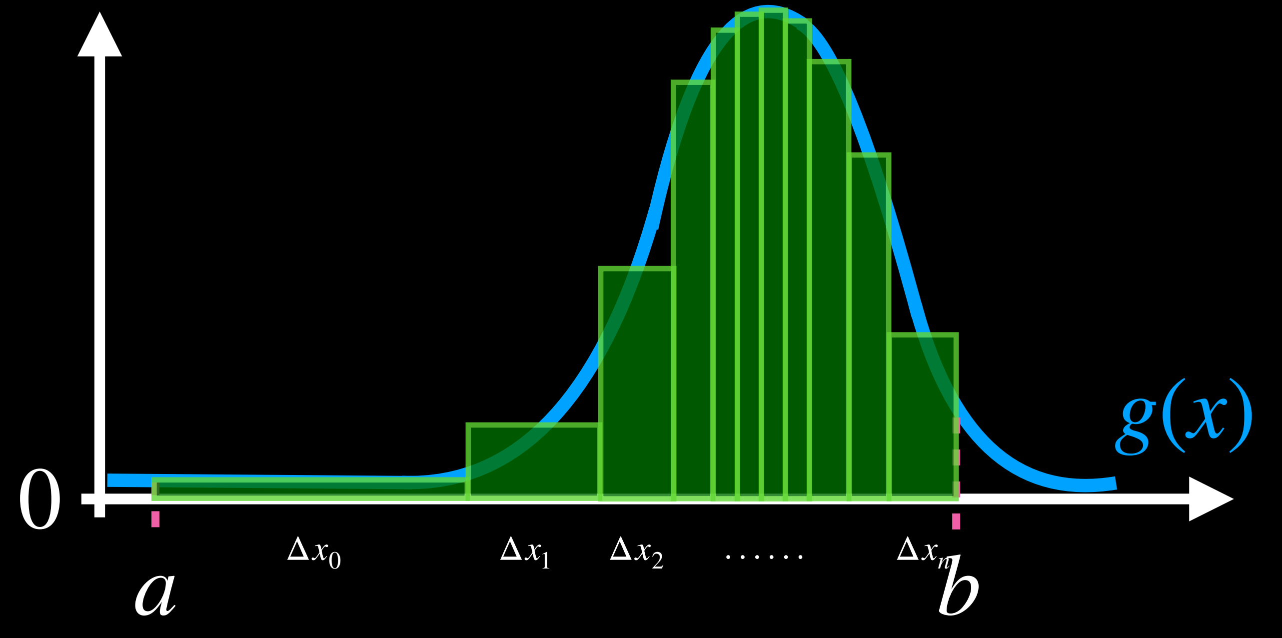
$$X \sim \mathcal{U}(a,b)$$

Monte Carlo integration (importance sampling)

intuition:

$$\int_a^b g(x)\,dx \approx \sum_{i=0}^{n-1} g(x_i)\,\Delta x_i = \sum_{i=0}^{n-1} g(x_i)\,\frac{1}{\underbrace{n \cdot \frac{importance(x_i)}{b-a}}_{=f_X(x_i)}} = \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(x_i)}{f_X(x_i)} \approx \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(X_i)}{f_X(X_i)}$$



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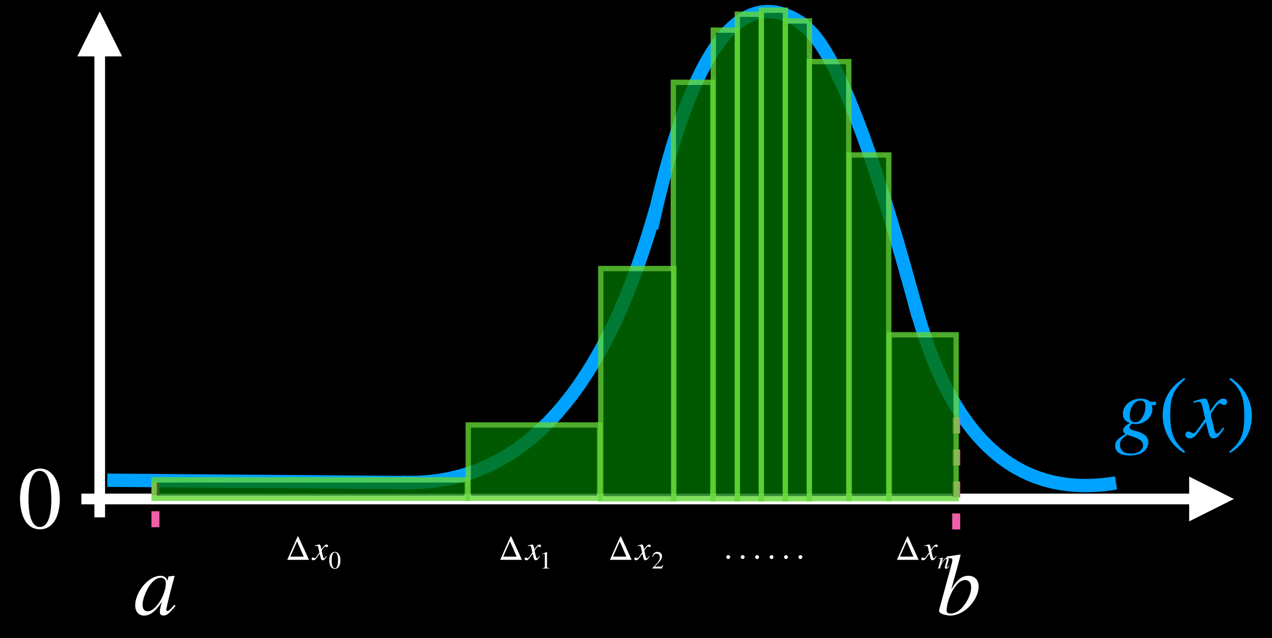

$\cancel{X \sim \mathcal{U}(a,b)}$

X distributed with PDF f_X

Monte Carlo integration (importance sampling)

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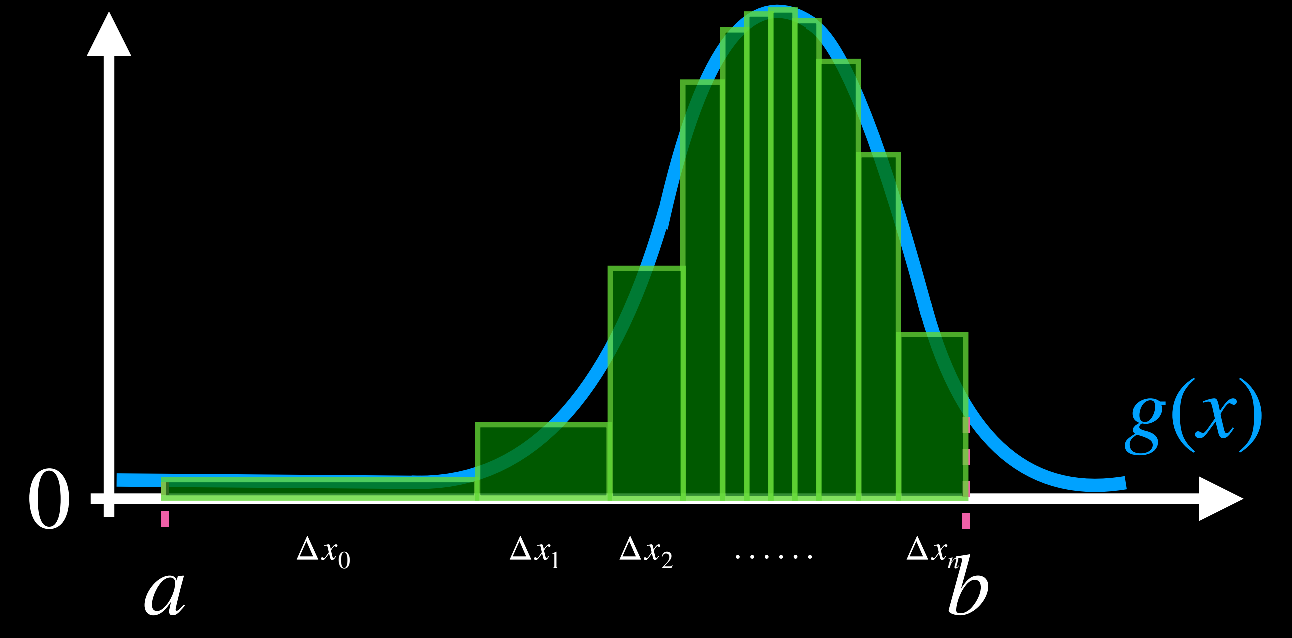

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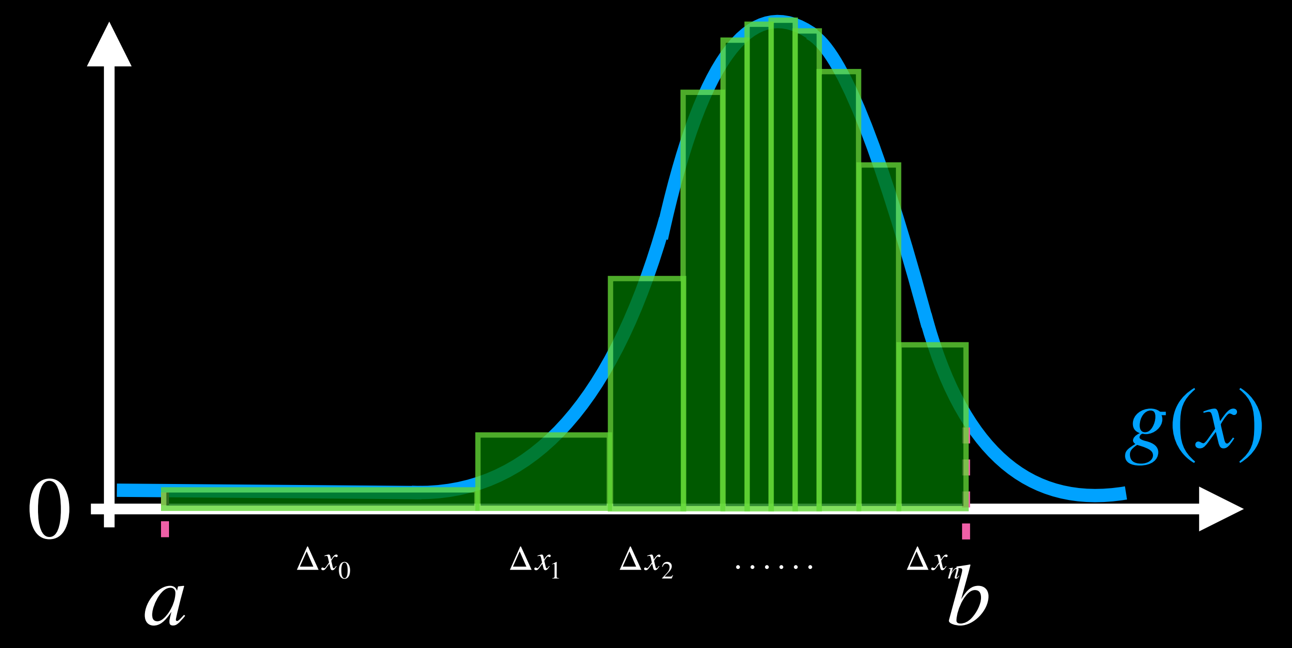
X distributed with PDF f_X

$$\Delta x = \frac{b-a}{n} \qquad importance(x_i) := \frac{\Delta x}{\Delta x_i} = \frac{b-a}{n \cdot \Delta x_i}$$

derivation:

Monte Carlo integration (importance sampling)



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$$\int_a^b g(x)\,dx \;\approx \sum_{i=0}^{n-1} g(x_i)\,\Delta x_i = \sum_{i=0}^{n-1} g(x_i)\frac{1}{n \cdot \underbrace{\dfrac{importance(x_i)}{b-a}}_{=f_X(x_i)}} = \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(x_i)}{f_X(x_i)} \approx \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(X_i)}{f_X(X_i)}$$

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derivation:

$\cancel{X \sim \mathcal{U}(a,b)}$

X distributed with PDF f_X

expectation

$$\mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x) f_X(x)\,dx$$

SLLN

$$P\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) = \mathbb{E}[g(X)]\right) = 1$$

Monte Carlo integration (importance sampling)

intuition:

$$\int_a^b g(x)\,dx \approx \sum_{i=0}^{n-1} g(x_i)\,\Delta x_i = \sum_{i=0}^{n-1} g(x_i)\,\frac{1}{\underbrace{n\cdot\frac{importance(x_i)}{b-a}}_{=f_X(x_i)}} = \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(x_i)}{f_X(x_i)} \approx \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(X_i)}{f_X(X_i)}$$

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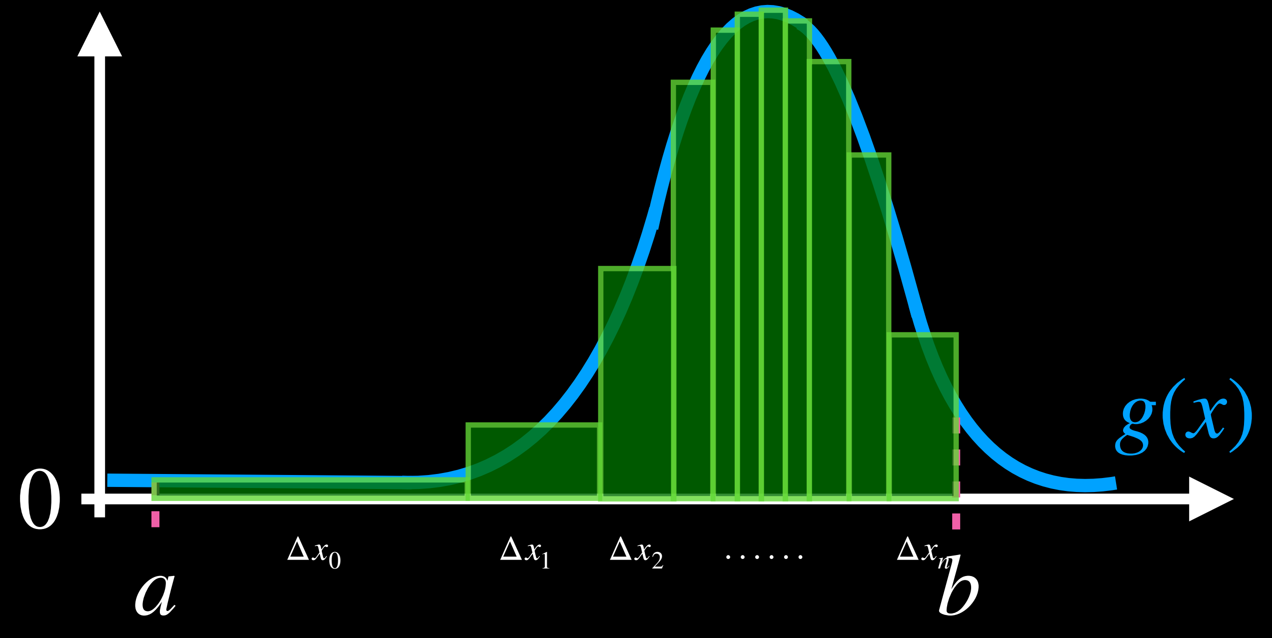
derivation:

$$\int_{\mathbb{S}} g(x)\,dx$$



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Monte Carlo integration (importance sampling)

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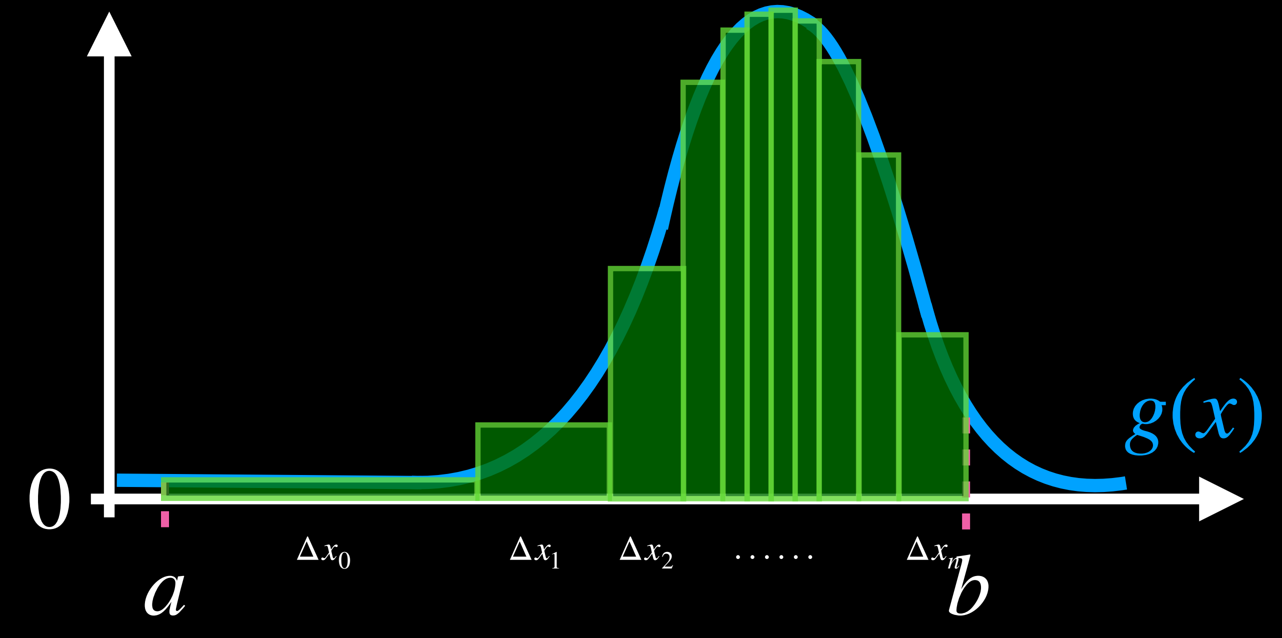

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$\cancel{X \sim \mathcal{U}(a,b)}$

X distributed with PDF f_X

derivation:

$$\int_{\mathbb{S}} g(x)\,dx \;= \int_{\mathbb{S}} g(x)\,1\,dx$$

expectation

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SLLN

$$P\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) = \mathbb{E}[g(X)]\right) = 1$$

Monte Carlo integration (importance sampling)

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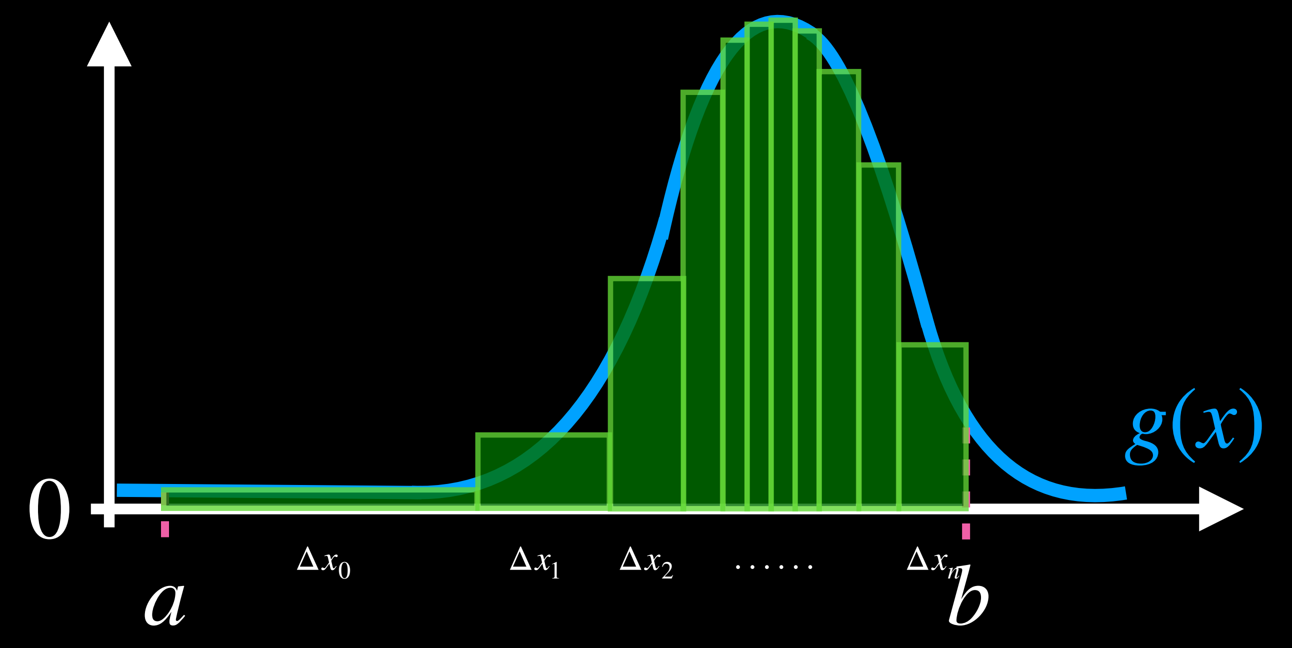

$\cancel{X \sim \mathcal{U}(a,b)}$

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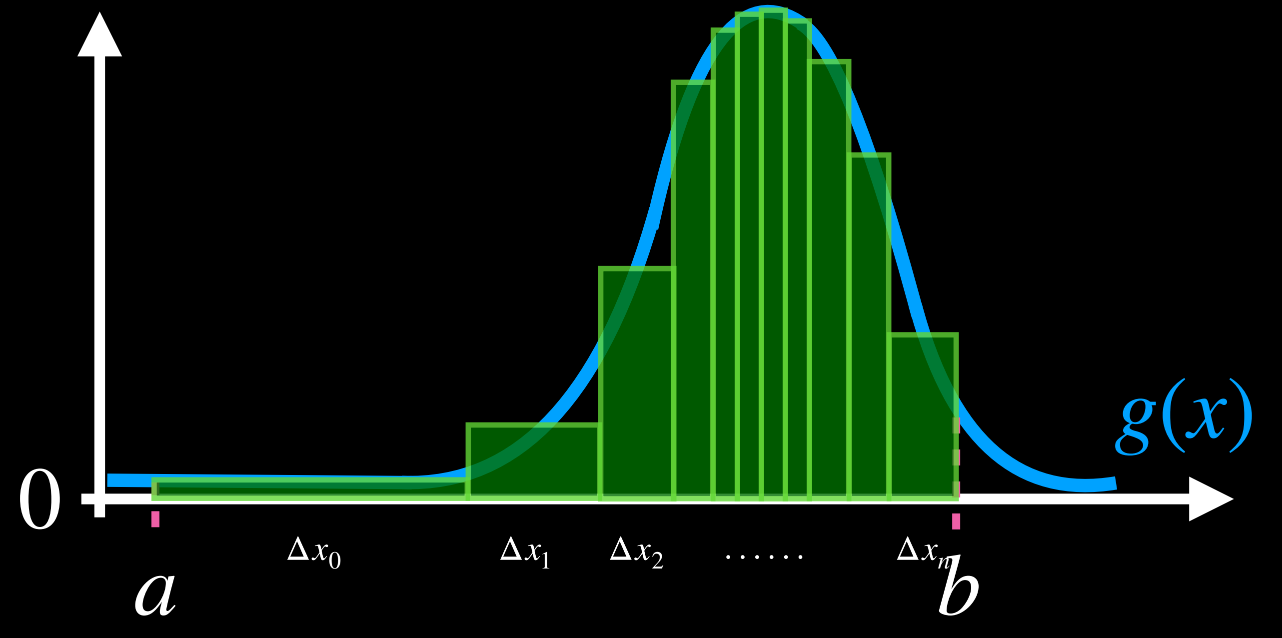

~~$X \sim \mathcal{U}(a,b)$~~

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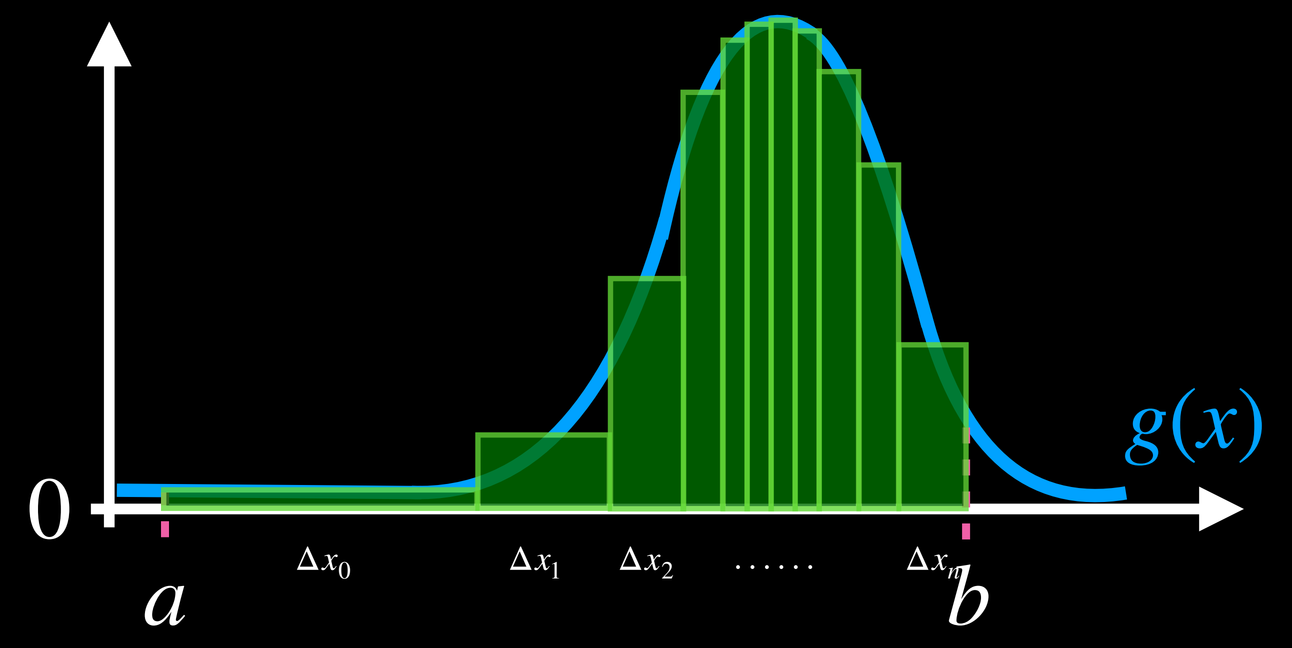

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$\cancel{X \sim \mathcal{U}(a,b)}$

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$$\int_{\mathbb{S}} g(x)\,dx = \int_{\mathbb{S}} g(x)\,1\,dx = \int_{\mathbb{S}} g(x)\frac{f_X(x)}{f_X(x)}\,dx = \int_{\mathbb{S}}\frac{g(x)}{f_X(x)}f_X(x)\,dx = \mathbb{E}\left[\frac{g(X)}{f_X(X)}\right]$$

expectation

$$\mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x)f_X(x)\,dx$$

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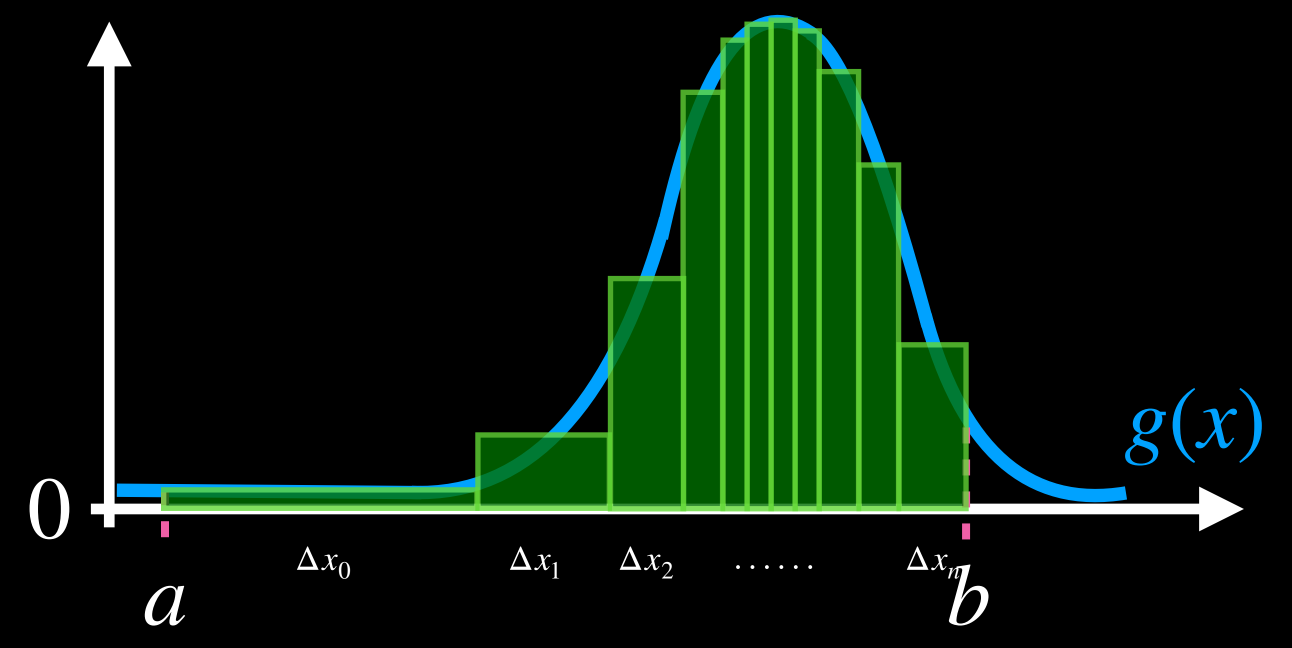

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$=$
almost
always

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SLLN

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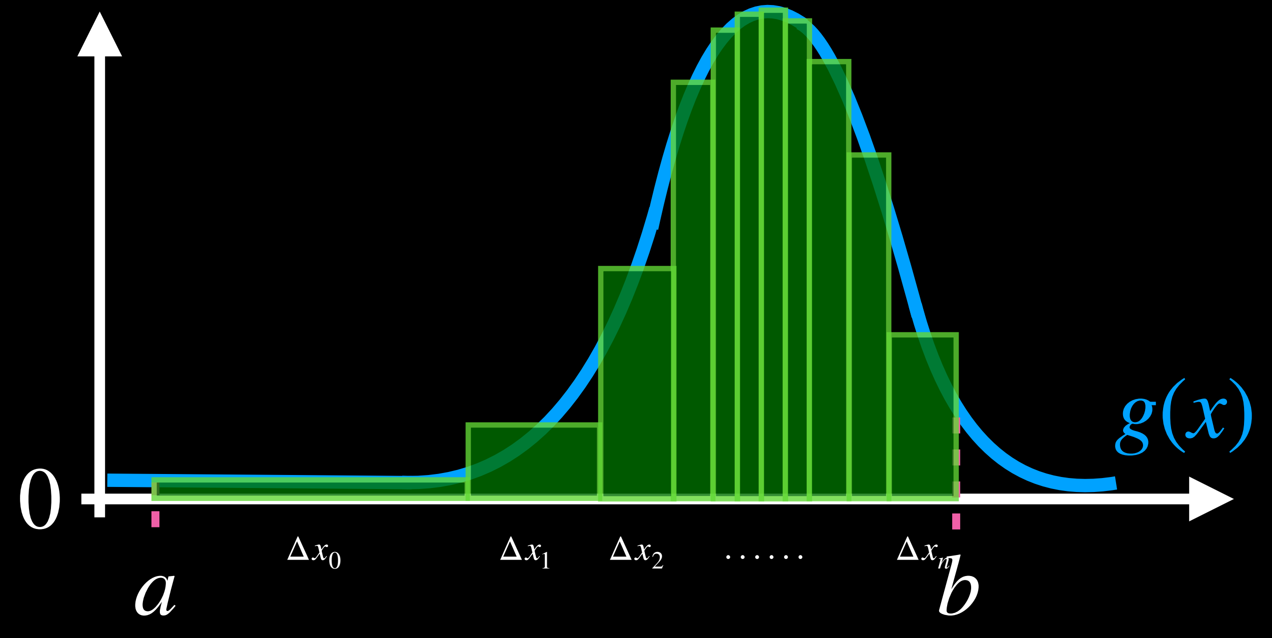

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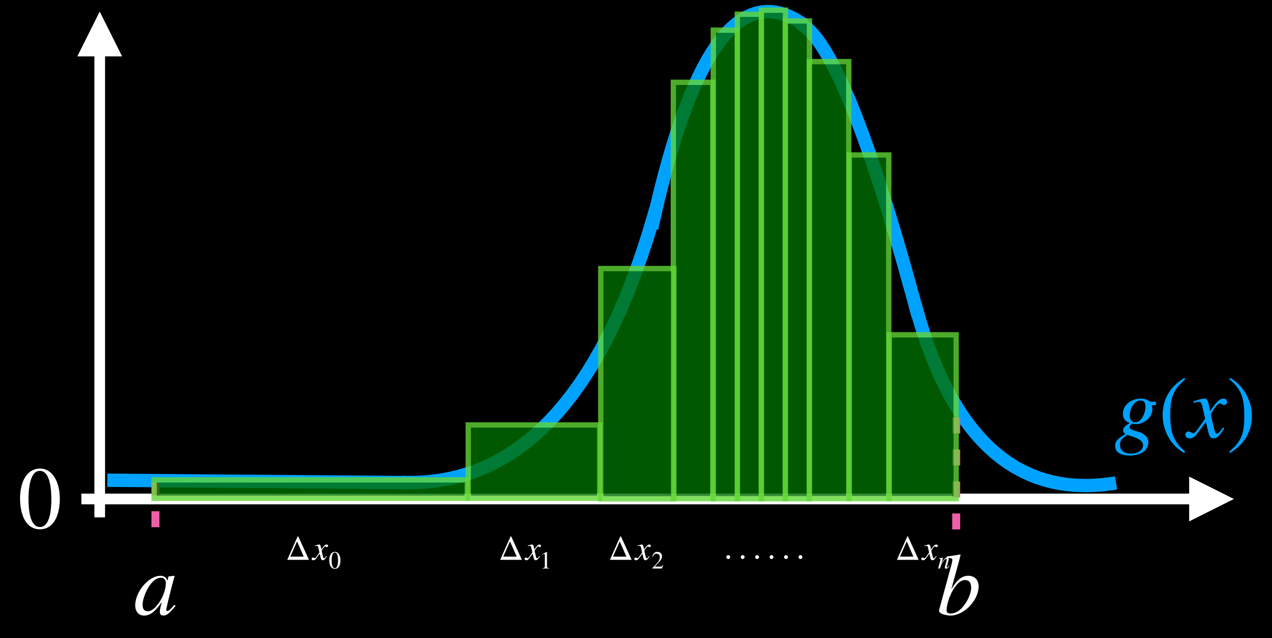

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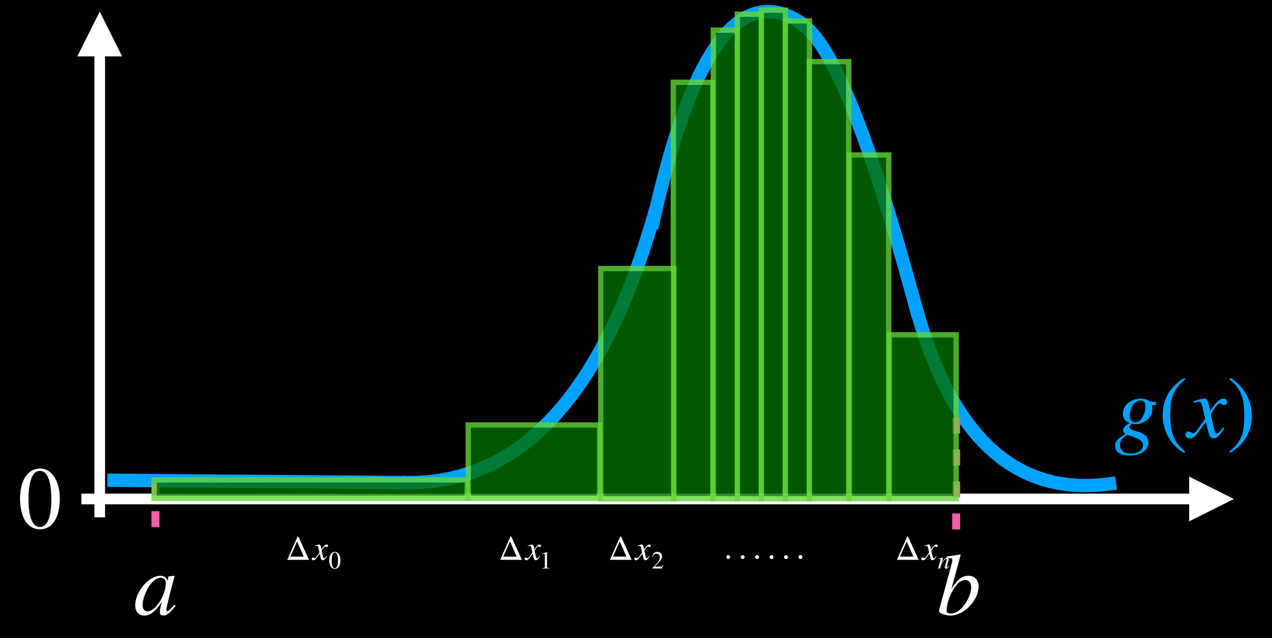

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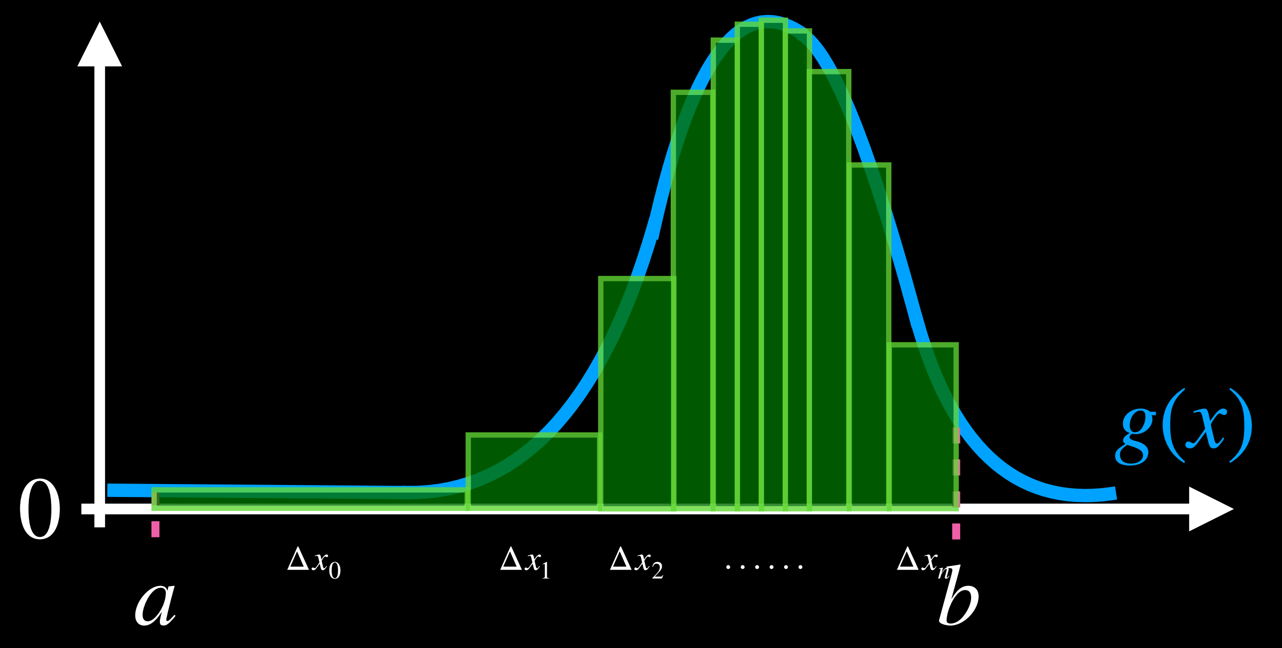

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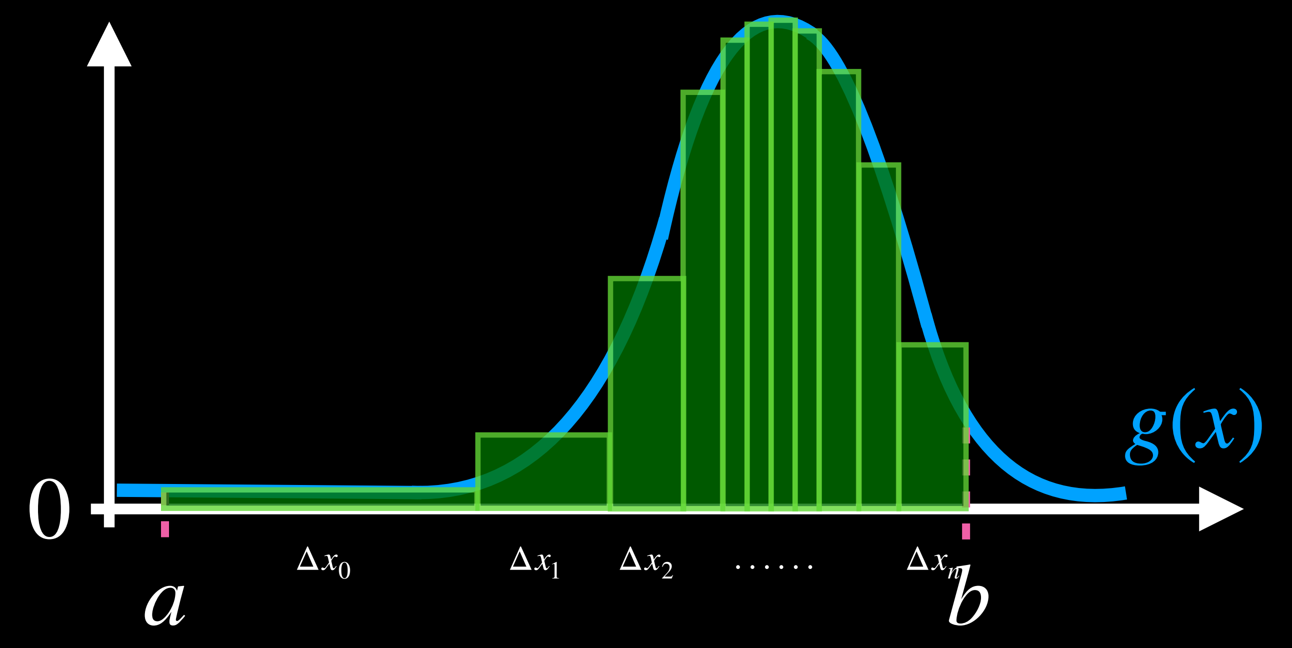

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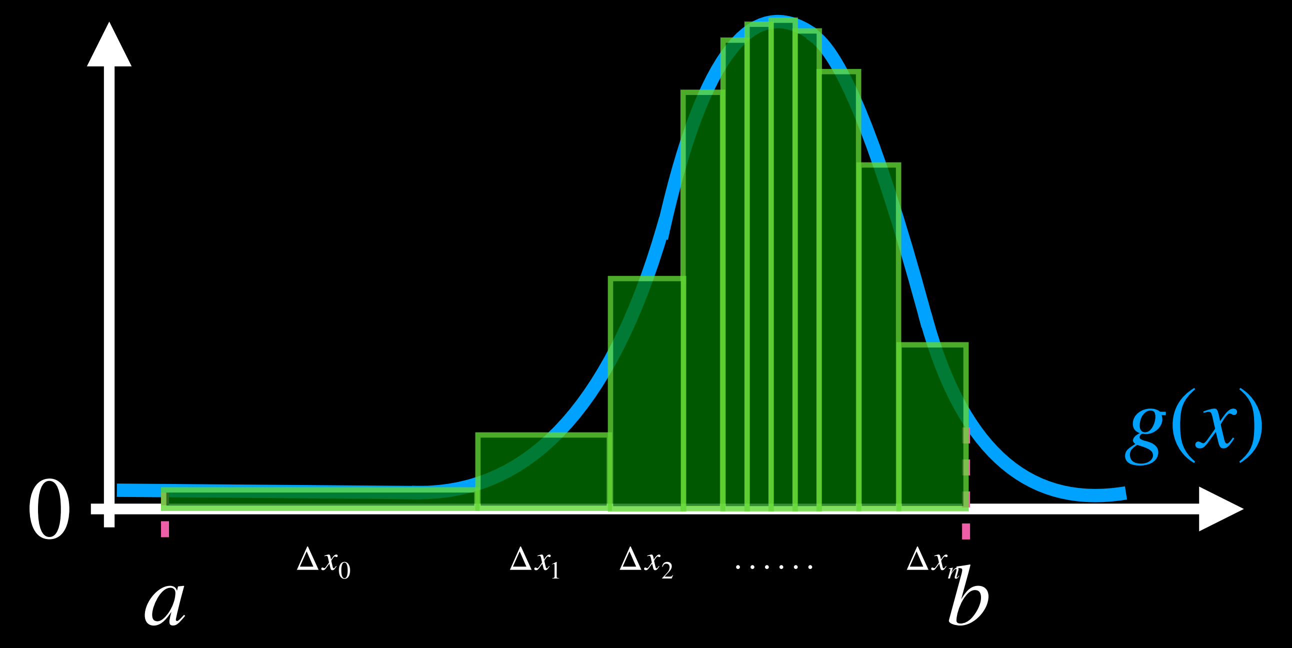
Monte Carlo integration (importance sampling)



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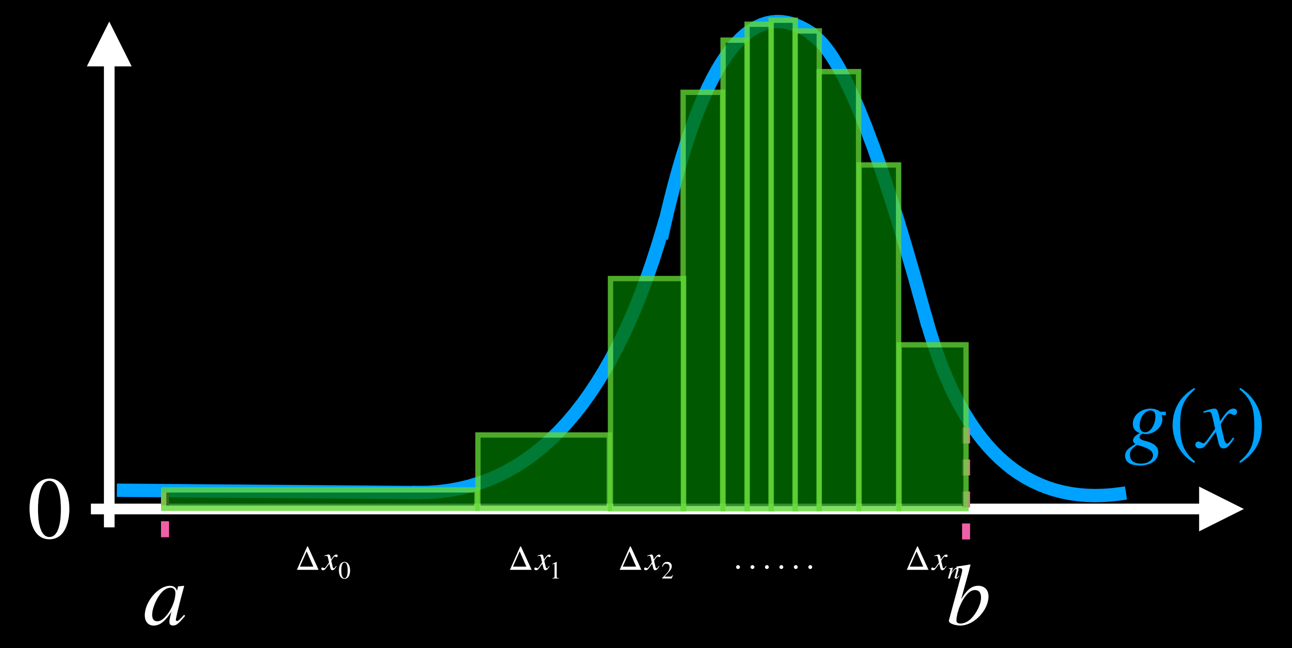

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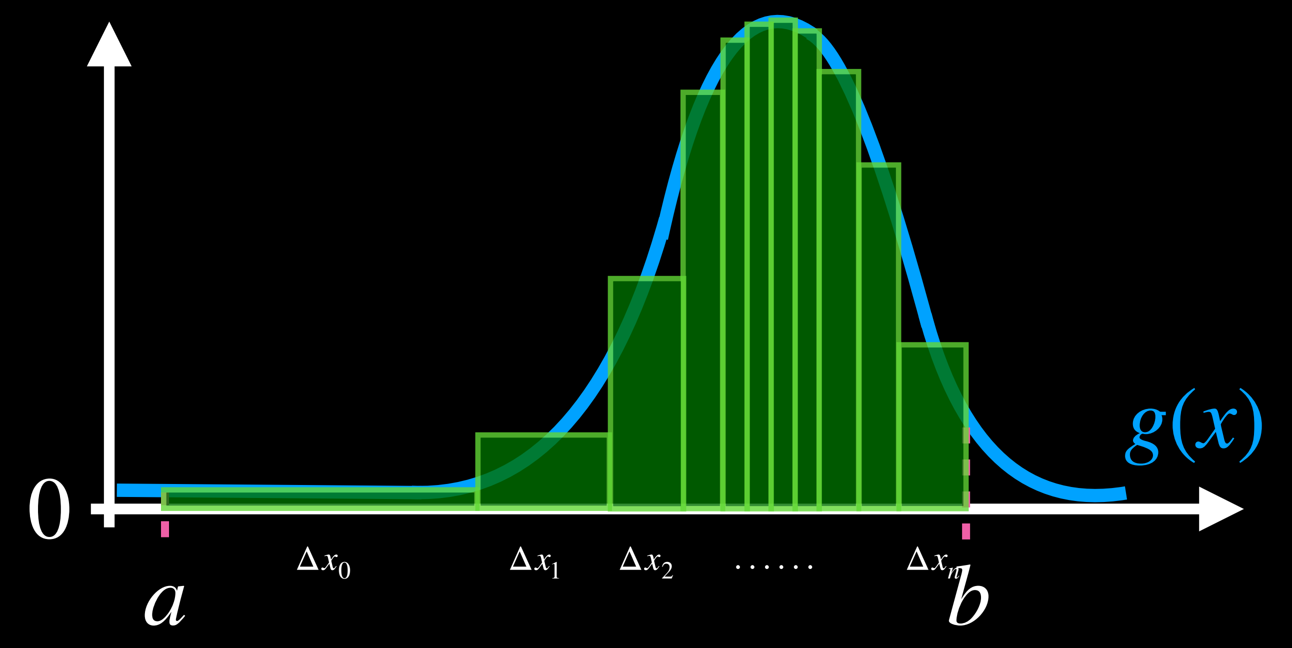

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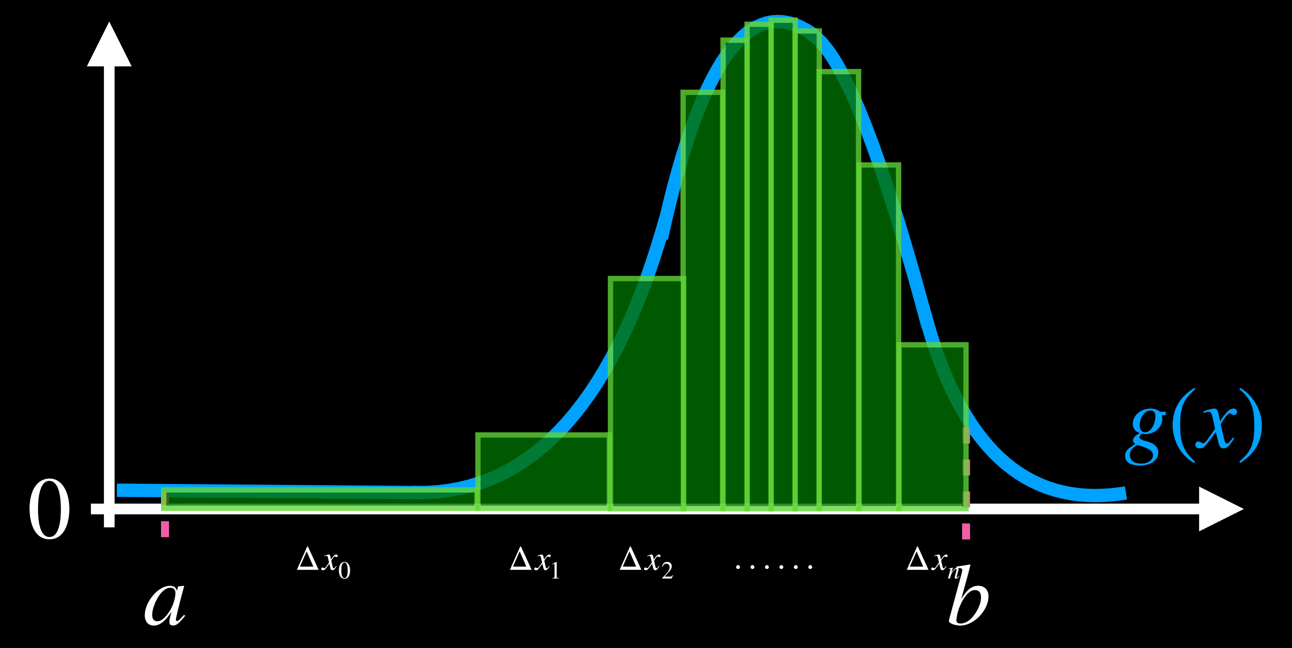

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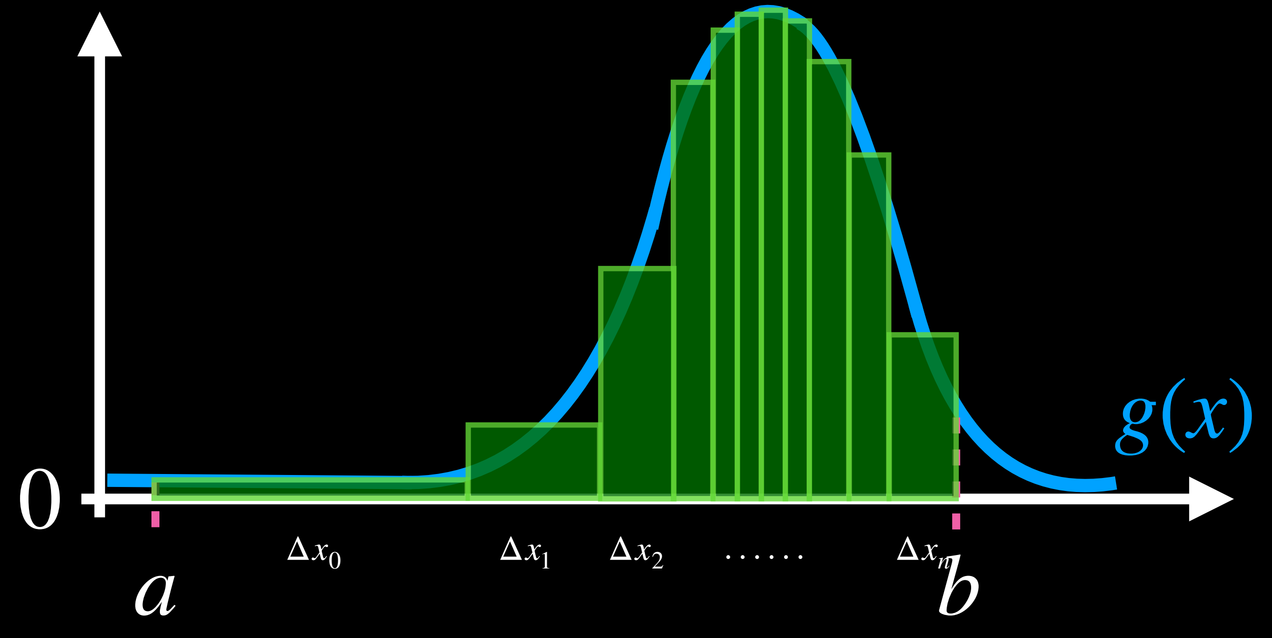

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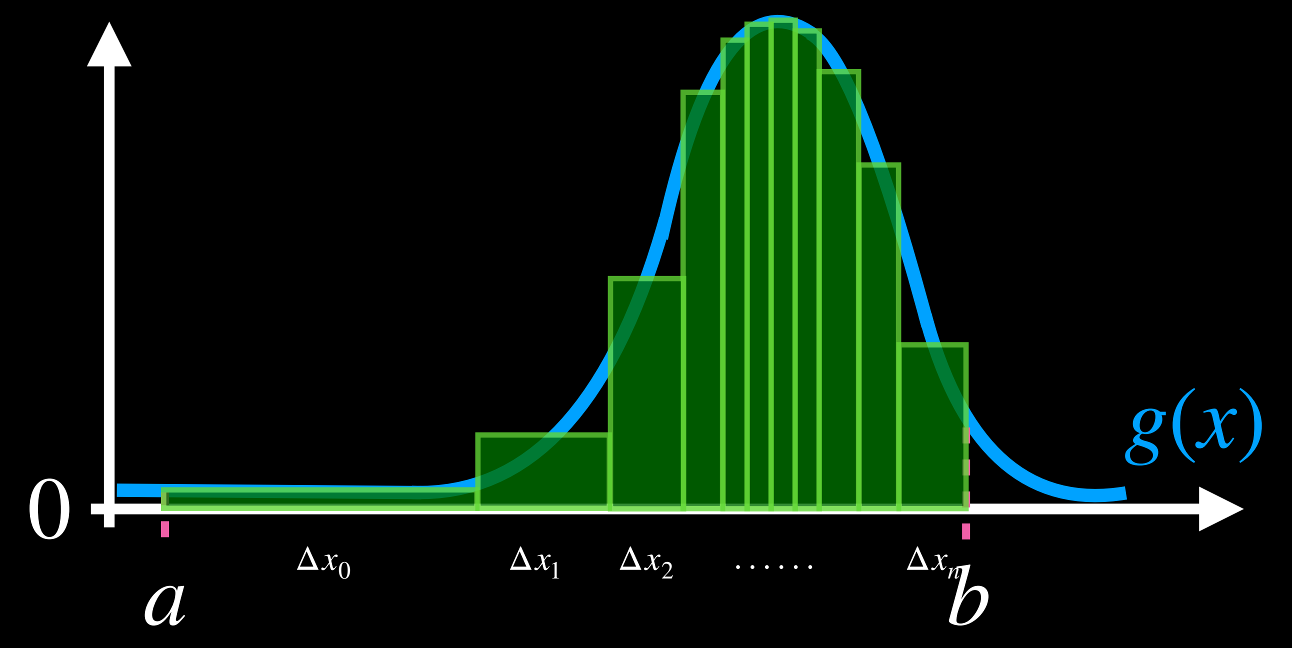

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$$error_{\vartheta_n} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(\frac{g(X)}{f_X(X)}\right)}$$

expectation

$$\mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x) f_X(x)\,dx$$

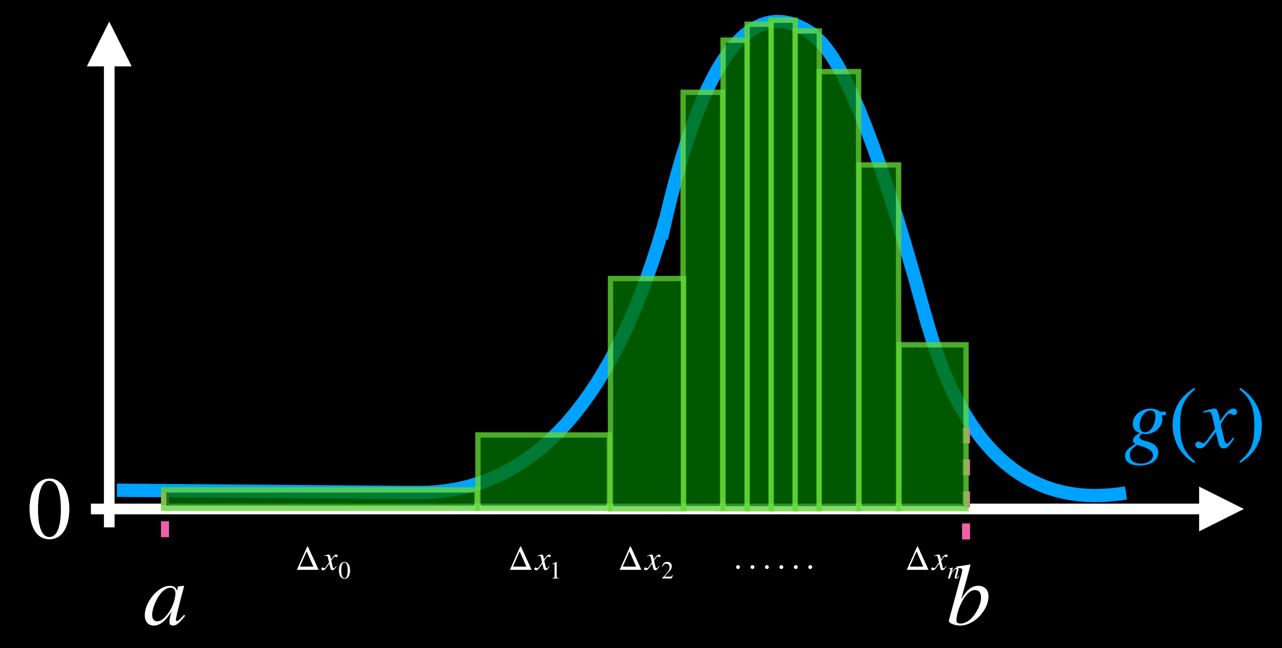
SLLN

$$P\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) = \mathbb{E}[g(X)]\right) = 1$$

choice of $f_X(x)$:

$$f_X(x) = g(x)\cdot k \qquad 1 = \int_{\mathbb{S}} f_X(x)\,dx = \int_{\mathbb{S}} g(x)\cdot k\,dx \qquad \frac{1}{k} = \int_{\mathbb{S}} g(x)\,dx \rightarrow$$

Monte Carlo integration (importance sampling)



intuition:

$$\int_a^b g(x)\,dx \approx \sum_{i=0}^{n-1} g(x_i)\,\Delta x_i = \sum_{i=0}^{n-1} g(x_i) \frac{1}{n \cdot \underbrace{\frac{importance(x_i)}{b-a}}_{=f_X(x_i)}} = \frac{1}{n}\sum_{i=0}^{n-1} \frac{g(x_i)}{f_X(x_i)} \approx \frac{1}{n}\sum_{i=0}^{n-1} \frac{g(X_i)}{f_X(X_i)}$$

$$\Delta x = \frac{b-a}{n} \qquad importance(x_i) := \frac{\Delta x}{\Delta x_i} = \frac{b-a}{n \cdot \Delta x_i}$$

$\cancel{X \sim \mathcal{U}(a,b)}$

X distributed with PDF f_X

derivation:

$$\int_{\mathbb{S}} g(x)\,dx = \int_{\mathbb{S}} g(x)\,1\,dx = \int_{\mathbb{S}} g(x)\frac{f_X(x)}{f_X(x)}\,dx = \int_{\mathbb{S}} \frac{g(x)}{f_X(x)} f_X(x)\,dx = \mathbb{E}\left[\frac{g(X)}{f_X(X)}\right]$$

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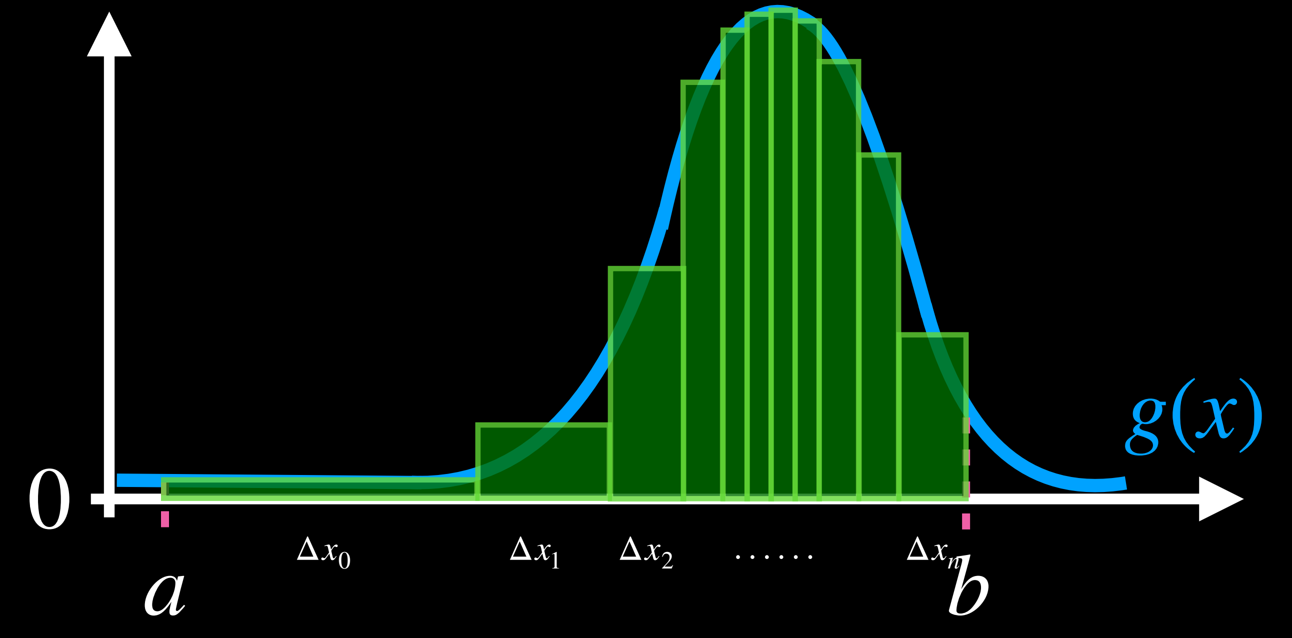
Monte Carlo integration (importance sampling)

intuition:

$$\int_a^b g(x)\,dx \approx \sum_{i=0}^{n-1} g(x_i)\,\Delta x_i = \sum_{i=0}^{n-1} g(x_i) \frac{1}{\underbrace{n \cdot \frac{importance(x_i)}{b-a}}_{=f_X(x_i)}} = \frac{1}{n}\sum_{i=0}^{n-1} \frac{g(x_i)}{f_X(x_i)} \approx \frac{1}{n}\sum_{i=0}^{n-1} \frac{g(X_i)}{f_X(X_i)}$$



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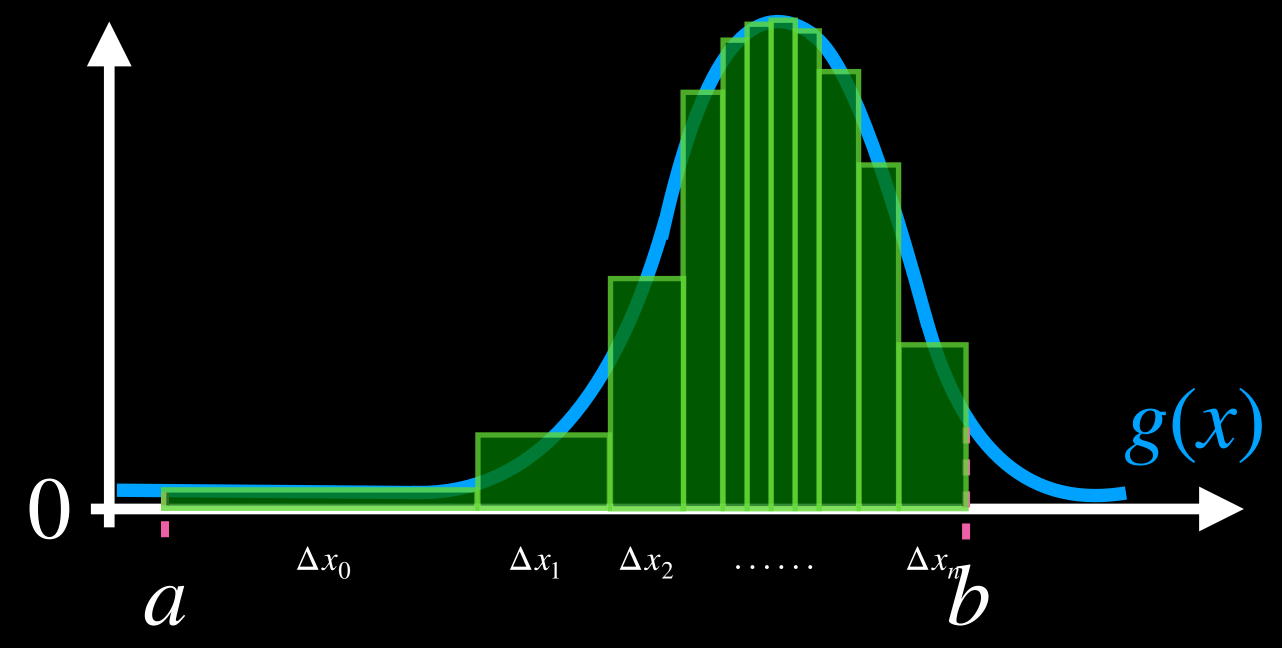
Monte Carlo integration (importance sampling)

intuition:

$$\int_a^b g(x)\,dx \approx \sum_{i=0}^{n-1} g(x_i)\,\Delta x_i = \sum_{i=0}^{n-1} g(x_i)\frac{1}{n\cdot\underbrace{\frac{importance(x_i)}{b-a}}_{=f_X(x_i)}} = \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(x_i)}{f_X(x_i)} \approx \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(X_i)}{f_X(X_i)}$$



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$\rightarrow$ $f_X(x)$ similar to $g(x)$

Monte Carlo integration (example)

Monte Carlo integration (example)

moments

$$\mu := \mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x) f_X(x)\, dx$$

$$Var(X) = \mathbb{E}\left[X^2\right] - \mu^2$$

MC integration

$$\int_a^b g(x)\, dx \approx (b-a)\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) =: \vartheta_n$$

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MC integration importance sampling

$$\int_{\mathbb{S}} g(x)\, dx \approx \frac{1}{n}\sum_{i=0}^{n-1} \frac{g(X_i)}{f_X(X_i)}$$

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx$$

moments

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2$$

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \;\approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n'$$

moments

$$\mu := \quad \mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x) f_X(x)\,dx$$

$$Var(X) \;= \mathbb{E}\left[X^2\right] - \mu^2$$

MC integration

$$\int_a^b g(x)\,dx \;\approx (b-a)\,\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) \quad =: \vartheta_n$$

$$error_{\vartheta_n} \;\approx \frac{b-a}{\sqrt{n}}\sqrt{Var\left(g(X)\right)}$$

MC integration importance sampling

$$\int_{\mathbb{S}} g(x)\,dx \;\approx \frac{1}{n}\sum_{i=0}^{n-1} \frac{g(X_i)}{f_X(X_i)}$$

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \;\approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n' \qquad\qquad X_i \sim \mathcal{U}(0,1)$$

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MC integration importance sampling

$$\int_{\mathbb{S}} g(x)\,dx \;\approx \frac{1}{n}\sum_{i=0}^{n-1} \frac{g(X_i)}{f_X(X_i)}$$

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n' \qquad\qquad X_i \sim \mathcal{U}(0,1)$$

$$error_{\vartheta_n'} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(X^2\right)}$$

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \;\approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n' \qquad\qquad X_i \sim \mathcal{U}(0,1)$$

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moments

$$\mu := \;\mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x)\,f_X(x)\,dx$$

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MC integration importance sampling

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n' \qquad\qquad X_i \sim \mathscr{U}(0,1)$$

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$$\mathbb{E}[X^4]$$

moments

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MC integration importance sampling

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$$error_{\vartheta_n} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(\frac{g(X)}{f_X(X)}\right)}$$

Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta'_n \qquad X_i \sim \mathcal{U}(0,1)$$

$$error_{\vartheta'_n} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(X^2\right)} = \frac{1}{\sqrt{n}}\sqrt{\mathbb{E}[X^4] - \mathbb{E}[X^2]^2}$$

$$\mathbb{E}[X^4] = \int_0^1 x^4\,dx$$

moments

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MC integration importance sampling

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n' \qquad X_i \sim \mathcal{U}(0,1)$$

$$error_{\vartheta_n'} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(X^2\right)} = \frac{1}{\sqrt{n}}\sqrt{\mathbb{E}[X^4] - \mathbb{E}[X^2]^2}$$

$$\mathbb{E}[X^4] = \int_0^1 x^4\,dx = \frac{1}{5}$$

moments

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MC integration importance sampling

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$$\mathbb{E}[X^4] = \int_0^1 x^4\,dx = \frac{1}{5} \qquad \mathbb{E}[X^2] = \int_0^1 x^2\,dx$$

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MC integration

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n' \qquad\qquad X_i \sim \mathcal{U}(0,1)$$

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$$\mathbb{E}[X^4] = \int_0^1 x^4\,dx = \frac{1}{5} \qquad\qquad \mathbb{E}[X^2] = \int_0^1 x^2\,dx = \frac{1}{3}$$

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MC integration importance sampling

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n' \qquad\qquad X_i \sim \mathcal{U}(0,1)$$

$$error_{\vartheta_n'} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(X^2\right)} \;= \frac{1}{\sqrt{n}}\sqrt{\mathbb{E}[X^4]-\mathbb{E}[X^2]^2} \;= \frac{1}{\sqrt{n}}\sqrt{\tfrac{1}{5}-\tfrac{1}{9}}$$

$$\mathbb{E}[X^4] = \int_0^1 x^4\,dx = \frac{1}{5} \qquad\qquad \mathbb{E}[X^2] = \int_0^1 x^2\,dx = \frac{1}{3}$$

moments

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$$Var(X) = \mathbb{E}\left[X^2\right] - \mu^2$$

MC integration

$$\int_a^b g(x)\,dx \approx (b-a)\,\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) \quad =: \vartheta_n$$

$$error_{\vartheta_n} \approx \frac{b-a}{\sqrt{n}}\sqrt{Var\left(g(X)\right)}$$

MC integration importance sampling

$$\int_{\mathbb{S}} g(x)\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(X_i)}{f_X(X_i)}$$

$$error_{\vartheta_n} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(\frac{g(X)}{f_X(X)}\right)}$$

Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta'_n \qquad\qquad X_i \sim \mathcal{U}(0,1)$$

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$$\mathbb{E}[X^4] = \int_0^1 x^4\,dx = \frac{1}{5} \qquad\qquad \mathbb{E}[X^2] = \int_0^1 x^2\,dx = \frac{1}{3}$$

moments

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Monte Carlo integration with importance sampling

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Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n' \qquad X_i \sim \mathcal{U}(0,1)$$

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$$\int_0^1 x^2\,dx$$

X_i is distributed with PDF $f_X(x) := 2x$

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Monte Carlo integration (example)

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Monte Carlo integration with importance sampling

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$\mathbb{E}[X^2]$

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Monte Carlo integration (example)

Monte Carlo integration

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Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \qquad = \vartheta_n' \qquad\qquad X_i \sim \mathcal{U}(0,1)$$

$$error_{\vartheta_n'} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(X^2\right)} = \frac{1}{\sqrt{n}}\sqrt{\mathbb{E}[X^4]-\mathbb{E}[X^2]^2} = \frac{1}{\sqrt{n}}\sqrt{\frac{1}{5}-\frac{1}{9}} \approx \frac{0.298}{\sqrt{n}}$$

$$\mathbb{E}[X^4] = \int_0^1 x^4\,dx = \frac{1}{5} \qquad\qquad \mathbb{E}[X^2] = \int_0^1 x^2\,dx = \frac{1}{3}$$

Monte Carlo integration with importance sampling

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1}\frac{X_i^2}{2X_i} = \vartheta_n'$$

X_i is distributed with PDF $f_X(x) := 2x$

$$error_{\vartheta_n'} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(\frac{X^2}{2X}\right)} = \frac{1}{2\sqrt{n}}\sqrt{Var\,(X)} = \frac{1}{2\sqrt{n}}\sqrt{\mathbb{E}[X^2]-\mathbb{E}[X]^2}$$

$$\mathbb{E}[X^2] = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 2x^3\,dx = \frac{1}{2} \qquad\qquad \mathbb{E}[X]$$

moments

$$\mu := \mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x) f_X(x)\,dx$$

$$Var(X) = \mathbb{E}\left[X^2\right] - \mu^2$$

MC integration

$$\int_a^b g(x)\,dx \approx (b-a)\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) =: \vartheta_n$$

$$error_{\vartheta_n} \approx \frac{b-a}{\sqrt{n}}\sqrt{Var\left(g(X)\right)}$$

MC integration importance sampling

$$\int_{\mathbb{S}} g(x)\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(X_i)}{f_X(X_i)}$$

$$error_{\vartheta_n} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(\frac{g(X)}{f_X(X)}\right)}$$

Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 \quad = \vartheta_n' \qquad X_i \sim \mathcal{U}(0,1)$$

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Monte Carlo integration with importance sampling

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Monte Carlo integration (example)

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Monte Carlo integration with importance sampling

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MC integration importance sampling

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Monte Carlo integration (example)

Monte Carlo integration

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Monte Carlo integration (example)

Monte Carlo integration

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$$\mathbb{E}[X^2] = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 2x^3\,dx = \frac{1}{2} \qquad \mathbb{E}[X] = \int_0^1 x f_X(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}$$

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end