stochastics and probability

Lecture 1

Dr. Johannes Pahlke

indroduction

stochastic model $\underline{x}, t$

stochastic simulation $\underline{x}_0, \underline{x}_1, \underline{x}_2, \cdots, \underline{x}_T$

$$\forall t = 0,\ldots,T: \quad \underline{x}_t \sim P(\underline{x}, t \,|\, \underline{x}_0, t_0)$$

problems

1) $P(\,.\,)$ not known in closed form

$\longrightarrow$ Master Equations

2) generating random numbers $\sim P(\,.\,)$ is not easy/ posible

$\longrightarrow$ sampling

3) time is continuous $\underline{x}(t)$

$\longrightarrow$ stochastic calculus

4) $P(\,.\,)$ changes as a function of history

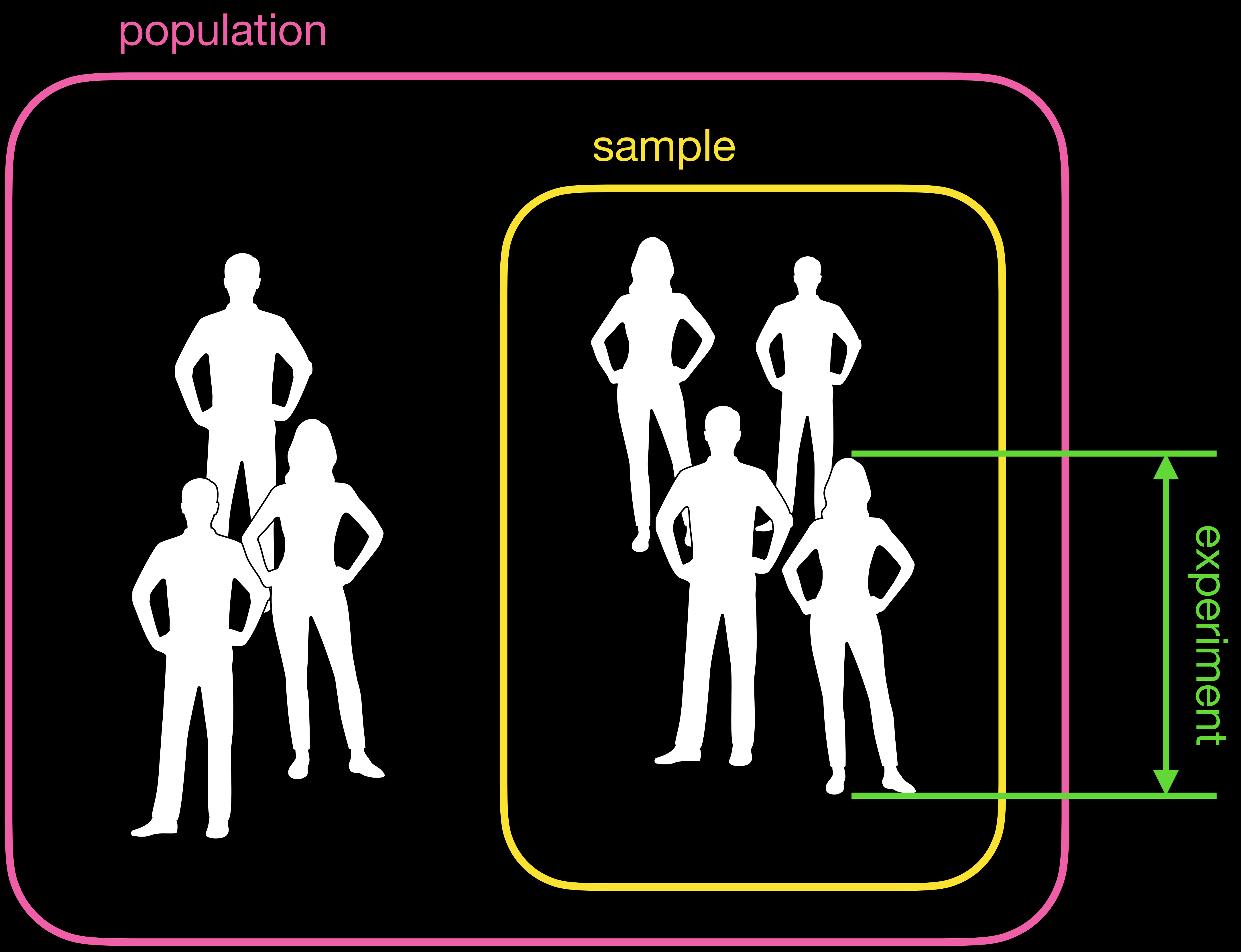
$\longrightarrow$ non-Markovian processes

elementary probability

definitions:

population:	a collection of objects
sample:	a subset of a population
experiment:	measuring something of a sample
event:	a concrete set of outcomes of an experiment
sample space:	space of all possible events Ω
probability:	likelihood of an event in a sample space $P(A)$ for the event $A \subseteq \Omega$

example elementary probability



event

140
190
175
130

Population: a collection of objects

Sample: a subset of a population

Experiment: measuring something of a sample

Event: a concrete set of outcomes of an experiment

Sample space: space of all possible events Ω

Probability: likelihood of an event in a sample space
$P(A)$ for the event $A \subseteq \Omega$

sample space

$$\Omega = [0,300]$$

probability

$$P(X \in [50,200]) = 0.98$$

Kolmogorov axioms

1) $P(A) \geq 0 \quad \forall A \subseteq \Omega$

2) $P(\Omega) = 1$

3) $P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

implications

$$P(\varnothing) = 0$$

$$P(\overline{A}) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

exercice

Axioms:

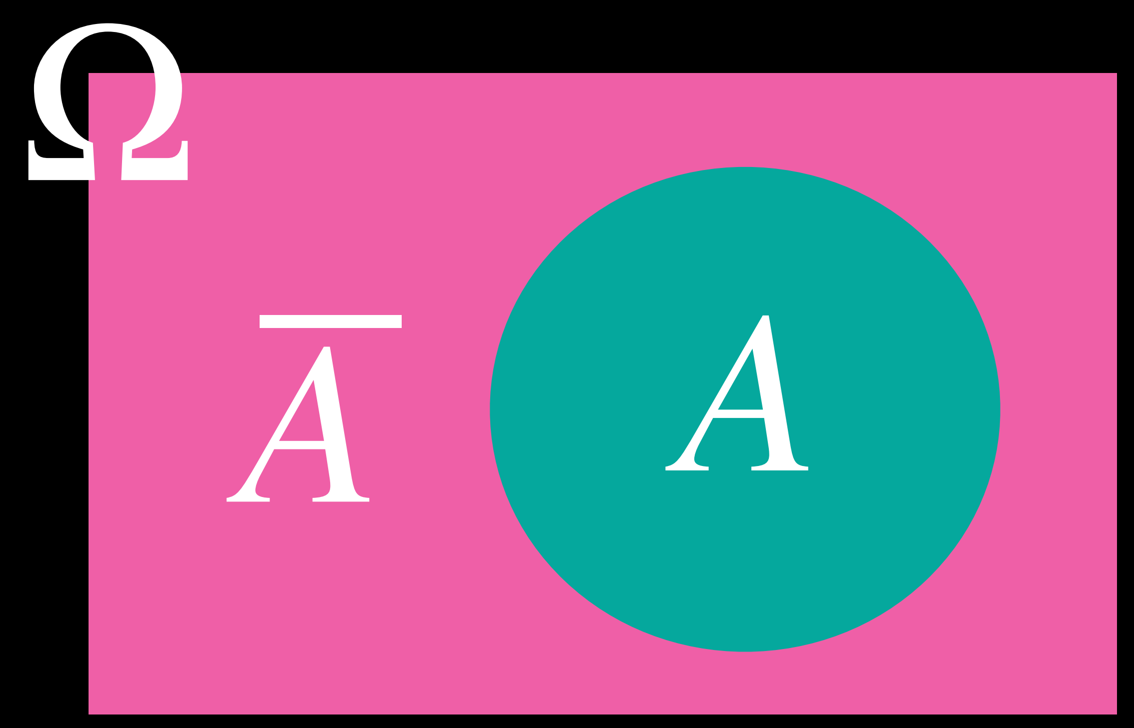
$\forall A : P(A) \geq 0$

$P(\Omega) = 1$

$P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

Prove: $P(\bar{A}) = 1 - P(A)$

$$\bar{A} = A^c = A' = \neg A := \Omega \backslash A = \Omega - A$$



$$\begin{aligned} 1 &= P(\Omega) \\ &= P(\Omega \backslash A \cup A) \\ &= P(\bar{A} \cup A) \\ &= P(\bar{A}) + P(A) \end{aligned}$$

$$\bar{A} \cap A = \varnothing$$

$$\longrightarrow \quad P(\bar{A}) = 1 - P(A) \qquad \square$$

conditional probabilities

definitions:

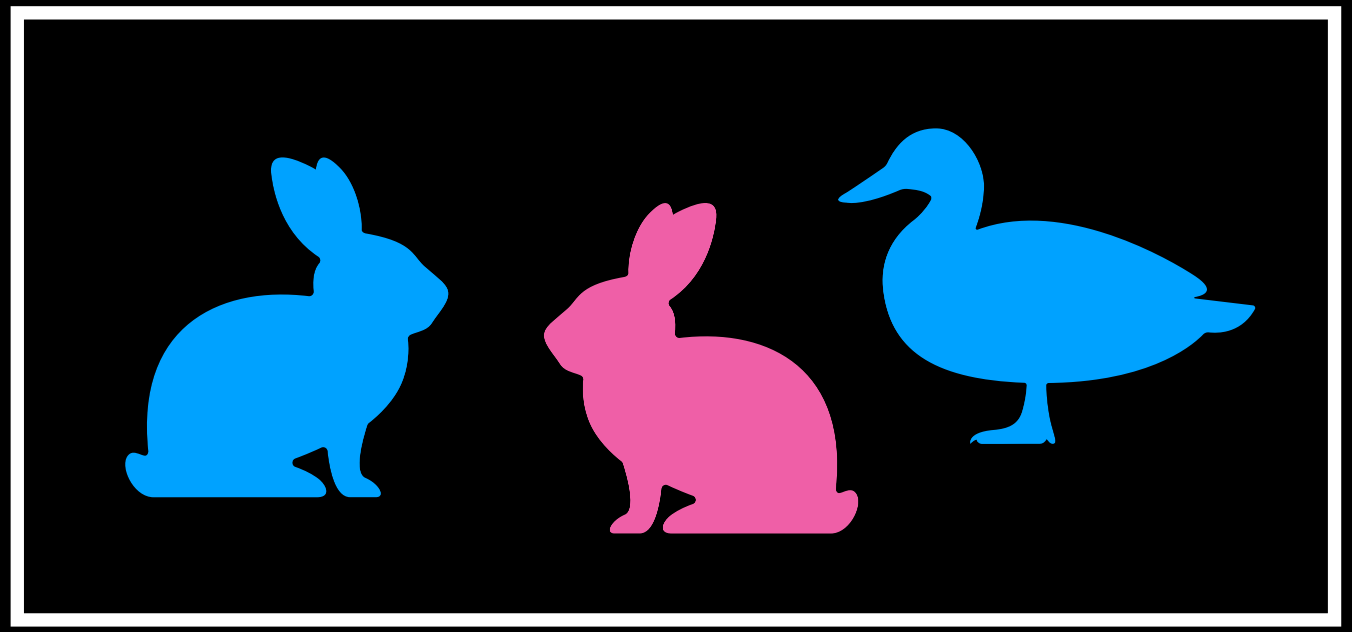
conditional probabilities $P(A \mid B) := \dfrac{P(A \cap B)}{P(B)}$ "A given B"

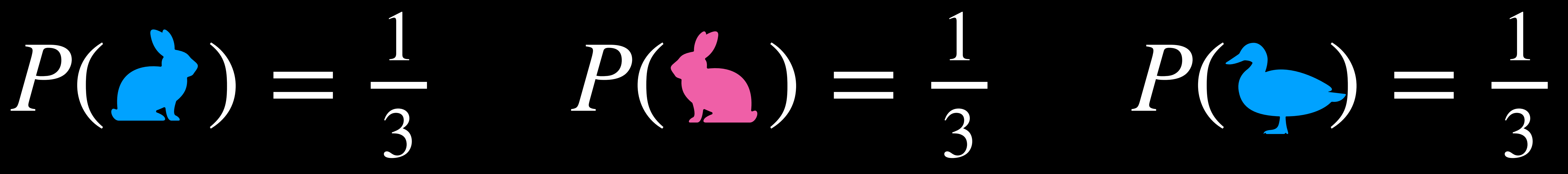
example conditional probabilities

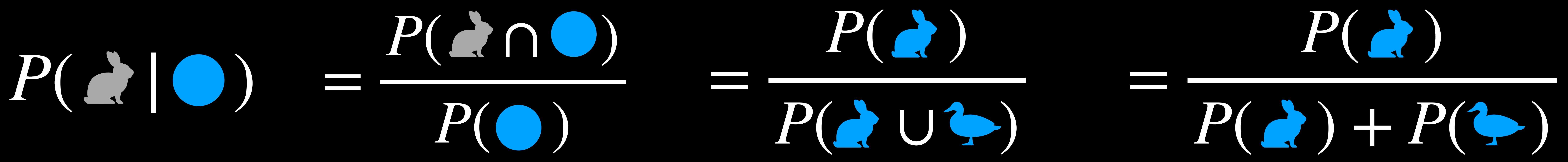
$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

$$P(A \cup B) = P(A) + P(B)$$
if $\;P(A \cap B) = 0$

draw







$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} \quad = \frac{1}{2}$$

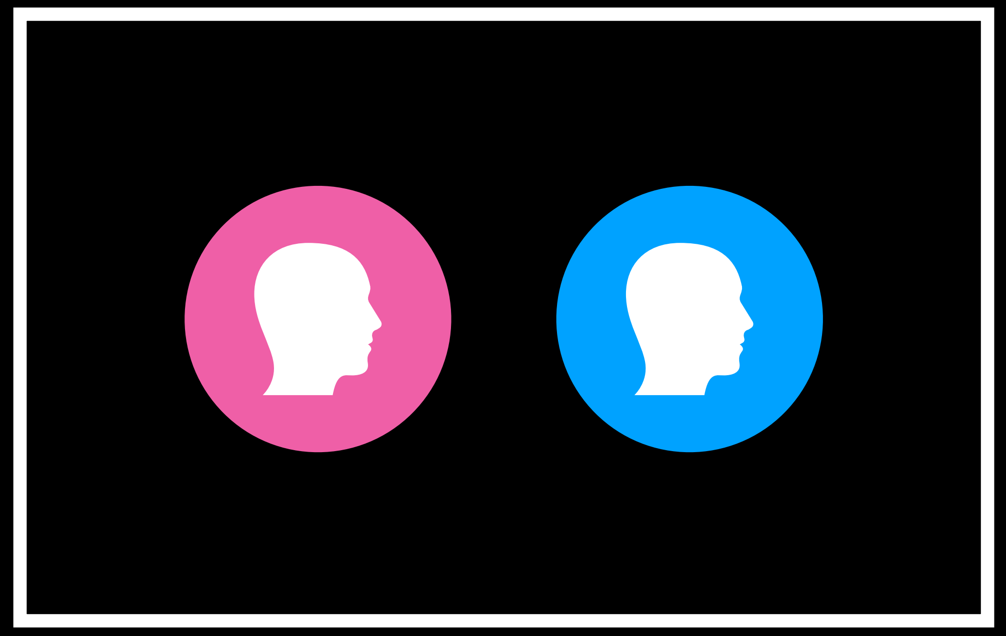
conditional probabilities

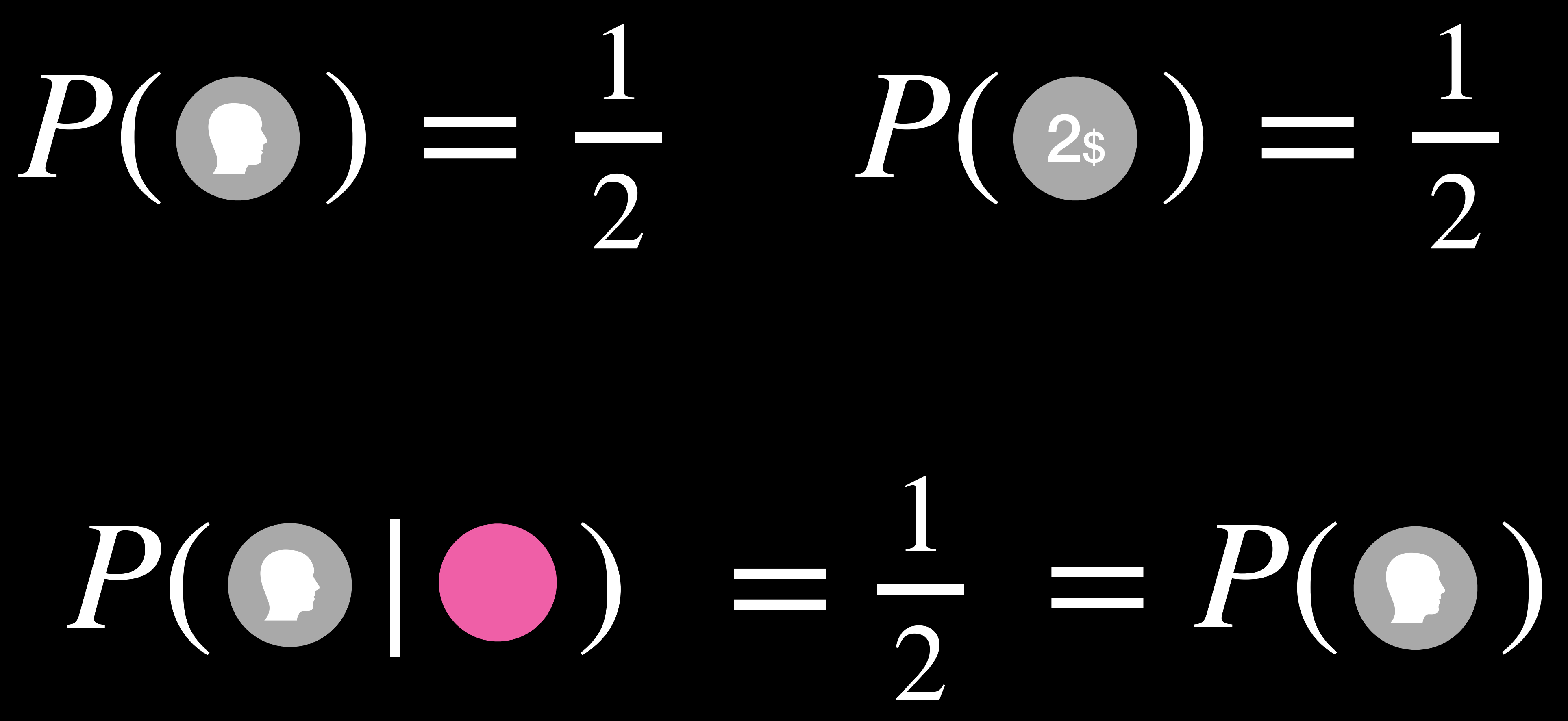
definitions:

An event A is independent of event B iff $P(A \mid B) = P(A)$.

$$\longrightarrow \quad P(A \cap B) = P(A) \cdot P(B)$$

draw and flip





exercice

Prove: $P(A \cap B) = P(A) \cdot P(B)$

for A, B independent

conditional probabilities:

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

A independent of B:

$$P(A \mid B) = P(A)$$

$$\begin{aligned} P(A \cap B) &= P(A \mid B) \cdot P(B) \\ &= P(A) \cdot P(B) \end{aligned}$$

$\square$

Bayes' theorem

posterior likelihood prior

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

marginal

exercice

conditional probabilities:

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

prove Bayes' theorem:

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

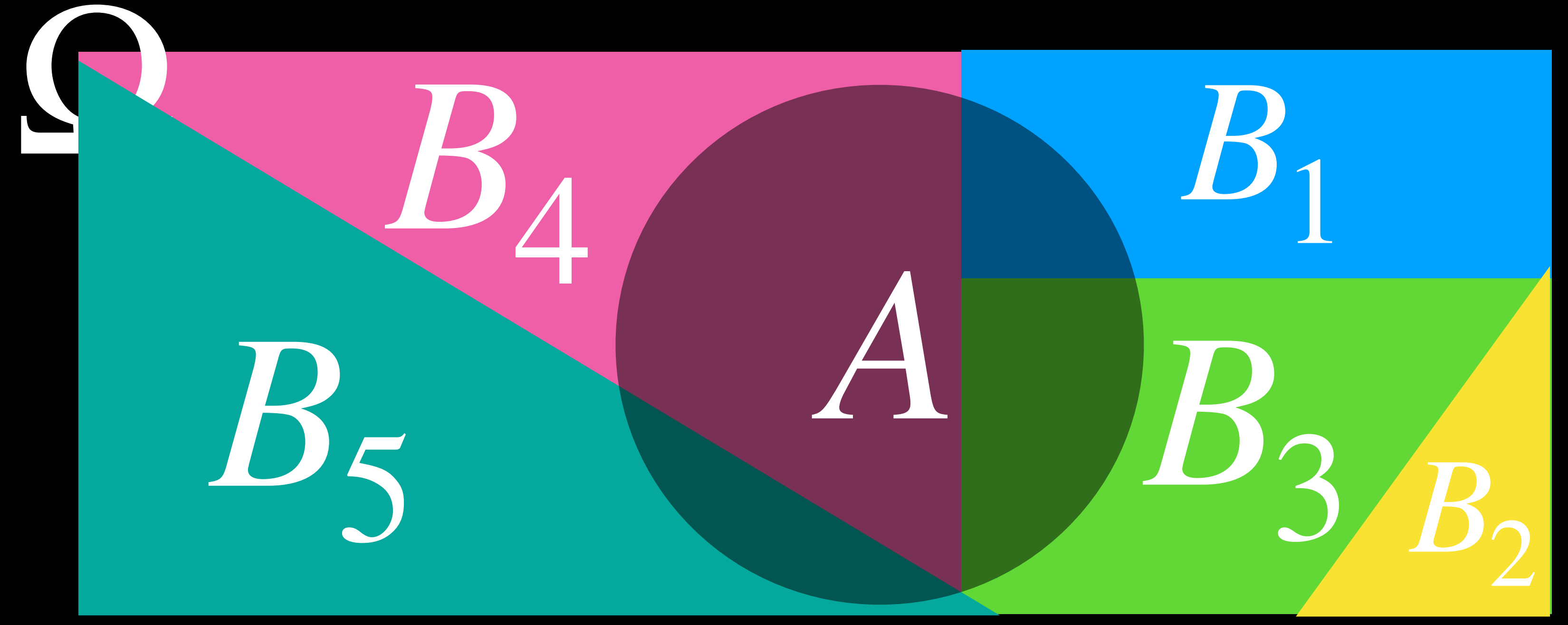
$$P(B \cap A) = P(B \mid A) \cdot P(A)$$

$$P(A \cap B) = P(B \cap A)$$

$$P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A) \quad \longrightarrow \quad P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

$\square$

Law of total probability



$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i) \qquad \bigcup_{i=1}^{n} B_i = \Omega \qquad B_i \cap B_j = \varnothing$$

prove

$$P(A \cup B) = P(A) + P(B) \quad \text{if} \quad P(A \cap B) = 0$$

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

$$P(A) = P(A \cap \Omega) = P\left(A \cap \bigcup_{i=1}^{n} B_i\right) = P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$$

probability expansion

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$
$$= P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot \ldots \cdot P(A_n | A_1 \cap \ldots \cap A_{n-1})$$

prove idea

$$\boxed{P(A \cap B) = P(A | B) \cdot P(B)}$$

$$B_{n-1} := A_1 \cap \ldots \cap A_{n-1}$$

$$\begin{aligned} P(A_1 \cap \ldots \cap A_n) &= P(A_n \cap B_{n-1}) = P(A_n | B_{n-1}) \cdot P(B_{n-1}) \\ &= P(A_1 \cap \ldots \cap A_{n-1}) \cdot P(A_n | A_1 \cap \ldots \cap A_{n-1}) \\ &= \ldots \end{aligned}$$

probability expansion example

draw 4 cards from 52 cards deck

What is the probability that all 4 aces are drawn?

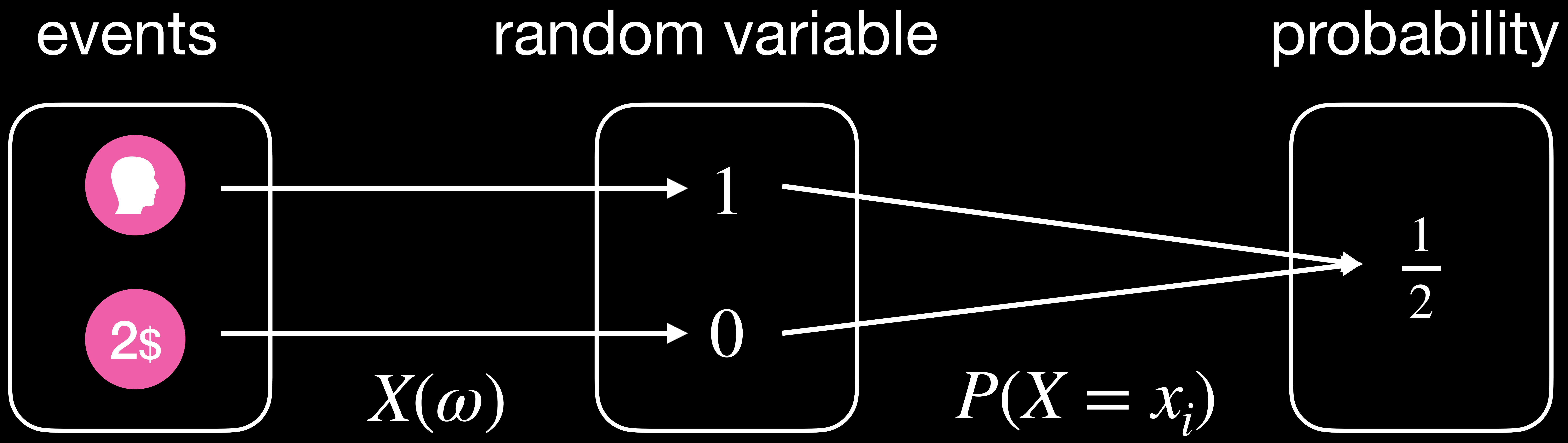
A_i : draw an aces in i-th draw

$$P(A_1) = \frac{4}{52}, \quad P(A_2 \mid A_1) = \frac{3}{51}, \quad P(A_3 \mid A_1 \cap A_2) = \frac{2}{50}, \quad P(A_4 \mid A_1 \cap A_2 \cap A_3) = \frac{1}{49}$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49}$$

random variable

A random variable is a number associated to an event.



Random variable

A random variable is a number associated to an event.

binary random variable/ indicator variable/ Bernoulli random variables

$$\mathbb{B} = \{0,1\}$$

$$I : \Omega \to \mathbb{B}$$

$$I(A) = \begin{cases} 1 & \text{if } A = \tilde{A} \\ 0 & \text{else} \end{cases}$$

$$\tilde{A} \subseteq \Omega$$

Random variable

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$\mathbb{B} = \{0,1\}$ $\quad I : \Omega \to \mathbb{B}$ $\quad I(A) = \begin{cases} 1 & \text{if } A = \tilde{A} \\ 0 & \text{else} \end{cases}$ $\quad \tilde{A} \subseteq \Omega$

discreate random variable

$S := \{x_1, x_2, x_3, \ldots\}$ finite or countable infinite

$X : \Omega \to S \qquad X(\omega) = x_\omega \in S$

$P(X = x_i) = ?$

Random variable

A random variable is a number associated to an event.

binary random variable/ indicator variable/ Bernoulli random variables

$\mathbb{B} = \{0,1\}$ $\quad I : \Omega \to \mathbb{B}$ $\quad I(\omega) = \begin{cases} 1 & \text{if } \omega = \tilde{\omega} \\ 0 & \text{else} \end{cases}$ $\quad \tilde{\omega} \in \Omega$

discreate random variable

$S := \{x_1, x_2, x_3, \ldots\}$ $\quad X : \Omega \to S$ $\quad X(\omega) = x_\omega \in S$

continuous random variables

$$J \subseteq \mathbb{R}$$

$$X : \Omega \to J$$

$$P(X = x_i) = 0$$

$$P(X > x_i) = ?$$

end