stochastics and probability

Lecture 3

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accept-reject sampling

want:

X with PDF $f_X(x)$

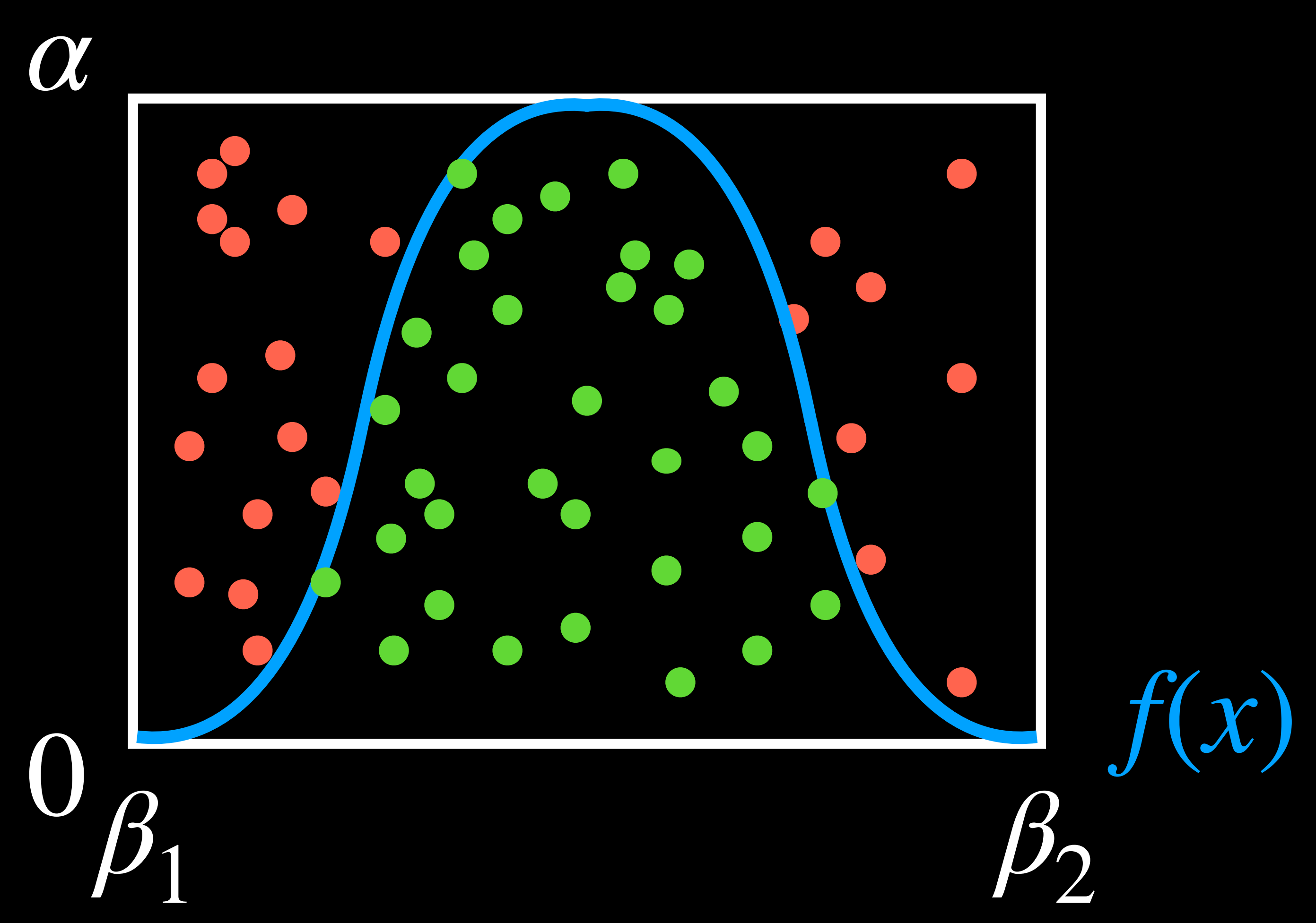
use:

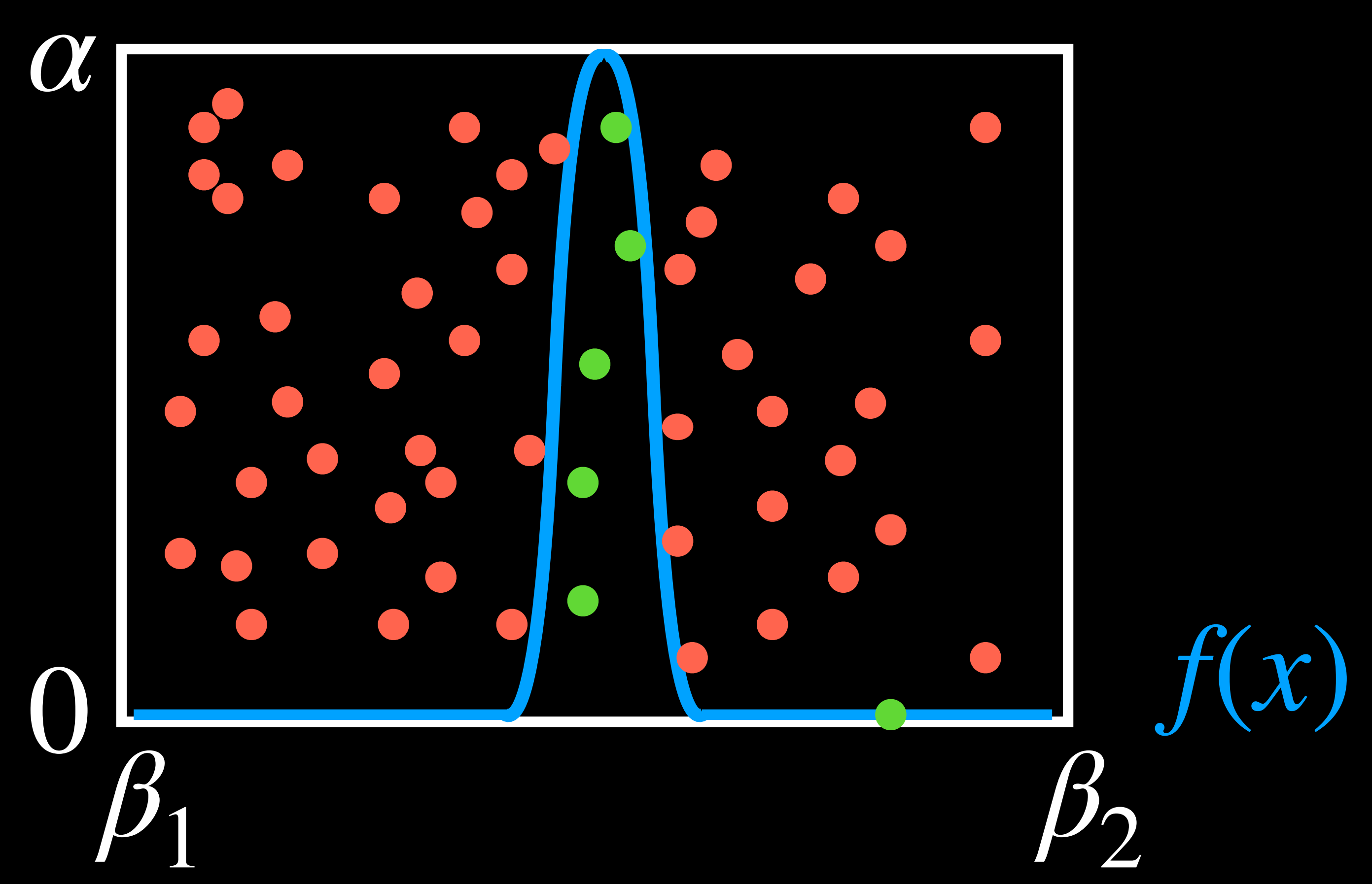
$f_X(x) \leq \alpha \quad Range(X) \subseteq [\beta_1, \beta_2]$

algorithm:

1.) draw $U_1 \sim \mathcal{U}(\beta_1, \beta_2), \quad U_2 \sim \mathcal{U}(0,\alpha)$

2.) if $U_2 \leq f_X(U_1)$, then $X = U_1$, else go to 1.)

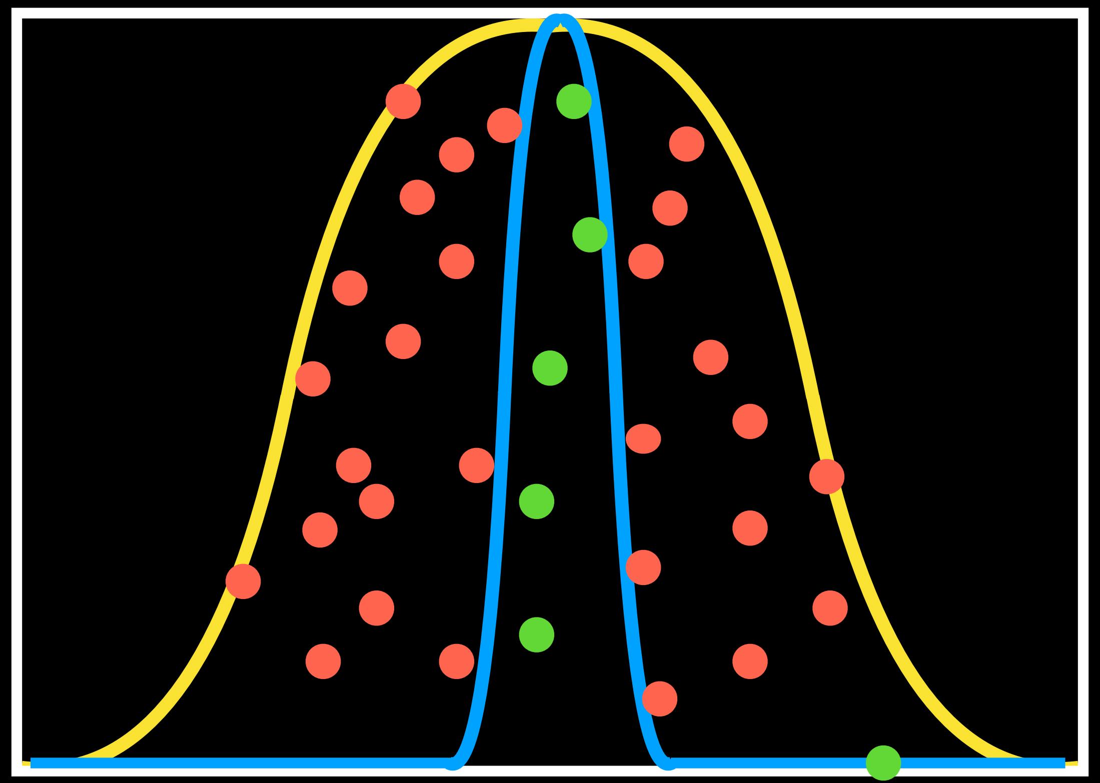




inefficient:

- peaky densities
- high dimensions
- long tails/infinite Range

accept-reject sampling



want:

X with PDF $f_X(x)$

use:

$G \sim \mathscr{G}$ with PDF $g_G(x)$ and $f_X(x) \leq c\,g_G(x)$ for all x

algorithm:

1.) draw G from $\mathscr{G}$ and U from $\mathscr{U}(0,1)$

2.) if $U \leq \dfrac{f_X(G)}{c\,g_G(G)}$, then $X = G$, else go to 1.)

discrete-time stochastic processes

$n \in \mathbb{N}$ discrete Time

$X_0, X_1, X_2, \ldots$ "sequence of random variables"

$\neq x_0, x_1, x_2, \ldots$ "sequence of random numbers"

$\{X_n : n \in \mathbb{N}\}$ discrete-time stochastic process

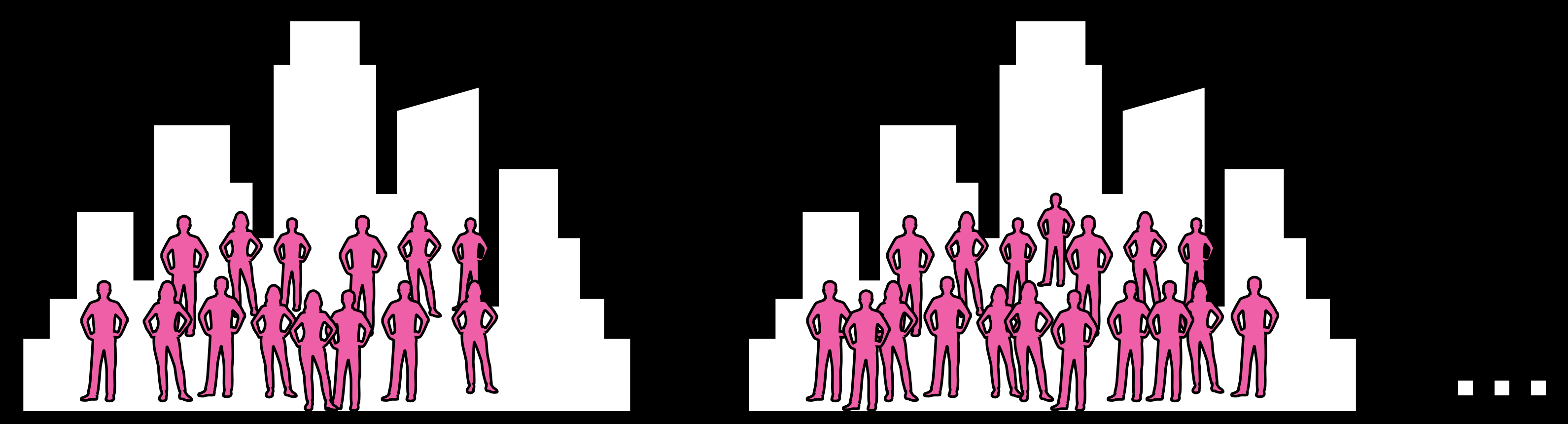
x_n value of random variable X_n

$X_n : \Omega \to \mathbb{S}$ state space $\mathbb{S}$

$x_n \in \mathbb{S}$ (discrete or continuous)

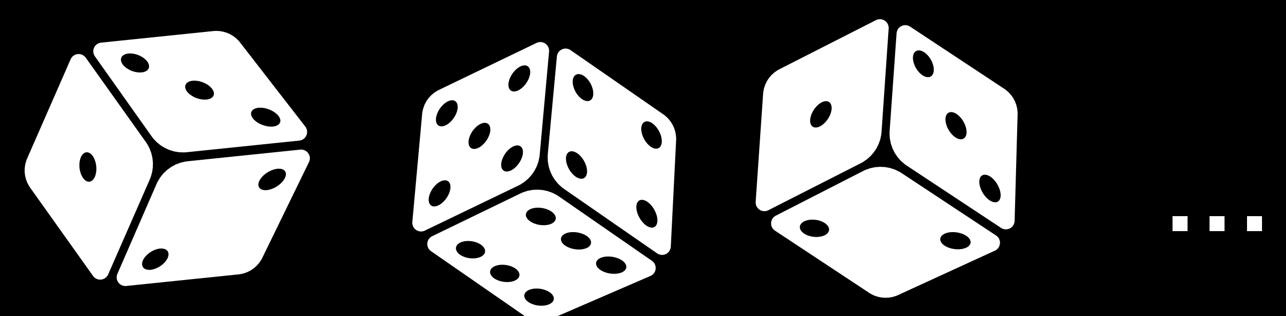
Examples:

population



2022 2023

rolling dice



1st 2nd 3rd

independent identical distributed (i.i.d.) process

$\{X_n : n \in \mathbb{N}\}$ stochastic process

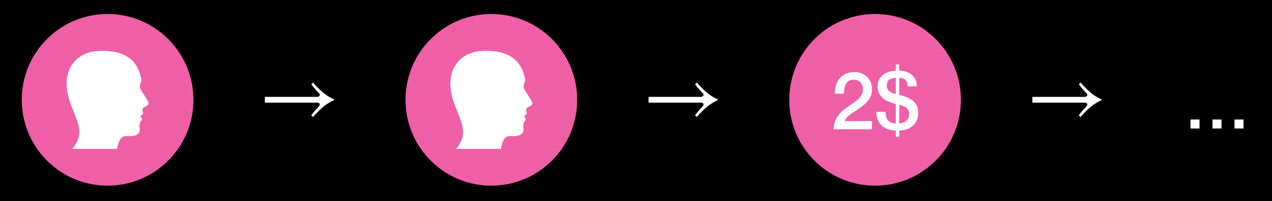
$F_{X_0}(x) = F_{X_1}(x) = F_{X_2}(x) = \dots$ identical distributed

$F_{X_0,\dots,X_n}(x_0, \dots, x_n) = F_{X_0}(x_0) \cdot \dots \cdot F_{X_n}(x_n)$ independent

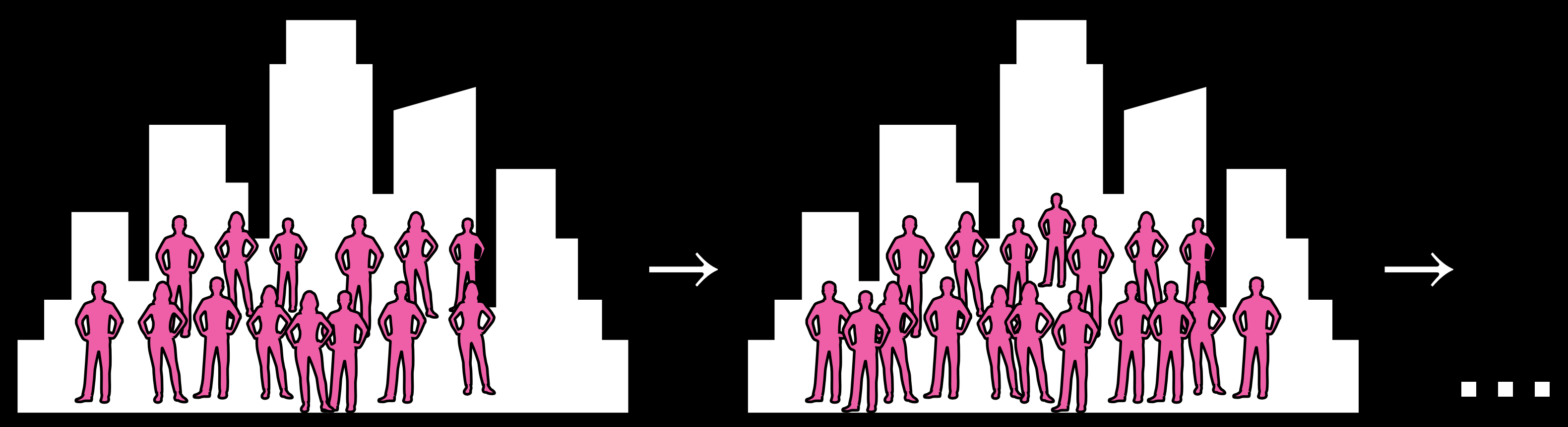
exercise:

Which system can be captured by an i.i.d. process?

coin-flip



population development



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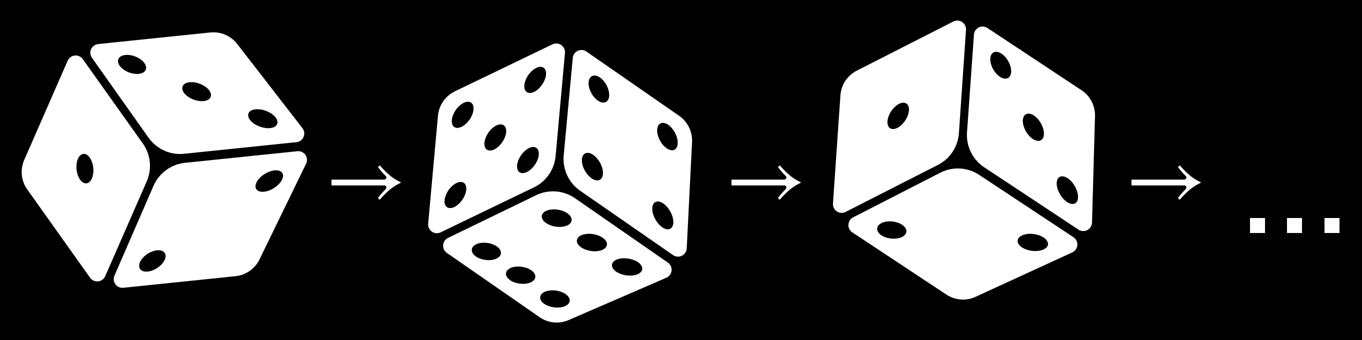


Markov chains

$$\{X_n : n \in \mathbb{N}\}$$

stochastic process

$$P(X_{n+1} = x_j \,|\, X_n = x_i,\, X_{n-1} = x_{i-1}, \ldots, X_0 = x_0)$$
$$= P(X_{n+1} = x_j \,|\, X_n = x_i) \;=: P_{ij}$$

(one-step transition probability)

Markov property

(X_{n+1} only depends on X_n)

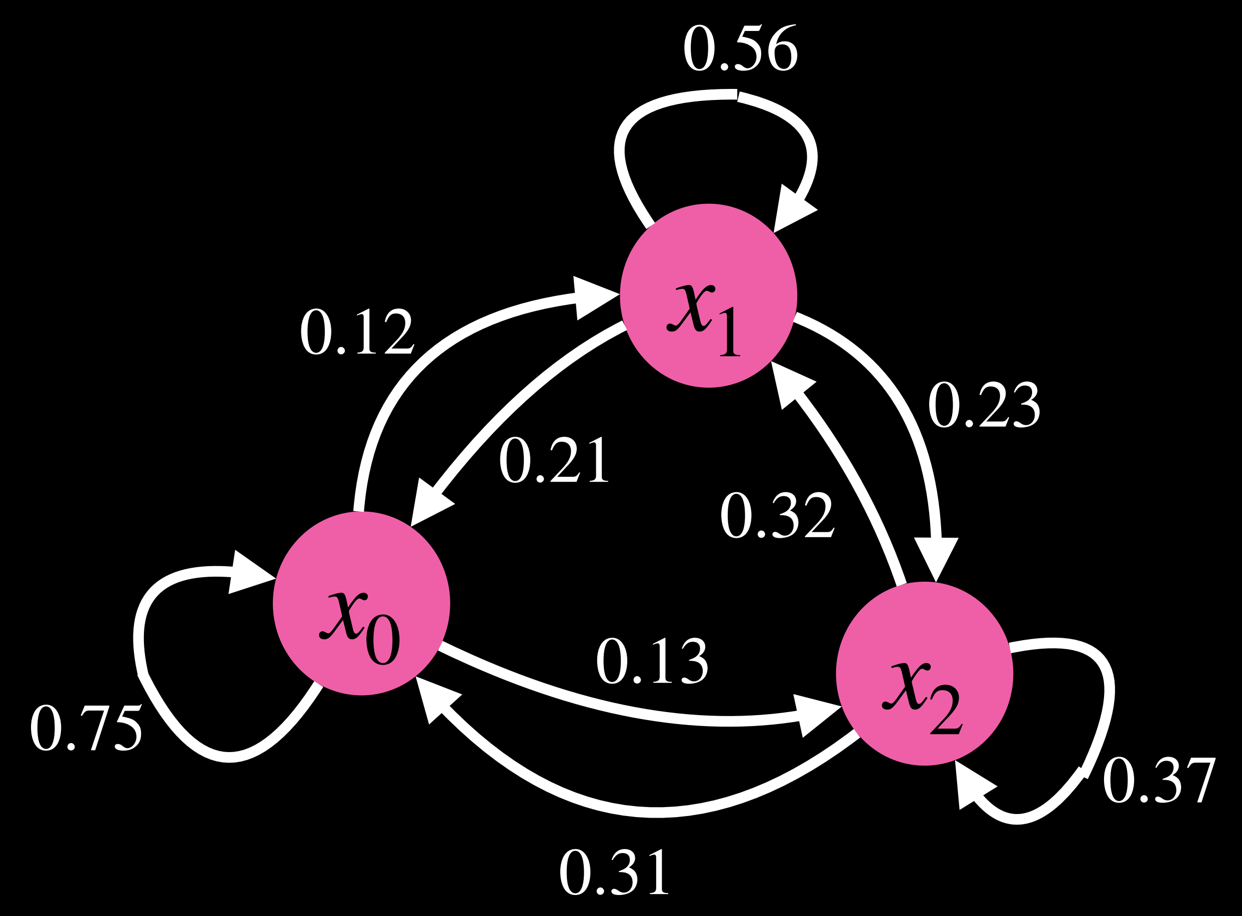
$$P_{ij}^{(n)} := P\left(X_{n+m} = x_j \,|\, X_m = x_i\right)$$

(n-step transition probability)

$$\underline{\underline{P}} := (P_{ij}) \;= \begin{pmatrix} P_{00} & \cdots & P_{0n} \\ \vdots & \ddots & \vdots \\ P_{n0} & \cdots & P_{nn} \end{pmatrix} \quad \text{for } |\mathbb{S}| < \infty$$

(one-step transition matrix)

example:



$$\underline{\underline{P}} = \begin{pmatrix} 0.75 & 0.12 & 0.13 \\ 0.21 & 0.56 & 0.23 \\ 0.31 & 0.32 & 0.37 \end{pmatrix}$$

Markov chains as recursions

theorem 1:

Let $g(x, u) \in \mathbb{R}$ be a function and $\{U_n : n \in \mathbb{N}\}$ an i.i.d. process,

then the recursion $X_{n+1} = g(X_n, U_n)$ is a Markov chain.

theorem 2:

Every Markov chain can be represented as recursion $X_{n+1} = g(X_n, U_n)$

where $\{U_n : n \in \mathbb{N}\}$ is an i.i.d. process with $U_n \sim \mathcal{U}(0,1)$.

theorem 3:

Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ and its one-step transition matrix $\underline{\underline{P}}$,

then its n-step transition matrix $\underline{\underline{P}}^{(n)} = \underline{\underline{P}}^n = \underline{\underline{P}} \cdot \ldots \cdot \underline{\underline{P}}$.

properties of Markov chains

def.: Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \to \mathbb{S}$.

Let $\mathbb{C} \subseteq \mathbb{S}$, then $\mathbb{C}$ is closed iff $\forall x_i \in \mathbb{C}\ \forall x_j \in \mathbb{S} \backslash \mathbb{C} : P_{ij} = 0.$

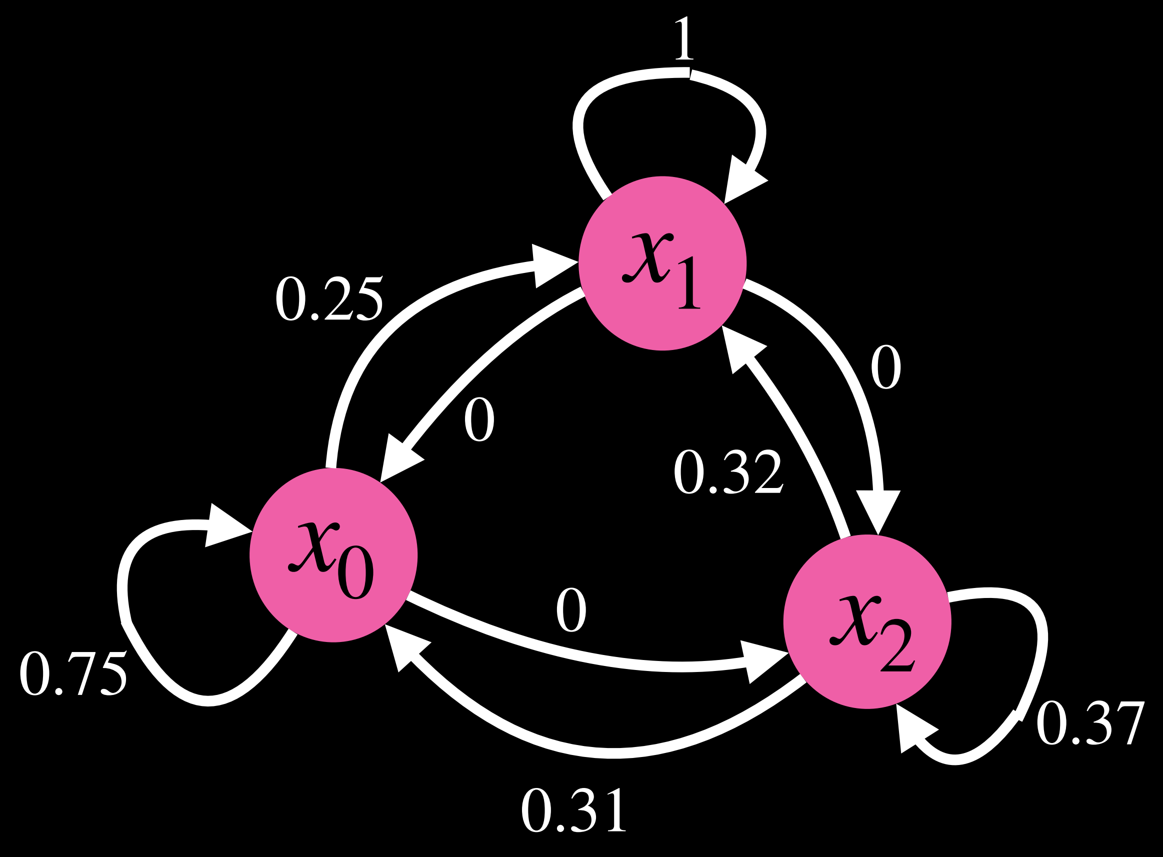
(form a state of $\mathbb{C}$ no state outside of $\mathbb{C}$ can be reached)

If $\{x_i\} = \mathbb{C}$, then x_i is absorbing state.

$\longrightarrow\ P_{ii} = 1$

exercise:

consier:



What are the closed sets? $\mathbb{C}' = \{x_0, x_1\},\ \mathbb{C}'' = \{x_1\},\ \mathbb{C}''' = \mathbb{S}$

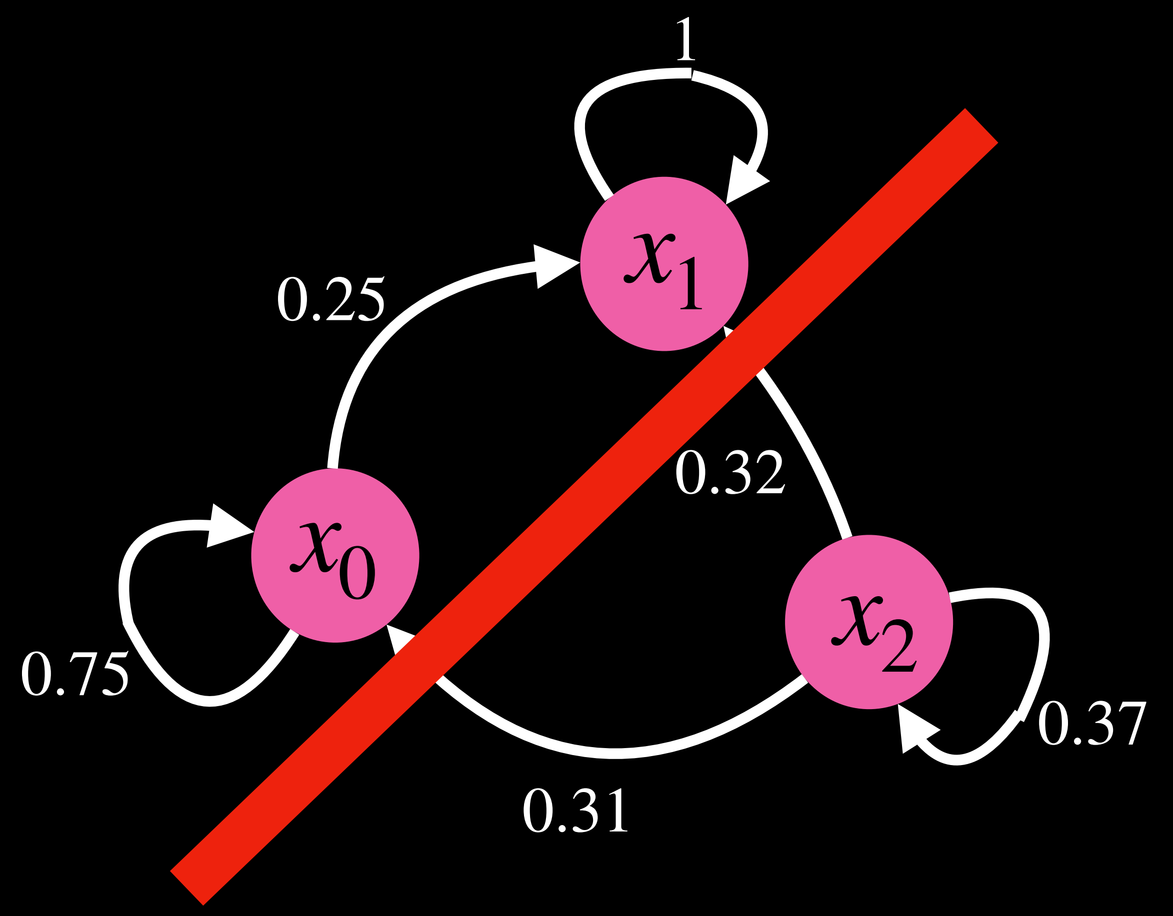
What are the absorbing states? x_1

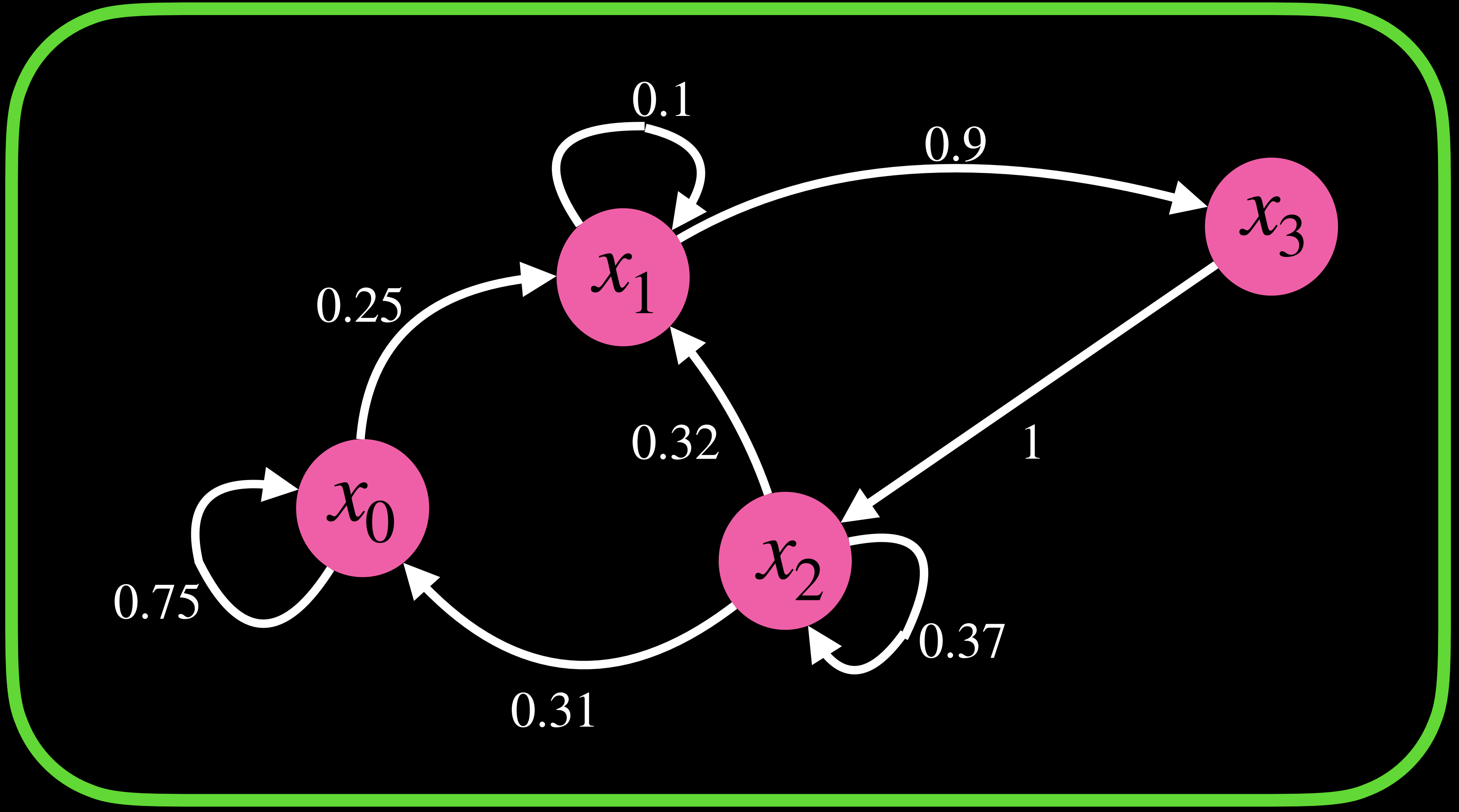
properties of Markov chains

def.: A Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \to \mathbb{S}$

is irreducible iff $\nexists \mathbb{C} \subset \mathbb{S} : \mathbb{C}$ is closed ($\mathbb{S}$ is the only closed set)

exercise: Which Markov chain is irreducible?





properties of Markov chains

def.: Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \to \mathbb{S}$.

Be $N_i := \{n \in \mathbb{N}_{>0} : P_{ii}^{(n)} > 0\}$ set of number of transitions to reach x_i from x_i,

then $d_i := gcd(N_i)$ period of x_i.

(greatest common divisor)

If $1 < d_i < \infty$, then x_i is periodic,

if $d_i = 1$, then x_i is aperiodic.

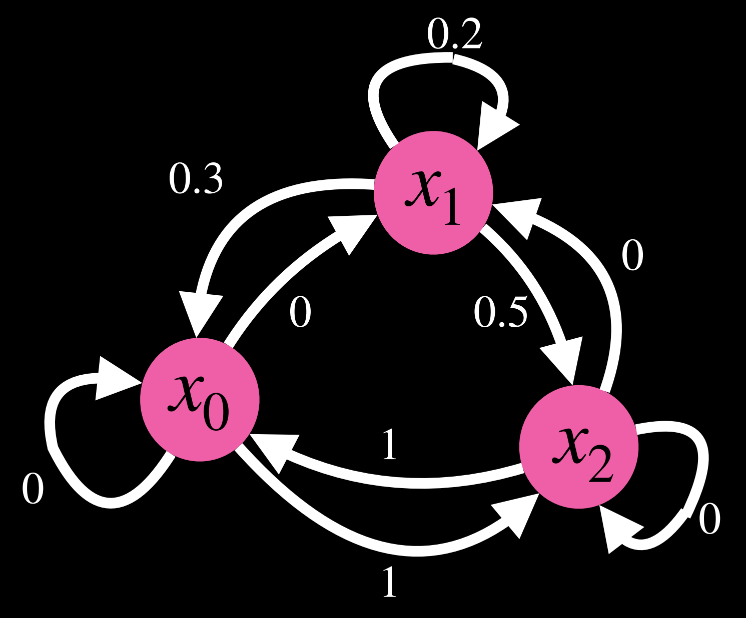
If $\forall x_i, x_j \in \mathbb{S} : 1 < d_i = d_j < \infty$,
then $\{X_n : n \in \mathbb{N}\}$ is periodic,

(The Markov chain is periodic if all states are periodic with the same period.)

if $\forall x_i \in \mathbb{S} : 1 = d_i$, then $\{X_n : n \in \mathbb{N}\}$ is aperiodic.
(The Markov chain is aperiodic if all states are aperiodic.)

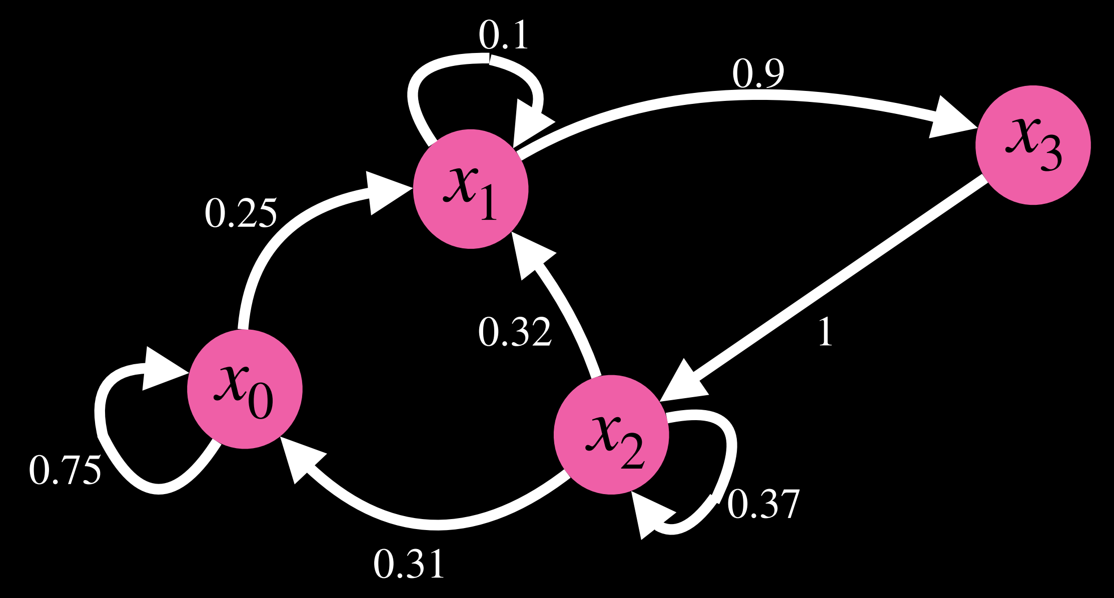
exercise:

Which states/ Markov chains are periodic?

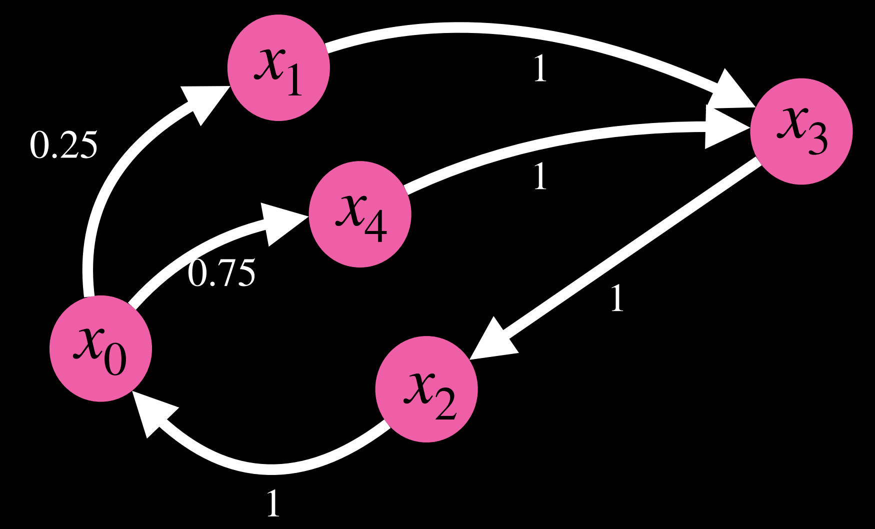


x_0, x_2 are periodic
$d_i = 2$

x_1 is aperiodic



all x_i are aperiodic



all x_i are periodic
$d_i = 4$

properties of Markov chains

def.: Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \to \mathbb{S}$.

If $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$ then x_i is transient,

(A state is transient iff there is a non-zero probability that it is never reached again and recurrent otherwise.)

if $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$ then x_i is recurrent.

If x_i is recurrent and aperiodic, then x_i is ergodic,

if $\forall x_i \in \mathbb{S} : x_i$ is ergodic, then $\{X_n : n \in \mathbb{N}\}$ is ergodic.

(The Markov chain is ergodic if all states are ergodic.)

properties of Markov chains

def.: Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \to \mathbb{S}$.

stationary distribution: $$P_k := \lim_{n\to\infty} P_{jk}^{(n)} \qquad \forall x_j \in \mathbb{S}$$

$$\sum_{x_i \in \mathbb{S}} P_i = 1$$

theorem 4:

A Markov chain has a stationary distribution iff it is ergodic.

example:

Every i.i.d. process is a Markov chain with $g(x, U) := U \qquad U \sim \mathcal{U}(0,1)$

and the stationary distribution is $\mathcal{U}(0,1)$.

Markov chains for random number gernation

praxis: $\cancel{n \to \infty}$ $\boxed{n >> 1}$ $\to$ "mixing time"

(In praxis, statistical tests decide if the stationary distribution is reached.)

Markov chain

$\{X_n : n \in \mathbb{N}\}$ $\quad X_n : \Omega \to \mathbb{S}$

stationary distribution

$P_k := \lim_{n\to\infty} P_{jk}^{(n)} \quad \forall x_j \in \mathbb{S}$

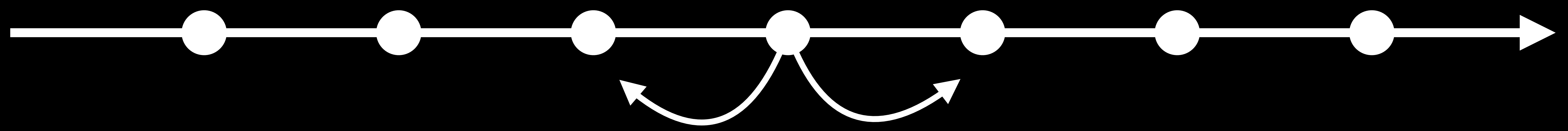
example:

Random Walk

Increments $\{D_n : n \in \mathbb{N}\}$ is i.i.d. process

$X_n := \sum_{i=1}^{n} D_i, \quad x_0 := 0 \quad \to \quad X_{n+1} = g(X_n, D_n) := X_n + D_n$ is a Markov chain.

1d discreat space



$P\left(D_i = 1\right) = P\left(D_i = -1\right) = \frac{1}{2}$ $\quad \to$ stionary distribution is Gausian end