stochastics and probability

Lecture 1

Dr. Johannes Pahlke

indroduction

indroduction

$$\underline{x}, t$$

indroduction

$$\underline{x}, t \,|\, \underline{x}_0, t_0$$

indroduction

$$P(\underline{x}, t \,|\, \underline{x}_0, t_0)$$

indroduction

stochastic model

$$P(\underline{x}, t \,|\, \underline{x}_0, t_0)$$

indroduction

stochastic model

$$P(\underline{x}, t \,|\, \underline{x}_0, t_0)$$

$$\underline{x}_0, \underline{x}_1, \underline{x}_2, \cdots, \underline{x}_T$$

indroduction

stochastic model $P(\underline{x}, t \,|\, \underline{x}_0, t_0)$

stochastic simulation $\underline{x}_0, \underline{x}_1, \underline{x}_2, \cdots, \underline{x}_T$

indroduction

stochastic model

$$P(\underline{x}, t \,|\, \underline{x}_0, t_0)$$

stochastic simulation

$$\underline{x}_0, \underline{x}_1, \underline{x}_2, \cdots, \underline{x}_T$$

$$\underline{x}_t \sim P(\underline{x}, t \,|\, \underline{x}_0, t_0)$$

indroduction

stochastic model $P(\underline{x}, t \,|\, \underline{x}_0, t_0)$

stochastic simulation $\underline{x}_0, \underline{x}_1, \underline{x}_2, \cdots, \underline{x}_T$

$$\forall t = 0,...,T: \quad \underline{x}_t \sim P(\underline{x}, t \,|\, \underline{x}_0, t_0)$$

problems

problems

1) $P(\,.\,)$ not known in closed form

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$\longrightarrow$ Master Equations

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4) $P(\,.\,)$ changes as a function of history

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3) time is continuous $\underline{x}(t)$

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4) $P(\,.\,)$ changes as a function of history

$\longrightarrow$ non-Markovian processes

elementary probability

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definitions:

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example elementary probability

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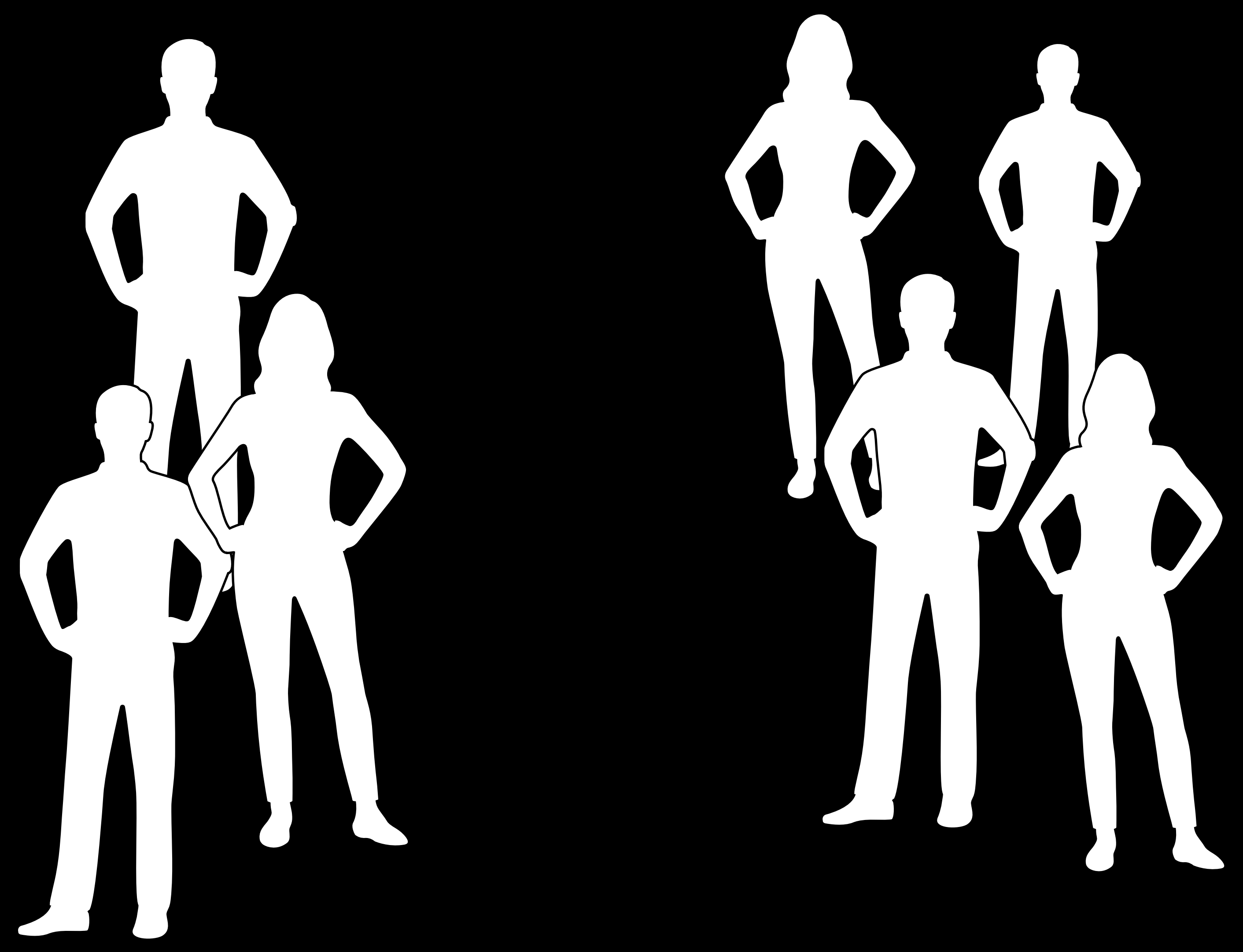

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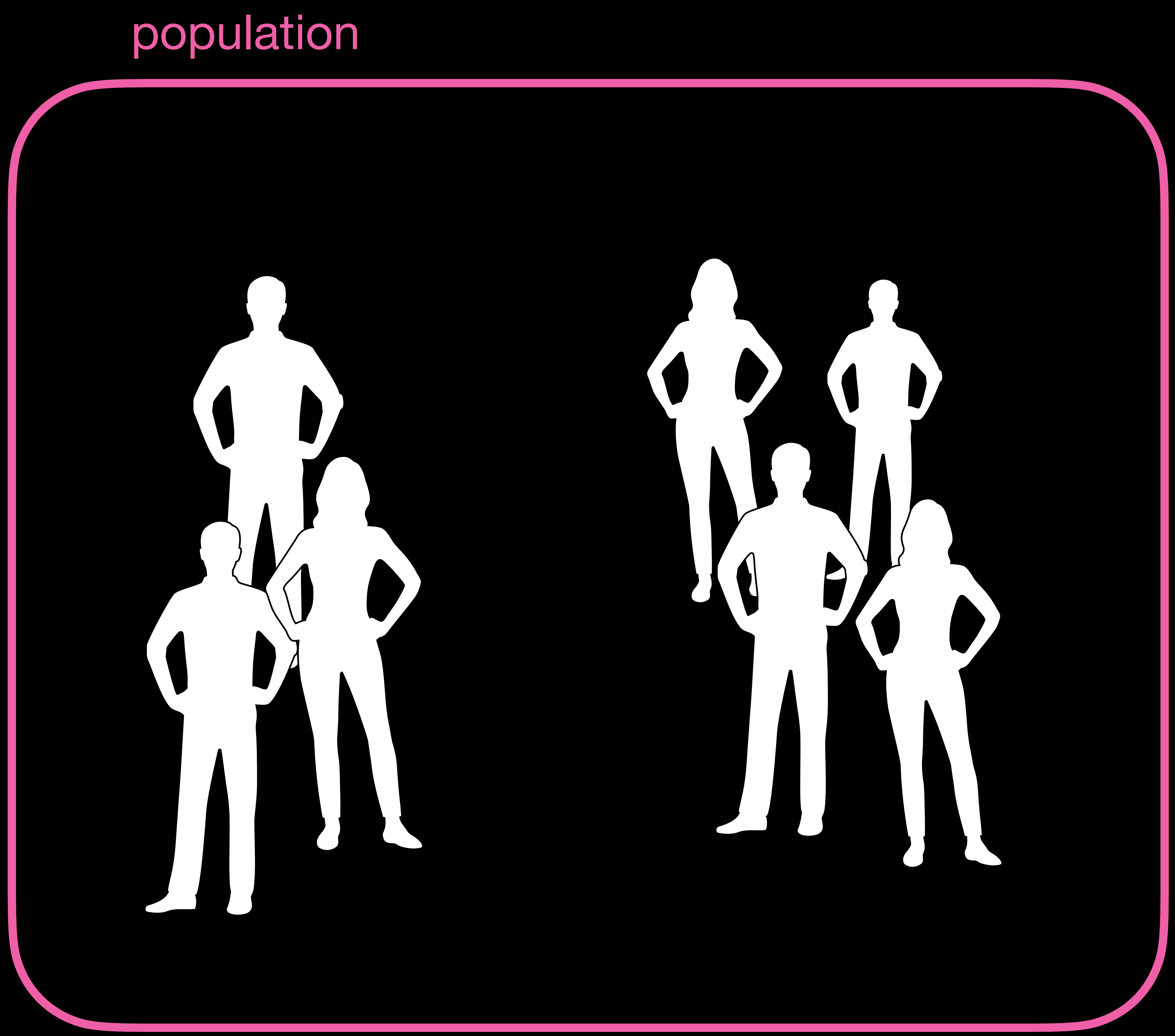

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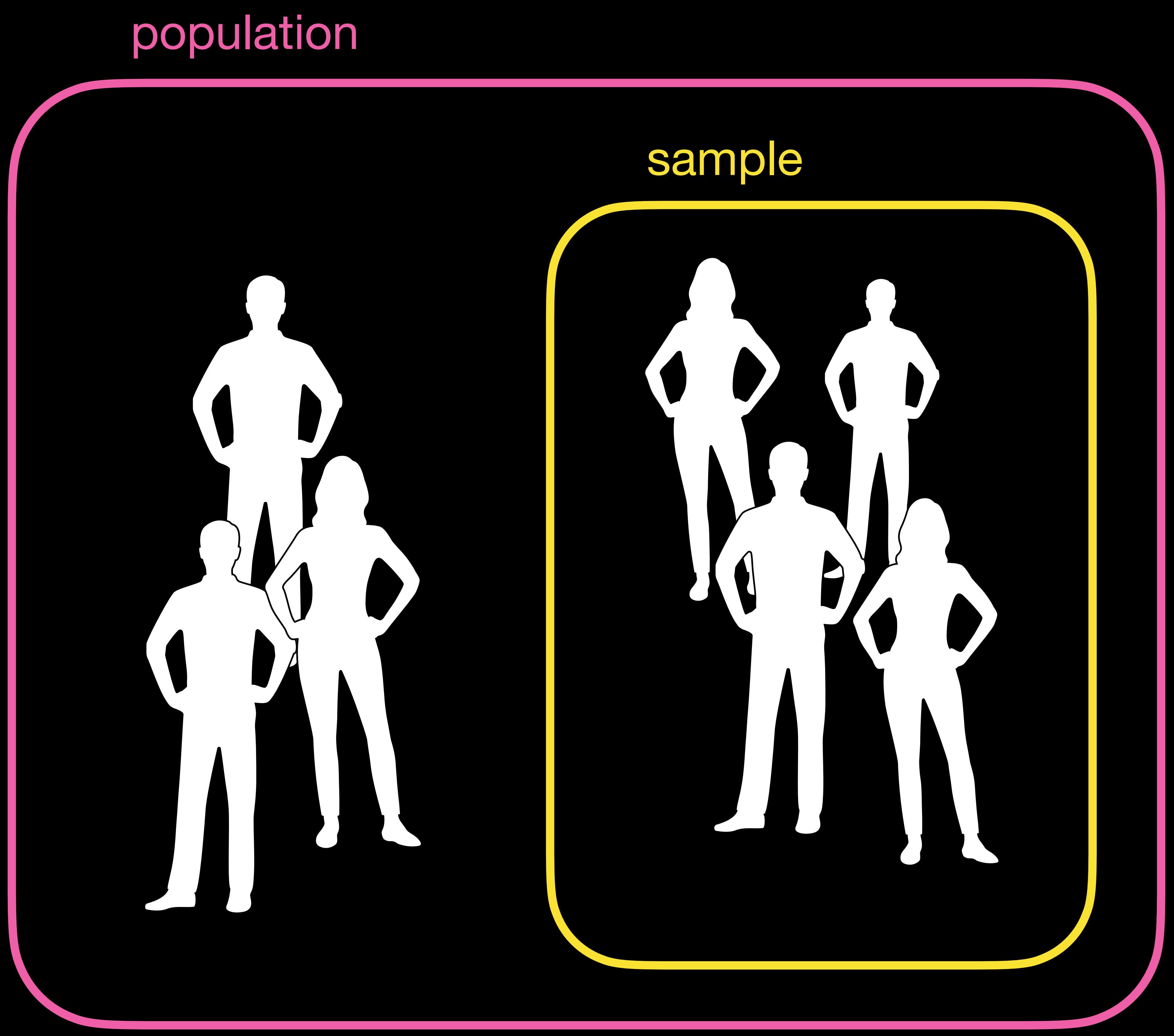

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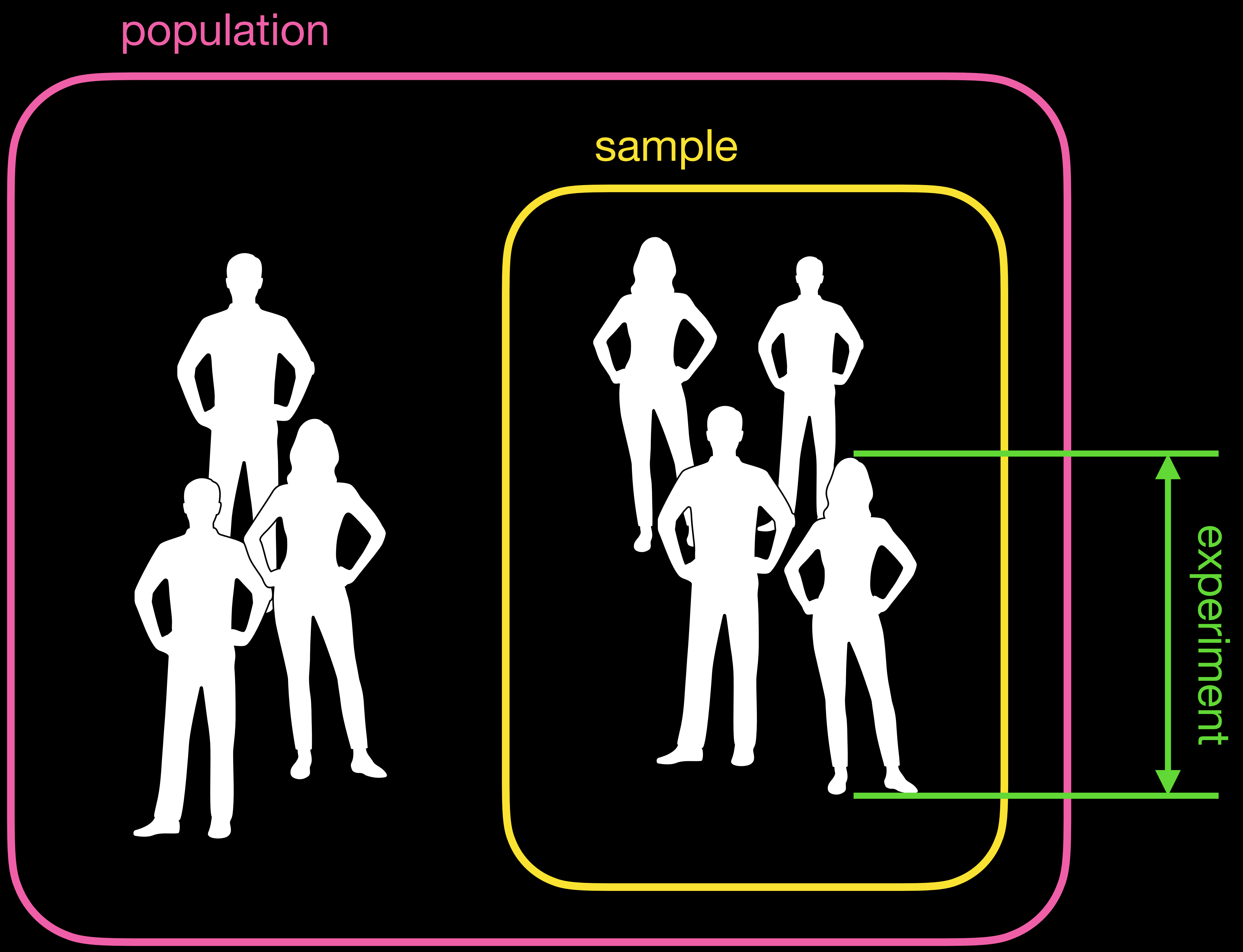

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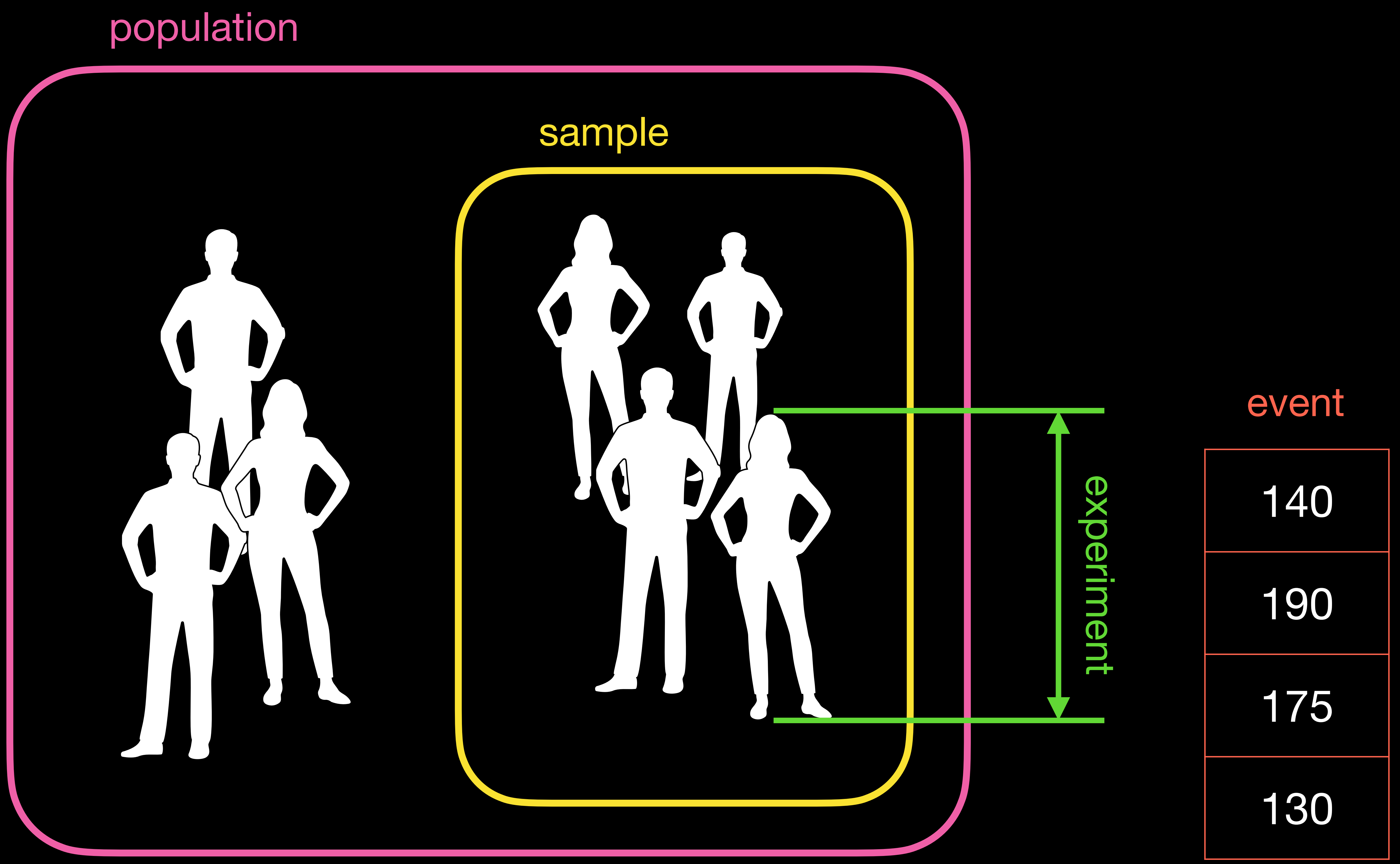

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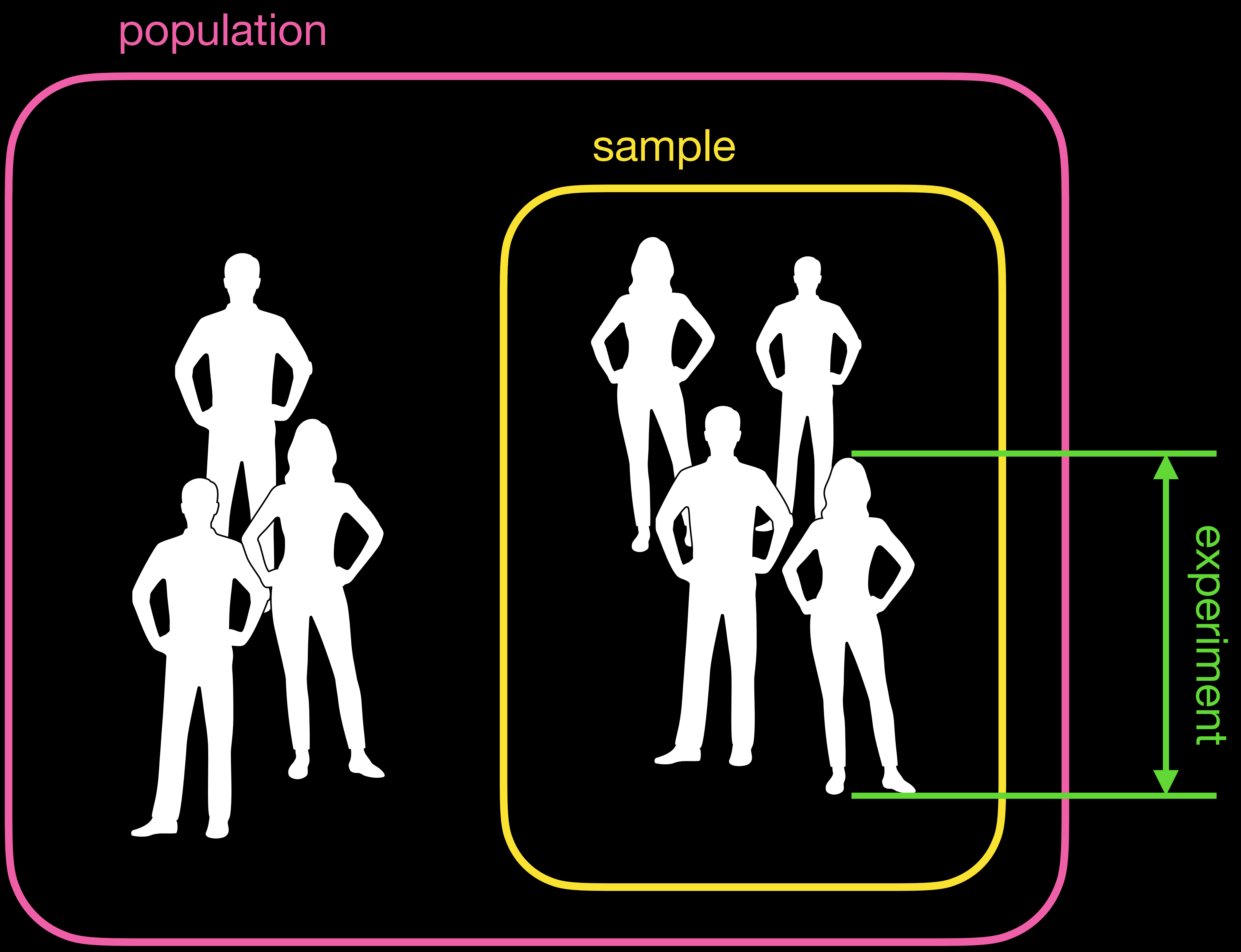

event
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190
175
130

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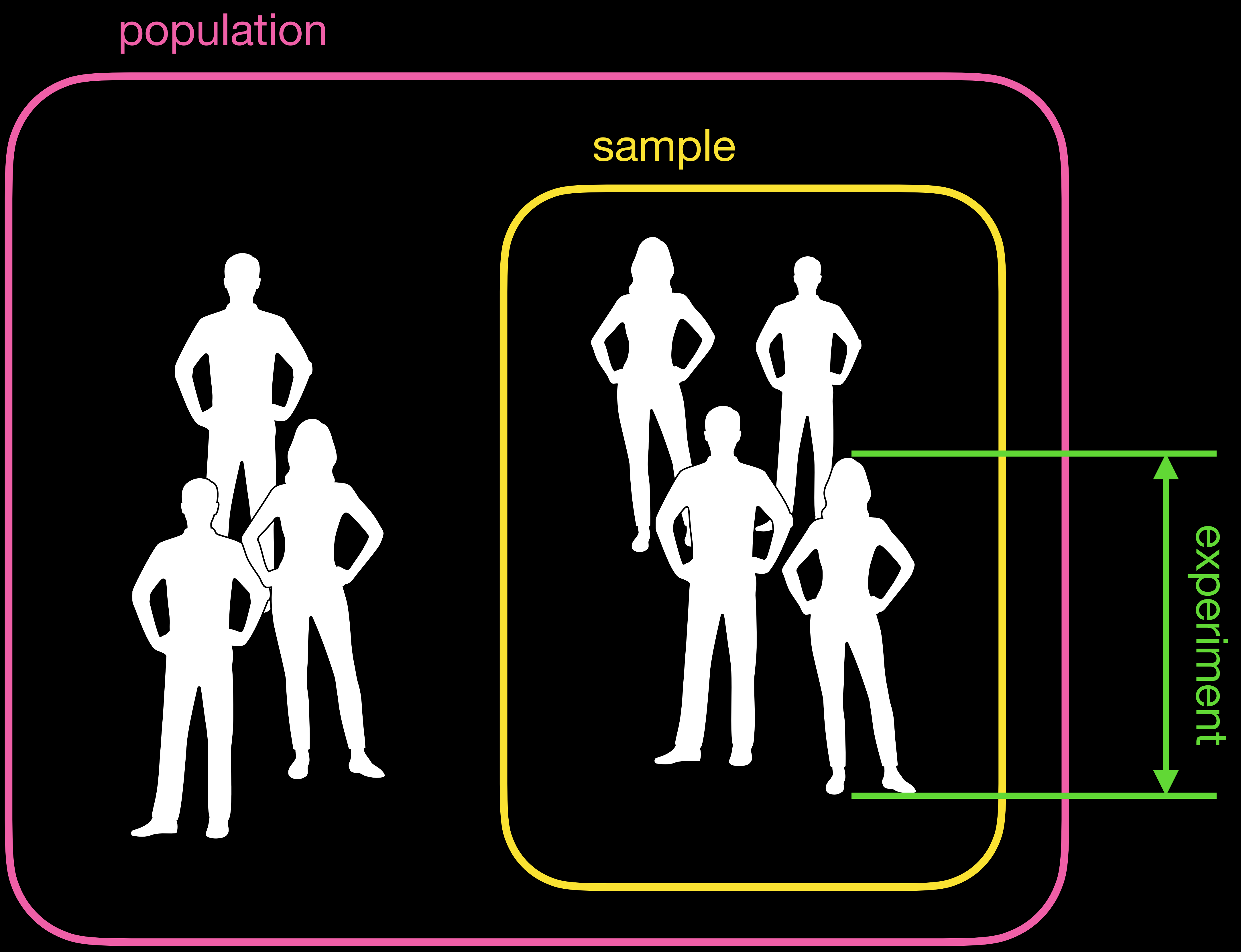
sample space

$$\Omega = [0,300]$$

example elementary probability



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Population: a collection of objects

Sample: a subset of a population

Experiment: measuring something of a sample

Event: a concrete set of outcomes of an experiment

Sample space: space of all possible events Ω

Probability: likelihood of an event in a sample space
$P(A)$ for the event $A \subseteq \Omega$

sample space

$$\Omega = [0,300]$$

probability

$$P(X \in [50,200]) = 0.98$$

Kolmogorov axioms

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2) $P(\Omega) = 1$

3) $P(A_1 \cup A_2 \cup \ldots) = \sum P(A_i)$ if $A_i \cap A_j = \varnothing$ for all $i \neq j$

implications

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$$P(\varnothing) = 0$$

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$$P(\overline{A}) = 1 - P(A)$$

implications

$$P(\varnothing) = 0$$

$$P(\overline{A}) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

exercice

Axioms:

$\forall A : P(A) \geq 0$

$P(\Omega) = 1$

$P(A \cup B) = P(A) + P(B) \quad$ if $\; P(A \cap B) = 0$

Prove: $P(\bar{A}) = 1 - P(A)$

$$\bar{A} = A^c = A' = \neg A \; := \Omega \backslash A = \Omega - A$$

exercice

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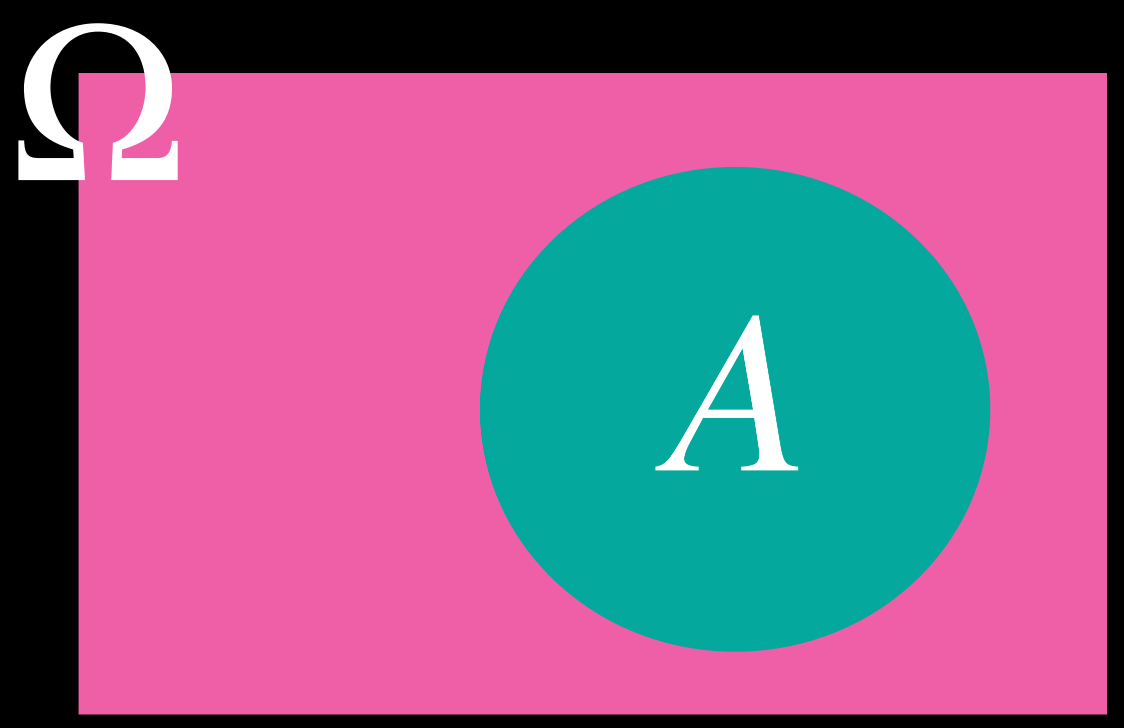

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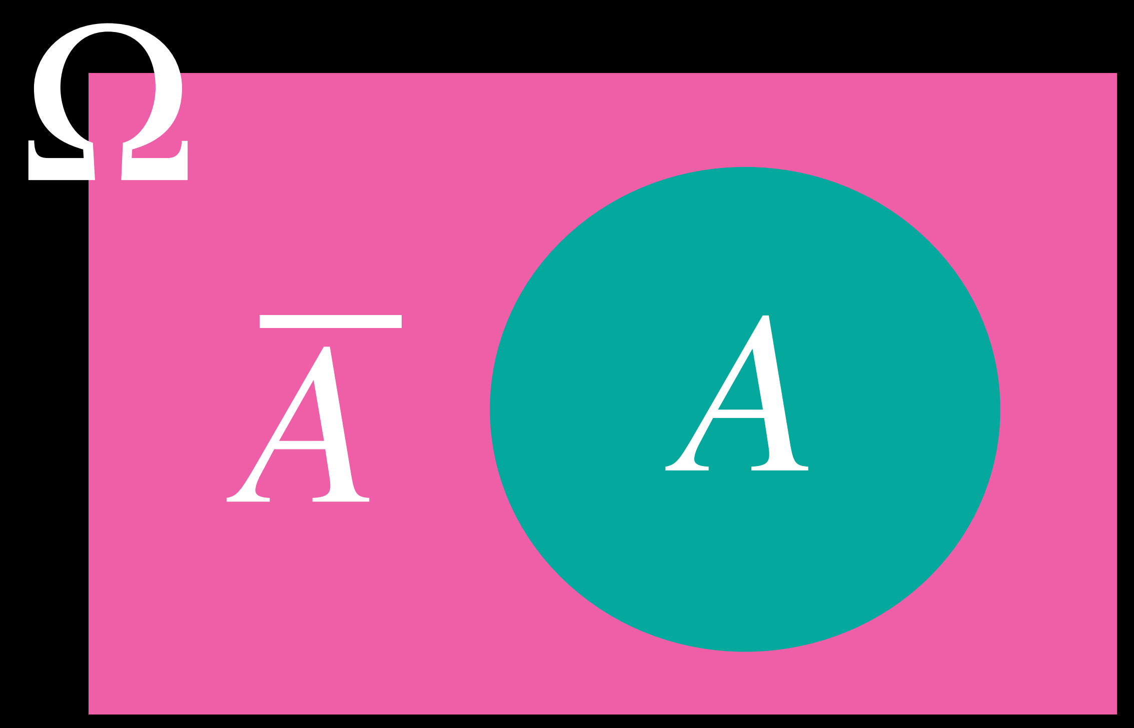

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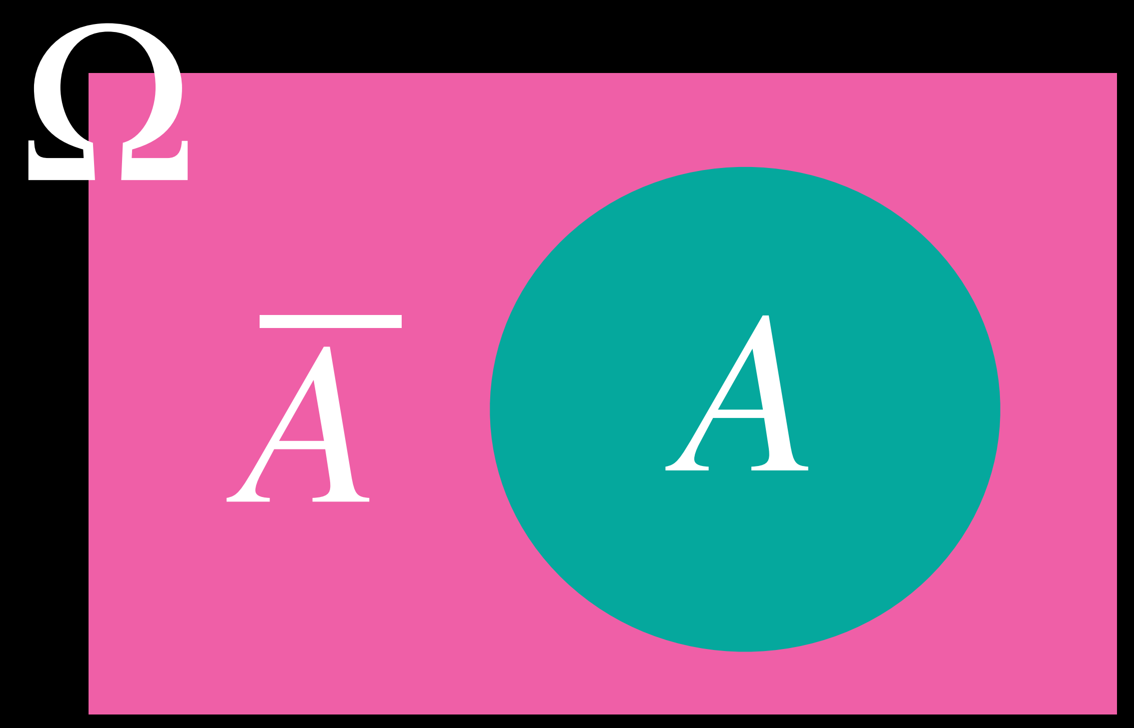

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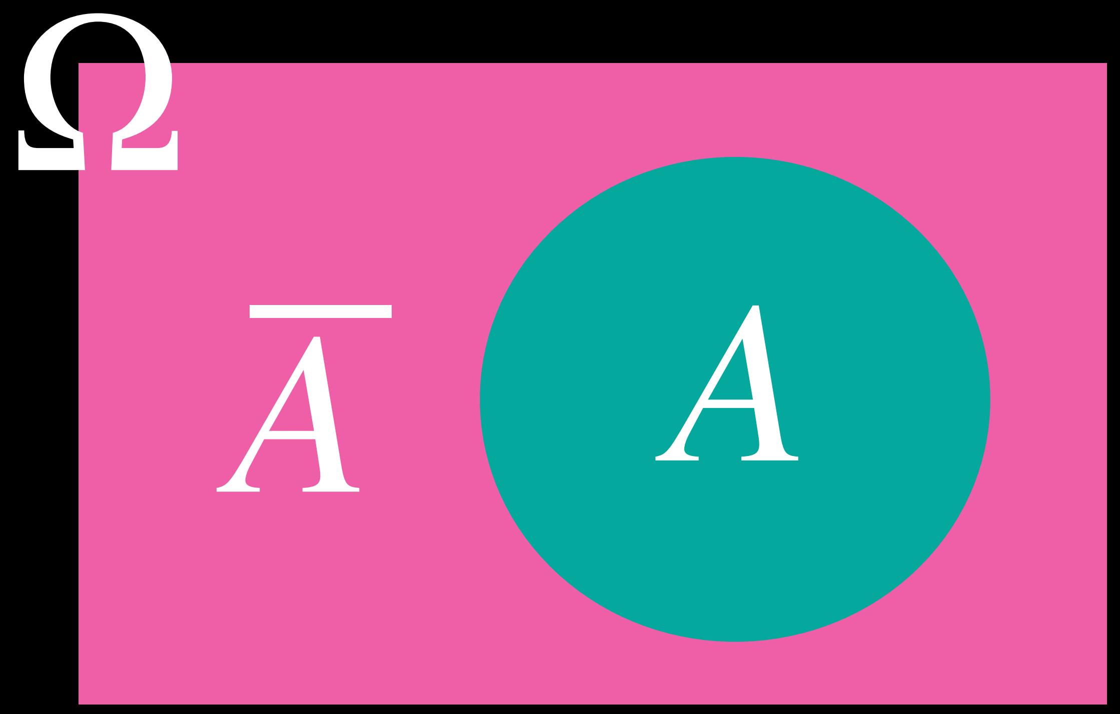

$\bar{A} \cap A = \varnothing$

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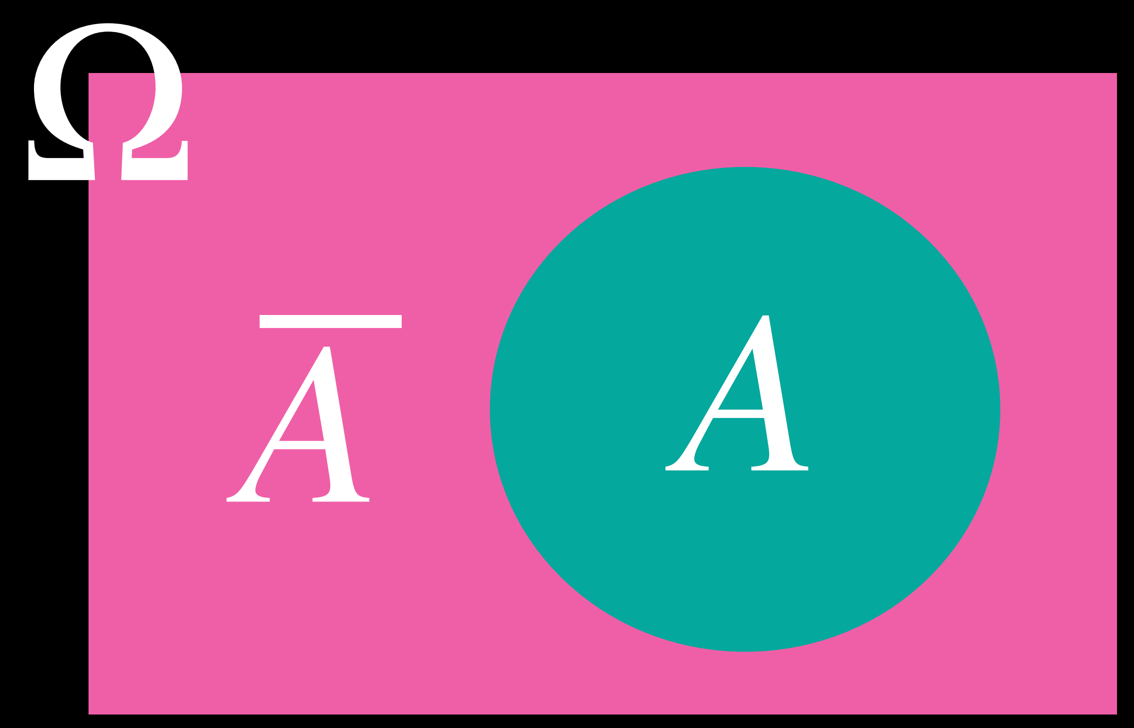

$$\bar{A} \cap A = \emptyset$$

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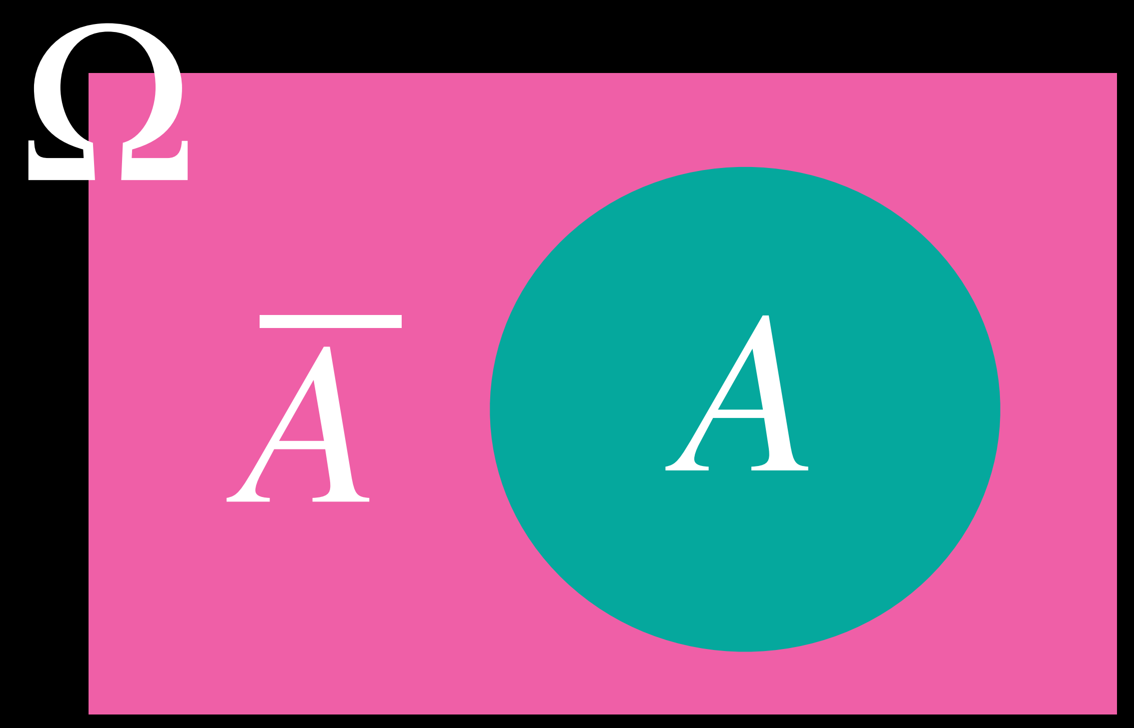

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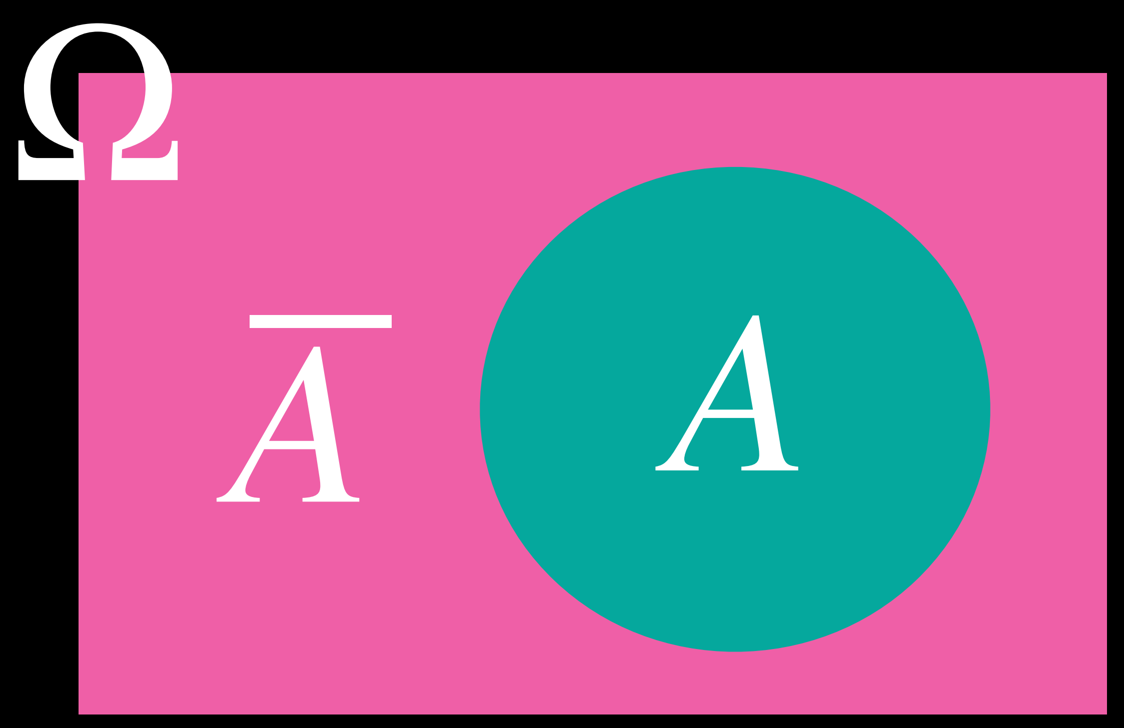

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$\longrightarrow \ P(\bar{A}) = 1 - P(A) \qquad \square$

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conditional probabilities $$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

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conditional probabilities $$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$ "A given B"

example conditional probabilities

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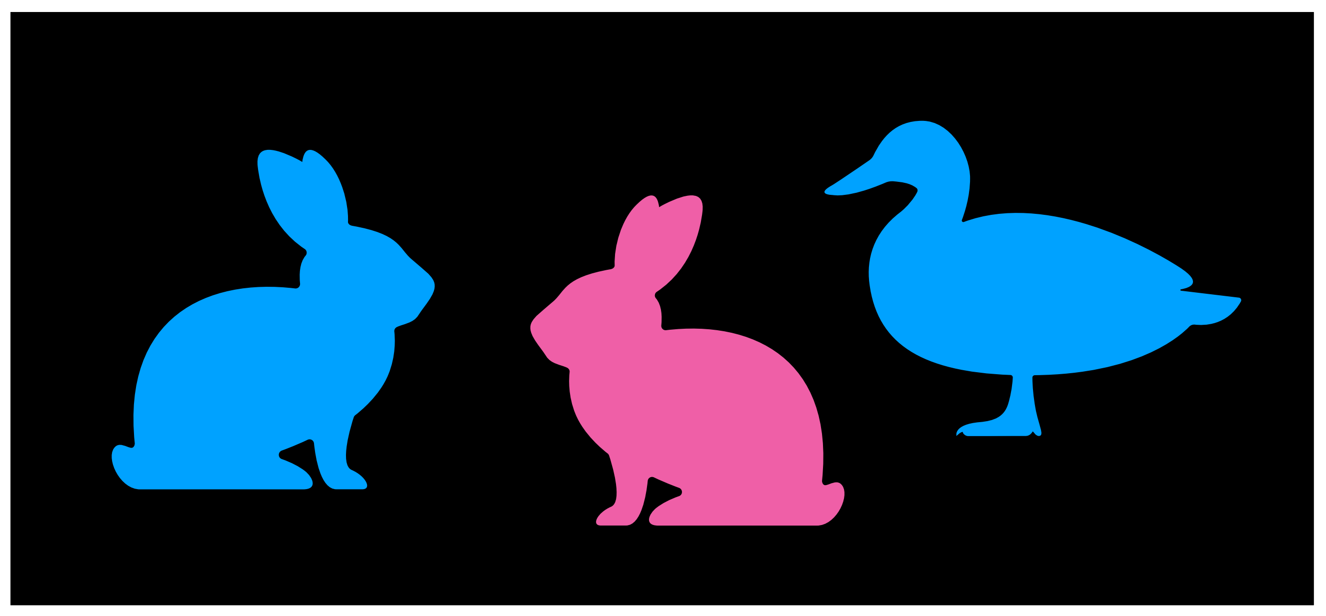

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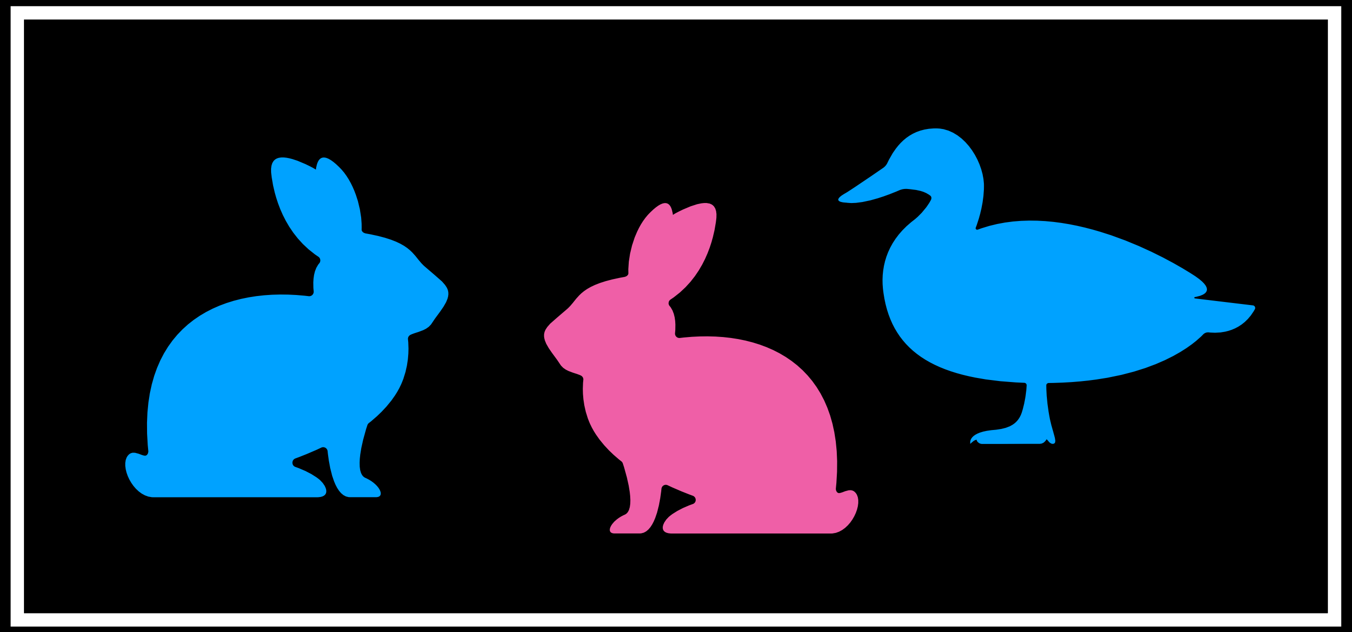
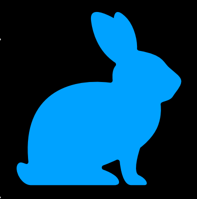
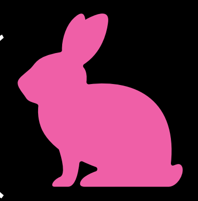
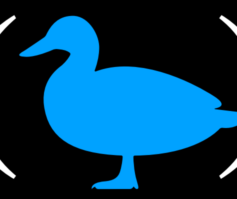


$P($$) = \frac{1}{3}$ $P($$) = \frac{1}{3}$ $P($$) = \frac{1}{3}$

example conditional probabilities

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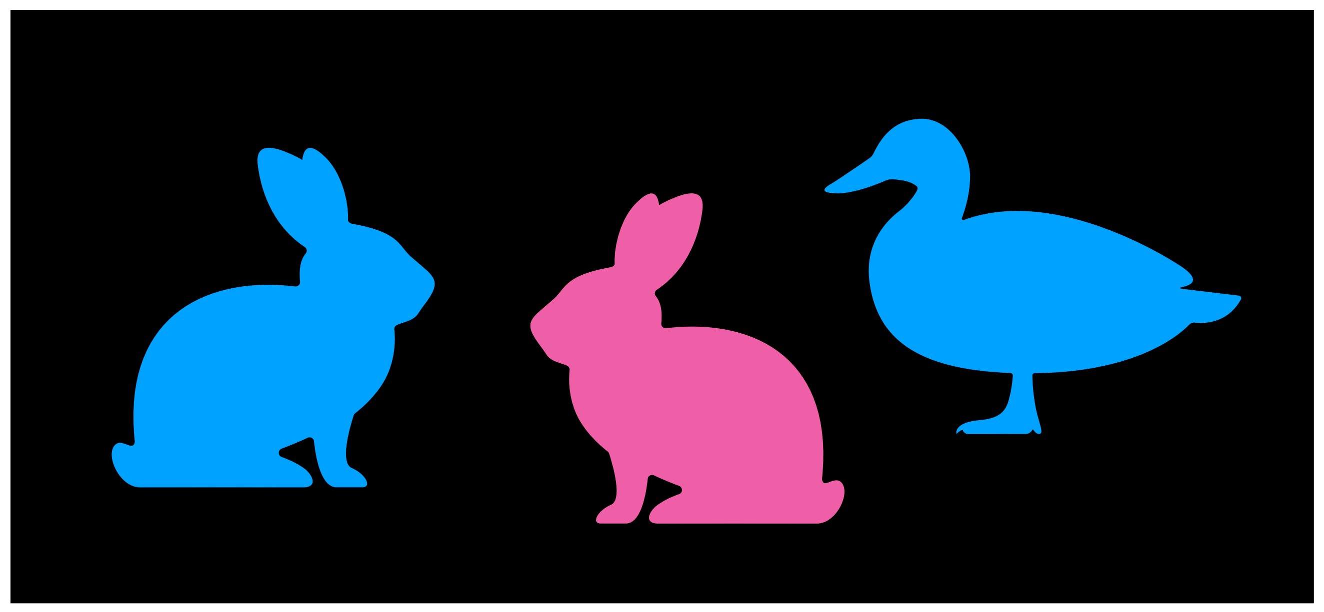
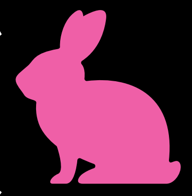
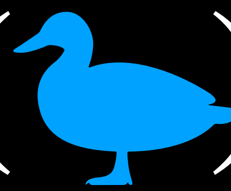
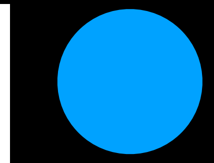


$P($  $) = \frac{1}{3}$ $P($  $) = \frac{1}{3}$ $P($  $) = \frac{1}{3}$

$P($  $\mid$  $)$

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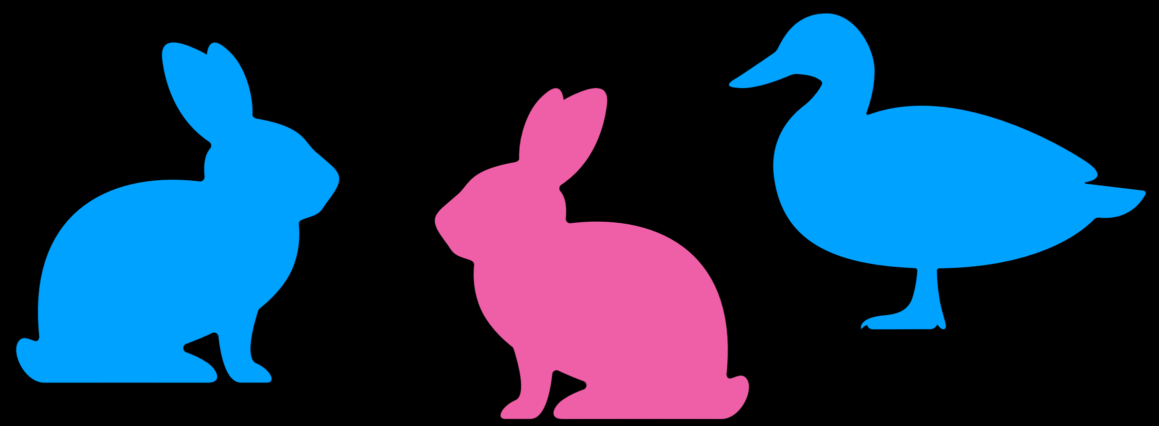
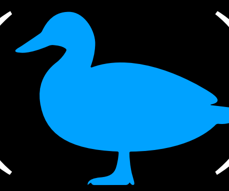
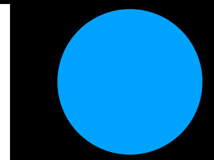
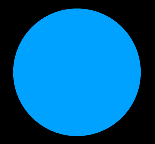
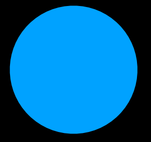


$P($  $) = \frac{1}{3}$ $P($  $) = \frac{1}{3}$ $P($  $) = \frac{1}{3}$

$P($  $|$  $) =$ $P($  $\cap$  $)$ / $P($  $)$

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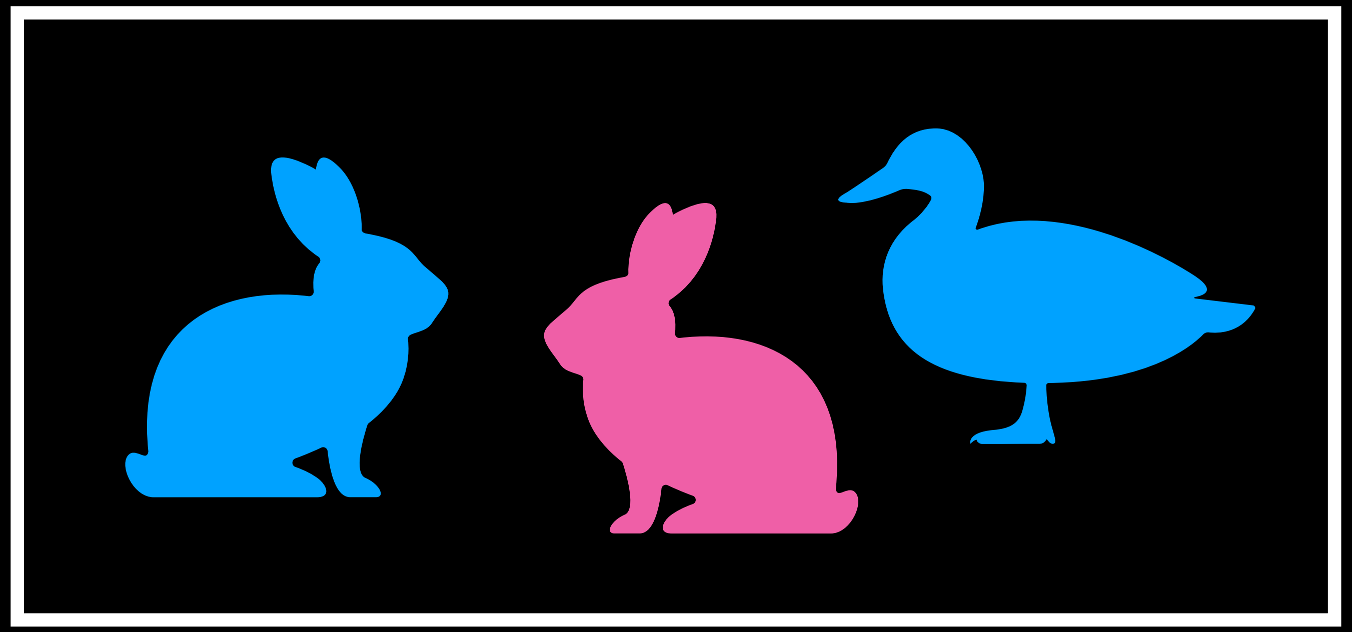
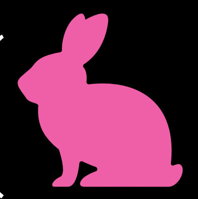
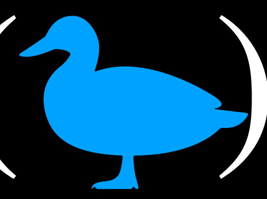
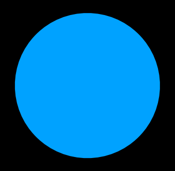
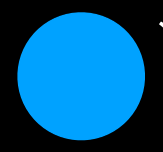
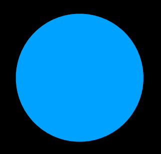
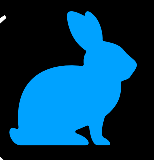
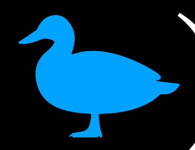
draw



$P($  $) = \frac{1}{3}$ $P($  $) = \frac{1}{3}$ $P($  $) = \frac{1}{3}$

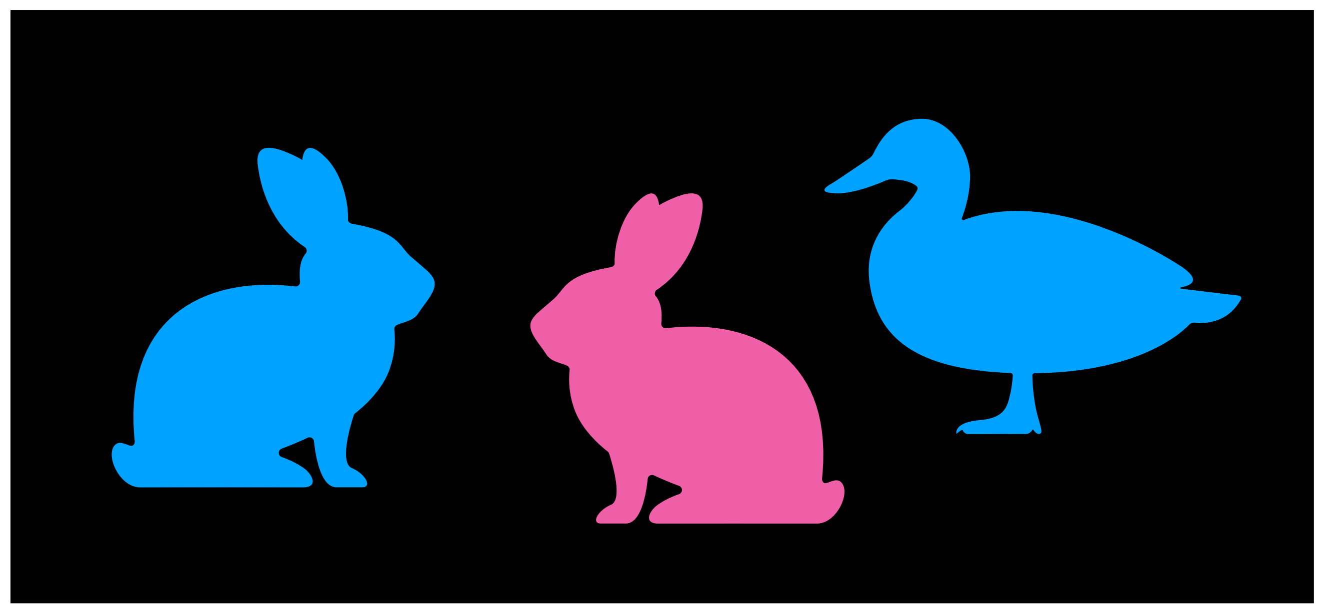
$P($  $\mid$  $)$ $=$ $P($  $\cap$  $)$ / $P($  $)$ $=$ $P($  $)$ / $P($  $\cup$  $)$

example conditional probabilities

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

$$P(A \cup B) = P(A) + P(B)$$
if $P(A \cap B) = 0$

draw



$$P(\ ) = \frac{1}{3} \qquad P(\ ) = \frac{1}{3} \qquad P(\ ) = \frac{1}{3}$$

$$P(\ \mid\ ) = \frac{P(\ \cap\ )}{P(\ )} = \frac{P(\ )}{P(\ \cup\ )} = \frac{P(\ )}{P(\ ) + P(\ )}$$

example conditional probabilities

$$P(A\,|\,B) := \frac{P(A \cap B)}{P(B)}$$

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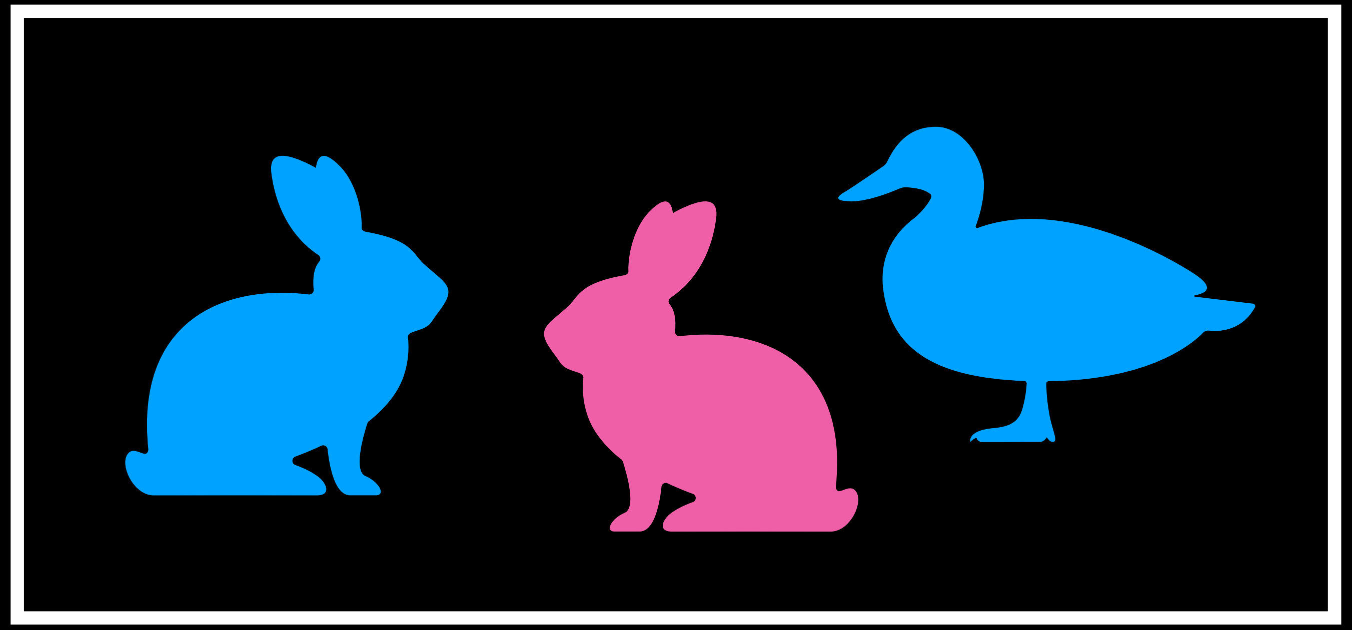
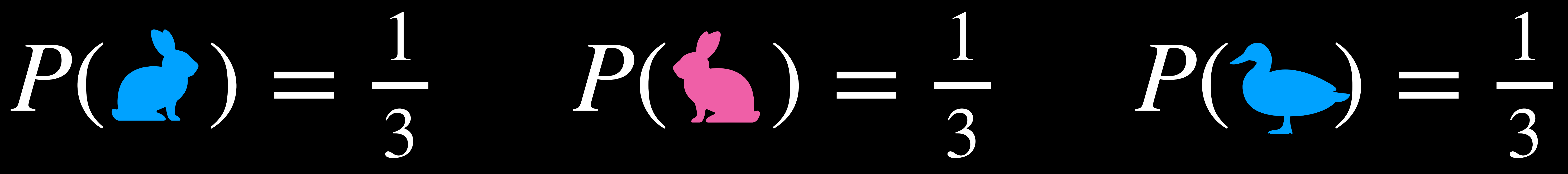
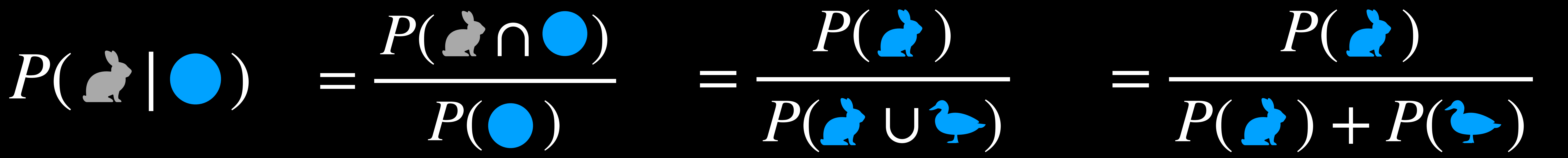
draw







$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}}$$

example conditional probabilities

$$P(A\,|\,B) := \frac{P(A \cap B)}{P(B)}$$

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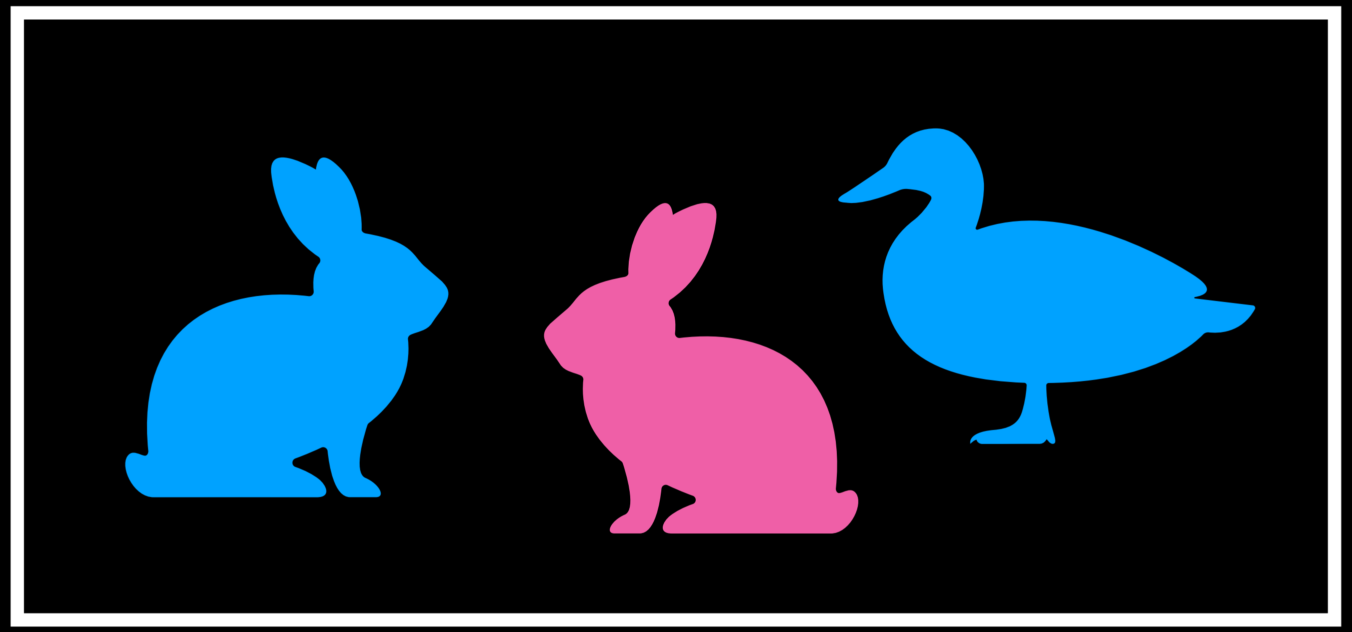
draw



$$P(\ ) = \frac{1}{3} \qquad P(\ ) = \frac{1}{3} \qquad P(\ ) = \frac{1}{3}$$

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$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} = \frac{1}{2}$$

conditional probabilities

conditional probabilities

definitions:

conditional probabilities

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An event A is independent of event B iff $P(A \mid B) = P(A)$.

conditional probabilities

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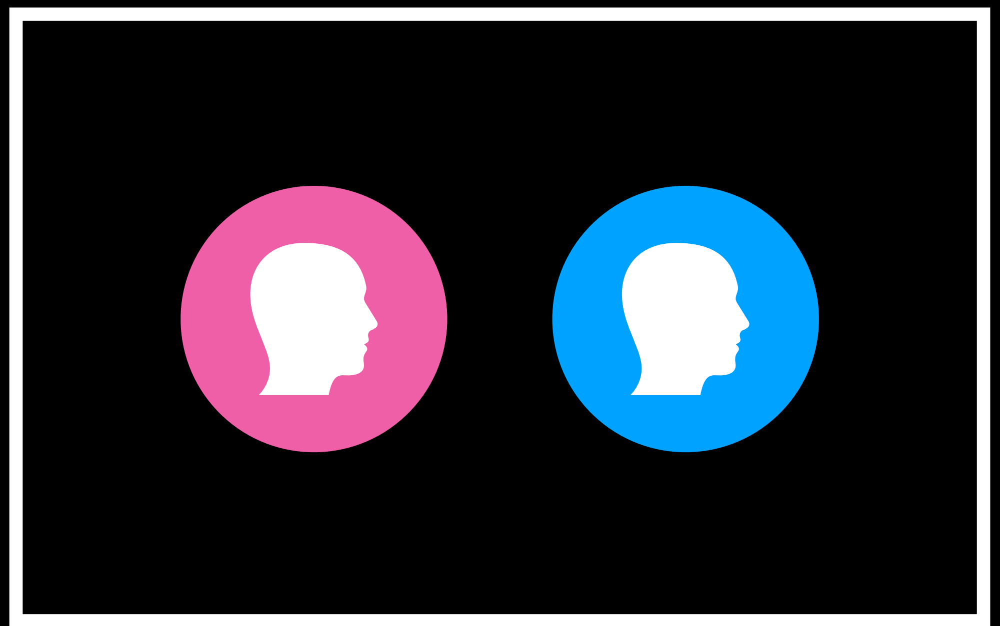
draw and flip



conditional probabilities

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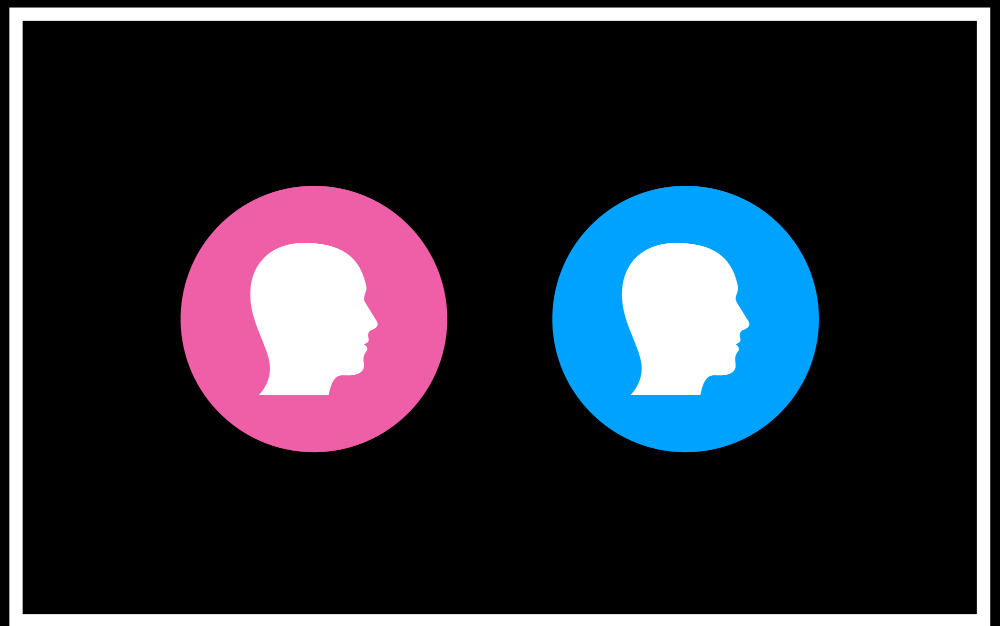
draw and flip



$P($$) = \frac{1}{2}$ $P($

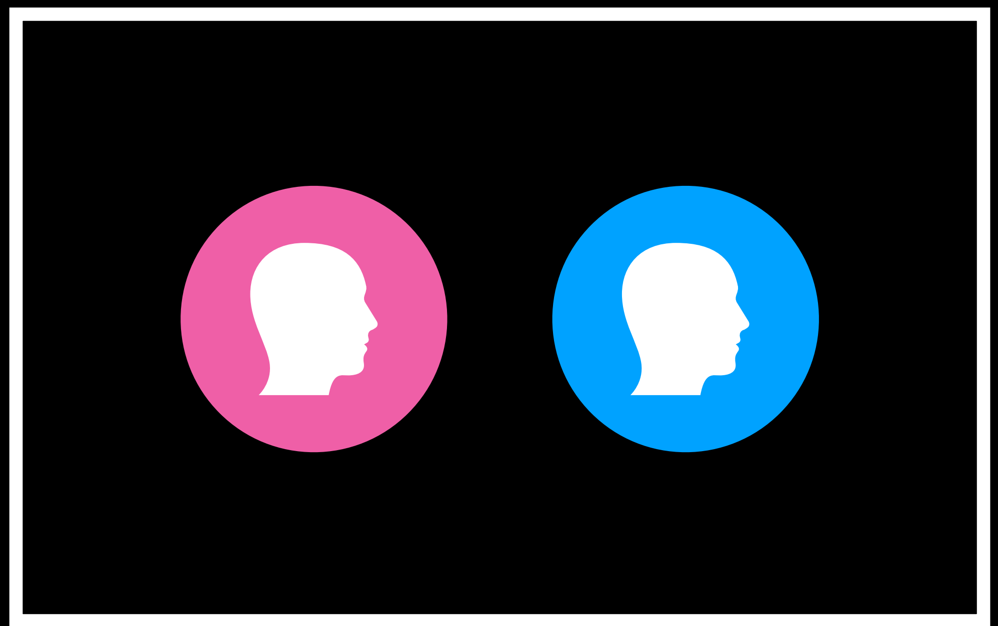
$) = \frac{1}{2}$

conditional probabilities

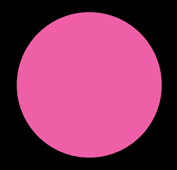
definitions:

An event A is independent of event B iff $P(A\,|\,B) = P(A)$.

draw and flip



$P($$) = \frac{1}{2}$ $P($$) = \frac{1}{2}$

$P($$\,|\,$$)$

conditional probabilities

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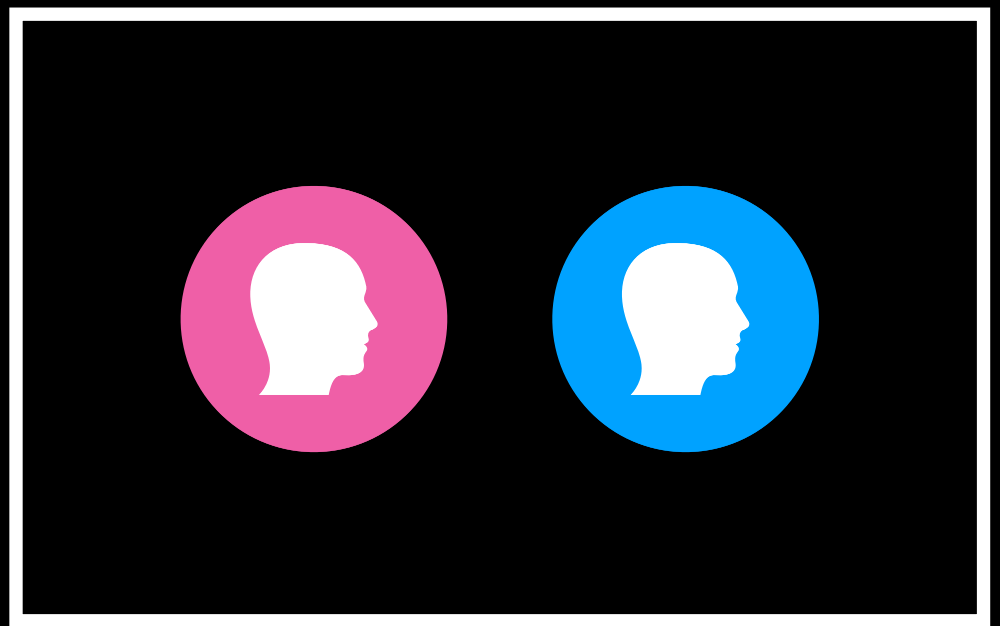
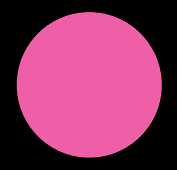


$P($$) = \frac{1}{2}$ $P($$) = \frac{1}{2}$

$P($$\mid$$) = \frac{1}{2}$

conditional probabilities

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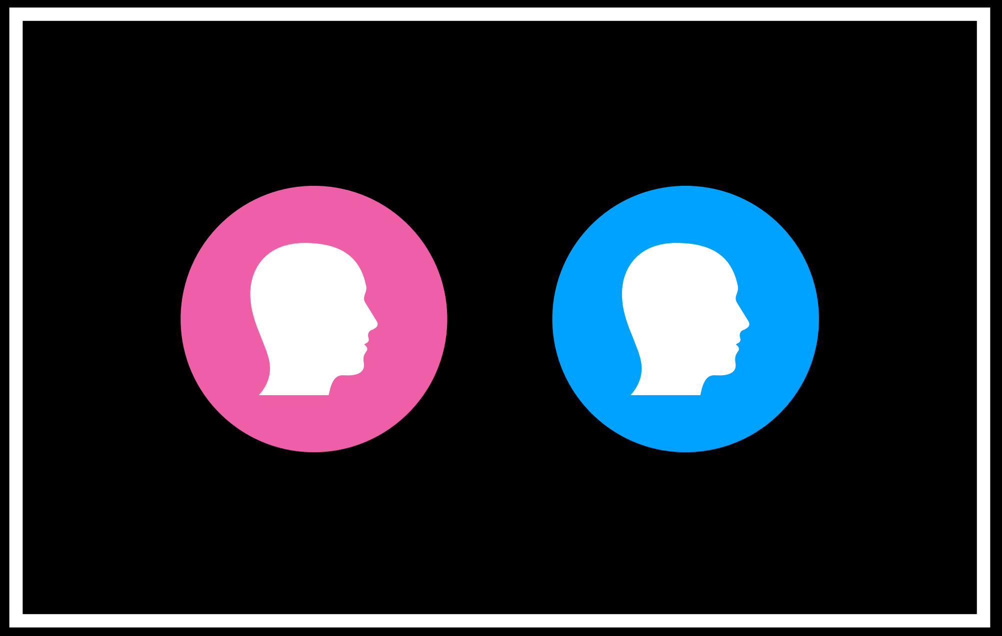
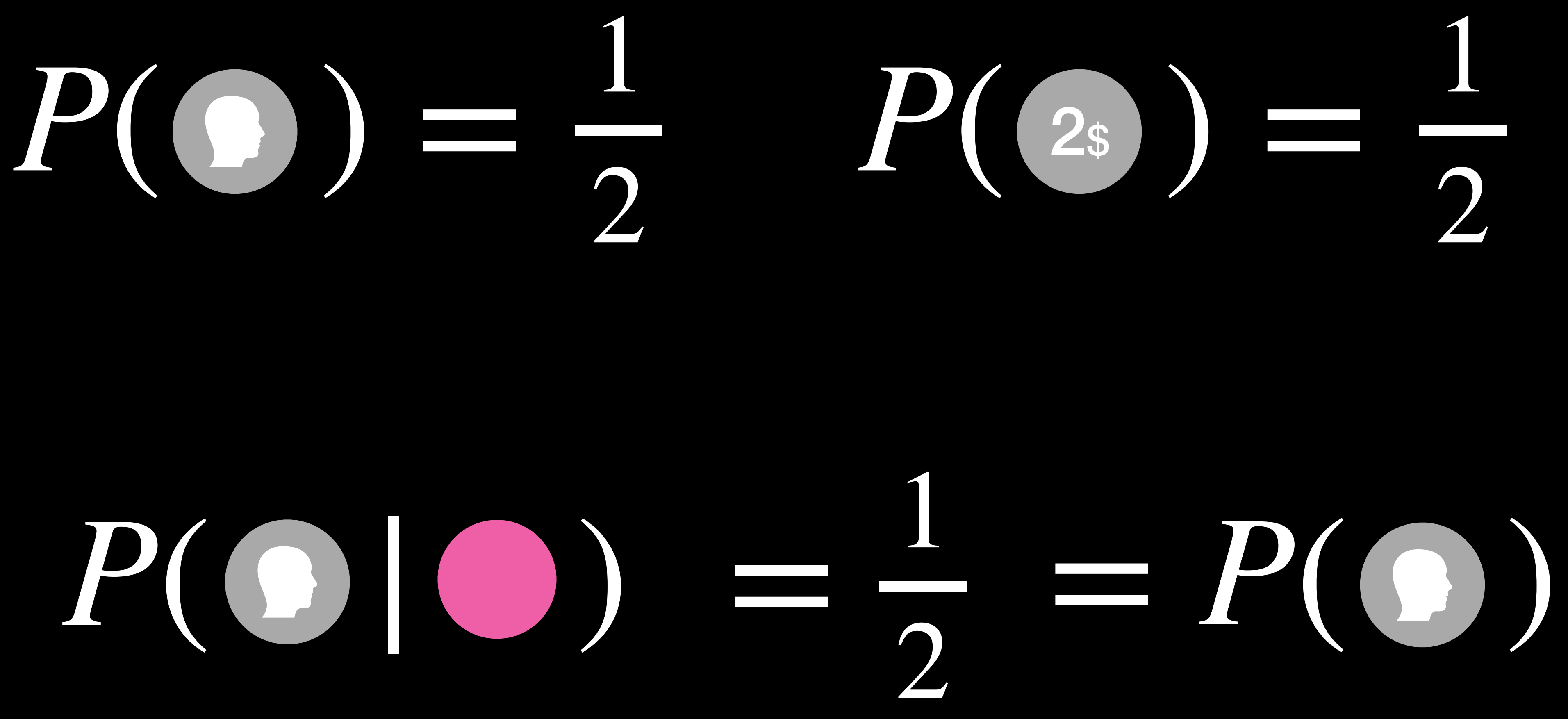
draw and flip





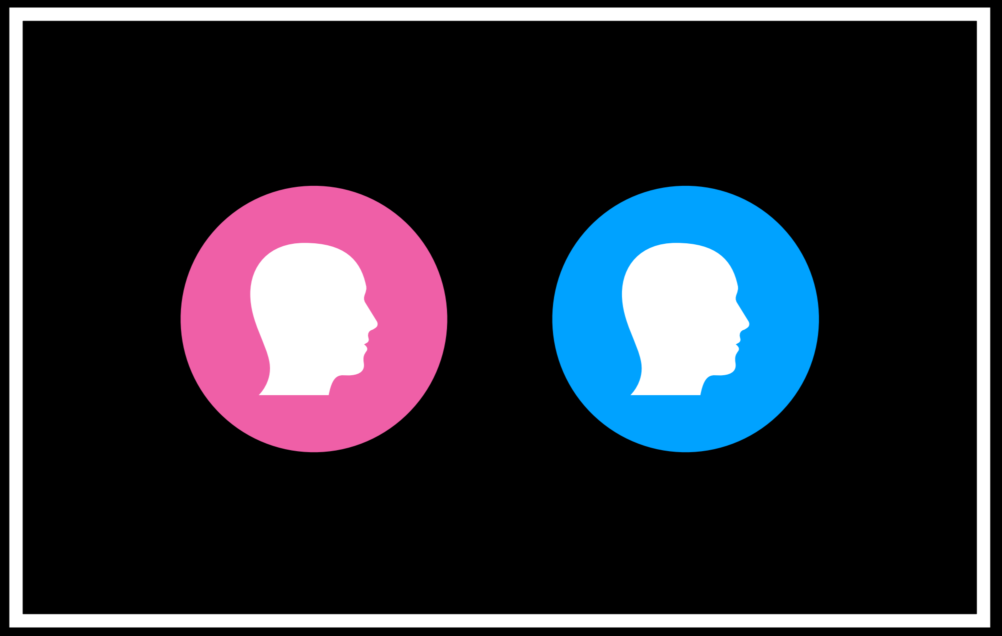
conditional probabilities

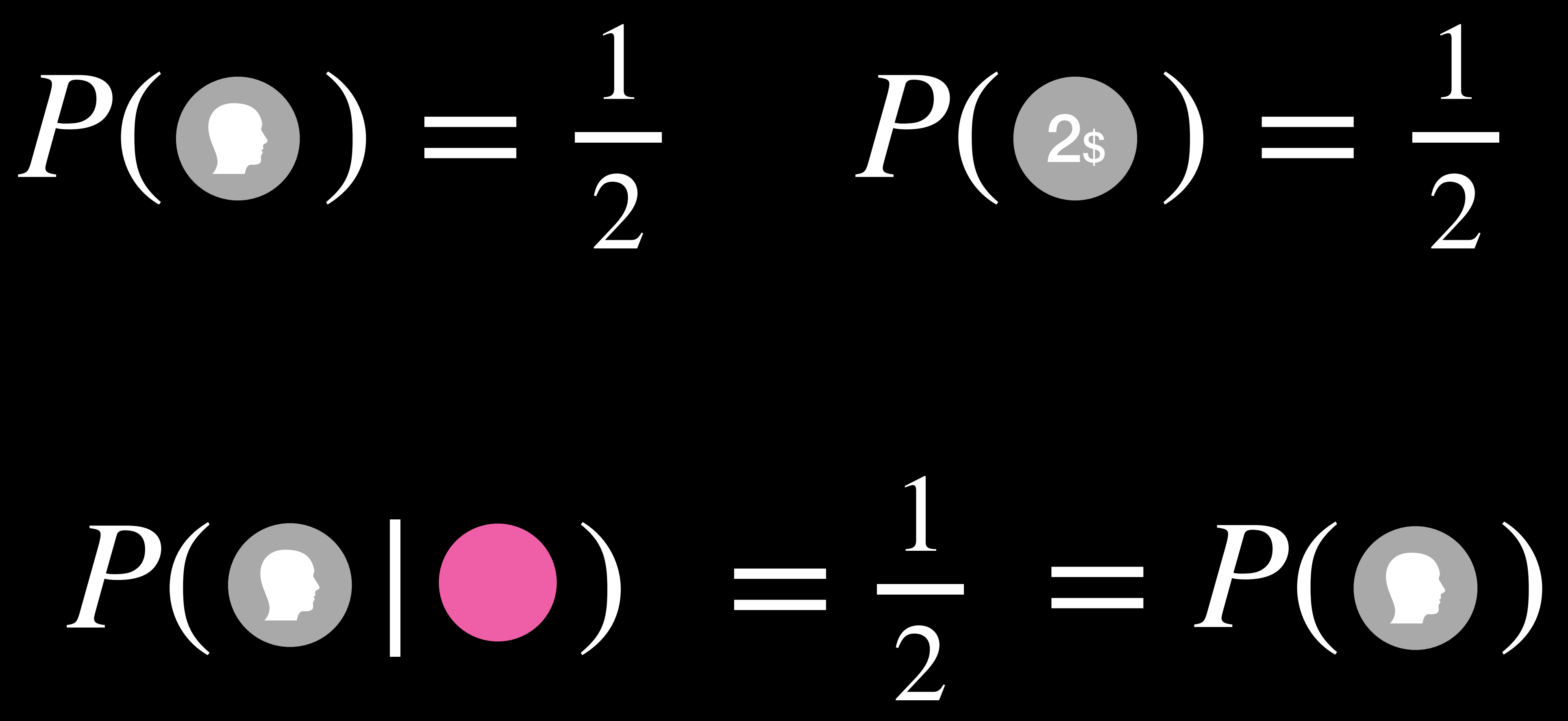
definitions:

An event A is independent of event B iff $P(A \mid B) = P(A)$.

$$\longrightarrow \quad P(A \cap B) = P(A) \cdot P(B)$$

draw and flip





exercice

Prove: $P(A \cap B) = P(A) \cdot P(B)$

for A, B independent

conditional probabilities:

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

A independent of B:

$$P(A \mid B) = P(A)$$

exercice

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for A, B independent

$$\begin{aligned} P(A \cap B) &= P(A \mid B) \cdot P(B) \\ &= P(A) \cdot P(B) \end{aligned}$$

exercice

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$\square$

Bayes' theorem

Bayes' theorem

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

Bayes' theorem

posterior

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

Bayes' theorem

posterior

prior

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

Bayes' theorem

posterior

prior

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

marginal

Bayes' theorem

$$\overbrace{P(A\,|\,B)}^{\text{posterior}} = \frac{\overbrace{P(B\,|\,A)}^{\text{likelihood}} \cdot \overbrace{P(A)}^{\text{prior}}}{\underbrace{P(B)}_{\text{marginal}}}$$

conditional probabilities:

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

exercice

prove Bayes' theorem:

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

exercice

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$$P(A \,|\, B) = \frac{P(B \,|\, A) \cdot P(A)}{P(B)}$$

$$P(A \cap B) = P(A \,|\, B) \cdot P(B)$$

conditional probabilities:

$$P(A \,|\, B) := \frac{P(A \cap B)}{P(B)}$$

exercice

prove Bayes' theorem:

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$$P(A\,|\,B) \cdot P(B) = P(B\,|\,A) \cdot P(A)$$

exercice

conditional probabilities:

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$$P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A) \quad \longrightarrow \quad P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

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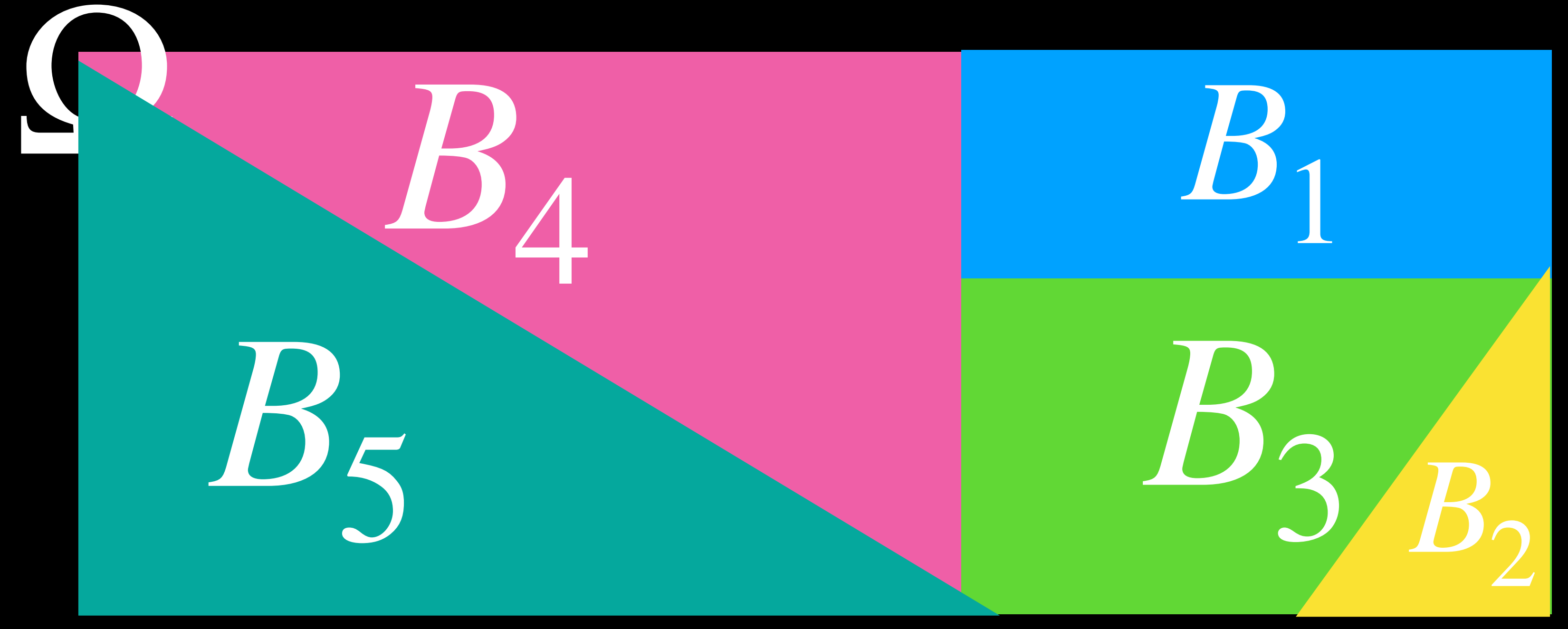
$$P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A) \quad \longrightarrow \quad P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

$\square$

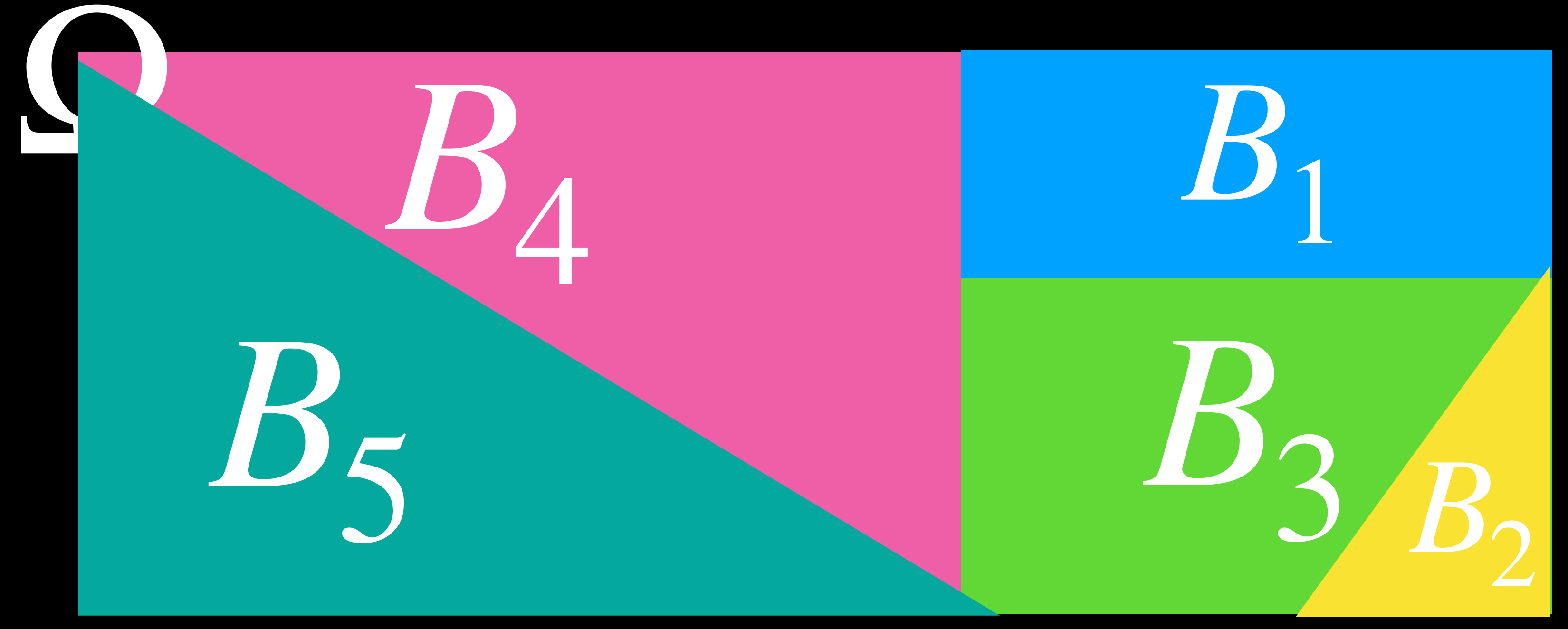
Law of total probability

Law of total probability

Law of total probability

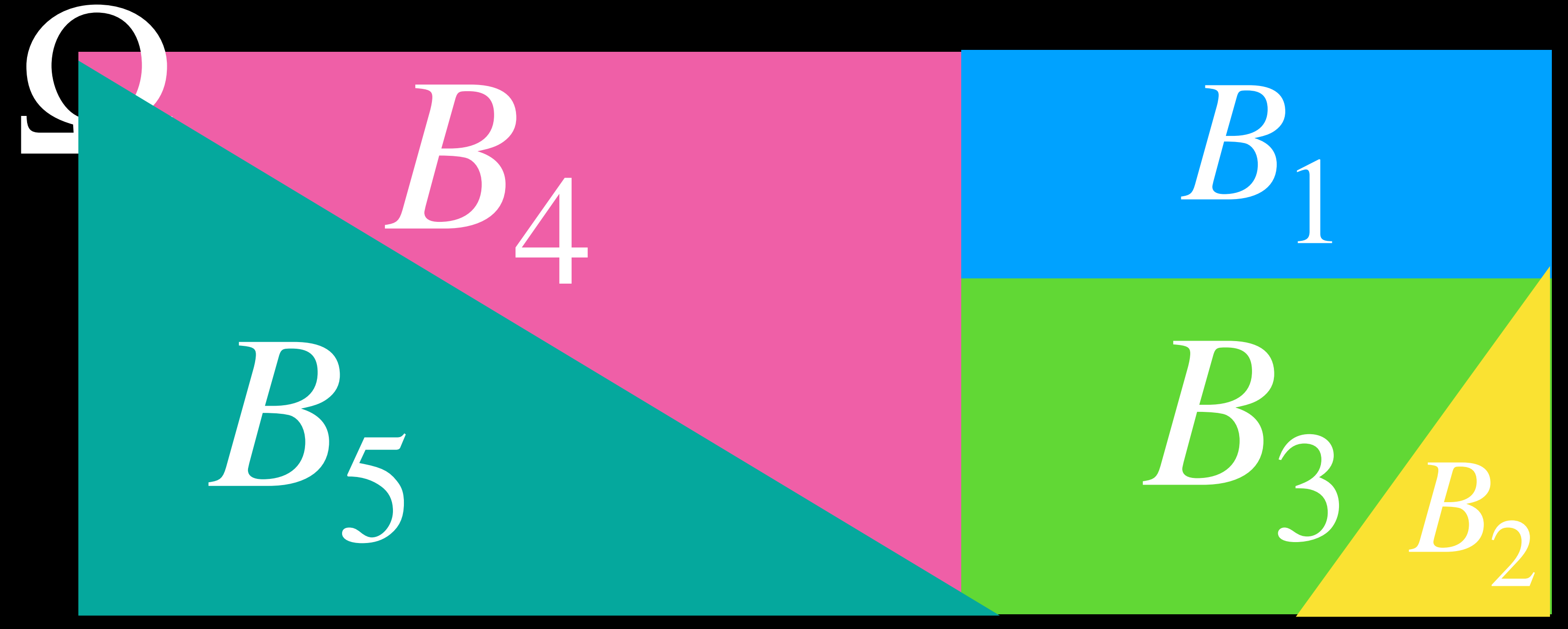


Law of total probability



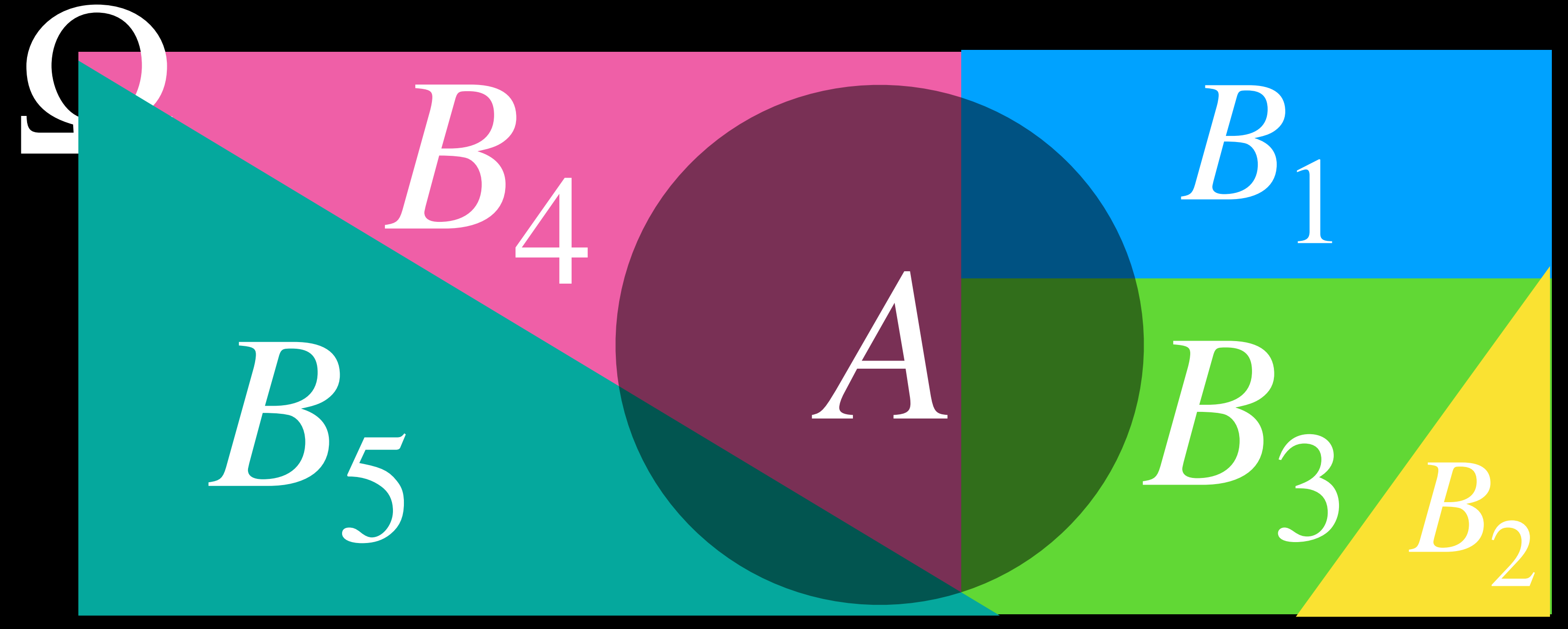
$$\bigcup_{i=1}^{n} B_i = \Omega$$

Law of total probability



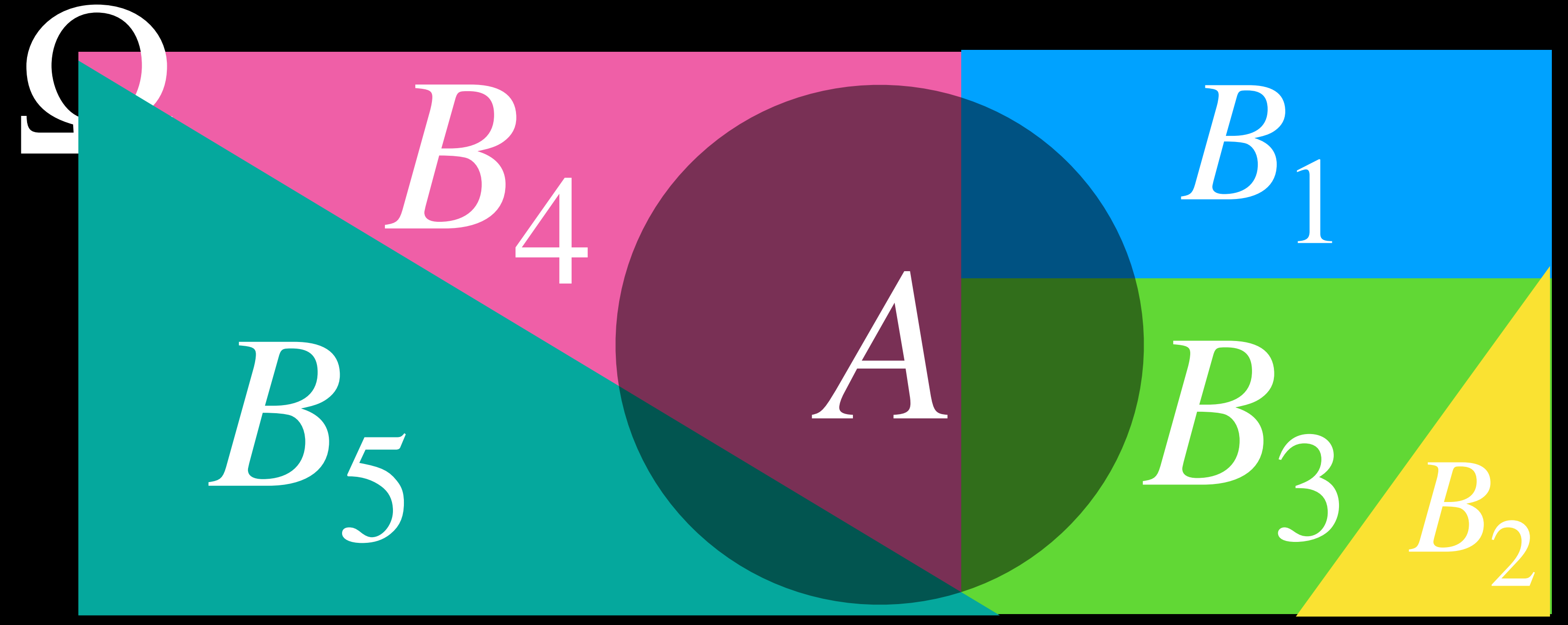
$$\bigcup_{i=1}^{n} B_i = \Omega \qquad B_i \cap B_j = \varnothing$$

Law of total probability



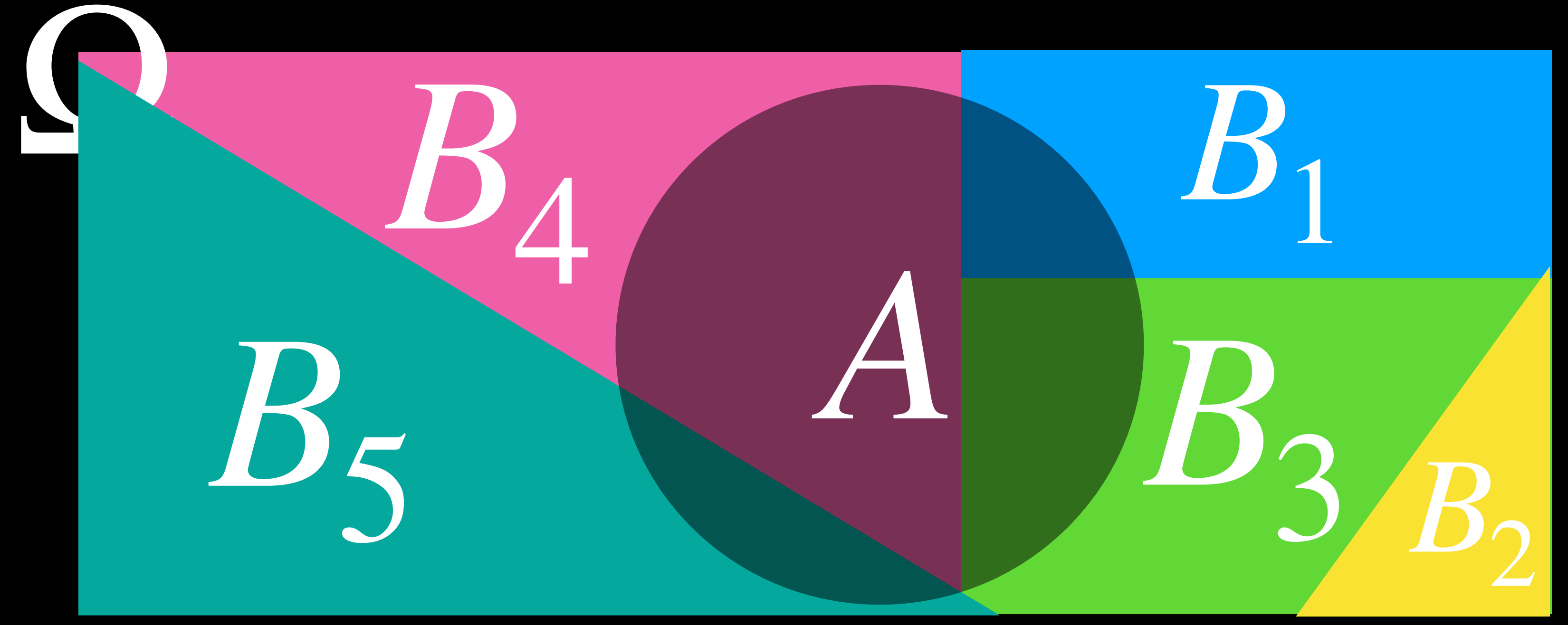
$$\bigcup_{i=1}^{n} B_i = \Omega \qquad B_i \cap B_j = \emptyset$$

Law of total probability



$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i) \qquad \bigcup_{i=1}^{n} B_i = \Omega \qquad B_i \cap B_j = \varnothing$$

Law of total probability



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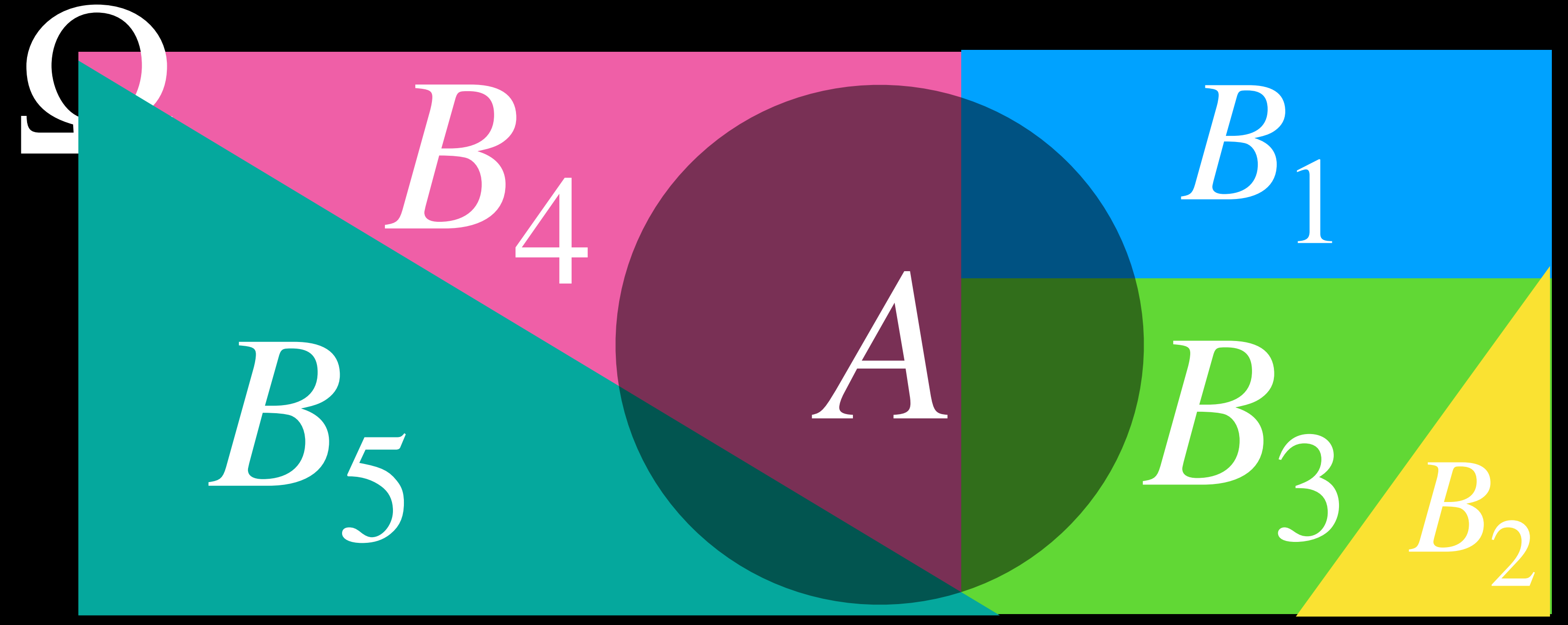
prove

Law of total probability



$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$$

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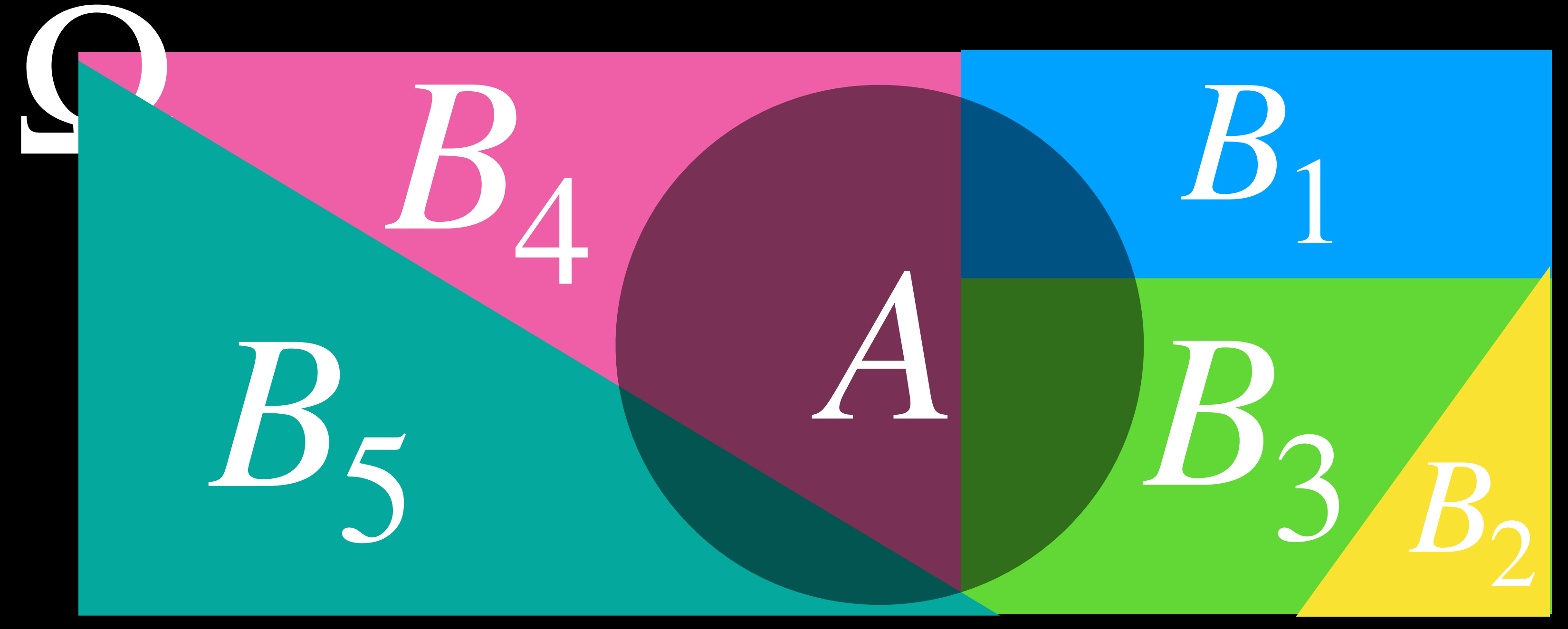
$$B_i \cap B_j = \varnothing$$

prove

$$P(A \cup B) = P(A) + P(B) \quad \text{if} \quad P(A \cap B) = 0$$

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

Law of total probability



$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$$

$$\bigcup_{i=1}^{n} B_i = \Omega$$

$$B_i \cap B_j = \varnothing$$

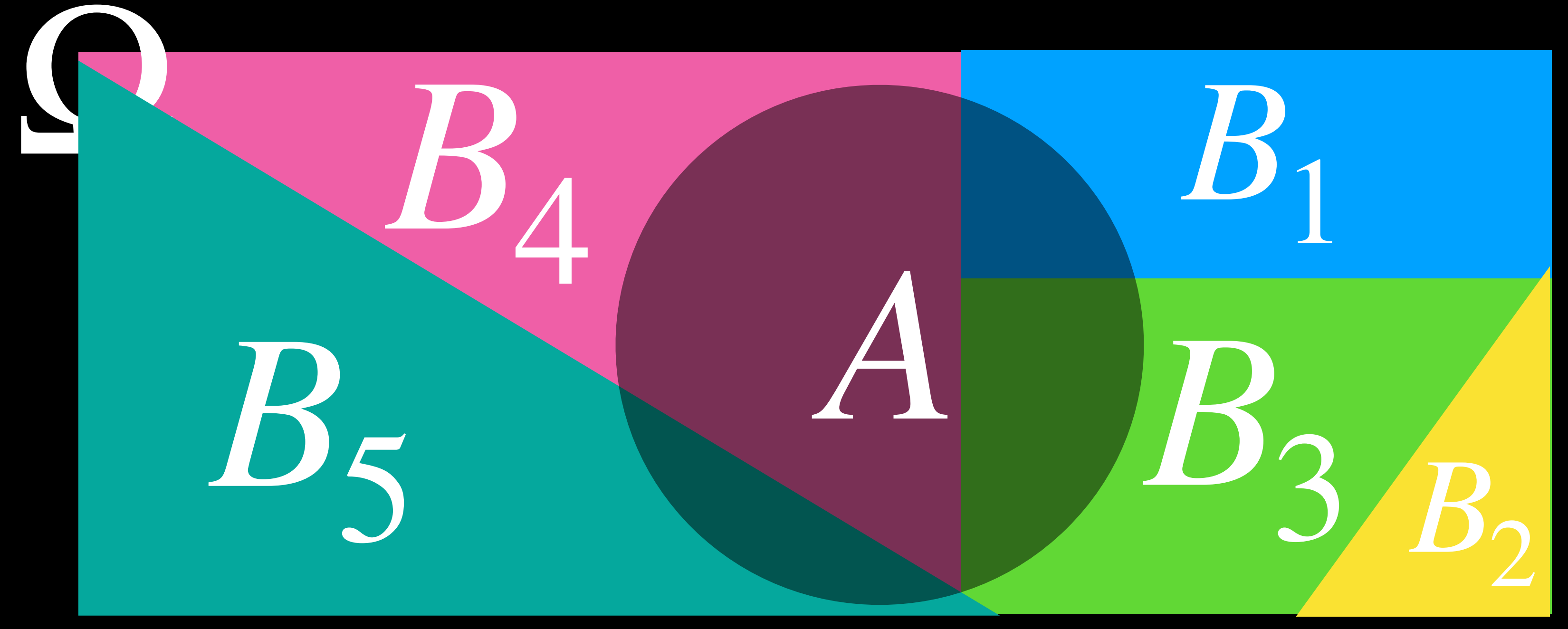
$P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

$P(A \cap B) = P(A \mid B) \cdot P(B)$

prove

$$P(A) = P\left(A \cap \Omega\right)$$

Law of total probability



$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$$

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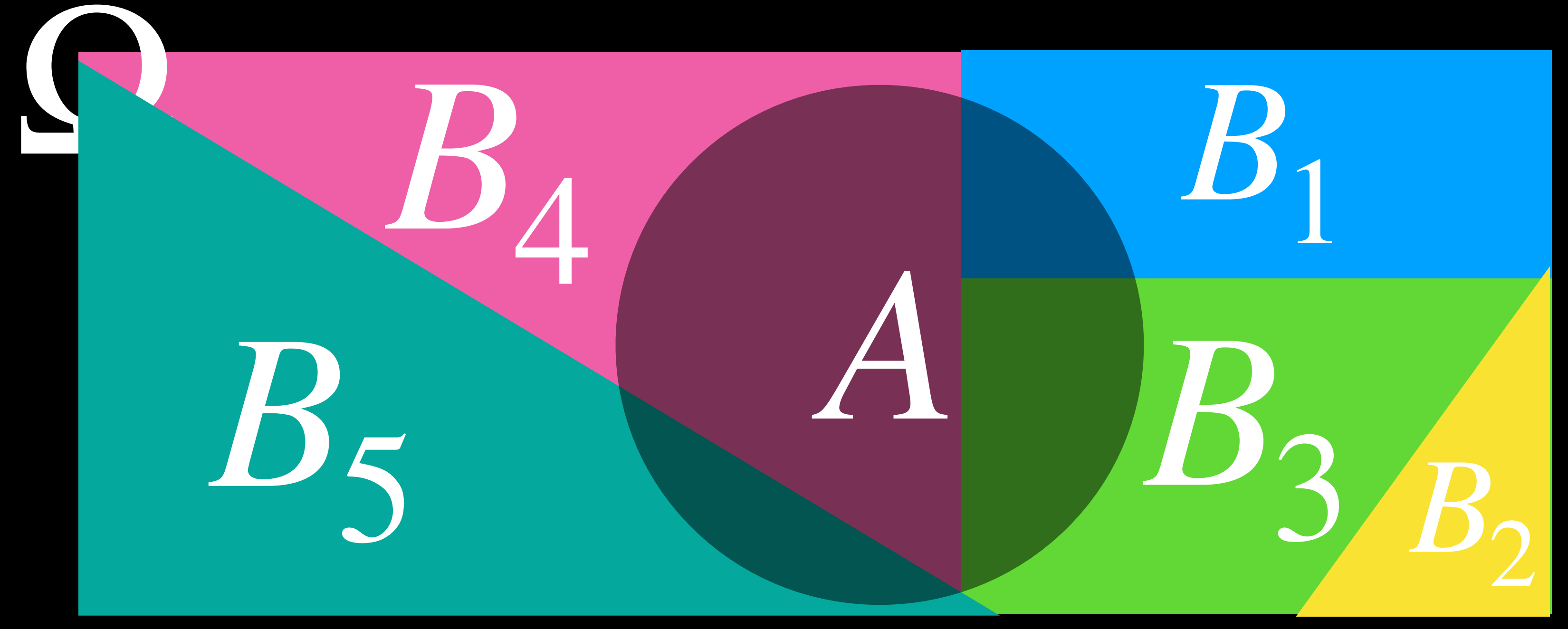
prove

$$P(A) = P(A \cap \Omega) \quad = P\left(A \cap \bigcup_{i=1}^{n} B_i\right)$$

Law of total probability



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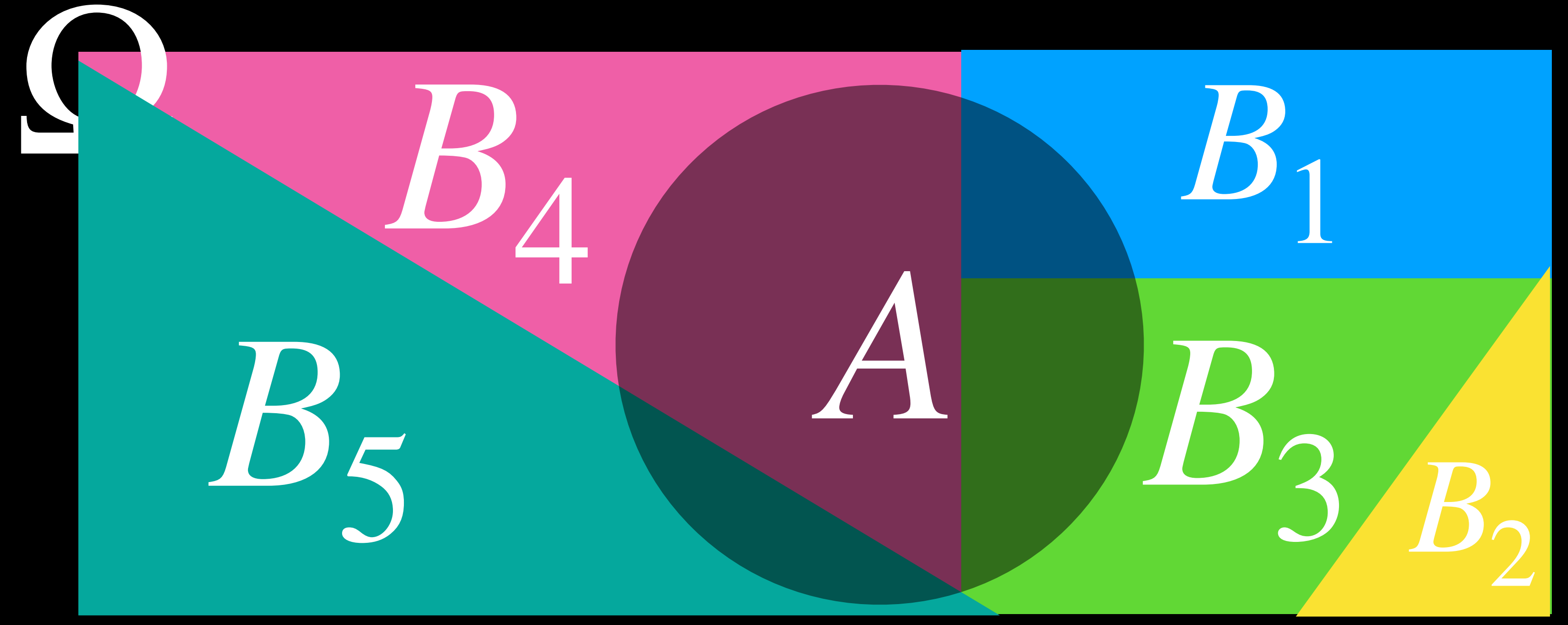
prove

$$P(A) = P(A \cap \Omega) = P\left(A \cap \bigcup_{i=1}^{n} B_i\right) = P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right)$$

Law of total probability



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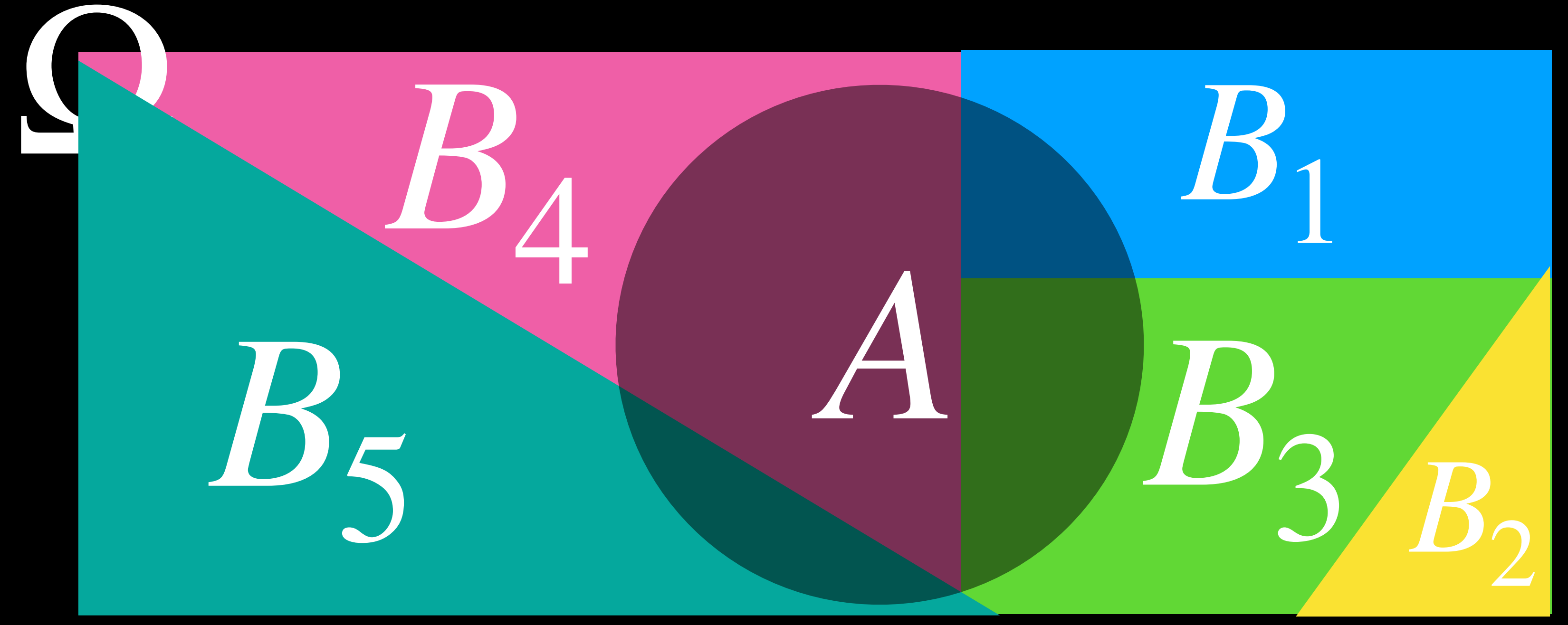
$P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

$P(A \cap B) = P(A \mid B) \cdot P(B)$

prove

$$P(A) = P(A \cap \Omega) = P\left(A \cap \bigcup_{i=1}^{n} B_i\right) = P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right) = \sum_{i=1}^{n} P(A \cap B_i)$$

Law of total probability



$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i) \qquad \bigcup_{i=1}^{n} B_i = \Omega \qquad B_i \cap B_j = \varnothing$$

$P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

$P(A \cap B) = P(A \mid B) \cdot P(B)$

prove

$$P(A) = P\,(A \cap \Omega) \quad = P\left(A \cap \bigcup_{i=1}^{n} B_i\right) \quad = P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right) \quad = \sum_{i=1}^{n} P(A \cap B_i) \quad = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$$

probability expansion

probability expansion

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

probability expansion

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$
$$= P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot \ldots \cdot P(A_n | A_1 \cap \ldots \cap A_{n-1})$$

probability expansion

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prove idea

$$\boxed{P(A \cap B) = P(A | B) \cdot P(B)}$$

probability expansion

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

$$= P(A_1) \cdot P(A_2 \,|\, A_1) \cdot P(A_3 \,|\, A_1 \cap A_2) \cdot \ldots \cdot P(A_n \,|\, A_1 \cap \ldots \cap A_{n-1})$$

prove idea

$$\boxed{P(A \cap B) = P(A \,|\, B) \cdot P(B)}$$

$$B_{n-1} := A_1 \cap \ldots \cap A_{n-1}$$

probability expansion

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$
$$= P(A_1) \cdot P(A_2 \,|\, A_1) \cdot P(A_3 \,|\, A_1 \cap A_2) \cdot \ldots \cdot P(A_n \,|\, A_1 \cap \ldots \cap A_{n-1})$$

prove idea

$$\boxed{P(A \cap B) = P(A \,|\, B) \cdot P(B)}$$

$$B_{n-1} := A_1 \cap \ldots \cap A_{n-1}$$
$$P(A_1 \cap \ldots \cap A_n)$$

probability expansion

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$
$$= P(A_1) \cdot P(A_2 \,|\, A_1) \cdot P(A_3 \,|\, A_1 \cap A_2) \cdot \ldots \cdot P(A_n \,|\, A_1 \cap \ldots \cap A_{n-1})$$

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$$B_{n-1} := A_1 \cap \ldots \cap A_{n-1}$$
$$P(A_1 \cap \ldots \cap A_n) \qquad = P(A_n \cap B_{n-1})$$

probability expansion

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

$$= P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 \cap A_2) \cdot \ldots \cdot P(A_n \mid A_1 \cap \ldots \cap A_{n-1})$$

prove idea

$$\boxed{P(A \cap B) = P(A \mid B) \cdot P(B)}$$

$$B_{n-1} := A_1 \cap \ldots \cap A_{n-1}$$

$$P(A_1 \cap \ldots \cap A_n) \quad = P(A_n \cap B_{n-1}) \quad = P(A_n \mid B_{n-1}) \cdot P(B_{n-1})$$

probability expansion

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$
$$= P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot \ldots \cdot P(A_n | A_1 \cap \ldots \cap A_{n-1})$$

prove idea

$$\boxed{P(A \cap B) = P(A | B) \cdot P(B)}$$

$$B_{n-1} := A_1 \cap \ldots \cap A_{n-1}$$

$$P(A_1 \cap \ldots \cap A_n) \quad = P(A_n \cap B_{n-1}) \quad = P(A_n | B_{n-1}) \cdot P(B_{n-1})$$
$$= P(A_1 \cap \ldots \cap A_{n-1}) \cdot P(A_n | A_1 \cap \ldots \cap A_{n-1})$$

probability expansion

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$$= \ldots$$

probability expansion example

probability expansion example

draw 4 cards from 52 cards deck

probability expansion example

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What is the probability that all 4 aces are drawn?

probability expansion example

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A_i : draw an aces in i-th draw

probability expansion example

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What is the probability that all 4 aces are drawn?

A_i : draw an aces in i-th draw

$$P(A_1) = \frac{4}{52},$$

probability expansion example

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What is the probability that all 4 aces are drawn?

A_i : draw an aces in i-th draw

$$P(A_1) = \frac{4}{52}, \qquad P(A_2 \mid A_1) = \frac{3}{51},$$

probability expansion example

draw 4 cards from 52 cards deck

What is the probability that all 4 aces are drawn?

A_i : draw an aces in i-th draw

$$P(A_1) = \frac{4}{52}, \quad P(A_2 | A_1) = \frac{3}{51}, \quad P(A_3 | A_1 \cap A_2) = \frac{2}{50},$$

probability expansion example

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What is the probability that all 4 aces are drawn?

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$$P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

probability expansion example

draw 4 cards from 52 cards deck

What is the probability that all 4 aces are drawn?

A_i : draw an aces in i-th draw

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$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49}$$

random variable

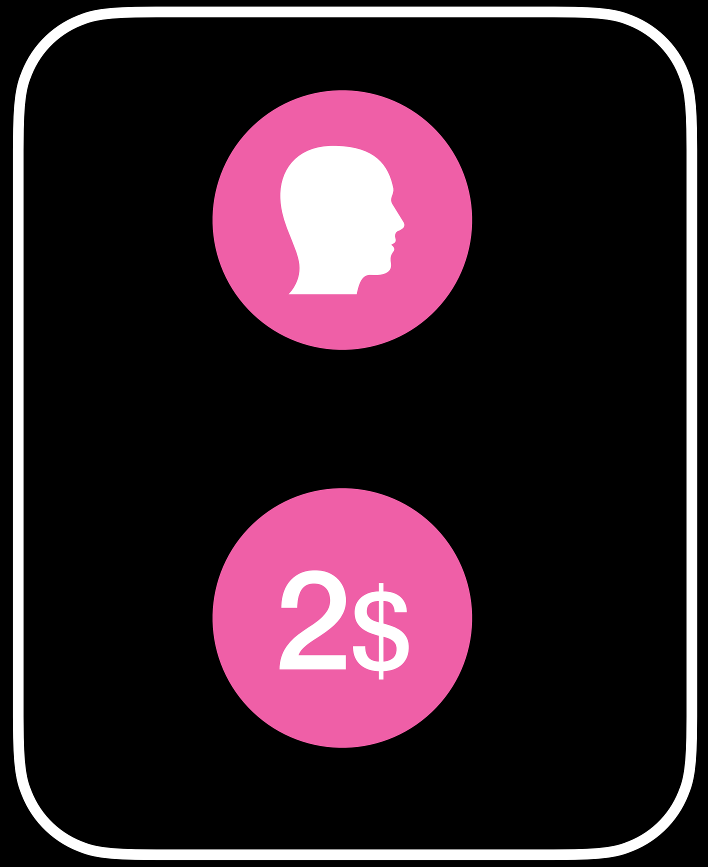
random variable

A random variable is a number associated to an event.

random variable

A random variable is a number associated to an event.

events



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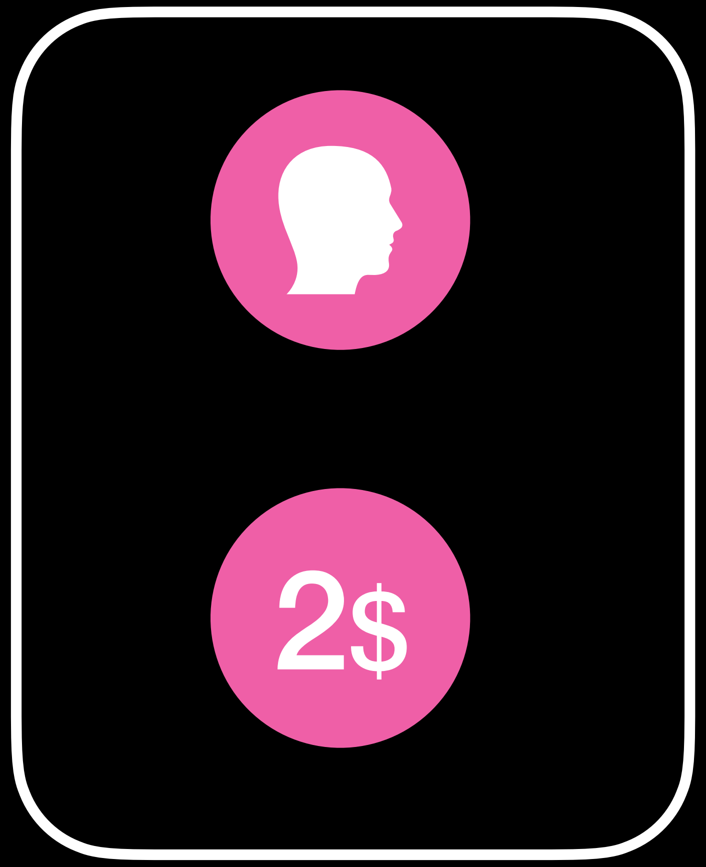

random variable

1

0

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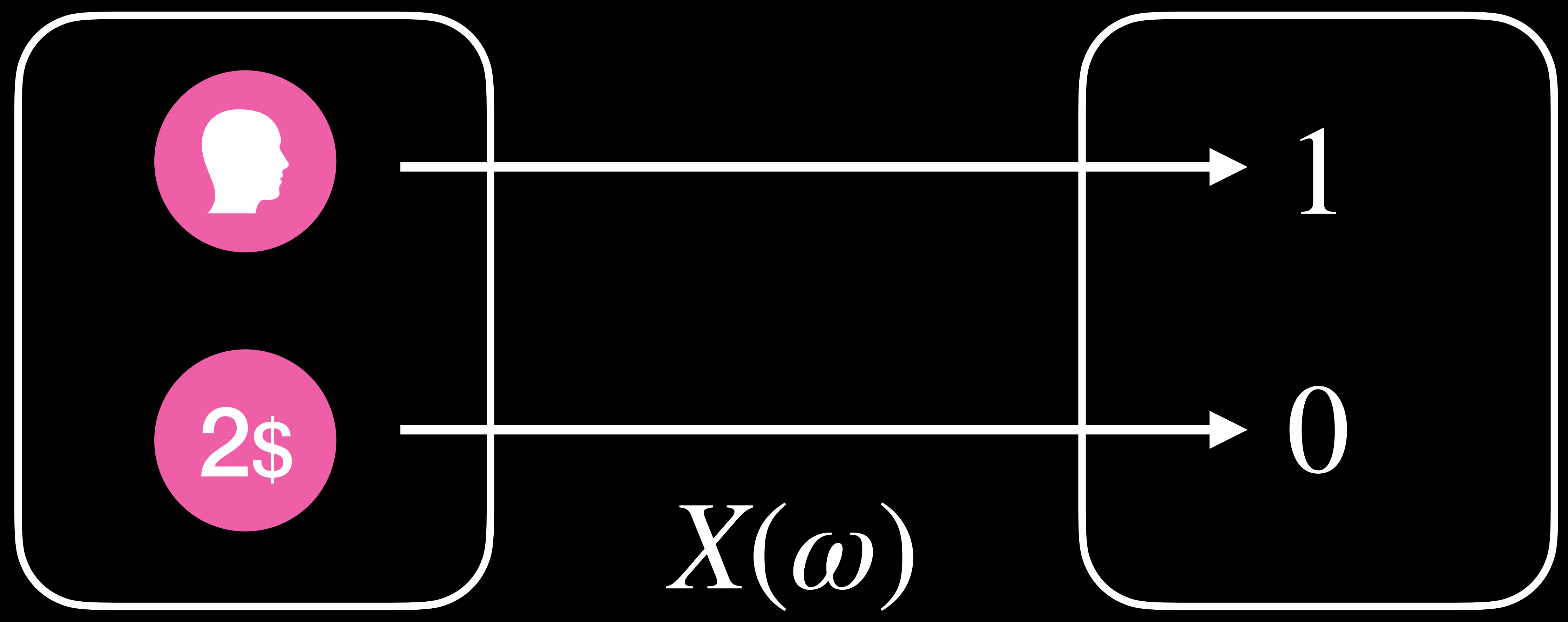

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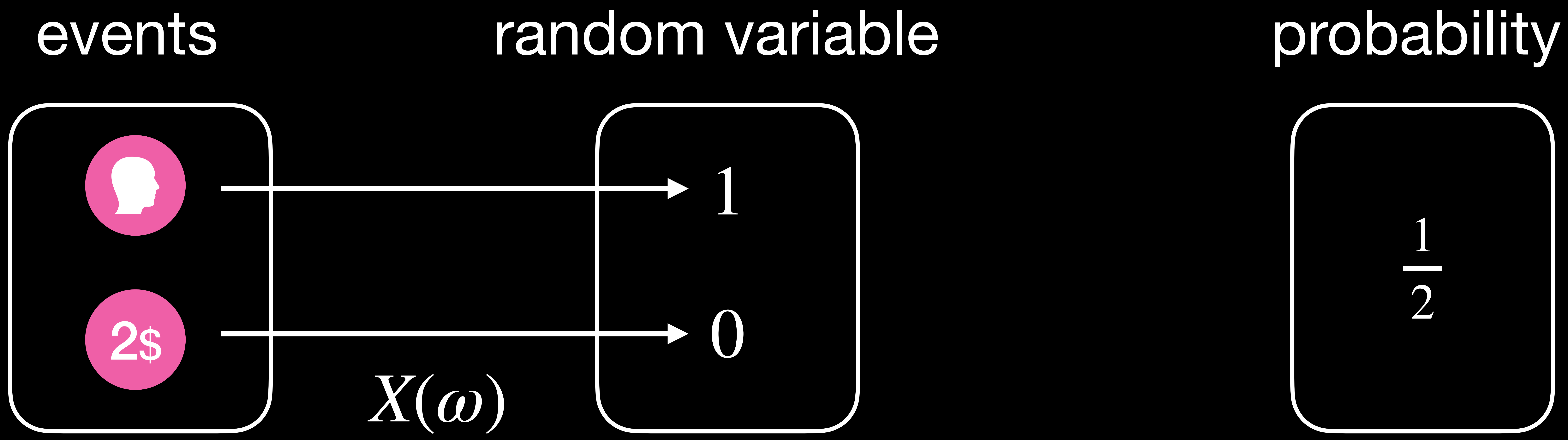

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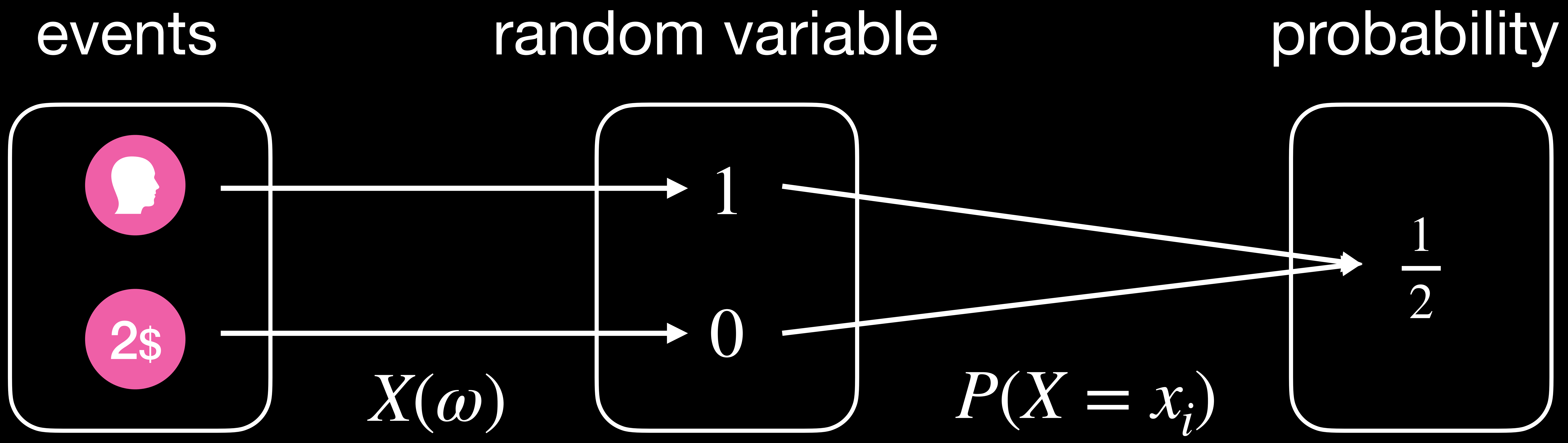

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binary random variable/ indicator variable/ Bernoulli random variables

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$$\mathbb{B} = \{0,1\}$$

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$$I(\omega) = \begin{cases} 1 & \text{if } \omega = \tilde{\omega} \\ 0 & \text{else} \end{cases}$$

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discreate random variable

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discreate random variable

$$S := \{x_1, x_2, x_3, \dots\}$$

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$S := \{x_1, x_2, x_3, \ldots\}$ finite or countable infinite

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finite or countable infinite

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$S := \{x_1, x_2, x_3, \ldots\}$ finite or countable infinite

$X : \Omega \to S \qquad X(\omega) = x_\omega \in S$

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$P(X = x_i) = ?$

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continuous random variables

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discreate random variable

$S := \{x_1, x_2, x_3, \dots\}$ $\quad X : \Omega \to S$ $\quad X(\omega) = x_\omega \in S$

continuous random variables

$J \subseteq \mathbb{R}$

Random variable

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discreate random variable

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continuous random variables

$$J \subseteq \mathbb{R}$$

$$X : \Omega \to J$$

Random variable

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continuous random variables

$J \subseteq \mathbb{R}$ $\quad\quad P(X = x_i) = 0$

$X : \Omega \to J$

Random variable

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continuous random variables

$J \subseteq \mathbb{R}$

$X : \Omega \to J$

$P(X = x_i) = 0$

$P(X > x_i) = ?$

end