

stochastics and probability

Lecture 3

Dr. Johannes Pahlke

accept-reject sampling

want:

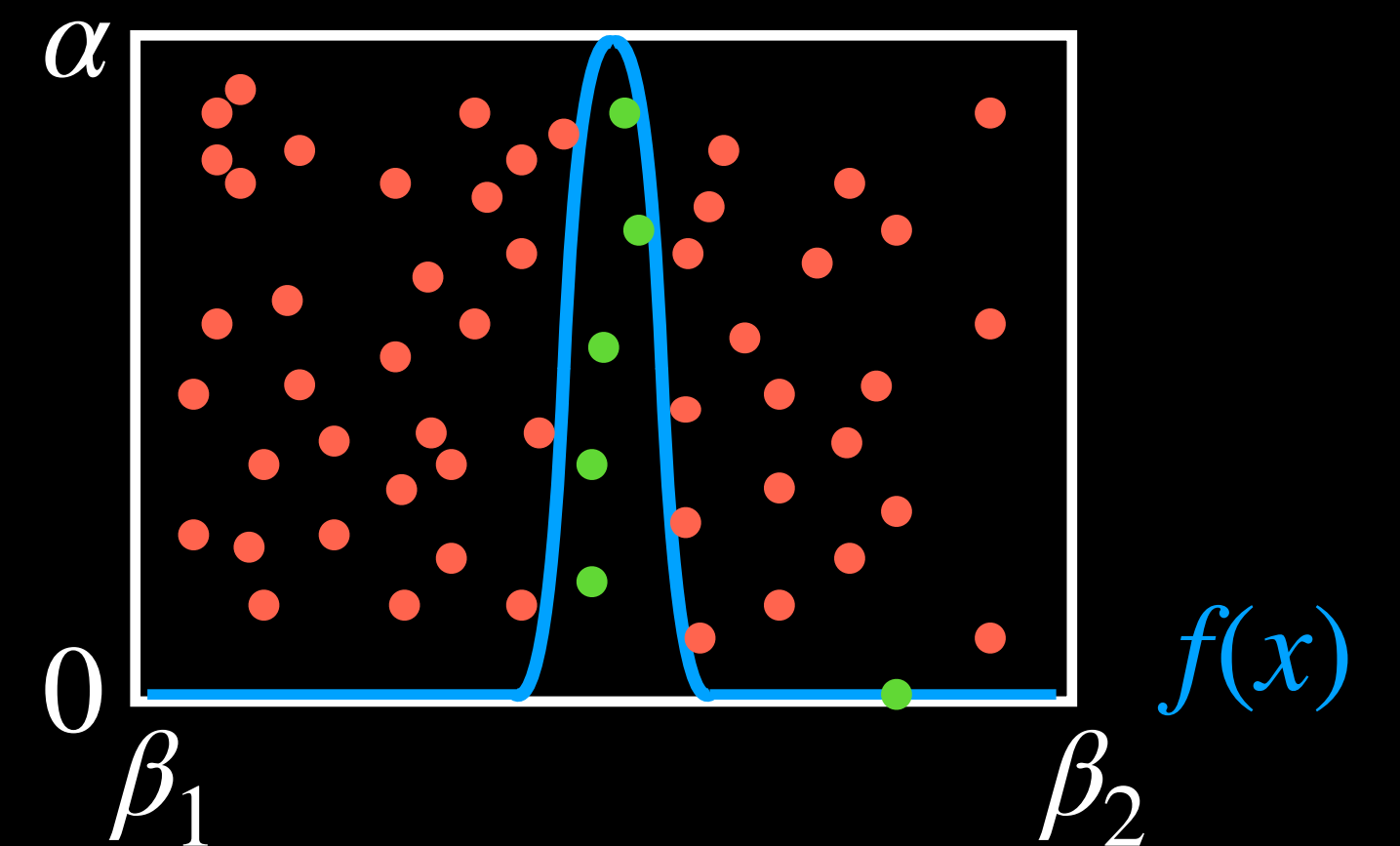
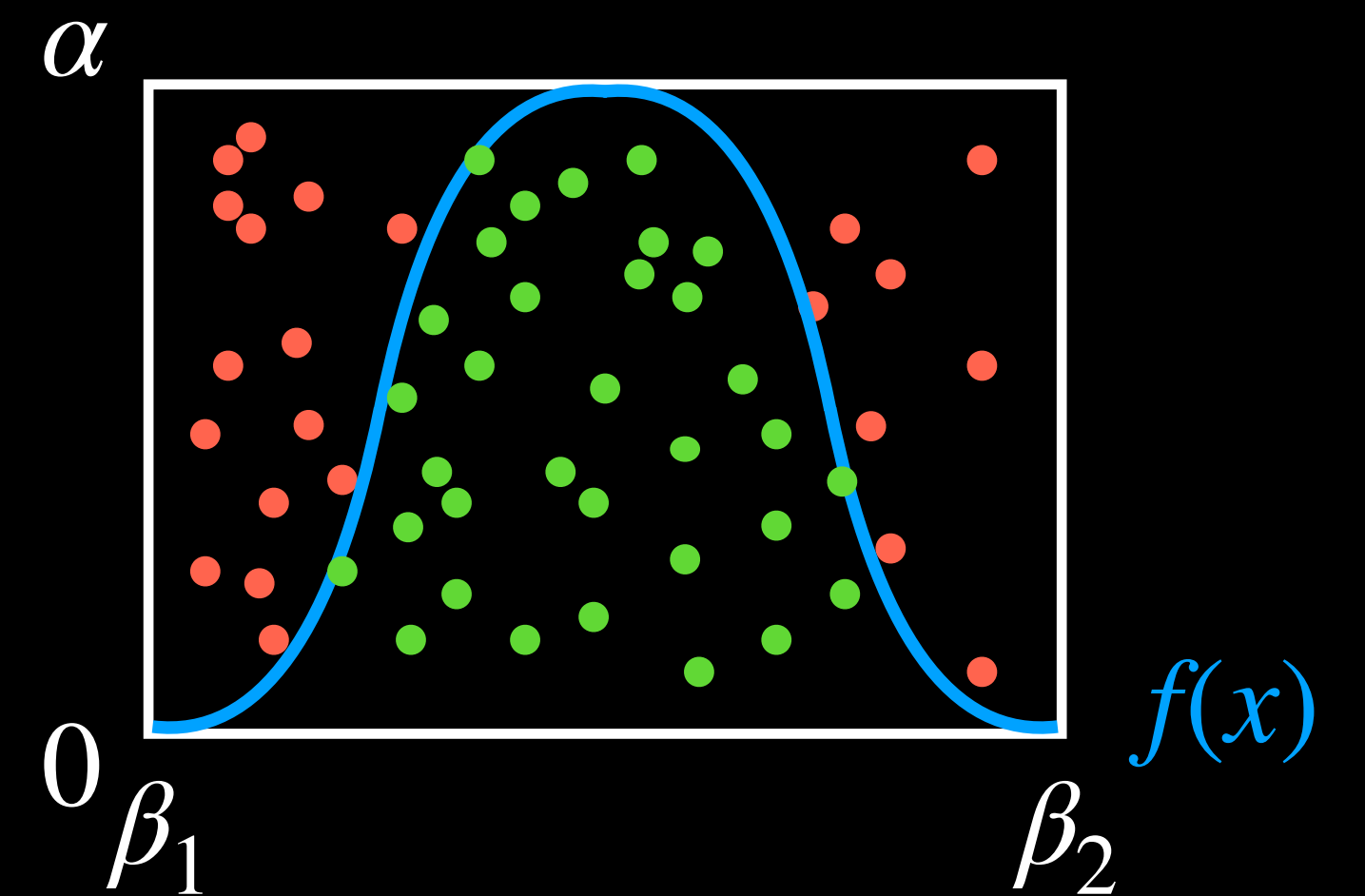
X with PDF $f_X(x)$

use:

$$f_X(x) \leq \alpha \quad \text{Range}(X) \subseteq [\beta_1, \beta_2]$$

algorithm:

- 1.) draw $U_1 \sim \mathcal{U}(\beta_1, \beta_2)$, $U_2 \sim \mathcal{U}(0, \alpha)$
- 2.) if $U_2 \leq f_X(U_1)$, then $X = U_1$, else go to 1.)



inefficient:

- peaky densities
- high dimensions
- long tails/infinite Range

accept-reject sampling

want:

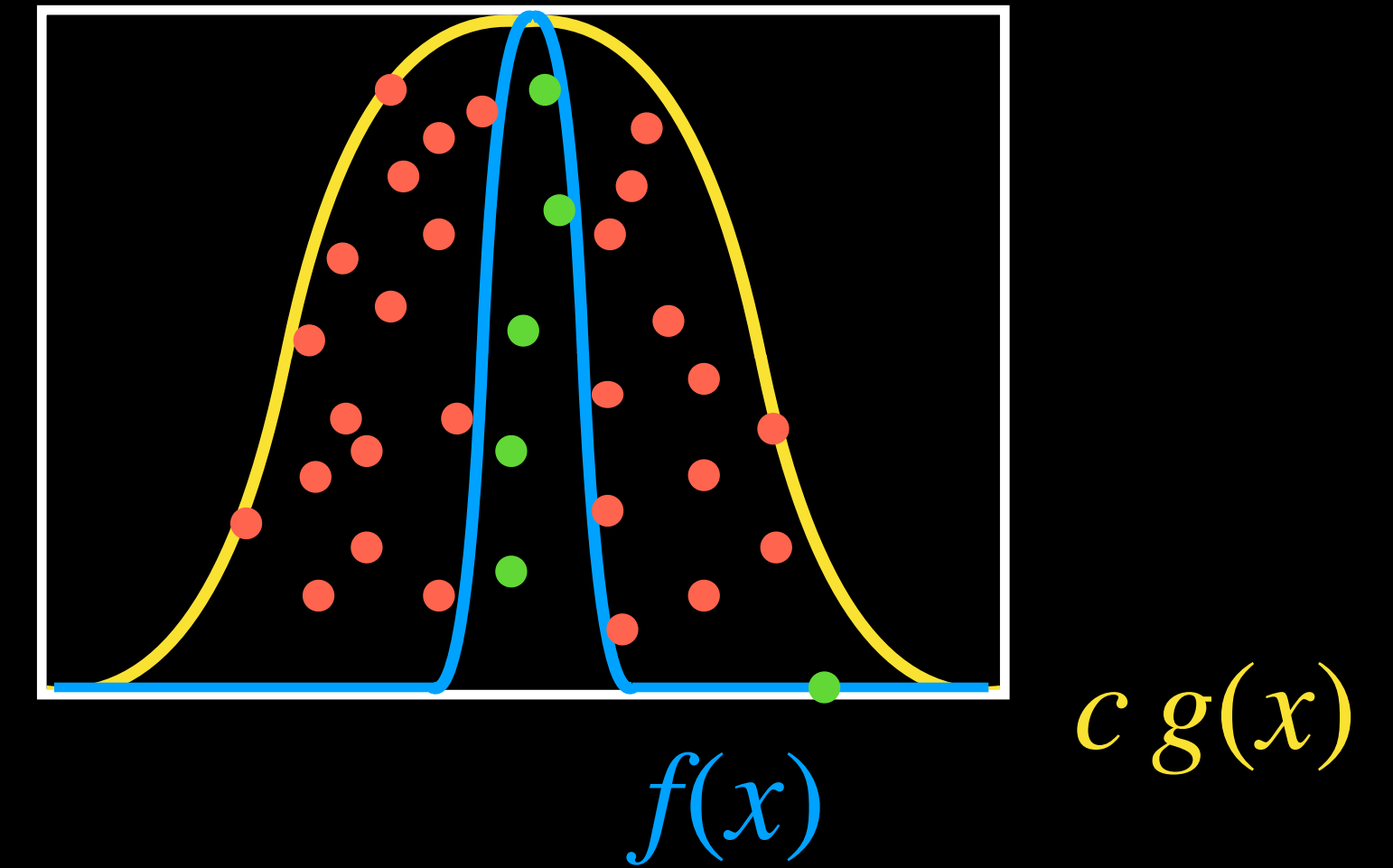
X with PDF $f_X(x)$

use:

$G \sim \mathcal{G}$ with PDF $g_G(x)$ and $f_X(x) \leq c g_G(x)$ for all x

algorithm:

- 1.) draw G from \mathcal{G} and U from $\mathcal{U}(0,1)$
- 2.) if $U \leq \frac{f_X(G)}{c g_G(G)}$, then $X = G$, else go to 1.)



discrete-time stochastic processes

$n \in \mathbb{N}$ discrete Time

X_0, X_1, X_2, \dots "sequence of random variables"

$\neq x_0, x_1, x_2, \dots$ "sequence of random numbers"

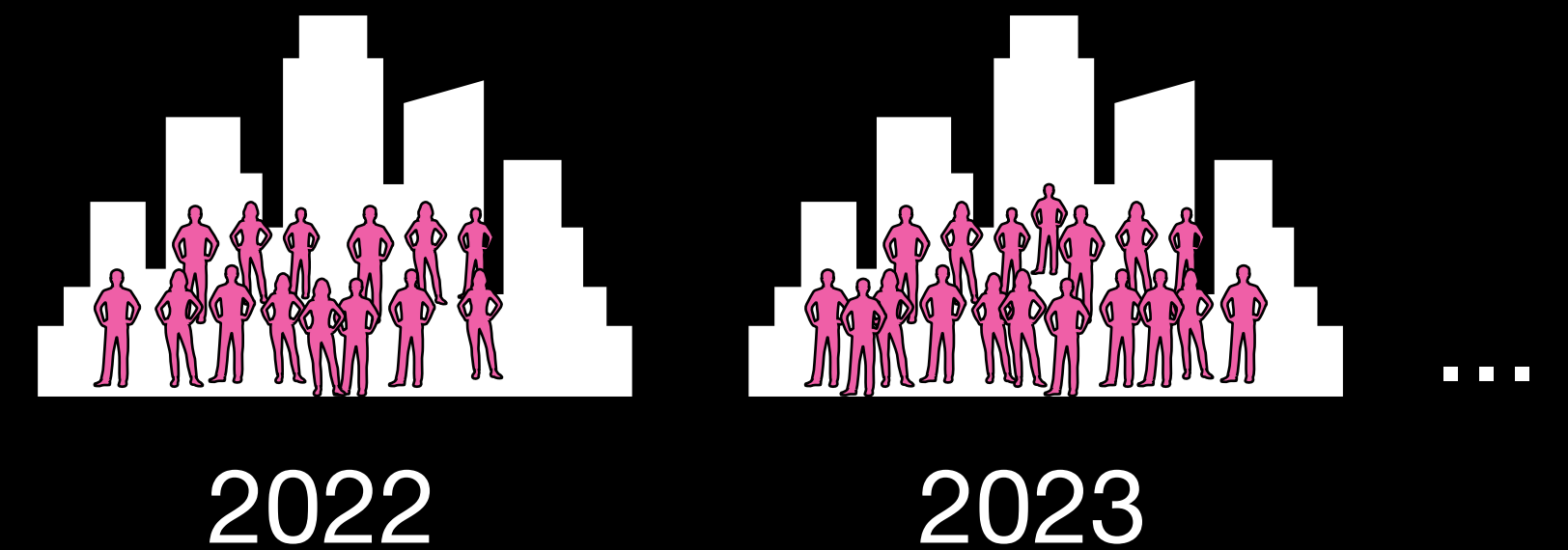
$\{X_n : n \in \mathbb{N}\}$ discrete-time stochastic process

x_n value of random variable X_n

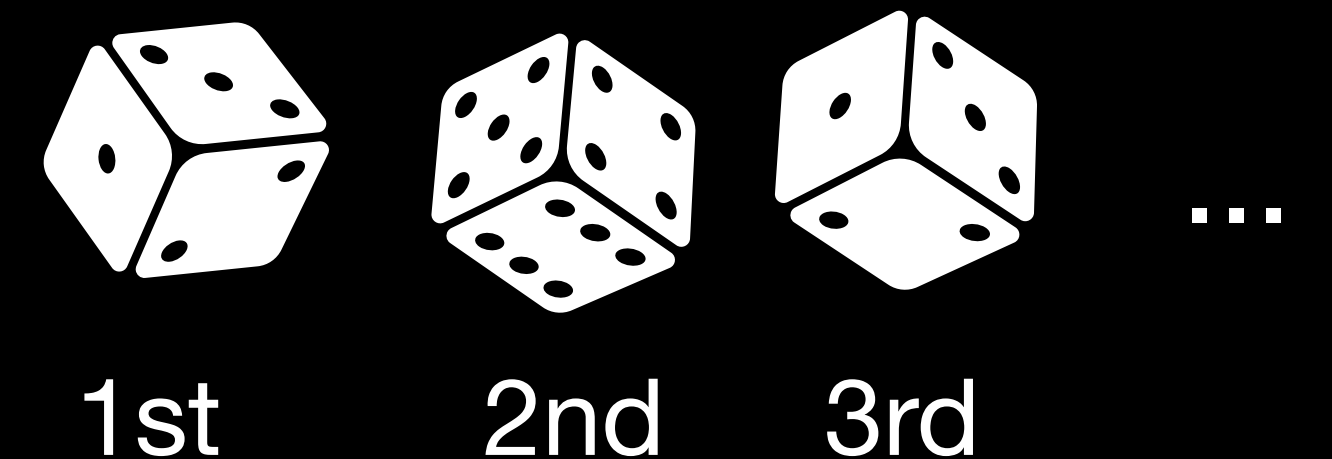
$X_n : \Omega \rightarrow \mathbb{S}$ state space \mathbb{S}
 $x_n \in \mathbb{S}$ (discrete or continuous)

Examples:

population



rolling dice



independent identical distributed (i.i.d.) process

$\{X_n : n \in \mathbb{N}\}$ stochastic process

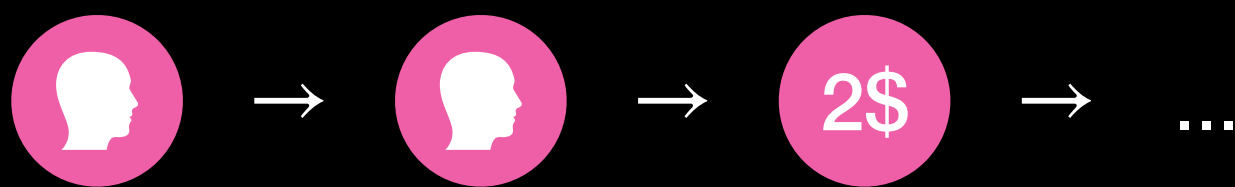
$F_{X_0}(x) = F_{X_1}(x) = F_{X_2}(x) = \dots$ identical distributed

$F_{X_0, \dots, X_n}(x_0, \dots, x_n) = F_{X_0}(x_0) \cdot \dots \cdot F_{X_n}(x_n)$ independent

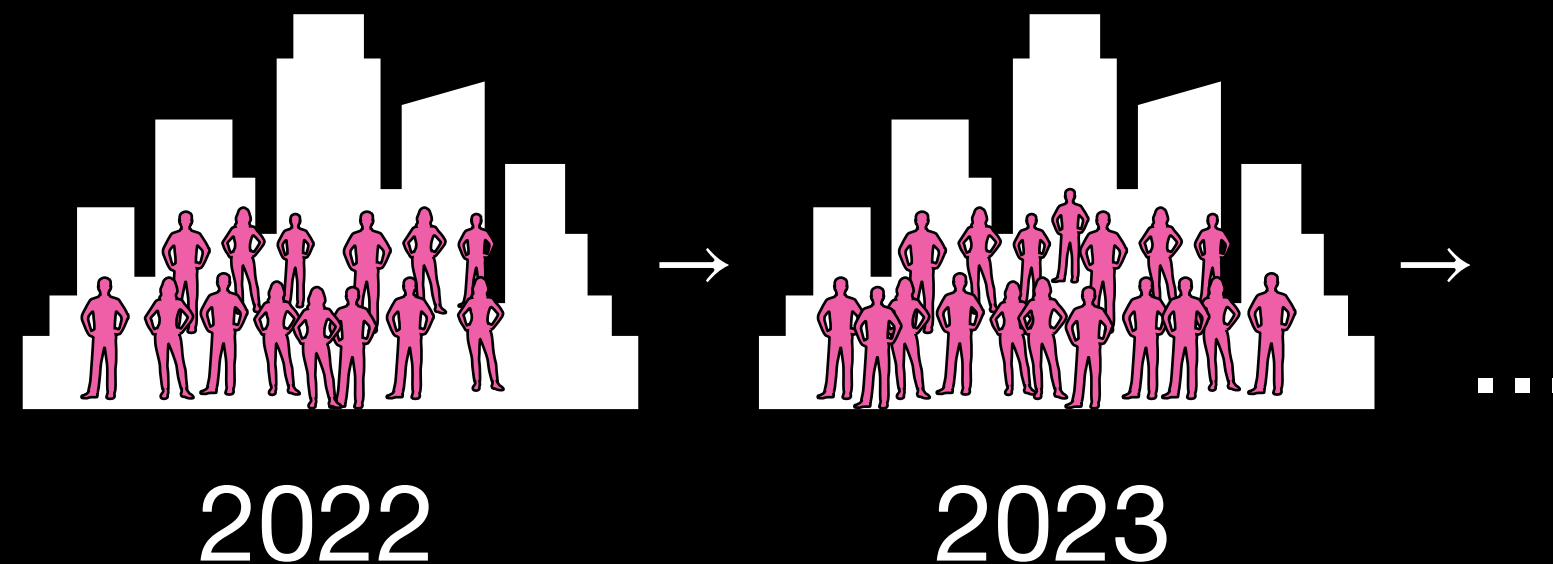
exercise:

Which system can be captured by an i.i.d. process?

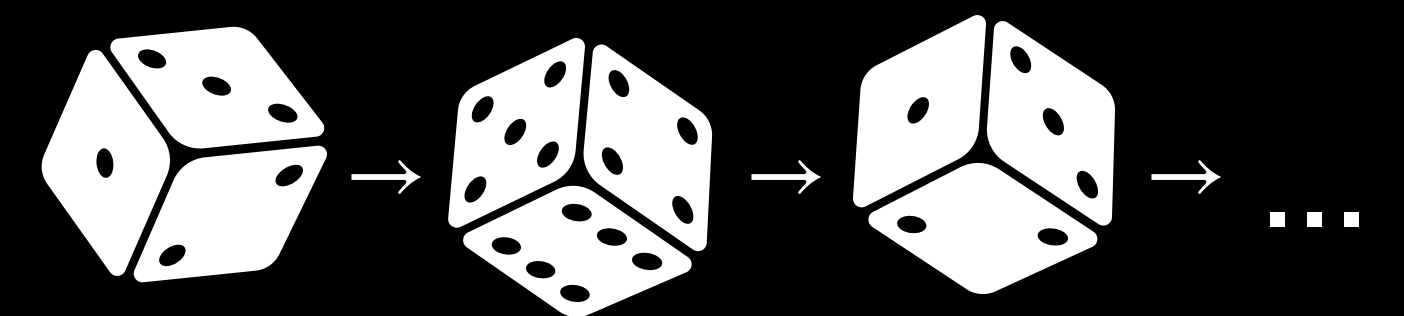
coin-flip



population development



rolling dice



Markov chains

$$\{X_n : n \in \mathbb{N}\}$$

stochastic process

$$\begin{aligned} P(X_{n+1} = x_j | X_n = x_i, X_{n-1} = x_{i-1}, \dots, X_0 = x_0) \\ = P(X_{n+1} = x_j | X_n = x_i) =: P_{ij} \end{aligned}$$

↑
(one-step transition probability)

Markov property
(X_{n+1} only depends on X_n)

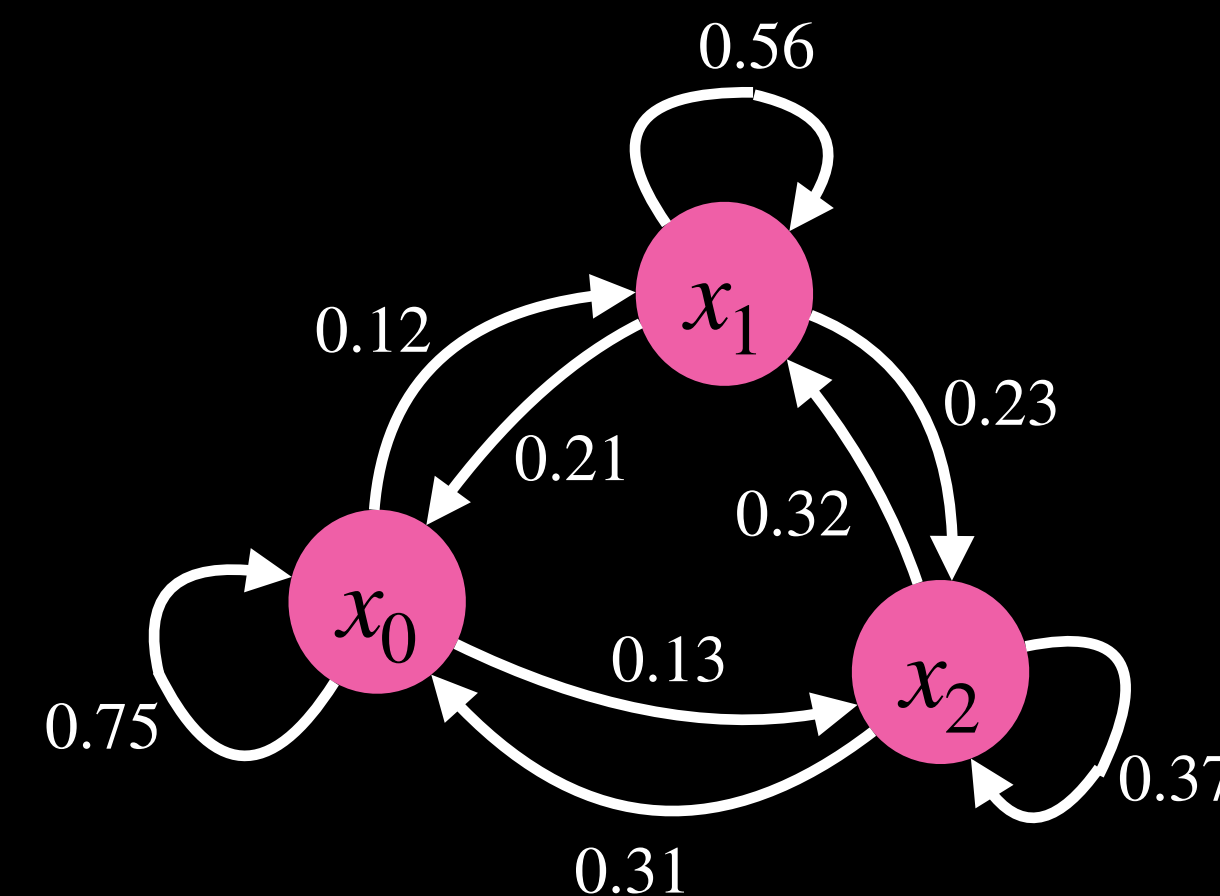
$$P_{ij}^{(n)} := P(X_{n+m} = x_j | X_m = x_i)$$

(n -step transition probability)

$$\underline{P} := (P_{ij}) = \begin{pmatrix} P_{00} & \dots & P_{0n} \\ \vdots & \ddots & \vdots \\ P_{n0} & \dots & P_{nn} \end{pmatrix} \text{ for } |\mathcal{S}| < \infty$$

(one-step transition matrix)

example:



$$\underline{P} = \begin{pmatrix} 0.75 & 0.12 & 0.13 \\ 0.21 & 0.56 & 0.23 \\ 0.31 & 0.32 & 0.37 \end{pmatrix}$$

Markov chains as recursions

theorem 1:

Let $g(x, u) \in \mathbb{R}$ be a function and $\{U_n : n \in \mathbb{N}\}$ an i.i.d. process,
then the recursion $X_{n+1} = g(X_n, U_n)$ is a Markov chain.

theorem 2:

Every Markov chain can be represented as recursion $X_{n+1} = g(X_n, U_n)$
where $\{U_n : n \in \mathbb{N}\}$ is an i.i.d. process with $U_n \sim \mathcal{U}(0,1)$.

theorem 3:

Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ and its one-step transition matrix $\underline{\underline{P}}$,
then its n -step transition matrix $\underline{\underline{P}}^{(n)} = \underline{\underline{P}}^n = \underline{\underline{P}} \cdot \dots \cdot \underline{\underline{P}}$.

properties of Markov chains

def.: Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \rightarrow \mathbb{S}$.

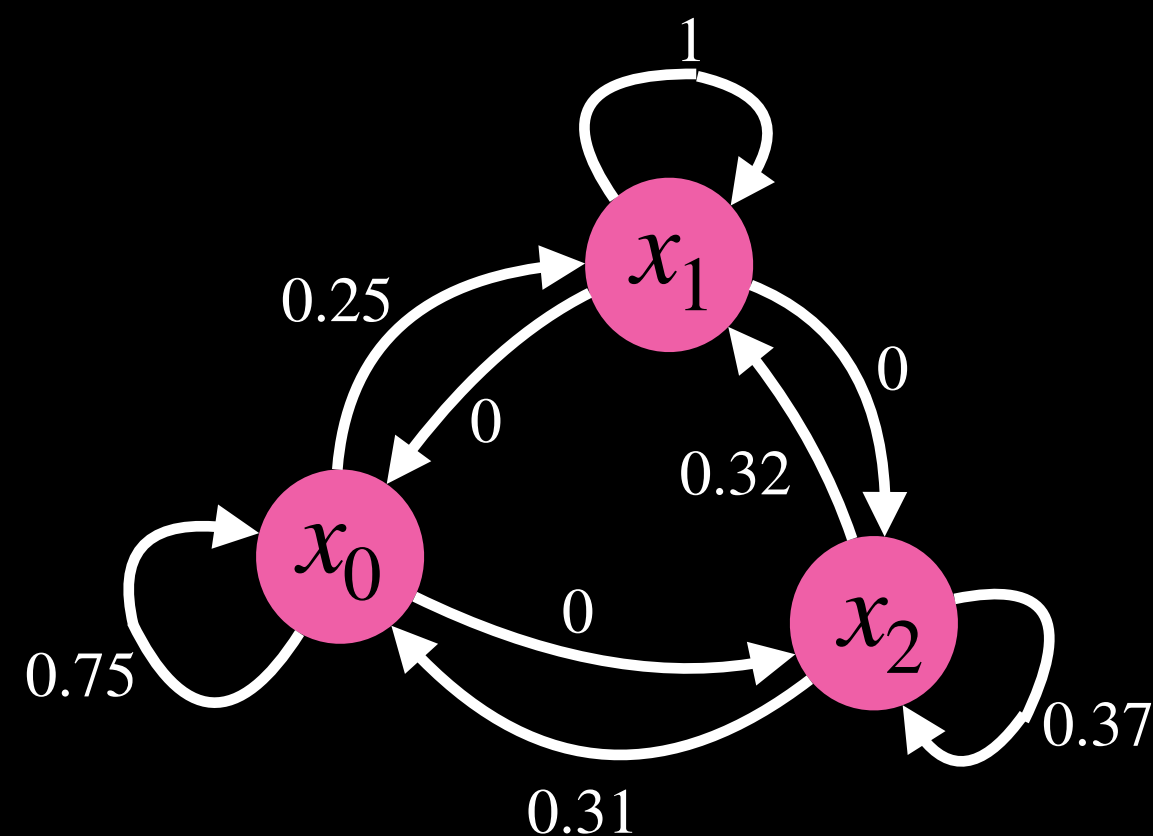
Let $\mathbb{C} \subseteq \mathbb{S}$, then \mathbb{C} is **closed** iff $\forall x_i \in \mathbb{C} \forall x_j \in \mathbb{S} \setminus \mathbb{C} : P_{ij} = 0$.

(from a state of \mathbb{C} no state outside of \mathbb{C} can be reached)

If $\{x_i\} = \mathbb{C}$, then x_i is **absorbing state**.
 $\longrightarrow P_{ii} = 1$

exercise:

consier:



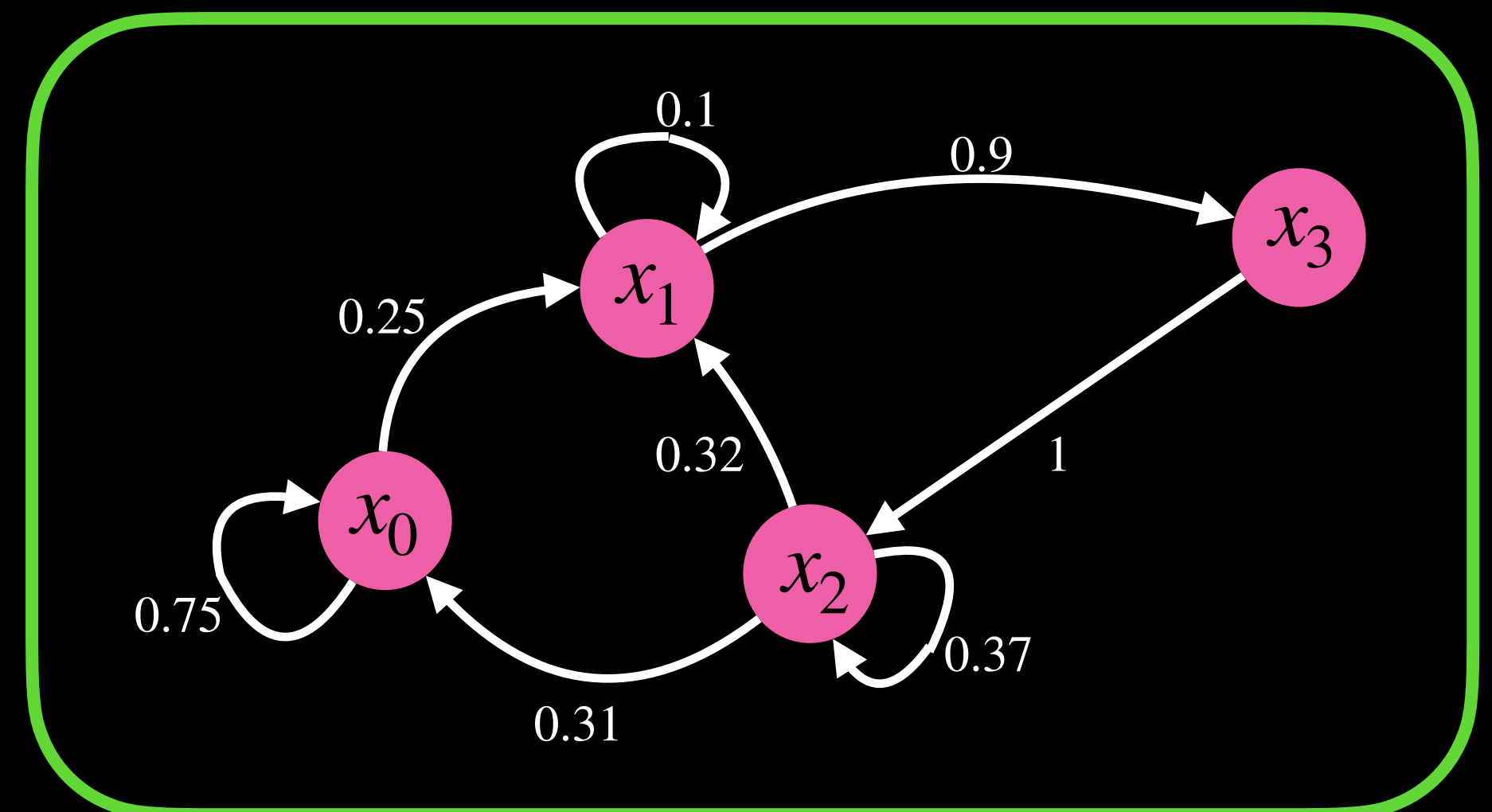
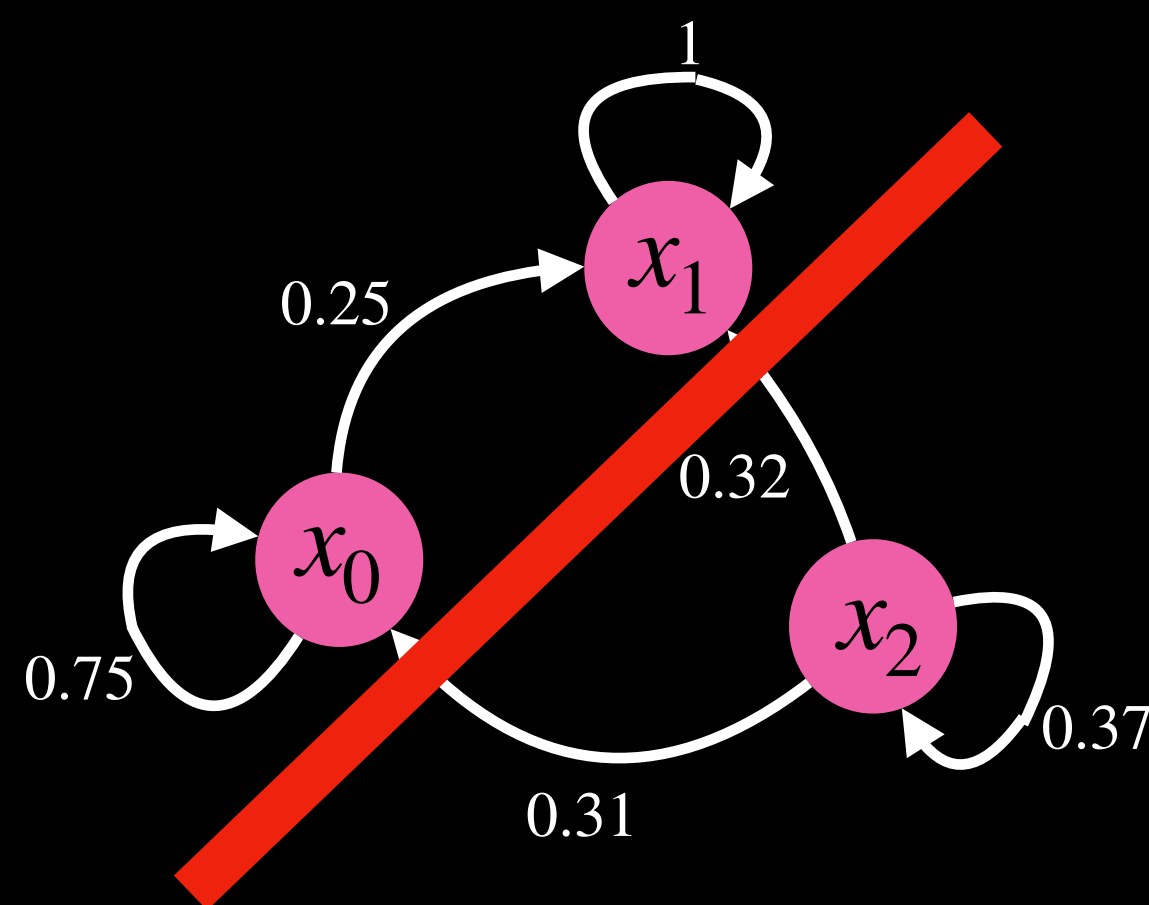
What are the closed sets? $\mathbb{C}' = \{x_0, x_1\}$, $\mathbb{C}'' = \{x_1\}$, $\mathbb{C}''' = \mathbb{S}$

What are the absorbing states? x_1

properties of Markov chains

def.: A Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \rightarrow \mathbb{S}$
is **irreducible** iff $\nexists \mathbb{C} \subset \mathbb{S} : \mathbb{C} \text{ is closed}$ (\mathbb{S} is the only closed set)

exercise: Which Markov chain is irreducible?



properties of Markov chains

def.: Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \rightarrow \mathbb{S}$.

Be $N_i := \{n \in \mathbb{N}_{>0} : P_{ii}^{(n)} > 0\}$ set of number of transitions to reach x_i from x_i ,

then $d_i := \gcd(N_i)$ **period** of x_i .

(greatest common divisor)

exercise:

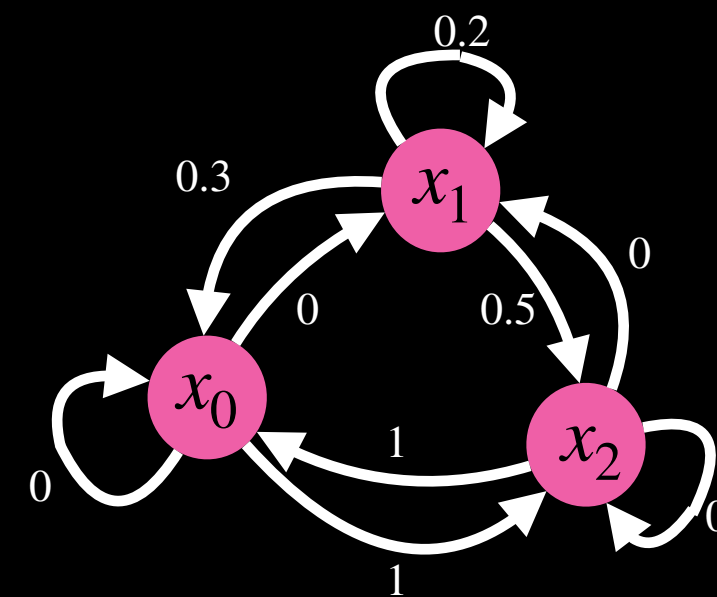
Which states/ Markov chains are periodic?

If $1 < d_i < \infty$, then x_i is **periodic**,
if $d_i = 1$, then x_i is **aperiodic**.

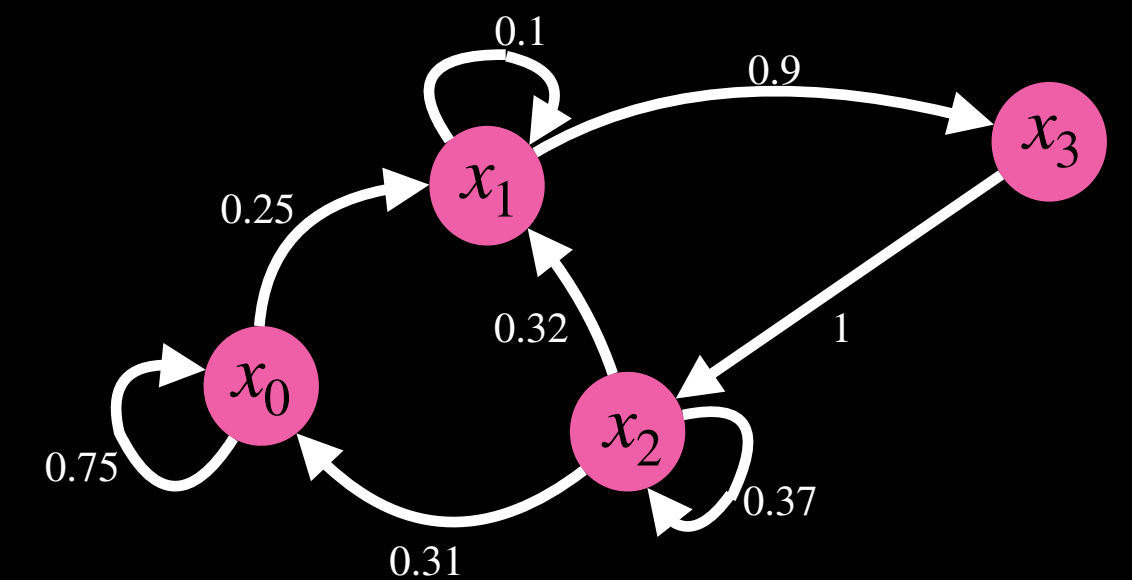
If $\forall x_i, x_j \in \mathbb{S} : 1 < d_i = d_j < \infty$,
then $\{X_n : n \in \mathbb{N}\}$ is **periodic**,

(The Markov chain is **periodic** if all states are periodic with the same period.)

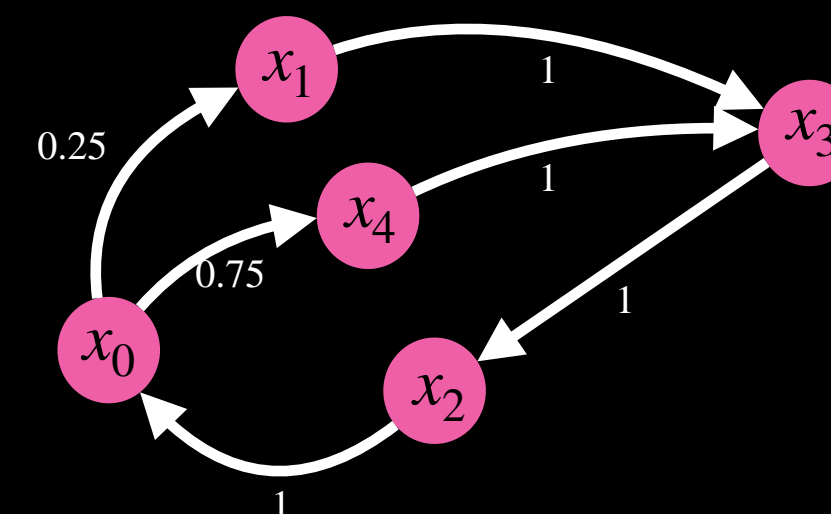
if $\forall x_i \in \mathbb{S} : 1 = d_i$, then $\{X_n : n \in \mathbb{N}\}$ is **aperiodic**.
(The Markov chain is **aperiodic** if all states are aperiodic.)



x_0, x_2 are periodic
 $d_i = 2$
 x_1 is aperiodic



all x_i are aperiodic



all x_i are periodic
 $d_i = 4$

properties of Markov chains

def.: Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \rightarrow \mathbb{S}$.

If $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$ then x_i is **transient**, (A state is **transient** iff there is a non-zero probability that it is never reached again and **recurrent** otherwise.)

if $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$ then x_i is **recurrent**.

The Markov chain $\{X_n : n \in \mathbb{N}\}$ is **ergodic**, iff it is irreducible and aperiodic.

properties of Markov chains

def.: Given a Markov chain $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \rightarrow \mathbb{S}$.

limiting distribution: $P_k := \lim_{n \rightarrow \infty} P_{jk}^{(n)} \quad \forall x_j \in \mathbb{S}$

$$\sum_{x_i \in \mathbb{S}} P_i = 1$$

theorem 4:

A Markov chain has a limiting distribution iff it is ergodic.

example:

Every i.i.d. process is a Markov chain with $g(x, U) := U \quad U \sim \mathcal{U}(0,1)$

and the limiting distribution is $\mathcal{U}(0,1)$.

Markov chains for random number generation

Markov chain

$$\{X_n : n \in \mathbb{N}\} \quad X_n : \Omega \rightarrow \mathbb{S}$$

limiting distribution

$$P_k := \lim_{n \rightarrow \infty} P_{jk}^{(n)} \quad \forall x_j \in \mathbb{S}$$

praxis: ~~$n \rightarrow \infty$~~ $n \gg 1$ \rightarrow "mixing time"

(In praxis, statistical tests decide if the limiting distribution is reached.)

end