stochastics and probability

Lecture 4

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moments

def.: Be $X : \Omega \to \mathbb{S}$ a discrete random variable and $p_X(x)$ its probability mass function (PMF).

$$\mu := \mathbb{E}[X] := \sum_{x_i \in \mathbb{S}} x_i\, p_X(x_i)$$ expectation/ expected value $$\left[\quad \mathbb{E}[g(X)] := \sum_{x_i \in \mathbb{S}} g(x_i)\, p_X(x_i) \quad\right]$$

exercise: $x \in \mathbb{S} := (1,2,\ldots,7)$, $p_X(x) := \frac{1}{7}$, $g(x) := \begin{cases} 0 & \text{if } x \in \{1,\ldots,5\} \\ 1 & \text{if } x \in \{6,7\} \end{cases}$

$$\mathbb{E}[X] = 1 \cdot p_X(1) + \ldots + 7 \cdot p_X(7) = (1 + \ldots + 7) \cdot \frac{1}{7} = \frac{28}{7} = 4$$

$$\mathbb{E}[g(X)] = g(1) \cdot p_X(1) + \ldots + g(7) \cdot p_X(7) = (0 + 0 + 0 + 0 + 0 + 1 + 1) \cdot \frac{1}{7} = \frac{2}{7}$$

Be $X : \Omega \to \mathbb{S}$ a continuous random variable and $f_X(x)$ its probailty density function (PDF).

$$\mu := \mathbb{E}[X] := \int_{\mathbb{S}} x\, f_X(x)\, dx$$ expectation/ expected value $$\left[\quad \mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x)\, f_X(x)\, dx \quad\right]$$

$$\mu_k := \mathbb{E}\left[(X - \mu)^k\right]$$ higher-order central moments

$$Var(X) := \mu_2 = \mathbb{E}\left[(X - \mu)^2\right] = \mathbb{E}\left[X^2 - 2\,\mu X + \mu^2\right] = \mathbb{E}\left[X^2\right] - \mu^2$$ variance (2nd central moments)

laws of large numbers

def.: Given an a stochastic process $\{X_n : n \in \mathbb{N}\}$ with $X_n : \Omega \to \mathbb{S}$.

Be $\mu = \mathbb{E}[X_0] = \mathbb{E}[X_1] = \ldots.$ and $\bar{X}_n := \frac{1}{n}\sum_{i=0}^{n-1} X_i$.

$\{X_n : n \in \mathbb{N}\}$ follows the weak law of large numbers iff, $\lim_{n\to\infty} P\left(-\epsilon < \left(\bar{X}_n - \mu\right) < \epsilon\right) = 1 \quad \forall \epsilon > 0.$

(WLLN)

(convergence of probabilities)

$\{X_n : n \in \mathbb{N}\}$ follows the strong law of large numbers iff, $P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1.$

(SLLN)

(behavior at the limit)

theorem:

The strong law of large numbers implies the weak law of large numbers. (SLLN $\to$ WLLN)

theorem:

If $\{X_n : n \in \mathbb{N}\}$ is an i.i.d. process and $\mathbb{E}(\bar{X}_n)$ exists then strong law of large numbers holds.

Monte Carlo integration (intuition)

$$\int_a^b g(x)\,dx$$

$$= \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} g(x_i)\,\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

$$= \lim_{n \to \infty} \sum_{i=0}^{n-1} g(x_i)\,\frac{b-a}{n}$$

$$= (b-a)\,\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(x_i)$$

$$= (b-a)\,\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(X_i)$$

$$\approx (b-a)\,\frac{1}{n} \sum_{i=0}^{n-1} g(X_i)$$

$$\bar{Y}_n := \frac{1}{n}\sum_{i=1}^{n} Y_i \qquad Y := g(X) \qquad \mu = \mathbb{E}[Y] = \mathbb{E}[g(X)] \qquad X \sim \mathcal{U}(a,b)$$

(SLLN) $$1 = P\left(\lim_{n\to\infty} \bar{Y}_n = \mu\right) = P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} Y_i = \mu\right) = P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} g(X_i) = \mathbb{E}[g(X)]\right)$$

Monte Carlo integration (derivation)

$$\int_a^b g(x)\,dx \;=\; \int_a^b g(x)\,1\,dx \;=\; \int_a^b g(x)\,\frac{b-a}{b-a}\,dx \;=\; (b-a)\int_a^b g(x)\,\overbrace{\frac{1}{b-a}}^{=f_X(x)}\,dx \qquad X \sim \mathcal{U}(a,b)$$

$$\left[\quad \mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x) f_X(x)\,dx \quad\right]$$

$$= (b-a)\,\mathbb{E}[g(X)]$$

SLLN: $P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} g(X_i) = \mathbb{E}[g(X)]\right) = 1$

$$= (b-a) \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} g(X_i)$$

almost always

$$\approx (b-a)\,\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) \quad =: \vartheta_n$$

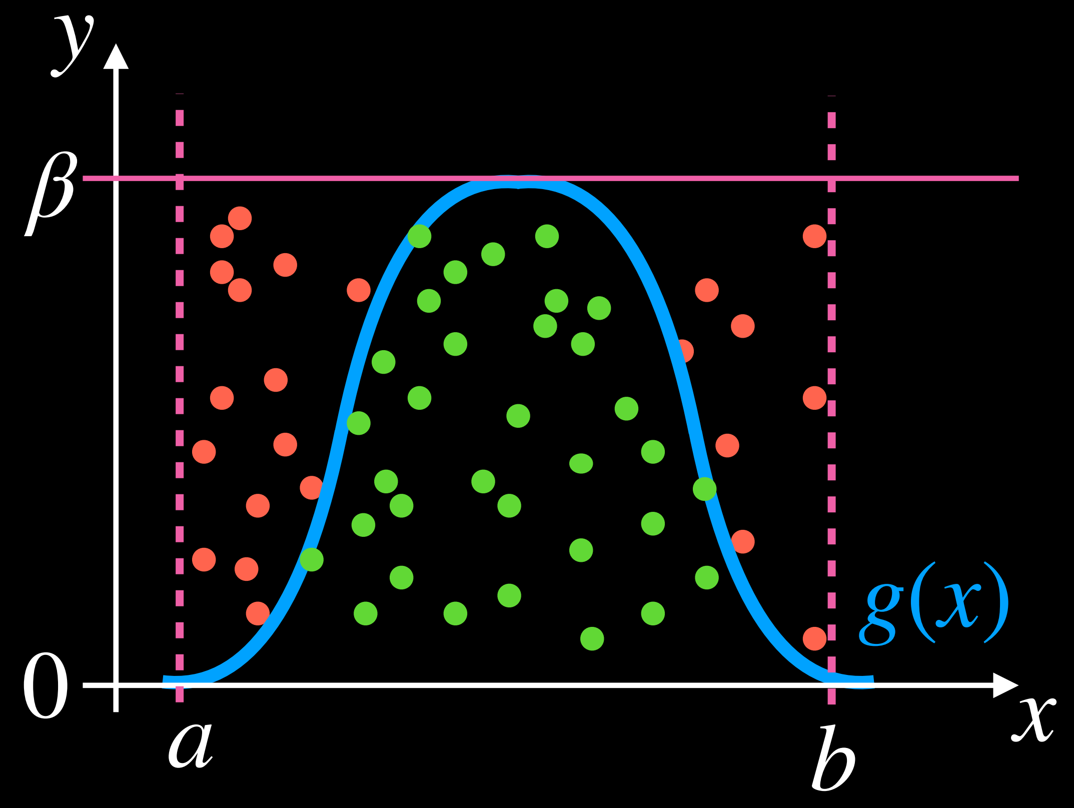
"Monte Carlo estimator"

multi-dimensional

(volume)

$$\int_{\mathbb{S}} g(\underline{x})\,d\underline{x} \approx V\,\frac{1}{n}\sum_{i=0}^{n-1} g(\underline{X}_i) \quad =: \vartheta_n$$

Monte Carlo integration (alternative like accept-reject sampling)



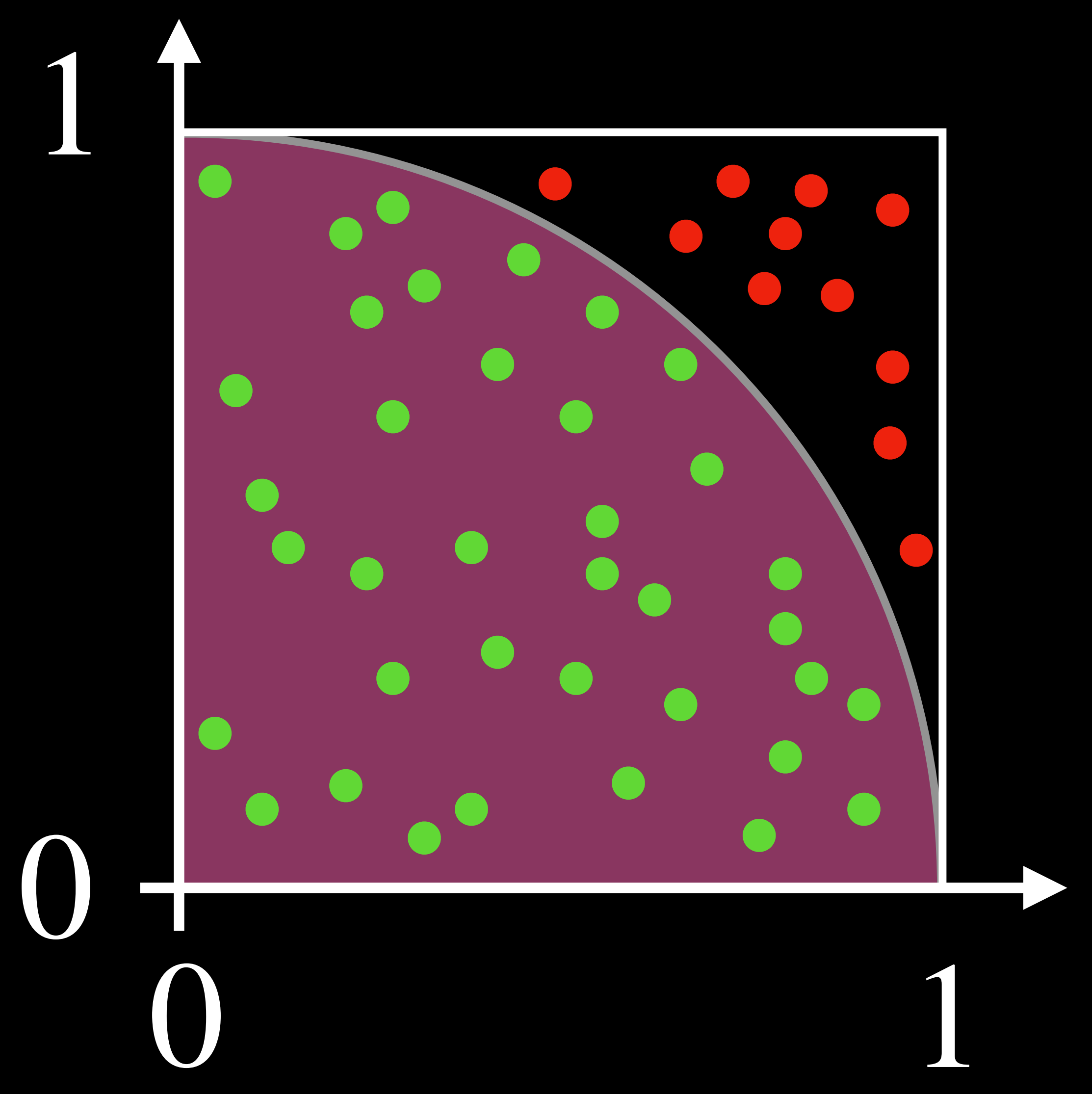
$$x \in [a,b] \qquad g(x) \in [0,\beta]$$

$$\mathbb{1}_g(x,y) := \begin{cases} 1 & \text{if } y \le g(x) \\ 0 & \text{else} \end{cases}$$

$$\int_a^b g(x)\,dx = \int_0^\beta \int_a^b \mathbb{1}_g(x,y)\,dx\,dy = \int_0^\beta \int_a^b \mathbb{1}_g(x,y)\,1\,dx\,dy = \int_0^\beta \int_a^b \mathbb{1}_g(x,y)\,\frac{(\beta-0)(b-a)}{(\beta-0)(b-a)}\,dx\,dy$$

$$= \beta(b-a)\int_0^\beta \int_a^b \mathbb{1}_g(x,y)\,\underbrace{\frac{1}{\beta(b-a)}}_{=f_{XY}(x,y)}\,dx\,dy = \beta(b-a)\;\mathbb{E}[\mathbb{1}_g(X,Y)] \approx \beta(b-a)\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{1}_g(X_i,Y_i) \qquad \begin{matrix} X_i \sim \mathcal{U}(a,b) \\ Y_i \sim \mathcal{U}(0,\beta) \end{matrix}$$

example:



$$x \in [0,1] \qquad g(x) \in [0,1]$$

$$\mathbb{1}_g(x,y) := \begin{cases} 1 & \text{if } x^2+y^2 \le 1 \\ 0 & \text{else} \end{cases}$$

$$Y_i, X_i \sim \mathcal{U}(0,1)$$

$$\int_0^1 g(x)\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1}\mathbb{1}_g(X_i,Y_i) \approx \frac{\pi}{4}$$

Monte Carlo integration (error)

$$error_{\vartheta_n} \approx \sqrt{Var(\vartheta_n)} \quad = \sqrt{Var\left((b-a)\frac{1}{n}\sum_{i=0}^{n-1} g(X_i)\right)}$$

$$= \frac{b-a}{n}\sqrt{Var\left(\sum_{i=0}^{n-1} g(X_i)\right)}$$

X_i i.i.d.

$$= \frac{b-a}{n}\sqrt{\sum_{i=0}^{n-1} Var\left(g(X_i)\right)}$$

$$= \frac{b-a}{n}\sqrt{n\ Var\left(g(X)\right)}$$

$$= \frac{b-a}{\sqrt{n}}\sqrt{Var\left(g(X)\right)} \quad \propto \frac{1}{\sqrt{n}}$$

higher dimension error

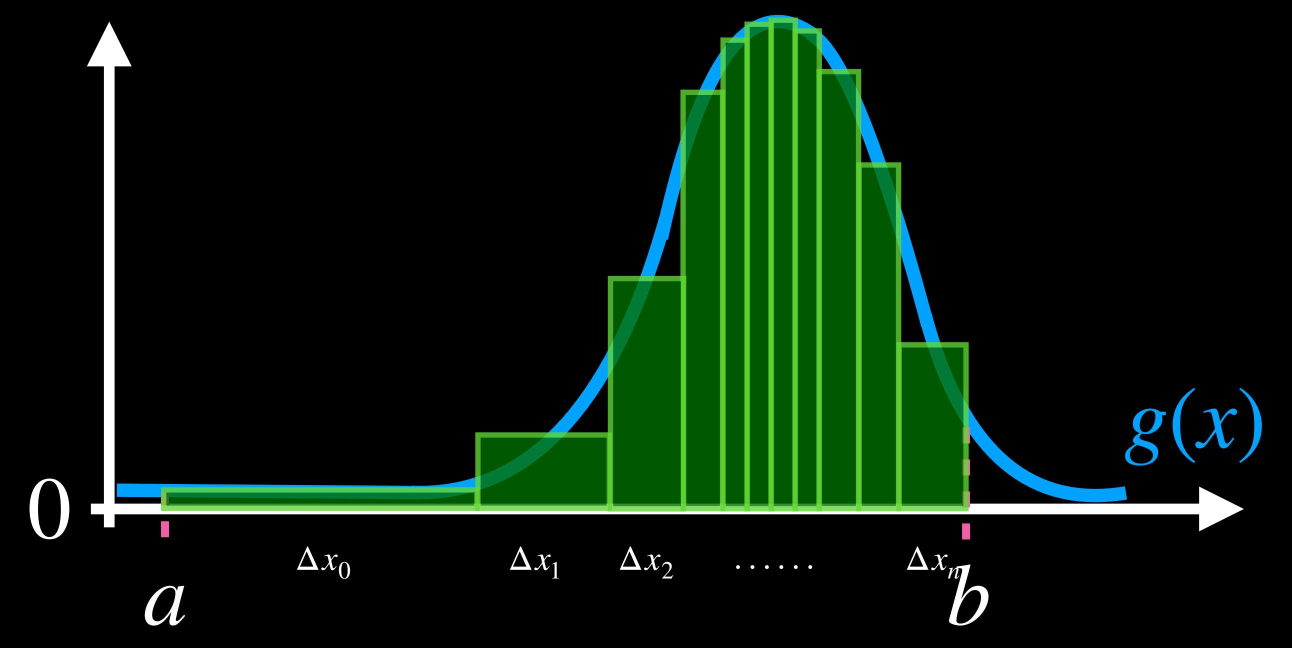
(volume)

$$error_{\vartheta_n} \approx \frac{V}{\sqrt{n}}\sqrt{Var\left(g(X)\right)}$$

Monte Carlo integration (importance sampling)

intuition:

$$\int_a^b g(x)\,dx \approx \sum_{i=0}^{n-1} g(x_i)\,\Delta x_i = \sum_{i=0}^{n-1} g(x_i)\frac{1}{n\cdot\underbrace{\frac{importance(x_i)}{b-a}}_{=f_X(x_i)}} = \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(x_i)}{f_X(x_i)} \approx \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(X_i)}{f_X(X_i)}$$



$$\Delta x = \frac{b-a}{n} \qquad importance(x_i) := \frac{\Delta x}{\Delta x_i} = \frac{b-a}{n\cdot\Delta x_i}$$

$\cancel{X \sim \mathcal{U}(a,b)}$

X distributed with PDF f_X

derivation:

$$\int_{\mathbb{S}} g(x)\,dx = \int_{\mathbb{S}} g(x)\,1\,dx = \int_{\mathbb{S}} g(x)\frac{f_X(x)}{f_X(x)}\,dx = \int_{\mathbb{S}} \frac{g(x)}{f_X(x)} f_X(x)\,dx = \mathbb{E}\left[\frac{g(X)}{f_X(X)}\right]$$

$$\underset{\text{almost always}}{=} \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\frac{g(X_i)}{f_X(X_i)} \approx \frac{1}{n}\sum_{i=0}^{n-1}\frac{g(X_i)}{f_X(X_i)}$$

$$error_{\vartheta_n} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(\frac{g(X)}{f_X(X)}\right)}$$

expectation

$$\mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x) f_X(x)\,dx$$

SLLN

$$P\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) = \mathbb{E}[g(X)]\right) = 1$$

choice of $f_X(x)$:

$$f_X(x) = g(x)\cdot k \qquad 1 = \int_{\mathbb{S}} f_X(x)\,dx = \int_{\mathbb{S}} g(x)\cdot k\,dx \qquad \frac{1}{k} = \int_{\mathbb{S}} g(x)\,dx$$

$\rightarrow$ $f_X(x)$ similar to $g(x)$

Monte Carlo integration (example)

Monte Carlo integration

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} X_i^2 = \vartheta_n' \qquad X_i \sim \mathcal{U}(0,1)$$

$$error_{\vartheta_n'} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(X^2\right)} = \frac{1}{\sqrt{n}}\sqrt{\mathbb{E}[X^4] - \mathbb{E}[X^2]^2} = \frac{1}{\sqrt{n}}\sqrt{\frac{1}{5}-\frac{1}{9}} \approx \frac{0.298}{\sqrt{n}}$$

$$\mathbb{E}[X^4] = \int_0^1 x^4\,dx = \frac{1}{5} \qquad \mathbb{E}[X^2] = \int_0^1 x^2\,dx = \frac{1}{3}$$

Monte Carlo integration with importance sampling

$$\int_0^1 x^2\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} \frac{X_i^2}{2X_i} = \vartheta_n'$$

X_i is distributed with PDF $f_X(x) := 2x$

$$error_{\vartheta_n'} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(\frac{X^2}{2X}\right)} = \frac{1}{2\sqrt{n}}\sqrt{Var\,(X)} = \frac{1}{2\sqrt{n}}\sqrt{\mathbb{E}[X^2] - \mathbb{E}[X]^2} = \frac{1}{2\sqrt{n}}\sqrt{\frac{1}{2}-\frac{4}{9}} \approx \frac{0.118}{\sqrt{n}}$$

$$\mathbb{E}[X^2] = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 2x^3\,dx = \frac{1}{2} \qquad \mathbb{E}[X] = \int_0^1 x f_X(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}$$

moments

$$\mu := \mathbb{E}[g(X)] := \int_{\mathbb{S}} g(x) f_X(x)\,dx$$

$$Var(X) = \mathbb{E}\left[X^2\right] - \mu^2$$

MC integration

$$\int_a^b g(x)\,dx \approx (b-a)\frac{1}{n}\sum_{i=0}^{n-1} g(X_i) =: \vartheta_n$$

$$error_{\vartheta_n} \approx \frac{b-a}{\sqrt{n}}\sqrt{Var\left(g(X)\right)}$$

MC integration importance sampling

$$\int_{\mathbb{S}} g(x)\,dx \approx \frac{1}{n}\sum_{i=0}^{n-1} \frac{g(X_i)}{f_X(X_i)}$$

$$error_{\vartheta_n} \approx \frac{1}{\sqrt{n}}\sqrt{Var\left(\frac{g(X)}{f_X(X)}\right)}$$

end