stochastics and probability

Lecture 2

Dr. Johannes Pahlke

random variable

random variable

random variables

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random variables

$$X : \Omega \to J$$

random variable

random variables

$$X : \Omega \to J$$

$$X(\omega) = x$$

random variable

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event: $A \subseteq \Omega$

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random variable

example:

random variables

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example:

Ω

Mon.
Tue.
Wed.
Thu.
Fri.
Sat.
Sun.

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example:

Ω

Mon.
Tue.
Wed.
Thu.
Fri.
Sat.
Sun.

$\mathbb{B}$

0
1

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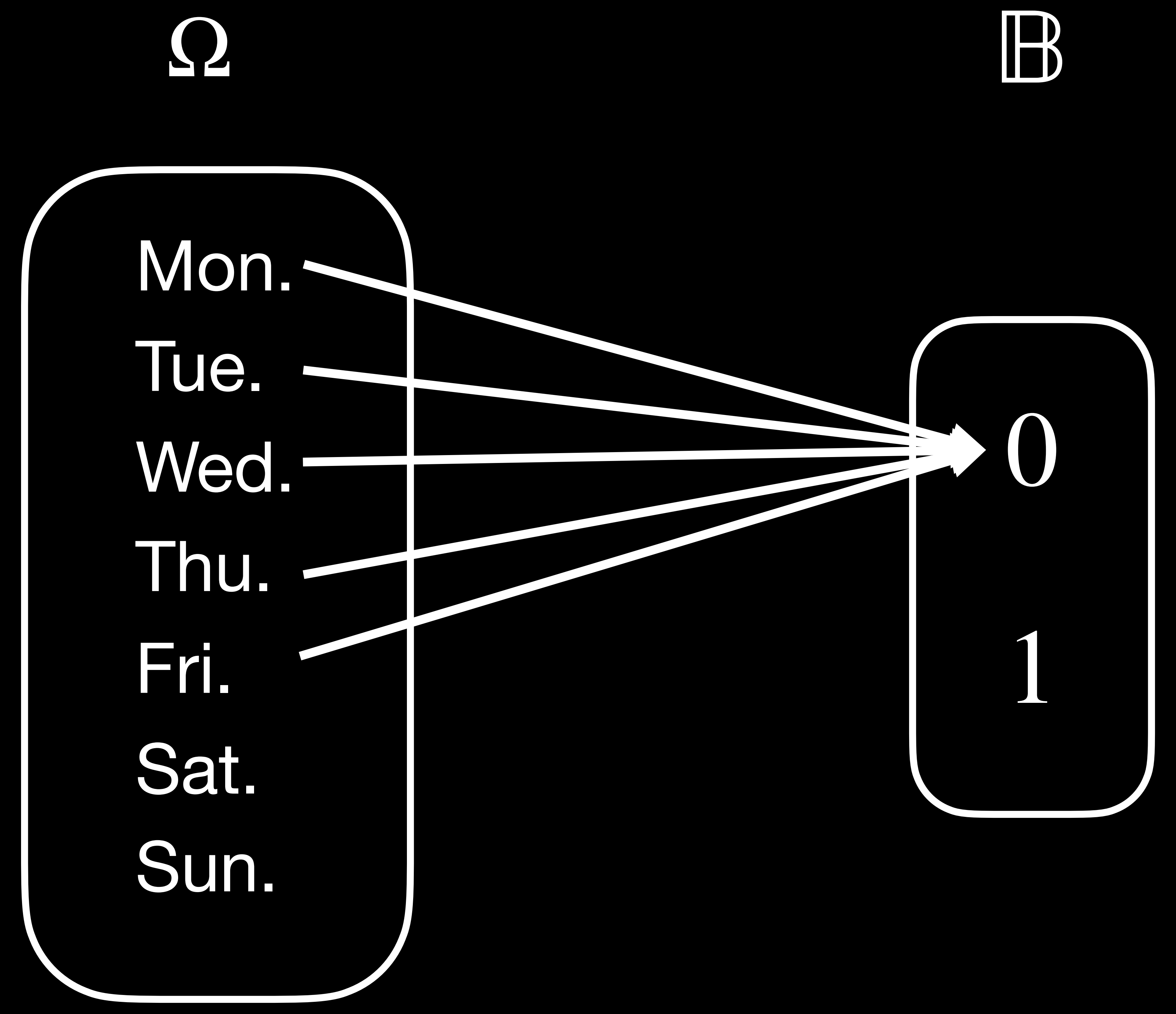

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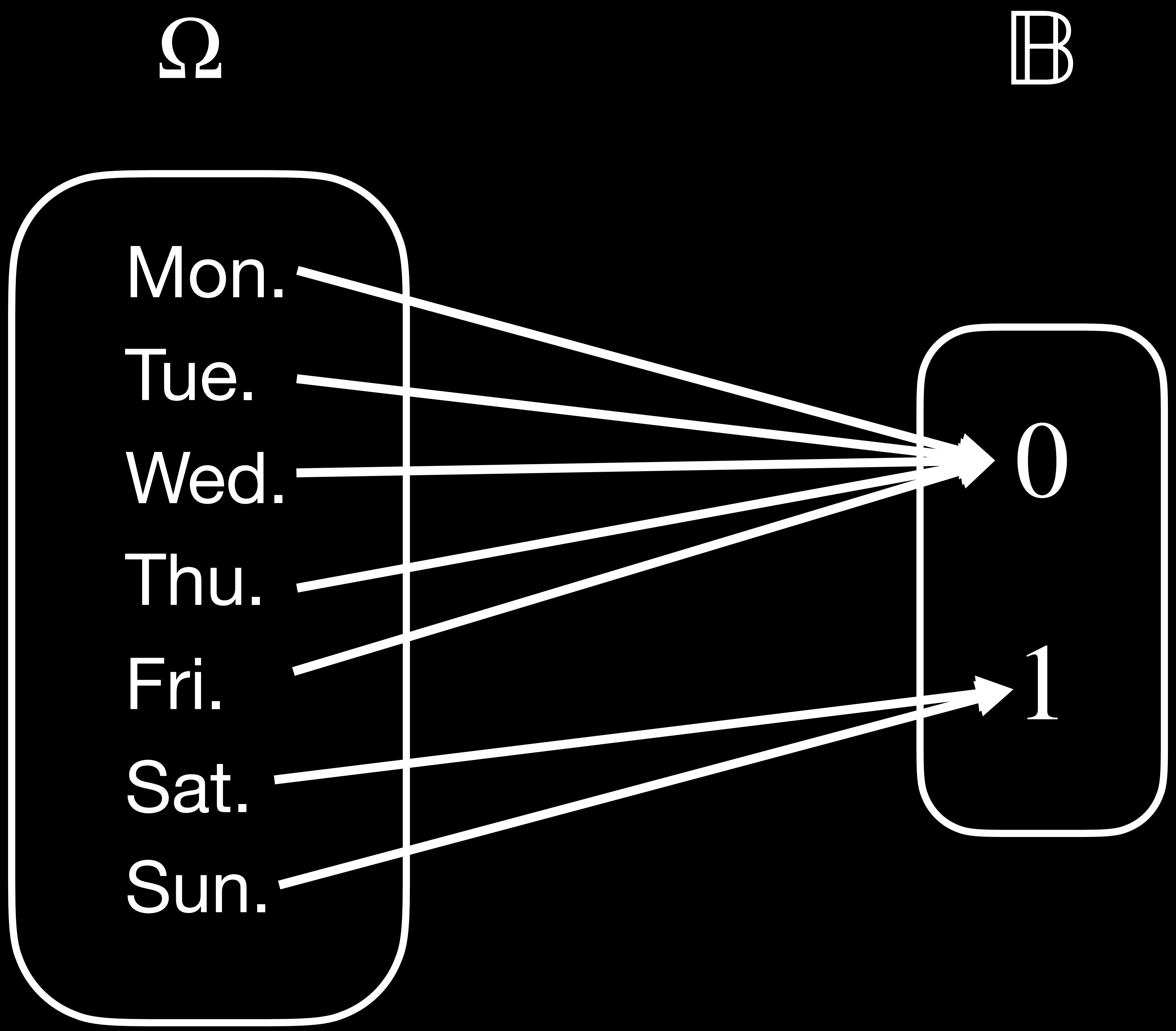

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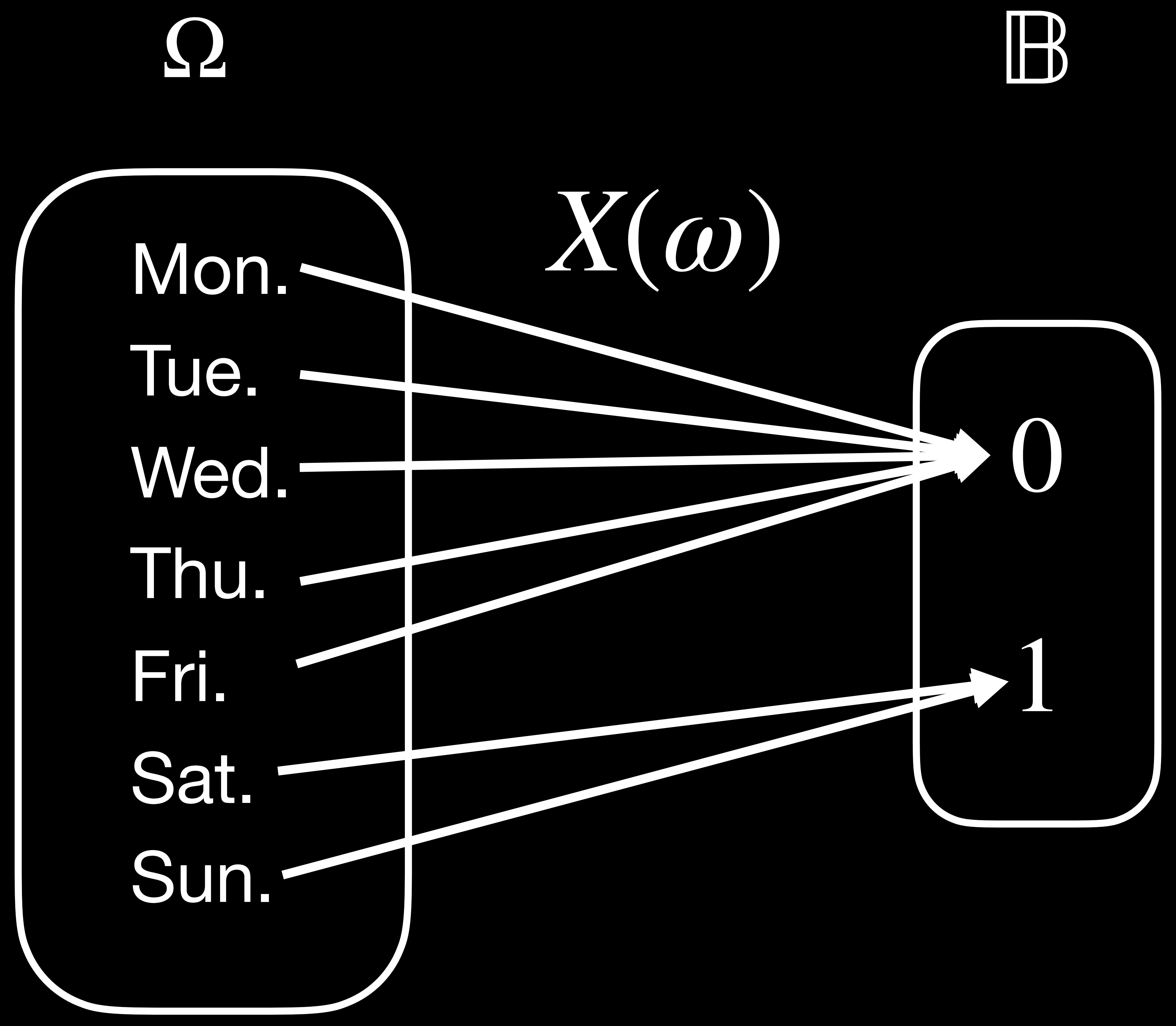

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range of X:

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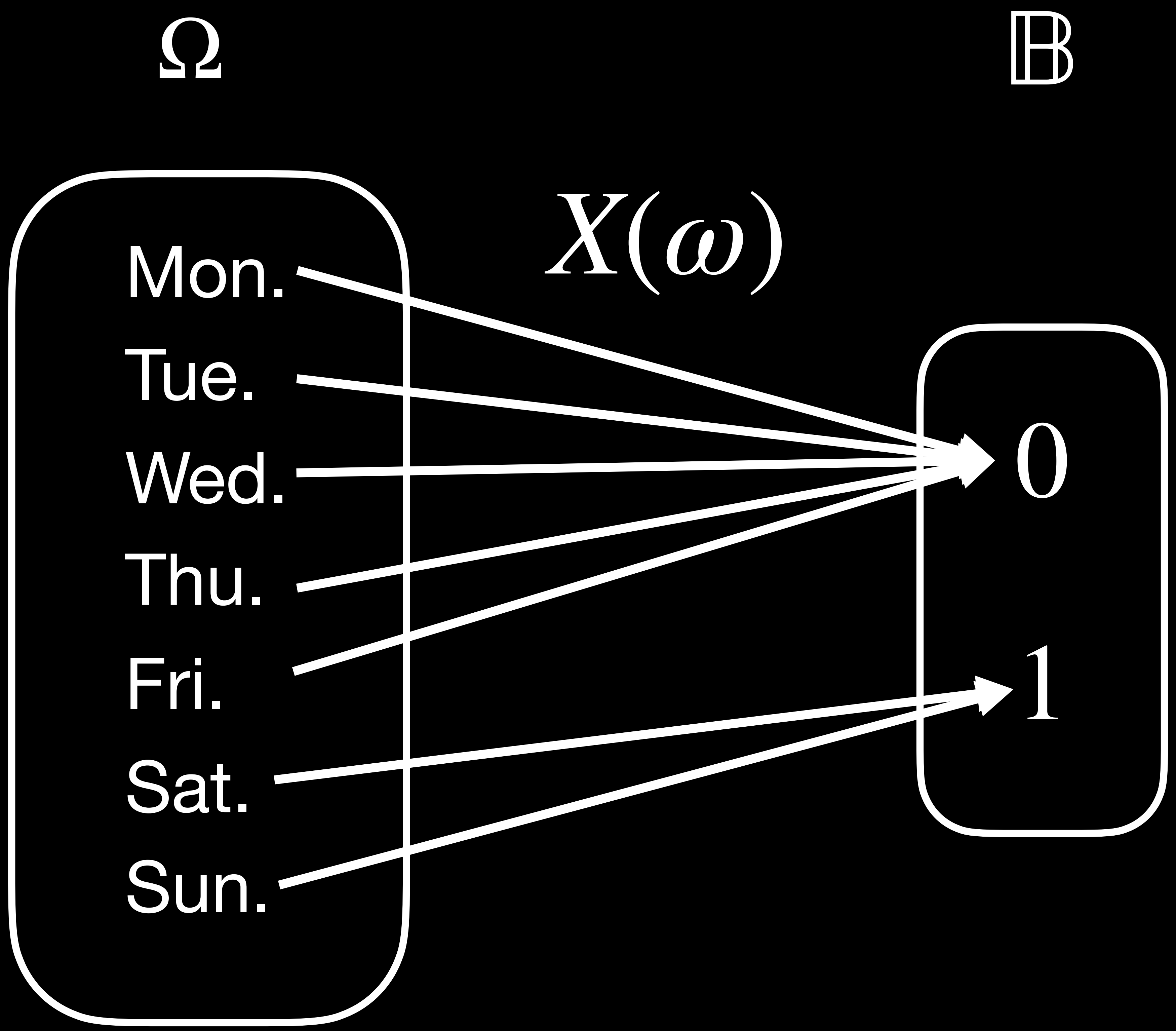

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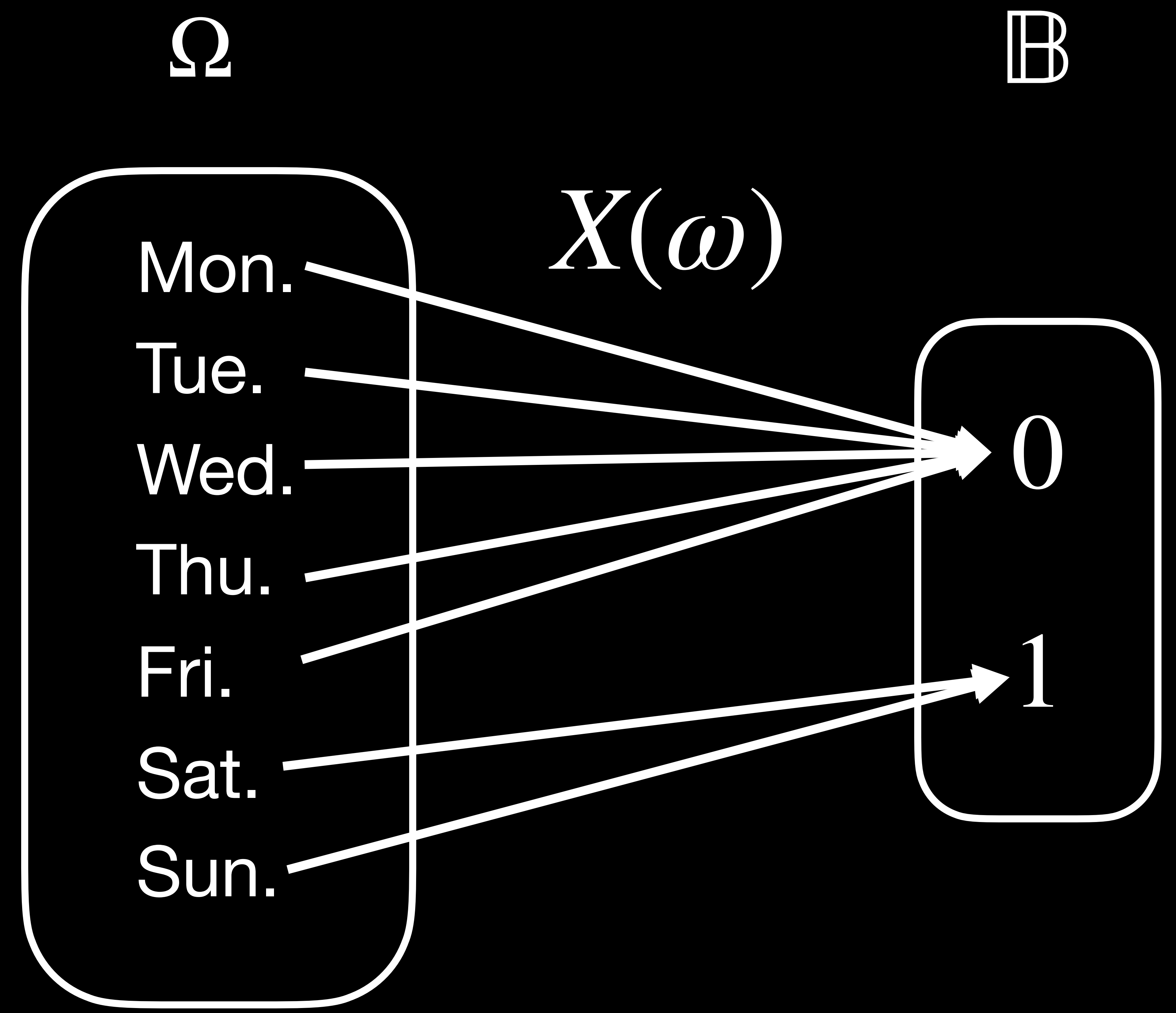

probability mass function

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$$p_X(x) = P(X = x)$$

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$$\sum_{x \in J} p_X(x) = 1$$

exercise: probability mass function

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$P(\Omega) = 1$

$P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

exercise: probability mass function

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proof:

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proof: $\sum_{x \in J} p_X(x)$

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$f : \mathbb{R} \to [0,\infty]$

probability density function (PDF)

$$P(X = x) = 0 \qquad\qquad f : \mathbb{R} \to [0, \infty]$$

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx$$

cumulative distribution function (CDF)

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discrete

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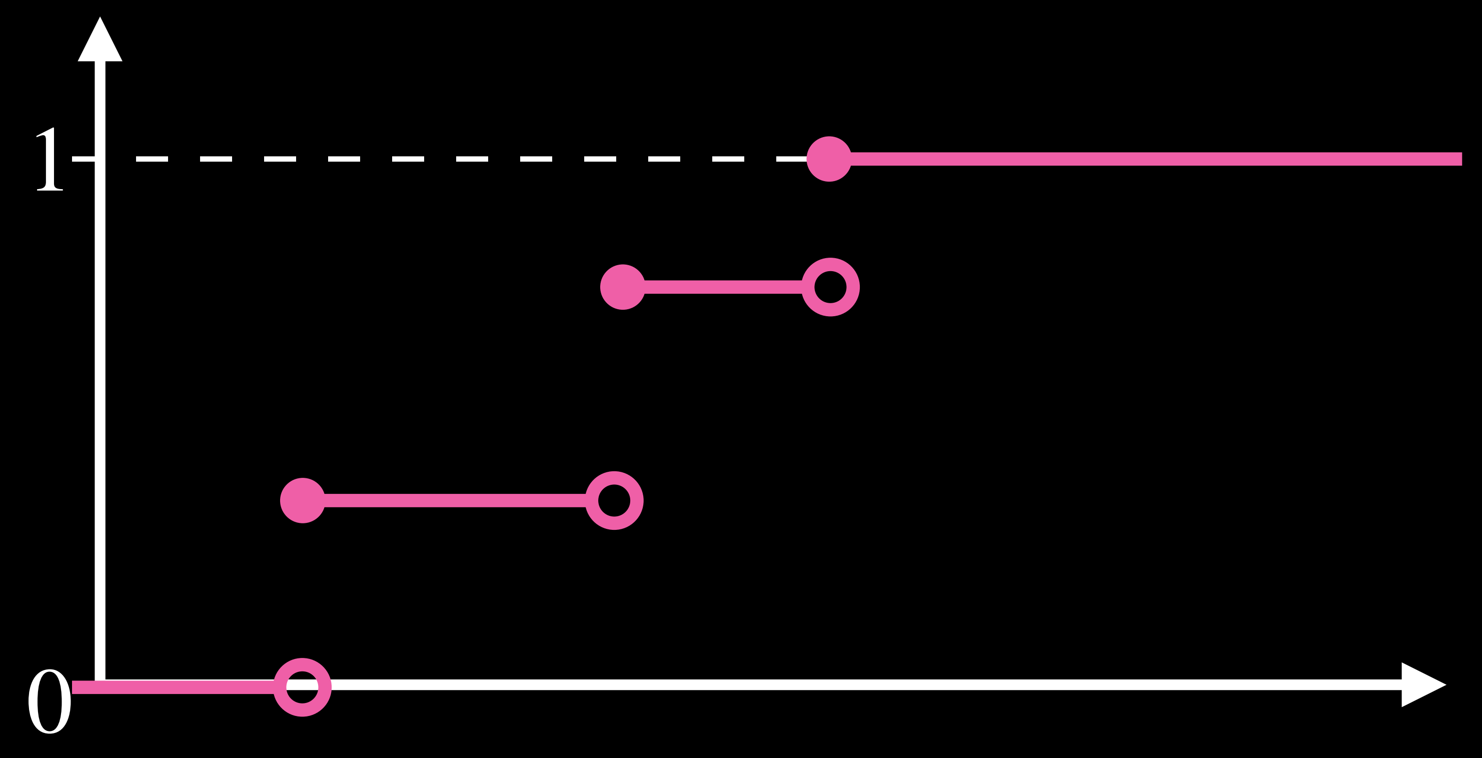

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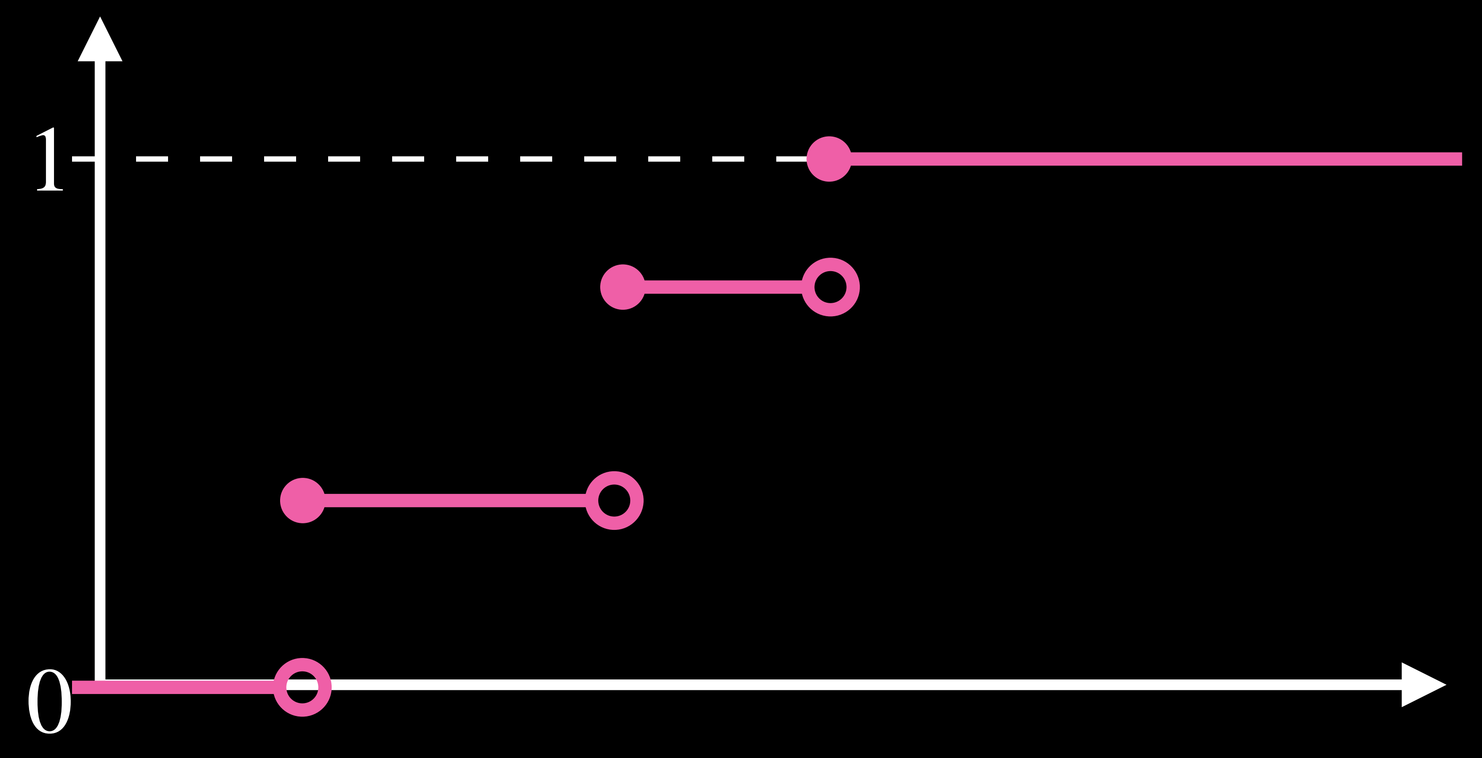

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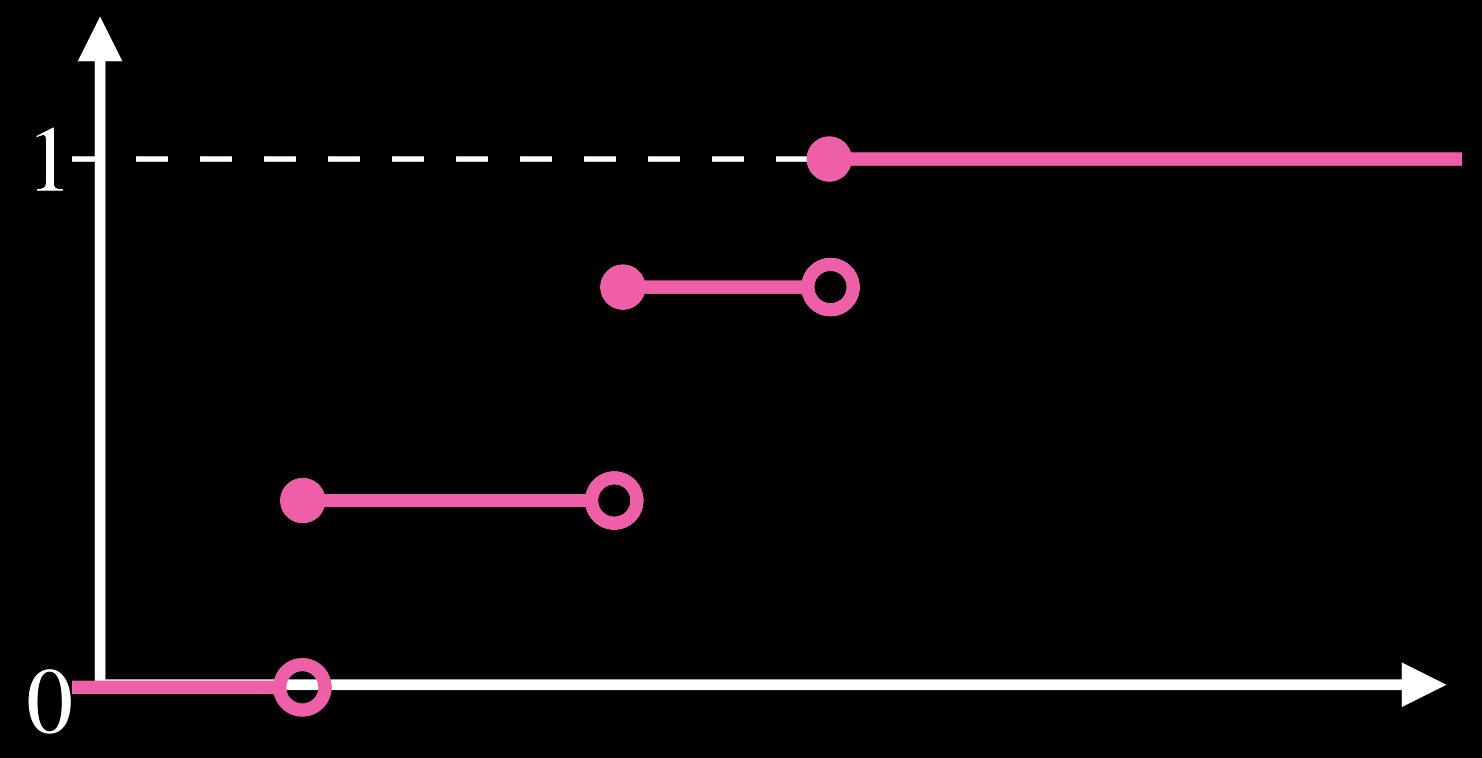

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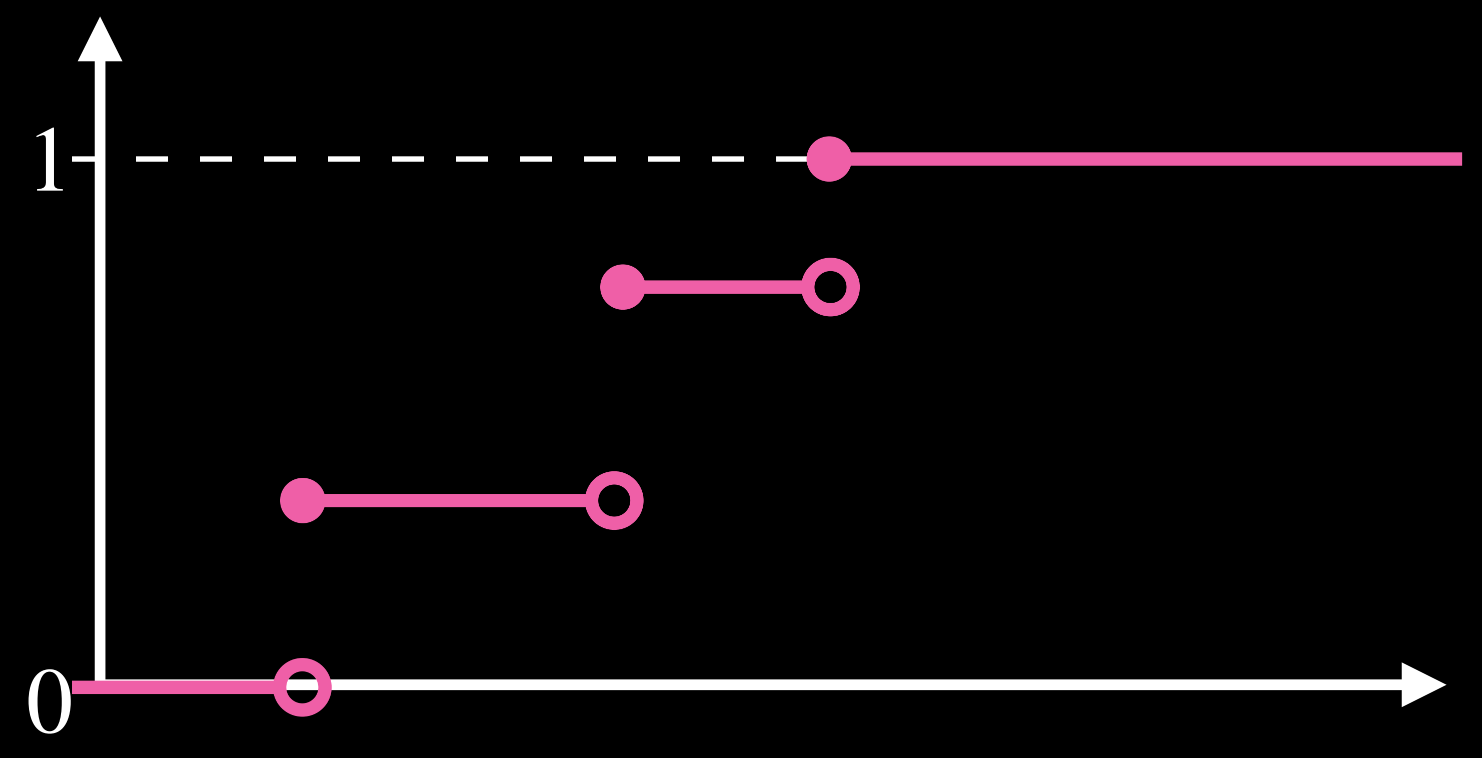

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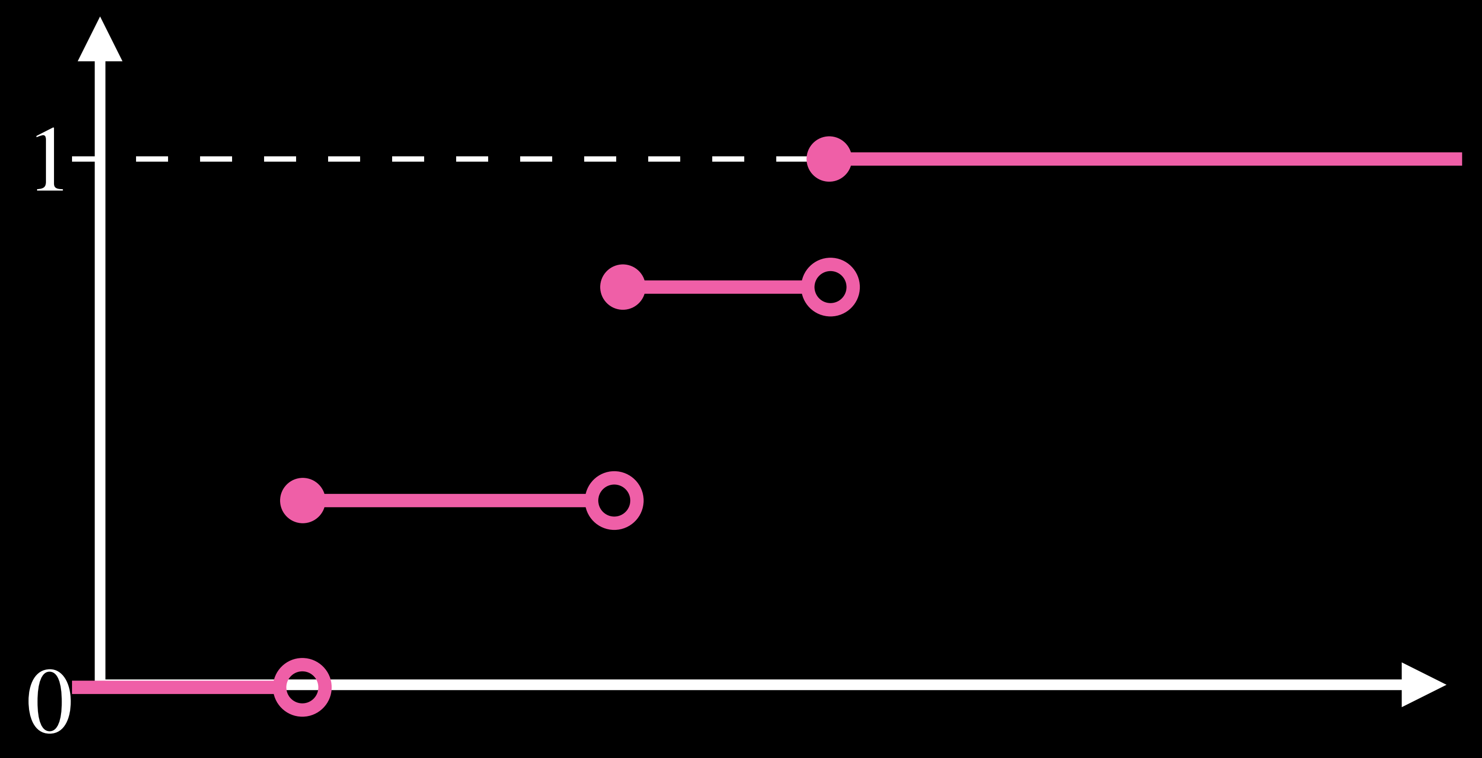

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$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\, dt$$

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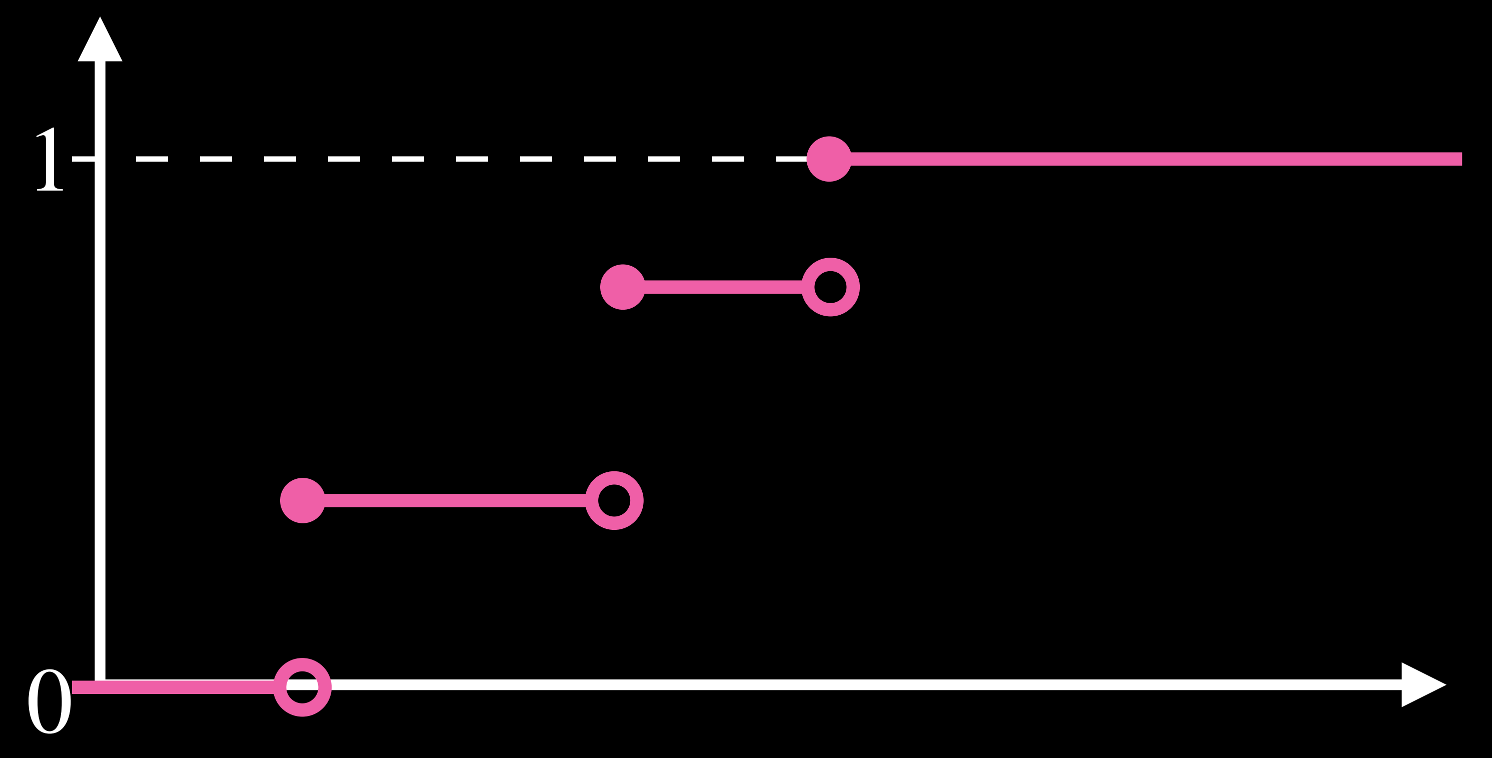

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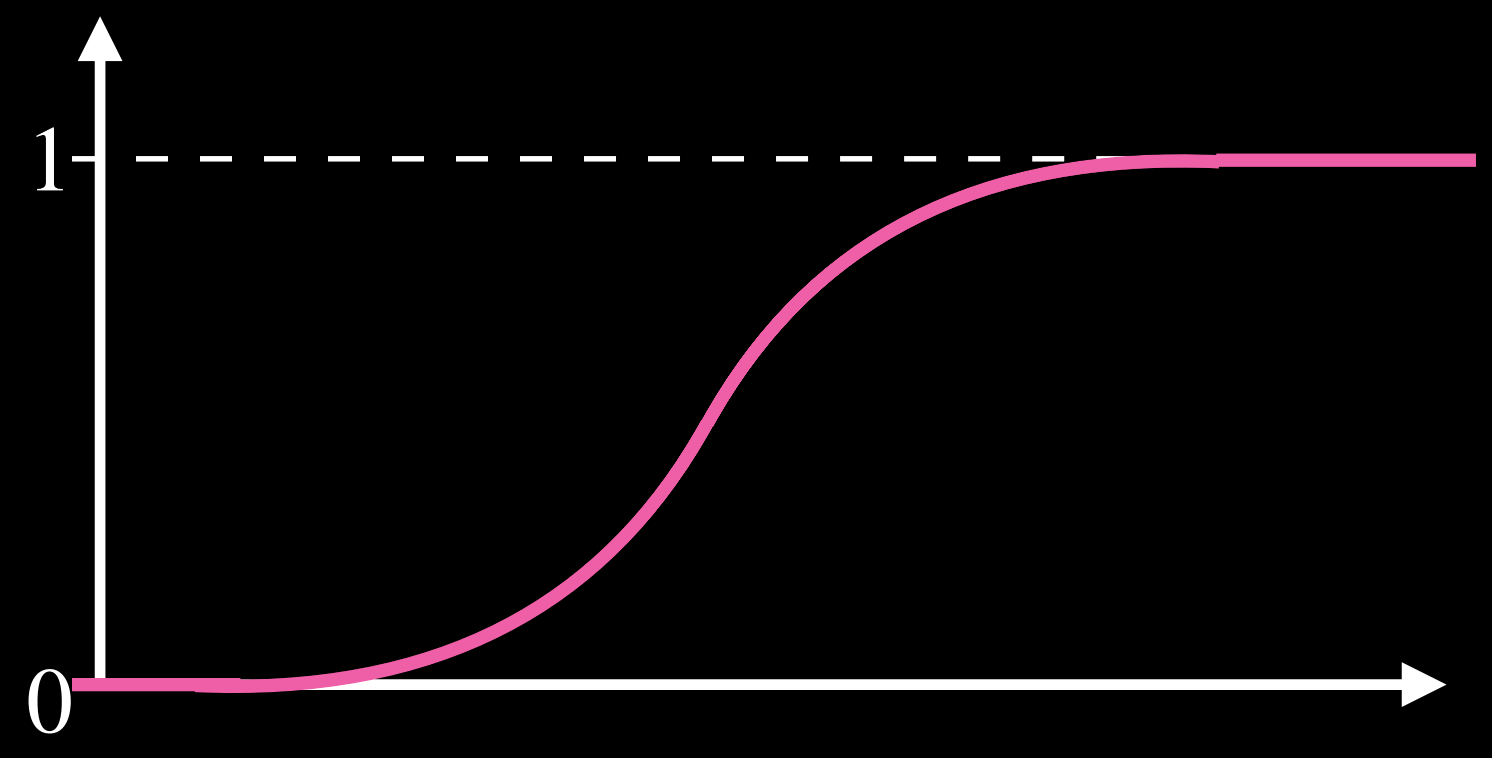

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$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\ dt$$

$$Range(X) := \{x \in \mathbb{R} : x = X(\omega), \omega \in \Omega\}$$

exercice

How to calculate the PDF $f_X(x)$ given the CDF $F_X(x)$?

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$$f_X(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

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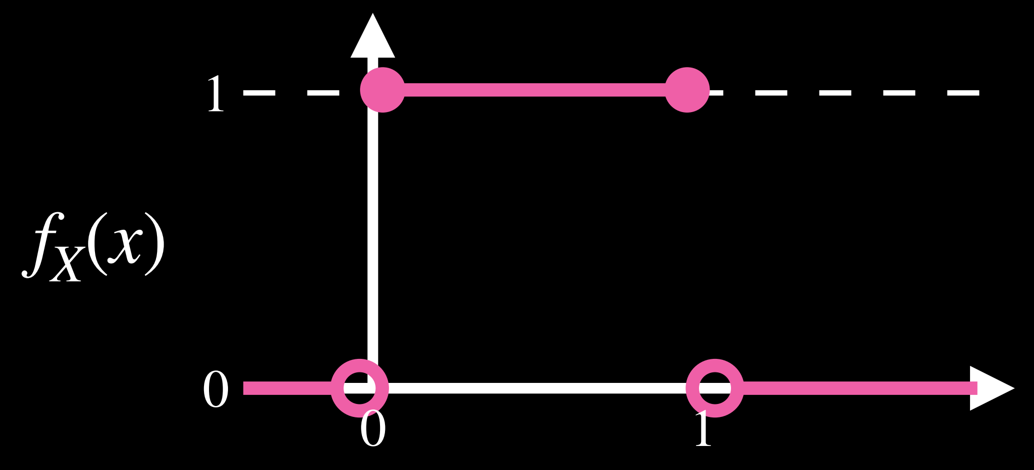

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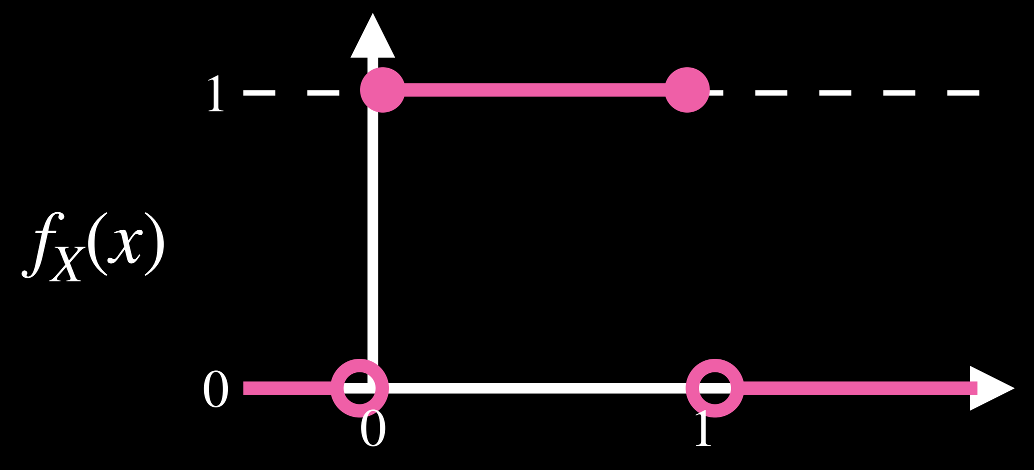

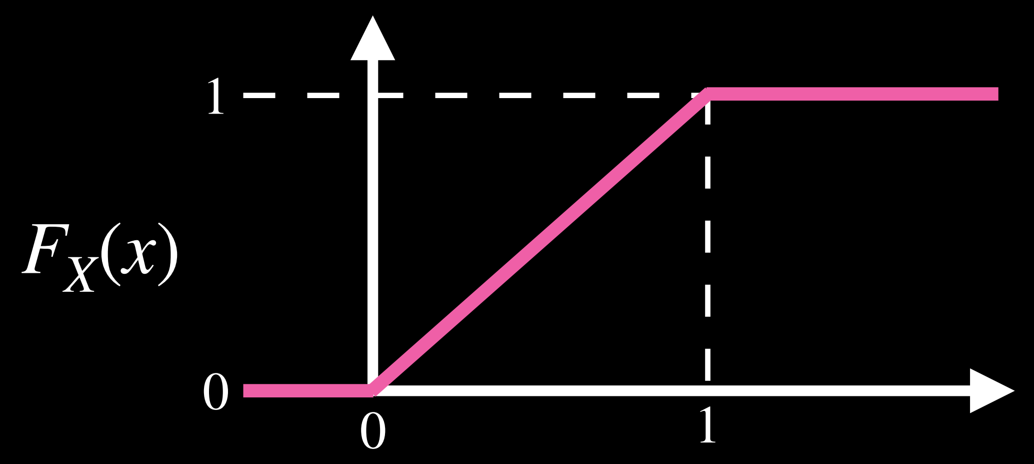

uniform random number generation

uniform random number generation

data-driven

uniform random number generation

data-driven

algorithmic

uniform random number generation

data-driven $\left\{ \right.$

algorithmic

uniform random number generation

data-driven $\left\{ \begin{array}{l} \text{not statistical random} \\ \\ \end{array} \right.$

algorithmic

uniform random number generation

data-driven $\begin{cases} \text{not statistical random} \\ \text{not reproducable} \end{cases}$

algorithmic

uniform random number generation

data-driven
- not statistical random
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algorithmic

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algorithmic $\begin{cases} \text{statistical random} \\ \end{cases}$

uniform random number generation

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$$\text{algorithmic} \begin{cases} \text{statistical random} \\ \text{reproducable} \end{cases}$$

uniform random number generation

data-driven $\left\{ \begin{array}{l} \text{not statistical random} \\ \text{not reproducable} \end{array} \right.$

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random numbers

uniform random number generation

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random numbers $x_1, \ldots, x_n \in [0,1]$

uniform random number generation

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require

uniform random number generation

data-driven $\begin{cases} \text{not statistical random} \\ \text{not reproducable} \end{cases}$

algorithmic $\begin{cases} \text{statistical random} \\ \text{reproducable} \end{cases}$

random numbers $x_1, \ldots, x_n \in [0,1]$

$$_n\hat{F}_U(x) := \frac{\left|\{x_i \in \{x_1, \ldots, x_n\} : x_i \leq x\}\right|}{n}$$

require $\lim_{n\to\infty} \left|{}_n\hat{F}_U(x) - F_U(x)\right|$

uniform random number generation

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require $$\lim_{n\to\infty} \left| {}_n\hat{F}_U(x) - F_U(x) \right| = \lim_{n\to\infty} \left| {}_n\hat{F}_U(x) - x \right|$$

uniform random number generation

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$$ {}_n\hat{F}_U(x) := \frac{\left|\left\{x_i \in \{x_1, \ldots, x_n\} : x_i \le x\right\}\right|}{n} $$

require $\lim_{n\to\infty} \left| {}_n\hat{F}_U(x) - F_U(x) \right| = \lim_{n\to\infty} \left| {}_n\hat{F}_U(x) - x \right|$ $\qquad x \in [0,1]$

uniform random number generation

data-driven $\begin{cases} \text{not statistical random} \\ \text{not reproducable} \end{cases}$

algorithmic $\begin{cases} \text{statistical random} \\ \text{reproducable} \end{cases}$

random numbers $x_1, \ldots, x_n \in [0,1]$

$$_n\hat{F}_U(x) := \frac{\left|\left\{x_i \in \{x_1, \ldots, x_n\} : x_i \leq x\right\}\right|}{n}$$

require $\lim_{n\to\infty} \left|_n\hat{F}_U(x) - F_U(x)\right| = \lim_{n\to\infty} \left|_n\hat{F}_U(x) - x\right| \qquad x \in [0,1]$

$= 0$

pseudo-random number generation

pseudo-random number generation

Lehmer generator /a Linear congruential generator (LCG)

pseudo-random number generation

Lehmer generator /a Linear congruential generator (LCG)

$$z_{i+1} = a z_i \mod m$$

pseudo-random number generation

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$$z_{i+1} = az_i \mod m \qquad\qquad i = 1,2,3,..$$

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$$x_i = \frac{z_i}{m}$$

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Need to choose:

pseudo-random number generation

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Need to choose:

$$m \in \mathbb{N}_{>0}$$

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$$x_i = \frac{z_i}{m} \quad \in [0,1)$$

Need to choose:

$$m \in \mathbb{N}_{>0} \qquad 0 < a < m$$

pseudo-random number generation

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Need to choose:

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seed:

pseudo-random number generation

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seed: $z_1 \in \mathbb{N}_{>0}$

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cycle length:

pseudo-random number generation

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cycle length:

$x_1, x_2, \ldots, x_T$

pseudo-random number generation

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$$x_i = \frac{z_i}{m} \quad \in [0,1)$$

Need to choose:

$$m \in \mathbb{N}_{>0} \qquad 0 < a < m$$

seed: $z_1 \in \mathbb{N}_{>0}$

cycle length:

$$x_1, x_2, \ldots, x_T \qquad\qquad x_1 = x_T$$

pseudo-random number generation

Lehmer generator /a Linear congruential generator (LCG)

$$z_{i+1} = az_i \mod m \qquad i = 1,2,3,..$$

$$x_i = \frac{z_i}{m} \quad \in [0,1)$$

Need to choose:

$m \in \mathbb{N}_{>0}$ $\qquad 0 < a < m$ $\qquad m = 2^{31} - 1$

seed: $z_1 \in \mathbb{N}_{>0}$

cycle length:

$x_1, x_2, \ldots, x_T \qquad x_1 = x_T$

pseudo-random number generation

Lehmer generator /a Linear congruential generator (LCG)

$$z_{i+1} = az_i \mod m \qquad\qquad i = 1,2,3,..$$

$$x_i = \frac{z_i}{m} \quad \in [0,1)$$

Need to choose:

$m \in \mathbb{N}_{>0} \qquad 0 < a < m$

seed: $z_1 \in \mathbb{N}_{>0}$

$m = 2^{31} - 1$
$a = 48271$

cycle length:

$x_1, x_2, \ldots, x_T \qquad\qquad x_1 = x_T$

pseudo-random number generation

pseudo-random number generation

generators with lager cycle length T

pseudo-random number generation

generators with lager cycle length T

Mersenne Twister (1998)

pseudo-random number generation

generators with lager cycle length T

Mersenne Twister (1998)

Park-Miller (1988)

pseudo-random number generation

generators with lager cycle length T

- Mersenne Twister (1998)
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pseudo-random number generation

generators with lager cycle length T

Mersenne Twister (1998)

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XoroShiro (2018)

pseudo-random number generation

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parallel random number generators

pseudo-random number generation

generators with lager cycle length T

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parallel random number generators

$z_1(\text{proc id})$

pseudo-random number generation

generators with lager cycle length T

- Mersenne Twister (1998)
- Park-Miller (1988)
- XOR-Shift (2003)
- XoroShiro (2018)

parallel random number generators

$z_1(\text{proc id})$

$$T_{parallel} = \frac{T}{\#\text{proc}}$$

quasi-RNG (low discrepancy sequence)

quasi-RNG (low discrepancy sequence)

pseudo-RNG

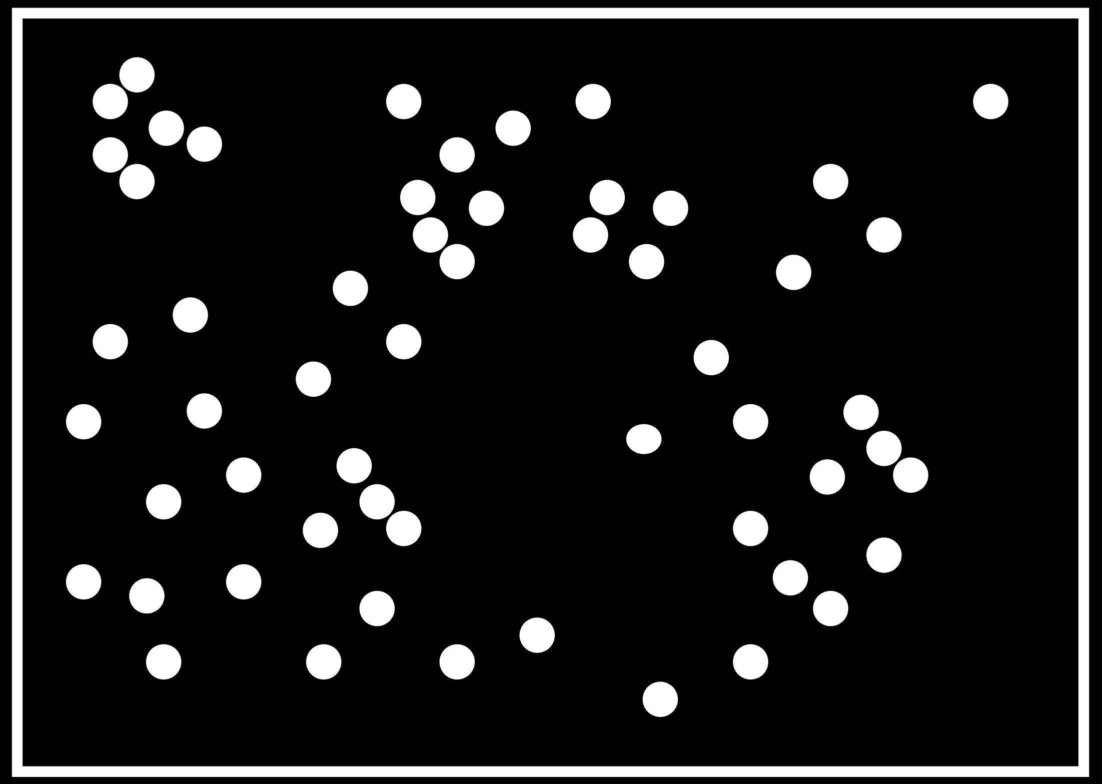
quasi-RNG (low discrepancy sequence)

pseudo-RNG

quasi-RNG

quasi-RNG (low discrepancy sequence)

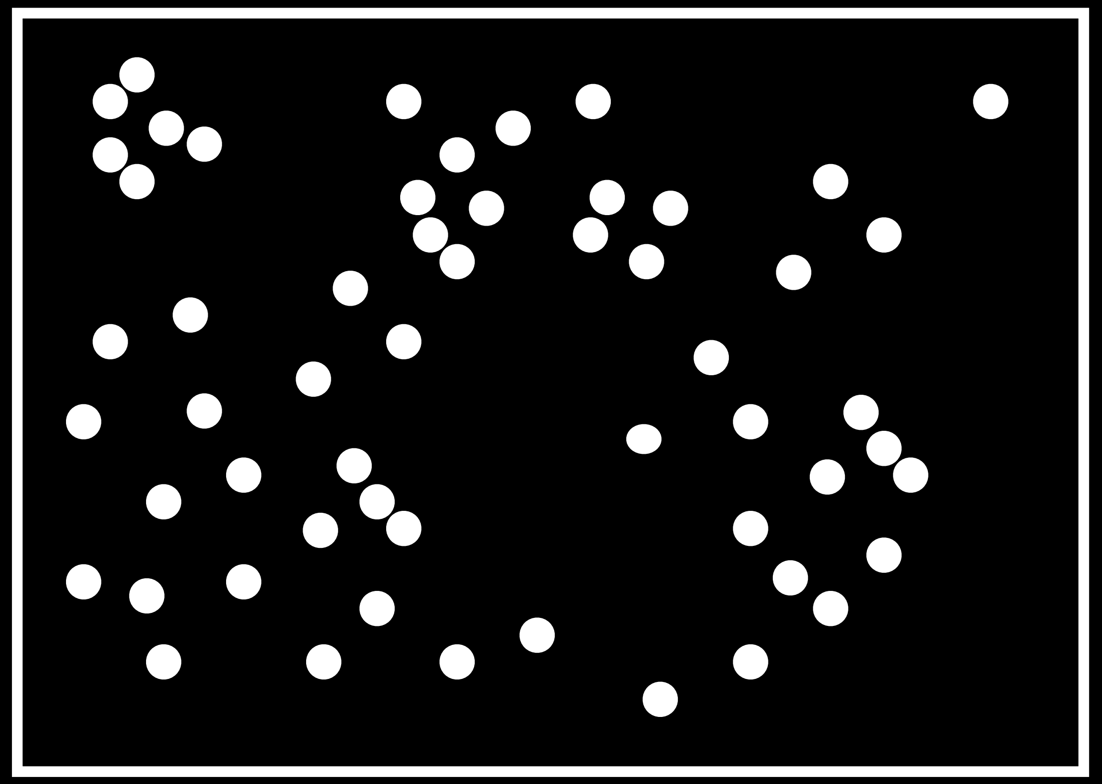
pseudo-RNG



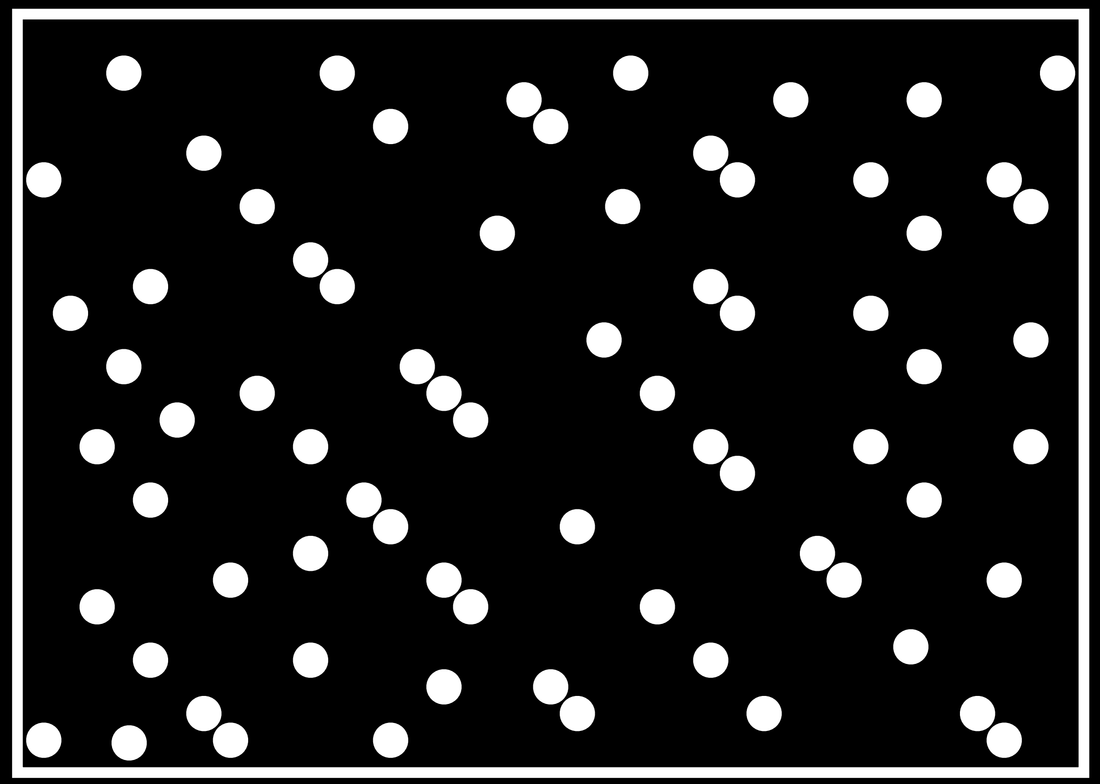
quasi-RNG

quasi-RNG (low discrepancy sequence)

pseudo-RNG

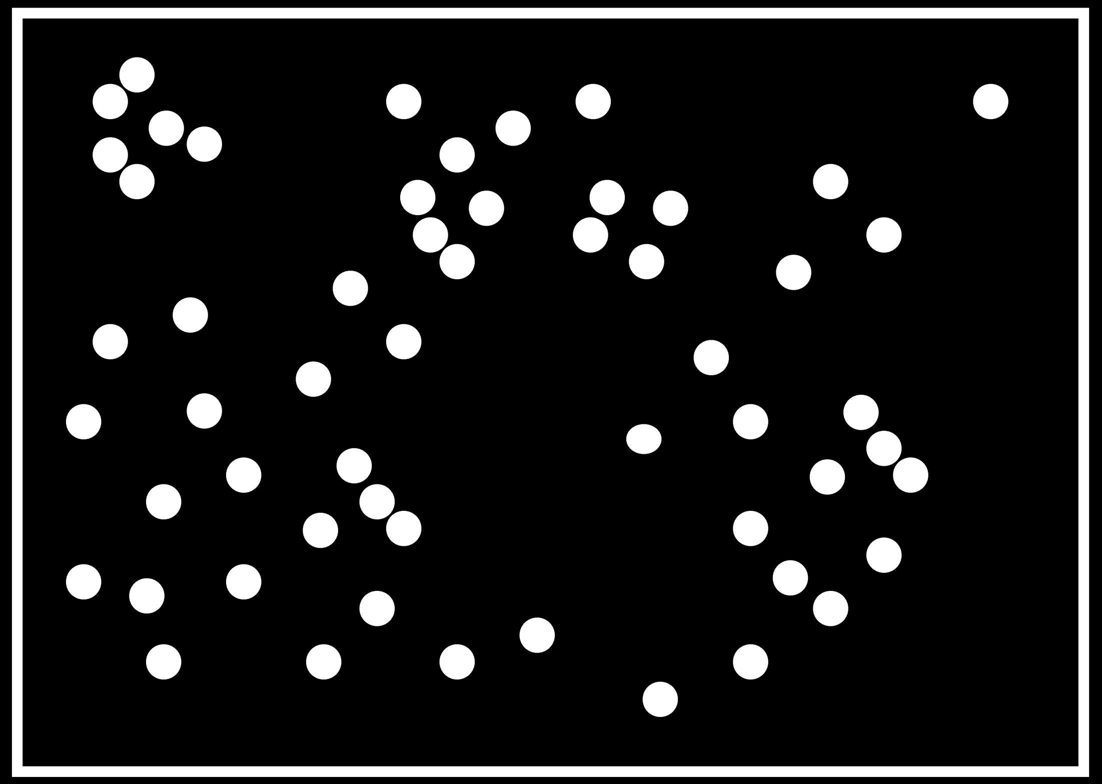


quasi-RNG

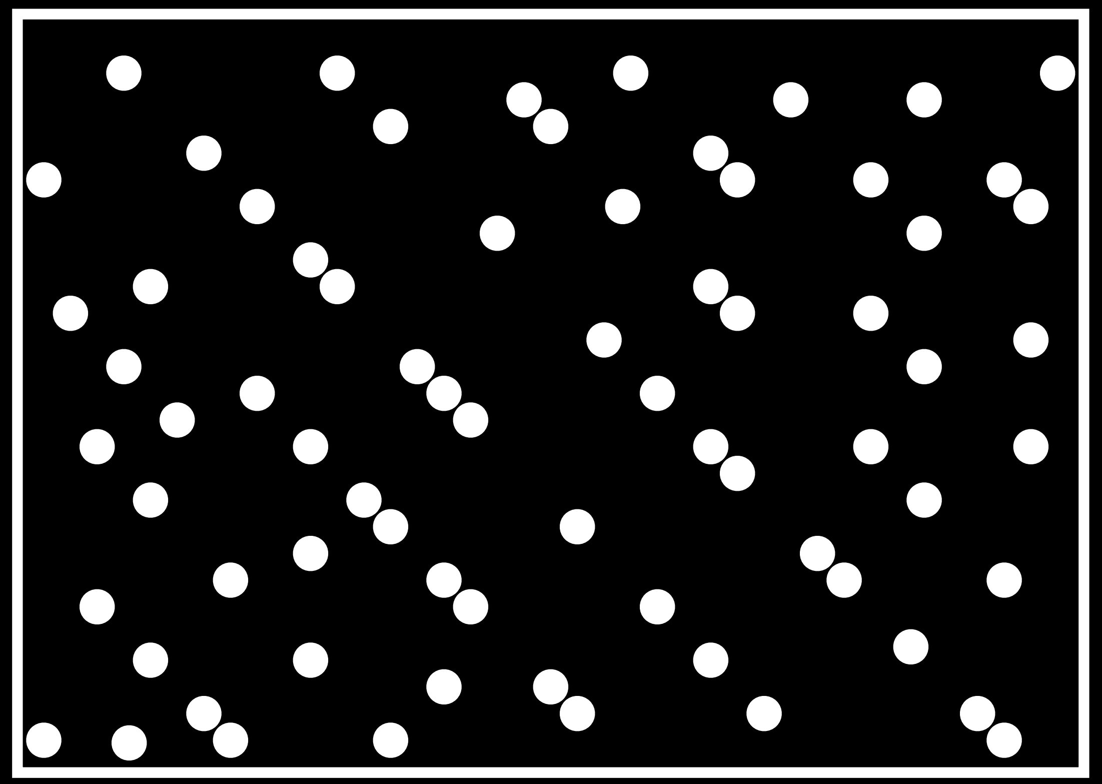


quasi-RNG (low discrepancy sequence)

pseudo-RNG



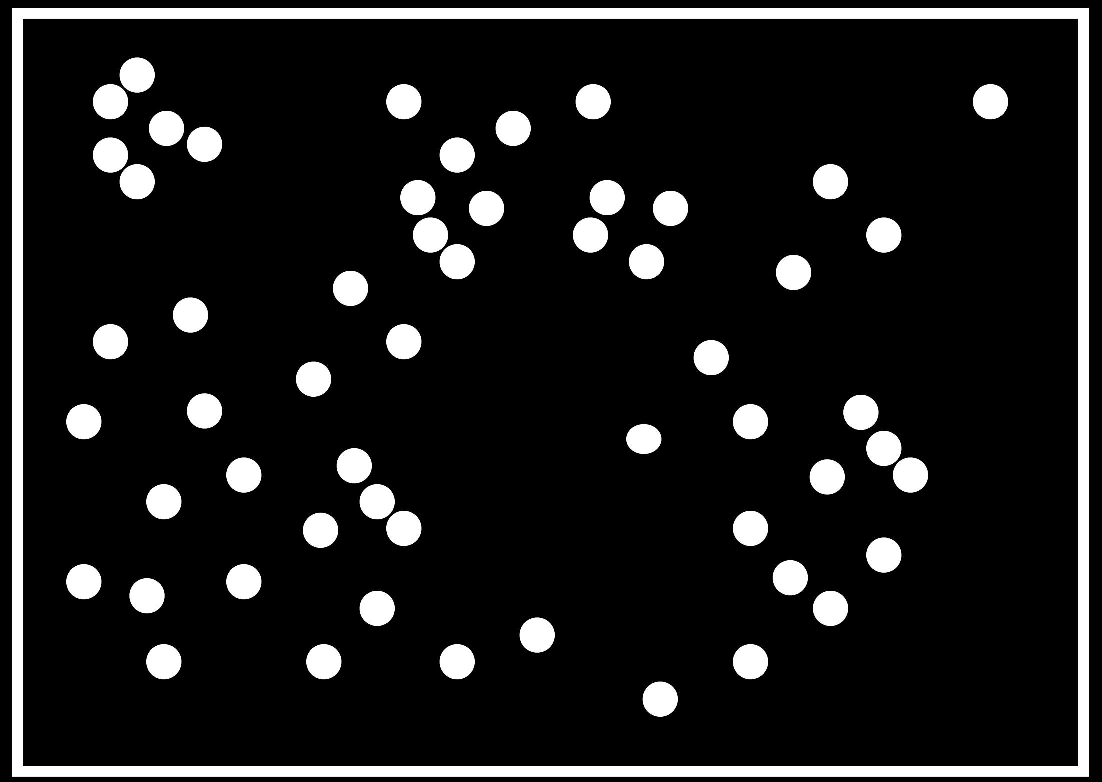
quasi-RNG



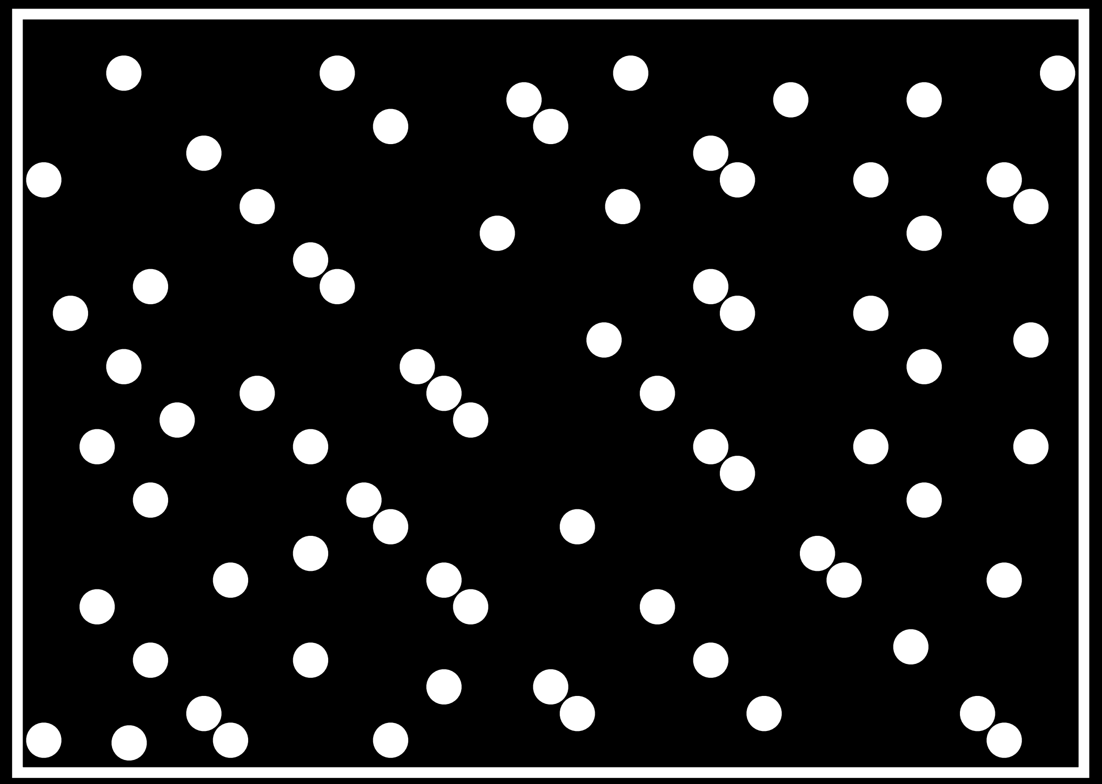
Additive recurrence sequence:

quasi-RNG (low discrepancy sequence)

pseudo-RNG



quasi-RNG

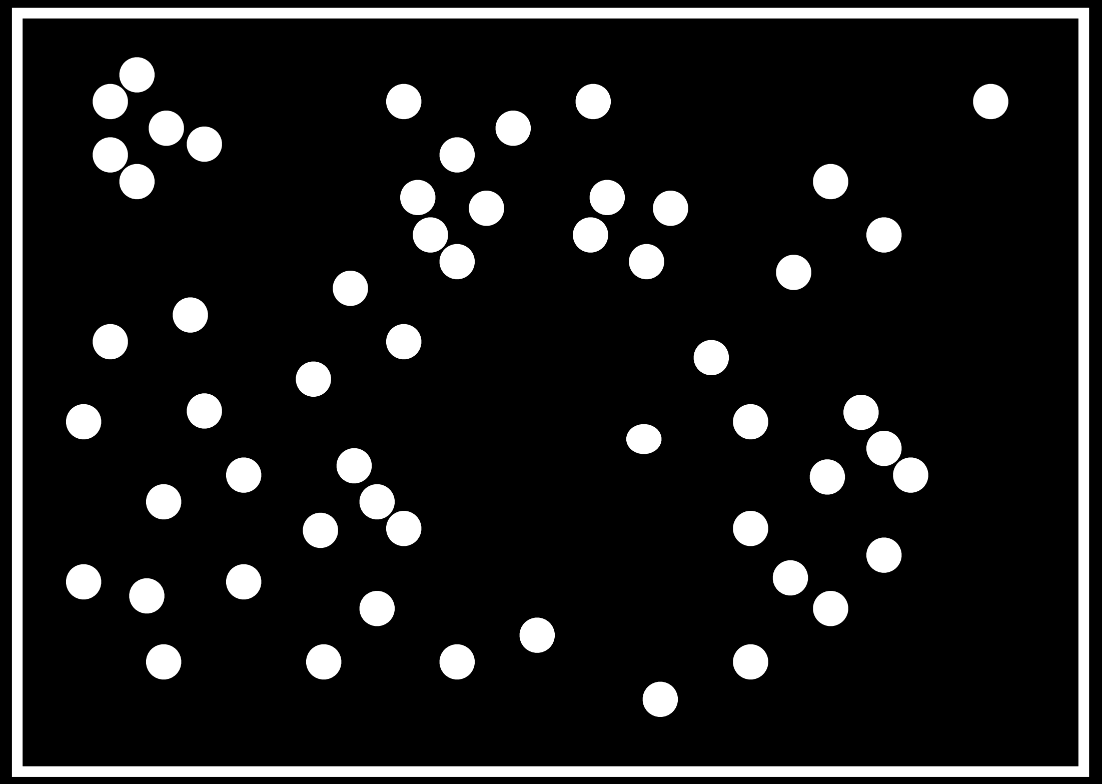


Additive recurrence sequence:

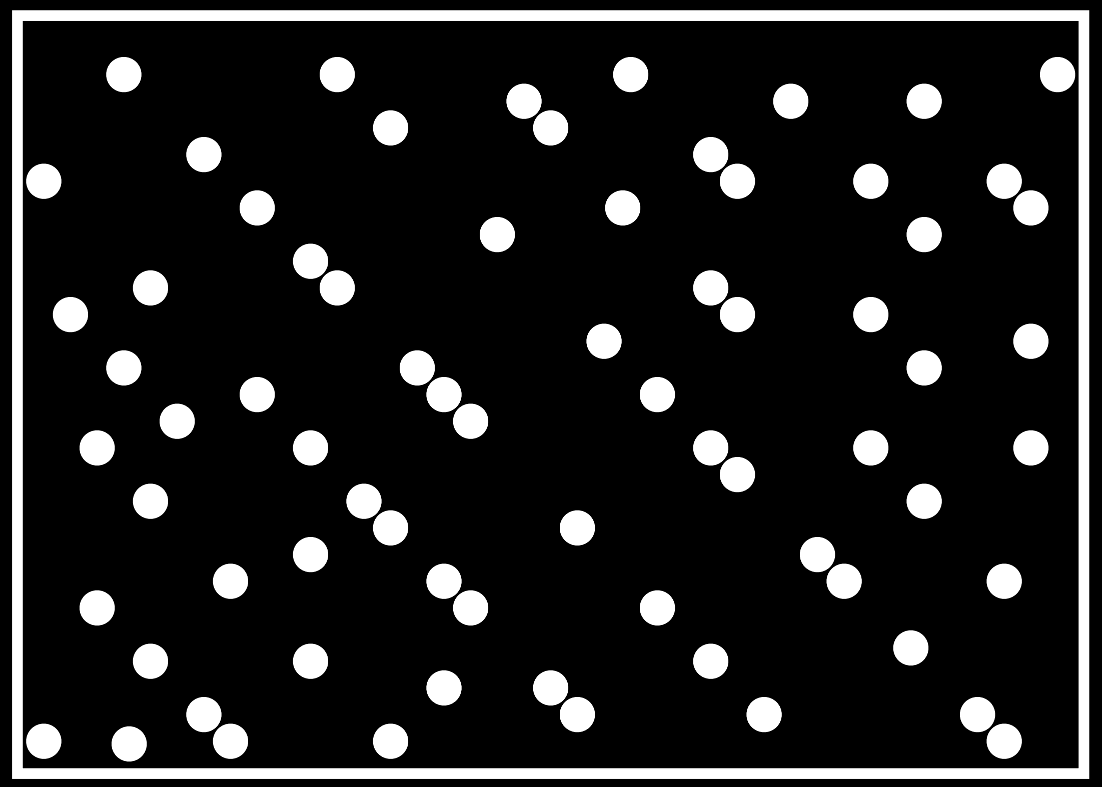
$$x_{i+1} = (x_i + \alpha) \mod 1$$

quasi-RNG (low discrepancy sequence)

pseudo-RNG



quasi-RNG



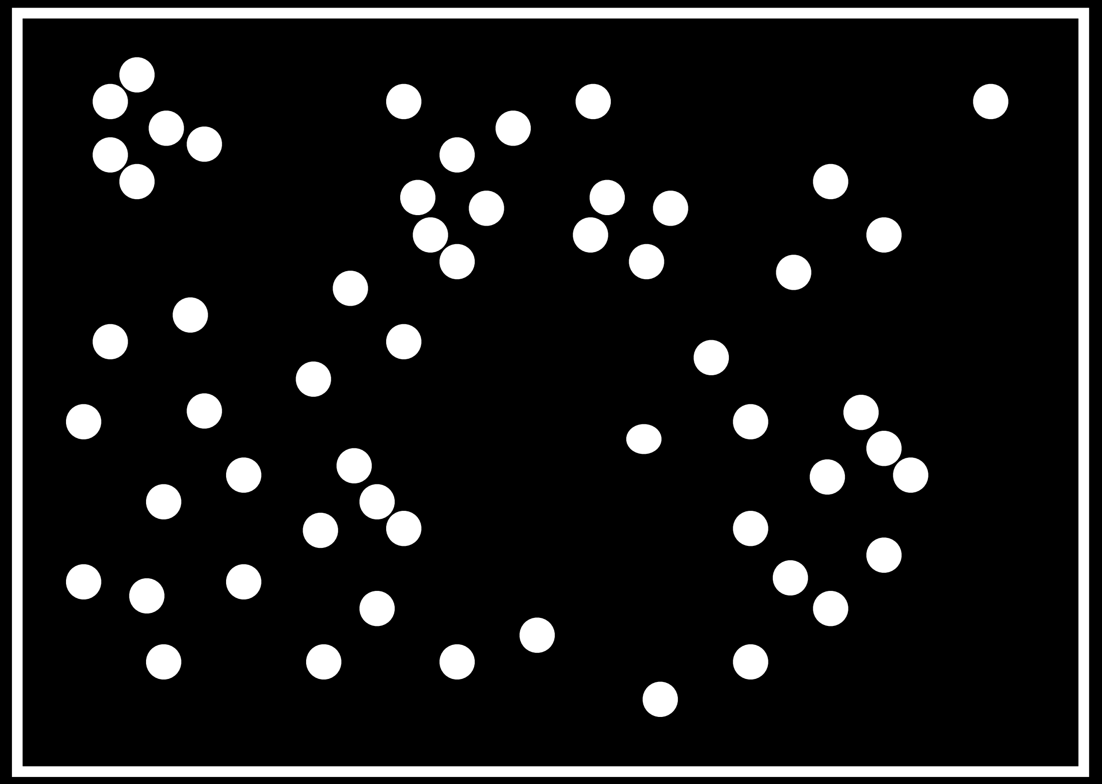
Additive recurrence sequence:

$$x_{i+1} = (x_i + \alpha) \mod 1$$

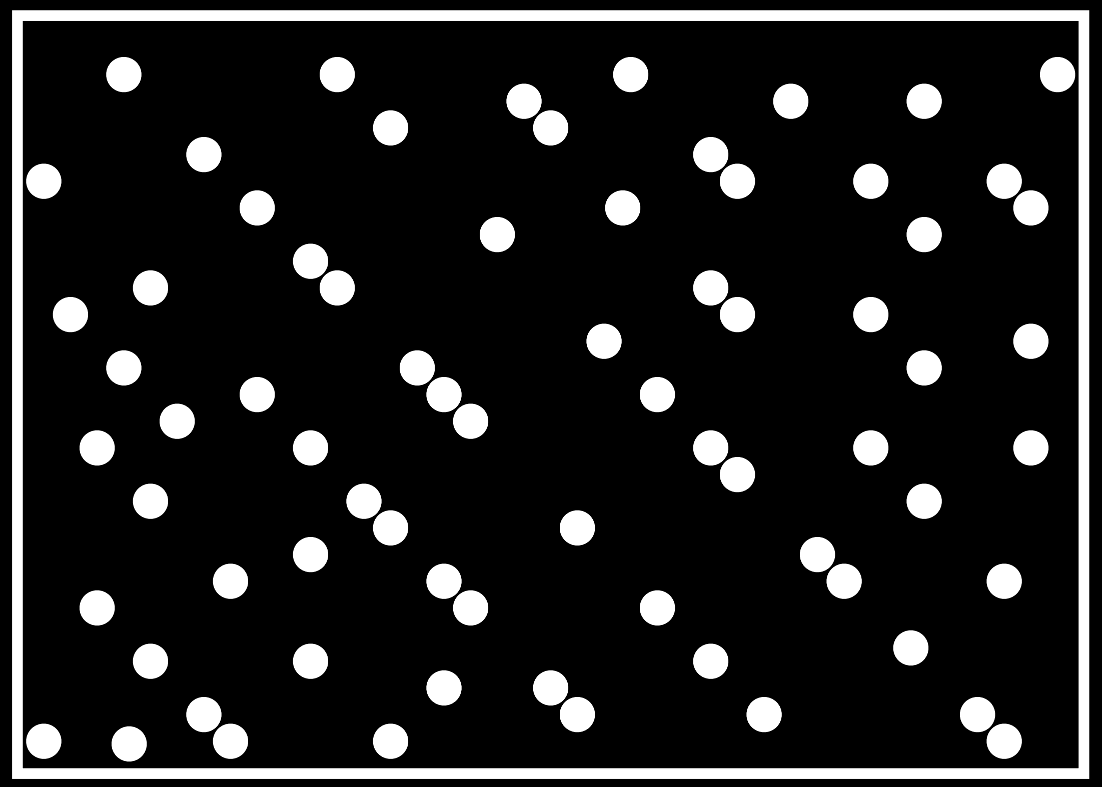
α irrational

quasi-RNG (low discrepancy sequence)

pseudo-RNG



quasi-RNG



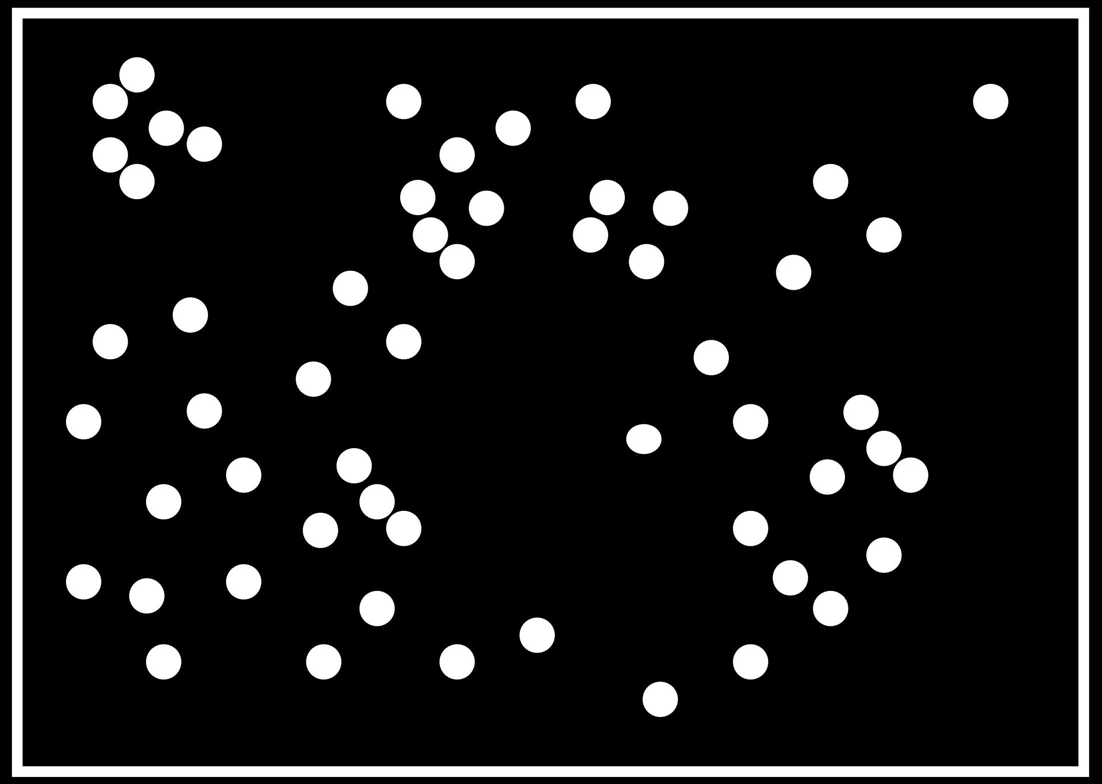
Additive recurrence sequence:

$$x_{i+1} = (x_i + \alpha) \mod 1$$

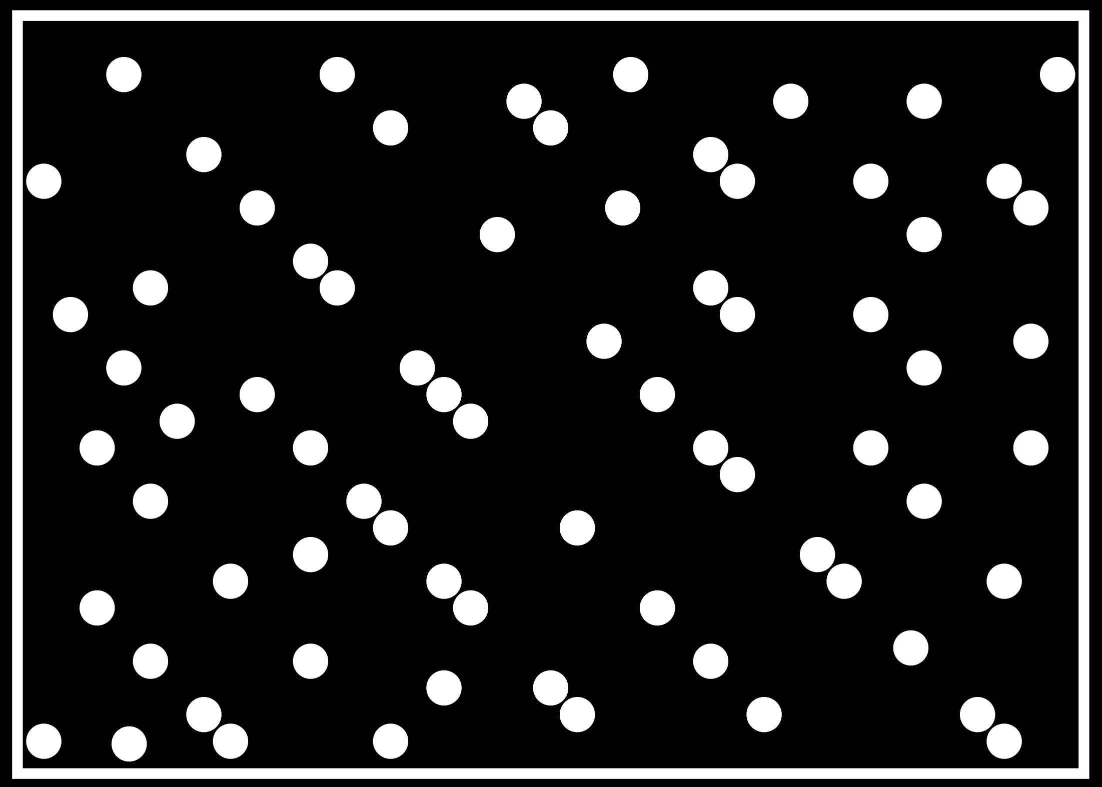
α irrational $\qquad \alpha = \sqrt{2} - 1$

quasi-RNG (low discrepancy sequence)

pseudo-RNG



quasi-RNG



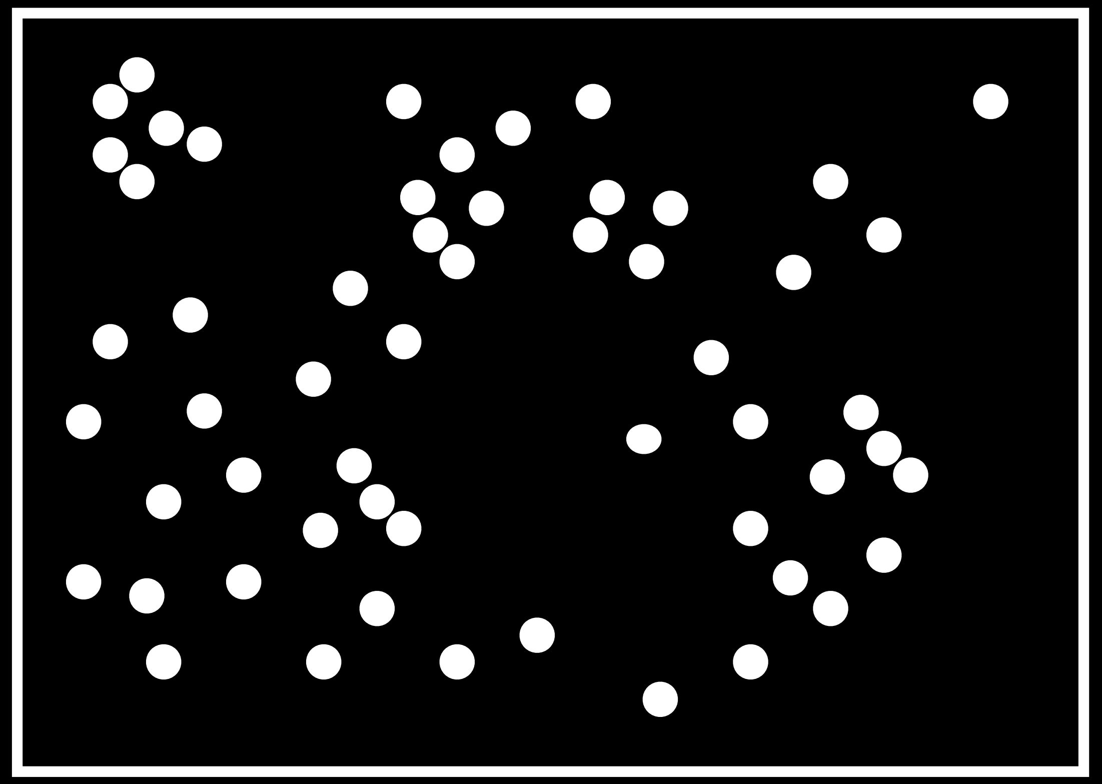
Additive recurrence sequence:

$$x_{i+1} = (x_i + \alpha) \mod 1$$

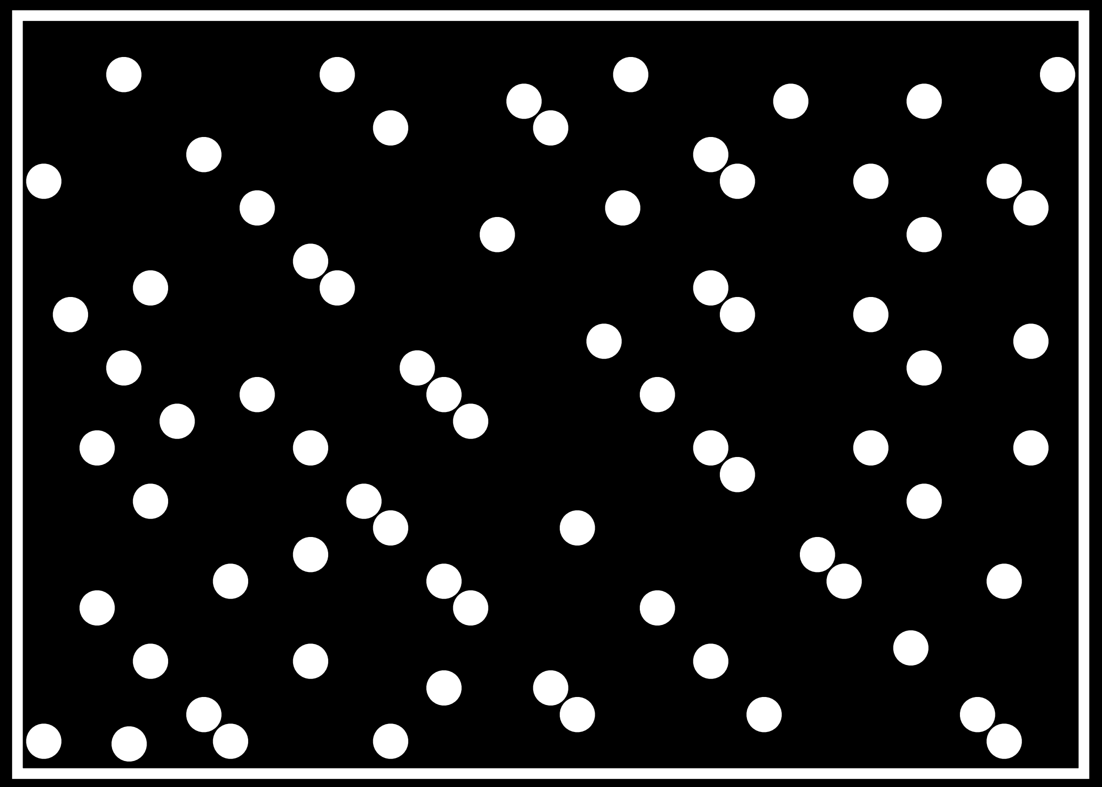
α irrational $\quad \alpha = \sqrt{2} - 1 \quad \alpha = \frac{\sqrt{5} - 1}{2}$

quasi-RNG (low discrepancy sequence)

pseudo-RNG



quasi-RNG



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more modern: Sobol - sequence

transform random variables

transform random variables

$$Y = g(X)$$

transform random variables

$$Y = g(X) \qquad\qquad g : J \to J'$$

transform random variables

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$$g : \mathbb{R} \to \mathbb{R}$$

transform random variables

$$Y = g(X) \qquad g : J \to J' \qquad Domain(g) := J$$

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$$Range(X) \subseteq Domain(g)$$

transform random variables

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$$g : \mathbb{R} \to \mathbb{R}$$

$$Range(X) \subseteq Domain(g)$$

inverse

transform random variables

$$Y = g(X) \qquad g : J \to J' \qquad Domain(g) := J$$

$$g : \mathbb{R} \to \mathbb{R}$$

$$Range(X) \subseteq Domain(g)$$

inverse $\quad g^{-1}(y) := \{x \in J : g(x) = y\}$

transform discrete random variables

$$g^{-1}(y) := \{x \in J : g(x) = y\}$$

$$P(A \cup B) = P(A) + P(B) \quad \text{if} \quad P(A \cap B) = 0$$

transform discrete random variables

given

$$g^{-1}(y) := \{x \in J : g(x) = y\}$$

$$P(A \cup B) = P(A) + P(B) \quad \text{if} \quad P(A \cap B) = 0$$

transform discrete random variables

given $p_X,$

$$g^{-1}(y) := \{x \in J : g(x) = y\}$$

$$P(A \cup B) = P(A) + P(B) \quad \text{if} \quad P(A \cap B) = 0$$

transform discrete random variables

given $p_X,$ $Y = g(X)$

$$g^{-1}(y) := \{x \in J : g(x) = y\}$$

$$P(A \cup B) = P(A) + P(B) \quad \text{if} \quad P(A \cap B) = 0$$

transform discrete random variables

given p_X, $Y = g(X)$

What is p_Y ?

$$g^{-1}(y) := \{x \in J : g(x) = y\}$$

$$P(A \cup B) = P(A) + P(B) \quad \text{if} \quad P(A \cap B) = 0$$

transform discrete random variables

given p_X, $Y = g(X)$

$g^{-1}(y) := \{x \in J : g(x) = y\}$

$P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

What is p_Y ?

$p_Y(y)$

transform discrete random variables

given p_X, $Y = g(X)$

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$$p_Y(y) = P(Y = y)$$

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$g^{-1}(y) := \{x \in J : g(x) = y\}$

$P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

What is p_Y ?

$$p_Y(y) = P(Y = y) = P(g(X) = y) = P\left(X \in g^{-1}(y)\right)$$

transform discrete random variables

given p_X, $Y = g(X)$

$$g^{-1}(y) := \{x \in J : g(x) = y\}$$

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$$= P\left(\bigcup_{x \in g^{-1}(y)} \{X = x\}\right)$$

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given p_X, $\quad Y = g(X)$

$$g^{-1}(y) := \{x \in J : g(x) = y\}$$

$$P(A \cup B) = P(A) + P(B) \quad \text{if} \quad P(A \cap B) = 0$$

What is p_Y ?

$$p_Y(y) = P(Y = y) = P(g(X) = y) = P\left(X \in g^{-1}(y)\right)$$

$$= P\left(\bigcup_{x \in g^{-1}(y)} \{X = x\}\right) = \sum_{x \in g^{-1}(y)} p_X(x)$$

exercise: transform discrete random variables

exercise: transform discrete random variables

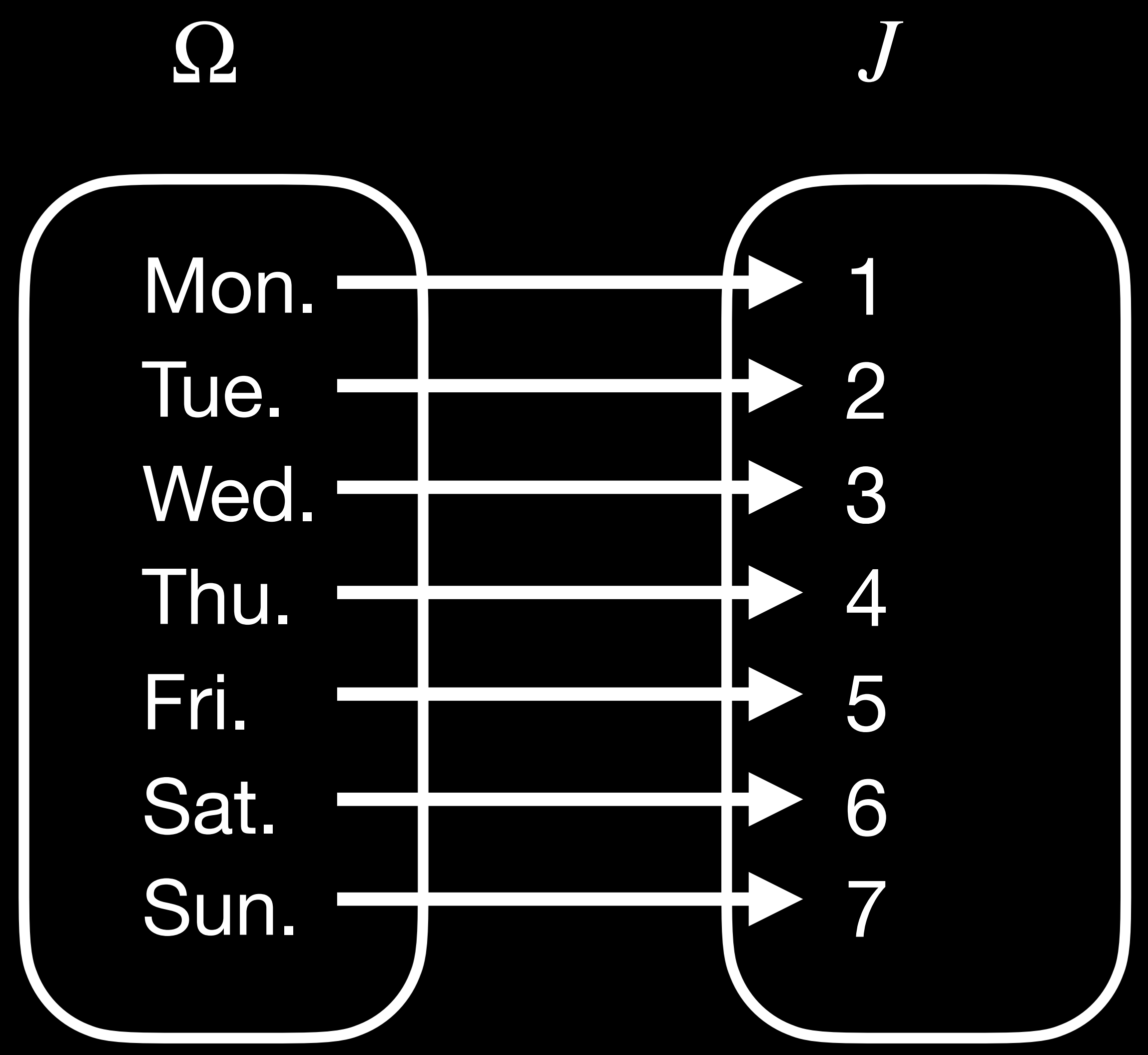
Ω

Mon.
Tue.
Wed.
Thu.
Fri.
Sat.
Sun.

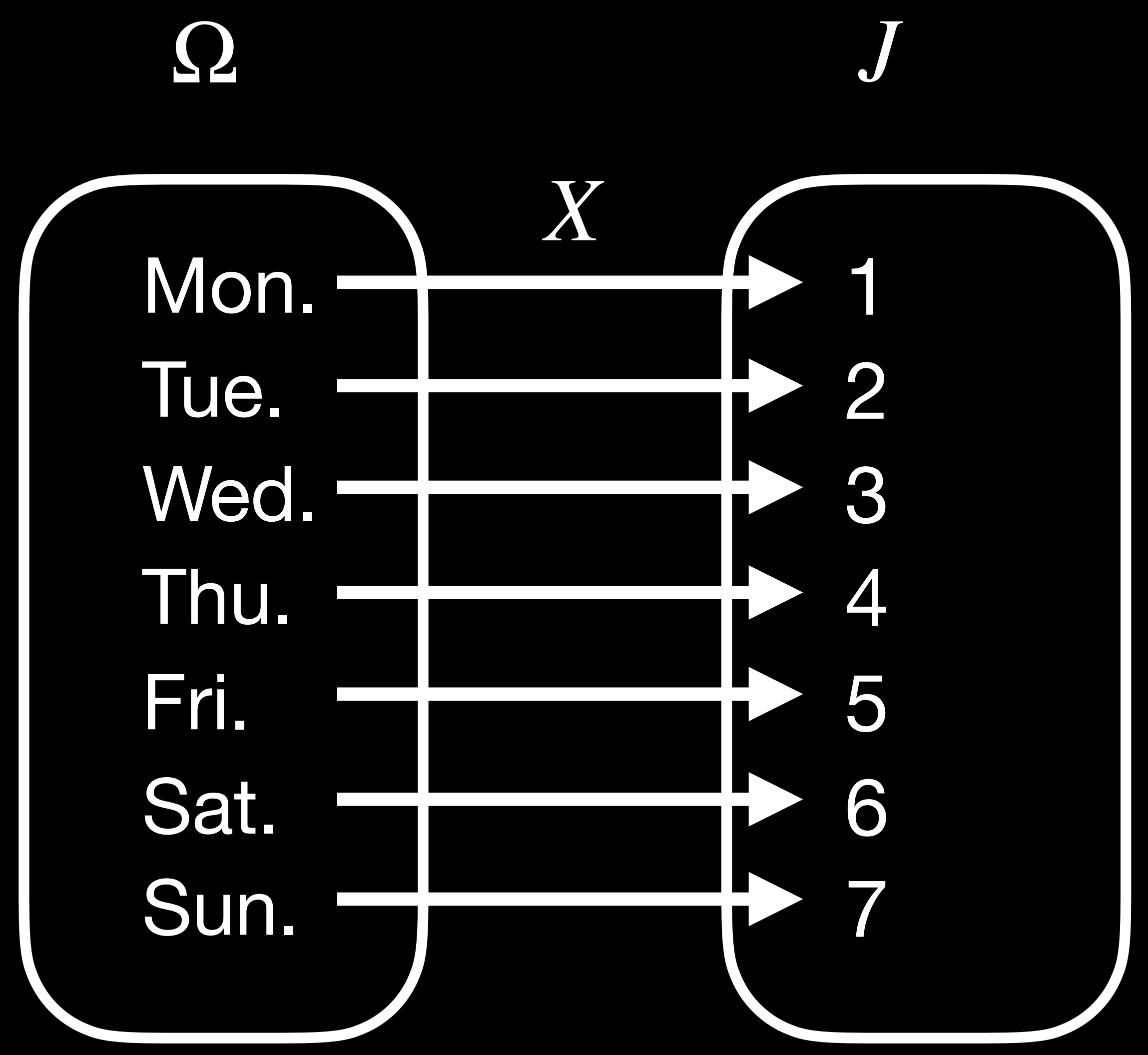
exercise: transform discrete random variables

Ω	J
Mon.	1
Tue.	2
Wed.	3
Thu.	4
Fri.	5
Sat.	6
Sun.	7

exercise: transform discrete random variables



exercise: transform discrete random variables



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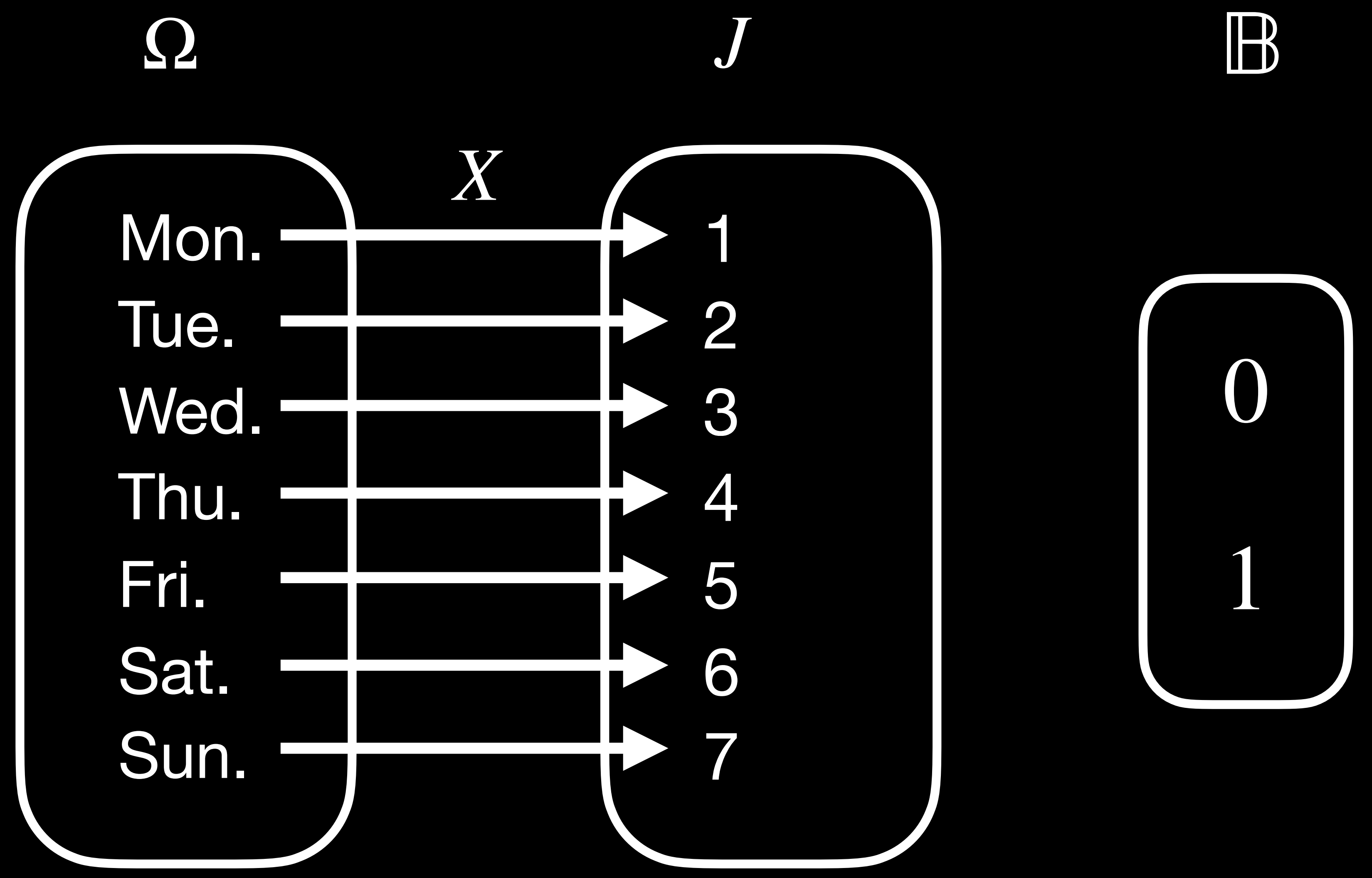

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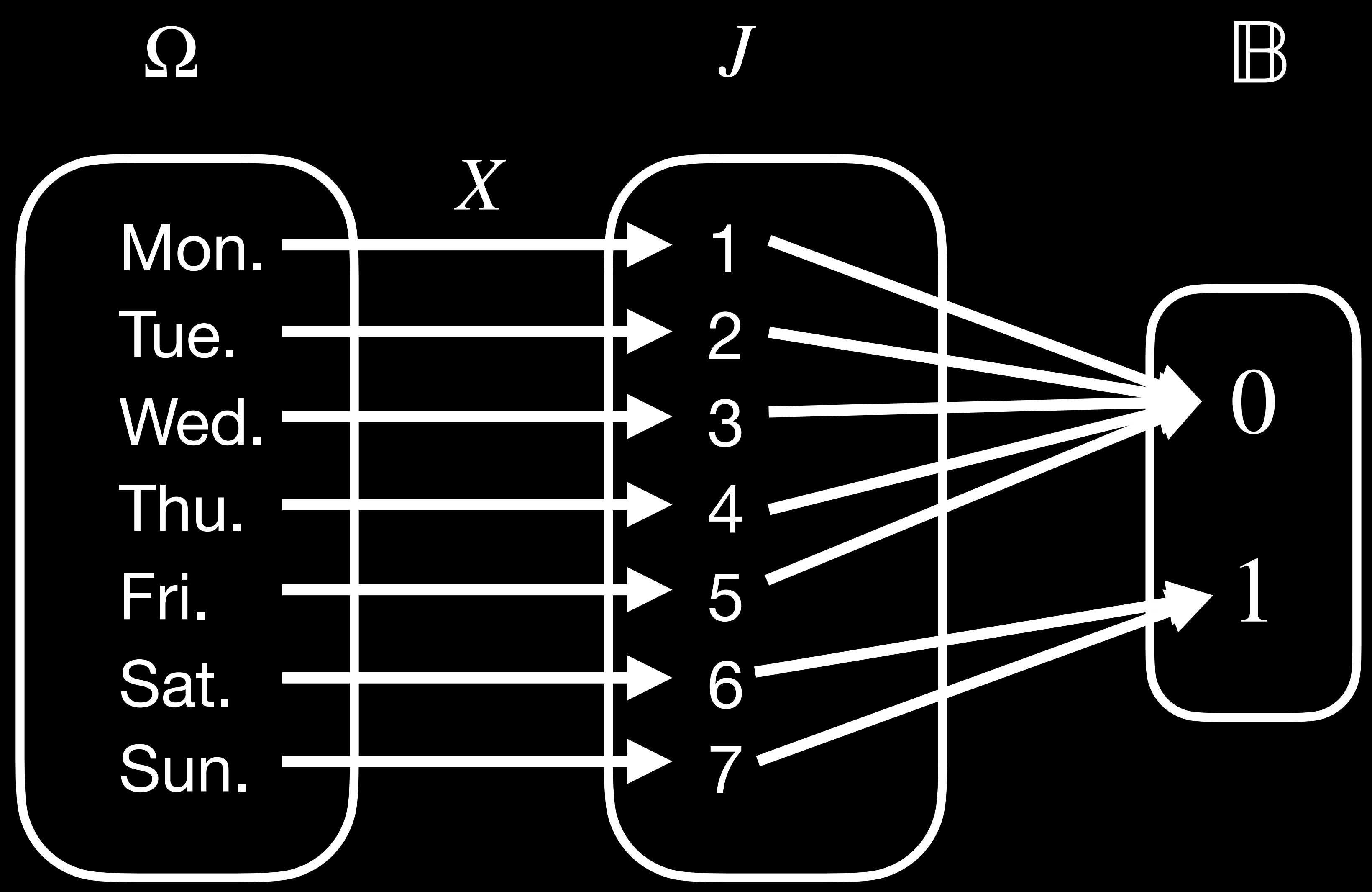

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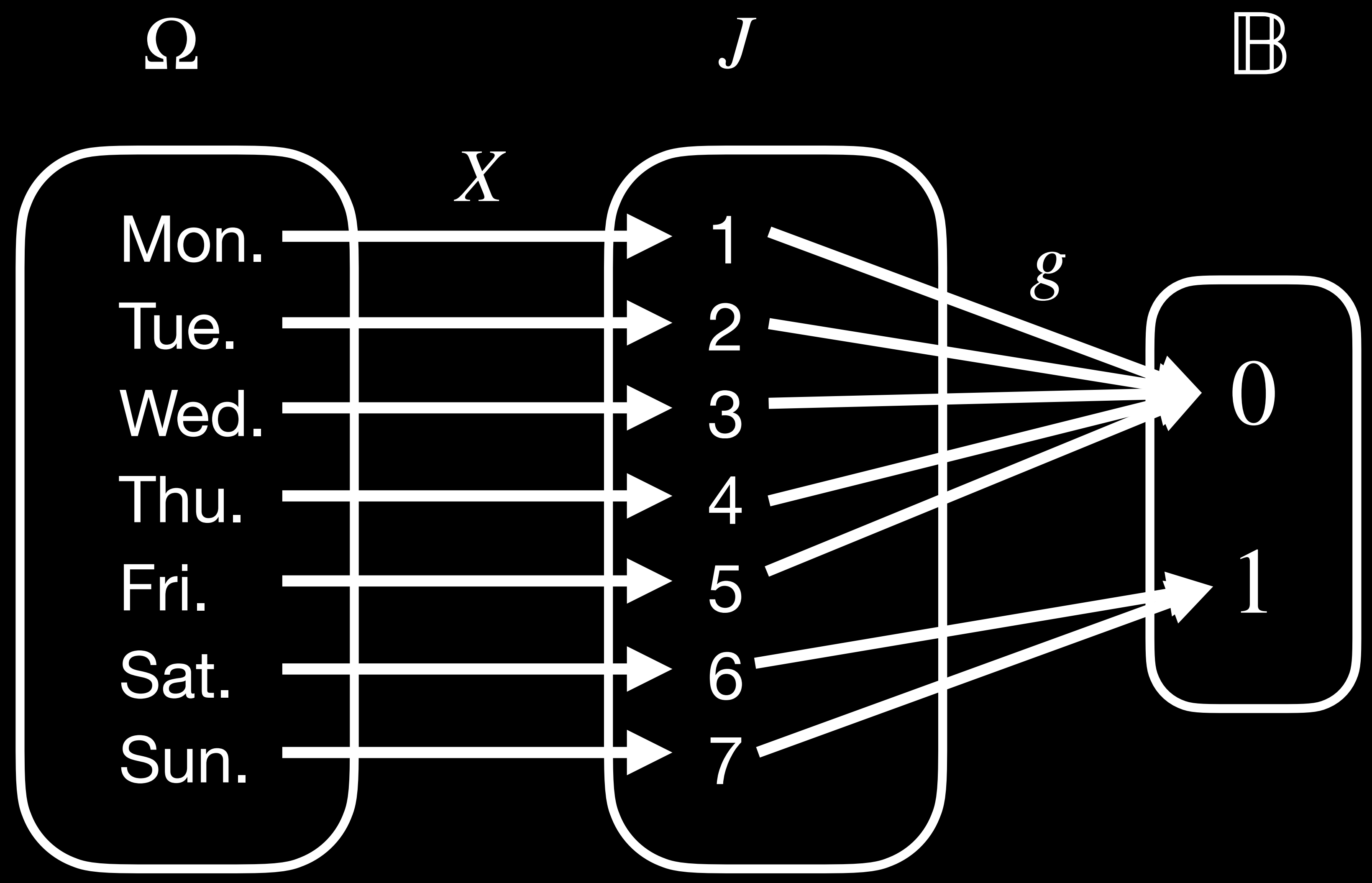
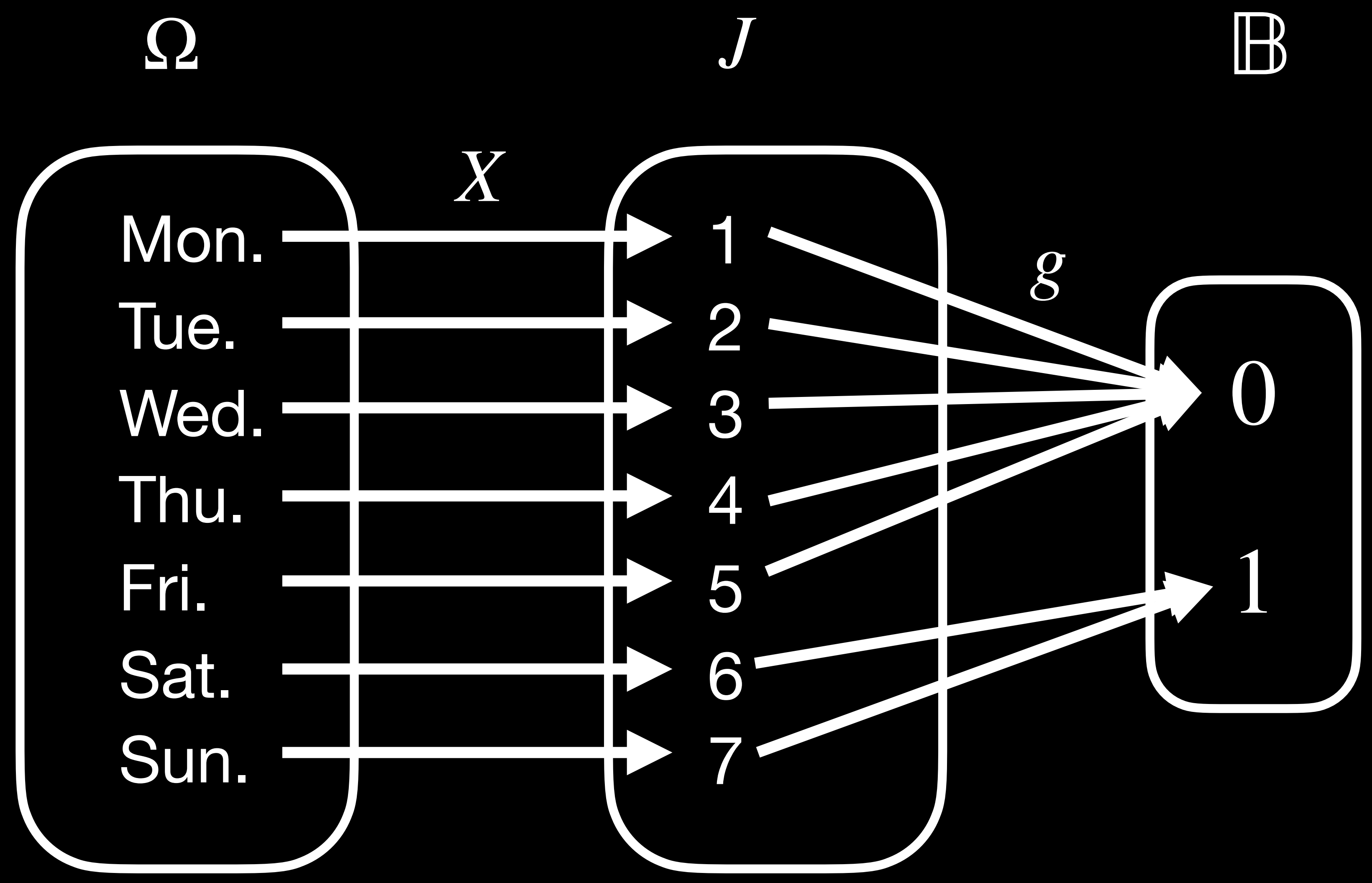
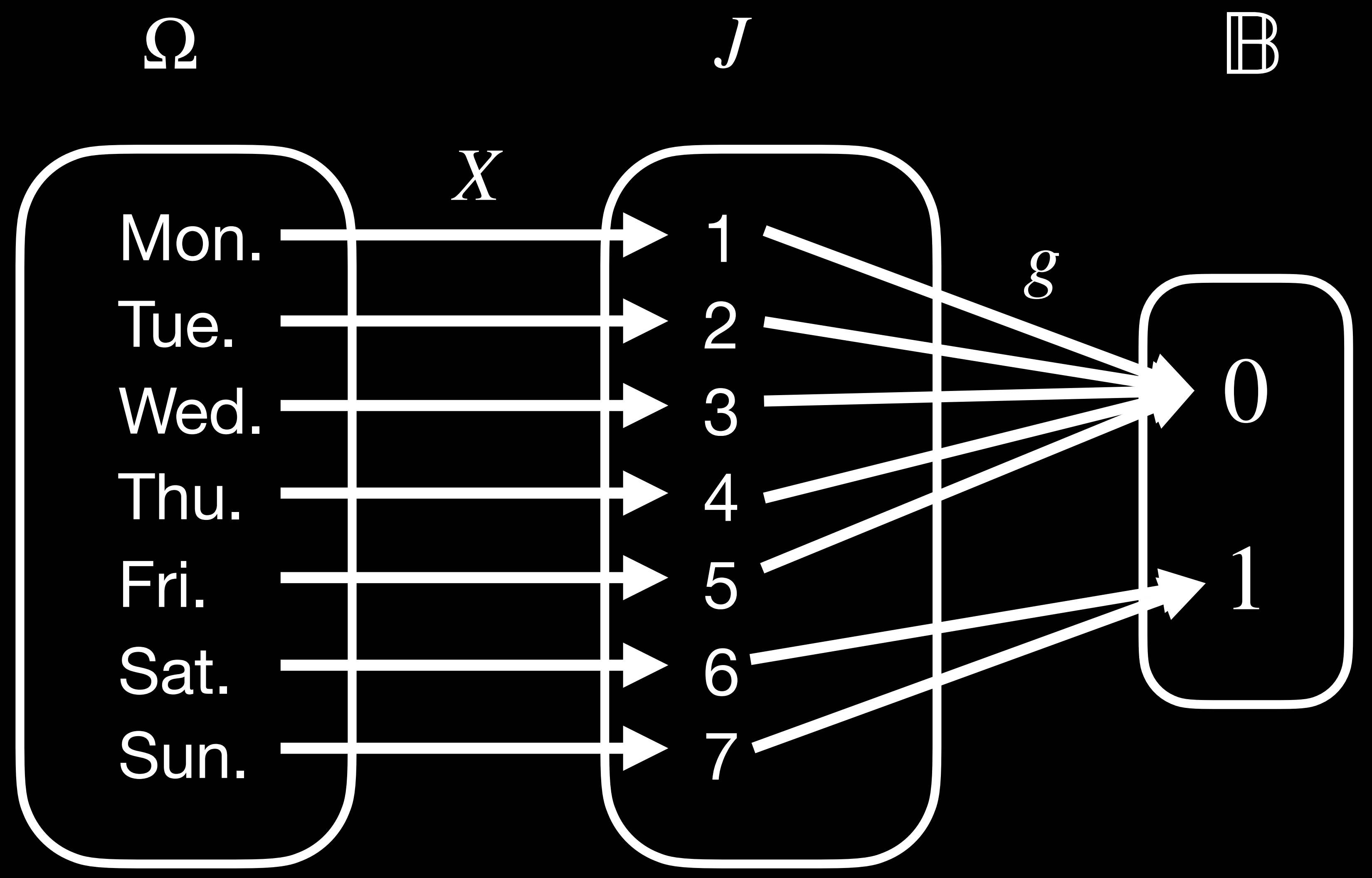


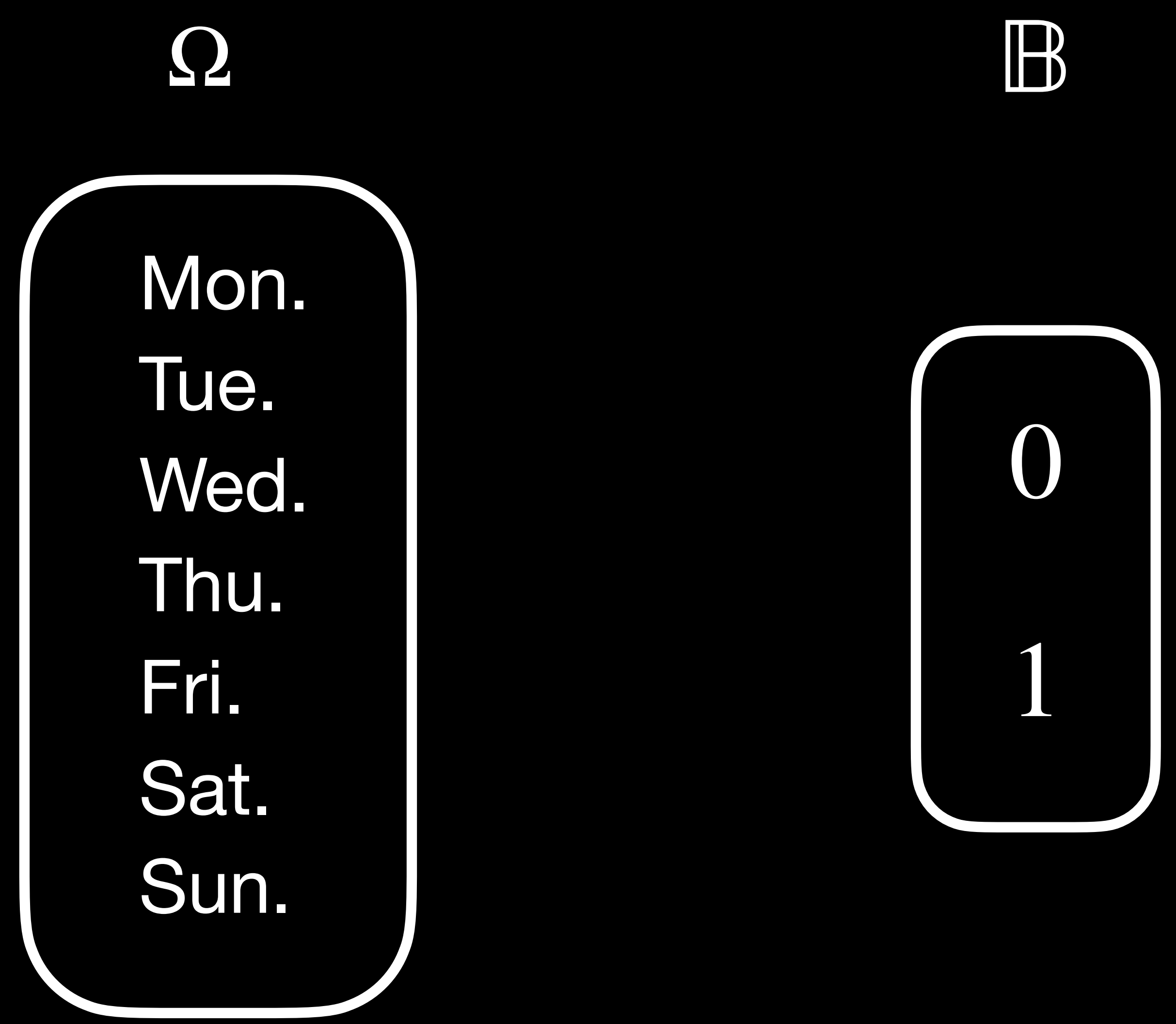
exercise: transform discrete random variables



$$g(x) = \begin{cases} 0 & \text{for } x \in \{1,\ldots,5\} \\ 1 & \text{for } x \in \{6,7\} \end{cases}$$

exercise: transform discrete random variables





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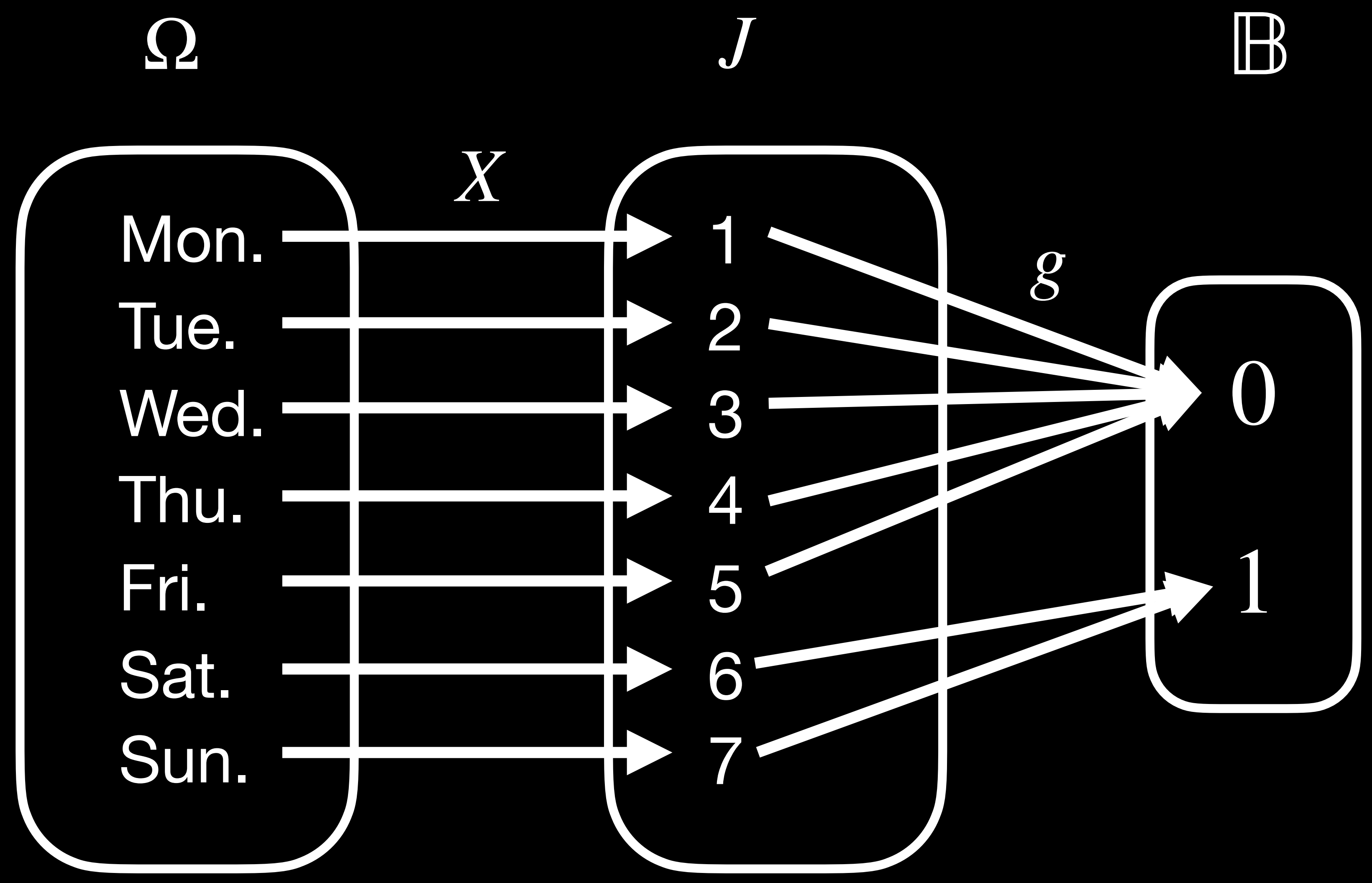

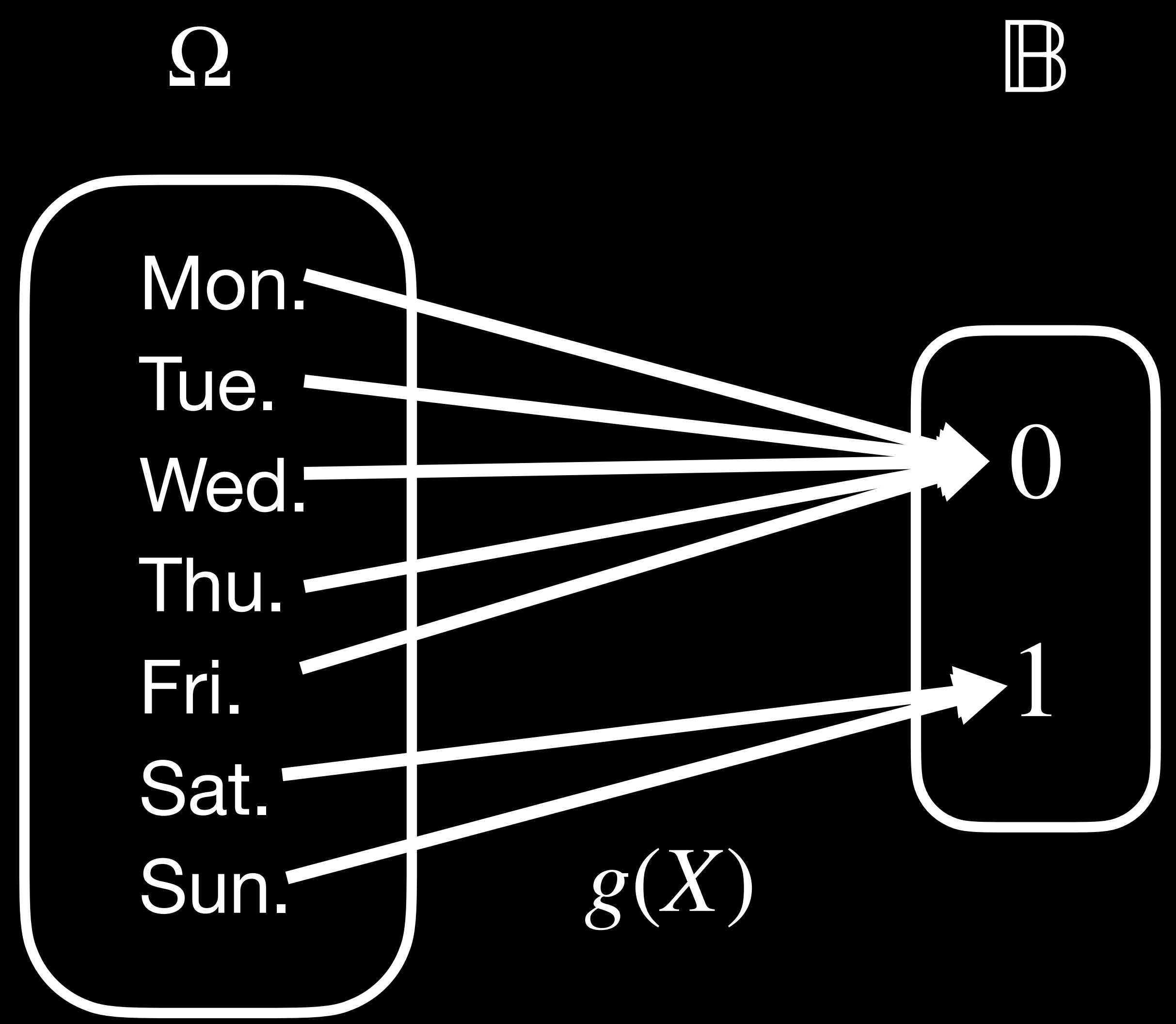

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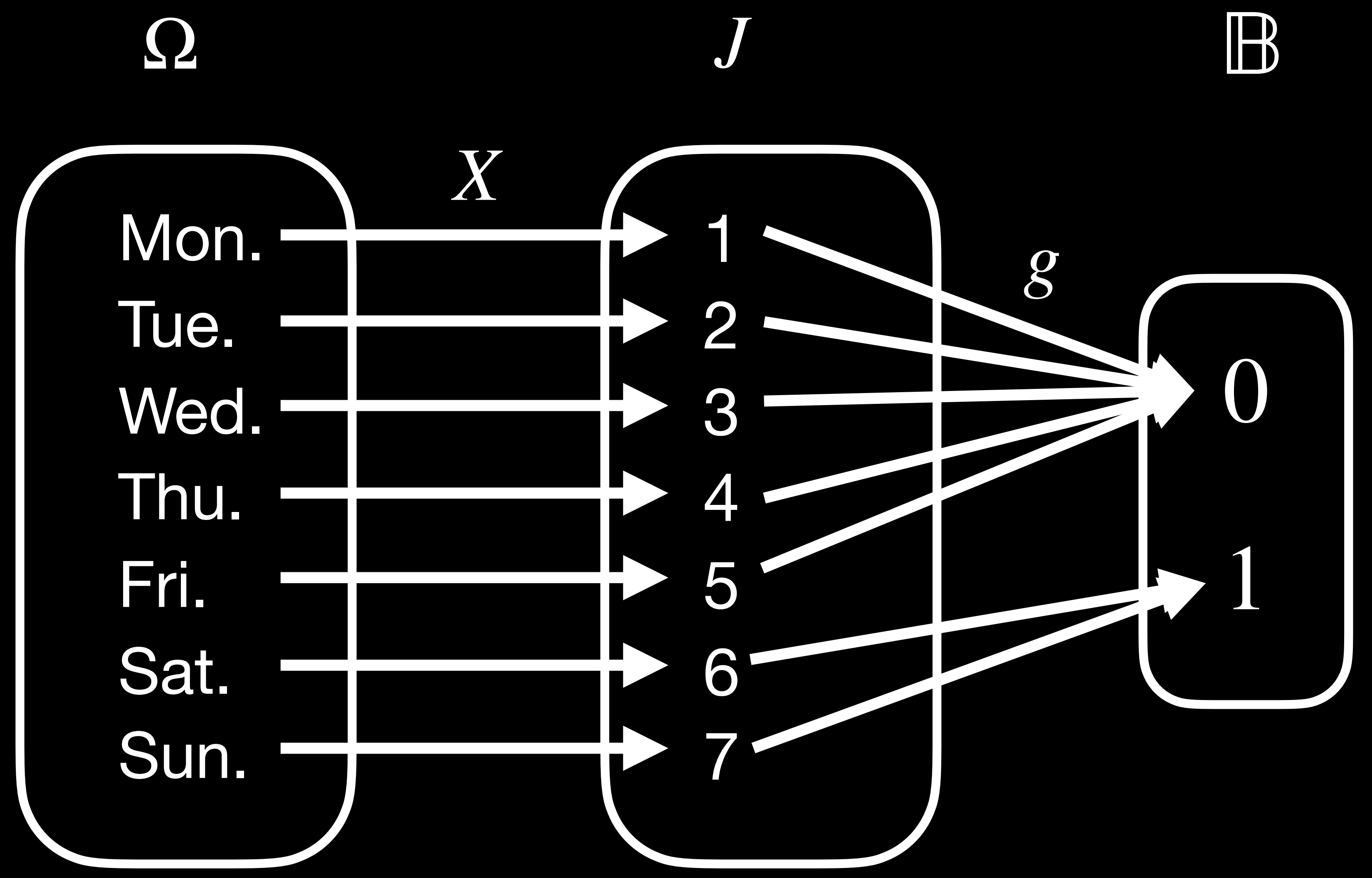

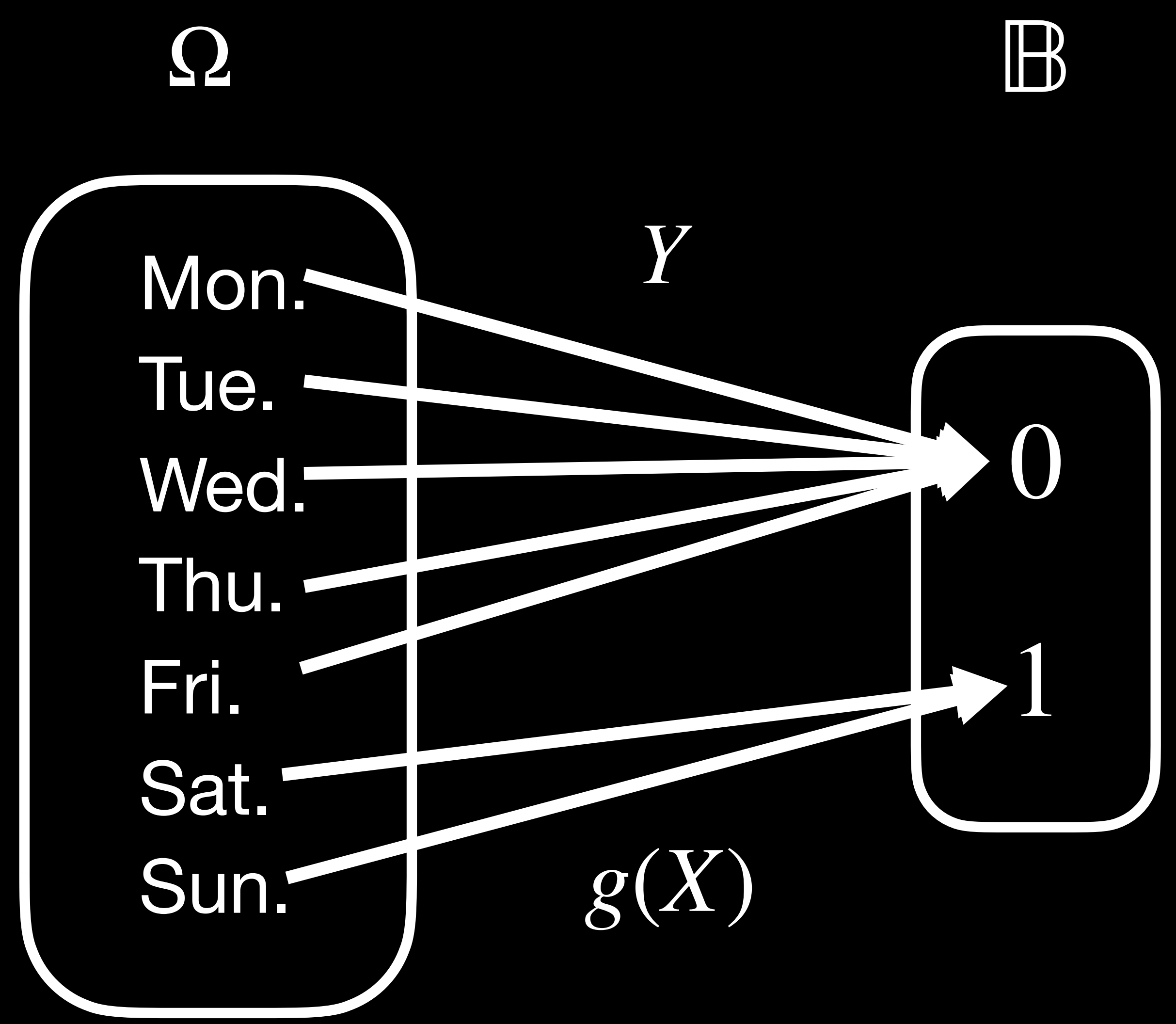

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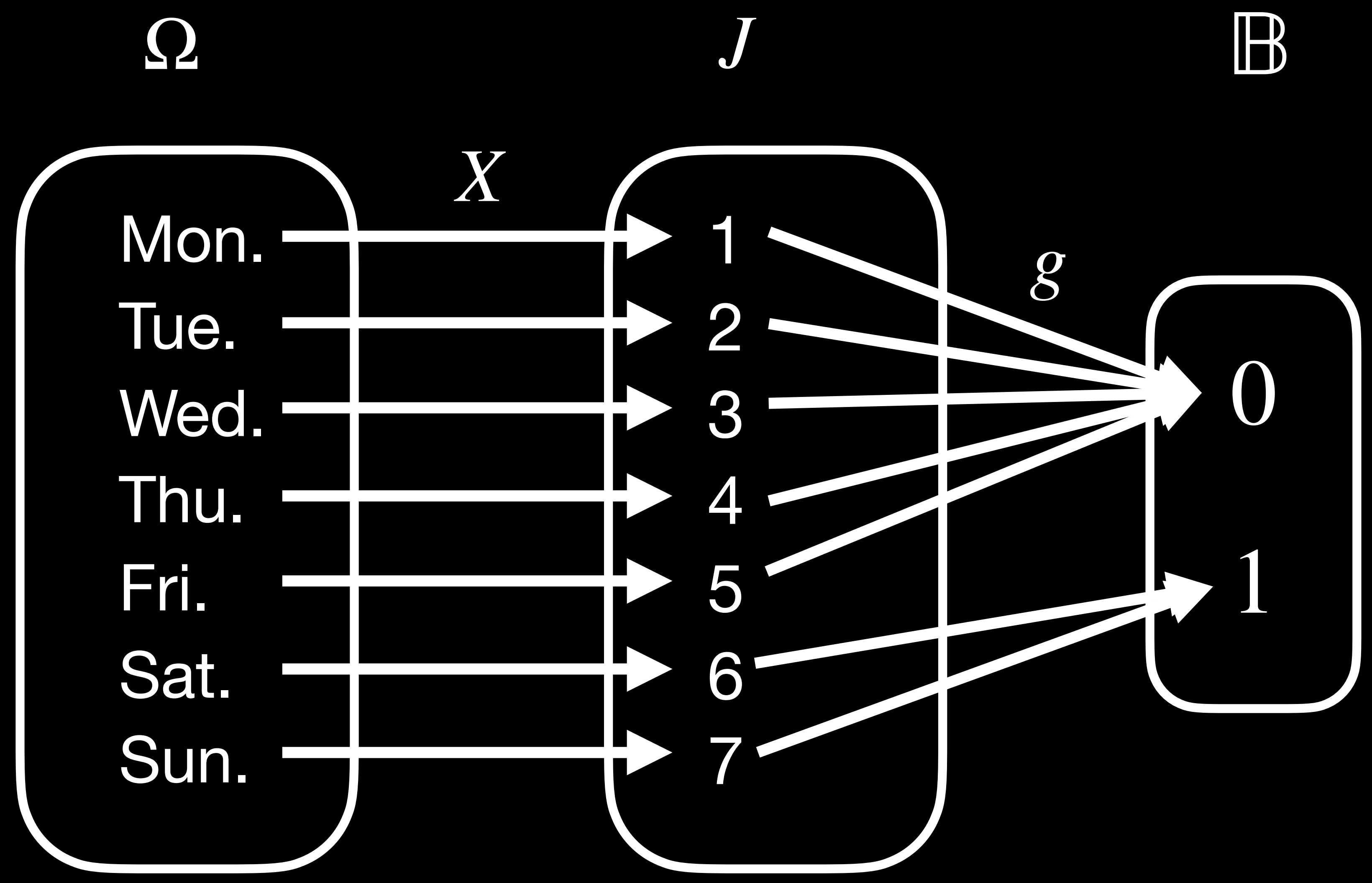
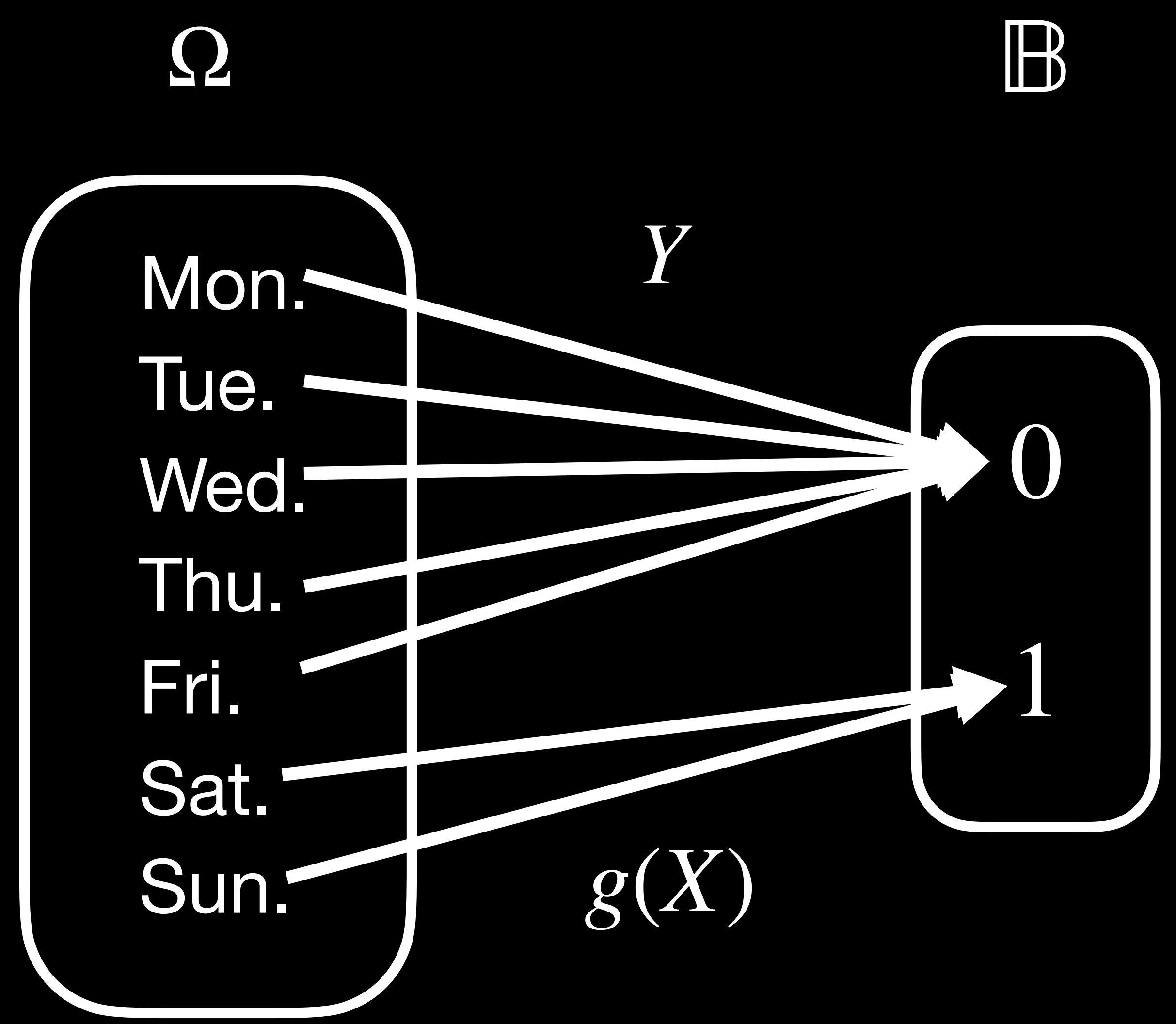
exercise: transform discrete random variables





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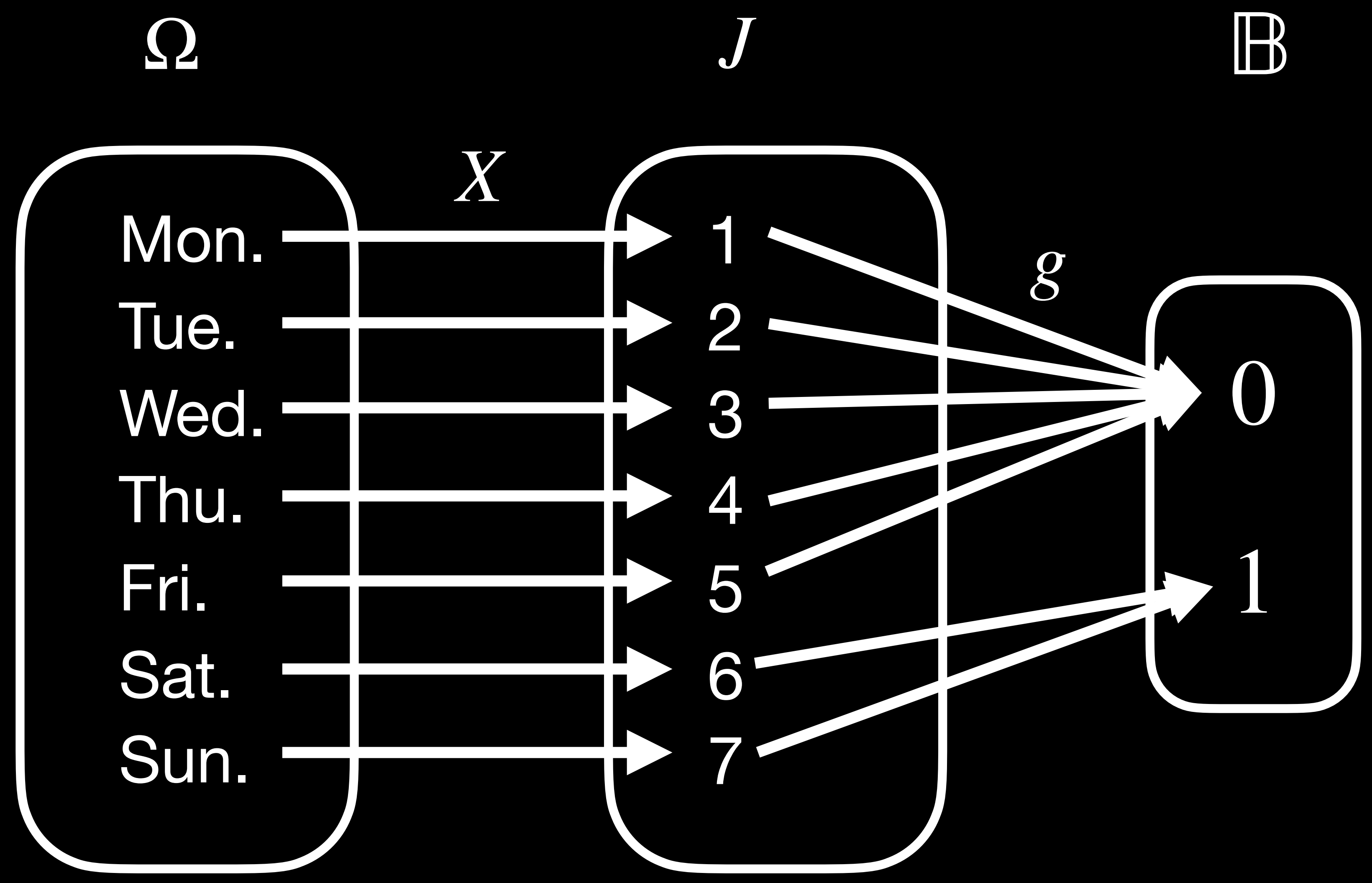

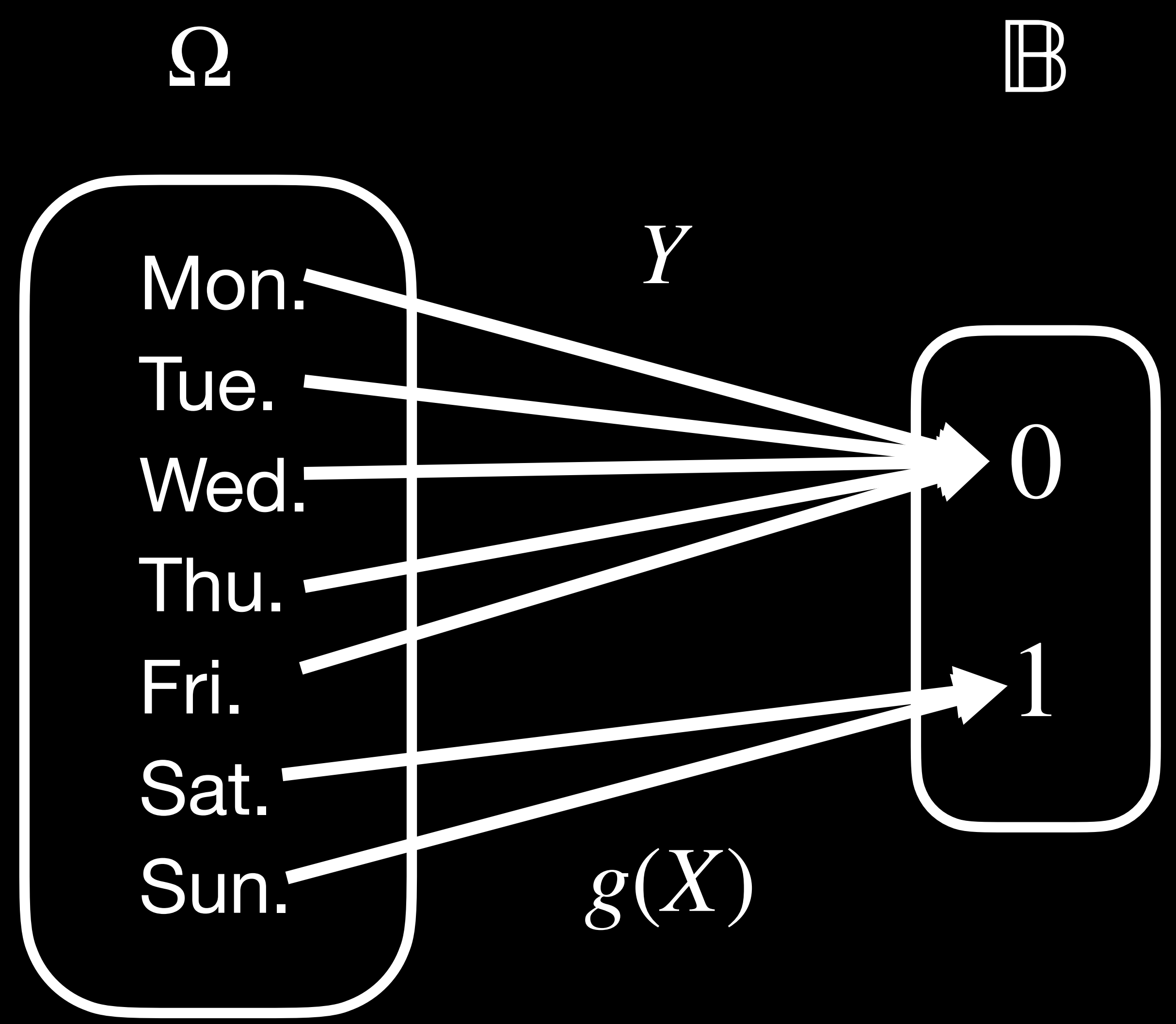

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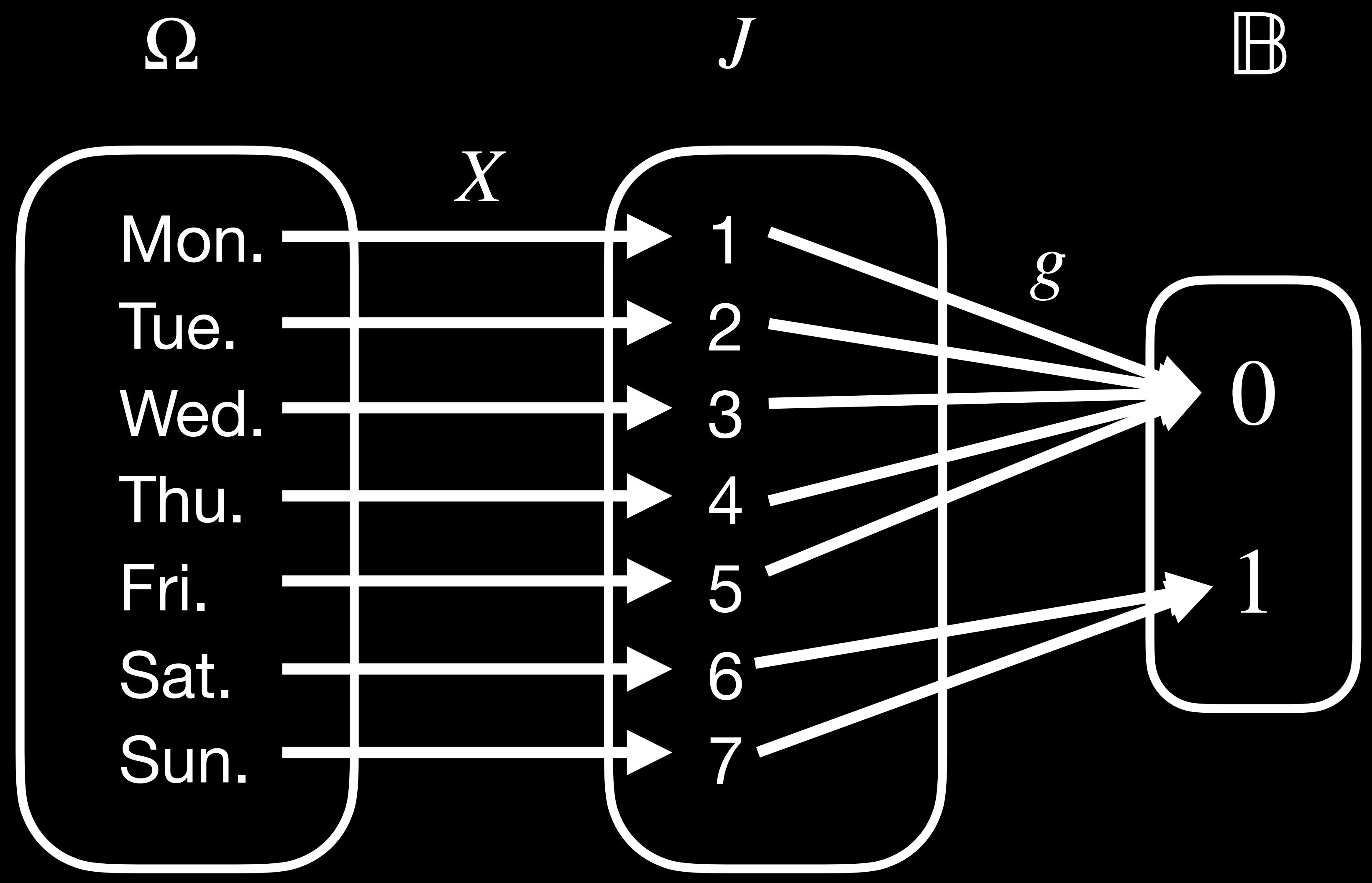

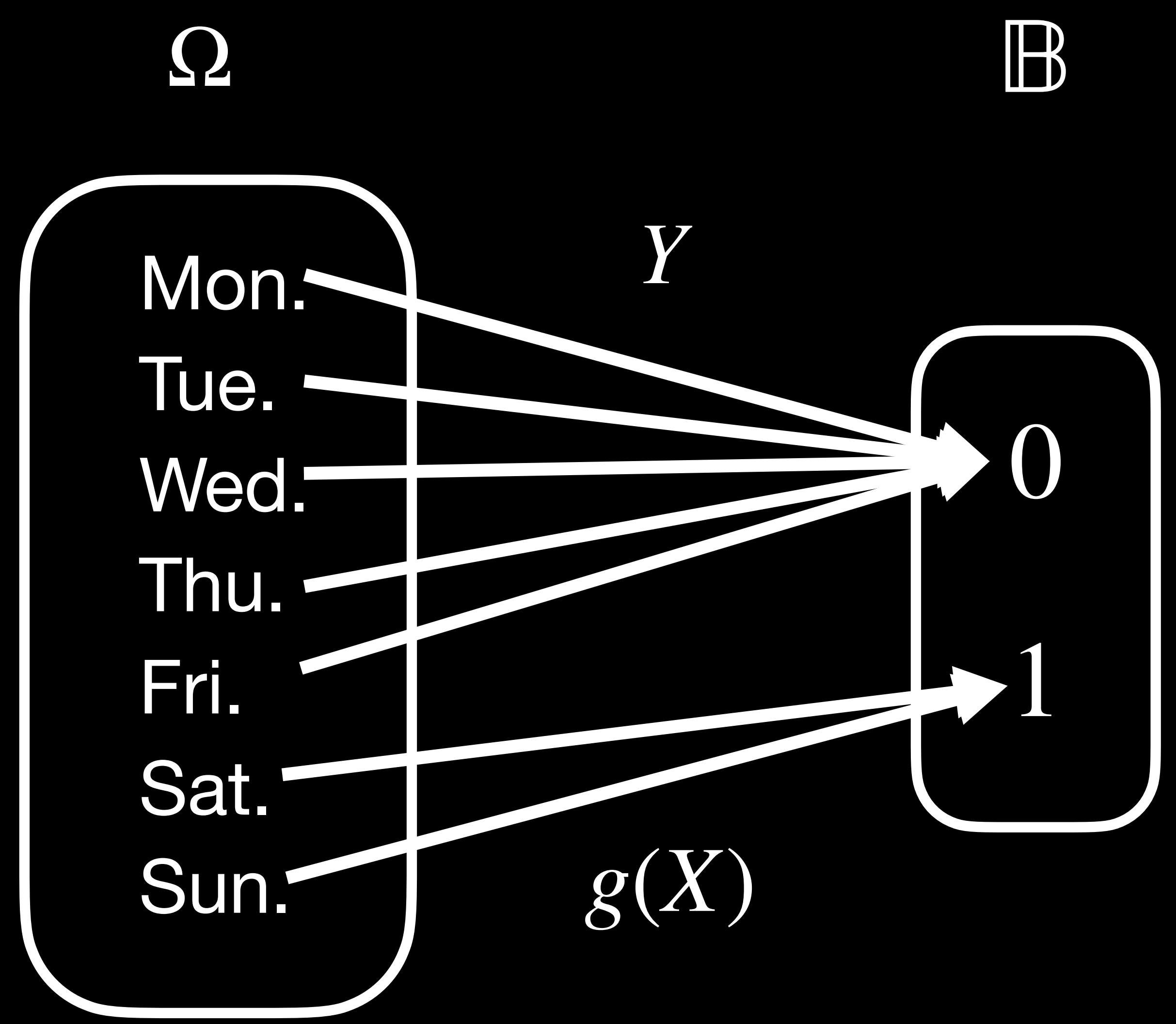

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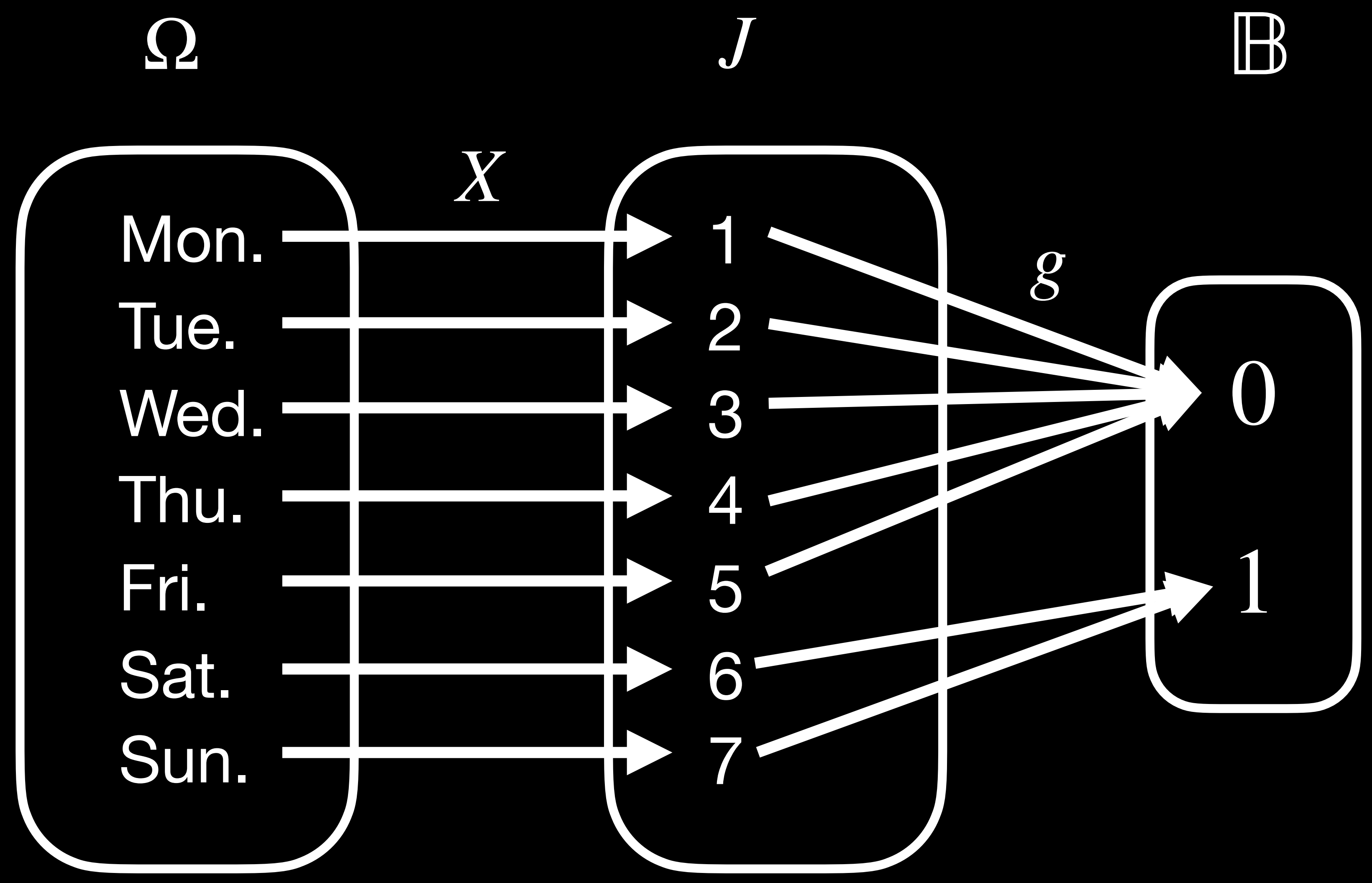
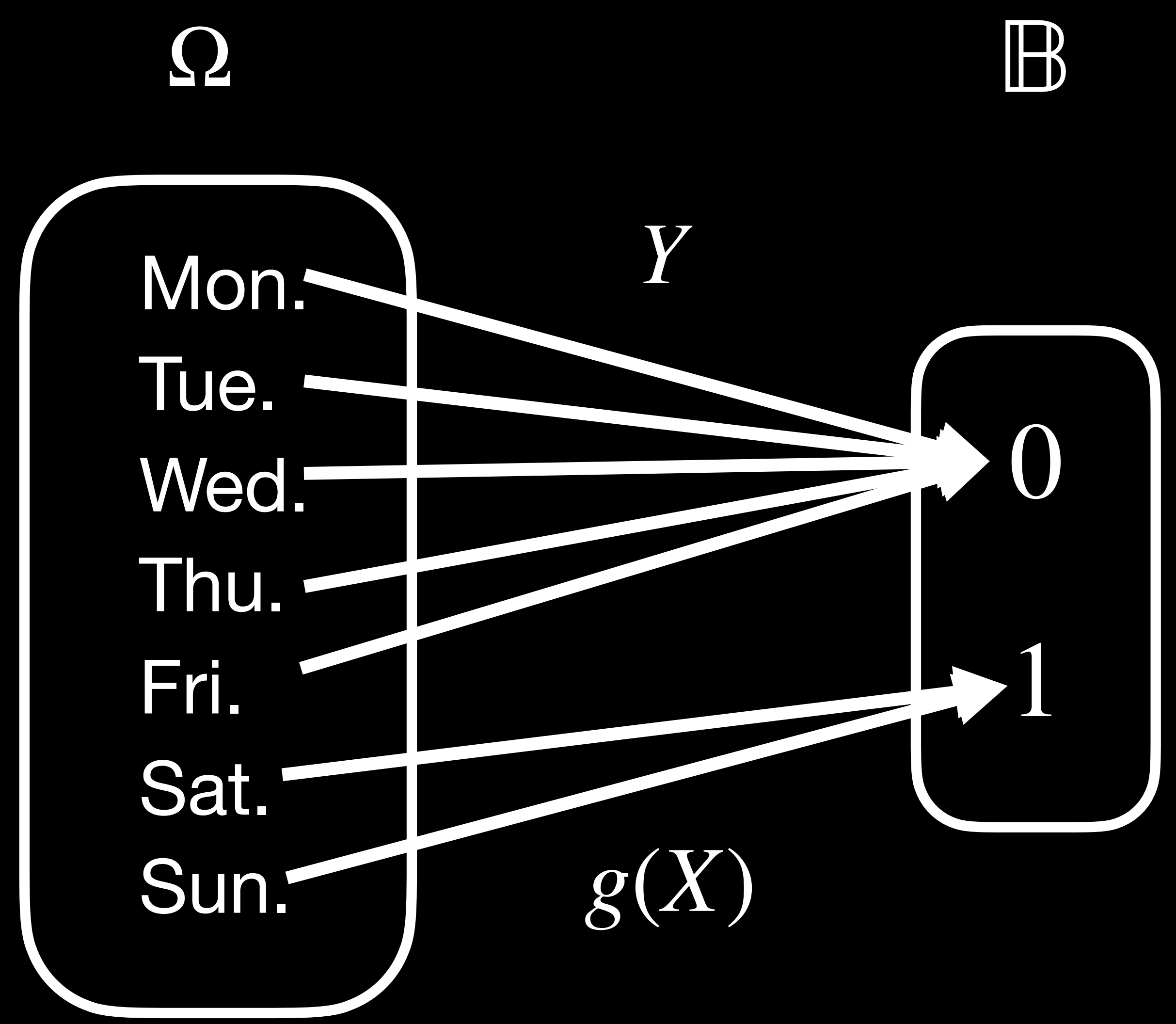
derive: $p_Y(1) =$

exercise: transform discrete random variables





$$g(x) = \begin{cases} 0 & \text{for } x \in \{1,\ldots,5\} \\ 1 & \text{for } x \in \{6,7\} \end{cases}$$

$$p_Y(y) = \sum_{x \in g^{-1}(y)} p_X(x)$$

$$g^{-1}(y) := \{x \in J : g(x) = y\}$$

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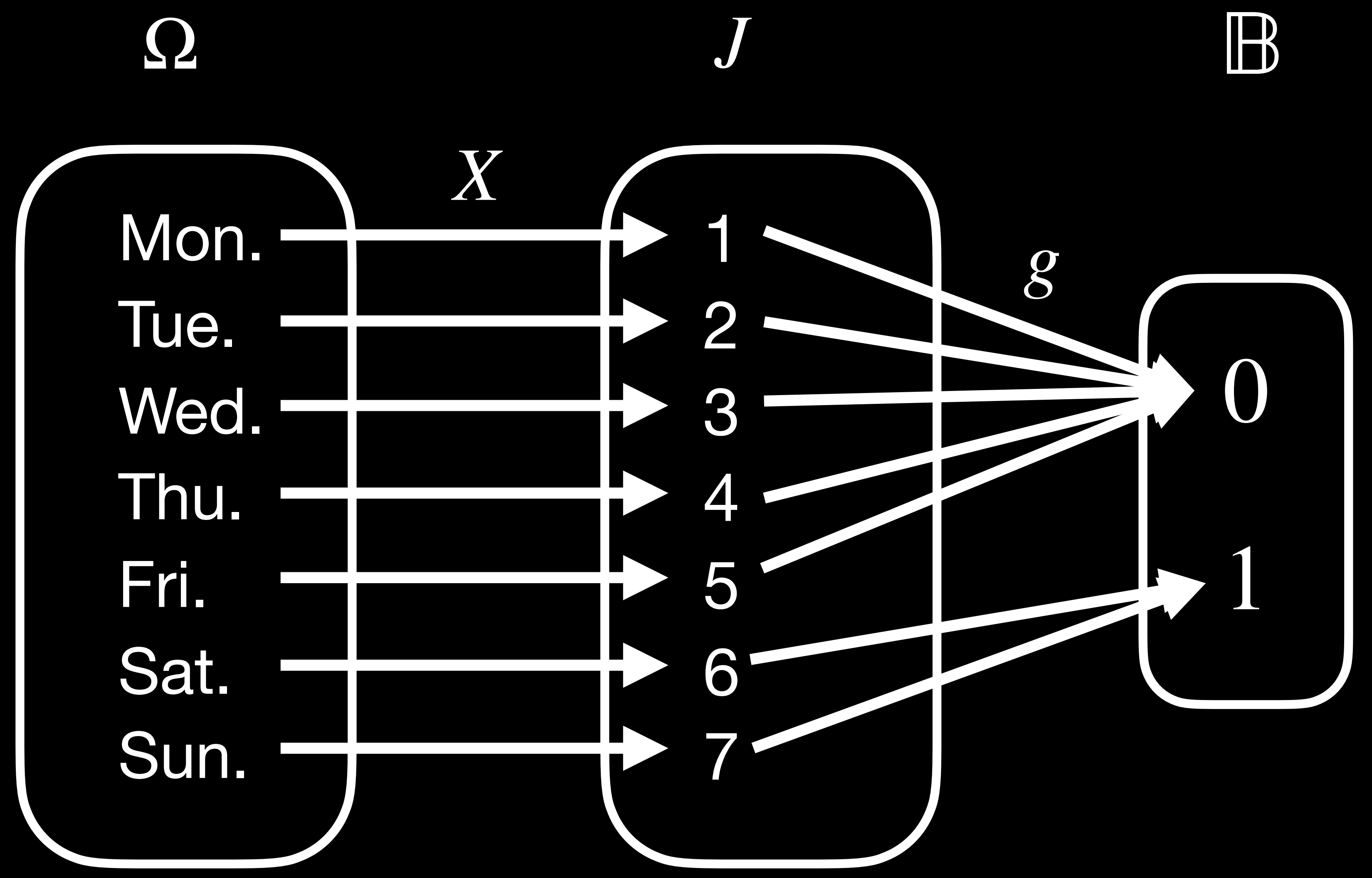
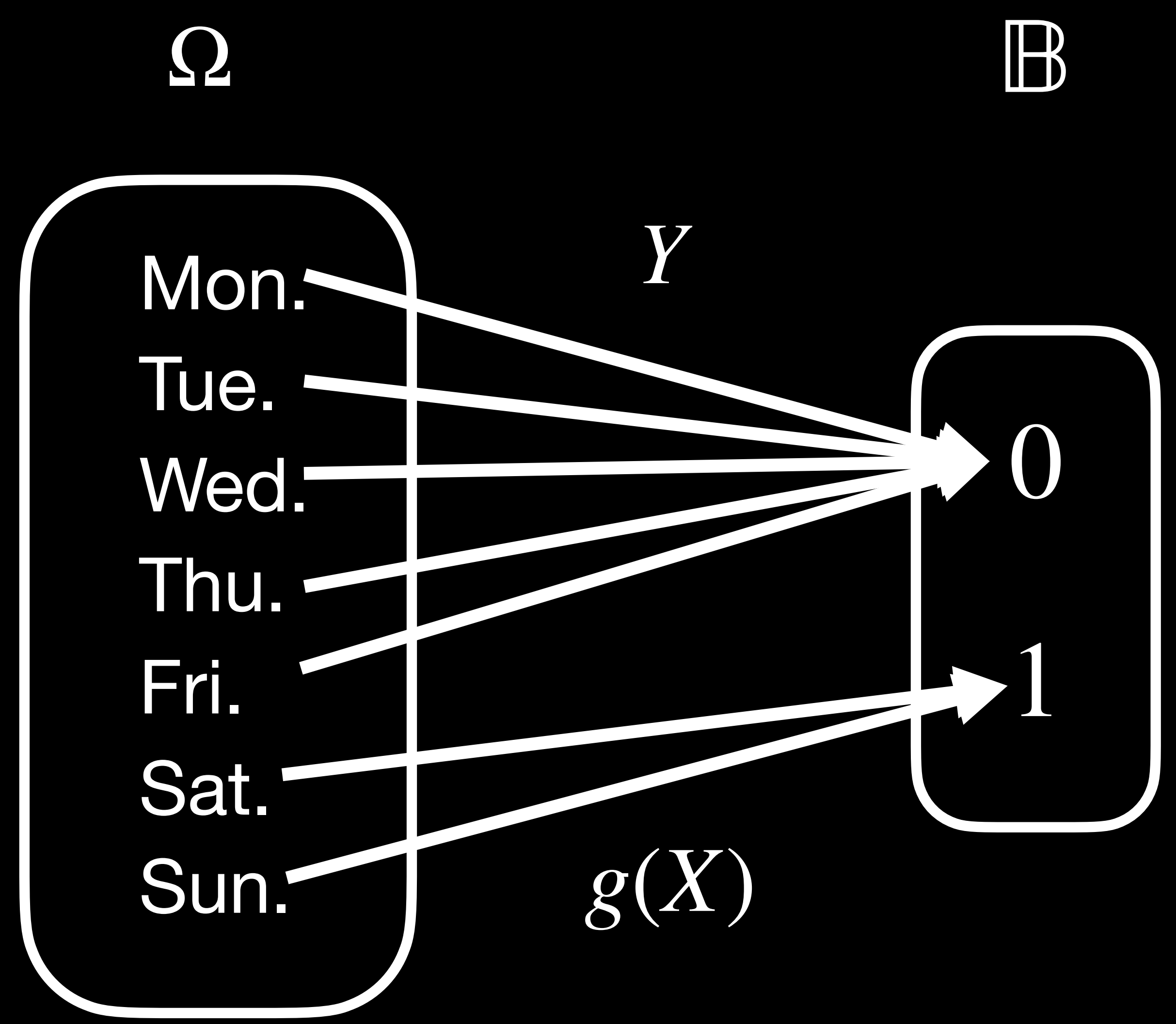
exercise: transform discrete random variables





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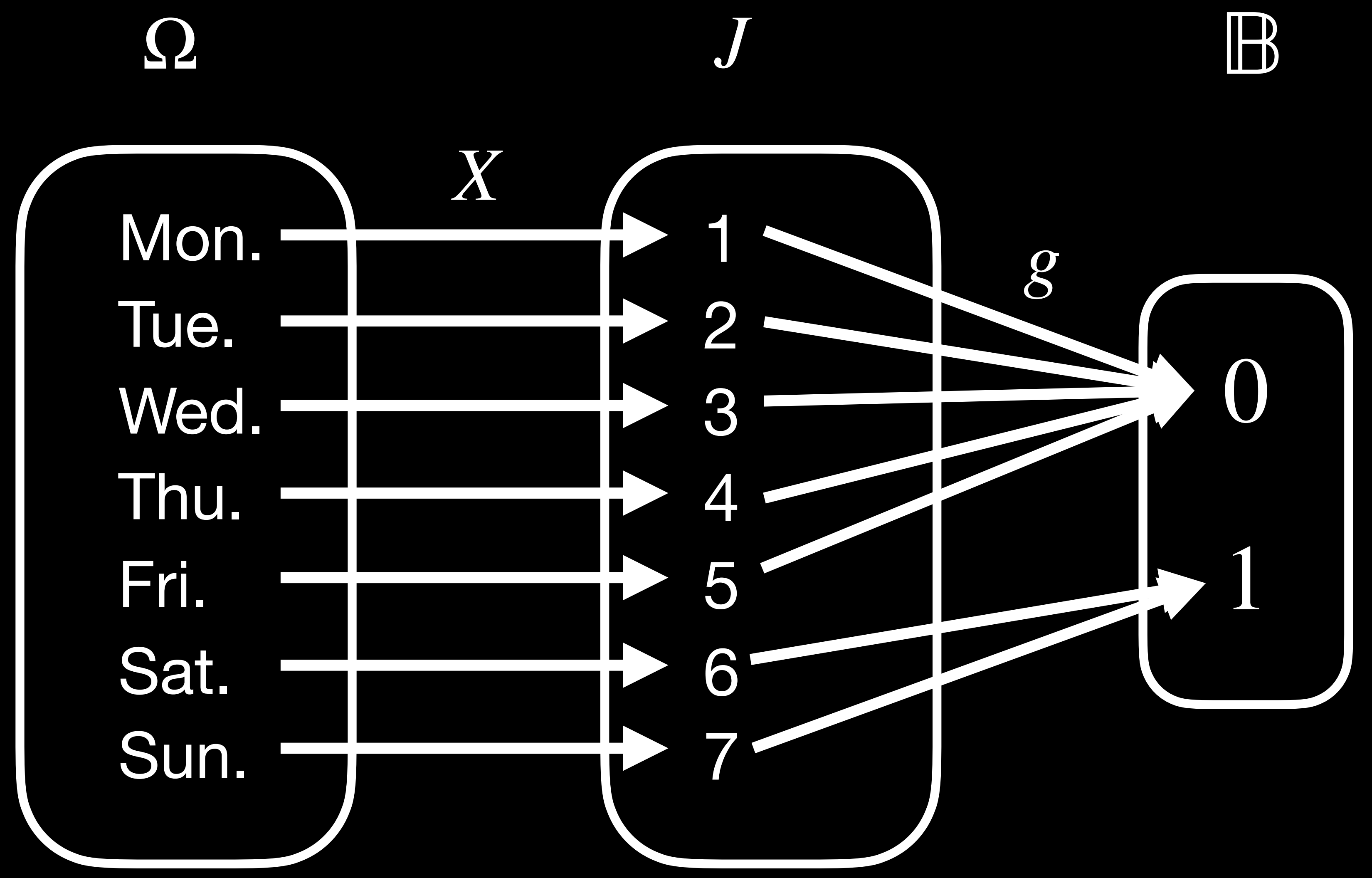
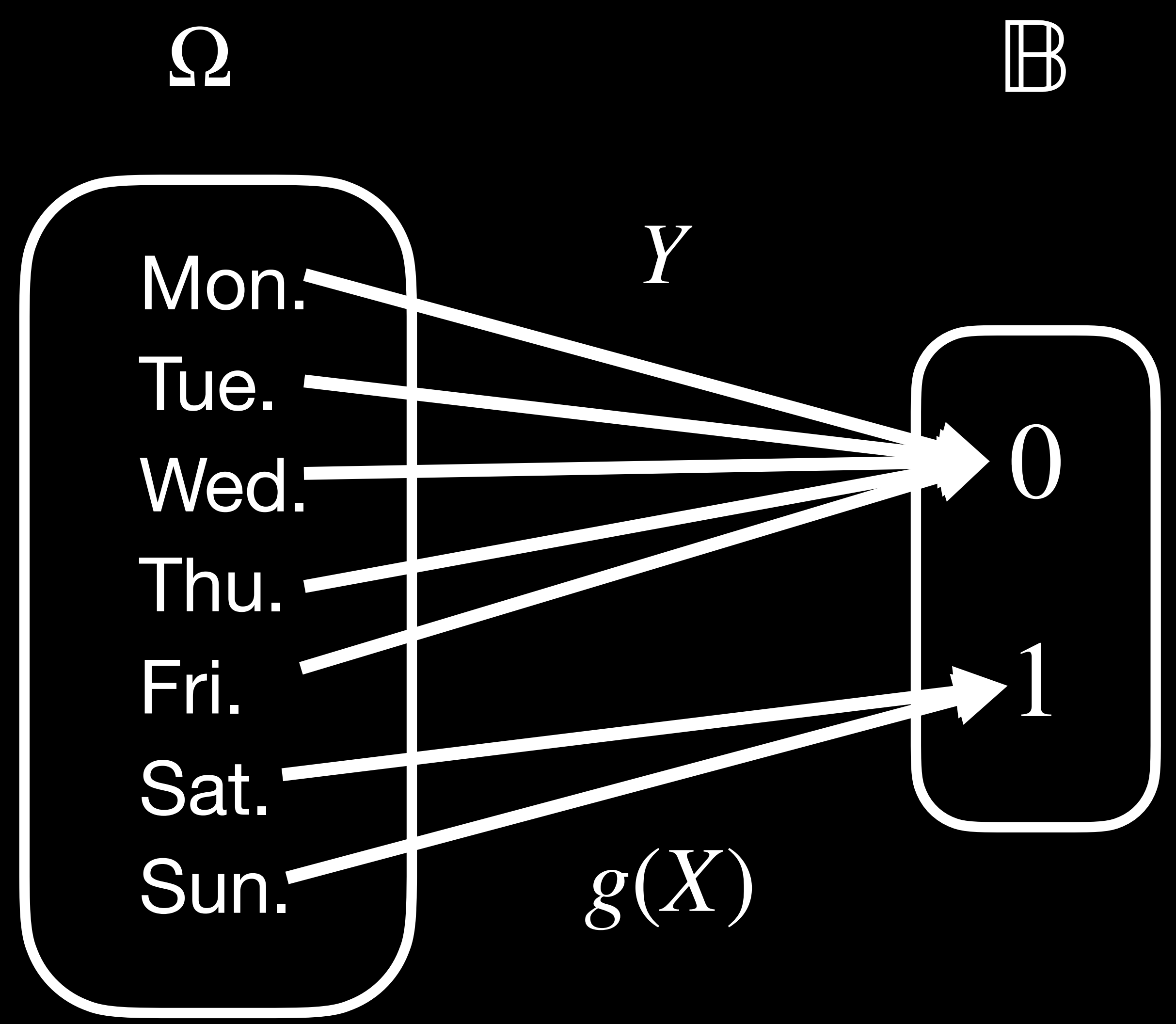
$$\boxed{\begin{array}{c} p_Y(y) = \sum_{x \in g^{-1}(y)} p_X(x) \\ g^{-1}(y) := \{x \in J : g(x) = y\} \end{array}}$$

$$p_X(x) = \frac{1}{7} \qquad x \in \{1,\ldots,7\}$$

derive: $p_Y(1) = \sum_{x \in g^{-1}(1)} p_X(x)$

exercise: transform discrete random variables





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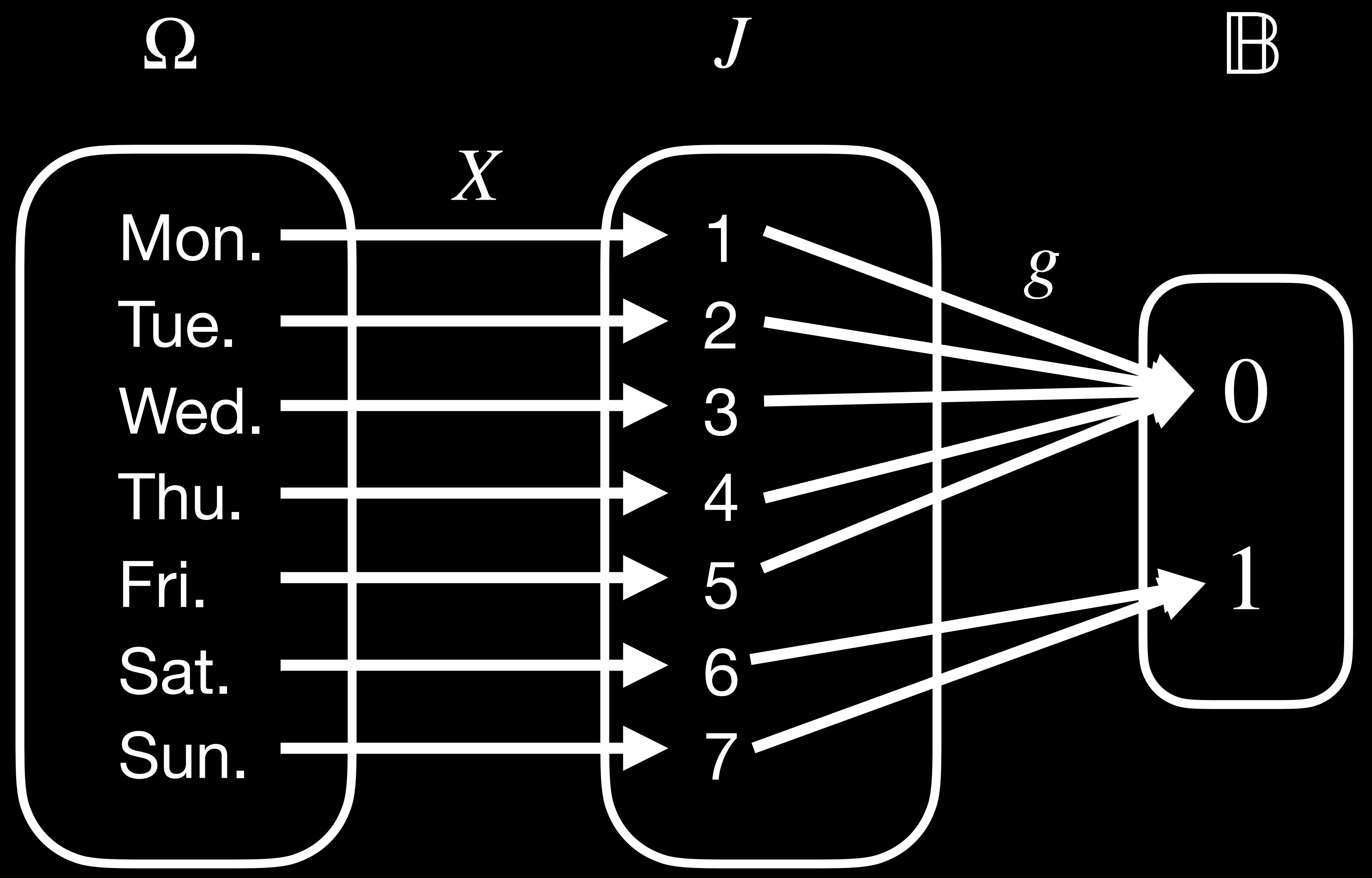
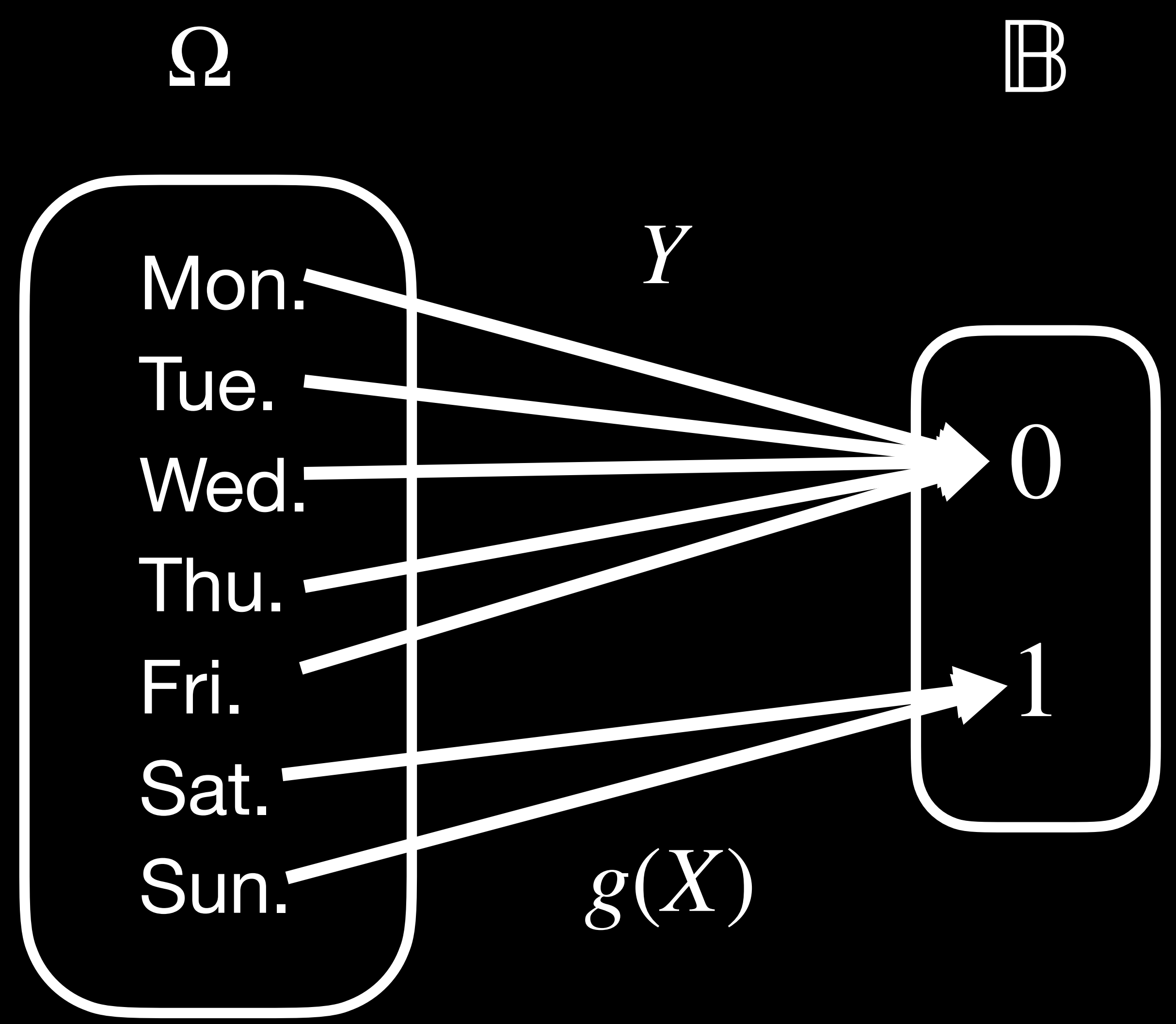
derive: $$p_Y(1) = \sum_{x \in g^{-1}(1)} p_X(x) = \sum_{x \in \{6,7\}} p_X(x)$$

exercise: transform discrete random variables





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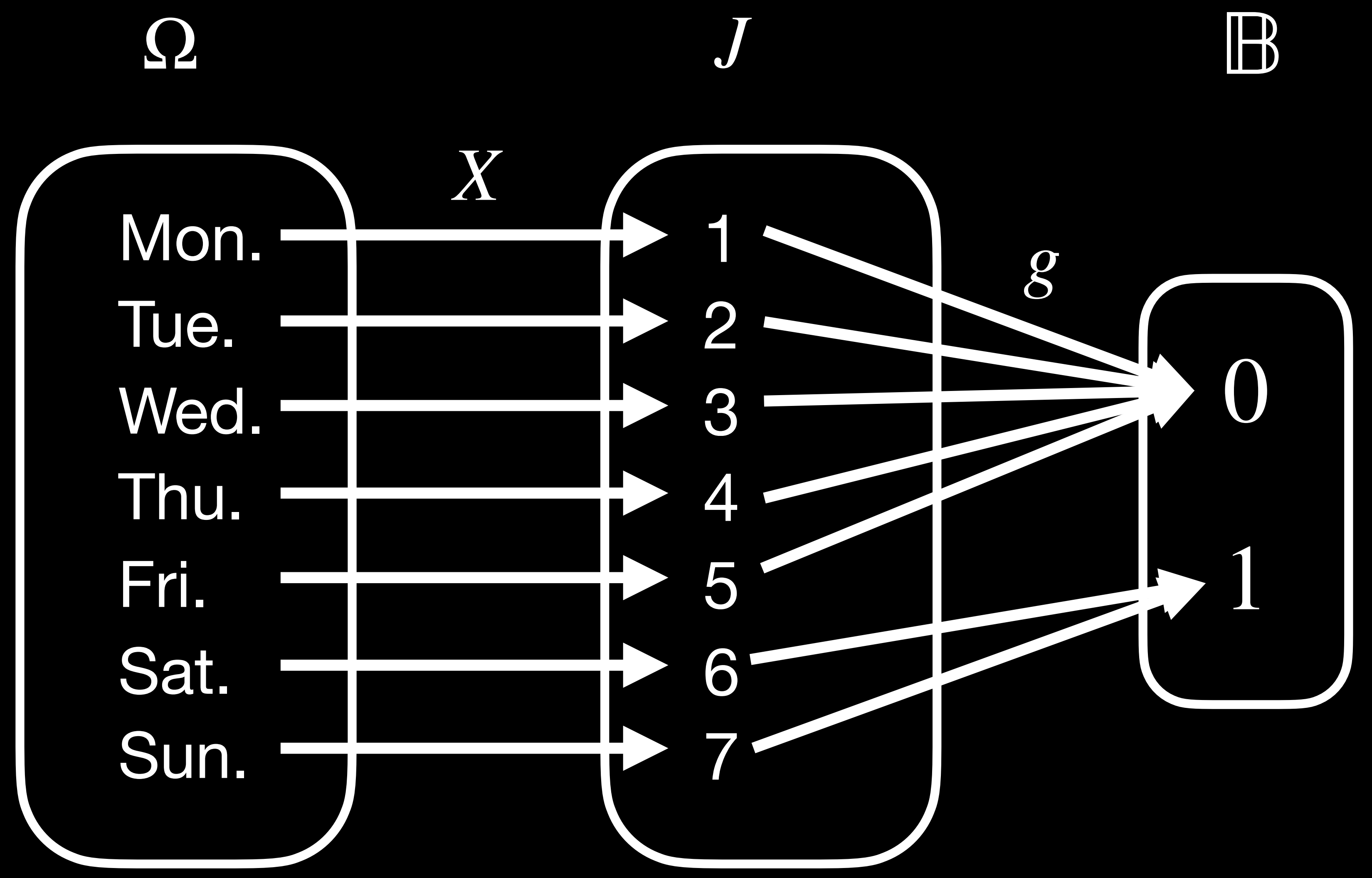
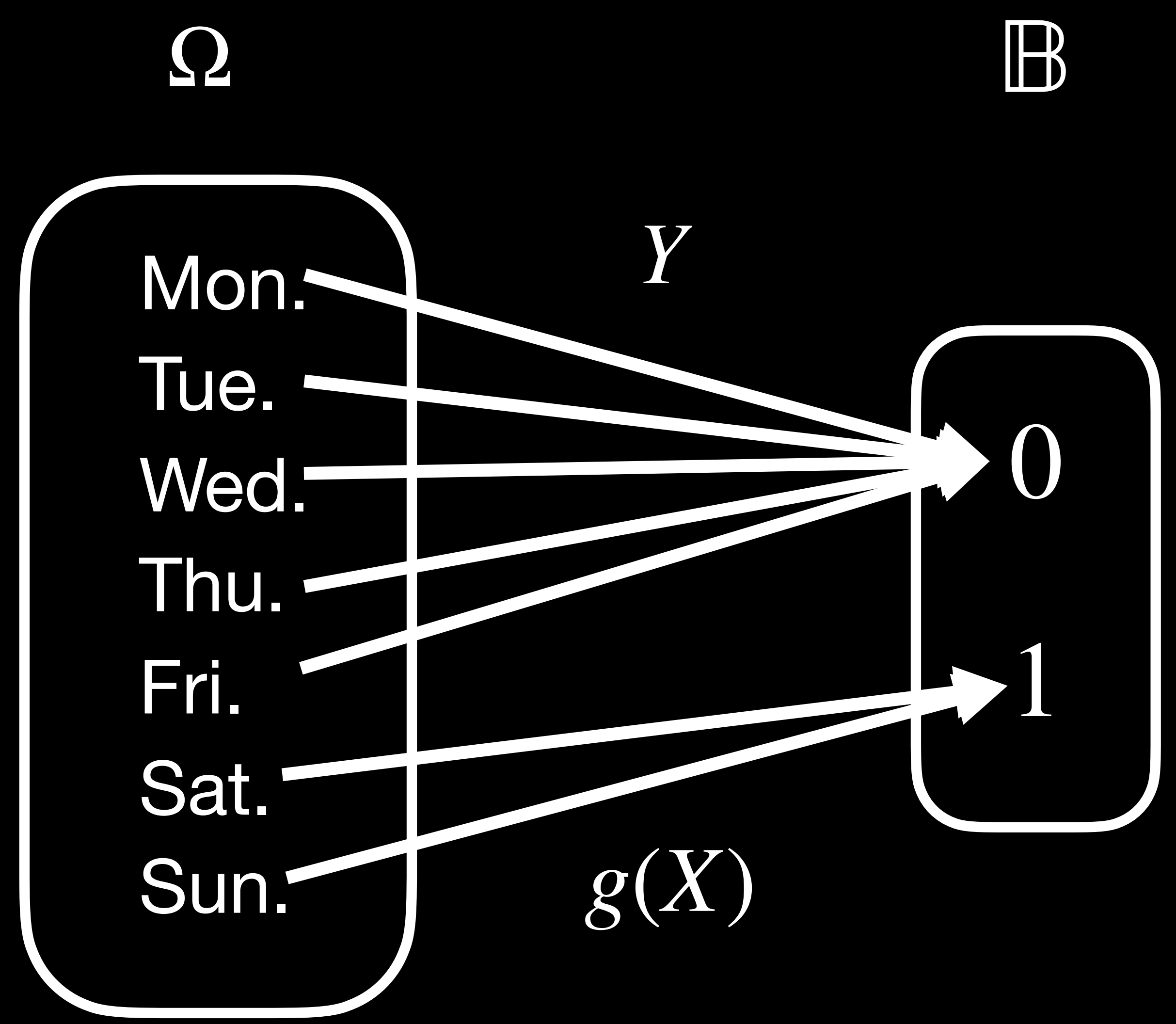
derive:

$$p_Y(1) = \sum_{x \in g^{-1}(1)} p_X(x) = \sum_{x \in \{6,7\}} p_X(x) = p_X(6) + p_X(7)$$

exercise: transform discrete random variables





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transform continuous random variables

transform continuous random variables

given

transform continuous random variables

given f_X,

transform continuous random variables

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transform continuous random variables

given f_X, $Y = g(X)$ with g increasing in $Range(X)$

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$\longrightarrow g^{-1}$ is a function * not the full story

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$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P\left(X \geq g^{-1}(y)\right)$$

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exercise: transform continuous random variables

$X \sim \mathcal{U}(0,1)$

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derive:

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derive:

$g^{-1}(y) = 1 - y$

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inversion transform

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$$Y = g(X) \; := F_X(X) \qquad\qquad F_X : \mathbb{R} \to [0,1]$$

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$F_Y(y)$

inversion transform

$$Y = g(X) \; := F_X(X) \qquad\qquad F_X : \mathbb{R} \to [0,1]$$

$$F_Y(y) = P(Y \leq y)$$

inversion transform

$$Y = g(X) \ := F_X(X) \qquad\qquad F_X : \mathbb{R} \to [0,1]$$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P\left(F_X(X) \leq y\right) \end{aligned}$$

inversion transform

$$Y = g(X) \ := F_X(X) \qquad\qquad F_X : \mathbb{R} \to [0,1]$$

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&= y
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inversion transform

$$Y = g(X) \ := F_X(X) \qquad\qquad F_X : \mathbb{R} \to [0,1]$$

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P\left(F_X(X) \le y\right) \\ &= P\left(X \le F_X^{-1}(y)\right) \\ &= F_X\left(F_X^{-1}(y)\right) \\ &= y \qquad \longrightarrow \qquad Y \sim \mathcal{U}(0,1) \end{aligned}$$

inversion transform

$Y = g(X) \ := F_X(X)$ $\qquad$ $F_X : \mathbb{R} \to [0,1]$

$$
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&= P\left(F_X(X) \leq y\right) \\
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&= y
\end{aligned}
$$

$\longrightarrow \quad Y \sim \mathcal{U}(0,1)$

$U \sim \mathcal{U}(0,1)$

inversion transform

$$Y = g(X) \; := F_X(X) \qquad\qquad F_X : \mathbb{R} \to [0,1]$$

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&= P\left(F_X(X) \leq y\right) \\
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\end{aligned}$$

$\longrightarrow \quad Y \sim \mathcal{U}(0,1)$

$$U \sim \mathcal{U}(0,1)$$

$$X = F_X^{-1}\,(U)$$

inversion transform

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&= y
\end{aligned}
\qquad \longrightarrow \qquad Y \sim \mathcal{U}(0,1)$$

$$U \sim \mathcal{U}(0,1)$$

$X = F_X^{-1}(U)$ $\longrightarrow$ X has the CDF $F_X(x)$

non-uniform distributions

non-uniform distributions

exponential distribution $Exp(\lambda)$

non-uniform distributions

exponential distribution $Exp(\lambda)$

$$y = F_X(x)$$

non-uniform distributions

exponential distribution $Exp(\lambda)$

$$y = F_X(x) \quad = 1 - e^{-\lambda x}$$

non-uniform distributions

exponential distribution $Exp(\lambda)$

$$y = F_X(x) \quad = 1 - e^{-\lambda x} \qquad\qquad x \geq 0$$

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non-uniform distributions

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non-uniform distributions

exponential distribution $Exp(\lambda)$

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$Y \sim \mathcal{U}(0,1) \longrightarrow U := 1 - Y \sim \mathcal{U}(0,1)$

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$$X = -\frac{1}{\lambda} \ln(U)$$

non-uniform distributions

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non-uniform distributions

exponential distribution $Exp(\lambda)$

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$$Y \sim \mathscr{U}(0,1) \qquad \longrightarrow \qquad U := 1 - Y \sim \mathscr{U}(0,1)$$

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$$\longrightarrow \quad X \sim Exp(\lambda)$$

non-uniform distributions

non-uniform distributions

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

non-uniform distributions

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

non-uniform distributions

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

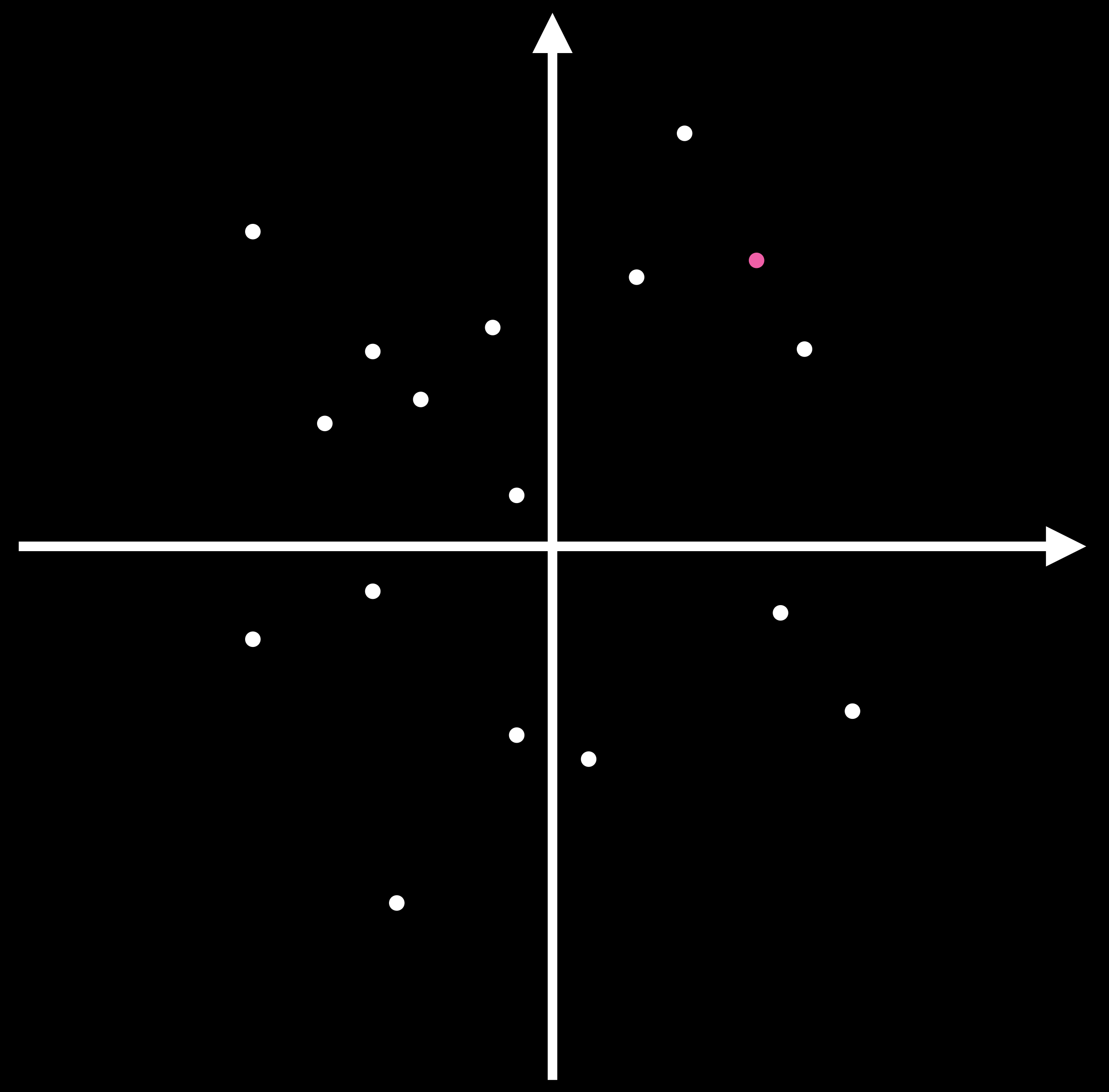
$$X_1, X_2 \sim \mathcal{N}(0,1)$$

non-uniform distributions

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$

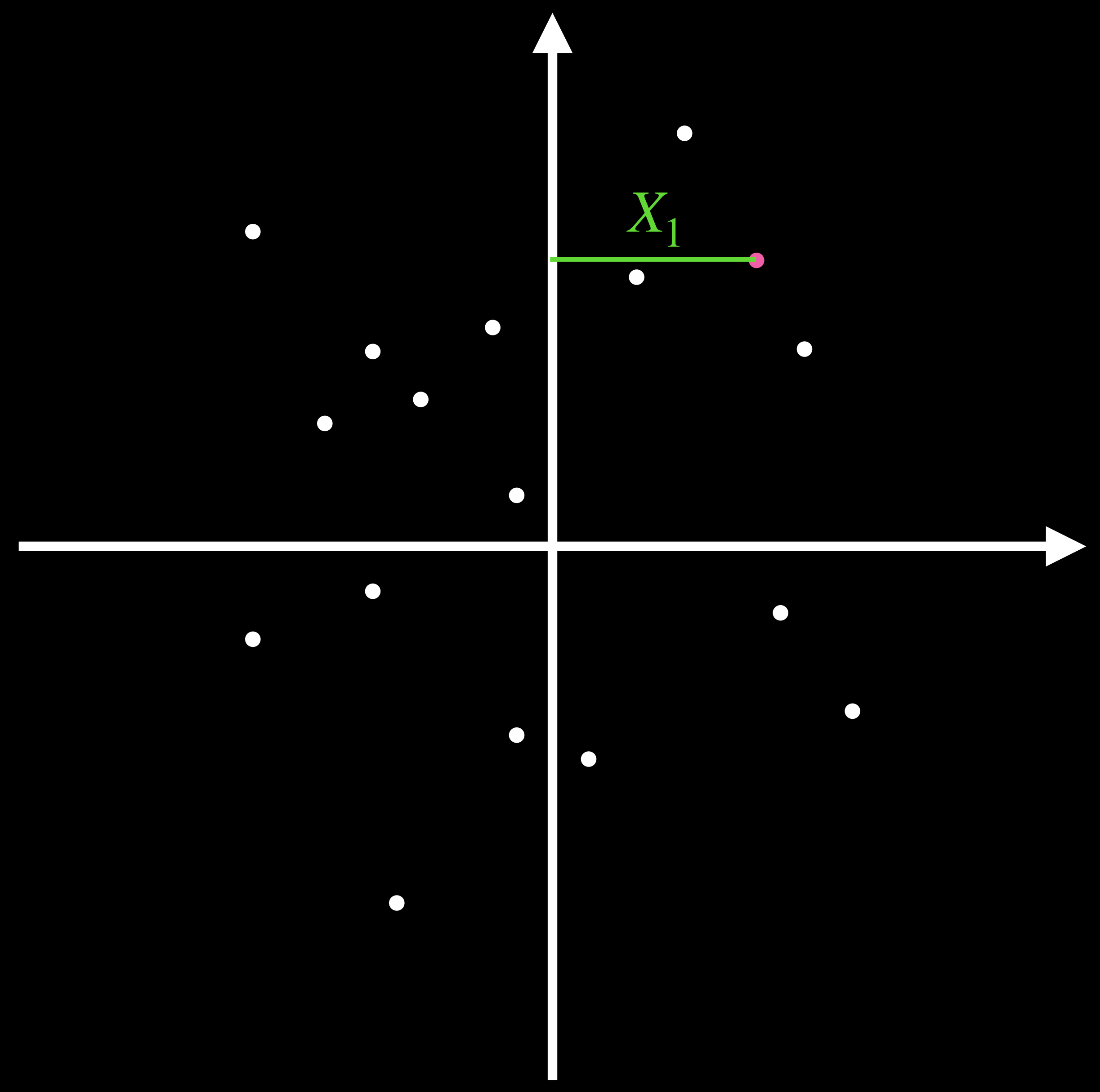


non-uniform distributions

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$



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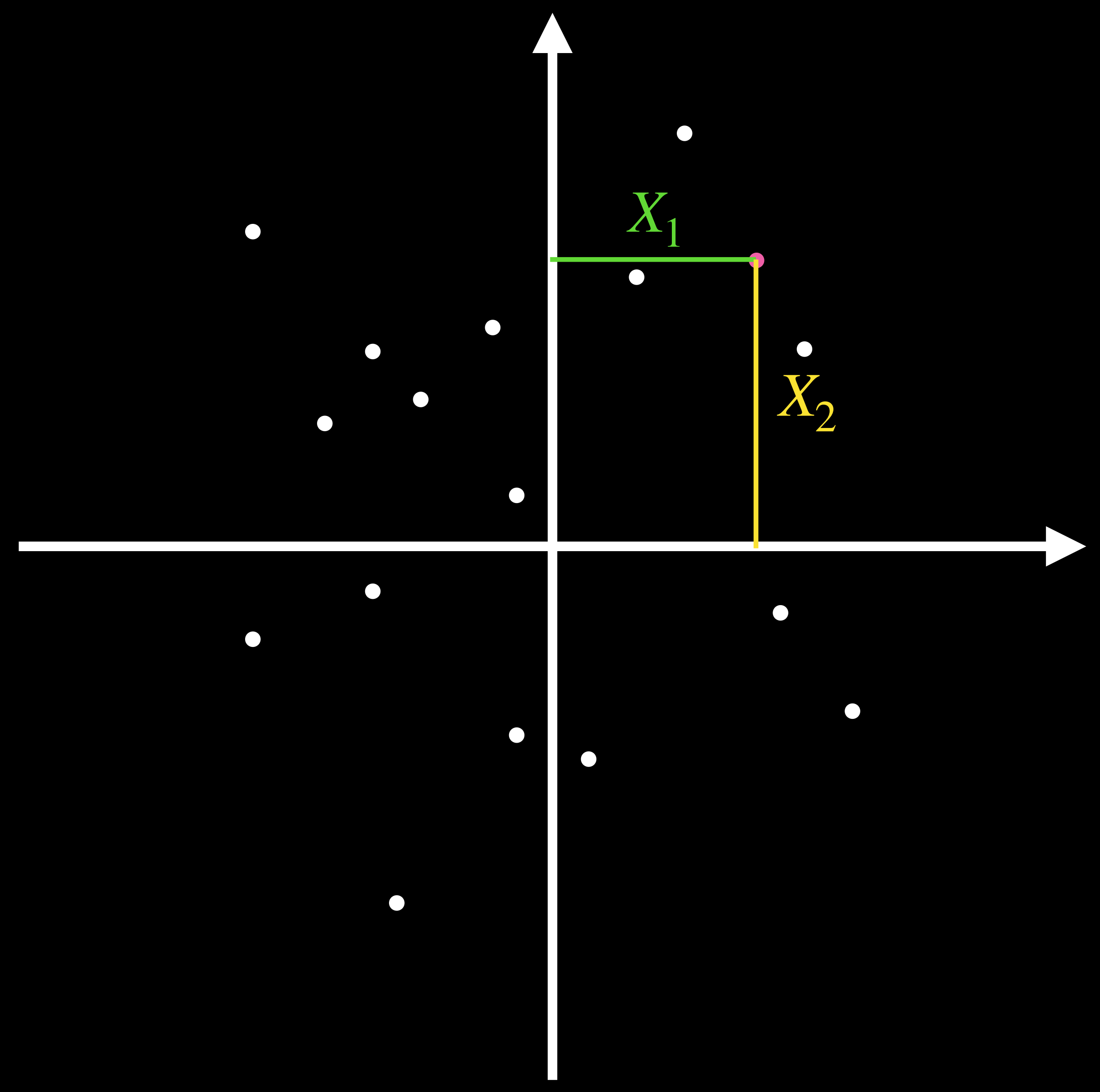
Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$



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standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

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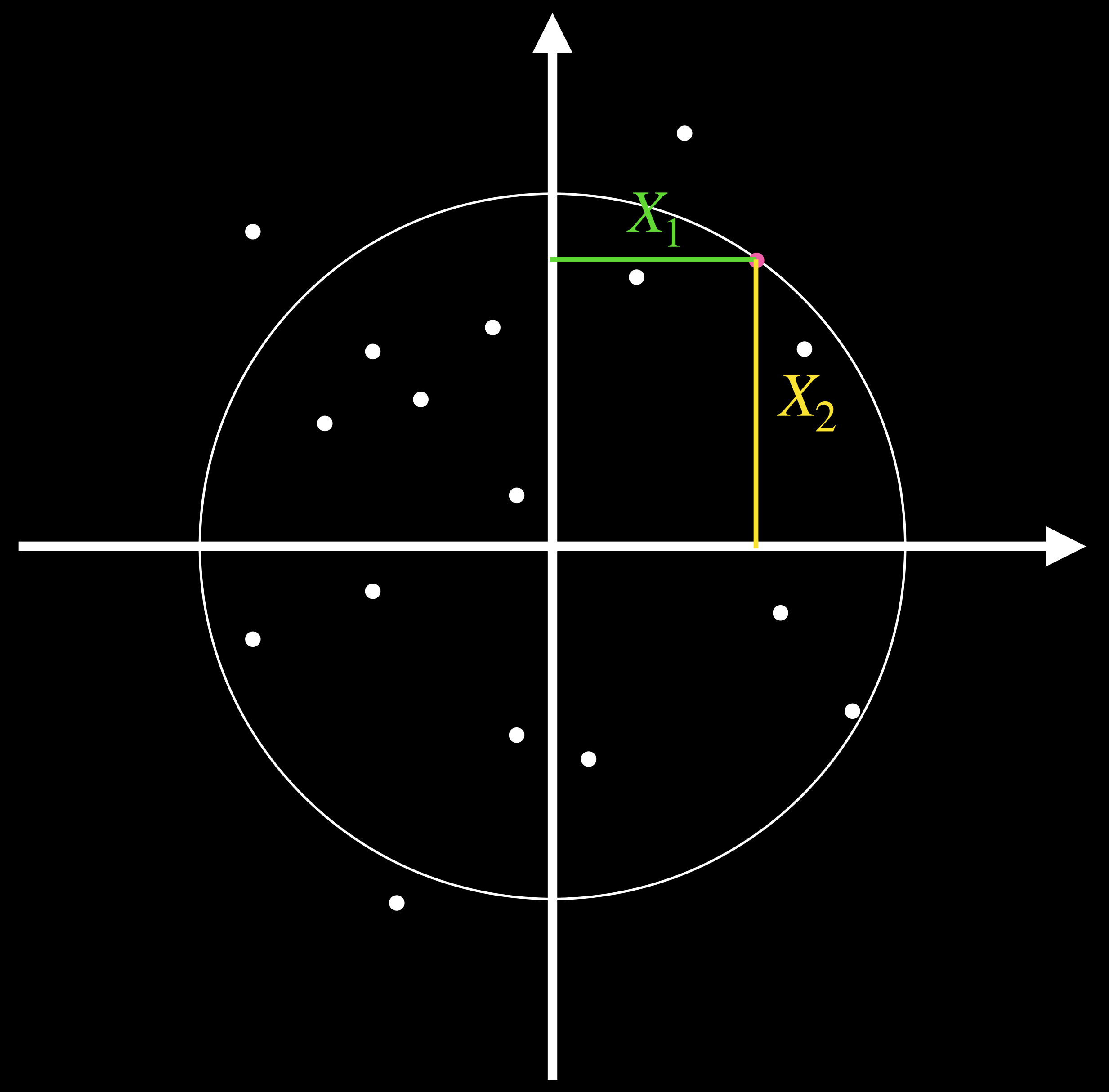

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Box-Muller Transform

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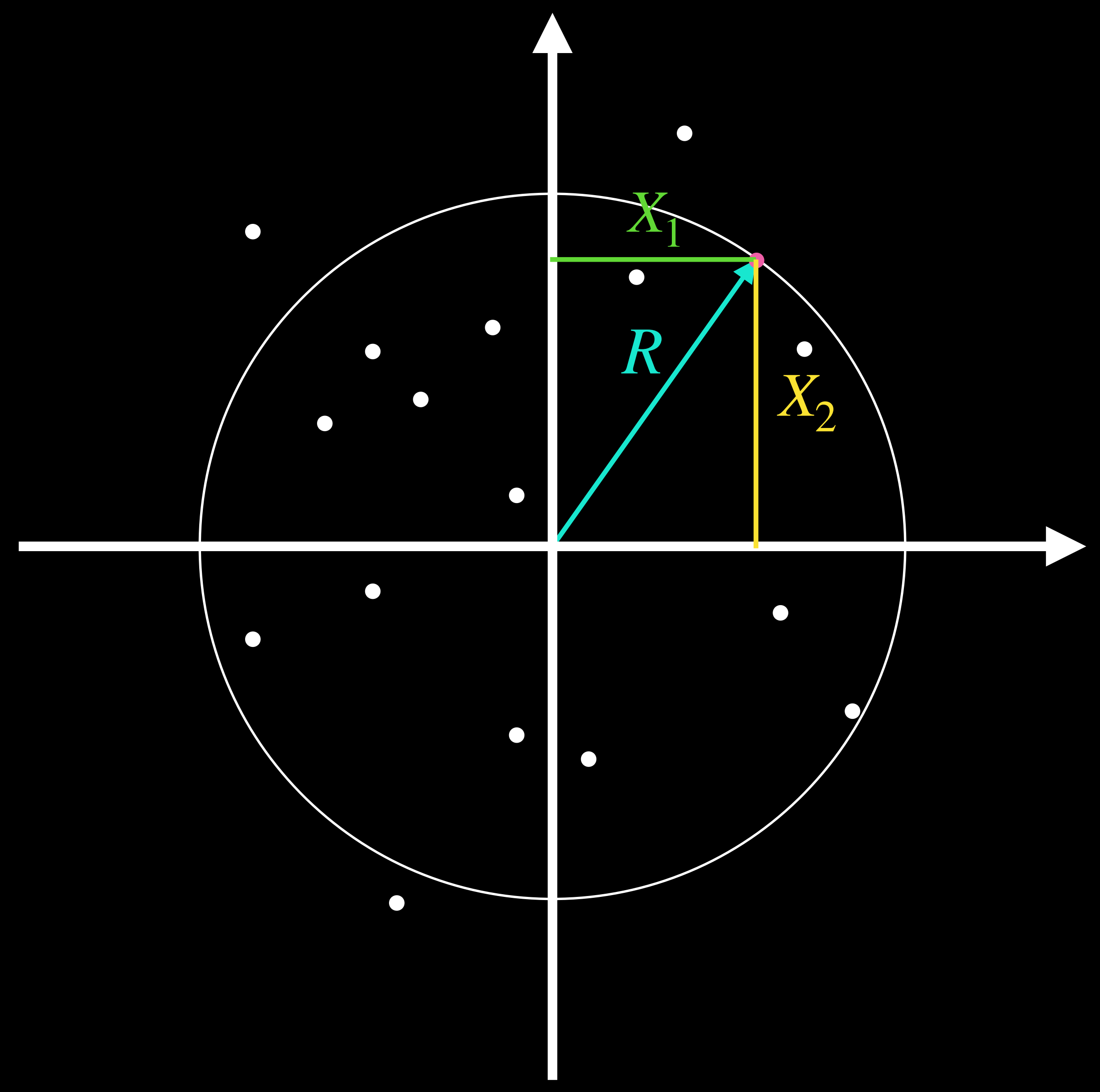

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Box-Muller Transform

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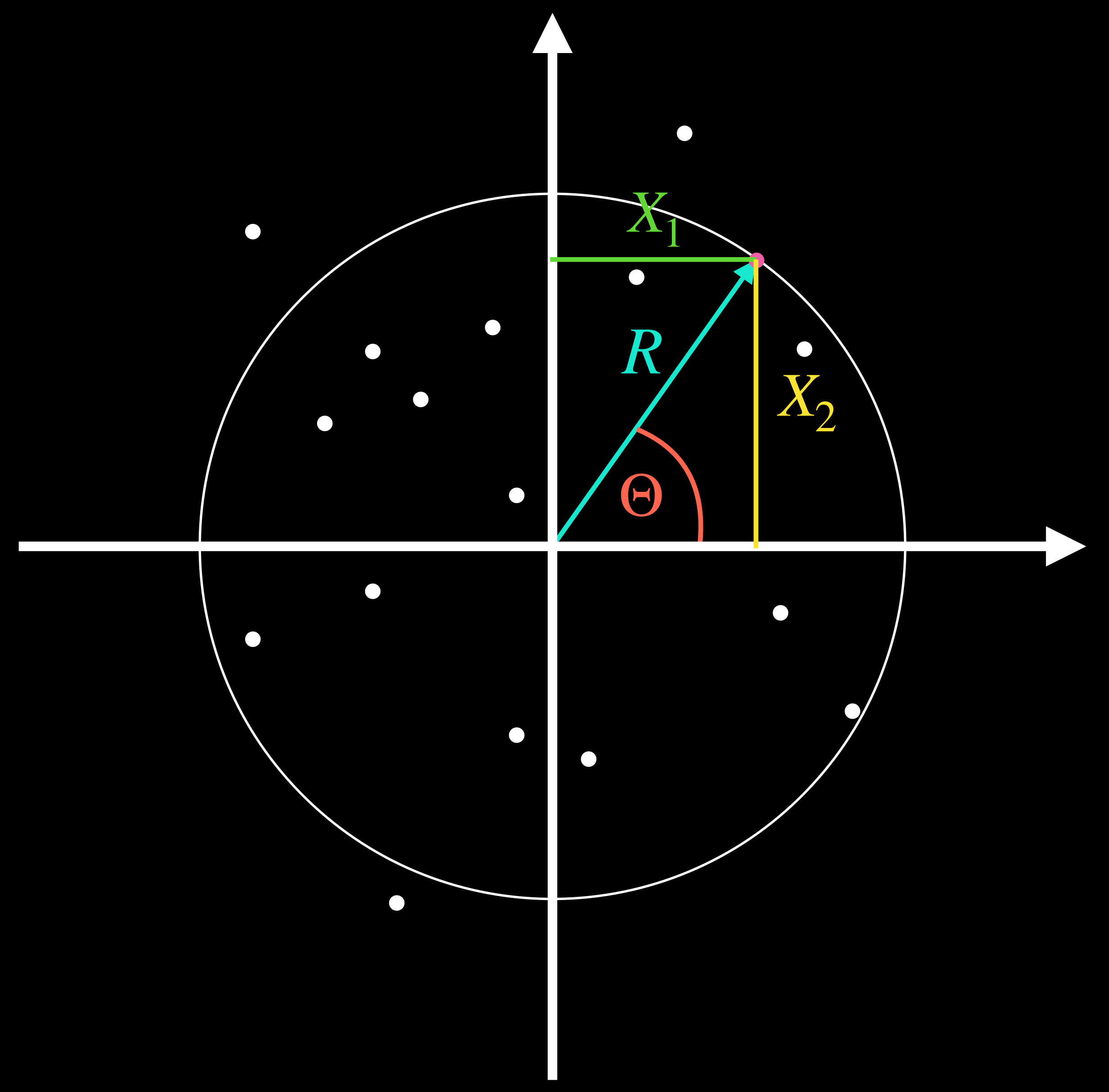
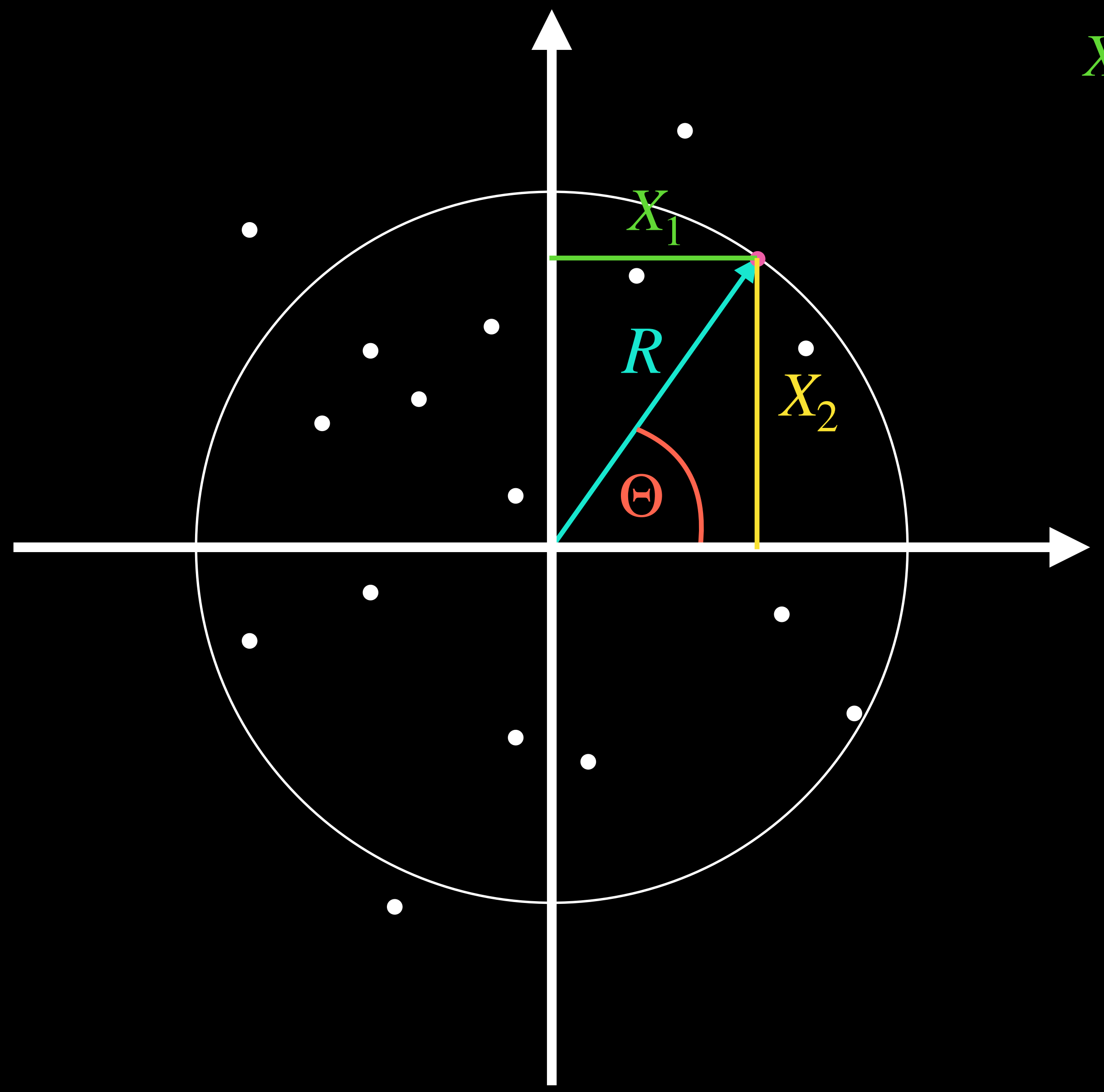


non-uniform distributions

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$



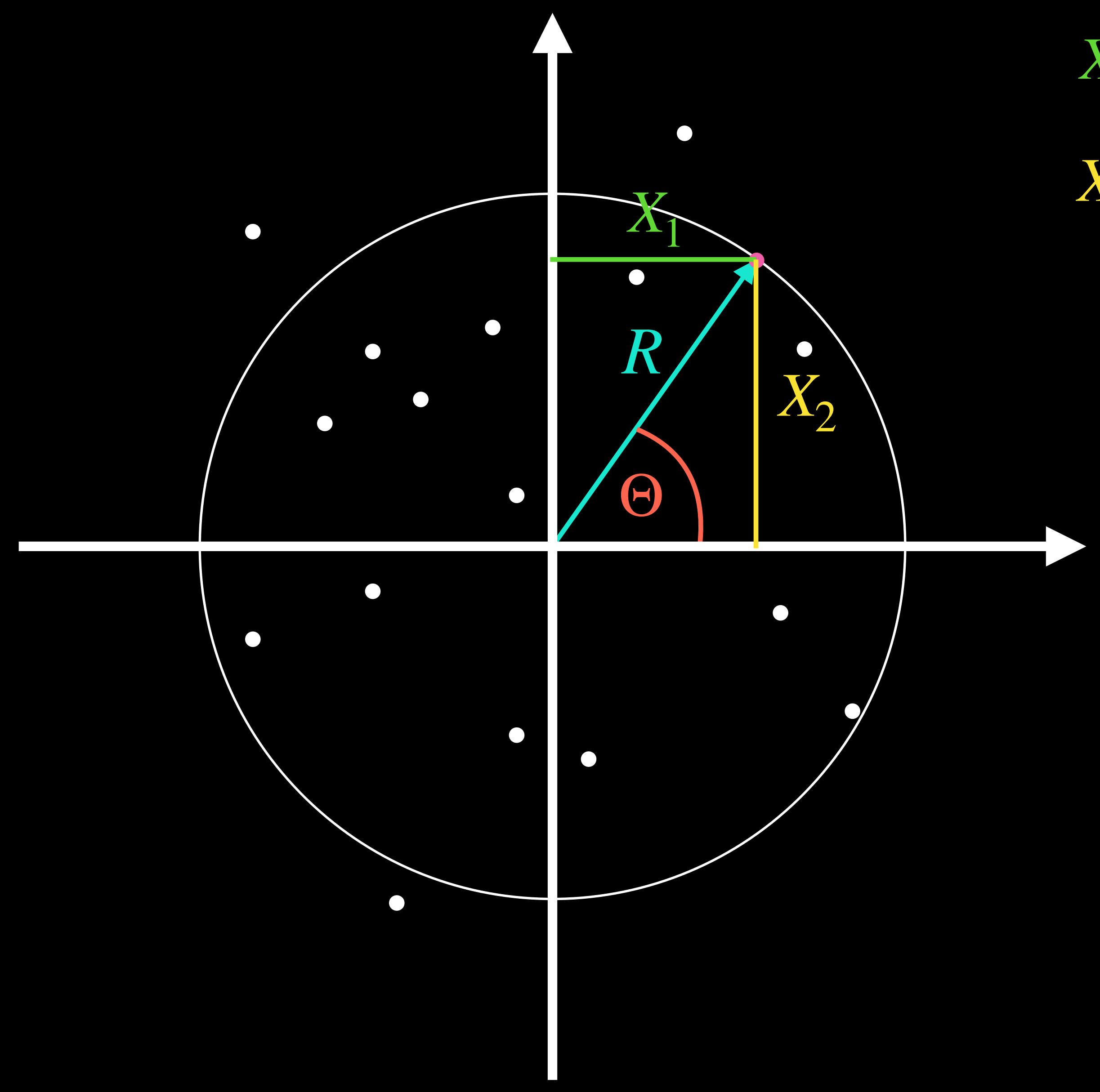
$X_1 = cos(\Theta) \cdot R$

non-uniform distributions

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$



$X_1 = cos(\Theta) \cdot R$

$X_2 = sin(\Theta) \cdot R$

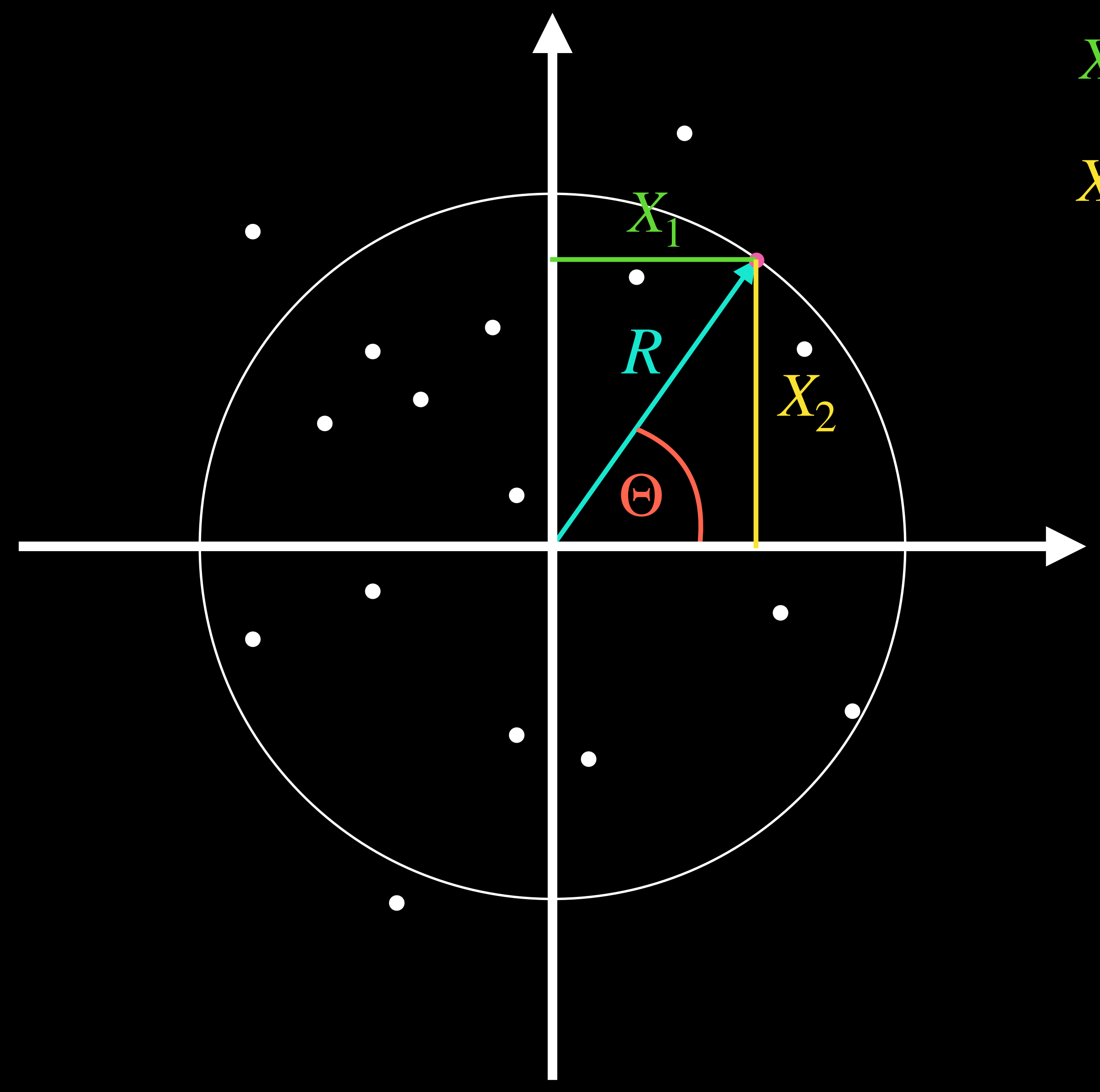
non-uniform distributions

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$

$$R^2 = X_1^2 + X_2^2 \sim Exp\left(\frac{1}{2}\right)$$



$X_1 = cos(\Theta) \cdot R$

$X_2 = sin(\Theta) \cdot R$

non-uniform distributions

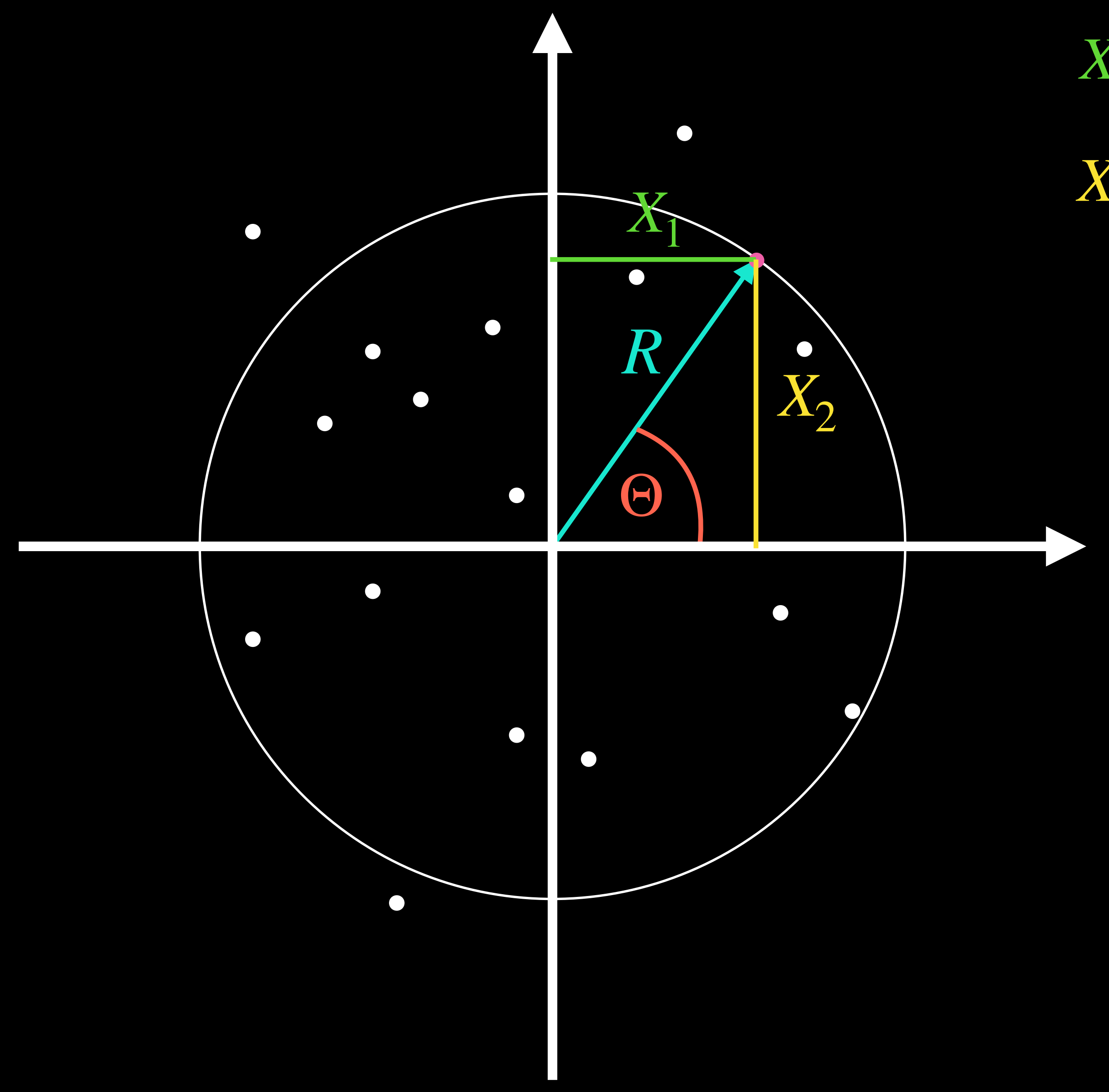
standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$

$$R^2 = X_1^2 + X_2^2 \sim Exp\left(\frac{1}{2}\right)$$

$$\Theta \sim \mathcal{U}(0,2\pi)$$



$X_1 = cos(\Theta) \cdot R$

$X_2 = sin(\Theta) \cdot R$

non-uniform distributions

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

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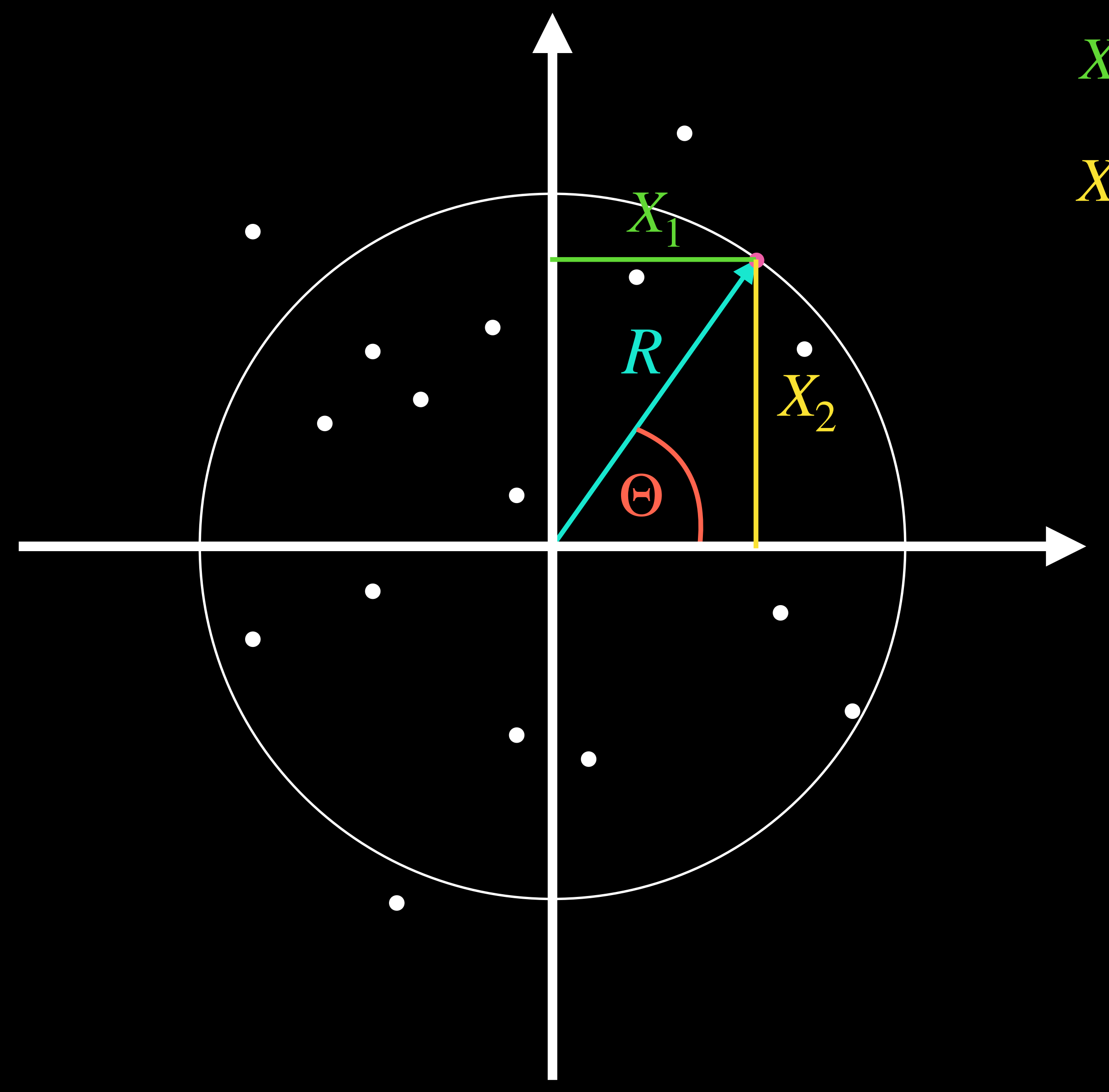


$X_1 = cos(\Theta) \cdot R$

$X_2 = sin(\Theta) \cdot R$

exercise:

non-uniform distributions

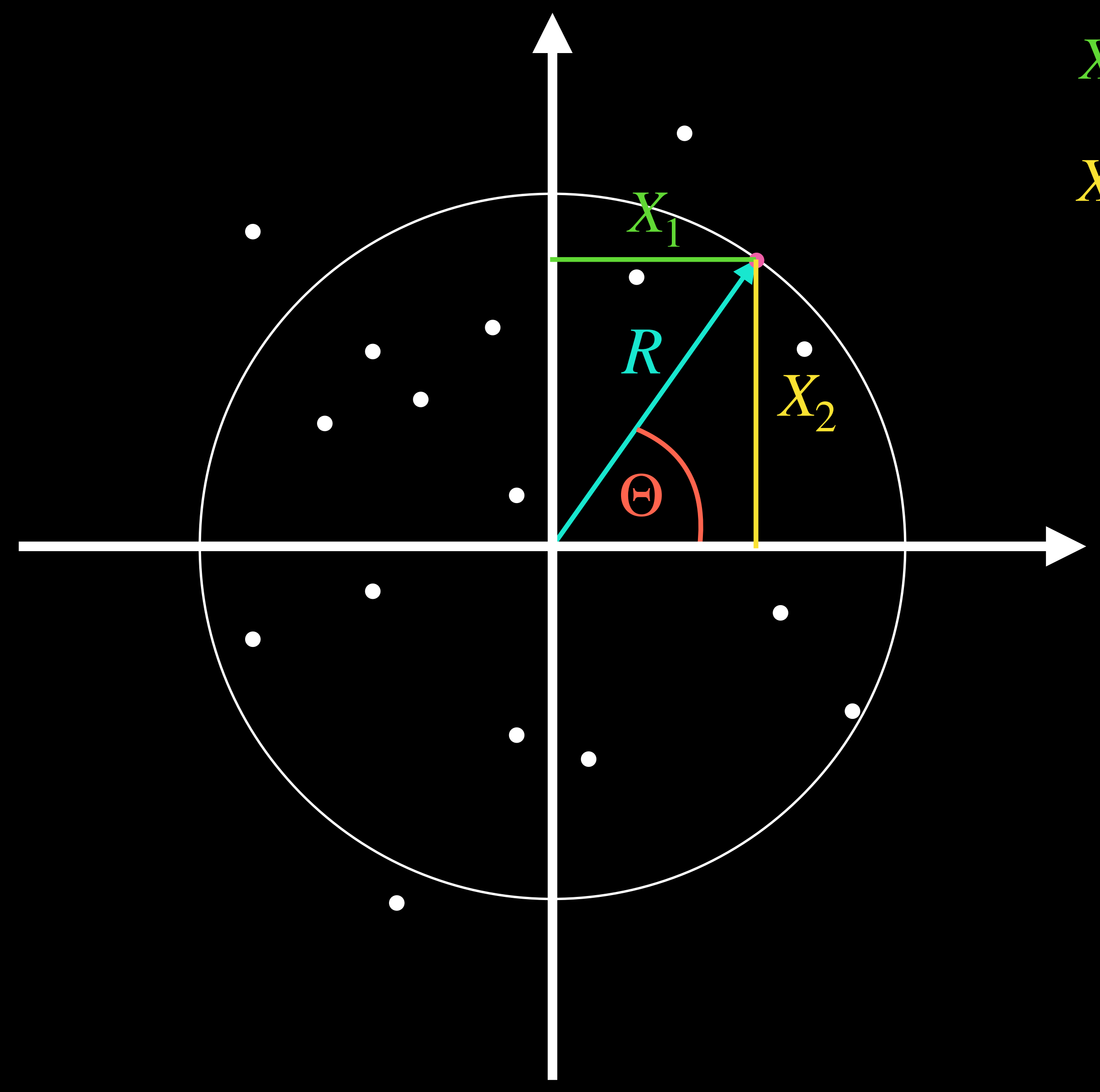
standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$

$$R^2 = X_1^2 + X_2^2 \sim Exp\left(\frac{1}{2}\right)$$

$$\Theta \sim \mathcal{U}(0,2\pi)$$



$X_1 = cos(\Theta) \cdot R$

$X_2 = sin(\Theta) \cdot R$

exercise: write X_1, X_2 using $U_1, U_2 \sim \mathcal{U}(0,1)$

non-uniform distributions

$$X = -\frac{1}{\lambda}\ln(U) \qquad U \sim \mathcal{U}(0,1) \qquad X \sim Exp(\lambda)$$

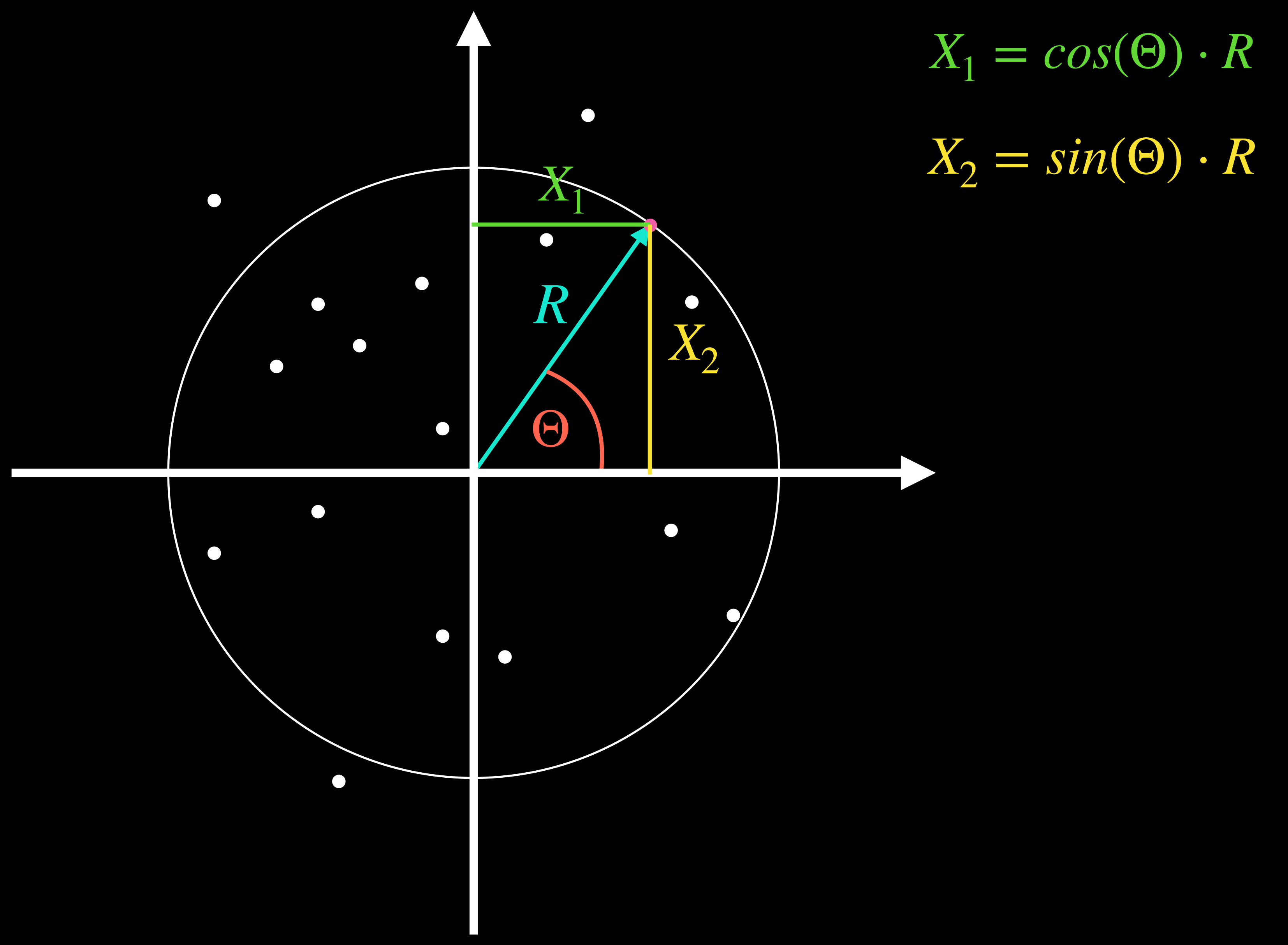
standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$

$$R^2 = X_1^2 + X_2^2 \sim Exp\left(\frac{1}{2}\right)$$

$$\Theta \sim \mathcal{U}(0,2\pi)$$



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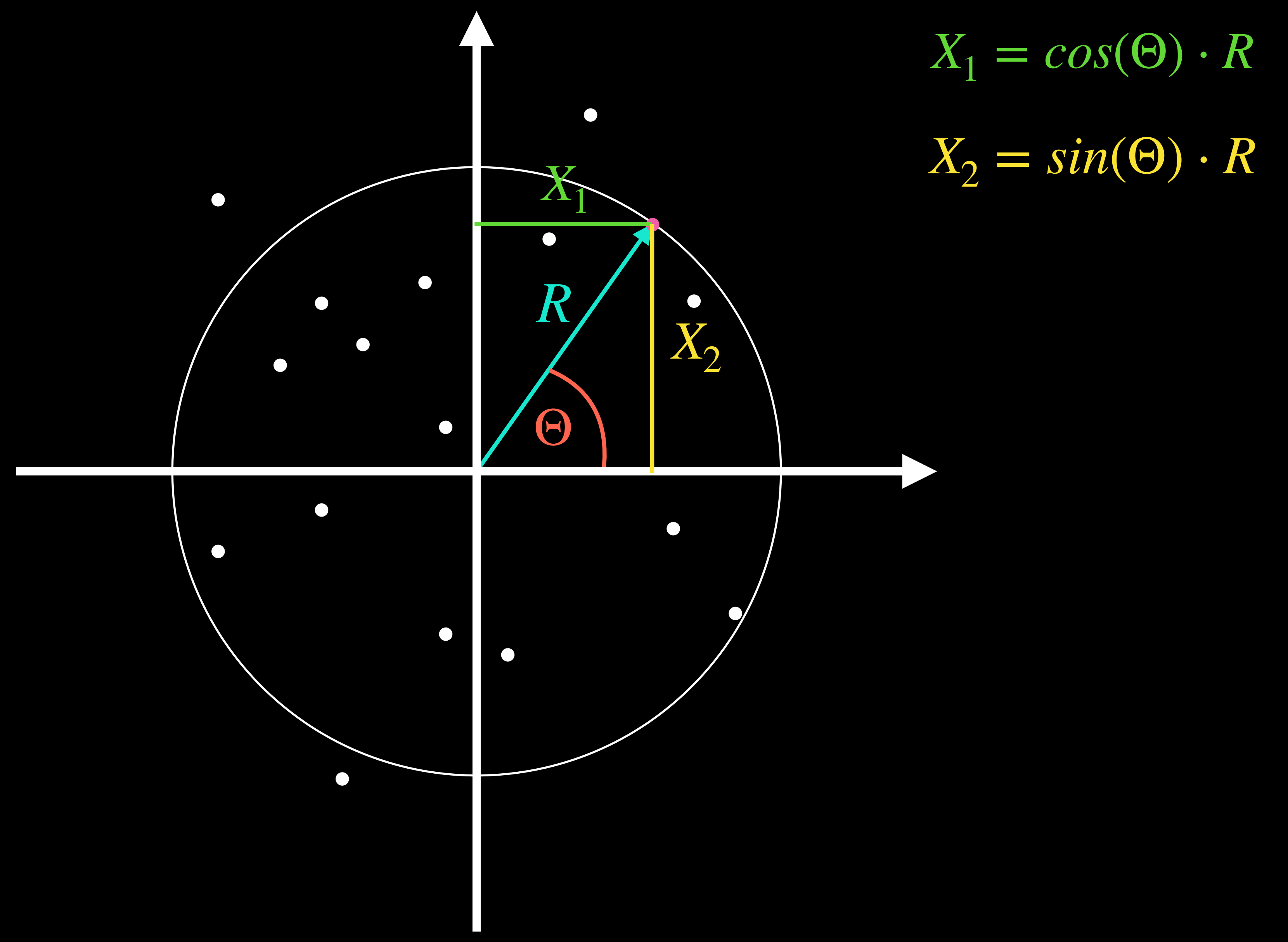


exercise: write X_1, X_2 using $U_1, U_2 \sim \mathcal{U}(0,1)$

$$R^2 = -2\ln(U_1)$$

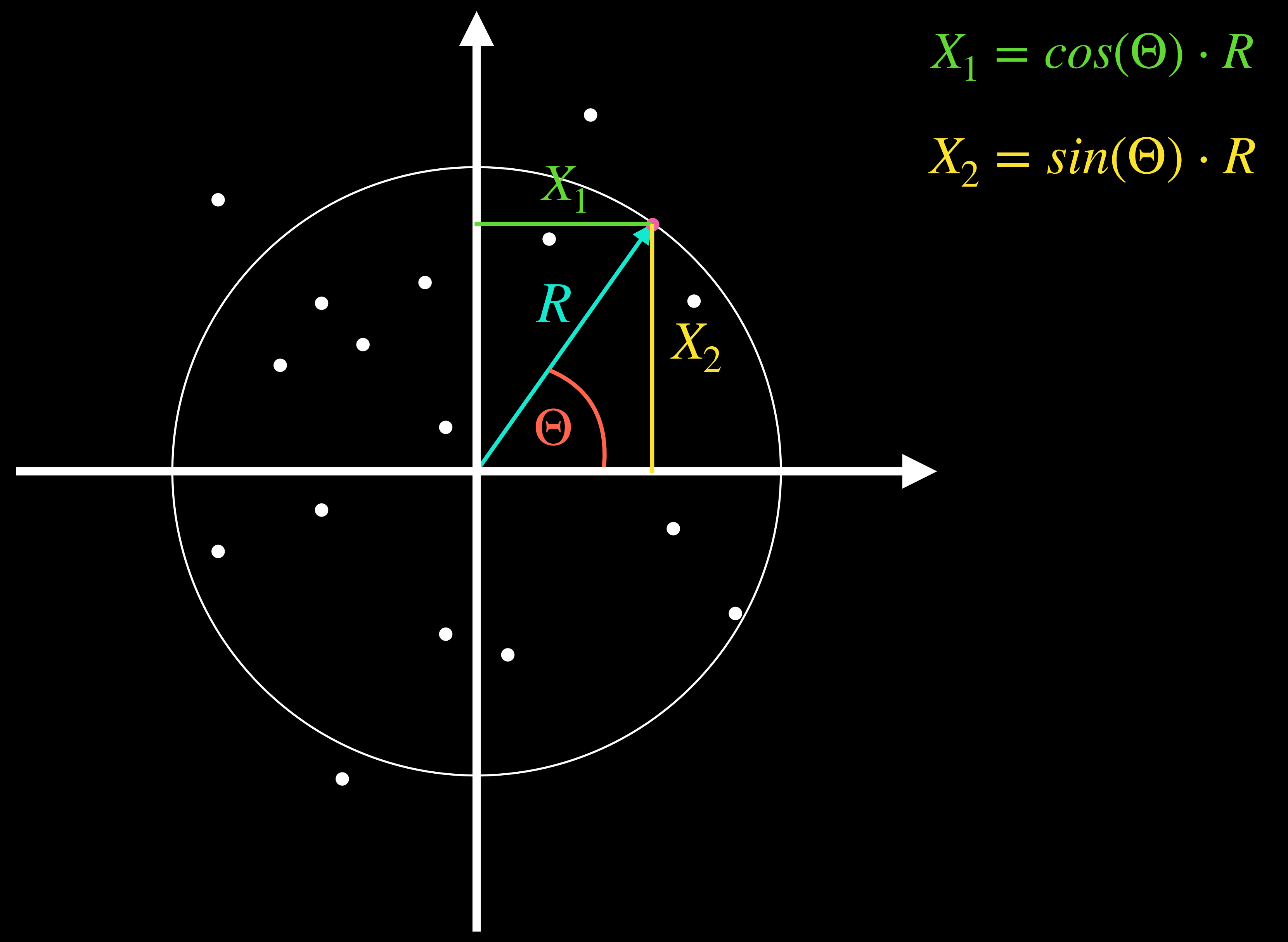
non-uniform distributions

$$X = -\frac{1}{\lambda}\ln(U) \qquad U \sim \mathcal{U}(0,1) \qquad X \sim Exp(\lambda)$$

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$
$$R^2 = X_1^2 + X_2^2 \sim Exp\left(\frac{1}{2}\right)$$
$$\Theta \sim \mathcal{U}(0,2\pi)$$



exercise: write X_1, X_2 using $U_1, U_2 \sim \mathcal{U}(0,1)$

$$R^2 = -2\ln(U_1)$$
$$\Theta = 2\pi U_2$$

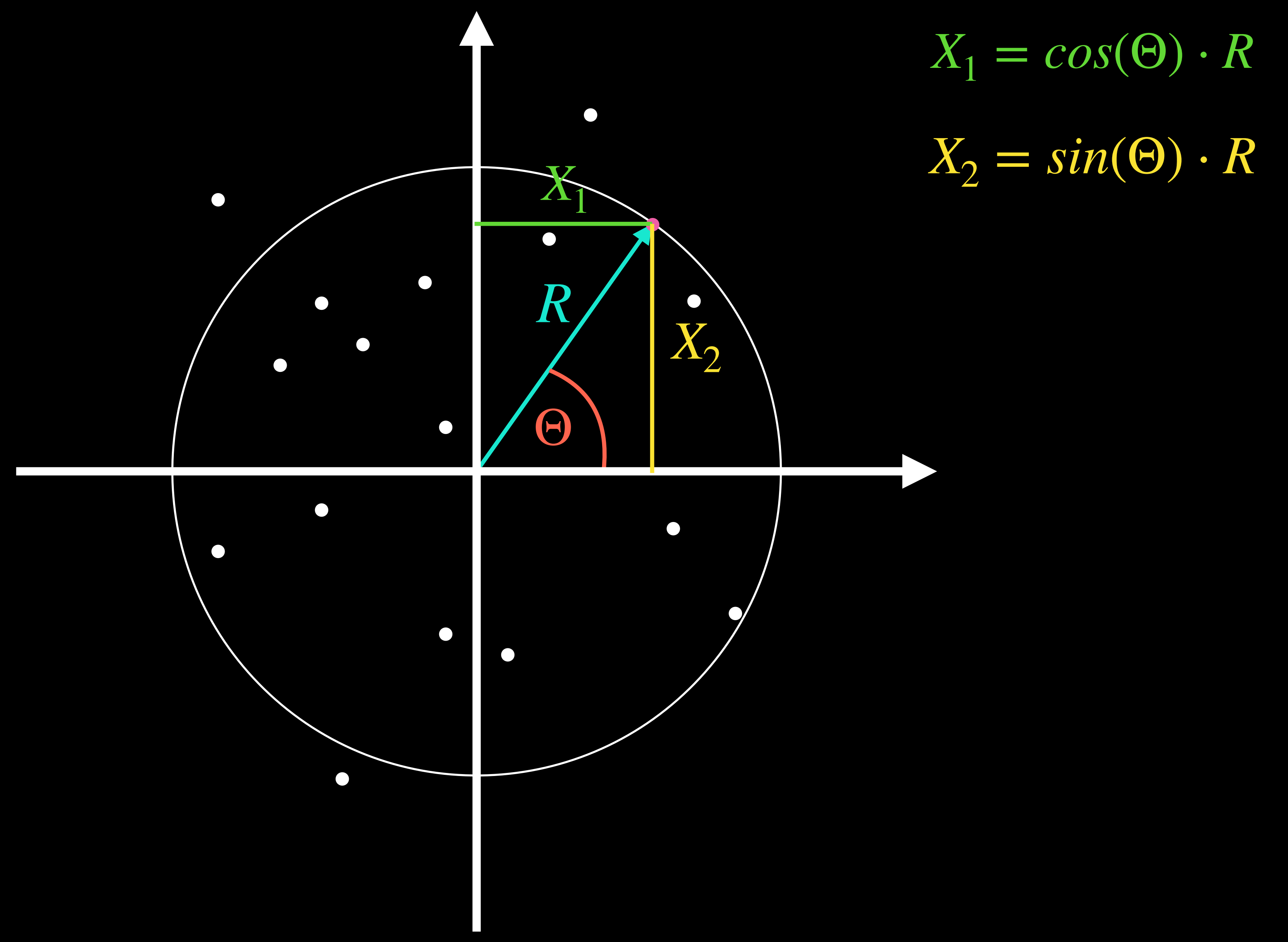
non-uniform distributions

$$X = -\frac{1}{\lambda}\ln(U) \qquad \begin{matrix} U \sim \mathscr{U}(0,1) \\ X \sim Exp(\lambda) \end{matrix}$$

standard Gaussian/ normal distribution $\mathscr{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathscr{N}(0,1)$$
$$R^2 = X_1^2 + X_2^2 \sim Exp\left(\frac{1}{2}\right)$$
$$\Theta \sim \mathscr{U}(0,2\pi)$$



exercise: write X_1, X_2 using $U_1, U_2 \sim \mathscr{U}(0,1)$

$$R^2 = -2\ln(U_1)$$
$$\Theta = 2\pi U_2$$

$$X_1 = cos(2\pi U_2) \cdot \sqrt{-2\ln(U_1)}$$

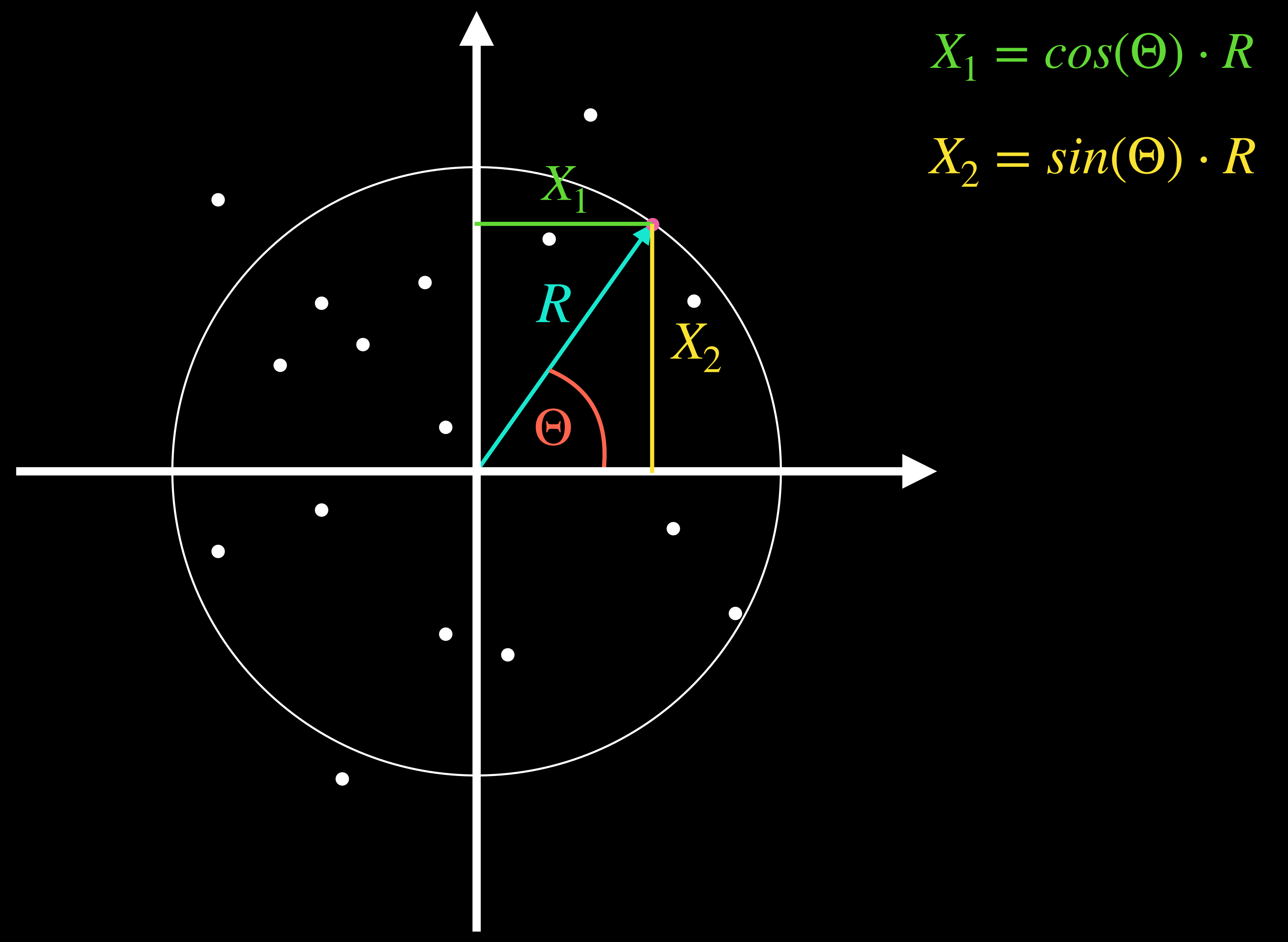
non-uniform distributions

$$X = -\frac{1}{\lambda}\ln(U) \qquad U \sim \mathcal{U}(0,1) \qquad X \sim Exp(\lambda)$$

standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$
$$R^2 = X_1^2 + X_2^2 \sim Exp\left(\frac{1}{2}\right)$$
$$\Theta \sim \mathcal{U}(0,2\pi)$$



exercise: write X_1, X_2 using $U_1, U_2 \sim \mathcal{U}(0,1)$

$$R^2 = -2\ln(U_1)$$
$$\Theta = 2\pi U_2$$

$$X_1 = cos(2\pi U_2) \cdot \sqrt{-2\ln(U_1)}$$
$$X_2 = sin(2\pi U_2) \cdot \sqrt{-2\ln(U_1)}$$

end