stochastics and probability

Lecture 2

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random variable

random variables

$$X : \Omega \to J$$

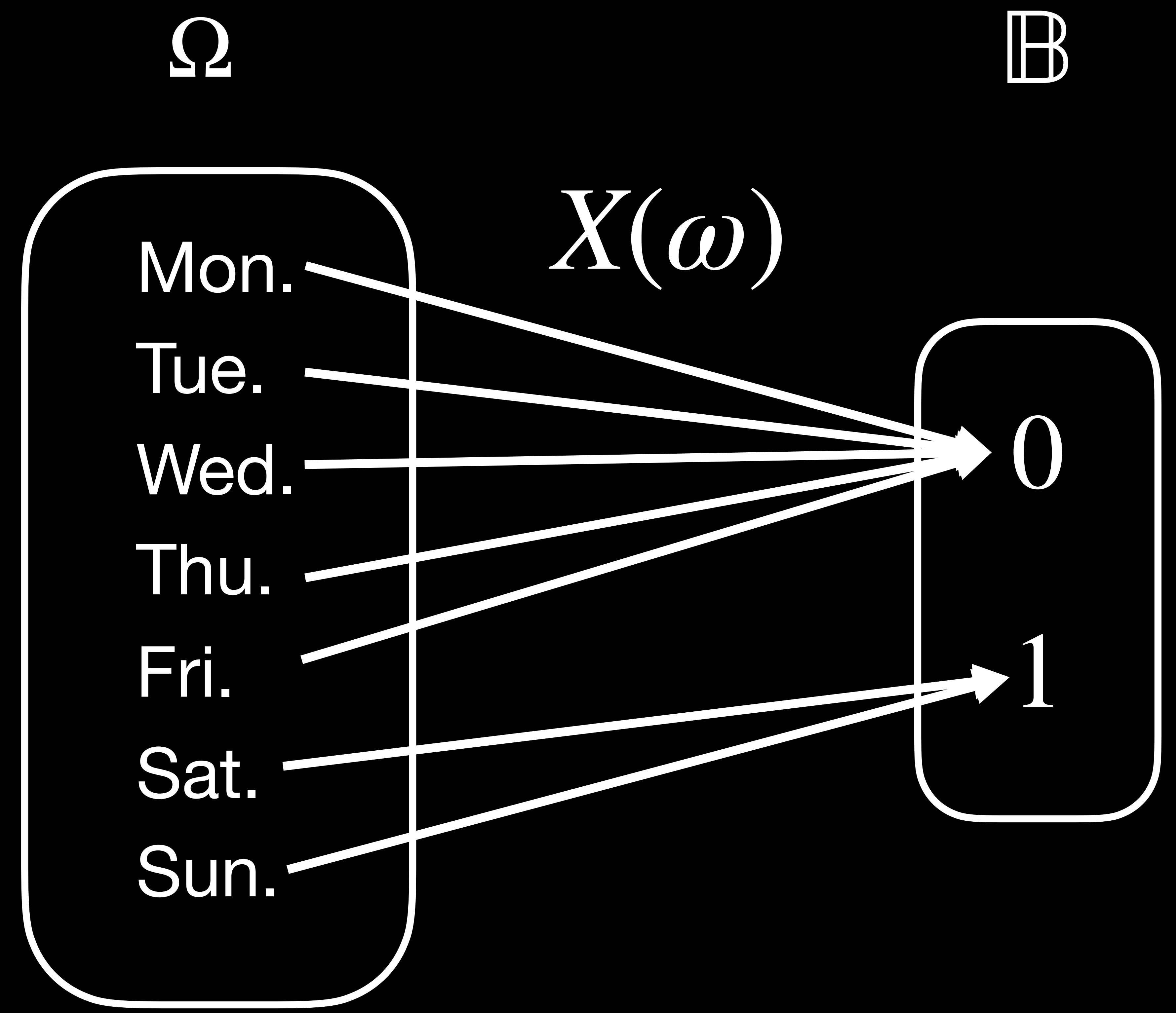
$$X(\omega) = x \qquad \omega \in \Omega, \quad x \in J$$

event: $A \subseteq \Omega$

$$P(X > x_i) := P(\{\omega \in \Omega : X(\omega) > x_i\})$$

range of X: $Range(X) := \{x \in \mathbb{R} : x = X(\omega), \omega \in \Omega\}$

example:



probability mass function

$$p_X(x) = P(X = x) \qquad p : J \to [0,1]$$

$$\sum_{x \in J} p_X(x) = 1$$

exercise: probability mass function

$X : \Omega \to J$

$P(X > x_i) := P(\{\omega \in \Omega : X(\omega) > x_i\})$

$p_X(x) = P(X = x)$

$P(\Omega) = 1$

$P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

proof: $$\sum_{x \in J} p_X(x) = \sum_{x \in J} P(X = x) = \sum_{x \in J} P(\{\omega \in \Omega : X(\omega) = x\})$$

$$= P\left(\bigcup_{x \in J} \{\omega \in \Omega : X(\omega) = x\}\right) = P(\Omega) = 1$$

probability density function (PDF)

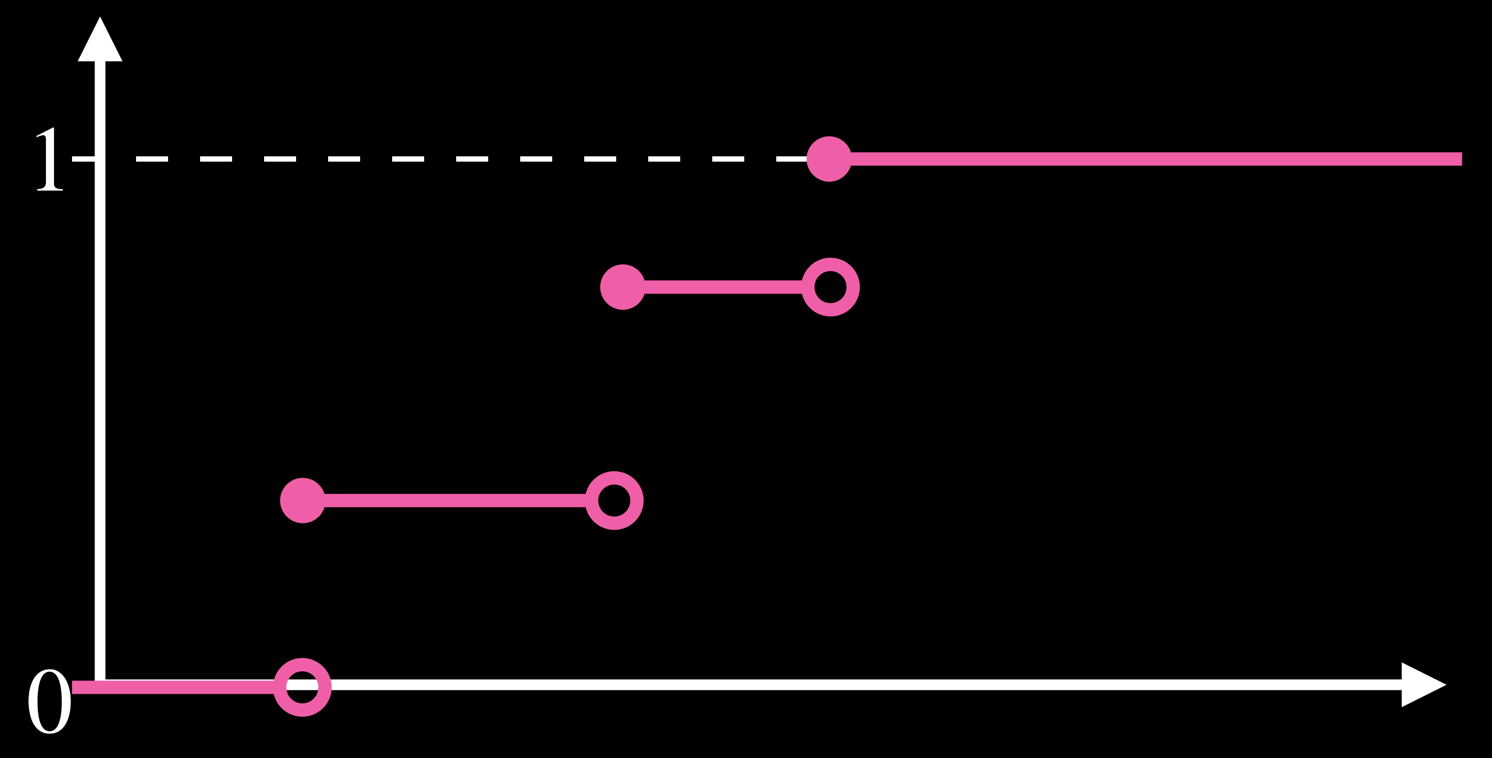
$$P(X = x) = 0 \qquad\qquad f : \mathbb{R} \to [0,\infty]$$

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx$$

cumulative distribution function (CDF)

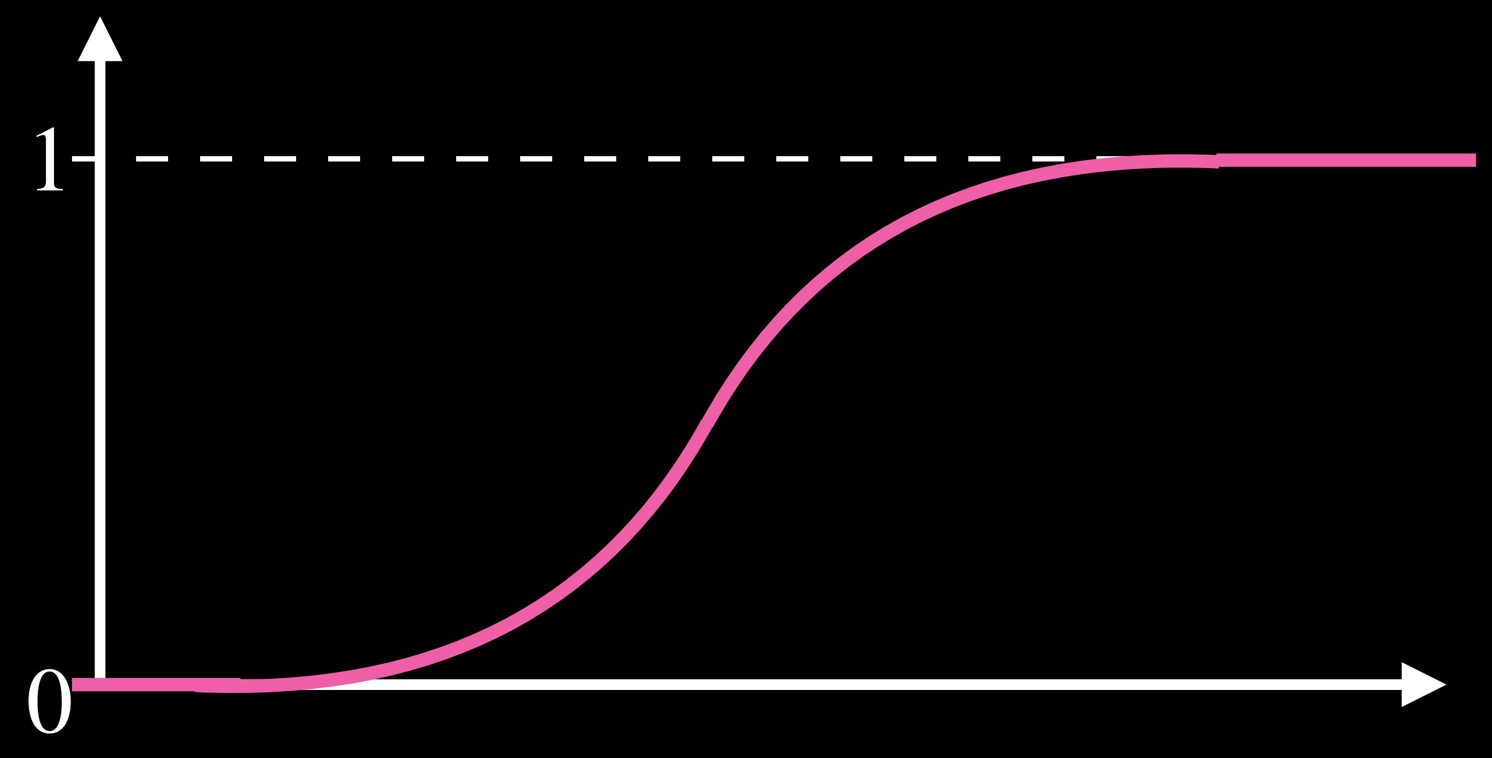
discrete

$$F_X(x) = P(X \leq x) = \sum_{t \leq x} P(X = t) = \sum_{t \leq x} p_X(t)$$



continuous

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\, dt$$



exercice

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\ dt$$

$$Range(X) := \{x \in \mathbb{R} : x = X(\omega), \omega \in \Omega\}$$

How to calculate the PDF $f_X(x)$ given the CDF $F_X(x)$?

$$f_X(x) = \frac{d}{dx} F_X(x)$$

What is the range of X ?

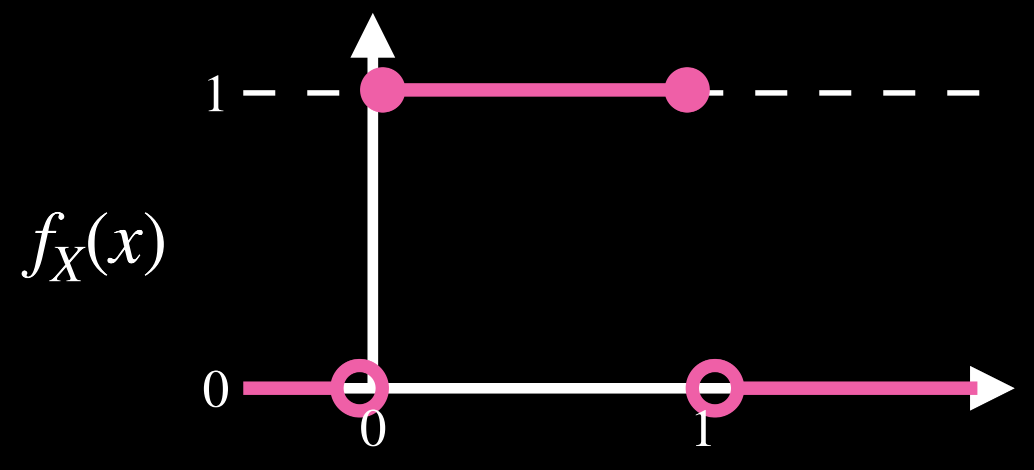
$X : \Omega \to \mathbb{R}$ $\qquad \Omega := [0cm, 100cm]$ $\qquad X(\alpha\text{cm}) = \frac{\alpha}{100}$

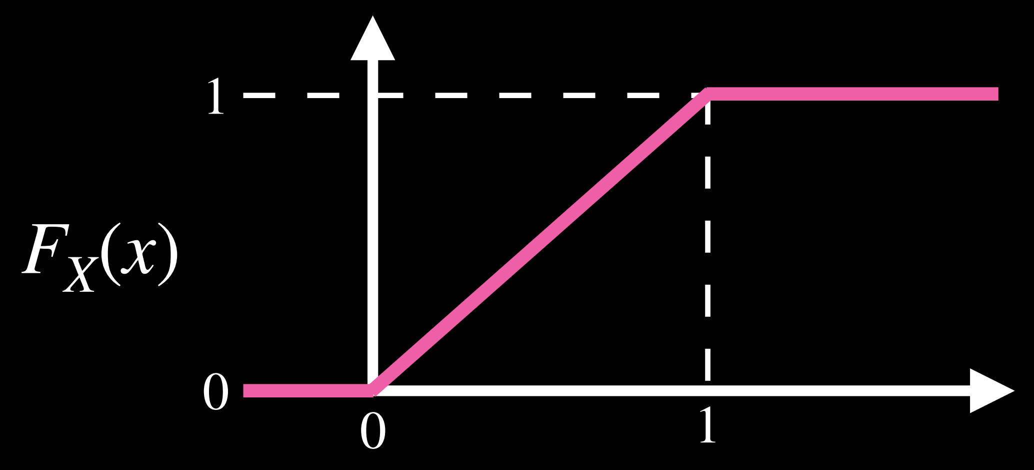
$$Range(X) = [0,1]$$

What is $F_X(x)$ of the uniform distribution $\mathcal{U}(0,1)$?

$$f_X(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ 1 & \text{else} \end{cases}$$





uniform random number generation

data-driven $\begin{cases} \text{not statistical random} \\ \text{not reproducable} \end{cases}$

algorithmic $\begin{cases} \text{statistical random} \\ \text{reproducable} \end{cases}$

random numbers $x_1, \ldots, x_n \in [0,1]$

$$ {}_n\hat{F}_U(x) := \frac{\left| \left\{ x_i \in \{x_1, \ldots, x_n\} : x_i \leq x \right\} \right|}{n} $$

require $$\lim_{n\to\infty} \left| {}_n\hat{F}_U(x) - F_U(x) \right| = \lim_{n\to\infty} \left| {}_n\hat{F}_U(x) - x \right| = 0 \qquad x \in [0,1]$$

pseudo-random number generation

Lehmer generator /a Linear congruential generator (LCG)

$$z_{i+1} = az_i \mod m \qquad i = 1,2,3,..$$

$$x_i = \frac{z_i}{m} \quad \in [0,1)$$

Need to choose:

$m \in \mathbb{N}_{>0} \qquad 0 < a < m$

seed: $z_1 \in \mathbb{N}_{>0}$

$m = 2^{31} - 1$
$a = 48271$

cycle length:

$x_1, x_2, \ldots, x_T \qquad x_1 = x_T$

pseudo-random number generation

generators with lager cycle length T

Mersenne Twister (1998)

Park-Miller (1988)

XOR-Shift (2003)

XoroShiro (2018)

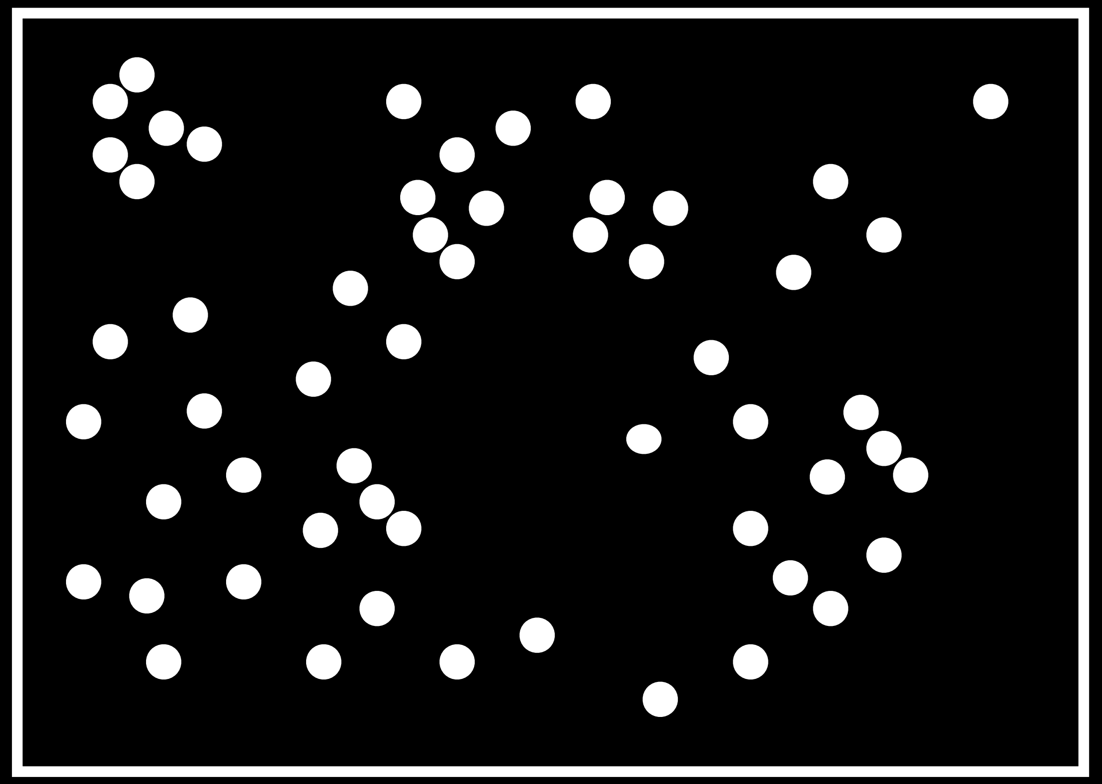
parallel random number generators

$z_1(\text{proc id})$

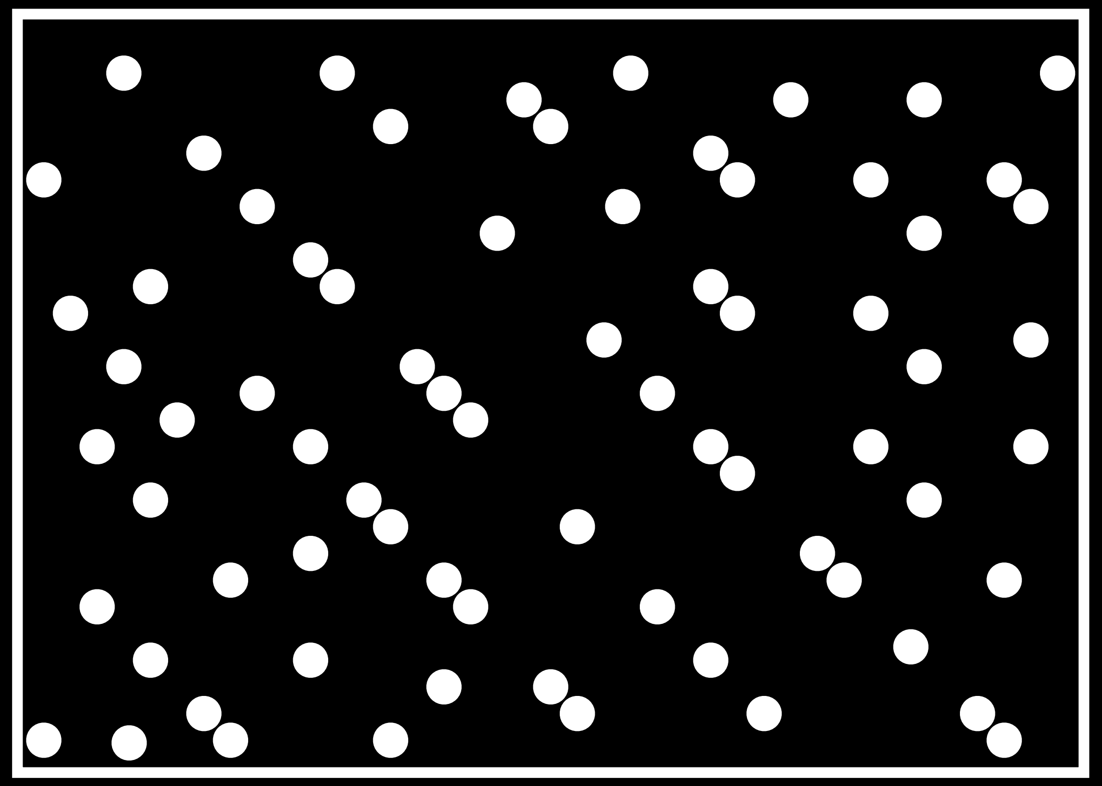
$$T_{parallel} = \frac{T}{\#\text{proc}}$$

quasi-RNG (low discrepancy sequence)

pseudo-RNG



quasi-RNG



Additive recurrence sequence:

$$x_{i+1} = (x_i + \alpha) \mod 1$$

α irrational $\quad \alpha = \sqrt{2} - 1 \quad \alpha = \frac{\sqrt{5} - 1}{2}$

more modern: Sobol - sequence

transform random variables

$$Y = g(X) \qquad g : J \to J' \qquad Domain(g) := J$$

$$g : \mathbb{R} \to \mathbb{R}$$

$$Range(X) \subseteq Domain(g)$$

inverse $\quad g^{-1}(y) := \{x \in J : g(x) = y\}$

transform discrete random variables

given p_X, $Y = g(X)$

$g^{-1}(y) := \{x \in J : g(x) = y\}$

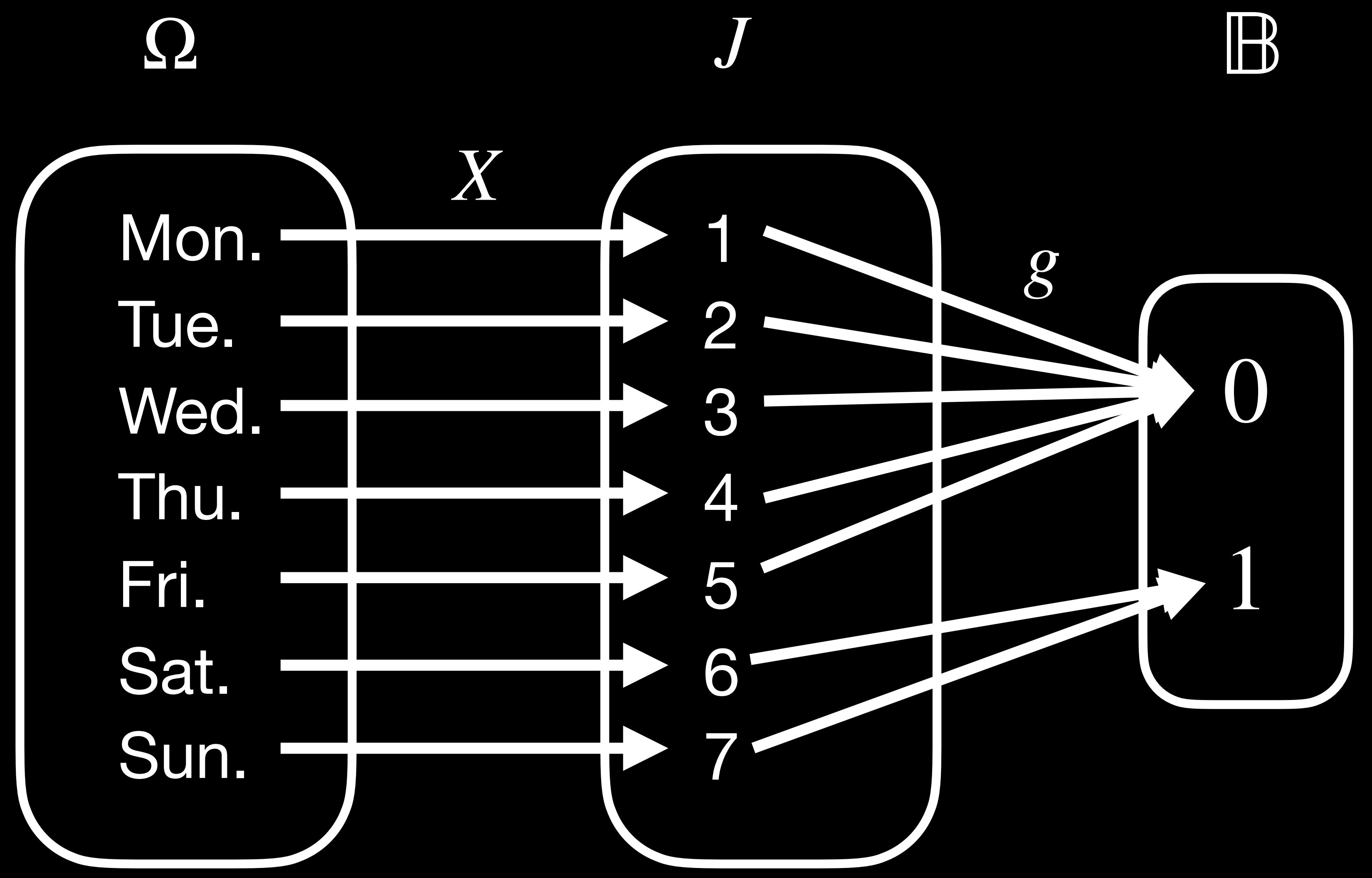
$P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$

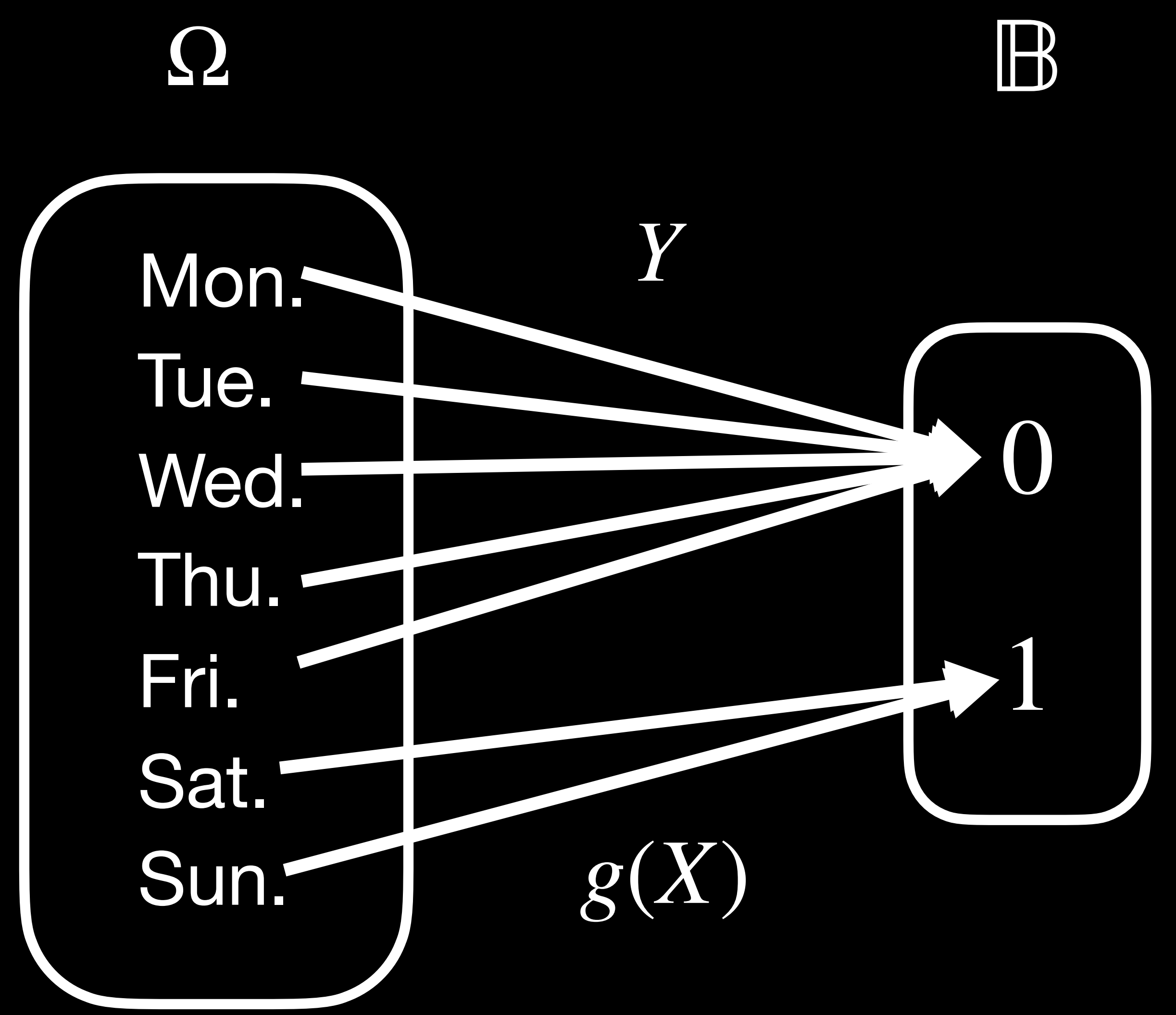
What is p_Y ?

$$p_Y(y) = P(Y = y) = P(g(X) = y) = P\left(X \in g^{-1}(y)\right)$$

$$= P\left(\bigcup_{x \in g^{-1}(y)} \{X = x\}\right) = \sum_{x \in g^{-1}(y)} p_X(x)$$

exercise: transform discrete random variables





$$g(x) = \begin{cases} 0 & \text{for } x \in \{1,\ldots,5\} \\ 1 & \text{for } x \in \{6,7\} \end{cases}$$

$$\boxed{\begin{aligned} p_Y(y) &= \sum_{x \in g^{-1}(y)} p_X(x) \\ g^{-1}(y) &:= \{x \in J : g(x) = y\} \end{aligned}}$$

$$p_X(x) = \frac{1}{7} \qquad x \in \{1,\ldots,7\}$$

derive:

$$p_Y(1) = \sum_{x \in g^{-1}(1)} p_X(x) = \sum_{x \in \{6,7\}} p_X(x) = p_X(6) + p_X(7) = \frac{2}{7}$$

transform continuous random variables

given f_X, $Y = g(X)$ with g increasing in $Range(X)$

$\longrightarrow g^{-1}$ is a function * not the full story

What is f_Y ?

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P\left(X \leq g^{-1}(y)\right)$$

$$= F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(g^{-1}(y)) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$$

transform continuous random variables

given f_X, $Y = g(X)$ with g decreasing in $Range(X)$

$\longrightarrow g^{-1}$ is a function * not the full story

What is f_Y ?

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P\left(X \geq g^{-1}(y)\right)$$

$$= 1 - P\left(X \leq g^{-1}(y)\right) \quad = 1 - F_X(g^{-1}(y))$$

$$\boxed{P(\bar{A}) = 1 - P(A)}$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = -\frac{d}{dy}F_X(g^{-1}(y)) = -f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$$

exercise: transform continuous random variables

$X \sim \mathcal{U}(0,1)$

$$f_X(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

increasing g

$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$

decreasing g

$f_Y(y) = -f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$

$Y = g(X) = 1 - X$

derive:

$g^{-1}(y) = 1 - y$

$$f_Y(y) = -f_X(1-y)\frac{d}{dy}(1-y) = -f_X(1-y)(-1)$$

$$= \begin{cases} 1 & \text{for } 0 \le 1-y \le 1 \\ 0 & \text{else} \end{cases} \quad = \begin{cases} 1 & \text{for } 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

inversion transform

$Y = g(X) \ := F_X(X)$ $\qquad F_X : \mathbb{R} \to [0,1]$

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P\left(F_X(X) \leq y\right) \\
&= P\left(X \leq F_X^{-1}(y)\right) \\
&= F_X\left(F_X^{-1}(y)\right) \\
&= y
\end{aligned}
$$

$\longrightarrow$ $Y \sim \mathcal{U}(0,1)$

$U \sim \mathcal{U}(0,1)$

$X = F_X^{-1}(U)$ $\longrightarrow$ X has the CDF $F_X(x)$

non-uniform distributions

exponential distribution $Exp(\lambda)$

$$y = F_X(x) \quad = 1 - e^{-\lambda x} \qquad\qquad x \geq 0$$

$$e^{-\lambda x} = 1 - y$$

$$-\lambda x = \ln(1 - y)$$

$$x = -\frac{1}{\lambda} \ln(1 - y)$$

$$Y \sim \mathscr{U}(0,1) \qquad \longrightarrow \qquad U := 1 - Y \sim \mathscr{U}(0,1)$$

$$X = -\frac{1}{\lambda} \ln(U) \qquad U \sim \mathscr{U}(0,1)$$

$$\longrightarrow \quad X \sim Exp(\lambda)$$

non-uniform distributions

$$X = -\frac{1}{\lambda}\ln(U) \qquad U \sim \mathcal{U}(0,1) \qquad X \sim Exp(\lambda)$$

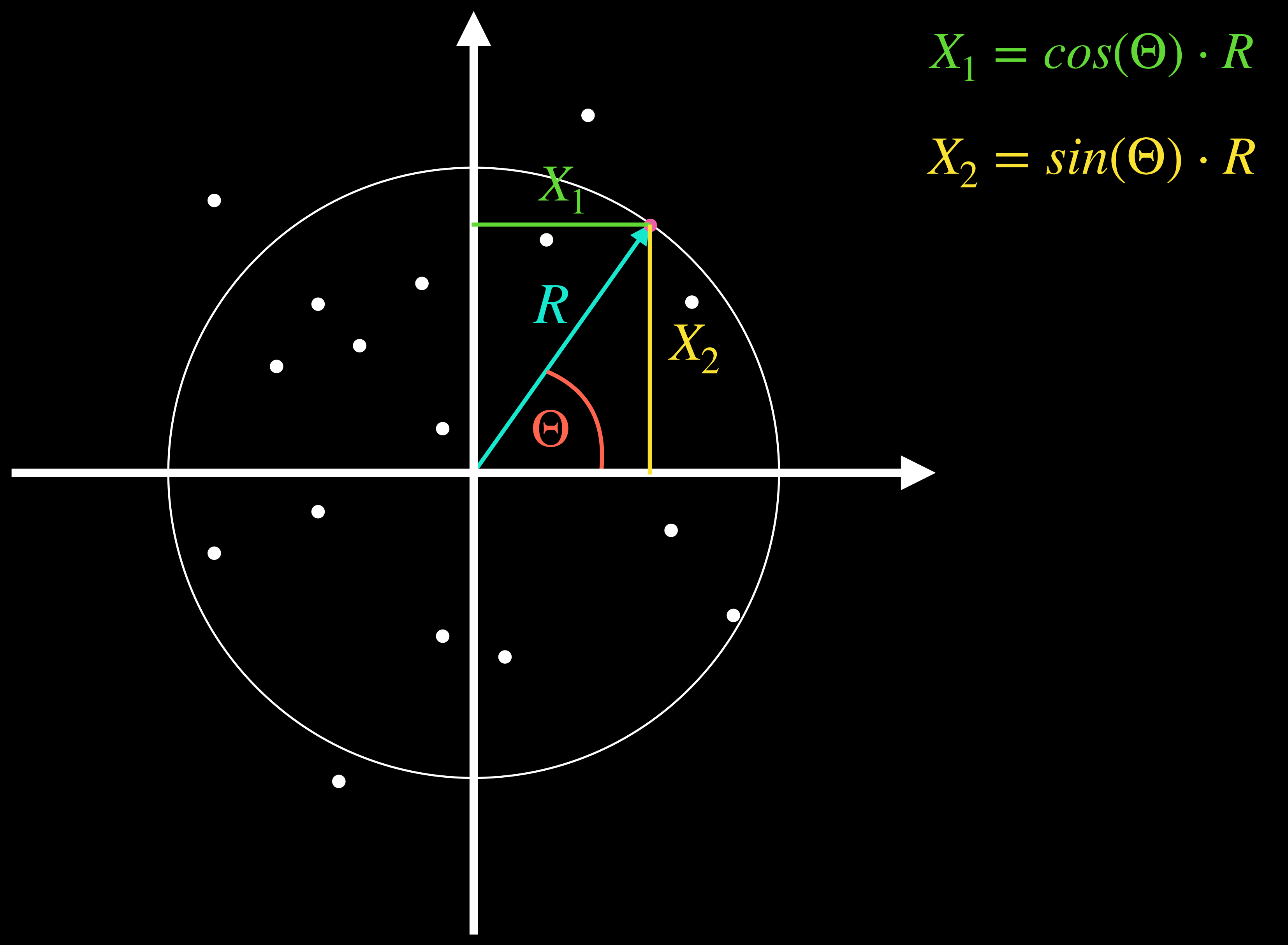
standard Gaussian/ normal distribution $\mathcal{N}(0,1)$

Box-Muller Transform

$$X_1, X_2 \sim \mathcal{N}(0,1)$$

$$R^2 = X_1^2 + X_2^2 \sim Exp\left(\frac{1}{2}\right)$$

$$\Theta \sim \mathcal{U}(0,2\pi)$$



exercise: write X_1, X_2 using $U_1, U_2 \sim \mathcal{U}(0,1)$

$$R^2 = -2\ln(U_1)$$

$$\Theta = 2\pi U_2$$

$$X_1 = cos(2\pi U_2) \cdot \sqrt{-2\ln(U_1)}$$

$$X_2 = sin(2\pi U_2) \cdot \sqrt{-2\ln(U_1)}$$

end