Tensor Categories (Spring 22)

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preliminary version 0.0

These notes were created for the course Tensor Categories at RWTH Aachen University in Spring 2022.

The idea was to give an introduction to the topic with minimal prerequisites using the book

> [EGNO] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, Victor Ostrik:
> **Tensor Categories.**
> American Mathematical Soc. 2016.

as the main reference.

The main goal was to explain Deligne's theorem on tensor categories and to include some current research directions.

I am happy to hear about any comments on these notes.

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Content I

Content II

6. Braided categories

[EGNO 8.1 – 8.2, 7.13, 8.5, 8.10] braided categories, symmetric braiding, monoidal center, Yetter–Drinfeld modules, Ribbon category, knot invariants, link invariants, graphical calculus

7. Tensor categories

[EGNO 4.1, 4.6 – 4.11, 4.13] semisimple fusion categories, Grothendieck rings, pivotal category, spherical category, trace, dimension, negligible morphisms, semisimplification, Verlinde category

8. Hopf algebras & supergroups

[EGNO 1.9, 5.2 – 5.3, 5.5 – 5.6, 5.9 – 5.10] algebras, coalgebras, bialgebras, Hopf algebras, antipode, smash product, Gabriel–Cartier–Kostant theorem, Taft Hopf algebra, supergroups

9. Tannaka–Krein reconstruction

[EGNO 1.10 – 1.11, 5.1 – 5.4] representations of Hopf algebras, Tannaka duality, Tannaka–Krein, reconstruction

10. Deligne's theorem

[EGNO 9.9 – 9.11] Deligne's theorem, symmetric fusion categories, positive fusion categories, supergroups, subexponential growth, Schur functors, symmetric and exterior powers

0. 🚀 Introduction

Welcome! 😃

An intuition for tensor categories

- Tensor categories are like rings in the world of categories.
- Tensor categories are categories with $\oplus$, $\otimes$, 0, 1.

“Adding” and “multiplying” vector spaces and linear maps

Let’s say, finite-dimensional vector spaces:

- add/multiply dimensions
- $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ (or $\ldots = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$...)
- $A \otimes B = \left(A_{ij}B\right)_{i,j} = \begin{pmatrix} A_{11}B & A_{12}B & \ldots \\ A_{21}B & A_{22}B & \\ \vdots & & \ddots \end{pmatrix}$ (*Kronecker product*) (or $\ldots = (AB_{ij})_{i,j}$...)

where “or” means: depending on the choice of a basis, and even more choices are possible!

Tensor categories are like rings

- Given a tensor category $\mathcal{C}$, we can compute its *Grothendieck ring* $K_0(\mathcal{C})$.
- Given a ring R, we can look for a tensor category $\mathcal{C}$ such that $K_0(\mathcal{C}) \cong R$. This is called *categorification*. (And for many rings, this is an interesting question.)

Representations form tensor categories

G a group with modules $(V_1, \phi_1), (V_2, \phi_2)$, that is, V_i are (finite-dimensional) vector spaces and $\phi_i \colon G \to Aut(V_i)$ group homomorphisms
$\Rightarrow V_1 \oplus V_2$ is a G-module with $\phi_\oplus(g) = \phi_1(g) \oplus \phi_2(g)$.
$\Rightarrow V_1 \otimes V_2$ is a G-module with $\phi_\otimes(g) = \phi_1(g) \otimes \phi_2(g)$.

> The collection of all G-modules forms a tensor category.

Note: If G is the one element-group, this is exactly the collection of (finite-dimensional) vector spaces.

Many tensor categories are "representation theoretic"

Deligne's theorem. Over the complex numbers, every symmetric tensor category $\mathcal{C}$ of subexponential growth is equivalent to the representation category of a supergroup G.

The supergroup G can be *reconstructed* from the category $\mathcal{C}$.

Beyond Deligne's theorem...

there are new worlds to discover!
We'll see examples of tensor categories, even over the complex numbers, which are **not** representation categories of anything.
👽

Let's jump right into it!

	Topic
0	Intro (these slides)
1	➡️ Categories
2	Functors
3	Additive Categories
4	Abelian Categories
5	Monoidal Categories & Grothendieck rings
6	Braided Categories
7	Tensor Categories
8	Hopf Algebras & Supergroups
9	Tannaka–Krein Reconstruction
10	Deligne's Theorem
11	Deligne's Interpolation Categories
12	Tensor Categories in Positive Characteristic

1. 🌌 Categories

- http://128.2.67.219/joyalscatlab/published/Categories
- https://ncatlab.org/nlab/show/category

Definition of categories

A locally small (this will be implied throughout) *category* is

1. a collection $\mathrm{Ob}(\mathcal{C})$ of *objects*,
2. sets of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$,
3. distinguished *"identity"* morphisms $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, $\forall X \in \mathrm{Ob}(\mathcal{C})$, and
4. *composition* maps $\forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z), \quad (g, f) \mapsto g \circ f,$$

such that $\forall X, Y, Z, W, f \in \mathrm{Hom}_{\mathcal{C}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z), h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$

- *associativity:* $h \circ (g \circ f) = (h \circ g) \circ f$ and
- $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$.

Notation: Some people (and we may) write $End(X)$ for $Hom_{\mathcal{C}}(X, X)$, $\mathcal{C}$ or $\mathcal{C}_0$ for $\mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}(X, Y)$, $\mathcal{C}(X, Y)$, $\mathrm{Mor}(X, Y)$, or $\mathcal{C}_1(X, Y)$ for $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, X for id_X, or omit the composition symbol $\circ$, as in gf.

Category of sets

$\mathcal{C} = Set$

- objects: sets
- morphisms: maps
- identity morphisms: $X \to X, x \mapsto x$
- composition: $(g \circ f)(x) = g(f(x))$

✓

Category of vector spaces

$\Bbbk$ a field.

$\mathcal{C} = \mathit{Vec}_{\Bbbk}$

- objects: $\Bbbk$-vector spaces
- morphisms: $\Bbbk$-linear maps
- identity morphisms, composition: as in *Set*

✓

$\mathit{FVec}_{\Bbbk} :=$ category of finite-dimensional $\Bbbk$-vector spaces ✓

Sets as categories

For any set S we have a *discrete category* with

- objects: X_s for $s \in S$
- morphisms: only identity morphisms id_{X_s}
- composition: no two distinct morphisms are compatible!

✔ 😴

If $S = \emptyset$, this is called the *empty category*.

Monoids $\leftrightarrow$ categories with 1 object

A *monoid* M is a set with an operation $M \times M \to M, (a, b) \mapsto ab$ and an identity element $e \in M$, i.e. $em = m = me$ for all $m \in M$.
It gives rise to a category $\mathcal{C}_M$ with

- one object X
- $\mathrm{Hom}(X, X) = M$
- $\mathrm{id}_X = e$,
- composition as in M.

✓

This yields a 1-1 correspondence between monoids and categories with 1 object.

Easy to see: Groups $\leftrightarrow$ categories with 1 object in which each morphism is invertible

Posets as categories

A *poset* P is a set with a binary relation $\leq$ which is

- *reflexive:* $p \leq p$,
- *transitive:* $p \leq q \leq r \Rightarrow p \leq r$, and
- *antisymmetric:* $p \leq q \leq p \Rightarrow p = q$ for all $p, q, r \in P$.

It gives rise to a category $\mathcal{C}_P$ with

- objects: $\{X_p\}_{p \in P}$
- morphisms: $\mathrm{Hom}(X_p, X_q) = \begin{cases} \{f_{p,q}\} & p \leq q \\ \emptyset & \text{else} \end{cases}$
- identity morphisms: $\mathrm{id}_{X_p} = f_{p,p}$
- composition: $f_{q,r} \circ f_{p,q} = f_{p,r}$

✓

$\mathcal{C}_P$ has the property that every morphism space has at most one element. (Is there a 1-1 correspondence between such categories and posets ?)

Note that discrete categories are special cases of the above categories. (How ?)

Categories of graphs

$\mathcal{C} = \mathit{Graph}$

- objects: simple (undirected, no loops, no multi-edges) graphs
- morphisms: maps between sets of vertices which preserve adjacency
- identity morphisms: identity maps on sets of vertices
- composiiton: as in *Set*

✓

Similarly, we could allow loops, multi-edges, directed edges here.

> The examples $\mathit{Graph}, \mathcal{C}_M, \mathcal{C}_P$ show: it doesn't generally make sense to talk about "elements" of an object in a category!

Quivers and categories

A *quiver* Q is a set of vertices Q_0 together with a multiset Q_1 of edges $e \in Q_0 \times Q_0$. (It's a directed graph, and now, loops and multi-edges are okay!)

1. To "any" category $\mathcal{C}$ we can associate a quiver $Q(\mathcal{C})$.
2. To any quiver Q we can associate a *free* category $\mathcal{C}_Q$.

? **Problem 1:** How do these constructions work, and what is their relation?

"any" above really means "any category we currently have in mind" or "any category which will be relevant for us". There is a set theory issue which we ignore for now.

Let's not confuse this with the category **of** quivers (in the sense of category of graphs above)!

Opposite categories & product categories

Any category $\mathcal{C}$ gives rise to an *opposite category* $\mathcal{C}^{op}$ with

- objects and identity morphisms: same as $\mathcal{C}$
- morphisms: $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$
- composition: $g \circ_{op} f := f \circ g$

✓

Any two categories $\mathcal{C}, \mathcal{D}$ give rise to the *product category* $\mathcal{C} \times \mathcal{D}$ with

- objects: (X, Y) for $X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D})$
- morphisms: (f, f') for f, f' morphisms in $\mathcal{C}, \mathcal{D}$, resp.
- identity morphisms: (id_X, id_Y) for $X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D})$
- composition: $(g, g') \circ (f, f') = (g \circ f, g' \circ f')$

✓

A selection of categories

$\mathbb{k}$ is a field.

category	objects	morphisms
Set	sets	maps
$\mathit{Vec}_{\mathbb{k}}$	$\mathbb{k}$-vector spaces	$\mathbb{k}$-linear maps
Grp	groups	group homomorphisms
Ab	abelian groups	group homomorphisms
Ring	rings	ring homomorphism
Top	topological spaces (sets with a topology)	continuous maps
Man	smooth manifols	smooth $(=C^{\infty}$-)maps
$\mathit{Ban}_{\mathbb{k}}$	$\mathbb{k}$-Banach spaces	bounded operators
$\mathit{Hilb}_{\mathbb{k}}$	$\mathbb{k}$-Hilbert spaces	linear maps
$\mathit{Var}_{\mathbb{k}}$	varieties over $\mathbb{k}$	morphisms of varieties

Special morphisms

A morphism f is called

- *invertible* or *isomorphism* if $\exists f^{-1}$ such that $f^{-1} \circ f$, $f \circ f^{-1}$ are identity morphisms
- *monomorphism* if $f \circ g = f \circ g' \Rightarrow g = g'$ for all compatible parallel morphisms g, g'
- *epimorphism* if $g \circ f = g' \circ f \Rightarrow g = g'$ for all compatible parallel morphisms g, g'

> Identity morphisms are isomorphisms, and all isomorphisms are both monomorphisms and epimorphisms.

We denote by $Aut_{\mathcal{C}}(X) \subset End_{\mathcal{C}}(X)$ the set of *automorphisms*, i.e., invertible endomorphisms.

Problem 2: Complete the following table.

category	isomorphisms	monomorphisms	epimorphisms
Set			
Vec			
Top			
$\mathcal{C}_M$, M a monoid			
$\mathcal{C}_P$, P a poset			
$\mathcal{C}_Q$, Q a quiver			
$\mathcal{C}^{op}$			
$\mathcal{C} \times \mathcal{D}$			

Intuitively:

- $\mathit{Vect}_{\Bbbk} \subset \mathit{Ab} \subset \mathit{Grp} \subset \mathit{Set}$
- $\mathit{Man} \subset \mathit{Top} \subset \mathit{Set}$
- $\mathit{Hilb}_{\Bbbk} \subset \mathit{Ban}_{\Bbbk} \subset \mathit{Top} \subset \mathit{Set}$

But what does $\subset$ mean for categories? To make this precise, and more generally, to relate categories to each other, we use functors.

Up next: **functors**. 😃

2. Functors

- http://128.2.67.219/joyalscatlab/published/
- https://en.wikipedia.org/wiki/Adjoint_functors
- https://en.wikipedia.org/wiki/Yoneda_lemma

oldie but goldie 😃

Definition of functors

> "Functors are homomorphisms for categories."

A *functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is

1. a mapping $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and
2. a mapping $F = F_{X,Y}\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$

such that (*functoriality:*)

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$
- $F(g \circ f) = F(g) \circ F(f)$ for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$

Special functors

F is called

- *faithful*, *full*, or *fully faithful* if $F_{X,Y}$ is injective, surjective, or bijective for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, respectively,
- *essentially surjective* if for every $Y \in \mathrm{Ob}(\mathcal{D})$ there is an $X \in \mathrm{Ob}(\mathcal{C})$ such that $F(X) \cong Y$
- an *equivalence* if F is all of the above (and we write $\mathcal{C} \simeq \mathcal{D}$)
- an *endofunctor* if $\mathcal{C} = \mathcal{D}$
- an *autoequivalence* if all of the above

Examples of functors

- *identity functors*, sending all objects and morphisms to themselves

> A functor F from a discrete category $\mathcal{C}$ to any category $\mathcal{D}$ is the same as a selection $F(X) \in \mathrm{Ob}(\mathcal{D})$ for each $X \in \mathrm{Ob}(\mathcal{C})$.

Such a functor is faithful, but generally not full or essentially surjective.

- If $F\colon \mathcal{C} \to \mathcal{D}$ is a faithful functor and $\mathcal{D}$ is discrete, then $\mathcal{C}$ is discrete. Discrete categories are equivalent if their collections of objects are in bijection.
- functors between $\mathcal{C}_M$ and $\mathcal{C}_{M'}$ for M, M' monoids (or groups) are exactly morphisms of monoids (groups)

Examples of functors (cont'd)

- $\mathcal{D}$ any category, $\mathcal{C}$ a category with
  - objects: any collection of objects from $\mathcal{D}$
  - morphisms: all morphisms in $\mathcal{D}$ between the objects in $\mathcal{C}$
  - composition & identity morphisms: as in $\mathcal{D}$

  Then there is an obvious functor $F\colon \mathcal{C} \to \mathcal{D}$. This functor is fully faithful, but in general not essentially surjective.
  $\mathcal{C}$ is called *full subcategory* of $\mathcal{D}$

Skeletal & essentially small category

FSet the category of finite sets, FSet^0 the **full subcategory** with objects $X_i := \{1, \ldots, i\}$ for all $i \geq 0$. Let $F\colon \mathit{FSet}^0 \to \mathit{FSet}$ be the obvious functor. It is fully faithful, and essentially surjective! So $\mathit{FSet} \simeq \mathit{FSet}^0$.

- Categories in which all isos are identity morphisms are called *skeletal* (like FSet^0, discrete categories, posets as categories, free categories over quivers, ...).
- Categories which are equivalent to a category with a set (not a "collection") of objects are called *essentially small* (like *FSet*).

Can you see how to repeat the above construction for *FVec*?

Almost all categories we consider are essentially small, so they have a **set** of isomorphism classes of objects. Often, we can work with isomorphism classes of objects, that is, an equivalent skeletal version of a given category.

Forgetful functors

Every vector space is a set. So there is a well-defined functor

$$F\colon \mathit{Vec}_{\mathbb{k}} \to \mathit{Set}, \quad F(X) = X, \quad F(f) = f.$$

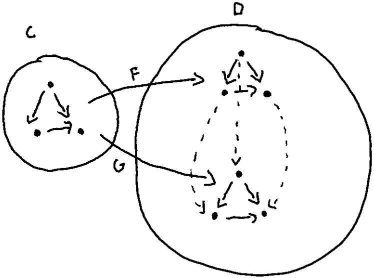
✓ functoriality

- "Functors which forget a part of the structure."
- faithful, but generally not full or essentially surjective

Similarly, there are forgetful functors for the "inclusions" $\mathit{Vec}_{\mathbb{k}} \subset \mathit{Ab} \subset \mathit{Gr} \subset \mathit{Set}$ or $\mathit{Man} \subset \mathit{Top} \subset \mathit{Set}$.

Natural transformations

> "Natural transformations are homomorphisms for functors."



(jasdev.me)

Natural transformations

Consider categories $\mathcal{C}, \mathcal{D}$ and functors $F, G\colon \mathcal{C} \to \mathcal{D}$.
A *natural transformation* $\alpha\colon F \Rightarrow G$ is a collection of morphisms $(\alpha_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X)))_{X \in \mathrm{Ob}(\mathcal{C})}$ such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$,

$$\alpha_Y \circ F(f) = G(f) \circ \alpha_X$$

There is a diagram for this (thanks, nlab):

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"'] & F(y) \arrow[d, "\alpha_y"] \\ G(x) \arrow[r, "G(f)"'] & G(y) \end{tikzcd}$$

The natural transformation are denoted by $Nat(F, G)$. α is called a *natural isomorphism* if α_X is an isomorphism for each X.

Functor category

For categories $\mathcal{C}, \mathcal{D}$, the *functor category* denoted *Fun*$(\mathcal{C}, \mathcal{D})$ (or $[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$) is the category with

- objects: functors from $\mathcal{C}$ to $\mathcal{D}$
- morphisms: natural transformations of functors from $\mathcal{C}$ to $\mathcal{D}$
- identity morphisms: *identity transformations* $(\mathrm{id}_{F(X)})_{X \in \mathrm{Ob}(\mathcal{C})}$
- composition: composition in $\mathcal{D}$

Isomorphic functors

Isomorphisms in $Fun(\mathcal{C}, \mathcal{D})$ are exactly natural isomorphisms.

Example: Pick an automorphism σ_X for each $X \in \mathrm{Ob}(\mathcal{C})$. Then

$$F\colon \mathcal{C} \to \mathcal{C}, \quad X \mapsto X, \quad \mathrm{Hom}_{\mathcal{C}}(X, Y) \ni f \mapsto \sigma_Y \circ f \circ \sigma_X^{-1},$$

is isomorphic to the identity functor. (What is α?)

Equivalences revisited

$F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that FG and GF are isomorphic to identity functors.

if: $F_{X,Y}$ is bijective with an inverse given by $G_{FY,FX}$, so F is fully faithful. For every $Y \in \mathcal{D}$, $Y \cong F(X)$ with $X := G(Y)$, so F is essentially surjective.

only if (idea): choose isomorphims $\beta_Y\colon Y \to F(X_Y)$ for all $Y \in \mathrm{Ob}(\mathcal{D})$. Define $G(Y) := Y_X$ and $G(f)$ using the β_Y.

G for F is not unique!

Adjoint functors

Consider $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$. We say (F, G) is an *adjoint pair* of functors if there is a natural isomorphism $\Phi\colon \operatorname{Hom}_{\mathcal{D}}(F-, -) \to \operatorname{Hom}_{\mathcal{C}}(-, G-)$, viewed as functors $\mathcal{C}^{op} \times \mathcal{D} \to Set$. In this case, we also say F is *left adjoint* to G, or G is *right adjoint* to F.

Examples:

- equivalences provide adjoint functors
- free and forgetful functors
- hom-tensor adjunction

Usually, if (F, G) is an adjoint pair, this doesn't mean (G, F) is one, too!

Free functors

Example: Let $G\colon \mathit{Vec} \to \mathit{Set}$ be the forgetful functor. Let $F\colon \mathit{Set} \to \mathit{Vec}$ be the "free vector space" functor. Then (F, G) is an adjoint pair.

Problem 1:

1. Can you write down F, and prove the statement?
2. Does G also have a right adjoint?

Representable functors

For any category $\mathcal{C}$ (assumed as always to be locally small) and $A \in \mathrm{Ob}(\mathcal{C})$, there is a functor

$$h_A = \mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathit{Set}, \quad h_A(X) = \mathrm{Hom}_{\mathcal{C}}(A, X), \quad h_A(f)(g) = f \circ g.$$

functoriality:

- $h_A(\mathrm{id}_X) = (g \mapsto g) = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,X)}$
- $h_A(f' \circ f)(g) = f' \circ f \circ g = h_A(f')(h_A(f)(g))$ ✓

Functors of this form are called *representable*.

Tensor-hom adjunction

$\mathcal{C} = Vec_{\Bbbk}$, $A \in \mathcal{C}$.
$h_A := \mathrm{Hom}_{\Bbbk}(A, -)$, $t_A := (- \otimes_{\Bbbk} A)$ are functors $\mathcal{C} \to \mathcal{C}$.
✓ functoriality for t_A

Define $\alpha_{X,Y}\colon \mathrm{Hom}_{\Bbbk}(t_A(X), Y) \to \mathrm{Hom}_{\Bbbk}(X, h_A(Y))$ as follows:

$$\mathrm{Hom}_{\Bbbk}(X \otimes A, Y) \ni f \mapsto (x \mapsto (a \mapsto f(x \otimes a))) \in \mathrm{Hom}_{\Bbbk}(X, \mathrm{Hom}_{\Bbbk}(A, Y)).$$

Then $\alpha_{X,Y}$ is an iso with inverse $g \mapsto (x \otimes a \mapsto g(x)(a))$, and $\alpha_{X,Y}$ is functorial in X and Y.

(t_A, h_A) is an adjoint pair.

Here, hom-spaces are considered as objects in the same category. This is usually not possible, and even if possible, this might not be a good idea!

Contravariant and covariant functors

A functor $F\colon \mathcal{C}^{op} \to \mathcal{D}$ is sometimes called a *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$.

Example: $\mathcal{C}$ a category, $A \in \mathcal{C}$. $F = h'_A = \mathrm{Hom}_{\mathcal{C}}(-, A)$.

In this terminology, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *covariant*.

Functoriality for a contravariant functor F means that it maps identities to identities and $F(g \circ f) = F(f) \circ F(g)$.

Yoneda lemma

Consider a category $\mathcal{C}$ and $A, B \in \mathrm{Ob}(\mathcal{C})$.

For every functor $F\colon \mathcal{C} \to \mathit{Set}$, the following is a bijection

$$\mathit{Nat}(h_A, F) \to F(A), \quad \alpha \mapsto \alpha_A(\mathrm{id}_A).$$

In particular, $\mathit{Nat}(h_A, h_B) \simeq \mathrm{Hom}_{\mathcal{C}}(B, A)$, and $h_A \cong h_B$ if and only if $A \cong B$, i.e., representable functors are isomorphic if and only the representing objects are.

Problem 2: Prove this.

It's time we get a $\oplus$ for our categories.

Up next: **additive categories**.

3. ✨ Additive categories

- [EGNO 1.1 – 1.2]
- `https://ncatlab.org/nlab/show/additive+category`

R-linear categories

Let R be a commutative ring.
A category $\mathcal{C}$ is called *R-linear* if its hom-spaces are R-modules, i.e., we can **add** morphisms and **multiply** them with scalars from R, and the composition is R-linear in each component.
A functor F between R-linear categories is called *R-linear* if the maps $F_{X,Y}$ on hom-spaces are R-linear for all X, Y. Categories of R-linear functors are denoted Fun_R.

- any R, even $R = \mathbb{Z}$: hom-spaces are abelian groups (with $+$)
- $R = \Bbbk$: hom-spaces are $\Bbbk$-vector spaces

In an R-linear category, every hom-space contains a distinguished element 0.

One can also say $\mathcal{C}$ is *enriched* over the category of R-modules. $\mathbb{Z}$-linear categories are also called *preadditive*.

Examples of R-linear categories

- Ab for $R = \mathbb{Z}$
- Vec, $FVec$, $Hilb$, Ban, ... for any ring $\mathbb{Z} \subset R \subset \Bbbk$
- a discrete category is $\mathbb{Z}$-linear iff it has one object
- $\mathcal{C}_M$ if $M = R$ is viewed as a monoid with $+$
- **non-examples:** Set, $FSet$, Top, ... (why❓)

Additive categories

Let $\mathcal{C}$ be an R-linear category, and consider $X_1, X_2 \in \mathcal{C}$.
An object $Y \in \mathcal{C}$ is called...

1. *zero object* if $End(Y) = \{0\}$ (the trivial R-module)
2. *direct sum* of X_1, X_2 if there are morphisms

$$i_1 : X_1 \to Y, \quad i_2 : X_2 \to Y, \quad p_1 : Y \to X_1, \quad p_2 : Y \to X_2$$

such that $p_1 i_1 = \mathrm{id}_{X_1}, \quad p_2 i_2 = \mathrm{id}_{X_2}, \quad i_1 p_1 + i_2 p_2 = \mathrm{id}_Y$.

(Note the $+$ sign makes sense!)
An R-linear category $\mathcal{C}$ is called *additive R-linear* if it has a zero object and direct sums.

> We say a category is *additive* if it is additive $\mathbb{Z}$-linear. But very often, we want additive $\Bbbk$-linear categories anyway, which are then additive in particular.

Universal property of the zero object

$\mathcal{C}$ additive, $Y, Y' \in \mathcal{C}$ zero objects, $X \in \mathcal{C}$.

1. $\mathcal{C}(X, Y) = \{0\} = \mathcal{C}(Y, X)$
2. $Y \cong Y'$ via a unique isomorphism

Due to 2., we write 0 for the isomorphism class of Y (or Y'), and use it as an object wherever the result does not depend on the actual object, but only on its isomorphism class.

1. above implies that zero objects are *initial* and *terminal* objects at once. We won't need these words.

Universal property of the direct sum

$\mathcal{C}$ additive, $X_1, X_2 \in \mathcal{C}$, $(Y, i_1, i_2, p_1, p_2), (Y', i'_1, i'_2, p'_1, p'_2)$ direct sums of X_1, X_2.

1. For all $Z \in \mathcal{C}$, $g_1 \in \mathcal{C}(Z, X_1)$, $g_2 \in \mathcal{C}(Z, X_2)$, there is a unique $g \in \mathcal{C}(Z, Y)$ such that $g_1 = p_1 g$ and $g_2 = p_2 g$.
2. For all $Z \in \mathcal{C}$, $f_k \in \mathcal{C}(X_1, Z)$, $f_2 \in \mathcal{C}(X_2, Z)$, there is a unique $f \in \mathcal{C}(Y, Z)$ such that $f_1 = f i_1$ and $f_2 = f i_2$.
3. $Y \cong Y'$ via a unique isomorphism

Due to 3., we write $X_1 \oplus X_2$ for the isomorphism class of Y (or Y'), and use it as an object whenever the result does not depend on the actual object, but only its isomorphism class.

1. above implies that $X_1 \oplus X_2$ is a *product*, 2. above implies it is a *coproduct*, and both together imply it is a *biproduct*. We won't need these words.

Problem 1

🤔 Prove the above universal properties of the zero object and the direct sum.

Morphisms between direct sums

Consider objects $X_1, \dots X_n, Y_1, \dots, Y_m$ in an additive $\Bbbk$-linear category $\mathcal{C}$. The following is a direct consequence of the universal property of the direct sum:

Then $\mathcal{C}(\bigoplus_k X_k, \bigoplus_\ell Y_\ell) \cong \bigoplus_{k,\ell} \mathcal{C}(X_k, Y_\ell)$ as $\Bbbk$-vector spaces via the map $f \mapsto (p_\ell f i_k)_{k,\ell}$, and in the other direction, $(f_{k\ell})_{k,\ell} \mapsto \sum_{k,\ell} i_\ell f_{k\ell} p_k$.

In particular, for $f_1 \in \mathcal{C}(X_1, Y_1)$ and $f_2 \in \mathcal{C}(X_2, Y_2)$, we write $f_1 \oplus f_2$ for the morphism $i_1^{(Y)} f_1 p_1^{(X)} + i_2^{(Y)} f_2 p_2^{(X)} \colon X_1 \oplus X_2 \to Y_1 \oplus Y_2$.

The direct sum can be viewed as a functor $\oplus \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (a *bifunctor*).

Additive Grothendieck groups (I)

$\mathcal{C}$ additive.
$X \in \mathcal{C}$ is called *indecomposable* if $X \not\cong 0$ and $X \cong Y_1 \oplus Y_2$ implies $Y_1 \cong 0$ or $Y_2 \cong 0$.
The *additive Grothendieck group*, $K^{\oplus}(\mathcal{C})$, is the abelian group (with $+$) generated by symbols $[X]$ representing isomorphism classes of objects $X \in \mathcal{C}$ with the relations $[X_1 \oplus X_2] = [X_1] + [X_2]$ for all $X_1, X_2 \in \mathcal{C}$.

Example: $FVec_{\Bbbk}$ has only one indecomposable object up to isomorphism: $\Bbbk$. $K^{\oplus}(FVec_{\Bbbk})$ is generated by symbols $[\Bbbk^i]$ for $i \geq 0$, and relations $[\Bbbk^{i+j}] = [\Bbbk^i] + [\Bbbk^j]$. Hence, $K^{\oplus}(FVec_{\Bbbk}) \cong \mathbb{Z}$.

Additive Grothendieck groups (II)

$K^{\oplus}$ is always generated by the isomorphism classes of indecomposable objects.

The example *FVec* suggests that Grothendieck groups are even **freely** generated (without further relations) by the isomorphism classes of indecomposable object. Wrong! 😮 For instance, it can happen that $Y_1 \oplus Y_2 \cong Y_3 \oplus Y_4$ for non-isomorphic indecomposables $Y_1, \ldots, Y_4$. However, it is possible to show this won't happen under certain assumptions...

Krull–Schmidt categories

An additive R-linear category is called *Krull–Schmidt* if

- its end-rings are *Artinian*, e.g., finite rings or finite-dimensional $\mathbb{k}$-algebras, and
- it is *pseudo-abelian*, i.e., for all idempotent endomorphisms e there are morphisms p, i such that $e = ip$ and pi is an identity.

> More generally, the end-rings being semiperfect would suffice here, but we won't need that version.

Krull–Schmidt theorem

$\mathcal{C}$ a Krull–Schmidt category.

Krull–Schmidt theorem. For every $X \in \mathcal{C}$, there exist $n \geq 0$, $Y_1, \ldots, Y_n \in \mathcal{C}$ indecomposable such that $X \cong Y_1 \oplus \cdots \oplus Y_n$. Furthermore, for a fixed X, n is unique, and the isomorphism classes of $Y_1, \ldots, Y_n$ are unique up to re-ordering.

$\Rightarrow K^{\oplus}(\mathcal{C}) \cong \mathbb{Z}^{Indec(\mathcal{C})}$, where $Indec(\mathcal{C})$ is the set of isomorphism classes of indecomposable objects in $\mathcal{C}$.

Example of Krull–Schmidt categories

$\mathit{FVec}_{\Bbbk}$ has up to isomorphism exactly one indecomposable object, $\Bbbk$. $\mathit{FVec}_{\Bbbk}$ is Krull–Schmidt:

- end-rings are finite-dimensional $\Bbbk$-algebras
- $\mathit{FVec}_{\Bbbk}$ has split idempotents: for any vector space endomorphism e, let i and p be the inclusion of and any projection onto the image of e, resp.

(Can you verify these ?)
Hence, $K^{\oplus}(\mathit{FVec}_{\Bbbk}) \cong \mathbb{Z}$.

Application: Classification of finite abelian groups

FAb, category of finite abelian groups

🤔 **Problem 2:**

- Show that *FAb* is ($\mathbb{Z}$-linear) Krull–Schmidt.
- Show that $X \in FAb$ is indecomposable if and only if X is a cyclic group whose order is a prime power.

> **Classification.** Any finite abelian group G is a direct sum (a.k.a. direct product) of a unique set of groups of the form $\mathbb{Z}/p^n\mathbb{Z}$ for p prime and $n \geq 1$.
> $K^{\oplus}(FAbb)$ is isomorphic to a free abelian group with generators $[X_{p,n}]$ for all primes p and $n \geq 1$.

Proof idea of Krull–Schmidt theorem

Consider $X \in \mathcal{C}$

1. As $R = End(X)$ is **Artinian**, it has orthogonal idempotents $e_1, \dots, e_m$ (i.e. $e_k e_\ell = \delta_{k\ell} e_k$), such that $e_1 + \dots + e_m = 1$ and $R_k := e_k R e_k$ is a local ring (i.e. has a unique maximal left/right ideal).
2. Let $p_k\colon X \to X_k$, $i_k\colon X_k \to X$ be morphisms **splitting** e_k. Then $X \cong X_1 \oplus \dots \oplus X_m$ and every X_k is indecomposable, as its end-ring R_k is local.
3. Any choice of orthogonal idempotents $(e'_k)_k$ as above must be such that possibly after reordering, $e'_k \in R_k$. Then e_k and e'_k are conjugate, so X_k is well-defined up to isomorphism.

Technique: Reduction modulo Jacobson radical. Simple Artinian rings = matrix rings over skew-fields.

Additive pseudo-abelian envelopes

$\mathcal{C}$ any R-linear category (not nec. additive!).
We can then embed $\mathcal{C}$ into a "minimal" additive pseudo-abelian category, in the following sense:

1. There exists an additive R-linear pseudo-abelian category $\mathcal{C}'$ and a fully faithful functor $F\colon \mathcal{C} \to \mathcal{C}'$, such that for any R-linear addititive pseudo-abelian category $\mathcal{D}$, the following is an equivalence: $Fun_R(\mathcal{C}', \mathcal{D}) \to Fun_R(\mathcal{C}, \mathcal{D})$, $G \mapsto G \circ F$.
2. $\mathcal{C}'$ is unique up to equivalence.
3. If $\mathcal{C}$ has Artinian end-rings, then $\mathcal{C}'$ is Krull–Schmidt.

We call $\mathcal{C}'$ the *additive pseudo-abelian envelope* of $\mathcal{C}$ – it will be needed! 🗒️

Construction additive pseudo-abelian envelopes (I)

$\mathcal{C}$ R-linear

$\mathcal{C}_1 :=$ "category of formal direct sums" with

- objects: $(X_1, \ldots, X_n)$ for $X_1, \ldots, X_n \in \mathcal{C}$ for $n \geq 0$
- morphisms: $\mathcal{C}_1((X_i)_i, (Y_j)_j) = \{(f_{ij})_{i,j} : f_{ij} \in \mathcal{C}(X_i, Y_j)\}$ ("matrices with entries in $\mathcal{C}$")
- identity morphisms: $(\delta_{ij} id_{X_i})_{i,j}$
- composition: $(g_{ij})_{i,j} \circ (f_{ij})_{i,j} = (\sum_j g_{kj} f_{ik})_{i,j}$ ("matrix multiplication")

> $\mathcal{C}_1$ is additive R-linear and there is a fully faithful functor $\mathcal{C} \to \mathcal{C}_1$, $X \mapsto (X_1 = X)$.

Construction additive pseudo-abelian envelopes (II)

$\mathcal{C}' := \mathcal{C}_2 :=$ "category with formal images of idempotents" with

- objects: (X, e) for $X \in \mathcal{C}_1$, $e \in End(X)$, $e^2 = e$
- morphisms: $\mathcal{C}_2((X, e), (X, e')) = \{e' fe : f \in \mathcal{C}_1(X, X')\}$
- identity morphisms: $\mathrm{id}_{(X,e)} = e$
- composition: as in $\mathcal{C}_1$

$\mathcal{C}_2$ is additive R-linear pseudo-abelian and there is a fully faithful functor $\mathcal{C} \to \mathcal{C}_2$, $X \mapsto ((X_1 = X), \mathrm{id})$.

Every idempotent $e \in End(X)$ is split by $p = e \colon (X, \mathrm{id}) \to (X, e)$ and $i = e \colon (X, e) \to (X, \mathrm{id})$.

Examples of additive pseudo-abelian envelopes

- $\mathbb{k}$ a field.
  $\mathcal{C} := \mathcal{C}_{\mathbb{k}}$ for $\mathbb{k}$ as a monoid, i.e., $\mathcal{C}$ has one object with end-ring $\mathbb{k}$.
  Then $\mathcal{C}_2 \simeq \mathcal{C}_1 \simeq \mathit{FVec}_{\mathbb{k}}$.
- $\mathbb{k}$ a field.
  $\mathcal{C} := \mathit{FVec}_{\mathbb{k}}$.
  Then $\mathcal{C}_2 \simeq \mathcal{C}_1 \simeq \mathit{FVec}_{\mathbb{k}}$.

Why the heck should we call anything “pseudo-abelian”?

Up next: abelian categories.

4. ★ Abelian categories

- [EGNO 1.3 – 1.5, 1.8]

Kernels and cokernels

$\mathcal{C}$ an additive category, $f\colon X \to Y$ a morphism.

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & \\
 & K \arrow[u, "k"] \arrow[r, "0"] & Y \\
K' \arrow[uur, "k'"] \arrow[ur, dashed, "u"] \arrow[urr, "0"'] & &
\end{tikzcd}$$

- $(K, k\colon K \to X)$ is a *kernel* of f if $fk = 0$ and for all $(K', k'\colon K' \to X)$ with $fk' = 0$, there is a unique morphism $u\colon K' \to K$ such that $k' = ku$.

> "The kernel is the largest part of X on which f vanishes."

- $(C, c\colon Y \to C)$ is a *cokernel* of f if $cf = 0$ and for all $(C', c'\colon Y \to C')$ with $c'f = 0$, there is a unique morphism $u\colon C \to C'$ such that $c' = uc$.

> "The cokernel is Y 'modulo' the image of f."

> If they exists, kernel and cokernels are unique up to unique isomorphism. So we may say "the" kernel $\ker f$ and "the" cokernel $\operatorname{coker} f$.

Examples

- *Vec*: f a morphism in $Vec_{\Bbbk}$, i.e., f a linear map. Then $K := \{x \in X : f(x) = 0\}$ with the obvious inclusion k is a kernel, and $C := X/\{f(x) : x \in X\}$ with the obvious quotient map c is a cokernel. (Note: these are objects and morphisms in *Vec*!)

✔ Verify this.

- Similarly: *FVec*, *Ab*, *FAb*.
- **Non-examples:** *Top* doesn't have a zero object! (It's not additive.)

Abelian categories

An additive category $\mathcal{C}$ is called *abelian* if

- every morphism $f\colon X \to Y$ has a kernel, a cokernel, and an *image* (more precisely: an *image factorization*); that is, there are objects and morphism

$$K \xrightarrow{k} X \xrightarrow{f_1} I \xrightarrow{f_2} Y \xrightarrow{c} C$$

  such that
- $f = f_2 f_1$
- $(K, k) = \ker f$, $(C, c) = \operatorname{coker} f$
- $(I, f_1) = \operatorname{coker} k$, $(I, f_2) = \ker c$

In this case, images are also unique up to unique isomorphism, so let's say "the" image $\operatorname{im} f = (I, f_1, f_2)$.

First examples of abelian categories

- In $\mathit{Vec}_{\Bbbk}$, $\operatorname{im} f$ is the object $I = \{f(x) : x \in X\}$ with the morphisms $f_1 = f\colon X \to I$ and $f_2 : I \to Y$ the obvious inclusion. (Note: these are objects and morphisms in *Vec*!)

  ✔ Verify these!
- *FVec*, *Ab*, *FAb* (we already saw *FAb* is pseudo-abelian)

Example: Representation categories

Recall that a *group* is a set G with an operation $G \times G \to G$, an identity element e, and an inverse element g^{-1} for each $g \in G$.

> $Aut(X)$ is a group for each object X of any category.

For any field $\Bbbk$ and group G, a *representation* of G over $\Bbbk$ is an object $V \in Vec_{\Bbbk}$ with a group homomorphisms $\phi \colon G \to Aut_{Vec_{\Bbbk}}(V)$.
Let $\mathcal{C} := Rep(G, \Bbbk)$ be the category with

$$\begin{tikzcd} V \arrow[r, "\phi(g)"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ V' \arrow[r, "\phi'(g)"'] & V' \end{tikzcd}$$

- objects: (V, ϕ) as above
- morphisms: $\mathcal{C}((V, \phi), (V', \phi)) = \{f \in Vec_{\Bbbk}(V, V') : f\phi(g) = \phi'(g)f \; \forall g \in G\}$.
- identity morphisms & composition: as in $Vec_{\Bbbk}$

Example: Representation categories (cont'ed)

$\mathit{Rep}(G, \mathbb{k})$ is a $\mathbb{k}$-linear abelian category, with

- 0 as in $\mathit{Vec}_{\mathbb{k}}$
- $(V, \phi) \oplus (V', \phi') = (V \oplus V',\ g \mapsto \phi(g) \oplus \phi'(g))$
- $\ker$, coker, im as in $\mathit{Vec}_{\mathbb{k}}$: in particular, the objects and morphisms involved are in $\mathit{Rep}(G, \mathbb{k})$ in a natural way!

🤔 **Problem 1:** Verify this.

📖 Side note: Freyd–Mitchell embedding theorem

More generally: For any $\mathbb{k}$-algebra A, a representation is a vector space V with an algebra homomorphism $A \to End_{Vec_{\mathbb{k}}}(V)$. The category $Rep(A, \mathbb{k})$ of such representations is $\mathbb{k}$-linear abelian (completely as for groups).

✔ Verity this.

> Any small abelian $\mathbb{k}$-linear category is equivalent to a full subcategory $Rep(A, \mathbb{k})$ for some $\mathbb{k}$-algebra A.

> More generally, this works with rings, if we don't require the category to be $\mathbb{k}$-linear. We will see something like this for tensor categories, "Deligne's theorem".

Morita equivalence

> Non-isomorphic rings might have equivalent categories of modules. In this case, they are called *Morita equivalent*. In particular, the algebra in the embedding theorem is not unique or unique up to isomorphism.

🤔 Can you find two non-isomorphic, Morita equivalent rings?

Morphisms in abelian categories

In an abelian category:

- $\{$kernel maps$\}$ = $\{$monomorphisms$\}$ = $\{f\colon \ker f = 0\}$
- $\{$cokernel maps$\}$ = $\{$epimorphisms$\}$ = $\{f\colon \operatorname{coker} f = 0\}$

Hence for any $f\colon X \to Y$:

- $\{$image factorizations of $f\}$ = $\{$epi-mono-factorizations of $f\}$
- f mono / epi $\Rightarrow \operatorname{im} f \cong X$ or Y, resp.
- f mono & epi $\Rightarrow$ f iso

Also:

Every abelian category is pseudo-abelian.

The image factorization $e = ip$ of any idempotent e satisfies $ipip = e^2 = e = ip = i(\mathrm{id})p$, so $pi = \mathrm{id}$, as i is a mono and p is an epi.

Subobjects & quotients

X an object in an abelian category.

- A *subobject* of X is an object Y with a monomorphism $i\colon Y \to X$. Two subobjects (Y, i) and (Y', i') are the said to be equal if there is an isomorphism $\phi\colon Y \to Y'$ such that $i' = \phi i$.
- A *quotient* of X is an object Y with an epimorphism $X \to Y$. (Equality is similar as for subobjects.)
- A *subquotient* of X is quotient of a subobject of X.

$\{$subobjects of $X\} \leftrightarrow \{$quotients of $X\}$ via $\mathrm{ker}, \mathrm{coker}$

Simple & indecomposable objects

X is called *simple* $:\Leftrightarrow$ it has no non-trivial subobjects $\Leftrightarrow$ it has no non-trivial quotients.

> Here, trivial means $\cong 0$ or $\cong X$.

Recall: X is *indecomposable* (in an additive category) if $X \cong X_1 \oplus X_2$ implies $X_1 \cong 0$ or $X_2 \cong 0$.

> In an abelian category, simple objects are indecomposable.

An abelian category is called *semisimple* if all indecomposable objects are simple.

(Non-)Examples of semisimple categories

- $Vec_{\mathbb{k}}, FVec_{\mathbb{k}}$ has one indecomposable object $\mathbb{k}$ which is simple $\Rightarrow$ $Vec_{\mathbb{k}}, FVec_{\mathbb{k}}$ are semisimple abelian categories
- $\mathbb{Z}/4\mathbb{Z}$ is an indecomposable finite abelian group with a non-trivial subgroup (which ?) $\Rightarrow$ *Ab*, *FAb* are non-semisimple abelian categories
- $Rep(G, \mathbb{k})$ may or may not be semisimple, see below...

Schur's lemma

X, Y simple in an abelian category $\mathcal{C}$.

1. $\forall\ f \colon X \to Y$: $f = 0$ or f an iso
2. $\mathcal{C}$ $\Bbbk$-linear $\Rightarrow$ $End(X)$ is a division algebra over $\Bbbk$
3. $\Bbbk$ is algebraically closed, $\mathcal{C}$ hom-finite $\Bbbk$-linear $\Rightarrow$ $End(X) \cong \Bbbk$

Pf.

- $\ker f$ is a subobject of X, $\operatorname{im} f$ is a subobject of Y. Hence those are isomorphic to 0 or X, or to 0 or Y, respectively.
- So if $f \neq 0$, then $\ker f \cong 0$ and $\operatorname{im} f \cong Y$.
- This means f is a mono and an epi, so it's an iso.
- The rest follows directly upon using the fact that over an algebraically closed field, there are no non-trivial finite-dimensional division algebras.

Non-semisimple representation categories

$G = \{1, \tau : \tau^2 = 1\}$, the group with 2 elements, $\Bbbk$ a field of characteristic 2, $\mathcal{C} = Rep(G, \Bbbk)$.

$X := \Bbbk^2$, $\phi\colon G \to Aut(X), \tau \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow (X, \phi) \in \mathcal{C}$. ✓

The subobjects of X are: 0, X, and $\Bbbk\binom{1}{0}$.

Pf. Y any subobject of X. $Y \ni v = \binom{v_1}{v_2}$ with $v_2 \neq 0$. Then $Y \ni \phi(\tau)(v) = \binom{v_1+v_2}{v_2}$, so $Y \ni v + \phi(\tau)(v) = \binom{v_2}{0}$. As $v_2 \neq 0$, this means $Y \ni \binom{v_2}{0}/v_2 = \binom{1}{0}$, and $Y \ni (v - v_1\binom{1}{0})/v_2 = \binom{0}{1}$, so $Y = X$. On the other hand, $\phi(\tau)(\binom{1}{0}) = \binom{1}{0}$, so $\Bbbk\binom{1}{0}$ is a subobject.

X is indecomposable & not simple. $\mathcal{C}$ is not semisimple.

Maschke's theorem

G a finite group, $\Bbbk$ a field.

If $\Bbbk$ has charactersitic 0 or relative prime to $|G|$, then $\mathcal{C} = Rep(G, \Bbbk)$ is semisimple.

Pf.

- Assume X is indecomposable and $0 \ncong Y \ncong X$ is a subobject, i.e. there is a mono $i\colon Y \to X$ in $\mathcal{C}$.
- In particular, this is an injective linear map, so (by completing a basis,) there is a surjective linear map $p_0 : X \to Y$ such that $p_0 i = \mathrm{id}_Y$.
- *averaging trick:* Let $p\colon X \to Y$ be the linear map $\frac{1}{|G|}\sum_{g\in G}\phi_Y(g)p_0\phi_X(g^{-1})$. Then p is a morphism in $\mathcal{C}$ and $pi = \mathrm{id}_Y$, in particular, p is an epi.
- So ip and $1 - ip$ are idempotents in $End_{\mathcal{C}}(X)$. This implies $X \cong \mathrm{im}\, ip \oplus \mathrm{im}(1 - ip)$, but $\mathrm{im}\, ip \cong Y \ncong 0, X$.
- This is a contradiction, as X is indecomposable.

✓ Verify these!

Composition series

$\mathcal{C}$ abelian, $X \in \mathcal{C}$.
A finite *composition series* of X is a finite sequence of objects

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that X_i/X_{i-1} is simple for all $1 \leq i \leq n$.
Two composition series $(X_i)_{0 \leq i \leq n}$ and $(X'_i)_{0 \leq i \leq n'}$ are called *equivalent* if $n = n'$ and there is a permutation $\sigma \in S_n$ such that $X_i/X_{i-1} \cong X'_{\sigma(i)}/X'_{\sigma(i)-1}$ for all $1 \leq i \leq n$.

Jordan–Hölder. If an object in an abelian category $\mathcal{C}$ has a finite composition series, then it is essentially unique, that is, any two finite composition series are equivalent.

In this case, the number n is called the *length* of X.
If every object has finite length, $\mathcal{C}$ is called a *length category*.

Problem 2

Let $G = \{1, \tau : \tau^2 = 1\}$ be the group with 2 elements, we consider $\mathcal{C} = Rep(G, \mathbb{k})$, the category of representations of G over $\mathbb{k}$

1. Classify the indecomposable objects in $\mathcal{C}$ if $\mathbb{k}$ is a field of characteristic not 2. **Hint:** Consider the eigenspaces of τ acting on $X \in \mathcal{C}$.
2. Assume now $\mathbb{k}$ is a field of characteristic 2. What is the length of $\mathbb{k}^2$ in $\mathcal{C}$, where τ acts as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$? (You may use the results on the previous slides.)

🤔 How can we generalize this, e.g. to finite cyclic groups, abelian groups, commutative subalgebras of an algebra, ...?

Exact sequences & Grothendieck groups

A *short exact sequence* is a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that f is a mono, g is an epi, and $\mathrm{im}\, f = \ker g$ (as subobjects of Y).

> Some people say Y is an *extension* of Z by X.

> **Example.** There is a short exact sequence as above if $Y \cong X \oplus Y$ (with f, g the structural map of the direct product), but not every short exact sequence comes from a direct product.

The *Grothendieck group* of an abelian category $\mathcal{C}$ is the abelian group generated by symbols representing the isomorphism classes $[X]$ of objects $X \in \mathcal{C}$ with relations

$$[Y] = [X] + [Z]$$

whenever there is an exact sequence $X \to Y \to Z$.

Grothendieck group of an abelian category

Assume $\mathcal{C}$ is an abelian length category, and $\{X_i\}_{i\in I}$ is a complete set of non-isomorphic simple objects in $\mathcal{C}$.

Then the following is an isomorphism of abelian groups:

$$K(\mathcal{C}) \to \mathbb{Z}^I, \quad [X] \mapsto (\dim \mathcal{C}(X, X_i))_{i\in I} = (\dim \mathcal{C}(X_i, X))_{i\in I}.$$

Proof: By Jordan–Hölder.

This can be viewed as an analog to the Krull–Schmidt theorem, especially using *exact categories*, that is, categories with a "notion of exact sequences".

For any abelian category $\mathcal{C}$, there is a ring map $K^{\oplus}(\mathcal{C}) \to K(\mathcal{C})$, which is an isomorphism if $\mathcal{C}$ is semisimple.

Examples of Grothendieck groups

$m \geq 1$, $C_m = \mathbb{Z}/m\mathbb{Z}$, $\mathcal{C} = Rep(C_m, \mathbb{k})$.

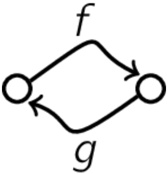
$\mathbb{k} = \overline{\mathbb{k}}$ and characteristic $\mathbb{k} \nmid m$:
$K(\mathcal{C}) \cong K^{\oplus}(\mathcal{C}) \cong \mathbb{Z}^m$ (Can you prove this ?) $K(Vec_{\mathbb{k}}) \cong K^{\oplus}(Vec_{\mathbb{k}}) \cong \mathbb{Z}$

$m = 2$, char $\mathbb{k} = 2$:
$\mathbb{Z}^2 \cong K(\mathcal{C})^{\oplus} \to K(\mathcal{C}) \cong \mathbb{Z}$ $(1,0) \mapsto 1$ $(0,1) \mapsto 2$

🤔 **Problem.** How does this generalize to $m > 2$ and $\mathbb{k} = \overline{\mathbb{k}}$ of characteristic m?

Blocks

A *block* is an additive (full) subcategory generated by a minimal set of indecomposable objects.
We can represent blocks using quivers:
Rep$(C_2, \overline{\mathbb{k}})$ in char. $\neq 2$:  semisimple, two indecomposables/simples



Rep$(C_2, \overline{\mathbb{k}})$ in char. 2: with $gf = 0$; two indecomposables, one simple

We now know categories with a $\oplus$. Meet categories with a $\otimes$.

Up next: monoidal categories.

5. 🌙 Monoidal categories

- [EGNO 2.1 - 2.5, 2.8 - 2.10]

Definition of monoidal categories

A *monoidal category* is a tuple $(\mathcal{C}, \mathbf{1}, \otimes, \lambda, \rho, \alpha)$, where

- $\mathcal{C}$ is a category, $\mathbf{1} \in \mathcal{C}$, $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor,
- $\lambda \colon \mathbf{1} \otimes X \to X$, $\rho \colon X \otimes \mathbf{1} \to X$, $\alpha \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ are natural isomorphisms (between the functors indicated using the placeholders X, Y, Z), satisfying

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbf{1},Y}"] \arrow[dr, "\rho_X \otimes Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

$$\begin{tikzcd}
& (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y \otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W \otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X \otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd}$$

α is called *associator*, λ, ρ are called *left* or *right associator*.

> 🤔 Why these diagrams?

Mac Lane's coherence theorem

Commutativity of the triangle and pentagon diagrams implies that **any** diagram involving

- tensor products of a fixed sequence of objects $X_1, \dots, X_n \in \mathcal{C}$ with possible insertions of unit objects and with any possible choice of parentheses as objects
- tensor products of identities, associators, left or right unitors as morphisms

is commutative in $\mathcal{C}$.

First examples

- any additive category with $\mathbf{1} = 0$, the zero object, and $X \otimes Y = X \oplus Y$
- *Set*, *Top* with $\mathbf{1} = \{*\}$ and $X \otimes Y = X \times Y$, the cartesian product (recall these are not additive categories)
- or *Set*, *Top* with $\mathbf{1} = \emptyset$ and $X \otimes Y = X \sqcup Y$, the disjoint union
- $Vec_{\Bbbk}$ with $\mathbf{1} = \Bbbk$ and $X \otimes Y$ the tensor product of vector spaces
- poset $\mathbb{Z} \geq 0$ as a category with $\mathbf{1} = X_0$ and $X_i \otimes X_j = X_{i+j}$ and $f_{i \leq i'} \otimes f_{j \leq j'} = f_{i+j \leq i'+j'}$
- $End(\mathcal{C})$, the category of endofunctors of a category $\mathcal{C}$, with $\mathbf{1} = Id$, the identity functor, and $\otimes$ the composition of functors

Representations form monoidal categories

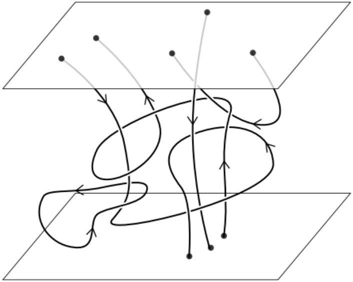
$\mathcal{C} = Rep(G, \Bbbk)$ is a monoidal category with

- $\mathbf{1} = \Bbbk$,
- $X \otimes Y$ as in $Vec_{\Bbbk}$, and with
- $\phi_{X \otimes Y}(g) = \phi_X(g) \otimes \phi_Y(g)$

(Verify this ✓)

Problem 1

- Verify that the category of tangles (EGNO, 2.3.14) is a monoidal category.
- Can you determine a skeleton of this category?



(Note that for the category in EGNO, all boundary points should have vanishing y-coordinates.) (from `http://science.mq.edu.au/~street/PilletRapport.pdf`)

Monoidal functors

A *monoidal functor* is a functor $F\colon \mathcal{C} \to \mathcal{D}$ between monoidal categories **together with**

- an isomorphism $\epsilon\colon \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ and
- a natural isomorphism $\mu\colon FX \otimes_{\mathcal{D}} FY \to F(X \otimes_{\mathcal{C}} Y)$ satisfying

$$\begin{tikzcd}
(FX \otimes FY) \otimes FZ \arrow[r, "\alpha^{(\mathcal{D})}"] \arrow[d, "\mu_{X,Y} \otimes FZ"'] & FX \otimes (FY \otimes FZ) \arrow[d, "FX \otimes \mu_{Y,Z}"] \\
F(X \otimes Y) \otimes FZ \arrow[d, "\mu_{X \otimes Y, Z}"'] & FX \otimes F(Y \otimes Z) \arrow[d, "\mu_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(\alpha^{(\mathcal{C})})"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{1} \otimes FX \arrow[r, "\epsilon \otimes FX"] \arrow[d, "\lambda^{(\mathcal{D})}_{FX}"'] & F(\mathbf{1}) \otimes FX \arrow[d, "\mu_{1,X}"] \\
FX & F(\mathbf{1} \otimes X) \arrow[l, "F(\lambda^{(\mathcal{C})}_X)"]
\end{tikzcd}
\quad
\begin{tikzcd}
FX \otimes \mathbf{1} \arrow[r, "FX \otimes \epsilon"] \arrow[d, "\rho^{(\mathcal{D})}_{FX}"'] & FX \otimes F(\mathbf{1}) \arrow[d, "\mu_{X,1}"] \\
FX & F(X \otimes \mathbf{1}) \arrow[l, "F(\rho^{(\mathcal{C})}_X)"]
\end{tikzcd}$$

Examples of monoidal functors

- forgetful functors like $\mathit{Top} \to \mathit{Set}$ or $\mathit{Rep}(G, \mathbb{k}) \to \mathit{Vec}_{\mathbb{k}}$
- $G_1 \subset G_2$ groups: $\mathit{Rep}(G_2, \mathbb{k}) \to \mathit{Rep}(G_1, \mathbb{k})$ *restriction functor*
- more generally: $\phi\colon G_1 \to G_2$, a group homomorphism, induces a monoidal functor $\mathit{Rep}(G_2, \mathbb{k}) \to \mathit{Rep}(G_1, \mathbb{k})$ (*pullback functor*)

A monoidal functor which is an equivalence is called *monoidal equivalence*.

Strict monoidal categories

A monoidal category is *strict* if its associators and unitors are identities.

Examples.

- $\overline{FSet}$:= full subcategory of $FSet$ with objects $(X_m := \{0, \dots, m-1\})_{m \geq 0}$, $X_m \otimes X_n := X_{mn}$ and

$$(f_1 \otimes f_2)(m_2 x_1 + x_2) = n_2 f_1(x_1) + f_2(x_2) \quad \forall x_i \in X_{m_i}$$

  for $f_i \colon X_{m_i} \to X_{n_i}$.
- $\overline{FVec_{\Bbbk}}$:= full subcategory of $FVec_{\Bbbk}$ with objects $(X_m := \Bbbk^m)_{m \geq 0}$, $X_m \otimes X_n := X_{mn}$ and $f_1 \otimes f_2$:= the Kronecker product.
- $End(\mathcal{C})$, the category of endofunctors of a category $\mathcal{C}$

Mac Lane's strictness theorem

Every monoidal category $\mathcal{C}$ is equivalent to a strict one.

So often, we may assume a given monoidal category is strict.
Examples. $\mathit{FSet} \simeq \overline{\mathit{FSet}}$, $\mathit{FVec}_{\Bbbk} \simeq \overline{\mathit{FVec}_{\Bbbk}}$.

Proof idea: Show that $\mathcal{C}$ is equivalent to a category of endofunctors (see [EGNO]).

- Every monoidal category is monoidally equivalent to a strict one.
- Every monoidal category is monoidally equivalent to a skeletal one.
- *Set*, *Vec* are monoidally equivalent to skeletal strict ones.
- ⚠️ Not every monoidal category is monoidally equivalent to a skeletal strict one.

Rigid categories

$\mathcal{C}$ monoidal, $X, Y \in \mathcal{C}$
Y is a *left dual* of X if there are morphisms $ev\colon Y \otimes X \to \mathbf{1}$, $coev\colon \mathbf{1} \to X \otimes Y$ satisfying

$$\left(X \xrightarrow{coev \otimes X} X \otimes Y \otimes X \xrightarrow{X \otimes ev} X\right) = id_X,$$

$$\left(Y \xrightarrow{Y \otimes coev} Y \otimes X \otimes Y \xrightarrow{ev \otimes Y} X\right) = id_Y,$$

where we use associators and unitors where necessary.
Similarly, Y is a *right dual* of X if there are morphisms $ev' : X \otimes Y \to \mathbf{1}$, $coev' : \mathbf{1} \to Y \otimes X$ satisfying

$$(ev' \otimes X)(X \otimes coev') = id_X, \quad (Y \otimes ev')(coev' \otimes Y) = id_Y.$$

$\mathcal{C}$ is called *rigid* (or: *rigid monoidal*) if all objects have left and right duals.

Examples of duals and rigid categories

- $FVec_{\Bbbk}$ is rigid with $X^* := \{f\colon X \to \Bbbk \text{ linear}\}$, $ev_X(f,x) := f(x)$, and $coev_X(1) := \sum_i f_i \otimes x_i$ for a pair of dual bases $(x_i)_i \in X$, $(f_i)_i \in X^*$.

> In this case, the right dual can be choosen to be isomorphic to the left dual, with analogous evaluation and coevaluation maps.

- $FRep(G,\Bbbk)$ is rigid with X^* as in $FVec_{\Bbbk}$ and $\phi_{X^*}(g)(f) = f \circ \phi_X(g^{-1})$. Evaluation and coevaluation maps are as in $FVec_{\Bbbk}$.
  ✓ Verify this!

> The coevaluation maps need finite-dimensional vector spaces!

- $End(\mathcal{C})$, $\mathcal{C}$ any category: left/right duals are left/right adjoints.

Properties of rigid categories

- If they exists, left or right duals are unique up to unique isomorphism. We denote them X^* and ${}^*X$, respectively.

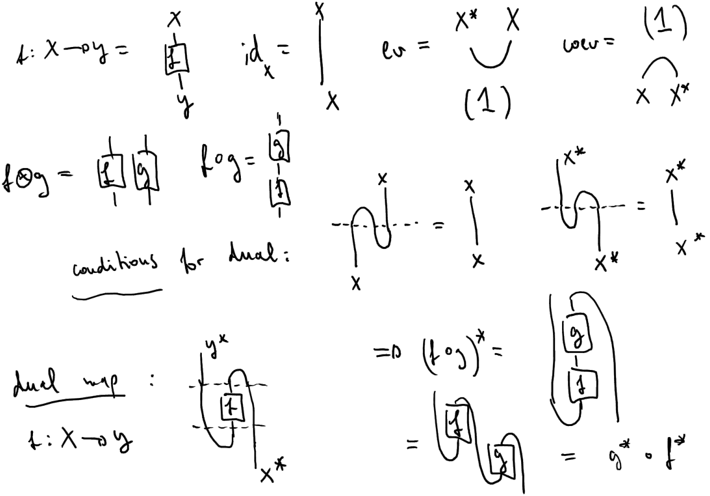
> Generally, $X^* \ncong {}^*X$!

- $(X \otimes Y)^* \cong X^* \otimes Y^*$ and ${}^*(X \otimes Y) \cong {}^*X \otimes {}^*Y$.
- We can *dualize morphisms*: for $f\colon X \to Y$, define

$$f^* := \left(Y^* \xrightarrow{Y^* \otimes \mathrm{coev}_X} Y^* \otimes X \otimes X^* \xrightarrow{Y^* \otimes f \otimes X^*} Y^* \otimes Y \otimes X^* \xrightarrow{\mathrm{ev}_Y \otimes X^*} X^*\right),$$

  and similarly for right duals.
- Then $(f \circ g)^* = g^* \circ f^*$ and ${}^*(f \circ g) = {}^*g \circ {}^*f$.

Graphical calculus for (rigid) monoidal categories



Tensor-hom adjunction revisited

$\mathcal{C}$ rigid monoidal.

The following are adjoint pairs of functors:

- $(-\otimes X, -\otimes X^*)$
- $(X^* \otimes -, X \otimes -)$
- $(-\otimes {}^*X, -\otimes X)$
- $(X \otimes -, {}^*X \otimes -)$

(The above functors can be viewed as partial specializations of the bifunctor $\otimes$.)

Example. For $\mathcal{C} = FVec_{\Bbbk}$, $Y \otimes X^* \cong \mathrm{Hom}_{\Bbbk}(X, Y)$, so we recover the tensor-hom adjunction $(-\otimes X, \mathrm{Hom}_{\Bbbk}(X, -))$.

$Hom_{\mathcal{C}}(X, -) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, -\otimes X^*)$ and similarly

Example. G a group. The G-module maps of the form $X \to Y$ are in correspondence with the G-invariants in $Y \otimes X^*$.

Additive monoidal categories

An *additive monoidal* category is a category that is additive and monoidal such that

$$(X \oplus Y) \otimes Z \cong (X \otimes Z) \oplus (Y \otimes Z) \quad \text{and} \quad X \otimes (Y \oplus Z) \cong (X \otimes Y) \oplus (X \otimes Z).$$

Examples.

- $\mathit{Vec}_{\Bbbk}$
- $\mathit{Rep}(G, \Bbbk)$

Grothendieck rings

Assume $\mathcal{C}$ is additive monoidal. Then the abelian Grothendieck group $K^{\oplus}(\mathcal{C})$ has a ring structure if we define the multiplication

$$[X] \cdot [Y] := [X \otimes Y] = \sum_i m_i [X_i]$$

if $X \otimes Y \cong \bigoplus X_i^{\oplus m_i}$.
We call this the *additive Grothendieck ring*.

$[\mathbb{1}]$ is the 1-element in this ring.

$\mathbb{1}$ might not be indecomposable in general. Also, the Grothendieck ring might not be commutative.

Example of a Grothendieck ring

$m \geq 1$, $G = C_m$, $\Bbbk = \overline{\Bbbk}$ of characteristic 0
Then $\mathcal{C} = Rep(G, \Bbbk)$ is semisimple (Maschke) with m simple objects
$X_0 = \Bbbk = [\mathbf{1}], X_1, \ldots, X_{m-1}$.

m=1. $\mathcal{C} = Vec_{\Bbbk}$, $K^{\oplus}(\mathcal{C}) \cong \mathbb{Z}$.

m=2. X_1 is a 1d-vector space, τ acts as -1. $X_0 \otimes X_i \cong X_i \cong X_i \otimes X_0$ and $X_1 \otimes X_1 \cong X_0$ So $K^{\oplus}(\mathcal{C}) \cong \mathbb{Z}[x_1]/(x_1^2 - 1)$.

🤔 What happens for $m > 2$?

Problem 2

- Verify briefly that $Rep(G, \mathbb{k})$ is a monoidal category.
- Assume $\mathbb{k} = \overline{\mathbb{k}}$ is of characteristic 0 and $G = C_3$. Decompose all tensor products of indecomposable objects in $Rep(G, \mathbb{k})$ as direct sums of indecomposable objects.

We observe that in $\mathit{Rep}(G, \Bbbk)$, $X \otimes Y \cong Y \otimes X$. This is also true in $\mathit{Vec}_{\Bbbk}$, Set, $\mathit{Top}, \ldots$, and in fact, in almost all examples we have seen so far. However, this does not follow from the definition of monoidal categories.

Up next: Braided categories.

6. 🪐 Braided categories

- [EGNO 8.1 – 8.2, 7.13, 8.5, 8.10]

Definition of braided monoidal categories

$\mathcal{C}$ monoidal. A *braiding* (or *commutativity constraint*) on $\mathcal{C}$ is a natural isomorphism $c\colon X \otimes Y \to Y \otimes X$ such that the following "hexagons" commute:

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "c_{X,Y\otimes Z}"] \arrow[dr, "c_{X,Y} \otimes Z"'] & & Y \otimes Z \otimes X \\ & Y \otimes X \otimes Z \arrow[ur, "Y \otimes c_{X,Z}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "c_{X\otimes Y,Z}"] \arrow[dr, "X \otimes c_{Y,Z}"'] & & Z \otimes X \otimes Y \\ & X \otimes Z \otimes Y \arrow[ur, "c_{X,Z} \otimes Y"'] & \end{tikzcd}$$

🤔 Why hexagons?

A *braided* (monoidal) category is a monoidal category with a braiding.

A braiding c is called *symmetric* if $c^2 = \mathrm{id}$.

First examples of braided monoidal categories

- direct products in additive categories $c := i_X p_X + i_Y p_Y \colon X \oplus Y \to Y \oplus X$
- *Set*, *Top*
  - with disjoint union: $c = \mathrm{id} \colon X \sqcup Y \to Y \sqcup X$
  - with cartesian product: $c \colon X \times Y \ni (x, y) \mapsto (y, x) \in Y \times X$
- *Vec*, $\mathit{Rep}(G, \Bbbk)$: $c \colon X \otimes Y \ni x \otimes y \mapsto y \otimes x \in Y \otimes X$

These are all symmetric braided categories. ✓

Graded vector spaces

G a finite group, $\Bbbk$ a field.
The *category of G-graded vector spaces* $Vec_G = Vec_{G,\Bbbk}$ is the monoidal category with

- objects: vector spaces V together with a collection of subspaces $(V_g)_{g\in G}$ such that $V = \bigoplus_{g\in G} V_g$
- $Vec_{G,\Bbbk}(V, V') := \{f \in Vec_\Bbbk(V, V') : f(V_g) \subset V'_g \quad \forall g \in G\}$
- tensor products as in $Vec_\Bbbk$ with $(V \otimes V')_g := \bigoplus_{g_1 g_2 = g} V_{g_1} \otimes V'_{g_2}$

Elements in the subspaces V_g are called *homogeneous*. For a homogeneous element $v \in V_g$, the *degree* $|v|$ is g.

Supervector spaces

C_2 the group with two elements, $\Bbbk$ a field. The *category of supervector spaces* $SVec_{\Bbbk}$ is the braided category which is $Vec_{C_2,\Bbbk}$ as a monoidal category with the braiding

$$c((v_0 + v_1) \otimes (v_0' + v_1')) = v_0' \otimes v_1' + v_0' \otimes v_1 + v_1' \otimes v_0 - v_1' \otimes v_1 .$$

Equivalently, the braiding can be defined as

$$c(v \otimes v') = (-1)^{|v||v'|} v' \otimes v$$

for **homogeneous** elements v, v'.

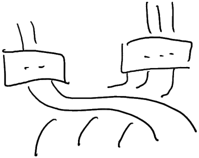
> Yet another symmetric braided category!

Category of braids

$\mathcal{B}$:= category with objects $\{n\}_{n\geq 0}$ and $\mathcal{B}(m,n)$:={braids with m upper and n lower points}/isotypy.

a braid on 6 strands

the braiding in $\mathcal{B}r$



 $\neq$ 

... is not symmetric!

$\mathcal{B}$ is braided but not symmetric. ✓

Yang–Baxter equation

In any strict monoidal category with a braiding c,

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$

on any tensor product $X \otimes Y \otimes Z$.

Braid group & symmetric group actions

$n \geq 0$, $B_n := \langle \sigma_1, \ldots, \sigma_n : \sigma_i\sigma_j = \sigma_j\sigma_i \ \forall |i-j| > 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$

$\mathcal{C}$ braided, $V \in \mathcal{C} \Rightarrow \phi \colon B_n \mapsto Aut_{\mathcal{C}}(V^{\otimes n})$, $\sigma_i \mapsto V^{\otimes(i-1)} \otimes c_{V,V} \otimes V^{\otimes(n-2)}$, is a group homo. ✓

$\mathcal{C} = \mathcal{B} \Rightarrow \phi \colon B_n \to Aut_{\mathcal{B}}(n) = End_{\mathcal{B}}(n)$ is a group iso. ✓

$\mathcal{C}$ symmetric $\Rightarrow$ ϕ induces a group homo $S_n \cong B_n/\langle \sigma_i^2 = 1 \rangle \to Aut_{\mathcal{C}}(V^{\otimes n})$. ✓

Mac Lane's braided coherence theorem

$\mathcal{C}$ braided.

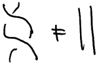
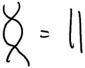
Consider objects $X_1, \dots, X_n \in \mathcal{C}$ and morphisms f, g

- between the tensor products $X_1 \otimes \cdots \otimes X_n$ with unit objects inserted and arbitrary parentheses
- obtained from associators, unitors, braidings, identity morphisms
- using tensor products and compositions.

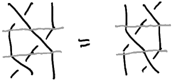
Then $f = g$ **if and only if** the braids defined by (the braidings in) f, g are isotypic.
(A coherence theorem on steroids!)

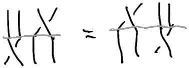
Graphical calculus for braided/symmetric monoidal categories

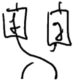
$c =$  or if c is symmetric: 

 but 

condition for braiding: 

Yang-Baxter equation/braid group relation: 

another braid group relation: 

$c \circ (f \otimes g) =$ 

...

Braided functors

A *braided functor* is a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}, \mathcal{D}$ are braided categories such that the following diagram commutes:

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "c_{FX,FY}"] \arrow[d, "\mu_{FX\otimes FY}"'] & FY \otimes FX \arrow[d, "\mu_{FY\otimes FX}"] \\
F(X \otimes Y) \arrow[r, "F(c_{X,Y})"'] & F(Y \otimes X)
\end{tikzcd}$$

where $\mu\colon F(X) \otimes F(Y) \to F(X \otimes Y)$ is the natural isomorphism belonging to F. A *braided equivalence* is a braided functor whose underlying functor is an equivalence.

Universal braided category

For any braided category $\mathcal{C}$ and $X \in \mathcal{C}$, there is a canonical braided functor $\mathcal{B} \to \mathcal{C}$ sending $1 \mapsto X$.
⚠️ $1 \ncong 0 \cong \mathbf{1}$ in $\mathcal{B}$

Proof idea:

- $n \mapsto X^{\otimes n}$
- any braid on n strands is sent to the corresponding endomorphism of $X^{\otimes n}$ which is a composition of braidings in $\mathcal{C}$

✓ Verify this is a braided functor.

Monoidal center

$\mathcal{C}$ monoidal. In the following we supress associators.
For $Z \in \mathcal{C}$, a *half-braiding* is a natural isomorphism $\gamma\colon -\otimes Z \to Z \otimes -$ such that $\gamma_{X\otimes Y} = (\gamma_X \otimes Y)(X \otimes \gamma_Y)$ for all $X, Y \in \mathcal{C}$.
The *monoidal center* (*Drinfeld center*) of $\mathcal{C}$ is the braided category $\mathcal{Z}(\mathcal{C})$ with

- objects (Z,γ), where $Z \in \mathcal{C}$ and γ is a half-braiding
- $\mathrm{Hom}((Z,\gamma),(Z',\gamma')) = \{f \in \mathcal{C}(Z,Z') : (f \otimes X)\gamma = \gamma'(X \otimes f)\ \forall X \in \mathcal{C}\}$ ("morphisms which commute with the half-braidings")
- tensor products:
  $$(Z,\gamma)\otimes(Z',\gamma') = (Z\otimes Z', X\otimes Z\otimes Z' \xrightarrow{\gamma\otimes Z'} Z\otimes X\otimes Z' \xrightarrow{Z\otimes\gamma'} Z\otimes Z'\otimes X)$$
- tensor unit $\mathbf{1}_{\mathcal{Z}(\mathcal{C})} = (\mathbf{1}_{\mathcal{C}}, X\otimes \mathbf{1} \xrightarrow{\cong} X \xrightarrow{\cong} \mathbf{1}\otimes X)$
- braiding: $c_{(Z,\gamma),(Z',\gamma')} = \gamma'_Z$
- If $\mathcal{C}$ is rigid, then $\mathcal{Z}(\mathcal{C})$ is rigid.

Yetter–Drinfeld categories

G a finite group, $\mathbb{k}$ a field.
A *Yetter–Drinfeld module* is a G-graded $\mathbb{k}$-vector space $V = \bigoplus_{g\in G} V_g$ with a G-module structure $\phi\colon G \to \mathit{Aut}_{\mathbb{k}}(V)$ such that

$$\phi(g)(V_h) \subset V_{ghg^{-1}}.$$

Note that a Yetter–Drinfeld module is an object both in $\mathit{Rep}(G, \mathbb{k})$ and in $\mathit{Vec}_{G,\mathbb{k}}$.
The *Yetter–Drinfeld category* $\mathit{YD} = \mathit{YD}(G, \mathbb{k})$ is the braided category with

- objects: Yetter–Drinfeld modules
- $\mathit{YD}((V, \phi), (V', \phi')) = \mathit{Rep}(G, \mathbb{k})(((V, \phi), (V', \phi'))) \cap \mathit{Vec}_{G,\mathbb{k}}(V, V')$
- tensor product as in $\mathit{Rep}(G, \mathbb{k})$ and $\mathit{Vec}_{G,\mathbb{k}}$ ✓ Verify this is compatible with the above YD-condition!
- braiding $c_{V\otimes V'}(v \otimes v') := \phi(|v|)(v') \otimes v$ (so $c^{-1}_{V\otimes V'}(v \otimes v') = v' \otimes \phi(|v'|)^{-1}(v)$)

Yetter–Drinfeld categories as monoidal centers

The forgetful functors induce braided equivalences $YD(G,\Bbbk) \to \mathcal{Z}(Rep(G,\Bbbk))$ and $YD(G,\Bbbk) \to \mathcal{Z}(Vec_{G,\Bbbk})$, respectively.

Pf.:

- We can verify the above functors are well-defined with $(V,\phi) \mapsto ((V,\phi), \gamma_X := c^{-1}_{V,X})$ and $(V,\phi) \mapsto (V, \gamma_X := c_{X,V})$, resp.
- We can construct quasi-inverse functors:
- Consider $(Z,\gamma) \in \mathcal{Z}(Rep(G,\Bbbk))$. Let $\Bbbk G$ be the group algebra, as a G-module by left multiplication. Define $Z_g := \{z \in Z : \gamma_{\Bbbk G}(1 \otimes z) = z \otimes g^{-1}\}$ for all $g \in G$.
- Consider $(Z,\gamma) \in \mathcal{Z}(Vec_{G,\Bbbk})$. Let $\Bbbk G$ be the group algebra, graded by $(\Bbbk G)_g = \Bbbk g$. Define an algebra map $\epsilon \colon \Bbbk G \to \Bbbk, g \mapsto 1$. Define $\phi(g)(z) = (Z \otimes \epsilon)\gamma_{\Bbbk G}(g \otimes z)$ for all $g \in G, z \in Z$.

Ribbon categories

$\mathcal{C}$ braided category.
A *twist* θ is an automorphism of the identity functor on $\mathcal{C}$ such that for all $X, Y \in \mathcal{C}$,

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) c_{Y,X} c_{X,Y}.$$

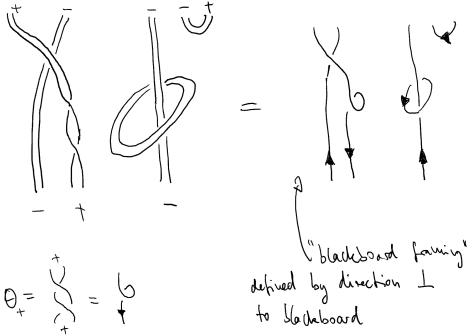
If $\mathcal{C}$ is rigid, a *ribbon structure* on $\mathcal{C}$ is a twist θ such that $\theta_X^* = \theta_{X^*}$ for all X.
A *ribbon category* is a rigid braided category with a ribbon structure.

Any rigid symmetric monoidal category is ribbon with $\theta_X = \mathrm{id}_X$.

For $\Bbbk$ of characterstic 0, $YD(G, \Bbbk)$ is ribbon, but not symmetric.

Category of framed tangles

Let *FT* be the category of *framed tangles*: objects are words in symbols $+$ and $-$ and morphisms are "tangles made of ribbons".



> *FT* is a ribbon category.

Universal ribbon category

For any ribbon category $\mathcal{C}$ and $X \in \mathcal{C}$, there is a canonical braided functor $FT \to \mathcal{C}$ sending $+ \mapsto X$ and preserving twists.

Proof idea:

- word in $+, -$ is sent to tensor product of X, X^*
- Any framed tangle can be written as a composition of braids, twists, "cup" and "cap" morphisms.
- Braids are sent to morphisms given by the braiding in $\mathcal{C}$,
- twists are sent to twists in $\mathcal{C}$,
- cups and caps are sent to evaluation and coevaluation morphisms.

If the twist is trivial on X, FT can be replaced by a category of tangles.

Knot invariants

Note: $\{$framed links$\} = \{$endomorphisms of $\mathbf{1} = 0$ in $FT\}$, where "link" means "knot with possibly several connected components" and "framed" means "made of ribbons".
Set $R := End_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}})$.

> Any ribbon category $\mathcal{C}$ with an object $X \in \mathcal{C}$ yields invariants in R for framed links via the canonical functor $FT \to \mathcal{C}$. These are called *Reshitikhin–Turaev invariants*. If the twist is trivial on X, this produces ordinary knot/link invariants.

Reference. Christian Kassel: Quantum groups, Thm. XIV.5.1.
(We'll discuss examples of this in the problem session.)

So many kinds of categories! Let's put it all together.
🍲

> Up next: **tensor categories**.

7. 🌍 Tensor categories

- [EGNO 4.1, 4.6 – 4.11, 4.13]

Definition of (pseudo-)tensor categories

$\Bbbk$ a field. A $\Bbbk$-*(pseudo-)tensor category* $\mathcal{C}$ is

- a $\Bbbk$-linear (pseudo-)abelian category and ($\rightarrow$ hom-spaces are vector spaces, direct sums $\oplus$, 0, images, kernels of all morphisms vs. just idempotent endomorphisms)
- a rigid monoidal category ($\rightarrow$ tensor product $\otimes$, $\mathbf{1}$, duals)

such that

- the dimensions of hom-spaces and the lengths of objects are finite
- *End*$(\mathbf{1}) \cong \Bbbk$
- the bifunctor $\otimes$ is bilinear on morphisms

- Krull–Schmidt and Jordan–Hölder holds in $\mathcal{C}$.
- The Grothendieck rings $K^{\oplus}(\mathcal{C})$ and $K(C)$ are $\mathbb{Z}$-spanned by the isomorphism classes of indecomposable/simple objects... ⚠ ...but there may be infinitely many!
- $\mathbf{1}$ is indecomposable & simple, corresponds to 1 in the Grothendieck rings.
- ⚠ $\mathcal{C}$ may or may not be semisimple.

If $\mathcal{C}$ is semisimple with finitely many simple (=indec.) objects, it is called *fusion*.

Examples

- $\mathit{FVec}_{\Bbbk} := \mathit{FVec}_{\Bbbk}$
- $\mathit{FRep}(G, \Bbbk) := \mathit{FRep}(G, \Bbbk)$ for G a group
- $\mathit{FVec}_{G,\Bbbk}$ for G a group

Convention: Let us redefine $\mathit{Vec} := \mathit{FVec}$ and $\mathit{Rep} := \mathit{FRep}$, and let us use $\mathbf{Vec}$ and $\mathbf{Rep}$ for the categories with infinite dimensional objects.

$\mathit{Vec}_{\Bbbk}$, $\mathit{Vec}_{G,\Bbbk}$ are fusion. $\mathit{Rep}(G, \Bbbk)$ is fusion if characteristic of $\Bbbk$ is 0 or not a divisor of $|G|$.

Semisimple tensor categories

$\mathcal{C}$ semisimple tensor category, $X, Y \in \mathcal{C}$ simple.

$\mathbf{1}$ is a direct summand in $X \otimes Y$ if and only if $X \cong {}^*Y$ if and only if $X \cong Y^*$. In this case, the multiplicity is 1. (**Cor.** $Y^* \cong {}^*Y$.)

- Let m be the multiplicity of $\mathbf{1}$ as a direct summand in $X \otimes Y$.
- Then $m = \dim \mathrm{Hom}(\mathbf{1}, X \otimes Y) = \dim \mathrm{Hom}({}^*Y, X)$ and $m = \dim \mathrm{Hom}(X \otimes Y, \mathbf{1}) = \dim \mathrm{Hom}(X, Y^*)$ (Week 4).
- Duals of simple objects are simple objects.
- So by Schur's lemma, the hom-spaces above are 1- or 0-dimensional, depending on whether or not the two objects are isomorphic or not (Week 5).

Grothendieck rings and categorification

For any semisimple tensor category (e.g. any fusion category), $K(\mathcal{C})$ is a *unital based ring*, i.e. a ring with a $\mathbb{Z}$-basis B and with an involution $*: B \to B$ such that

- $1 \in B$
- for all $b_1, b_2 \in B$, $b_1 b_2 = \sum_{b \in B} c_b b$ where $c_b \geq 0$ for all $b \in B$
- $b \mapsto b^*$ induces an anti-involution of $K(\mathcal{C})$
- the coefficient c_1 of $1 \in B$ in $b_1 b_2$ is 1 if $b_1^* = b_2$, and 0 otherwise

Question (*Categorification*). Given any unital based ring R, can we find a semisimple tensor category whose Grothendieck ring is R?

Example. $K(Vec_{G,\Bbbk}) \cong \mathbb{Z}G$, the group ring.

Pivotal categories

A *pivotal category* is a rigid monoidal category $\mathcal{C}$ with a natural isomorphism $a\colon X \to X^{**}$ satisfying

$$a_{X\otimes Y} = a_X \otimes a_Y \quad \forall X, Y \in \mathcal{C}.$$

a is called *pivotal structure*.

Examples.

- $Vec_{\Bbbk}$, $Vec_{G,\Bbbk}$, $Rep(G, \Bbbk)$ are pivotal with $V \to V^{**}, (v \mapsto (f \mapsto f(v)))$. ✔ Verify this. ⚠ Note (again) that for rigidity, we need finite-dimensional spaces.

Duals in pivotal categories

> Recall (Week 5) that the characterization of a
>
> - left dual X^* is via morphisms $ev_X \colon X^* \otimes X \to \mathbf{1}$, $coev_X \colon \mathbf{1} \to X \otimes X^*$
> - right dual ${}^*X$ is via morphisms $ev'_X \colon X \otimes {}^*X \to \mathbf{1}$, $coev'_X \colon \mathbf{1} \to {}^*X \otimes X$.

$\mathcal{C}$ pivotal with pivotal structure a, $X \in \mathcal{C}$.

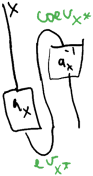
Define $ev' \colon = X \otimes X^* \xrightarrow{a_X \otimes X^*} X^{**} \otimes X^* \xrightarrow{ev_{X^*}} \mathbf{1}$ and

$coev' \colon = \mathbf{1} \xrightarrow{coev_{X^*}} X^* \otimes X^{**} \xrightarrow{X^* \otimes a_X^{-1}} X^* \otimes X$

> X^* is not just a left dual, but also a right dual of X (with ev', $coev'$)!

(Proof)

$(ev' \otimes X)(X \otimes coev')$

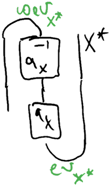
$=$ 

$=$ 

$=$ 

$= \mathrm{id}_X$

$(X^* \otimes ev')(wev' \otimes X^*)$

$=$ 

$=$ 

$= \mathrm{id}_{X^*}$

Traces and dimensions in pivotal categories

$\mathcal{C}$ a pivotal category with pivotal structure a.
For any $X \in \mathcal{C}, f \in End(X)$ we define

$$\mathrm{tr}^L(f) := ev'_X(f \otimes X^*)coev_X, \quad \mathrm{tr}^R(f) := ev_X(X^* \otimes f)coev'_X,$$
$$\dim^L(X) := \mathrm{tr}^L(\mathrm{id}_X), \quad \dim^R(X) := \mathrm{tr}^R(\mathrm{id}_X),$$

all as elements in $End(\mathbf{1}_{\mathcal{C}})$.
Both behave like like we know it from Vec:

- $\mathrm{tr}^*(f \oplus g) = \mathrm{tr}^*(f) + \mathrm{tr}^*(g)$ and $\dim^*(X \oplus Y) = \dim^*(X) + \dim^*(Y)$ if $\mathcal{C}$ is additive,
- $\mathrm{tr}^*(f) = \mathrm{tr}^*(f|_Y) + \mathrm{tr}^*(f_{X/Y})$ and $\dim^*(X) = \dim^*(Y) + \dim^*(X/Y)$ for any $Y \subset X$ if $\mathcal{C}$ is abelian, $f(Y) \subset Y$,
- $\mathrm{tr}^*(f \otimes g) = \mathrm{tr}^*(f)\,\mathrm{tr}^*(g)$ $\dim(X \otimes Y) = \dim(X) \otimes \dim(Y)$ ✓ Check these.

Spherical categories

A pivotal category $\mathcal{C}$ is called *spherical* if $\mathrm{tr}^L(f) = \mathrm{tr}^R(f)$ for all endomorphisms f. In this case, we just write tr and $\dim$.

Example. $Vec_{\Bbbk}$, $Vec_{G,\Bbbk}$, $Rep(G, \Bbbk)$.

Let $(v_i)_i$, $(f_i)_i$ be a pair of dual bases for V. Then for any $f \in End(V)$:
$\mathrm{tr}^L(f) = ev'(f \otimes V^*)coev = ev'(\sum_i f(v_i) \otimes f_i) = \sum_i f_i(f(v_i)) = \mathrm{tr}^R(f)$.

In any spherical category $\mathcal{C}$, $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ for all $X, Y \in \mathcal{C}$, $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, X)$. ✔ Verify this.

Semisimple spherical categories

$\mathcal{C}$ a semisimple spherical tensor category.

The dimension of any simple object is non-zero.

To argument to see that any simple object X has non-zero dimension is explained in [EGNO, Pf. of Prop. 4.8.4]:

- Assume the contrary, so $\mathbf{1} \xrightarrow{coev} X \otimes X^* \xrightarrow{ev'} \mathbf{1}$ is zero.
- As $\mathbf{1}$ is simple, the first map is a mono and the second is an epi.
- Hence, can view $\mathbf{1}$ as a subobject of $X \otimes X^*$ and "mod it out".
- The induced map $\overline{ev'} \colon X \otimes X^*/\mathbf{1} \to \mathbf{1}$ is still an epi, so non-zero.
- As everything is semisimple, this means $\mathbf{1}$ is a direct summand in $X \otimes X^*/\mathbf{1}$, so it appears at least twice as a direct summand in $X \otimes X^*$.
- This contradicts what we saw earlier in these slides!

Negligible morphisms

$\mathcal{C}$ a spherical $\Bbbk$-(pseudo-)tensor category.
We define $N_{\mathcal{C}}(X, Y) := \{f \in \mathcal{C}(X, Y) : \mathrm{tr}(fg) = 0 = \mathrm{tr}(gf)\ \forall g \in \mathcal{C}(Y, X)\}$ the *negligible morphisms* in $\mathcal{C}$.

> $N_{\mathcal{C}}$ is a *tensor ideal* in $\mathcal{C}$, i.e., $\alpha f + \beta f'$, fg, gf, $f \otimes g$, $g \otimes f$ are in $N_{\mathcal{C}}$ for all $\alpha, \beta \in \Bbbk$, $f, f' \in N_{\mathcal{C}}$, and all morphisms g in $\mathcal{C}$. ✔

Negligible morphisms in semisimple tensor categories

$\mathcal{C}$ a semisimple spherical $\Bbbk$-tensor category.

$N_{\mathcal{C}}(X, Y) = \{0\}$ for all $X, Y \in \mathcal{C}$.

- Consider $0 \neq f \in N_{\mathcal{C}}(X, Y)$. Write $X \cong \bigoplus X_i$ and $Y \cong \bigoplus Y_j$ for simple(=indecomposable) X_i and Y_j, and $f = (f_{ij})_{i,j}$.
- By Schur's lemma (Week 4), every f_{ij} is zero or an iso. $f \neq 0$ means $\exists i, j$ such that $f_{ij} \colon X_i \to Y_j$ is an iso.
- Let $\iota_i \colon X_i \to X$ and $\pi_j \colon Y \to Y_j$ be the obvious structure morphisms of the directs sums. Then, $\mathrm{tr}(\iota_i f_{ij}^{-1} \pi_j f) = \mathrm{tr}(f_{ij}^{-1} \pi_j f \iota_i) = \mathrm{tr}(\mathrm{id}_{X_i}) = \dim(X_i) \neq 0.$

Semisimplification

$\mathcal{C}$ a spherical $\mathbb{k}$-tensor category.
$\overline{\mathcal{C}}$:= the category with same objects as $\mathcal{C}$, but with morphism spaces $\overline{\mathcal{C}}(X, Y) := \mathcal{C}(X, Y)/N_{\mathcal{C}}(X, Y)$.
$\overline{\mathcal{C}}$ is called *semisimplification* of $\mathcal{C}$.

⚠️ This means, if $\mathrm{id}_X \in N_{\mathcal{C}}$, then $X \cong 0$ in $\overline{\mathcal{C}}$!

> $\overline{\mathcal{C}}$ is a semisimple $\mathbb{k}$-tensor category. Its simple objects correspond to those indecomposable(!) objects in $\mathcal{C}$ whose dimension is non-zero. If $\mathcal{C}$ is semisimple, then $\mathcal{C} \simeq \overline{\mathcal{C}}$.

Verlinde categories

Fix $\mathbb{k}$, an algebraically closed field of characteristic $p > 0$.
Set $\mathcal{C} := \mathit{Rep}(C_p, \mathbb{k})$, and $\mathit{Ver}_p := \overline{\mathcal{C}}$, the semisimplification.

> Ver_p is a symmetric fusion category. $\mathit{Ver}_p \not\simeq \mathit{Rep}(C_p, \mathbb{k})$.

🤔 **Problem.** Compute the simple objects in Ver_p, and their tensor product decompositions for $p = 2, 3, 5$.

Semisimplification of pseudo-tensor categories

$\mathcal{C}$ a spherical **pseudo**-tensor category. $\overline{\mathcal{C}}$ defined as in the abelian case.

If the trace of any nilpotent endomorphism in $\mathcal{C}$ is 0, then $\overline{\mathcal{C}}$ is a semisimple (and hence abelian) tensor category.

Etingof, Ostrik ('18'): On semisimplification of tensor categories.

In a tensor category, the trace of any nilpotent endomorphism is 0.

- Consider $f \in End(X)$ such that $f^n = 0$, i.e. $\mathrm{im}\, f \subset \ker f^{n-1}$.
- Then $\mathrm{tr}(f) = \mathrm{tr}(f|_{\ker f^{n-1}}) + \mathrm{tr}(f|_{X/\ker f^{n-1}}) = 0$, using an induction.

One of the most important results on tensor categories is *Deligne's theorem*: it relates a large class of tensor categories to a class of algebraic objects called supergroups. We will work our way towards this result.

Up next: **Hopf algebras & supergroups**.

8. 🔭 Hopf algebras & supergroups

- [EGNO 1.9, 5.2 – 5.3, 5.5 – 5.6, 5.9 – 5.10]

Algebras & Coalgebras

$\mathcal{C}$ a monoidal category.

An *algebra* in $\mathcal{C}$ is a triple $(A, \eta\colon \mathbf{1} \to A, \mu\colon A \otimes A \to A)$ in $\mathcal{C}$ satisfying: A *coalgebra* in $\mathcal{C}$ is a triple $(C, \varepsilon\colon C \to \mathbf{1}, \Delta\colon C \to C \otimes C)$ in $\mathcal{C}$ satisfying:

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes A"] \arrow[d, "A \otimes \mu"'] & A \otimes A \arrow[d, "\mu"'] & A \otimes \mathbf{1} \arrow[l, "A \otimes \eta"'] \arrow[dl, "\cong"] \\
A \otimes A \arrow[r, "\mu"] & A & \\
\mathbf{1} \otimes A \arrow[u, "\eta \otimes A"] \arrow[ur, "\cong"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta \otimes C"'] \arrow[r, "C \otimes \varepsilon"] & C \otimes \mathbf{1} \\
C \otimes C \arrow[u, "C \otimes \Delta"] \arrow[d, "\varepsilon \otimes C"'] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \arrow[ur, "\cong"'] \arrow[dl, "\cong"] & \\
\mathbf{1} \otimes C & &
\end{tikzcd}
$$

η/ε: *(co)unit*, μ/Δ: *(co)multiplication*, triangles: *(co)unitality*, squares: *(co)associativity*

1^{st} examples

- $\mathcal{C}$ any monoidal category. $\mathbf{1} \in \mathcal{C}$ is an algebra and a coalgebra where all structural morphisms are identities, up to unitors
- algebras in $\mathbf{Vec}$ are $\mathbb{k}$-algebras
- algebras in *Ab*, the category of abelian groups, are rings

If (A, η, μ) is an algebra in *Vec*, then (A^*, η^*, μ^*) is a coalgebra in *Vec*, and vice versa.

Bialgebras

$\mathcal{C}$ braided monoidal.
(A, η, μ), (A', η', μ') algebras in $\mathcal{C}$

$(A \otimes A', \eta \otimes \eta', (\mu \otimes \mu')(A \otimes c_{A',A} \otimes A'))$ is an algebra in $\mathcal{C}$.

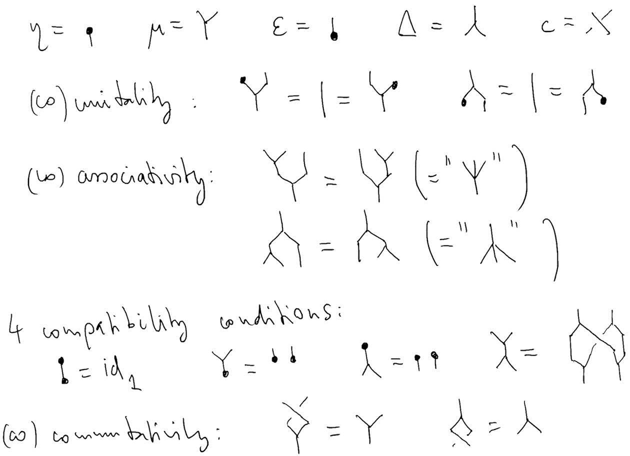
A *bialgebra* in $\mathcal{C}$ is a tuple $(H, \eta, \mu, \varepsilon, \Delta)$ such that (H, η, μ) is an algebra, (H, ε, Δ) is a coalgebra, and ε, Δ are "algebra maps", i.e.:

- $\varepsilon\eta = \eta_1 = \mathrm{id}_1, \quad \varepsilon\mu = \mu_1(\varepsilon \otimes \varepsilon) = \varepsilon \otimes \varepsilon$
- $\Delta\eta = \eta_{H\otimes H} = \eta \otimes \eta, \quad \Delta\mu = \mu_{H\otimes H}(\Delta \otimes \Delta) = (\mu \otimes \mu)(H \otimes c_{H,H} \otimes H)(\Delta \otimes \Delta)$

Equivalently, the these conditions mean that η, μ are "coalgebra maps", with a suitable coalgebra structure on $H \otimes H$.

An (co)algebra in $\mathcal{C}$ is called *(co)commutative* if $\mu = \mu c$ or $\Delta = c\Delta$, respectively.
Example. $1 \in \mathcal{C}$ is a commutative & cocommutative bialgebra.

A graphical calculus for bialgebras



Hopf algebras

$\mathcal{C}$ monoidal, A an algebra in $\mathcal{C}$, C a coalgebra in $\mathcal{C}$.

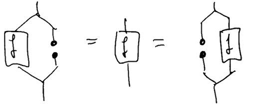
$\mathrm{Hom}(C, A)$ is a monoid with identity element $\eta\varepsilon$ with the operation (*convolution*) $f * g := \mu_A(f \otimes g)\Delta_C$.

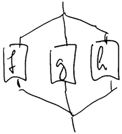
- associativity follows form (co)associativity of μ_A, Δ_C
- $\eta\varepsilon$ being the identity element follows from (co)unitality

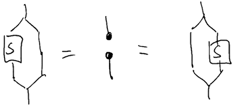
$\mathcal{C}$ braided. A *Hopf algebra* in $\mathcal{C}$ is a bialgebra with a convolution-inverse S (*antipode*) of id_H in $End(H)$ such that S is an isomorphism. (⚠️ For many folks, Hopf algebras can have non-invertible antipodes, as well.)
Example. $1 \in \mathcal{C}$ is a Hopf algebra.

If a bialgebra admits an antipode, then it is unique. S is an anti-algebra map, i.e. $S\mu = \mu c(S \otimes S)$.

Graphical calculus for Hopf algebras

$f * (\eta\varepsilon) =$  $= (\eta\varepsilon) * f$

$(f * g) * h =$  $= f * (g * h)$

S an antipode $\Longleftrightarrow$ 

<u>Problem</u>: Show that S is an anti-alg. map!

Group algebras

$\Bbbk$ a field, $\mathcal{C} = \mathbf{Vec}_{\Bbbk}$, G a group. Note $\mathcal{C}$ is symmetric with braiding $v \otimes w \mapsto w \otimes v$.
Set $H := \Bbbk G$, the free vector space with basis indexed by G.
The structure maps η, μ are uniquely defined as $\Bbbk$-linear maps satisfying

$$\eta(1) = 1 = e \in \Bbbk G \quad \mu(g \otimes g') = gg' \in \Bbbk G \qquad \forall\, g, g' \in G.$$

The structure maps ε, Δ, S are uniquely defined as $\Bbbk$-(anti)algebra maps satisfying

$$\varepsilon(g) = 1 \quad \Delta(g) = g \otimes g \quad S(g) = g^{-1} \qquad \forall\, g \in G.$$

H is a cocommutative Hopf algebra with $S^2 = \mathrm{id}_H$. H is commutative if and only if G is abelian. ✓ Check this.

Universal enveloping algebras of Lie algebras

Again $\mathcal{C} = Vec_{\mathbb{k}}$. L a Lie algebra. $H = U(L) = T(L)/(xy - yx - [x, y] : x, y \in L)$.
The structure maps ε, Δ, S are uniquely defined by

$$\varepsilon(x) = 0 \quad \Delta(x) = 1 \otimes x + x \otimes 1 \quad S(x) = -x \qquad \forall\, x \in L.$$

H is a cocommutative Hopf algebra with $S^2 = \mathrm{id}_H$. H is commutative if and only if G is abelian. ✔ Check this.

Smash products

G a group, K a Hopf algebra in $\mathbf{Rep}(G,\mathbb{k})$. In particular, K is an algebra and a coalgebra in $\mathbf{Vec}_{\mathbb{k}}$.
Define $\tilde{c} := \mathbb{k}G \otimes K \to K \otimes \mathbb{k}G, g \otimes k \mapsto \phi_K(g)(k) \otimes g$ in $\mathbf{Vec}_{\mathbb{k}}$. (Remember Yetter–Drinfeld modules from Week 6?)
Define $H := K \rtimes \mathbb{k}G := K \otimes \mathbb{k}G$ as a $\mathbb{k}$-algebra by

$$\eta_H := \eta_K \otimes \eta_{\mathbb{k}G} \quad \mu_H := (\mu_K \otimes \mu_{\mathbb{k}G})\tilde{c}$$

The structure maps ε, Δ, S are uniquely defined by

$$\varepsilon(k) = \varepsilon_K(k) \quad \Delta(k) = \Delta_K(k) \quad S(k) = S_K(k)$$
$$\varepsilon(k') = \varepsilon_{\mathbb{k}G}(k') \quad \Delta(k') = \Delta_{\mathbb{k}G}(k') \quad S(k') = S_{\mathbb{k}G}(k')$$

using the embeddings $K \ni k \mapsto k \otimes 1 \in H$ and $\mathbb{k}G \ni k' \mapsto 1 \otimes k' \in H$.

$K \rtimes \mathbb{k}G$ is a Hopf algebra in $\mathbf{Vec}_{\mathbb{k}}$.

Cartier–Gabriel–Kostant theorem

$\Bbbk$ an algebraically closed field of characteristic 0.

Any cocommutative Hopf algebra H in $\mathbf{Vec}_{\Bbbk}$ is isomorphic to $U(L) \rtimes \Bbbk G$ for some group G, some Lie algebra L, and some action of G on L (which makes $U(L)$ a Hopf algebra in $\mathbf{Rep}(G, \Bbbk)$).

More precisely:

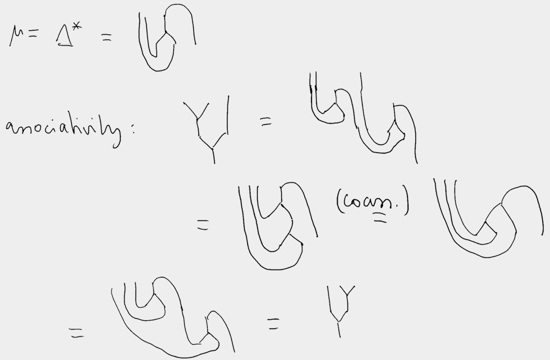
- Define $G := \{h \in H : \varepsilon(h) = 1, \Delta(h) = h \otimes h\}$, *grouplike* elements, $L := \{h \in H : \varepsilon(h) = 0, \Delta(h) = 1 \otimes h + h \otimes 1\}$, the *primitive* elements.
- Then G is a group with the multiplication in H, L is a Lie algebra with the commutator in H, and conjugation in H gives an action of G on L.
- This datum is the one in the theorem.

Dual of a Hopf algebra

Assume $(H, \eta, \mu, \varepsilon, \Delta, S)$ is a Hopf algebra in $Vec_{\Bbbk}$ (i.e. finite-dimensional).

> $(H^*, \varepsilon^*, \Delta^*, \eta^*, \mu^*)$ is a Hopf algebra in $Vec_{\Bbbk}$. It is commutative if and only if H is cocommutative, and vice versa. ($\Rightarrow$ Cartier–Gabriel—Kostant for duals of comm. Hopf algebras in $Vec_{\Bbbk}$!)

Proof can be done “graphically”, like this:



Taft Hopf algebras

$\mathbb{k}$ a field, $q \in \mathbb{k}$ a primitive n-th root of unity for $n \geq 2$.
$H := \langle g, x : g^n = 1, x^n = 0, gxg^{-1} = qx \rangle$ as a $\mathbb{k}$-algebra with structure ($\mathbb{k}$-algebra) map Δ defined by

$$\Delta(g) = g \otimes g \qquad \Delta(x) = 1 \otimes x + x \otimes g$$

✔ Verify that there is unique Hopf algebra structure on H with Δ as above. What is ε, S?

> H is a Hopf algebra (the *Taft Hopf algebra*) in $\mathbf{Vec}_{\mathbb{k}}$ or $\mathit{Vec}_{\mathbb{k}}$ which is not commutative and not cocommutative. The order of S (as an automorphism of H) is $2n$.

✔ Check this.

The quantum group $U_q(\mathfrak{sl}_2)$

Is a Hopf algebra ("quantum") deformation of $U(\mathfrak{sl}_2)$, the universal enveloping of the Lie algebra of 2×2-matrices with trace 0.

$q \in \mathbb{k}^\times \setminus \{\pm 1\}$

$$H := \langle E, F, K, K^{-1} : KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \rangle$$

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F$$

$$\varepsilon(K) = 1, \quad \varepsilon(E) = 0 = \varepsilon(F), \quad S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.$$

H is an infinite-dimensional Hopf algebra in $\mathbf{Vec}_{\mathbb{k}}$, which is not commutative or cocommutative.

Interpreting $K^{\pm 1}$ as $q^{\pm H}$ and taking the limit $q \to 1$, we recover $U(\mathfrak{sl}_2) = \langle E, F, H : [H, E] = E, [H, F] = -F, [E, F] = H \rangle$, where $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Supergroups

> In algebraic geometry, we have an equivalence of categories $\{$comm. algebra in $\mathbf{Vec}_{\Bbbk}\}^{op} \simeq \{$affine scheme over $\Bbbk\}$, where $A \mapsto Spec(A)$, the set of prime ideals, with inverse functor $X \mapsto \mathcal{O}(X)$, the *coordinate ring/ring of regular functions*. This equivalence induces one between $\{$comm. Hopf algebras in $\mathbf{Vec}_{\Bbbk}\}^{op}$ and $\{$affine group schemes over $\Bbbk\}$

For any symmetric monoidal category $\mathcal{C}$, we define the category of *affine group schemes* in $\mathcal{C}$ as the opposite of the category of commutative Hopf algebras in $\mathcal{C}$.

For $\mathcal{C} = \mathbf{SVec}_{\Bbbk}$, the category of supervector spaces, this is called the category of *supergroups*.

$GL(m|n)$

$m, n \geq 0$
We define $GL(m|n) := \{A = (a_{ij})_{i,j} \in \mathbb{k}^{(m+n)\times(m+n)} : \det A \neq 0\}$, the *general linear supergroup* with coordinate ring the supercommutative algebra $H = \mathbb{k}[a_{ij}, b]/(\det((a_{ij})_{i,j})b - 1)$, where the generators a_{ij} for $i \leq m < j$ or $j \leq m < i$ are defined as odd.

> H is a commutative Hopf algebra in $\mathbf{SVec}_{\mathbb{k}}$ with
>
> $$\varepsilon(a_{ij}) = \delta_{ij} \quad \Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj} \quad \varepsilon(b) = 1 \quad \Delta(b) = b \otimes b.$$

Similarly: $SL(m|n)$.

🤔 **Problem.** Verify the above defines a bialgebra structure and compute the antipode in these Hopf algebras.

So, how are all these Hopf algebras related to tensor categories again?

Up next: **Tannaka–Krein reconstruction theory**.

9. 📡 Tannaka–Krein reconstruction

- [EGNO 1.10 – 1.11, 5.1 – 5.4]

> ⚠️ The following slides may contain depictions of linear algebra.

Representations of algebras

We fix a field $\Bbbk$. All algebras, Hopf algebras, ... will be in $\mathbf{Vec} = \mathbf{Vec}_{\Bbbk}$.
For any algebra A in $\mathbf{Vec}$, $Rep(A)$:= the category with objects
$\{(V, \phi) : V \in Vec, \phi \in Alg(A, End(V))\}$ and morphisms
$\{f \in Vec(V, V') : f\phi_V(h) = \phi_{V'}(h)f \; \forall h \in H\}$

$Rep(A)$ is a $\Bbbk$-linear abelian category such that the forgetful functor $F\colon Rep(A) \to Vec$ is faithful and preserves images, kernels, and short exact sequences.

The proof works just as for group representations, which we have seen in Week 4.

Representations of Hopf algebras

H a Hopf algebra (with bijective antipode).

Rep(H) is a $\Bbbk$-tensor category such that the forgetful functor $F\colon Rep(H) \to Vec$ is (strict) monoidal and preserves duals and co-/evaluation morphisms, with

- $\phi_1 = \phi_\Bbbk = \varepsilon \qquad \phi_{V\otimes W} = (\phi_V \otimes \phi_W)\Delta$
- $\phi_{V^*}(h)(f) = f\phi_V(S(h)) \qquad \phi_{{}^*V}(h)(f) = f\phi_V(S^{-1}(h))$

- Δ is an algebra map, S is an anti-algebra map $\Rightarrow$ ϕ's above are algebra maps
- NTS: usual co-/evaluation maps are morphisms in *Rep*(H)

- write $\Delta(h) = \sum_i h_{i,1} \otimes h_{i,2}$, then

$$\begin{aligned}
ev \circ \phi_{V^*\otimes V}(h)(f\otimes v) &= ev \circ ((\phi_{V^*} \otimes \phi_V) \circ \Delta(h))(f\otimes v) \\
= \sum_i ev(f\phi_V(S(h_{i,1})) \otimes \phi_V(h_{i,2})(v) &= \sum_i f(\phi_V(S(h_{i,1}))\phi_V(h_{i,2})v) \\
= f(\phi_V(\mu(S\otimes \mathrm{id}_H)\Delta(h))(v)) &= f(\phi_V(\eta\varepsilon(h))(v)) \\
&= \epsilon(h)f(v) = \phi_\Bbbk(h) \circ ev(f\otimes v)
\end{aligned}$$

✓ (similarly for the other checks)

Examples

G a group, $H = \Bbbk G$. Then $\Delta(g) = g \otimes g$ and $S(g) = g^{-1}$ imply that

- $\phi_{V \otimes W}(g) = \phi_V(g) \otimes \phi_W(g)$
- $\phi_{V^*}(g)(f) = f\phi_V(g^{-1})$.

L a Lie algebra, $H = U(L)$. Then $\Delta(x) = x \otimes 1 + x \otimes x$ and $S(x) = -x$ imply that

- $\phi_{V \otimes W}(x) = \phi_V(x) \otimes W + V \otimes \phi_W(x)$
- $\phi_{V^*}(x)(f) = -f\phi_V(x)$.

Generally $V^* \ncong {}^*V$ in $Rep(H)$, but if $S^2 = \mathrm{id}$, this is true.

Tannaka duality for abelian categories

Can we recover the algebra from its representation category?

$\mathcal{C}$ $\Bbbk$-linear abelian with an exact faithful functor $F\colon \mathcal{C} \to \mathit{Vec}$.

Recall from Week 2 that a natural transformation $F \Rightarrow F$ (*endomorphism of* F) is a collection of endomorphisms $\alpha_X \in \mathit{End}(F(X))$ such that for all $f\colon X \to Y$ in $\mathcal{C}$, $F(f)\alpha_X = \alpha_Y F(f)$.

Then $\mathit{End}(F)$, the endomorphisms of F, form a $\Bbbk$-algebra with $1 =$ the identity transformation and with composition as multiplication.

(EGNO, Thm. 1.10.1) The assignments $A \mapsto (\mathit{Rep}(A), \mathit{Forget})$ and $(\mathcal{C}, F) \mapsto \mathit{End}(F)$ define mutually quasi-inverse equivalences between

- the category of $\Bbbk$-algebras A and
- the category of $\Bbbk$-linear abelian categories $\mathcal{C}$ with an exact faithful functor $F\colon \mathcal{C} \to \mathit{Vec}_{\Bbbk}$.

Example

$\Bbbk = \overline{\Bbbk}$ in characterstic 0. $m \geq 1$, $G = C_m$, $\mathcal{C} = Rep(G, \Bbbk)$.

> We know $\mathcal{C}$ is semisimple with simple objects $X_0, \dots, X_{m-1}$, all are 1d-vector spaces. So the forgetful functor $F : \mathcal{C} \to Vec$ is representable, namely, $F \cong \mathrm{Hom}_{\mathcal{C}}(P, -)$ where $P := X_0 \oplus \cdots \oplus X_{m-1}$.

> By the Yoneda lemma (Week 2), $End(F) \cong End_{\mathcal{C}}(P)^{op}$. $A := End_{\mathcal{C}}(P)$ is $\Bbbk^m$ (with componentwise operations), as $(X_i)_i$ are non-isomorphic simple objects and $\mathcal{C}$ is semisimple. Hence $End(F) \cong A^{op} = (\Bbbk^m)^{op} = \Bbbk^m$.

> 🤔 Can you determine $End(F)$ directly, without using the Yoneda lemma?

Example (cont'd)

[...] Hence $End(F) \cong \Bbbk^m$.

We claim that this is isomorphic to $\Bbbk C_m$: send a generator $\tau \in C_m$ to $(1, \xi, \xi^2, \dots, \xi^{m-1}) \in \Bbbk^m$, then the powers $\tau^0, \tau^1, \dots, \tau^{m-1}$ are linearly independent, because we can compute the Vandermonde-determinant

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \xi & \dots & \xi^{m-1} \\ \vdots & & \ddots & \\ 1 & \xi^{m-1} & \dots & \xi^{(m-1)^2} \end{pmatrix} = \prod_{0 \leq i < j \leq m-1} (\xi^i - \xi^j) \neq 0$$

😃

Example (cont'ed)

Indeed, $End(F) \cong \mathbb{k}^m \cong \mathbb{k}C_m$ as $\mathbb{k}$-algebras.

The last step is kind of arbitrary here; as an algebra without extra structure, $End(F)$ is simply $\mathbb{k}^m$ and $\mathcal{C}$ is equivalent to its representation category as a $\mathbb{k}$-linear abelian category. Only taking the extra structure on $\mathcal{C}$ into account (tensor products, duals, ...), $End(F)$ becomes an algebra with an extra structure which clarifies it should be $\mathbb{k}C_m$.

Problem 1

Again, assume $\Bbbk = \overline{\Bbbk}$ is of characteristic 0. Let $\mathcal{C}$ be a $\Bbbk$-linear semisimple ($\Rightarrow$ abelian) category with a faithful exact functor $F\colon \mathcal{C} \to \mathit{Vec}_{\Bbbk}$. Assume $\mathcal{C}$ has exactly 3 (non-isomorphic) simple objects whose images under F have the dimensions $1, 1, 2$.

- Can you determine $\mathit{End}(F)$?
- Can you find a finite group G such that $\mathit{End}(F) \cong \Bbbk G$ as $\Bbbk$-algebras?
- How should P be defined for any $\Bbbk$-linear semisimple category?

Non-semisimple example

$\Bbbk = \overline{\Bbbk}$ of char. 2, $\mathcal{C} = Rep(C_2, \Bbbk)$ with F the forgetful functor.

We have indecomposables $\mathbf{1} \cong X_0$ and X_1, corresponding to the Jordan blocks J_1, J_2 with eigenvalue 1 of size 1 and 2. ⚠ Now there are morphisms between them!

We compute $\dim \mathrm{Hom}(X_0, X_0) = 1$, $\dim \mathrm{Hom}(X_1, X_0) = 1$, and

$$\begin{aligned}\dim \mathrm{Hom}(X_1, X_1) &= \dim\{A \in \Bbbk^{2\times 2} : AJ_2 = J_2 A\} \\ &= \dim\{f(J_2) : f \in \Bbbk[x]\} = \dim span\{J_2, J_2^2 = J_2^0\} = 2\end{aligned}$$

using **(linear algebra:)** If minimial polynomial=characteristic polynomial of B, then the matrices commuting with B are the polynomials in B.

Hence, $F \cong \mathrm{Hom}(P, -)$ for $P = X_1$. ⚠ This is very different from the semisimple case, where we took $P = X_0 \oplus \cdots \oplus X_{m-1}$.

Non-semisimple example (cont'd)

$F \cong \mathrm{Hom}(P, -)$ for $P = X_1$.

Now $End(F) \cong End_{\mathcal{C}}(X_1)^{op}$, but $End_{\mathcal{C}}(X_1)$ is generated by J_2, so this is $\Bbbk[x]/(x^2 - 1) \cong \Bbbk C_2$.

$End(F) \cong \Bbbk C_2$ as in the semisimple example, but the computation looks quite different.

Problem 2

Can you generalize the previous example to $Rep(C_p, \mathbb{k})$ for $\mathbb{k} = \overline{\mathbb{k}}$ of characteristic p for $p = 3$? Or $p \geq 3$?
Hint. Note that for $p = 2$,

$$\begin{aligned}\dim Hom(X_1, X_0) &= \dim\{A \in \mathbb{k}^{1\times 2} : J_1 A = A J_2\} \\ &= \dim\{A \in \mathbb{k}^{1\times 2} : J_2 \begin{pmatrix} A \\ 0 \end{pmatrix} = \begin{pmatrix} A \\ 0 \end{pmatrix} J_2\}\end{aligned}$$

and now we can use the linear algebra result above to finish the computation.

$End(F)$ for a tensor category

$\mathcal{C}$ a $\Bbbk$-tensor category.
A *fiber functor* is a faithful exact monoidal functor $F\colon \mathcal{C} \to Vec_{\Bbbk}$ with $F(\mathbf{1}) = \Bbbk$.
Assume F is a fiber functor for $\mathcal{C}$.

(EGNO 5.2.1, 5.3.1) The algebra $H := End(F)$ is, in fact, a **Hopf algebra**.

For simplicity, let's assume F is strictly monoidal. Then for any $\alpha \in End(F)$,

- $\varepsilon(\alpha) = \alpha_{\mathbf{1}} \in End_{Vec}(\Bbbk) \cong \Bbbk$
- $\Delta(\alpha)_{X,Y} = \alpha_{X\otimes Y} \in End(F(X \otimes Y)) = End(F(X) \otimes F(Y))$ (using a suitable identification)
- $S(\alpha)_X = \alpha^*_{X^*}$

Reconstruction for tensor categories

(EGNO, Thm. 5.2.3) The assignments $H \mapsto (\mathit{Rep}(H), \mathit{Forget})$ and $(\mathcal{C}, F) \mapsto \mathit{End}(F)$ define mutually quasi-inverse equivalences between

- the category of Hopf algebras H over $\Bbbk$ and
- the category of $\Bbbk$-tensor categories $\mathcal{C}$ with a fiber functor F.

Furthermore,

- symmetric tensor categories with braided fiber functors correspond to cocommutative Hopf algebras,
- finite tensor categories correspond to finite-dimensional Hopf algebras

Example (cont'd)

$\Bbbk = \overline{\Bbbk}$ of char. 0, $\mathcal{C} = Rep(C_m, \Bbbk)$ with F the forgetful functor.
Recap. $End(F) \cong \Bbbk^m$.

Let $\alpha^i \in End(F)$ be the endomorphism $\alpha^i_{X_j} = \xi^{ij} \in End(\Bbbk) = End(F(X_j))$ for $0 \leq i, j < m$. Then

- $\alpha^0 = 1$, the identity transformation
- $(\alpha^i \alpha^j)_{X_k} = \xi^{ik}\xi^{jk} = \xi^{(i+j)k} = \alpha^{i+j}_{X_k} \in End(\Bbbk) = End(F(X_k))$
- $\varepsilon(\alpha^i) = \alpha^i_{\mathbb{1}} = \alpha^i_{X_0} = 1$
- $\Delta(\alpha_i)_{X_k, X_\ell} = \alpha^i_{X_k \otimes_{\Bbbk} X_\ell} = \alpha^i_{X_{k+\ell}} = \xi^{i(k+\ell)} = \xi^{ik}\xi^{i\ell} = (\alpha^i \otimes \alpha^i)_{X_k, X_\ell} \in End(\Bbbk) = End(F(X_k \otimes X_\ell))$.
- $S(\alpha^i)_{X_k} = (\alpha^i_{X_k^*})^* = (\alpha^i_{X_{-k}})^* = \xi^{-ik} = \alpha^{-i}_{X_k}$

We have a Hopf algebra isomorphism $\Bbbk C_m \to End(F), \tau^i \mapsto \alpha^i$.

So tensor categories with fiber functors are “the same as” Hopf algebras. But when do we have a fiber functor?

> Up next: **Deligne’s theorem.**

10. 🛰️ Deligne's theorem

- [EGNO 9.9 – 9.11]
- `https://ncatlab.org/nlab/show/Deligne%27s+theorem+on+tensor+categories`

Symmetric fusion categories

$\Bbbk$ a field, G a finite group, $z \in G$ central such that $z^2 = 1$.
$Rep(G, z) := Rep(G, \Bbbk)$ as a rigid monoidal abelian category, with a (symmetric) braiding given by

$$c(v \otimes w) = (-1)^{mn} w \otimes v \quad \forall v, w : \phi(z)v = (-1)^m v, \phi(z)w = (-1)^n w$$

Equivalently,
$Rep(G, z) := \{(V, \phi) \in SVec(G, \Bbbk) : \phi(z)v_i = (-1)^i v_i \; \forall i \in \{0, 1\}, v_i \in V_i\}$.

$Rep(G, z)$ is a symmetric fusion category.

We'll show: if $\Bbbk = \overline{\Bbbk}$ is of characteristic 0, then these are all symmetric fusion categories.

Reconstruction for symmetric fusion categories

From now on, all (super) fiber functors are assumed braided!

$\mathcal{C}$ a symmetric fusion category with a **braided** fiber functor $F\colon \mathcal{C} \to \mathit{Vec}$.

- By reconstruction theory (Week 9), $\mathcal{C} \simeq \mathit{Rep}(H)$ for a cocommutative finite-dimensional Hopf algebra $H = \mathit{End}(F)$.
- By Cartier–Gabriel–Kostant (Week 8), $H \cong \Bbbk G$ for a finite group G.
- This implies $G = \mathit{Aut}(F)$.

$\mathcal{C} \simeq \mathit{Rep}(\mathit{Aut}(F))$.
Similarly, if $\mathcal{C}$ admits a super fiber functor, then it is equivalent to $\mathit{Rep}(\mathit{Aut}(F), \theta)$ for θ the *parity involution*, i.e. $\theta_V(v_i) = (-1)^i v_i$.

It remains to determine, if a symmetric fusion category admits a fiber functor.

Symmetric fusion categories are spherical

A *pivotal structure* is a natural isomorphism $\theta\colon X \to X^{**}$ such that $\theta_{X\otimes Y} = \theta_X \otimes \theta_Y$. It is *spherical* if left- and right-traces coincide.

$\mathcal{C}$ a symmetric fusion category with symmetric braiding c.

Define $u_X : X \xrightarrow{X\otimes coev_{X^*}} X \otimes X^* \otimes X^{**} \xrightarrow{c_{X,X^*}\otimes X^{**}} X^* \otimes X \otimes X^{**} \xrightarrow{ev_X \otimes X^{**}} X^{**}$.

(EGNO 8.10.12) u_X is a spherical pivotal structure on $\mathcal{C}$ (which corresponds to a ribbon structure with trivial twist).

$\Rightarrow$ (coinciding left-/right-) traces and dimensions

Dimensions in symmetric fusion categories

- $\mathcal{C} = SVec_{\Bbbk}$,
- $X = \Bbbk^{m|n}$, the supervector space with basis $e_1, \ldots, e_m$ (even), $e_{m+1}, \ldots, e_{m+n}$ (odd).
- pick a basis $(e_i^*)_i$ in X^* dual to $(e_i)_i$,
- pick a basis $(\delta_i)_i$ in X^{**} dual to $(e_i^*)_i$.

Then

$$\begin{aligned}\dim X = ev'_X coev_X(1) &= ev_{X^*}(u_X \otimes X^*) coev_X(1) \\ = (ev_X \otimes ev_{X^*})(c_{X,X^*} \otimes X^{**} \otimes X^*) &\sum_{i,j} e_i \otimes e_j^* \otimes \delta_j \otimes e_i^* \\ &= \sum_{i,j} (-1)^{|e_i||e_j|} \delta_{ij}\delta_{ij} = m - n\end{aligned}$$

🤔 Verify that in $Rep(G,z)$ as above, $\dim X = \operatorname{tr} z|_X$, which is compatible with the (forgetful) functor $Rep(G,z) \to SVec$.

Symmetric and exterior powers

Recall that we have a group homomorphism $S_n \to End(X)$ for any X in a symmetric monoidal category, where $(i, i+1) \mapsto X^{\otimes(i-1)} \otimes c_{X,X} \otimes X^{\otimes(n-i-1)}$.

$\mathcal{C}$ symmetric fusion category, $X \in \mathcal{C}$.
For $n \geq 0$, the *n-th symmetric/exterior power* of X is defined as the maximal quotient of $X^{\otimes n}$ on which each $\sigma \in S_n$ acts as $1^\sigma := 1$ or $(-1)^\sigma := \operatorname{sign}\sigma$.

In characteristic 0, these objects are given by the images of the (anti-)symmetrizers

$$1/n! \sum_{\sigma \in S_n} (\pm 1)^\sigma \sigma \in End(X^{\otimes n}).$$

Dimensions in symmetric fusion categories (cont'ed)

$\mathcal{C}$ symmetric fusion in char. 0, $X \in \mathcal{C}$, $\alpha := \dim X$.

For all $n \geq 0$, $\dim S^n X = \binom{\alpha+n-1}{n}$ and $\dim \Lambda^n X = \binom{\alpha}{n}$.

Problem 1. Prove this (EGNO, Exercise 9.9.9).

The above implies: $\dim X \in \mathbb{Z}$.

- $\dim$ is a character on the finite-rank Grothendieck ring of $\mathcal{C}$, so its values are algebraic integers (the elements corresponding to $(X^{\otimes n})_n$ can not be all $\mathbb{Z}$-linearly independent).
- **Nice lemma.** If $\binom{\alpha+n-1}{n}$ and $\binom{\alpha}{n}$ are algebraic integers for all $n \geq 0$, then $\alpha \in \mathbb{Z}$.

Nice lemma

A the *algebraic integers*. So $\mathbb{Z} \subset \begin{cases} \mathbb{Q} \\ A \end{cases} \subset \mathbb{C}$, $\mathbb{Q} \cap A = \mathbb{Z}$.

> [Andruskiewitsch–Etingof–Gelaki: Triangular Hopf Algebras with the Chevalley Property, Sec. 7] $\{\binom{\alpha+n-1}{n}, \binom{\alpha}{n}\}_{n\geq 0} \subset A \Rightarrow \alpha \in \mathbb{Z}$.

- the assumptions imply $\{\binom{\alpha'+n-1}{n}, \binom{\alpha'}{n}\}_{n\geq 0} \subset A$ for all algebraic conjugates α' of α
- this implies $\{\frac{N(\alpha)\dots N(\alpha+n-1)}{(n!)^d}, \frac{N(\alpha)\dots N(\alpha-n+1)}{(n!)^d}\} \in A \cap \mathbb{Q} = \mathbb{Z}$, where d is the degree of the Q, the monic minimal poly. of α over $\mathbb{Z}$ and N is the norm function (product over all conjugates).
- $N(\alpha - n) = \pm Q(n)$, so $\{b_n := \frac{Q(0)\dots Q(n-1)}{(n!)^d}, \frac{Q(0)\dots Q(-n+1)}{(n!)^d}\} \in \mathbb{Z}$
- write $Q(x) = x^d - ax^{d-1} + \dots$ with $a \geq 0$ (wlog, otherwise replace $\alpha \to -\alpha$). Then $Q(n-1) = n^d - (a+d)n^{d-1} + \dots < n^d$ for $n \gg 0$, so $|b_n| < |b_{n+1}|$ for all $n \gg 0$.
- As $b_n \in \mathbb{Z}$, this means $b_n = 0$ for some $n > 0$, so $Q(n) = 0$ for some $n \geq 0$, so $\alpha = n \in \mathbb{Z}$.

Positive fusion categories

In the following, we will be in characteristic 0.

A symmetric fusion category $\mathcal{C}$ is called *positive* if $\dim X \geq 0$ for all $X \in \mathcal{C}$.

$\dim X = 0$ implies $X = 0$ for any object X in such a category.

X is a direct sum of simple objects, its dim. is the sum of their dim's. The dimension of a simple object is $\neq 0$ (Week 7), hence > 0.

Every **symmetric** fusion category is equivalent to a **positive** fusion category, possibly up to a modification of the braiding.
$\Rightarrow$ If every **positive** fusion category admits a fiber functor, then every **symmetric** fusion category admits a super fiber functor.

So we may focus on positive fusion categories.

Existence of fiber functors

[9.9.22(i)] Every positive fusion category $\mathcal{C}$ admits a fiber functor.

- For every algebra $A \in \mathcal{C}$, we have a category $\mathcal{C}_A$ of right A-modules in (the ind-completion of) $\mathcal{C}$, and a braided tensor functor $\mathcal{C} \ni X \mapsto X_A := X \otimes A \in \mathcal{C}_A$.
- **Crucial Proposition.** There is an algebra $A \in \mathcal{C}$ such that $X_A \cong (\mathbf{1}^{\oplus \dim X})_A$ for all $X \in \mathcal{C}$.
- Let $\mathcal{D} \subset \mathcal{C}_A$ be the full subcategory on finite direct sums of A. Then with A as above, we have a tensor functor $\mathcal{C} \ni X \mapsto X_A \in \mathcal{D}$, and $\mathcal{D} \simeq Mod^{fin.rank}(R)$ for $R := \mathcal{C}(\mathbf{1}, A)$. Now as R is a comm. $\mathbb{k}$-algebra and as $\mathbb{k} = \overline{\mathbb{k}}$, we may replace $\mathcal{D}$ by $Vec_{\mathbb{k}}$ by Hilbert's Nullstellensatz.

Crucial Proposition

[9.10.10] There is an algebra $A \in \mathcal{C}$ such that $X_A \cong (\mathbf{1}^{\oplus \dim X})_A$ for all $X \in \mathcal{C}$.

- It is enough to show for any simple $X_i \in \mathcal{C}$ there is an algebra $A_i \in \mathcal{C}$ such that $(X_i)_{A_i} \cong (\mathbf{1}^{d_i})_{A_i}$ where $d_i := \dim X_i$; then take $A := \bigotimes_i A_i$.
- **Crucial lemma.** (EGNO 9.10.8) $M \in \mathcal{C}_A$ rigid. Then $S^n M \neq 0$ in $\mathcal{C}_A$ for all $n \geq 0 \Leftrightarrow$ there is an algebra $B \in \mathcal{C}_A$ such that $\mathbf{1}$ is a direct summand in M_B.
- $X \in \mathcal{C}$ simple of dim. d. Induction + Crucial Lemma $\Rightarrow$ there is an algebra $A \in \mathcal{C}$ such that $X_A \cong (\mathbf{1}^{\oplus d})_A \oplus N$ for some $N \in \mathcal{C}_A$.
- Now

$$\Lambda^{d+1}(X) \cong \bigoplus_{i+j=d+1} \Lambda^i((\mathbf{1}^{\oplus d})_A) \otimes_A \Lambda^j(N)$$

contains $\Lambda^d((\mathbf{1}^{\oplus d})_A) \otimes_A \Lambda^1(N) \cong N$ as a direct summand, but is of dim. 0, hence is 0 as $\mathcal{C}$ is **positive**. So $N \cong 0$.

Crucial Lemma

$M \in \mathcal{C}_A$ rigid.

[9.10.8] Then $S^n M \neq 0$ in $\mathcal{C}_A$ for all $n \geq 0$ $\Leftrightarrow$ there is an algebra $B \in \mathcal{C}_A$ such that $\mathbf{1}$ is a direct summand in M_B.

- important for us is $\Rightarrow$
- Set $S := S(M \oplus M^*)$, $\overline{\delta} : \mathbf{1} \xrightarrow{coev} M \otimes_A M^* \to S \otimes_A S \xrightarrow{\mu} S$,
  $B := S/(\mathrm{im}(1 - \overline{\delta}))$, $\beta\colon = (M_B = M \otimes_A B \to B \otimes_A B \to B = \mathbf{1}_B)$,
  $\alpha := (\mathbf{1}_B = B \xrightarrow{coev \otimes B} M \otimes_A M^* \otimes_A B \to M \otimes_A B \otimes_A B \to M \otimes_A B = M_B)$.
- Then α, β exhibit $\mathbf{1}_B$ as a summand in M_B, once we show $B \neq 0$.
- Assume $B = 0 \Leftrightarrow p := \mu(S \otimes (1 - \overline{\delta}))\colon S = S \otimes_A \mathbf{1}_A \to S$ is surjective.
- As $A \in \mathcal{C}_A$ projective, $1\colon A \to S$ factors through p, so there is $x\colon \mathbf{1}_A \to S$ such that $1 = (1 - \overline{\delta})x$.
- This implies $x = \sum_{n \geq 0} \overline{\delta}^n$ and $\overline{\delta}^n = 0$ for some $n \geq 0$.
- But $\overline{\delta}^n$ is the coevaluation of $S^n M$. So $S^n M = 0$.

Deligne's theorem – Positive Fusion categories

$\mathbb{k} = \overline{\mathbb{k}}$ of characteristic 0.

We have correspondences between

- positive fusion categories $\mathcal{C} \leftrightarrow$ finite groups G,
- symmetric fusion categories $\mathcal{C} \leftrightarrow$ finite groups G with $z \in Z(G)$, $z^2 = 1$

given by Tannaka–Krein reconstruction via an essentially unique (super) fiber functor.

Schur functors

$\mathcal{C}$ a symmetric tensor category in characteristic 0.
For any partition λ of $n \geq 0$, let $P, Q \in S_n$ be the subgroups of permutations along the rows/columns of the standard Young tableaux of shape λ.
Set $e_\lambda(X) := \sum_{p \in P, q \in Q} (-1)^q pq \in End(X^{\otimes n})$ and $S^\lambda X := \mathrm{im}\, e_\lambda(X)$ for all $X \in \mathcal{C}$. S^λ is called a *Schur functor*.

Examples.

- $\lambda = n \Rightarrow S^\lambda = S^n$
- $\lambda = 1^n \Rightarrow S^\lambda = \Lambda^n$
- $V \in Vec$, $\dim V = d \Rightarrow \Lambda^n V = 0$ for all $n > d$.
- $V \in SVec$, $\dim V_0 = 0$, $\dim V_1 = d \Rightarrow S^n V = 0$ for all $n > d$.

In *SVec*, every object is annihilated by some Schur functor.

Problem 2. Show this! (See EGNO, Exercise 9.11.18.)

Growth-conditions

$\mathcal{C}$ a symmetric tensor category.
$\mathcal{C}$ is of *subexponential growth* if for all $X \in \mathcal{C}$ there is a number $C \geq 1$ such that the length of $X^{\otimes p}$ is at most C^p for all $p \geq 0$.
Example. $V = \mathbb{k}^n \in Vec$. Then the length of $V^{\otimes p}$ is n^p. $V = \mathbb{k}^{m|n} \in SVec$. Then the length of $V^{\otimes p}$ is $(m+n)^p$.

The following are equivalent:

- $\mathcal{C}$ is of subexponential growth.
- Every object in $\mathcal{C}$ is annihilated by some Schur functor.
- There exists a super fiber functor for $\mathcal{C}$.

Deligne's theorem – Tensor categories

$\mathbb{k} = \overline{\mathbb{k}}$ of characteristic 0.

We have correspondences between

- symmetric tensor categories **of subexponential growth** $\mathcal{C} \leftrightarrow$ supergroups G with $z \in Z(G)$, $z^2 = 1$

given by Tannaka–Krein reconstruction via an essentially unique super fiber functor.

Deligne's theorem is great, because it really covers a lot of the known symmetric tensor categories in characteristic 0. So are there interesting symmetric tensor categories in characteristic 0, which are not of subexponential growth?

> Up next: **Deligne's interpolation categories**.