Lawvere's Fixed Point Theorem

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Definition of Cartesian Closed Categories (CCCs)

Definition

A Category $\mathcal{C}$ is **Cartesian Closed** if it has (chosen) finite products and for all $A \in \mathrm{ob}(\mathcal{C})$ the functor $- \times A$ has a right adjoint $(-)^A$.

For $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, there is a natural bijection

$$\mathrm{curry} : \hom(X \times Y, Z) \to \hom(X, Z^Y)$$

We also get the following evaluation map:

$$\mathrm{ev}_{X,Y} = \mathrm{curry}^{-1}(\mathrm{id}_{X^Y}) : X^Y \times Y \to X$$

Implementation in Mathlib

In Mathlib CCCs are definded as monoidal closed categories, where the monoidal structure $(C, \top, \times)$ is given the Cartesian product:

```
abbrev CartesianClosed (C : Type u) [Category.{v} C]
  [ChosenFiniteProducts C] :=
  MonoidalClosed C
```

Here `MonoidalClosed` is defined as follows:

```
class Closed {C : Type u} [Category.{v} C]
  [MonoidalCategory.{v} C] (X : C) where
  rightAdj : C ⥤ C
  adj : tensorLeft X ⊣ rightAdj
class MonoidalClosed (C : Type u) [Category.{v} C]
  [MonoidalCategory.{v} C] where
  closed (X : C) : Closed X := by infer_instance
```

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(Weak) Point Surjectivity

Definition

A morphism $\Phi : X \to Y$ is **point surjective** (ps), if for every "point" $x : \top \to X$, there exists a "point" $y : \top \to Y$, such that $\Phi(x) = y$.

Definition

A morphism $\Phi : X \to Z^Y$ is **weakly point surjective** (wps), if for every $g : Y \to Z$, there exists a "point" $x : \top \to X$, such that for all $y : \top \to Y$ we have $g(y) = \Phi(x)(y)$.

Here $\Phi(x)$ denotes $\Phi \circ x$ and $\Phi(x)(y)$ denotes the morphism

$$\top \xrightarrow{\langle \Phi \circ x, y \rangle} Z^Y \times Y \xrightarrow{\mathrm{ev}_{Z,Y}} Z$$

Lawvere's fixed point theorem (LFPT)

Theorem

Let $\mathcal{C}$ be a CCC. If there is a weakly point surjective morphism $\Phi : A \to B^A$ in $\mathcal{C}$, then every morphism $f : B \to B$ has a fixed point (a $s : \top \to B$ such that $f \circ s = s$).

Theorem (alternative version)

Let $\mathcal{C}$ be a category with finite products. If there is a "weakly point surjective" morphism $\Phi : A \times A \to B$ in $\mathcal{C}$, then every morphism $f : B \to B$ has a fixed point (a $s : \top \to B$ such that $f \circ s = s$).

Proof of LFPT

Proof.

Let $g : A \to B$ be the composite morphism

$$A \xrightarrow{\Delta} A \times A \xrightarrow{\Phi \times \mathrm{id}_A} B^A \times A \xrightarrow{\mathrm{ev}_{B,A}} B \xrightarrow{f} B$$

Because Φ is wps, there exists a $p : \top \to A$ such that $g = \Phi(p)$. Now $\Phi(p)(p) = g(p)$ is the morphism

$$\top \xrightarrow{\langle \Phi \circ p, p \rangle} B^A \times A \xrightarrow{\mathrm{ev}_{B,A}} B \xrightarrow{f} B$$

Thus $\Phi(p)(p) = f \circ \Phi(p)(p)$. □

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Cantor's Theorem (in a topos)

We can use LFPT to prove a categorical version of Cantor's theorem.

Definition

A **topos** $\mathcal{T}$ is a CCC with finite limits and a subobject classifier Ω.

Topoi naturally give rise to an internal first-order theory.

Theorem

Let $\mathcal{T}$ be a topos. If there exists a $X \in \mathrm{ob}(\mathcal{T})$ and an epimorphism $\Phi : X \to \Omega^X$, then $\mathcal{T}$ is degenerate (i.e. the internal theory is inconsistent).

Self-referential Theorems

We can apply LFPT to several "self-referential" theorems in logic in the following way:

- For any L-theory T we construct a category with finite products and designated elements Ω and A.
- Morphisms $t : \top \to A^n$ correspond to classes of constant terms.
- Morphisms $\varphi : A^n \to \Omega$ correspond to classes of formulas with n free variables.
- Composition corresponds to substitution.
- There are morphisms true, false : $\top \to \Omega$
- A Gödel numbering can be regarded as a point surjection $g : A \to \Omega^A$ (Ω^A does not exist, but we can use currying)