

Lawvere's Fixed Point Theorem

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Definition of Cartesian Closed Categories (CCCs)

Definition

A Category \mathcal{C} is **Cartesian Closed** if it has (chosen) finite products and for all $A \in \text{ob}(\mathcal{C})$ the functor $- \times A$ has a right adjoint $(-)^A$.

For $X, Y, Z \in \text{ob}(\mathcal{C})$, there is a natural bijection

$$\text{curry} : \text{hom}(X \times Y, Z) \rightarrow \text{hom}(X, Z^Y)$$

We also get the following evaluation map:

$$\text{ev}_{X,Y} = \text{curry}^{-1}(\text{id}_{X^Y}) : X^Y \times Y \rightarrow X$$

Implementation in Mathlib

In Mathlib CCCs are defined as monoidal closed categories, where the monoidal structure (C, \top, \times) is given the Cartesian product:

```
abbrev CartesianClosed (C : Type u) [Category.{v} C]
  [ChosenFiniteProducts C] :=
  MonoidalClosed C
```

Here `MonoidalClosed` is defined as follows:

```
class Closed {C : Type u} [Category.{v} C]
  [MonoidalCategory.{v} C] (X : C) where
  rightAdj : C  $\Rightarrow$  C
  adj : tensorLeft X  $\dashv$  rightAdj
class MonoidalClosed (C : Type u) [Category.{v} C]
  [MonoidalCategory.{v} C] where
  closed (X : C) : Closed X := by infer_instance
```

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(Weak) Point Surjectivity

Definition

A morphism $\Phi : X \rightarrow Y$ is **point surjective** (ps), if for every “point” $x : \top \rightarrow X$, there exists a “point” $y : \top \rightarrow Y$, such that $\Phi(x) = y$.

Definition

A morphism $\Phi : X \rightarrow Z^Y$ is **weakly point surjective** (wps), if for every $g : Y \rightarrow Z$, there exists a “point” $x : \top \rightarrow X$, such that for all $y : \top \rightarrow Y$ we have $g(y) = \Phi(x)(y)$.

Here $\Phi(x)$ denotes $\Phi \circ x$ and $\Phi(x)(y)$ denotes the morphism

$$\top \xrightarrow{\langle \Phi \circ x, y \rangle} Z^Y \times Y \xrightarrow{\text{ev}_{Z,Y}} Z$$

Lawvere's fixed point theorem

Theorem

Let \mathcal{C} be a CCC. If there is a weakly point surjective morphism $\Phi : A \rightarrow B^A$ in \mathcal{C} , then every morphism $f : B \rightarrow B$ has a fixed point (a $s : \top \rightarrow B$ such that $f \circ s = s$).

Theorem (alternative version)

Let \mathcal{C} be a category with finite products. If there is a “weakly point surjective” morphism $\Phi : A \times A \rightarrow B$ in \mathcal{C} , then every morphism $f : B \rightarrow B$ has a fixed point (a $s : \top \rightarrow B$ such that $f \circ s = s$).

Proof.

Let $g : A \rightarrow B$ be the composite morphism

$$A \xrightarrow{\Delta} A \times A \xrightarrow{\Phi \times \text{id}_A} B^A \times A \xrightarrow{\text{ev}_{B,A}} B \xrightarrow{f} B$$

Because Φ is wps, there exists a $p : \top \rightarrow A$ such that $g = \Phi(p)$.
 Now $\Phi(p)(p) = g(p)$ is the morphism

$$\top \xrightarrow{\langle \Phi \circ p, p \rangle} B^A \times A \xrightarrow{\text{ev}_{B,A}} B \xrightarrow{f} B$$

Thus $\Phi(p)(p) = f \circ \Phi(p)(p)$. □