```
Exit[]
```

```
Moments = Table[W ^ n → Limit[D[Exp[t ^ 2 / 2], {t, n}], t -> 0], {n, 4, 1, -1}]
```

$\left\{W^4 \to 3, W^3 \to 0, W^2 \to 1, W \to 0\right\}$

```
Moments = {φ ^ 4 → kurt, φ ^ 3 → skew, φ ^ 2 → 1, φ → 0}
```

$\left\{\phi^4 \to \text{kurt}, \phi^3 \to \text{skew}, \phi^2 \to 1, \phi \to 0\right\}$

```
n = 4;
```

```
S[dt_] := S Exp[(μ - σ ^ 2 / 2) dt ^ 2 + σ φ dt];
```

```
dΠ = Series[V[t + dt ^ 2, S[dt]] - Δ S[dt], {dt, 0, n}] - (V[t, S] - Δ S)
```

$\left(-S\,\Delta\,\sigma\,\phi + S\,\sigma\,\phi\,V^{(0,1)}[t, S]\right)\,dt +$

$\qquad\left(-S\,\Delta\,\mu + \frac{1}{2}\,S\,\Delta\,\sigma^2 - \frac{1}{2}\,S\,\Delta\,\sigma^2\,\phi^2 + \left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}\,S\,\sigma^2\,\phi^2\right)\,V^{(0,1)}[t, S] +\right.$

$\qquad\left.\frac{1}{2}\,S^2\,\sigma^2\,\phi^2\,V^{(0,2)}[t, S] + V^{(1,0)}[t, S]\right)\,dt^2 +$

$\qquad\left(-S\,\Delta\,\mu\,\sigma\,\phi + \frac{1}{2}\,S\,\Delta\,\sigma^3\,\phi - \frac{1}{6}\,S\,\Delta\,\sigma^3\,\phi^3 + \left(S\,\mu\,\sigma\,\phi - \frac{1}{2}\,S\,\sigma^3\,\phi + \frac{1}{6}\,S\,\sigma^3\,\phi^3\right)\,V^{(0,1)}[t, S] +\right.$

$\qquad\left.S\,\sigma\,\phi\,\left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}\,S\,\sigma^2\,\phi^2\right)\,V^{(0,2)}[t, S] + \frac{1}{6}\,S^3\,\sigma^3\,\phi^3\,V^{(0,3)}[t, S] + S\,\sigma\,\phi\,V^{(1,1)}[t, S]\right)$

$\qquad dt^3 + \left(-\frac{1}{2}\,S\,\Delta\,\mu^2 + \frac{1}{2}\,S\,\Delta\,\mu\,\sigma^2 - \frac{1}{8}\,S\,\Delta\,\sigma^4 - \frac{1}{2}\,S\,\Delta\,\mu\,\sigma^2\,\phi^2 + \frac{1}{4}\,S\,\Delta\,\sigma^4\,\phi^2 - \frac{1}{24}\,S\,\Delta\,\sigma^4\,\phi^4 +\right.$

$\qquad\left(\frac{S\,\mu^2}{2} - \frac{1}{2}\,S\,\mu\,\sigma^2 + \frac{S\,\sigma^4}{8} + \frac{1}{2}\,S\,\mu\,\sigma^2\,\phi^2 - \frac{1}{4}\,S\,\sigma^4\,\phi^2 + \frac{1}{24}\,S\,\sigma^4\,\phi^4\right)\,V^{(0,1)}[t, S] +$

$\qquad\frac{1}{2}\,\left(\left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}\,S\,\sigma^2\,\phi^2\right)^2 + 2\,S\,\sigma\,\phi\,\left(S\,\mu\,\sigma\,\phi - \frac{1}{2}\,S\,\sigma^3\,\phi + \frac{1}{6}\,S\,\sigma^3\,\phi^3\right)\right)\,V^{(0,2)}[t, S] +$

$\qquad\frac{1}{2}\,S^2\,\sigma^2\,\phi^2\,\left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}\,S\,\sigma^2\,\phi^2\right)\,V^{(0,3)}[t, S] +$

$\qquad\frac{1}{24}\,S^4\,\sigma^4\,\phi^4\,V^{(0,4)}[t, S] + \left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}\,S\,\sigma^2\,\phi^2\right)\,V^{(1,1)}[t, S] +$

$\qquad\left.\frac{1}{2}\,S^2\,\sigma^2\,\phi^2\,V^{(1,2)}[t, S] + \frac{1}{2}\,V^{(2,0)}[t, S]\right)\,dt^4 + O[dt]^5$

```
A1 = Simplify[SeriesCoefficient[dΠ, 1]]
```

$S\,\sigma\,\phi\,\left(-\Delta + V^{(0,1)}[t, S]\right)$

```
A2 = Simplify[SeriesCoefficient[dΠ, 2]]
```

$\frac{1}{2}\,\left(-2\,S\,\Delta\,\mu + S\,\Delta\,\sigma^2 - S\,\Delta\,\sigma^2\,\phi^2 +\right.$

$\qquad\left.S\,\left(2\,\mu + \sigma^2\,\left(-1 + \phi^2\right)\right)\,V^{(0,1)}[t, S] + S^2\,\sigma^2\,\phi^2\,V^{(0,2)}[t, S] + 2\,V^{(1,0)}[t, S]\right)$

```
E2 = Series[Expand[Normal[dΠ ^ 2]] /. Moments, {dt, 0, n}];
```

```
E1 = Expand[Normal[dΠ]] /. Moments;
```

```
dVar = D[Simplify[E2 - E1 ^ 2], Δ]
```

$$2 \, S^2 \, \sigma^2 \, \left(\Delta - V^{(0,1)}[t, S]\right) \, dt^2 +$$
$$\left(S^2 \, \text{skew} \, \sigma^3 \, \left(\Delta - V^{(0,1)}[t, S]\right) - S^2 \, \text{skew} \, \sigma^3 \, \left(-\Delta + V^{(0,1)}[t, S] + S \, V^{(0,2)}[t, S]\right)\right) \, dt^3 +$$
$$\frac{1}{12} \, S^2 \, \sigma^2 \, \left(24 \, \Delta \, \mu - 15 \, \Delta \, \sigma^2 + 7 \, \text{kurt} \, \Delta \, \sigma^2 + \Delta \, \left(24 \, \mu - 15 \, \sigma^2 + 7 \, \text{kurt} \, \sigma^2\right) + \right.$$
$$2 \, \left(-24 \, \mu + 15 \, \sigma^2 - 7 \, \text{kurt} \, \sigma^2\right) V^{(0,1)}[t, S] - 6 \, S \, \left(4 \, \mu + 3 \, (-1 + \text{kurt}) \, \sigma^2\right) V^{(0,2)}[t, S] -$$
$$\left. 4 \, \text{kurt} \, S^2 \, \sigma^2 \, V^{(0,3)}[t, S] - 24 \, V^{(1,1)}[t, S]\right) \, dt^4 + O[dt]^5$$

```
dVarC = Series[CoefficientList[dVar, Δ], {dt, 0, n}]
```

$$\left\{-2 \, \left(S^2 \, \sigma^2 \, V^{(0,1)}[t, S]\right) \, dt^2 + \left(-2 \, S^2 \, \text{skew} \, \sigma^3 \, V^{(0,1)}[t, S] - S^3 \, \text{skew} \, \sigma^3 \, V^{(0,2)}[t, S]\right) \, dt^3 + \right.$$
$$\frac{1}{6} \, \left(-24 \, S^2 \, \mu \, \sigma^2 \, V^{(0,1)}[t, S] + 15 \, S^2 \, \sigma^4 \, V^{(0,1)}[t, S] - 7 \, \text{kurt} \, S^2 \, \sigma^4 \, V^{(0,1)}[t, S] - \right.$$
$$12 \, S^3 \, \mu \, \sigma^2 \, V^{(0,2)}[t, S] + 9 \, S^3 \, \sigma^4 \, V^{(0,2)}[t, S] - 9 \, \text{kurt} \, S^3 \, \sigma^4 \, V^{(0,2)}[t, S] -$$
$$2 \, \text{kurt} \, S^4 \, \sigma^4 \, V^{(0,3)}[t, S] - 12 \, S^2 \, \sigma^2 \, V^{(1,1)}[t, S]\right) \, dt^4 + O[dt]^5,$$
$$\left. 2 \, S^2 \, \sigma^2 \, dt^2 + 2 \, S^2 \, \text{skew} \, \sigma^3 \, dt^3 + \left(4 \, S^2 \, \mu \, \sigma^2 - \frac{5 \, S^2 \, \sigma^4}{2} + \frac{7}{6} \, \text{kurt} \, S^2 \, \sigma^4\right) \, dt^4 + O[dt]^5\right\}$$

```
fr = -dVarC[[1]] / dVarC[[2]]
```

$$V^{(0,1)}[t, S] + \left(-\text{skew} \, \sigma \, V^{(0,1)}[t, S] + \frac{2 \, S^2 \, \text{skew} \, \sigma^3 \, V^{(0,1)}[t, S] + S^3 \, \text{skew} \, \sigma^3 \, V^{(0,2)}[t, S]}{2 \, S^2 \, \sigma^2}\right) \, dt +$$
$$\left(\left(-2 \, \mu + \frac{5 \, \sigma^2}{4} - \frac{7 \, \text{kurt} \, \sigma^2}{12} + \text{skew}^2 \, \sigma^2\right) V^{(0,1)}[t, S] - \right.$$
$$\frac{1}{2 \, S^2 \, \sigma} \, \text{skew} \, \left(2 \, S^2 \, \text{skew} \, \sigma^3 \, V^{(0,1)}[t, S] + S^3 \, \text{skew} \, \sigma^3 \, V^{(0,2)}[t, S]\right) +$$
$$\frac{1}{12 \, S^2 \, \sigma^2} \, \left(24 \, S^2 \, \mu \, \sigma^2 \, V^{(0,1)}[t, S] - 15 \, S^2 \, \sigma^4 \, V^{(0,1)}[t, S] + 7 \, \text{kurt} \, S^2 \, \sigma^4 \, V^{(0,1)}[t, S] + \right.$$
$$12 \, S^3 \, \mu \, \sigma^2 \, V^{(0,2)}[t, S] - 9 \, S^3 \, \sigma^4 \, V^{(0,2)}[t, S] + 9 \, \text{kurt} \, S^3 \, \sigma^4 \, V^{(0,2)}[t, S] +$$
$$\left. \left. 2 \, \text{kurt} \, S^4 \, \sigma^4 \, V^{(0,3)}[t, S] + 12 \, S^2 \, \sigma^2 \, V^{(1,1)}[t, S]\right)\right) \, dt^2 + O[dt]^3$$

```
fr2 = -Series[Normal[dVarC[[1]]] / Normal[dVarC[[2]]], {dt, 0, n}]
```

$V^{(0,1)}[t, S] + \frac{1}{2} S \text{ skew } \sigma V^{(0,2)}[t, S] dt +$

$\frac{1}{12} \left(12 S \mu V^{(0,2)}[t, S] - 9 S \sigma^2 V^{(0,2)}[t, S] + 9 \text{ kurt } S \sigma^2 V^{(0,2)}[t, S] - \right.$

$\left. 6 S \text{ skew}^2 \sigma^2 V^{(0,2)}[t, S] + 2 \text{ kurt } S^2 \sigma^2 V^{(0,3)}[t, S] + 12 V^{(1,1)}[t, S]\right) dt^2 +$

$\frac{1}{24} \left(-48 S \text{ skew } \mu \sigma V^{(0,2)}[t, S] + 33 S \text{ skew } \sigma^3 V^{(0,2)}[t, S] - 25 \text{ kurt } S \text{ skew } \sigma^3 V^{(0,2)}[t, S] + \right.$

$\left. 12 S \text{ skew}^3 \sigma^3 V^{(0,2)}[t, S] - 4 \text{ kurt } S^2 \text{ skew } \sigma^3 V^{(0,3)}[t, S] - 24 \text{ skew } \sigma V^{(1,1)}[t, S]\right) dt^3 +$

$\frac{1}{144} \left(-288 S \mu^2 V^{(0,2)}[t, S] + 396 S \mu \sigma^2 V^{(0,2)}[t, S] - 300 \text{ kurt } S \mu \sigma^2 V^{(0,2)}[t, S] + \right.$

$432 S \text{ skew}^2 \mu \sigma^2 V^{(0,2)}[t, S] - 135 S \sigma^4 V^{(0,2)}[t, S] + 198 \text{ kurt } S \sigma^4 V^{(0,2)}[t, S] -$

$63 \text{ kurt}^2 S \sigma^4 V^{(0,2)}[t, S] - 288 S \text{ skew}^2 \sigma^4 V^{(0,2)}[t, S] + 192 \text{ kurt } S \text{ skew}^2 \sigma^4 V^{(0,2)}[t, S] -$

$72 S \text{ skew}^4 \sigma^4 V^{(0,2)}[t, S] - 48 \text{ kurt } S^2 \mu \sigma^2 V^{(0,3)}[t, S] + 30 \text{ kurt } S^2 \sigma^4 V^{(0,3)}[t, S] -$

$14 \text{ kurt}^2 S^2 \sigma^4 V^{(0,3)}[t, S] + 24 \text{ kurt } S^2 \text{ skew}^2 \sigma^4 V^{(0,3)}[t, S] - 288 \mu V^{(1,1)}[t, S] +$

$\left. 180 \sigma^2 V^{(1,1)}[t, S] - 84 \text{ kurt } \sigma^2 V^{(1,1)}[t, S] + 144 \text{ skew}^2 \sigma^2 V^{(1,1)}[t, S]\right) dt^4 + O[dt]^5$

```
hr = Normal[Series[Solve[Normal[dVar] == 0, Δ][[1, 1, 2]], {dt, 0, 4}]]
```

$V^{(0,1)}[t, S] + \frac{1}{2} dt S \text{ skew } \sigma V^{(0,2)}[t, S] +$

$\frac{1}{12} dt^2 \left(12 S \mu V^{(0,2)}[t, S] - 9 S \sigma^2 V^{(0,2)}[t, S] + 9 \text{ kurt } S \sigma^2 V^{(0,2)}[t, S] - \right.$

$\left. 6 S \text{ skew}^2 \sigma^2 V^{(0,2)}[t, S] + 2 \text{ kurt } S^2 \sigma^2 V^{(0,3)}[t, S] + 12 V^{(1,1)}[t, S]\right) + \frac{1}{24} dt^3$

$\left(-48 S \text{ skew } \mu \sigma V^{(0,2)}[t, S] + 33 S \text{ skew } \sigma^3 V^{(0,2)}[t, S] - 25 \text{ kurt } S \text{ skew } \sigma^3 V^{(0,2)}[t, S] + \right.$

$\left. 12 S \text{ skew}^3 \sigma^3 V^{(0,2)}[t, S] - 4 \text{ kurt } S^2 \text{ skew } \sigma^3 V^{(0,3)}[t, S] - 24 \text{ skew } \sigma V^{(1,1)}[t, S]\right) +$

$\frac{1}{144} dt^4 \left(-288 S \mu^2 V^{(0,2)}[t, S] + 396 S \mu \sigma^2 V^{(0,2)}[t, S] - 300 \text{ kurt } S \mu \sigma^2 V^{(0,2)}[t, S] + \right.$

$432 S \text{ skew}^2 \mu \sigma^2 V^{(0,2)}[t, S] - 135 S \sigma^4 V^{(0,2)}[t, S] + 198 \text{ kurt } S \sigma^4 V^{(0,2)}[t, S] -$

$63 \text{ kurt}^2 S \sigma^4 V^{(0,2)}[t, S] - 288 S \text{ skew}^2 \sigma^4 V^{(0,2)}[t, S] + 192 \text{ kurt } S \text{ skew}^2 \sigma^4 V^{(0,2)}[t, S] -$

$72 S \text{ skew}^4 \sigma^4 V^{(0,2)}[t, S] - 48 \text{ kurt } S^2 \mu \sigma^2 V^{(0,3)}[t, S] + 30 \text{ kurt } S^2 \sigma^4 V^{(0,3)}[t, S] -$

$14 \text{ kurt}^2 S^2 \sigma^4 V^{(0,3)}[t, S] + 24 \text{ kurt } S^2 \text{ skew}^2 \sigma^4 V^{(0,3)}[t, S] - 288 \mu V^{(1,1)}[t, S] +$

$\left. 180 \sigma^2 V^{(1,1)}[t, S] - 84 \text{ kurt } \sigma^2 V^{(1,1)}[t, S] + 144 \text{ skew}^2 \sigma^2 V^{(1,1)}[t, S]\right)$

```
Series[E1 /. Moments /. Δ → hr, {dt, 0, n}] -
 Series[(V[t, S] - hr S) (Exp[r dt ^ 2] - 1), {dt, 0, n}]
```

$$\left(-r\, V[t,\,S] + r\, S\, V^{(0,1)}[t,\,S] + \frac{1}{2}\, S^2\, \sigma^2\, V^{(0,2)}[t,\,S] + V^{(1,0)}[t,\,S]\right) dt^2\, +$$

$$\left(\frac{1}{2}\, r\, S^2\, \text{skew}\, \sigma\, V^{(0,2)}[t,\,S]\, +\right.$$

$$\left.\frac{1}{6}\, \left(-3\, S^2\, \text{skew}\, \mu\, \sigma\, V^{(0,2)}[t,\,S] + 3\, S^2\, \text{skew}\, \sigma^3\, V^{(0,2)}[t,\,S] + S^3\, \text{skew}\, \sigma^3\, V^{(0,3)}[t,\,S]\right)\right) dt^3\, +$$

$$\left(\frac{1}{12}\, \left(-6\, r^2\, V[t,\,S] + 6\, r^2\, S\, V^{(0,1)}[t,\,S] + 12\, r\, S^2\, \mu\, V^{(0,2)}[t,\,S]\, -\right.\right.$$

$$9\, r\, S^2\, \sigma^2\, V^{(0,2)}[t,\,S] + 9\, \text{kurt}\, r\, S^2\, \sigma^2\, V^{(0,2)}[t,\,S]\, -$$

$$\left.6\, r\, S^2\, \text{skew}^2\, \sigma^2\, V^{(0,2)}[t,\,S] + 2\, \text{kurt}\, r\, S^3\, \sigma^2\, V^{(0,3)}[t,\,S] + 12\, r\, S\, V^{(1,1)}[t,\,S]\right)\, +$$

$$\frac{1}{24}\, \left(-12\, S^2\, \mu^2\, V^{(0,2)}[t,\,S] + 42\, S^2\, \mu\, \sigma^2\, V^{(0,2)}[t,\,S] - 18\, \text{kurt}\, S^2\, \mu\, \sigma^2\, V^{(0,2)}[t,\,S]\, +\right.$$

$$12\, S^2\, \text{skew}^2\, \mu\, \sigma^2\, V^{(0,2)}[t,\,S] - 15\, S^2\, \sigma^4\, V^{(0,2)}[t,\,S] + 7\, \text{kurt}\, S^2\, \sigma^4\, V^{(0,2)}[t,\,S]\, -$$

$$2\, S^2\, \text{skew}^2\, \sigma^4\, V^{(0,2)}[t,\,S] + 12\, S^3\, \mu\, \sigma^2\, V^{(0,3)}[t,\,S] - 4\, \text{kurt}\, S^3\, \mu\, \sigma^2\, V^{(0,3)}[t,\,S]\, -$$

$$6\, S^3\, \sigma^4\, V^{(0,3)}[t,\,S] + 6\, \text{kurt}\, S^3\, \sigma^4\, V^{(0,3)}[t,\,S] + \text{kurt}\, S^4\, \sigma^4\, V^{(0,4)}[t,\,S]\, +$$

$$\left.\left.12\, S^2\, \sigma^2\, V^{(1,2)}[t,\,S] + 12\, V^{(2,0)}[t,\,S]\right)\right) dt^4 + O[dt]^5$$