

Exit[]

Moments = Table[W ^ n → Limit[D[Exp[t ^ 2 / 2], {t, n}], t -> 0], {n, 4, 1, -1}]

{W ^ 4 → 3, W ^ 3 → 0, W ^ 2 → 1, W → 0}

Moments = {φ ^ 4 → kurt, φ ^ 3 → skew, φ ^ 2 μ → e2μ,

φ ^ 2 σ ^ 2 → e2σ2, φ ^ 2 → σ ^ 2, φ μ → e1μ, φ σ ^ 2 → e1σ2, φ → 0}

{φ ^ 4 → kurt, φ ^ 3 → skew, μ φ ^ 2 → e2μ, σ ^ 2 φ ^ 2 → e2σ2, φ ^ 2 → σ ^ 2, μ φ → e1μ, σ ^ 2 φ → e1σ2, φ → 0}

n = 3;

S[dt_] := S Exp[(μ - σ ^ 2 / 2) dt ^ 2 + φ dt];

dΠ = Series[V[t + dt ^ 2, S[dt]] - Δ S[dt], {dt, 0, n}] - (V[t, S] - Δ S)

(-S Δ φ + S φ V^(0,1)[t, S]) dt +

$\frac{1}{2} (-2 S \Delta \mu + S \Delta \sigma^2 - S \Delta \phi^2 + 2 S \mu V^{(0,1)}[t, S] - S \sigma^2 V^{(0,1)}[t, S] +$

$S \phi^2 V^{(0,1)}[t, S] + S^2 \phi^2 V^{(0,2)}[t, S] + 2 V^{(1,0)}[t, S]) dt^2 +$

$\frac{1}{6} (-6 S \Delta \mu \phi + 3 S \Delta \sigma^2 \phi - S \Delta \phi^3 + 6 S \mu \phi V^{(0,1)}[t, S] - 3 S \sigma^2 \phi V^{(0,1)}[t, S] +$

$S \phi^3 V^{(0,1)}[t, S] + 6 S^2 \mu \phi V^{(0,2)}[t, S] - 3 S^2 \sigma^2 \phi V^{(0,2)}[t, S] +$

$3 S^2 \phi^3 V^{(0,2)}[t, S] + S^3 \phi^3 V^{(0,3)}[t, S] + 6 S \phi V^{(1,1)}[t, S]) dt^3 + O[dt]^4$

A1 = Simplify[SeriesCoefficient[dΠ, 1]]

S φ (-Δ + V^(0,1)[t, S])

A2 = Simplify[SeriesCoefficient[dΠ, 2]]

$\frac{1}{2} (-2 S \Delta \mu + S \Delta \sigma^2 - S \Delta \phi^2 + S (2 \mu - \sigma^2 + \phi^2) V^{(0,1)}[t, S] + S^2 \phi^2 V^{(0,2)}[t, S] + 2 V^{(1,0)}[t, S])$

E2 = Series[Expand[Normal[dΠ^2]] /. Moments, {dt, 0, n+1}]

$$\begin{aligned}
& (S^2 \Delta^2 \sigma^2 - 2 S^2 \Delta \sigma^2 V^{(0,1)}[t, S] + S^2 \sigma^2 V^{(0,1)}[t, S]^2) dt^2 + \\
& (2 e1\mu S^2 \Delta^2 - e1\sigma 2 S^2 \Delta^2 + S^2 \text{skew} \Delta^2 - 4 e1\mu S^2 \Delta V^{(0,1)}[t, S] + 2 e1\sigma 2 S^2 \Delta V^{(0,1)}[t, S] - \\
& 2 S^2 \text{skew} \Delta V^{(0,1)}[t, S] + 2 e1\mu S^2 V^{(0,1)}[t, S]^2 - e1\sigma 2 S^2 V^{(0,1)}[t, S]^2 + \\
& S^2 \text{skew} V^{(0,1)}[t, S]^2 - S^3 \text{skew} \Delta V^{(0,2)}[t, S] + S^3 \text{skew} V^{(0,1)}[t, S] V^{(0,2)}[t, S]) dt^3 + \\
& \frac{1}{12} (36 e2\mu S^2 \Delta^2 - 18 e2\sigma 2 S^2 \Delta^2 + 7 \text{kurt} S^2 \Delta^2 + 12 S^2 \Delta^2 \mu^2 - 12 S^2 \Delta^2 \mu \sigma^2 + 3 S^2 \Delta^2 \sigma^4 - \\
& 72 e2\mu S^2 \Delta V^{(0,1)}[t, S] + 36 e2\sigma 2 S^2 \Delta V^{(0,1)}[t, S] - 14 \text{kurt} S^2 \Delta V^{(0,1)}[t, S] - \\
& 24 S^2 \Delta \mu^2 V^{(0,1)}[t, S] + 24 S^2 \Delta \mu \sigma^2 V^{(0,1)}[t, S] - 6 S^2 \Delta \sigma^4 V^{(0,1)}[t, S] + \\
& 36 e2\mu S^2 V^{(0,1)}[t, S]^2 - 18 e2\sigma 2 S^2 V^{(0,1)}[t, S]^2 + 7 \text{kurt} S^2 V^{(0,1)}[t, S]^2 + \\
& 12 S^2 \mu^2 V^{(0,1)}[t, S]^2 - 12 S^2 \mu \sigma^2 V^{(0,1)}[t, S]^2 + 3 S^2 \sigma^4 V^{(0,1)}[t, S]^2 - \\
& 36 e2\mu S^3 \Delta V^{(0,2)}[t, S] + 18 e2\sigma 2 S^3 \Delta V^{(0,2)}[t, S] - 18 \text{kurt} S^3 \Delta V^{(0,2)}[t, S] + \\
& 36 e2\mu S^3 V^{(0,1)}[t, S] V^{(0,2)}[t, S] - 18 e2\sigma 2 S^3 V^{(0,1)}[t, S] V^{(0,2)}[t, S] + \\
& 18 \text{kurt} S^3 V^{(0,1)}[t, S] V^{(0,2)}[t, S] + 3 \text{kurt} S^4 V^{(0,2)}[t, S]^2 - \\
& 4 \text{kurt} S^4 \Delta V^{(0,3)}[t, S] + 4 \text{kurt} S^4 V^{(0,1)}[t, S] V^{(0,3)}[t, S] - 24 S \Delta \mu V^{(1,0)}[t, S] + \\
& 24 S \mu V^{(0,1)}[t, S] V^{(1,0)}[t, S] + 12 S^2 \sigma^2 V^{(0,2)}[t, S] V^{(1,0)}[t, S] + 12 V^{(1,0)}[t, S]^2 - \\
& 24 S^2 \Delta \sigma^2 V^{(1,1)}[t, S] + 24 S^2 \sigma^2 V^{(0,1)}[t, S] V^{(1,1)}[t, S]) dt^4 + O[dt]^5
\end{aligned}$$

E1 = Expand[Normal[dΠ]] /. Moments

$$\begin{aligned}
& -dt^3 e1\mu S \Delta + \frac{1}{2} dt^3 e1\sigma 2 S \Delta - \frac{1}{6} dt^3 S \text{skew} \Delta - dt^2 S \Delta \mu + dt^3 e1\mu S V^{(0,1)}[t, S] - \\
& \frac{1}{2} dt^3 e1\sigma 2 S V^{(0,1)}[t, S] + \frac{1}{6} dt^3 S \text{skew} V^{(0,1)}[t, S] + dt^2 S \mu V^{(0,1)}[t, S] + \\
& dt^3 e1\mu S^2 V^{(0,2)}[t, S] - \frac{1}{2} dt^3 e1\sigma 2 S^2 V^{(0,2)}[t, S] + \frac{1}{2} dt^3 S^2 \text{skew} V^{(0,2)}[t, S] + \\
& \frac{1}{2} dt^2 S^2 \sigma^2 V^{(0,2)}[t, S] + \frac{1}{6} dt^3 S^3 \text{skew} V^{(0,3)}[t, S] + dt^2 V^{(1,0)}[t, S]
\end{aligned}$$

Var = Simplify[E2 - E1^2]

$$\begin{aligned}
& S^2 \sigma^2 (\Delta - V^{(0,1)}[t, S])^2 dt^2 + S^2 (\Delta - V^{(0,1)}[t, S]) \\
& ((2 e1\mu - e1\sigma 2 + \text{skew}) \Delta + (-2 e1\mu + e1\sigma 2 - \text{skew}) V^{(0,1)}[t, S] - S \text{skew} V^{(0,2)}[t, S]) dt^3 + \\
& \frac{1}{12} S^2 ((36 e2\mu - 18 e2\sigma 2 + 7 \text{kurt} - 12 \mu \sigma^2 + 3 \sigma^4) V^{(0,1)}[t, S]^2 - \\
& 6 S \Delta (6 e2\mu - 3 e2\sigma 2 + 3 \text{kurt} - 2 \mu \sigma^2) V^{(0,2)}[t, S] + \\
& 3 S^2 (\text{kurt} - \sigma^4) V^{(0,2)}[t, S]^2 + \Delta (36 e2\mu - 18 e2\sigma 2 + 7 \text{kurt} - 12 \mu \sigma^2 + 3 \sigma^4) - \\
& 4 \text{kurt} S^2 V^{(0,3)}[t, S] - 24 \sigma^2 V^{(1,1)}[t, S]) - 2 V^{(0,1)}[t, S] \\
& (36 e2\mu \Delta - 18 e2\sigma 2 \Delta + 7 \text{kurt} \Delta - 12 \Delta \mu \sigma^2 + 3 \Delta \sigma^4 + 3 S (-6 e2\mu + 3 e2\sigma 2 - 3 \text{kurt} + 2 \mu \sigma^2) \\
& V^{(0,2)}[t, S] - 2 \text{kurt} S^2 V^{(0,3)}[t, S] - 12 \sigma^2 V^{(1,1)}[t, S]) dt^4 + O[dt]^5
\end{aligned}$$

dVar = Series[CoefficientList[D[Var, Δ], Δ], {dt, 0, n+1}]

$$\begin{aligned} & \left\{ -2 \left(S^2 \sigma^2 V^{(0,1)}[t, S] \right) dt^2 + \left(-4 e1\mu S^2 V^{(0,1)}[t, S] + \right. \right. \\ & \quad \left. \left. 2 e1\sigma^2 S^2 V^{(0,1)}[t, S] - 2 S^2 \text{skew } V^{(0,1)}[t, S] - S^3 \text{skew } V^{(0,2)}[t, S] \right) dt^3 + \right. \\ & \quad \left(-\frac{1}{6} S^2 \left(36 e2\mu - 18 e2\sigma^2 + 7 \text{kurt} - 12 \mu \sigma^2 + 3 \sigma^4 \right) V^{(0,1)}[t, S] - \right. \\ & \quad \left. \frac{1}{2} S^3 \left(6 e2\mu - 3 e2\sigma^2 + 3 \text{kurt} - 2 \mu \sigma^2 \right) V^{(0,2)}[t, S] - \right. \\ & \quad \left. \frac{1}{3} \text{kurt } S^4 V^{(0,3)}[t, S] - 2 S^2 \sigma^2 V^{(1,1)}[t, S] \right) dt^4 + O[dt]^5, \\ & \quad 2 S^2 \sigma^2 dt^2 + \left(4 e1\mu S^2 - 2 e1\sigma^2 S^2 + 2 S^2 \text{skew} \right) dt^3 + \\ & \quad \left(6 e2\mu S^2 - 3 e2\sigma^2 S^2 + \frac{7 \text{kurt } S^2}{6} - 2 S^2 \mu \sigma^2 + \frac{S^2 \sigma^4}{2} \right) dt^4 + O[dt]^5 \} \end{aligned}$$

fr = Normal[-Series[dVar[[1]] / dVar[[2]], {dt, 0, 1}]]

$$V^{(0,1)}[t, S] + \frac{dt S \text{skew } V^{(0,2)}[t, S]}{2 \sigma^2}$$

fra = V^(0,1)[t, S];

Simplify[Series[Normal[Var] /. Δ → fr, {dt, 0, n+1}]]

$$- \frac{\left(S^4 \left(\text{skew}^2 - \text{kurt } \sigma^2 + \sigma^6 \right) V^{(0,2)}[t, S]^2 \right) dt^4}{4 \sigma^2} + O[dt]^5$$

BS = Expand[Simplify[Normal[Series[E1 /. Δ → fra, {dt, 0, n}]] -

Series[(V[t, S] - fra S) (Exp[r dt ^ 2] - 1), {dt, 0, n}]]] / dt ^ 2]

$$\begin{aligned} & -r V[t, S] + r S V^{(0,1)}[t, S] + dt e1\mu S^2 V^{(0,2)}[t, S] - \frac{1}{2} dt e1\sigma^2 S^2 V^{(0,2)}[t, S] + \\ & \frac{1}{2} dt S^2 \text{skew } V^{(0,2)}[t, S] + \frac{1}{2} S^2 \sigma^2 V^{(0,2)}[t, S] + \frac{1}{6} dt S^3 \text{skew } V^{(0,3)}[t, S] + V^{(1,0)}[t, S] \end{aligned}$$

Collect[BS, S ^ 2 D[V[t, S], {S, 2}]]

$$\begin{aligned} & -r V[t, S] + r S V^{(0,1)}[t, S] + \\ & S^2 \left(dt e1\mu - \frac{dt e1\sigma^2}{2} + \frac{dt \text{skew}}{2} + \frac{\sigma^2}{2} \right) V^{(0,2)}[t, S] + \frac{1}{6} dt S^3 \text{skew } V^{(0,3)}[t, S] + V^{(1,0)}[t, S] \end{aligned}$$

simplify[2 / S ^ 2 (

$$\frac{1}{2} dt r S^2 \text{skew } \sigma V^{(0,2)}[t, S] - \frac{1}{2} dt S^2 \text{skew } \mu \sigma V^{(0,2)}[t, S] + \frac{1}{2} S^2 \sigma^2 V^{(0,2)}[t, S] + \frac{1}{2} dt S^2 \text{skew } \sigma^3 V^{(0,2)}[t, S] \Big)$$

$$\sigma \left(\sigma + dt \text{skew} (r - \mu + \sigma^2) \right) V^{(0,2)}[t, S]$$