```
<< NC`;
<< NCAlgebra`;
```

You are using the version of NCAlgebra which is found in:

d:\Users\Johannes\Documents\NC.

You can now use "<< NCAlgebra`" to load NCAlgebra or "<< NCGB`" to load NCGB.

You have already loaded NCAlgebra.m

```
exp[A_, m_] := 1 + Sum[Nest[(A ** # &), A, n] / (n + 1)!, {n, 0, m - 1}]

SetCommutative[h]; SNC[A]; SNC[B];

n = 2;

Series[NCExpand[exp[h (A + B), 10]] - NCExpand[exp[h A, n] ** exp[h B, n]], {h, 0, 3}]
```

$$\frac{1}{2}(-A ** B + B ** A)\, h^2 + \frac{1}{6}(A ** A ** A - 2 A ** A ** B + A ** B ** A - 2 A ** B ** B + B ** A ** A + B ** A ** B + B ** B ** A + B ** B ** B)\, h^3 + O[h]^4$$

```
Series[NCExpand[exp[h (A + B), n]] -
  NCExpand[exp[h A / 2, n] ** exp[h B, n] ** exp[h A / 2, n]], {h, 0, 3}]
```

$$\frac{1}{8}(-A ** A ** A - A ** A ** B - 2 A ** B ** A - 2 A ** B ** B - B ** A ** A - 2 B ** B ** A)\, h^3 + O[h]^4$$

■ Vergleich der Lösungen

$$\text{CN}[\text{A\_}] := \frac{(1 - A / 2)}{1 + A / 2}$$

■ Its second order, but only, because A and B commute

```
CN[h A] CN[h B] - CN[h (A + B)] + O[h]^4
```

$$\frac{1}{4}\left(A^2 B + A B^2\right) h^3 + O[h]^4$$

```
CN[h A / 2] CN[h B] CN[h A / 2] - CN[h (A + B)] + O[h]^4
```

$$\frac{1}{16}\left(A^3 + 4 A^2 B + 4 A B^2\right) h^3 + O[h]^4$$

■ Vergleich der Lösungen 1.ordnung (inhomogen)

$$\text{EU}[\text{A\_}, \text{h\_}, \text{t\_}] := \frac{\# - b[A, t]}{1 + h A} \;\&$$

```
Clear[b];

b[A_ + B_, t_] := b[A, t]
```

b[B, t] = 0

0

EU[B, h, t][EU[A, h, t][u0]] – EU[A + B, h, t][u0] + O[h]^4

$(-A\,B\,u0 + A\,B\,b[A,t])\,h^2 + \left(2\,A^2\,B\,u0 + 2\,A\,B^2\,u0 - 2\,A^2\,B\,b[A,t] - 2\,A\,B^2\,b[A,t]\right)\,h^3 + O[h]^4$

CN[A, h / 2, t + h / 2][CN[B, h, t][CN[A, h / 2, t][u0]]] – CN[A + B, h, t][u0] + O[h]^3

$O[h]^3 + CN\left[A, \frac{h}{2}, \frac{h}{2} + t\right]\left[CN[B, h, t]\left[CN\left[A, \frac{h}{2}, t\right][u0]\right]\right] - CN[A + B, h, t][u0]$

■ Vergleich der Lösungen (inhomogen): 1.

$$\mathbf{CN[A\_, h\_, t\_] := \frac{(1 - h\,A/2)\;\#-\frac{1}{2}\,h\,(b[A,t] + b[A, h+t])}{1 + h\,A/2}\;\&}$$

Clear[b];

b[A_ + B_, t_] := b[A, t] + b[B, t]

■ Due to the inhomogenity is the following approximation only first order (even with commuting A and B!)

CN[B, h, t][CN[A, h, t][u0]] – CN[A + B, h, t][u0] + O[h]^4

$\frac{1}{2}\,(B\,b[A,t] - A\,b[B,t])\,h^2 + \frac{1}{4}\left(A^2\,B\,u0 + A\,B^2\,u0 - B^2\,b[A,t] + A^2\,b[B,t] + 2\,A\,B\,b[B,t] + B\,b^{(0,1)}[A,t] - A\,b^{(0,1)}[B,t]\right)h^3 + O[h]^4$

CN[A, h / 2, t + h / 2][CN[B, h, t][CN[A, h / 2, t][u0]]] – CN[A + B, h, t][u0] + O[h]^3

$O[h]^3$

■ z.B. b[A]= 0

CN[A, h / 2, t + h / 2][CN[B, h, t][CN[A, h / 2, t][u0]]] /. b[A, t_] → 0

$$\frac{\left(1 - \frac{A\,h}{4}\right)\left(\frac{\left(1-\frac{A\,h}{4}\right)\left(1-\frac{B\,h}{2}\right)u0}{1+\frac{A\,h}{4}} - \frac{1}{2}\,h\,(b[B,t] + b[B, h+t])\right)}{\left(1 + \frac{A\,h}{4}\right)\left(1 + \frac{B\,h}{2}\right)}$$

■ Variante: keine b[,t+h/2]:

CN[A, h / 2, t + h / 2][CN[B, h, t][CN[A, h / 2, t][u0]]]

$$\frac{1}{1+\frac{A\,h}{4}}\left(-\frac{1}{4}\,h\,(2\,b[A,\,h+t])+1\Big/\left(1+\frac{B\,h}{2}\right)\left(1-\frac{A\,h}{4}\right)\right.$$
$$\left.\left(\frac{\left(1-\frac{B\,h}{2}\right)\left(\left(1-\frac{A\,h}{4}\right)u0-\frac{1}{4}\,h\,(b[A,\,t]\;2)\right)}{1+\frac{A\,h}{4}}-\frac{1}{2}\,h\,(b[B,\,t]+b[B,\,h+t])\right)\right)$$

$$\frac{1}{1+\frac{A\,h}{4}}\left(-\frac{1}{2}\,h\,b[A,\,h+t]+1\Big/\left(1+\frac{B\,h}{2}\right)\left(1-\frac{A\,h}{4}\right)\right.$$
$$\left.\left(\frac{\left(1-\frac{B\,h}{2}\right)\left(\left(1-\frac{A\,h}{4}\right)u0-\frac{1}{2}\,h\,b[A,\,t]\right)}{1+\frac{A\,h}{4}}-\frac{1}{2}\,h\,(b[B,\,t]+b[B,\,h+t])\right)\right)$$

```
% - CN[A + B, h, t][u0] + O[h]^3
```

$O[h]^3$

■ diese Variante noch allgemeiner:

```
Exit[]
```

$$\text{CN3}[A\_,\,\Delta\_] := \frac{(1-\,h\,/\,4\,A)\;\#-\frac{h}{2}\,b[A,\,t+\Delta\,h]}{1+\,h\,A\,/\,4}\;\&;$$

```
b[A_ + B_ + C_, t_] := b[A, t] + b[B, t] + b[C, t]

CN3[A, 1][CN3[ B, 1][CN[C, h, t][CN3[ B, 0][CN3[A, 0][u0]]]]] -
  CN[A + B + C, h, t][u0] + O[h]^3
```

$O[h]^3$

■ Vergleich der Gleichungen

```
A = A1 + A2;

a = (1 + h A / 2) u[t + h] - (1 - h A / 2) u[t]
```

$$-\left(1-\frac{1}{2}\,(A1+A2)\,h\right)u[t]+\left(1+\frac{1}{2}\,(A1+A2)\,h\right)u[h+t]$$

```
a - (D[u[t], t] + A u[t]) / 2 h - ((D[u[t], t] + A u[t]) / 2 /. t → t + h) h + O[h]^3
```

$O[h]^3$

```
b = (1 + h A1 / 4) (1 + h A2 / 2) (1 + h A1 / 4) u[t + h] -
  (1 - h A1 / 4) (1 - h A2 / 2) (1 - h A1 / 4) u[t]
```

$$-\left(1-\frac{A1\,h}{4}\right)^2\left(1-\frac{A2\,h}{2}\right)u[t]+\left(1+\frac{A1\,h}{4}\right)^2\left(1+\frac{A2\,h}{2}\right)u[h+t]$$

```
b - (D[u[t], t] + A u[t]) / 2 h - ((D[u[t], t] + A u[t]) / 2 /. t → t + h) h + O[h]^3
```

$O[h]^3$

```
v[h_] := u[0] + h a[u[0]]

u2[h_] := v[h] + h b[v[h]]

gl[v_] := Simplify[D[v[h], h] - a[v[h]] - b[v[h]] + O[h]^2]

gl[u2]
```

$(a[u[0]]\,(-a'[u[0]] + b'[u[0]]) - b[u[0]]\,(a'[u[0]] + b'[u[0]]))\,h + O[h]^2$

```
a[x_] := A x; b[x_] := B x;

Clear[a, b]

u1[h_] := u[0] + h / 2 a[u[0]]
v[h_] := u1[h] + h b[u1[h]]
u2[h_] := v[h] + h / 2 a[v[h]]

gl[v_] := Simplify[D[u2[h], h] - a[u[h]] - b[u[h]] + O[h]^3]

gl[u2]
```

$$\left(b[u[0]]\,a'[u[0]] + a[u[0]]\left(\frac{1}{2}\,a'[u[0]] + b'[u[0]]\right) - (a'[u[0]] + b'[u[0]])\,u'[0]\right) h + \frac{1}{16}\left(12\,b[u[0]]^2\,a''[u[0]] + 12\,a[u[0]]\,(a'[u[0]]\,b'[u[0]] + b[u[0]]\,a''[u[0]]) + 3\,a[u[0]]^2\,(a''[u[0]] + 2\,b''[u[0]]) - 8\left(u'[0]^2\,a''[u[0]] + u'[0]^2\,b''[u[0]] + (a'[u[0]] + b'[u[0]])\,u''[0]\right)\right) h^2 + O[h]^3$$

```
Series[1 + Inverse[g[x], x],

Inverse[g[x], x]
```

Inverse::nonopt : Options expected (instead of x) beyond position 1 in Inverse[g[x], x]. An option must be a rule or a list of rules. ≫

```
InverseFunction[g[x], x]
```