

```

Exit[]

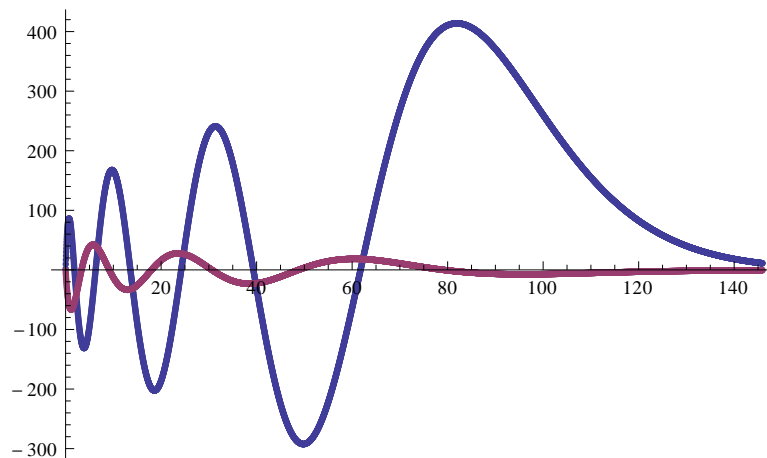
a = 7297352537.6 * 10^-12; M = 510998.910; Z = 1; k = -1;
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k^2 - (Z * a)^2]))^2]);
Table[N[Energie[i]], {i, 10}]

{13.6059, 3.40148, 1.51176, 0.850365,
 0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}

,

f[u_, r_] := Simplify[{(Z * a / r + 2 - Enn) * u[[2]] - k / r * u[[1]],
  k / r * u[[2]] + (Enn - Z * a / r) * u[[1]]}];
k = -1; Z = 1; U = .;
n = 5000;
h = 20000 / n;
Enn = 0.27766906844567757` / M;
u = {(91.35044102604739` (-3.662751763692355` + Enn) (1.6262886176197724` + Enn)) /
  ((-0.0109728221664999` + Enn) (181.38339842774778` + Enn)), -1};
r = 1; U = {{r, u}};
Do[
  k0 = h * f[u, r]; k1 = h * f[u + k0 / 2, r + h / 2];
  k2 = h * f[u + k1 / 2, r + h / 2]; k3 = h * f[u + k2, r + h];
  u += 1 / 6 * (k0 + 2 * k1 + 2 * k2 + k3); r += h;
  AppendTo[U, {r, u}], {n}]; x = .;
ListPlot[Table[{#[[1]] * a, (137^(i - 2) * #[[2, i]])^1} & /@ U[[1 ;; n]], {i, 2}] // N,
  PlotRange -> All]

```



```

lambda[n_] := Sqrt[1 - (1 - En[n] / M) ^ 2];
gamma := Sqrt[k ^ 2 - Z ^ 2 * a ^ 2];
f[r_, n_, k_] := -r ^ gamma * Exp[-lambda[n] * r] * (((n - 1 + gamma) / (1 - En[n] / M) - k) *
  Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r] +
  (n - 1) * Hypergeometric1F1[1 - (n - 1), 2 * gamma + 1, 2 * lambda[n] * r]);
g[r_, n_, k_] := Sqrt[2 * M / En[n] - 1] * r ^ gamma * Exp[-lambda[n] * r] *
  (((n - 1 + gamma) / (1 - En[n] / M) - k) *
  Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r] -
  (n - 1) * Hypergeometric1F1[1 - (n - 1), 2 * gamma + 1, 2 * lambda[n] * r]);

g[1, 1, -1] / f[1, 1, -1]

- 274.068

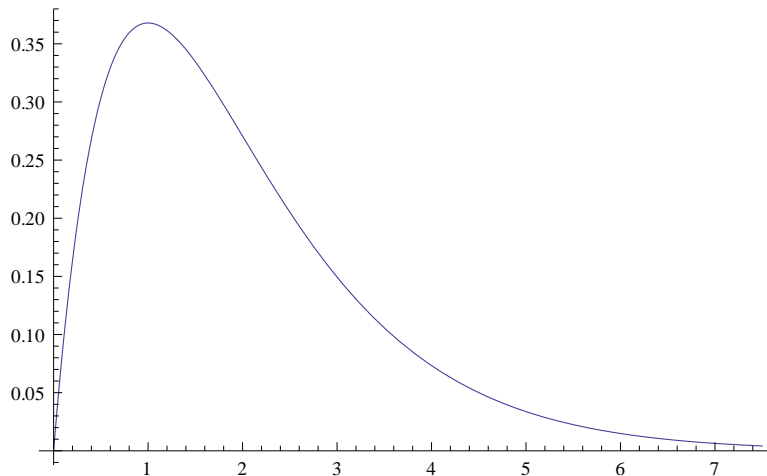
a = 7 297 352 537.6 * 10 ^ -12; M = 510 998.910; Z = 1; k = -1;
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
lambda[n_] := Sqrt[1 - (1 - Energie[n] / M) ^ 2];
gamma := Sqrt[k ^ 2 - Z ^ 2 * a ^ 2];

Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r / M / a] *
  r ^ gamma * Exp[-lambda[n] * r / M / a] /. n -> 1

1.0000000000000000 e-1.95695×10-6 r x0.999973

Plot[Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r / a] *
  r ^ gamma * Exp[-lambda[n] * r / a] /. n -> 1, {r, 0, 7.5}, PlotRange -> All]

```



# Randbedingungen

$r \ll 1$

```
Exit[]
```

```
a = 7 297 352 537.6 * 10 ^ -12; M = 510 998.910; k = -1; Z = 1;
```

```
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
Table[N[Energie[i]], {i, 10}]
```

```
{13.6059, 3.40148, 1.51176, 0.850365,
 0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}
```

```
e
```

```
f[u_, r_] := Simplify[{(Z * a / r + 2 - En) * u[[2]] - k / r * u[[1]],
  k / r * u[[2]] + (En - Z * a / r) * u[[1]]}];
```

```
L =  $\sqrt{2 \text{En} - \text{En}^2}$ ; L =.
```

```
u := x ^ (s + n) * {a[n], b[n]} * Exp[-x * L];
```

```
g1 = Collect[Simplify[(f[u, x] - D[u, x]) / x ^ (s - 1) / Exp[-x * L]], x]; g1 // MatrixForm
```

$$\begin{pmatrix} -x^{1+n} (-L a[n] + (-2 + \text{En}) b[n]) - x^n (k a[n] + n a[n] + s a[n] - a Z b[n]) \\ x^{1+n} (\text{En} a[n] + L b[n]) + x^n (-a Z a[n] + k b[n] - n b[n] - s b[n]) \end{pmatrix}$$

```
g2 = Table[Simplify[Sum[D[g1, {x, n2}] / n2!, {n, 0, 5}] /. x -> 0], {n2, 0, 5}];
```

```
g2 // MatrixForm
```

$-k a[0] - s a[0] + a Z b[0]$	$-a Z a[0] + (k - s) b[0]$
$L a[0] - (1 + k + s) a[1] + 2 b[0] - \text{En} b[0] + a Z b[1]$	$\text{En} a[0] - a Z a[1] + L b[0] - b[1] + k b[1]$
$L a[1] - (2 + k + s) a[2] + 2 b[1] - \text{En} b[1] + a Z b[2]$	$\text{En} a[1] - a Z a[2] + L b[1] - 2 b[2] + k b[2]$
$L a[2] - (3 + k + s) a[3] + 2 b[2] - \text{En} b[2] + a Z b[3]$	$\text{En} a[2] - a Z a[3] + L b[2] - 3 b[3] + k b[3]$
$L a[3] - (4 + k + s) a[4] + 2 b[3] - \text{En} b[3] + a Z b[4]$	$\text{En} a[3] - a Z a[4] + L b[3] - 4 b[4] + k b[4]$
$L a[4] - (5 + k + s) a[5] + 2 b[4] - \text{En} b[4] + a Z b[5]$	$\text{En} a[4] - a Z a[5] + L b[4] - 5 b[5] + k b[5]$

```
Det[{{-s - k, a Z}, {-a Z, -s + k}}]
```

```
 $-k^2 + s^2 + a^2 Z^2$ 
```

```
g4 = Simplify[Inverse[{{(n + s + k), -a}, {a, (n + s - k)}}].{{L, -En + 2}, {En, L}}]
```

$$\left\{ \left\{ \frac{a \text{En} + L (-k + n + s)}{a^2 - k^2 + (n + s)^2}, \frac{(-2 + \text{En}) k + a L - (-2 + \text{En}) (n + s)}{a^2 - k^2 + (n + s)^2} \right\}, \right.$$

$$\left. \left\{ \frac{-a L + \text{En} (k + n + s)}{a^2 - k^2 + (n + s)^2}, \frac{a (-2 + \text{En}) + L (k + n + s)}{a^2 - k^2 + (n + s)^2} \right\} \right\}$$

```
L =  $\sqrt{2 \text{En} - \text{En}^2}$ ; g5 = Simplify[Eigenvalues[g4]]; g5
```

$$\left\{ 0, \frac{2 (a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s))}{a^2 - k^2 + (n + s)^2} \right\}$$

```
g6 = Simplify[Transpose[Eigenvectors[g4]]]
```

$$\left\{ \left\{ \frac{a (-2 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (k + n + s)}{a \sqrt{-(-2 + \text{En}) \text{En}} - \text{En} (k + n + s)}, \frac{-a \text{En} + \sqrt{-(-2 + \text{En}) \text{En}} (k - n - s)}{a \sqrt{-(-2 + \text{En}) \text{En}} - \text{En} (k + n + s)} \right\}, \{1, 1\} \right\}$$

**Inverse** [{T1, T2}, {1, 1}]

$$\left\{ \left\{ \frac{1}{T1 - T2}, -\frac{T2}{T1 - T2} \right\}, \left\{ -\frac{1}{T1 - T2}, \frac{T1}{T1 - T2} \right\} \right\}$$

**Simplify** [Inverse[g6]].{0, 1}

$$\left\{ \frac{a \text{En} + \sqrt{-(-2 + \text{En}) \text{En}} (-k + n + s)}{2 (a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s))}, \frac{a (-2 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (k + n + s)}{2 (a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s))} \right\}$$

**Exit** []

$$\frac{a \text{En} + \sqrt{-(-2 + \text{En}) \text{En}} (-k + n + s)}{2 (a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s))} / \left( \frac{a \text{En} + \sqrt{-(-2 + \text{En}) \text{En}} (-k + n + s)}{2 (a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s))} /. n \rightarrow (n - 1) \right) * \frac{2 (a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s))}{a^2 - k^2 + (n + s)^2}$$

$$\left( 2 \left( 0.00729735 (-1 + \text{En}) + \sqrt{(2 - \text{En}) \text{En}} (-1 + n + s) \right) \left( 0.00729735 \text{En} + \sqrt{(2 - \text{En}) \text{En}} (1 + n + s) \right) \right) / \left( \left( 0.00729735 \text{En} + \sqrt{(2 - \text{En}) \text{En}} (n + s) \right) (-0.999947 + (n + s)^2) \right)$$

**Simplify** [Inverse[g6].g4.g6]

$$\left\{ \{0, 0\}, \left\{ 0, \left( 2 \left( a^2 (-1 + \text{En})^2 + 2 a (-1 + \text{En}) \sqrt{-(-2 + \text{En}) \text{En}} (n + s) - (-2 + \text{En}) \text{En} (n + s)^2 \right) \right) / \left( \left( a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s) \right) (a^2 - k^2 + (n + s)^2) \right) \right\} \right\}$$

$$\text{Limit} \left[ n \frac{a \text{En} + \sqrt{-(-2 + \text{En}) \text{En}} (-k + n + s)}{2 (a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s))} / \left( \frac{a \text{En} + \sqrt{-(-2 + \text{En}) \text{En}} (-k + n + s)}{2 (a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s))} /. n \rightarrow (n - 1) \right) * \frac{2 (a (-1 + \text{En}) + \sqrt{-(-2 + \text{En}) \text{En}} (n + s))}{a^2 - k^2 + (n + s)^2}, \{n \rightarrow \text{Infinity}\} \right]$$

$$\{ 2 \sqrt{-(-2 + \text{En}) \text{En}} \}$$

```
{0.000026626031501830738`, 6.656530030069163`*^-6,
 2.958447944512166`*^-6, 1.6641228148817078`*^-6, 1.0650367227027502`*^-6,
 7.39607883493143`*^-7, 5.433848546676501`*^-7, 4.1602871314605494`*^-7,
 3.2871384492150213`*^-7, 2.6625808169367104`*^-7}
```

```
#[[1, 2]] & /@ Solve[
  2 (a (-1 + En) + Sqrt[(2 - En) En] (-1 + n + s)) (a (-2 + En) + Sqrt[(2 - En) En] (k + n + s)) == 0,
  En] M /. s -> Sqrt[k^2 - (Z * a)^2] /. n -> 3
```

```
{6.04705, 1.022 × 106, 1.51176, 1.022 × 106}
```

```
#[[1, 2]] & /@ Solve[
  2 (a (-1 + En) + Sqrt[(2 - En) En] (-1 + n + s)) (a (-2 + En) + Sqrt[(2 - En) En] (k + n + s)) == 0,
  En] * M /. s -> Sqrt[k^2 - (Z * a)^2] /. n -> 2
```

$$\left\{ 2 M, \frac{2 a^2 M}{4 + a^2 + 4 k + 2 k^2 - a^2 Z^2 + 4 \sqrt{k^2 - a^2 Z^2} + 2 k \sqrt{k^2 - a^2 Z^2}}, \right.$$

$$\left( M \left( 2 + 2 a^2 + 4 \sqrt{k^2 - a^2 Z^2} + 2 (k^2 - a^2 Z^2) - \sqrt{\left( -4 a^2 \left( 1 + a^2 + k^2 - a^2 Z^2 + 2 \sqrt{k^2 - a^2 Z^2} \right) + \right.} \right. \right.$$

$$\left. \left. \left. \left( -2 - 2 a^2 - 4 \sqrt{k^2 - a^2 Z^2} - 2 (k^2 - a^2 Z^2) \right)^2 \right) \right) \right) /$$

$$\left( 2 \left( 1 + a^2 + k^2 - a^2 Z^2 + 2 \sqrt{k^2 - a^2 Z^2} \right) \right),$$

$$\left( M \left( 2 + 2 a^2 + 4 \sqrt{k^2 - a^2 Z^2} + 2 (k^2 - a^2 Z^2) + \sqrt{\left( -4 a^2 \left( 1 + a^2 + k^2 - a^2 Z^2 + 2 \sqrt{k^2 - a^2 Z^2} \right) + \right.} \right. \right.$$

$$\left. \left. \left. \left( -2 - 2 a^2 - 4 \sqrt{k^2 - a^2 Z^2} - 2 (k^2 - a^2 Z^2) \right)^2 \right) \right) \right) /$$

$$\left( 2 \left( 1 + a^2 + k^2 - a^2 Z^2 + 2 \sqrt{k^2 - a^2 Z^2} \right) \right) \}$$

```
s = Sqrt[k^2 - (Z * a)^2];
```

**En = .;**

**Simplify**[Inverse[{{Z \* a / En, (n + s - k) / En}, {(n + s + k) / (2 - En), Z \* a / (En - 2)}}]]

$$S[n_, En_] := \left\{ \left\{ \frac{a \, En \, Z}{n^2 + 2 \, n \sqrt{k^2 - a^2 \, Z^2}}, - \frac{(-2 + En) \left( -k + n + \sqrt{k^2 - a^2 \, Z^2} \right)}{n \left( n + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} \right\}, \right.$$

$$\left. \left\{ \frac{En \left( k + n + \sqrt{k^2 - a^2 \, Z^2} \right)}{n \left( n + 2 \sqrt{k^2 - a^2 \, Z^2} \right)}, \frac{a \, (-2 + En) \, Z}{n \left( n + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} \right\} \right\};$$

$$DS[n_, En_] := \left\{ \left\{ \frac{a \, Z}{n^2 + 2 \, n \sqrt{k^2 - a^2 \, Z^2}}, - \frac{\left( -k + n + \sqrt{k^2 - a^2 \, Z^2} \right)}{n \left( n + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} \right\}, \right.$$

$$\left. \left\{ \frac{\left( k + n + \sqrt{k^2 - a^2 \, Z^2} \right)}{n \left( n + 2 \sqrt{k^2 - a^2 \, Z^2} \right)}, \frac{a \, Z}{n \left( n + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} \right\} \right\};$$

**S[m, En] // MatrixForm**

**DS[m, En] // MatrixForm**

$$\left\{ \left\{ \frac{a \, En \, Z}{-k^2 + n^2 + 2 \, n \, s + s^2 + a^2 \, Z^2}, \frac{(-2 + En) \, (k - n - s)}{-k^2 + n^2 + 2 \, n \, s + s^2 + a^2 \, Z^2} \right\}, \right.$$

$$\left. \left\{ \frac{En \, (k + n + s)}{-k^2 + n^2 + 2 \, n \, s + s^2 + a^2 \, Z^2}, \frac{a \, (-2 + En) \, Z}{-k^2 + n^2 + 2 \, n \, s + s^2 + a^2 \, Z^2} \right\} \right\}$$

$$\left( \begin{array}{cc} \frac{a \, En \, Z}{m^2 + 2 \, m \sqrt{k^2 - a^2 \, Z^2}} & - \frac{(-2 + En) \left( -k + m + \sqrt{k^2 - a^2 \, Z^2} \right)}{m \left( m + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} \\ \frac{En \left( k + m + \sqrt{k^2 - a^2 \, Z^2} \right)}{m \left( m + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} & \frac{a \, (-2 + En) \, Z}{m \left( m + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} \end{array} \right)$$

$$\left( \begin{array}{cc} \frac{a \, Z}{m^2 + 2 \, m \sqrt{k^2 - a^2 \, Z^2}} & - \frac{-k + m + \sqrt{k^2 - a^2 \, Z^2}}{m \left( m + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} \\ \frac{k + m + \sqrt{k^2 - a^2 \, Z^2}}{m \left( m + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} & \frac{a \, Z}{m \left( m + 2 \sqrt{k^2 - a^2 \, Z^2} \right)} \end{array} \right)$$

**S[n, En].u**

$$S[n, En] \cdot \{x^{n+s} a[n], x^{n+s} b[n]\}$$

```

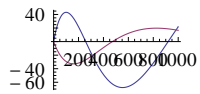
UN[R_, N_, En_] := Module[{u = {1, (k + s) / Z / a}, U = {1, (k + s) / Z / a} * R ^ s},
  For[n = 1, n < N, n++,
    u = S[n, En].u;
    U += u * R ^ (s + n);
  ];

  U]

U[r_, g_, En_] :=
Module[{u = {1, (k + s) / Z / a}, U = {0, 0}, DU = {0, 0}, du = {0, 0}, n = 0},
  Label[begin];
  U += u * r ^ (s + n);
  DU += du * r ^ (s + n);
  n++;
  du = DS[n, En].u + S[n, En].du;
  u = S[n, En].u;
  If[(#[[1]] > g || #[[2]] > g) & [Abs[u * R ^ (s + n) / U /. r -> R]], Goto[begin]];
  {n, U, DU}]

R = 1000; g = 0.01; rU = U[r, g, 1 / M];
Plot[{rU[[2, 1]], rU[[2, 2]] * 137}, {r, 0, R}, PlotRange -> All]

```



```

EN[iEn_, g2_] := Module[{rU, fU, n = 0, i, En = iEn, ll},
  Label[begin];
  fU = U[r, g, En];

  rU = fU /. r -> R;

  If[rU[[2, 1]] * rU[[2, 2]] > 0,

    En -= (rU[[2, 1]] + rU[[2, 2]]) / (rU[[3, 1]] + rU[[3, 2]]);

    n++
    Goto[begin];
  ];

  {n, En * M, Abs[rU[[2, 1]] - rU[[2, 2]]]}
]

```

```

R = 3000; g = 0.001; EN[13 / M, 0.1]
{8, 13.6059, 0.000441718}
- {0, 13.605873075061169

```

```
R = 2000; g = 0.001;
```

```
plot[{rU[[2, 1]], rU[[3, 1]]}, 100, R]
```

```
En = 4 / M; rU[[2, 1]]
```

```
129.072
```

```
n = 0; x
```

```
10
```

```
n = 0; While[x = n; x < 10, n++; Print[n]]
```

```
1
```

```
2
```

```
3
```

```
4
```

```
5
```

```
6
```

```
7
```

```
8
```

```
9
```

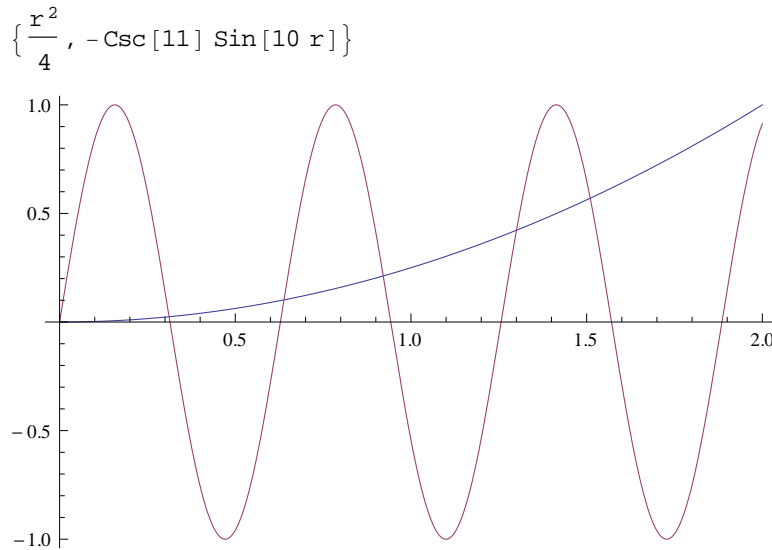
```
10
```



```

plot[liste_, R_] := Module[{nN = 100, table, max, st = {Red, Green, Blue}},
  liste / (Max[Abs[#]] & /@ (Table[# /. r -> i * R / nN, {i, 0, nN}] & /@ liste))
]
l1 = plot[{r ^ 2, Sin[10 * r]}, 2]
Plot[l1, {r, 0, 2}]

```



## r gegen Infinity

```

Exit[]

a = 7 297 352 537.6 * 10 ^ -12; M = 510 998.910; k = -1; Z = 1;

Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
Table[N[Energie[i]], {i, 10}]

{13.6059, 3.40148, 1.51176, 0.850365,
 0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}

$Assumptions = 1 > En > 0;

s = (En - 1) * Z * a / L; L := Sqrt[(2 - En) * En]; L =.; s =.

f[u_, r_] := Simplify[{(Z * a / r + 2 - En) * u[[2]] - k / r * u[[1]],
  k / r * u[[2]] + (En - Z * a / r) * u[[1]]}];

u =.

U = {(hb[r] - ha[r]) * L / En, hb[r] + ha[r]} * Exp[-r * L];

```

```
#[[2]] & /@ Simplify[Solve[Simplify[(f[U, r] - D[U, r]) / Exp[-r * L] * En * r] == 0,
  {ha'[r], hb'[r]}][[1]]] /. ha[r] -> u[[1]] /. hb[r] -> u[[2]] /. r -> rr
```

Part::partd: Part specification u[[1]] is longer than depth of object. >>

Part::partd: Part specification u[[2]] is longer than depth of object. >>

$$\left\{ \frac{1}{2 \text{En} L \text{rr}} \left( (\text{En}^3 \text{rr} + \text{En} L^2 \text{rr} + a L^2 Z - \text{En}^2 (2 \text{rr} + a Z)) u[[1]] + \right. \right. \\ \left. (\text{En}^3 \text{rr} + \text{En} L (2 k + L \text{rr}) - a L^2 Z - \text{En}^2 (2 \text{rr} + a Z)) u[[2]] \right), \\ \frac{1}{2 \text{En} L \text{rr}} \left( (-\text{En}^3 \text{rr} + \text{En} L (2 k - L \text{rr}) + a L^2 Z + \text{En}^2 (2 \text{rr} + a Z)) u[[1]] + \right. \\ \left. (-\text{En}^3 \text{rr} + 3 \text{En} L^2 \text{rr} - a L^2 Z + \text{En}^2 (2 \text{rr} + a Z)) u[[2]] \right) \}$$

$$\text{F}[u_, rr_] := \text{Simplify} \left[ \left\{ \frac{-a (-1 + \text{En}) Z u[[1]] + \left( \sqrt{-(-2 + \text{En}) \text{En}} k - a Z \right) u[[2]]}{\sqrt{-(-2 + \text{En}) \text{En} \text{rr}}}, \right. \right. \\ \left. \left( \left( \sqrt{-(-2 + \text{En}) \text{En}} k + a Z \right) u[[1]] + (4 \text{En} \text{rr} - 2 \text{En}^2 \text{rr} - a Z + a \text{En} Z) u[[2]] \right) / \right. \\ \left. \left( \sqrt{-(-2 + \text{En}) \text{En} \text{rr}} \right) \right\} \right]$$

```
r[x_] := 1/x; u := {a[n], b[n]} * x^(n+s)
```

```
g1 = Collect[Simplify[(F[u, r[x]] * D[r[x], x] - D[u, x]) * x^(2-s)], {a[n], b[n]}]; g1
```

$$\left\{ - \left( x^{1+n} \left( a (-1 + \text{En}) \sqrt{-(-2 + \text{En}) \text{En}} Z + L \left( \sqrt{-(-2 + \text{En}) \text{En}} n + a Z - a \text{En} Z \right) \right) a[n] \right) / \right. \\ \left( \sqrt{-(-2 + \text{En}) \text{En}} L \right) - \frac{x^{1+n} \left( \sqrt{-(-2 + \text{En}) \text{En}} k - a Z \right) b[n]}{\sqrt{-(-2 + \text{En}) \text{En}}}, \\ - \frac{x^{1+n} \left( \sqrt{-(-2 + \text{En}) \text{En}} k + a Z \right) a[n]}{\sqrt{-(-2 + \text{En}) \text{En}}} - \frac{1}{\sqrt{-(-2 + \text{En}) \text{En}} L} \\ x^n \left( -2 \text{En}^2 L + x \left( \sqrt{-(-2 + \text{En}) \text{En}} L n - a \sqrt{-(-2 + \text{En}) \text{En}} Z - a L Z \right) + \right. \\ \left. \text{En} \left( 4 L + a \sqrt{-(-2 + \text{En}) \text{En}} x Z + a L x Z \right) \right) b[n] \}$$

```
g2 := {x^n (-n) a[n] + x^n (-k + a Z / L) b[n],
  -x^{1+n} (k + a Z / L) a[n] + x^n (-n x - 2 s x + -2 * L) b[n]}; g2 // MatrixForm
```

$$\begin{pmatrix} -n x^n a[n] + x^n \left( -k + \frac{a Z}{L} \right) b[n] \\ -x^{1+n} \left( k + \frac{a Z}{L} \right) a[n] + x^n (-2 L - n x - 2 s x) b[n] \end{pmatrix}$$

```
a[n_] := (Z * a / L - k) / n * b[n]
```

```
g2[[2]]
```

$$x^n (-2 L - n x - 2 s x) b[n] - \frac{x^{1+n} \left( -k + \frac{a Z}{L} \right) \left( k + \frac{a Z}{L} \right) b[n]}{n}$$

```
M1 = {{L, 2 - En}, {En, L}}; Eigenvalues[M1]; B = Transpose[Eigenvectors[M1]];
```

`$Assumptions = 1 > En > 0`

`1 > En > 0`

`= >`

`Exit[]`

`a[n_] := (Z * a / L - k) / n * b[n]`

`b[n_] := b0 * Product[((k^2 - a^2 * Z^2 / L^2) / i - (i + 2 * s)) / 2 / L, {i, 1, n}]`

`b[4]`

$$\frac{1}{16 L^4} b_0 \left( -1 + k^2 - 2 s - \frac{a^2 Z^2}{L^2} \right) \left( -4 - 2 s + \frac{1}{4} \left( k^2 - \frac{a^2 Z^2}{L^2} \right) \right) \\ \left( -3 - 2 s + \frac{1}{3} \left( k^2 - \frac{a^2 Z^2}{L^2} \right) \right) \left( -2 - 2 s + \frac{1}{2} \left( k^2 - \frac{a^2 Z^2}{L^2} \right) \right)$$

`((k^2 - a^2 * Z^2 / L^2) / i - (i + 2 * s)) / 2 / L`

$$\frac{-i - 2 s + \frac{k^2 - \frac{a^2 Z^2}{L^2}}{i}}{2 L}$$

`Exit[];`

`En = .;`

$$\text{Solve} \left[ -i * \sqrt{(2 - \text{En}) \text{En}} - 2 a (-1 + \text{En}) + \frac{\sqrt{(2 - \text{En}) \text{En}} - \frac{a^2}{\sqrt{(2 - \text{En}) \text{En}}}}{i} == 0, \text{En} \right] // \text{MatrixForm}$$

$$\left( \begin{array}{l} \text{En} \rightarrow \frac{2 - 4 i^2 + 8 a^2 i^2 + 2 i^4 - \sqrt{(-2 + 4 i^2 - 8 a^2 i^2 - 2 i^4)^2 - 4 (1 - 2 i^2 + 4 a^2 i^2 + i^4) (a^2 + a^2 i^2 - 2 \sqrt{a^4 i^2 - a^6 i^2})}}{2 (1 - 2 i^2 + 4 a^2 i^2 + i^4)} \\ \text{En} \rightarrow \frac{2 - 4 i^2 + 8 a^2 i^2 + 2 i^4 + \sqrt{(-2 + 4 i^2 - 8 a^2 i^2 - 2 i^4)^2 - 4 (1 - 2 i^2 + 4 a^2 i^2 + i^4) (a^2 + a^2 i^2 - 2 \sqrt{a^4 i^2 - a^6 i^2})}}{2 (1 - 2 i^2 + 4 a^2 i^2 + i^4)} \\ \text{En} \rightarrow \frac{2 - 4 i^2 + 8 a^2 i^2 + 2 i^4 - \sqrt{(-2 + 4 i^2 - 8 a^2 i^2 - 2 i^4)^2 - 4 (1 - 2 i^2 + 4 a^2 i^2 + i^4) (a^2 + a^2 i^2 + 2 \sqrt{a^4 i^2 - a^6 i^2})}}{2 (1 - 2 i^2 + 4 a^2 i^2 + i^4)} \\ \text{En} \rightarrow \frac{2 - 4 i^2 + 8 a^2 i^2 + 2 i^4 + \sqrt{(-2 + 4 i^2 - 8 a^2 i^2 - 2 i^4)^2 - 4 (1 - 2 i^2 + 4 a^2 i^2 + i^4) (a^2 + a^2 i^2 + 2 \sqrt{a^4 i^2 - a^6 i^2})}}{2 (1 - 2 i^2 + 4 a^2 i^2 + i^4)} \end{array} \right)$$

,

```

ET[i_, k_, Z_, a_] := 
$$\left( 2 i^4 - 4 i^2 k^2 + 2 k^4 + 8 a^2 i^2 Z^2 - \sqrt{\left( (-2 i^4 + 4 i^2 k^2 - 2 k^4 - 8 a^2 i^2 Z^2)^2 - 4 (i^4 - 2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2) (a^2 i^2 Z^2 + a^2 k^2 Z^2 + 2 \sqrt{a^4 i^2 k^2 Z^4 - a^6 i^2 Z^6}) \right)} \right) / (2 (i^4 - 2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2))$$


Table[M * ET[n, -1, 1, a] - Energie[n+1], {n, 0, 10}]
{-5.67315 × 10-11, -1.9084 × 10-7, 5.30991 × 10-11,
-1.3482 × 10-10, 3.24187 × 10-11, -5.86517 × 10-11, 5.5719 × 10-11,
2.10913 × 10-11, 1.87879 × 10-11, 1.01339 × 10-10, -1.46405 × 10-12}

Series[M * ET[n, -1, 1, a] - Energie[n+1], {n, 0, 5}]
-5.67315 × 10-11 - 1.26477 × 10-10 n2 -
2.84217 × 10-14 n3 - 2.54019 × 10-10 n4 + 1.42109 × 10-14 n5 + O[n]6

M = 510 998.910;
s = (En - 1) * Z * a / L; L := Sqrt[(2 - En) * En];
a = 7 297 352 537.6 * 10 ^ -12; M = 510 998.910; Z = 1; k = -1;
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
Table[N[Energie[i]], {i, 10}]
{13.6059, 3.40148, 1.51176, 0.850365,
0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}

```

## Verhältnis bei r=0

```

a = 7 297 352 537.6 * 10 ^ -12; M = 510 998.910; k = -1; Z = 1;
s = Sqrt[k ^ 2 - (Z * a) ^ 2];

S[n_] := 
$$\left\{ \left\{ \frac{a \text{En} Z}{n^2 + 2 n \sqrt{k^2 - a^2 Z^2}}, -\frac{(-2 + \text{En}) \left( -k + n + \sqrt{k^2 - a^2 Z^2} \right)}{n \left( n + 2 \sqrt{k^2 - a^2 Z^2} \right)} \right\}, \right.$$


$$\left. \left\{ \frac{\text{En} \left( k + n + \sqrt{k^2 - a^2 Z^2} \right)}{n \left( n + 2 \sqrt{k^2 - a^2 Z^2} \right)}, \frac{a (-2 + \text{En}) Z}{n \left( n + 2 \sqrt{k^2 - a^2 Z^2} \right)} \right\} \right\} /. \text{En} \rightarrow \text{Enn};$$


S[
10]
{0.0000608115 Enn, -0.1 (-2 + Enn)}, {0.0833335 Enn, 0.0000608115 (-2 + Enn)}

```

```

Enn = .; u = {1, (k + s) / Z / a}; U = u;
For [n = 1, n < 3, n++,
  u = S[n].u;
  U = Simplify[U + u];
]; n = .;
Simplify[U[[1]] / U[[2]]]
- 
$$\frac{91.3504 (-3.66275 + \text{Enn}) (1.62629 + \text{Enn})}{(-0.0109728 + \text{Enn}) (181.383 + \text{Enn})}$$

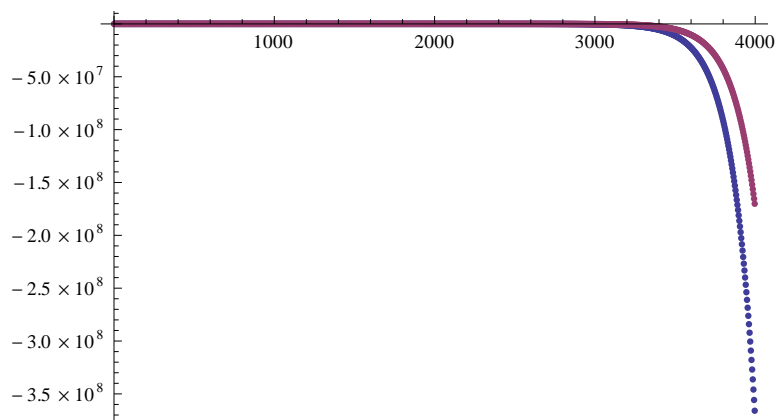

```

## Runge von links

```

f[u_, r_] := Simplify[{(Z * a / r + 2 - Enn) * u[[2]] - k / r * u[[1]],
  k / r * u[[2]] + (Enn - Z * a / r) * u[[1]]}];
k = -1; Z = 1; U = .;
n = 1000;
h = 4000 / n;
Enn = 13.605 / M;
u = {(91.35044102604739` (-3.662751763692355` + Enn) (1.6262886176197724` + Enn)) /
  ((-0.0109728221664999` + Enn) (181.38339842774778` + Enn)), -1};
r = 1; U = {{r, u}};
Do[
  k0 = h * f[u, r]; k1 = h * f[u + k0 / 2, r + h / 2];
  k2 = h * f[u + k1 / 2, r + h / 2]; k3 = h * f[u + k2, r + h];
  u += 1 / 6 * (k0 + 2 * k1 + 2 * k2 + k3); r += h;
  AppendTo[U, {r, u}], {n}]; x = .;
ListPlot[
  Table[{#[[1]], 137^(i - 2) * #[[2, i]]} & /@ U[[1 ;; n]], {i, 2}] // N, PlotRange -> All]

```



```
Sum[A[n] * r^n / n!, {n, 0, 10}]
```

$$\begin{aligned}
 & A[0] + r A[1] + \frac{1}{2} r^2 A[2] + \frac{1}{6} r^3 A[3] + \frac{1}{24} r^4 A[4] + \\
 & \frac{1}{120} r^5 A[5] + \frac{1}{720} r^6 A[6] + \frac{r^7 A[7]}{5040} + \frac{r^8 A[8]}{40320} + \frac{r^9 A[9]}{362880} + \frac{r^{10} A[10]}{3628800}
 \end{aligned}$$

**D[%, {r, 4}]**

$$A[4] + r A[5] + \frac{1}{2} r^2 A[6] + \frac{1}{6} r^3 A[7] + \frac{1}{24} r^4 A[8] + \frac{1}{120} r^5 A[9] + \frac{1}{720} r^6 A[10]$$

**Sum[A[n + 4] \* r^n / n!, {n, 0, 10}]**

$$A[4] + r A[5] + \frac{1}{2} r^2 A[6] + \frac{1}{6} r^3 A[7] + \frac{1}{24} r^4 A[8] + \\ \frac{1}{120} r^5 A[9] + \frac{1}{720} r^6 A[10] + \frac{r^7 A[11]}{5040} + \frac{r^8 A[12]}{40320} + \frac{r^9 A[13]}{362880} + \frac{r^{10} A[14]}{3628800}$$