```
Exit[]
```

```
n = 6;
```

$$\text{BSE} = -r\, V[t, S] + r\, S\, V^{(0,1)}[t, S] + \frac{1}{2} S^2 \sigma^2 V^{(0,2)}[t, S] + V^{(1,0)}[t, S] == 0;$$

```
VS = Join[Table[Solve[D[BSE, {S, nn - 2}], D[V[t, S], {S, nn}]][[1, 1]], {nn, n, 3, -1}],
   {Solve[D[BSE, t, {S, 2}], D[V[t, S], {S, 2}, {t, 2}]][[1, 1]]}];
VS = #[[1]] → Simplify[#[[2]] /. VS] & /@ VS;
VS = #[[1]] → Simplify[#[[2]] /. VS] & /@ VS;
SN[x_] := CDF[NormalDistribution[], x];
d[sk_, σ_, r_, T_] := (Log[sk] + (r + σ^2 / 2) T) / σ / Sqrt[T];
BS[σ_, SK_, r_, T_] :=
  SK SN[d[SK, σ, r, T]] - Exp[-r T] SN[d[SK, σ, r, T] - σ Sqrt[T]];
Moments = Table[W^nn → Limit[D[Exp[t^2 / 2], {t, nn}], t -> 0], {nn, 2 n, 1, -1}]
ExpValue[a_] := Simplify[a - a + Expand[Normal[a] /. Moments]
Cov[a_, b_] := Simplify[ExpValue[a b] - ExpValue[a] ExpValue[b]]
Var[a_] := Cov[a, a]
dX = μ dt^2 + σ W dt;
dS = S (Series[Exp[dX], {dt, 0, n}] - 1);
dV = Series[V[t + dt^2, S + dS], {dt, 0, n}] - V[t, S];
dP[Δ_] := dV - Δ dS - (V[t, S] - Δ S) (Exp[dt^2 r] - 1)
VarHedgingError[Δ_] := Var[dP[Δ]]
```

$$\{W^{12} \to 10\,395,\ W^{11} \to 0,\ W^{10} \to 945,\ W^9 \to 0,\ W^8 \to 105,$$
$$W^7 \to 0,\ W^6 \to 15,\ W^5 \to 0,\ W^4 \to 3,\ W^3 \to 0,\ W^2 \to 1,\ W \to 0\}$$

■ Analysis

```
Cov[dS, dV]
```

$$S^2 \sigma^2 V^{(0,1)}[t, S]\, dt^2 + \frac{1}{2} S^2 \sigma^2$$
$$\left(\left(4\mu + 3\sigma^2\right) V^{(0,1)}[t, S] + 2S\left(\mu + 2\sigma^2\right) V^{(0,2)}[t, S] + S^2 \sigma^2 V^{(0,3)}[t, S] + 2 V^{(1,1)}[t, S]\right)$$
$$dt^4 + \frac{1}{24} S^2 \sigma^2 \left(4\left(12\mu^2 + 18\mu\sigma^2 + 7\sigma^4\right) V^{(0,1)}[t, S] +$$
$$3S\left(20\mu^2 + 60\mu\sigma^2 + 41\sigma^4\right) V^{(0,2)}[t, S] + 12 S^2 \mu^2 V^{(0,3)}[t, S] + 96 S^2 \mu \sigma^2 V^{(0,3)}[t, S] +$$
$$121 S^2 \sigma^4 V^{(0,3)}[t, S] + 12 S^3 \mu \sigma^2 V^{(0,4)}[t, S] + 36 S^3 \sigma^4 V^{(0,4)}[t, S] +$$
$$3 S^4 \sigma^4 V^{(0,5)}[t, S] + 48 \mu V^{(1,1)}[t, S] + 36 \sigma^2 V^{(1,1)}[t, S] + 24 S \mu V^{(1,2)}[t, S] +$$
$$48 S \sigma^2 V^{(1,2)}[t, S] + 12 S^2 \sigma^2 V^{(1,3)}[t, S] + 12 V^{(2,1)}[t, S]\Big)\, dt^6 + O[dt]^8$$

```
Δ0 = Simplify[Cov[dS, dV] / Var[dS]]
```

$$V^{(0,1)}[t, S] + \left(S\left(\mu + 2\sigma^2\right) V^{(0,2)}[t, S] + \frac{1}{2} S^2 \sigma^2 V^{(0,3)}[t, S] + V^{(1,1)}[t, S]\right) dt^2 + \frac{1}{24}\left(3S\left(4\mu^2 + 16\mu\sigma^2 + 17\sigma^4\right) V^{(0,2)}[t, S] + S^2\left(12\mu^2 + 72\mu\sigma^2 + 103\sigma^4\right) V^{(0,3)}[t, S] + 3\left(4S^3\sigma^2\left(\mu + 3\sigma^2\right) V^{(0,4)}[t, S] + S^4\sigma^4 V^{(0,5)}[t, S] + 4\left(2S\left(\mu + 2\sigma^2\right) V^{(1,2)}[t, S] + S^2\sigma^2 V^{(1,3)}[t, S] + V^{(2,1)}[t, S]\right)\right)\right) dt^4 + O[dt]^6$$

```
var10[S1_, σ1_, r1_, μ1_, t1_, t_] :=
 Normal[Simplify[VarHedgingError[Δ0] /. VS] /. V → (BS[σ, #2, r, 1 - #1] &)] /. σ → σ1 /.
      r → r1 /. t → t1 /. μ → μ1 /. dt → t /. S → S1

var8BS[S1_, σ1_, r1_, μ1_, t1_, t_] :=
 Normal[Simplify[VarHedgingError[V^(0,1)[t, S]] /. VS] /. V → (BS[σ, #2, r, 1 - #1] &)] /.
       σ → σ1 /. r → r1 /. t → t1 /. μ → μ1 /. dt → t /. S → S1

σ1 = 0.2; μ1 = .05; r1 = -0.25; t1 = 0; S1 = 1;

v8BS = var8BS[S1, σ1, r1, μ1, t1, t]
```

$$0.00281611\, t^4 + 0.000368441\, t^6 + 0.000206457\, t^8$$

```
v8 = var8[S1, σ1, r1, μ1, t1, t]
```

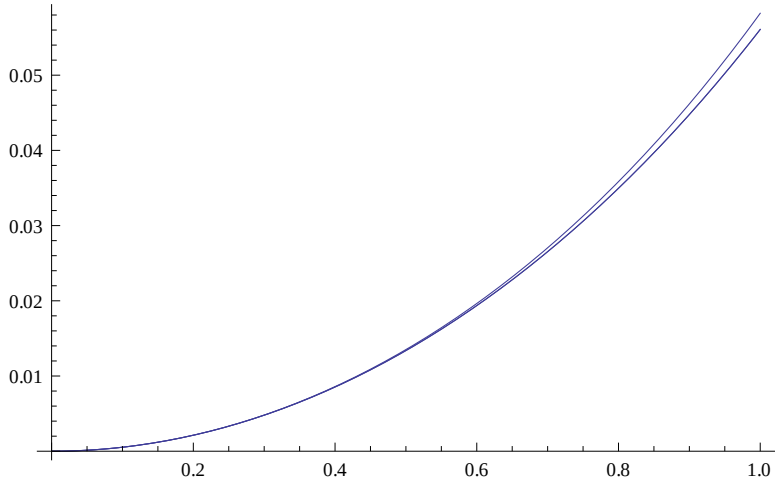
$$0.00281611\, t^4 + 0.000143152\, t^6 + 0.000185994\, t^8$$

```
v10 = var10[S1, σ1, r1, μ1, t1, t]
```

$$0.00281611\, t^4 + 0.000143152\, t^6 + 0.000185994\, t^8$$

```
BS[σ, S, r, 1 - t] /. σ → σ1 /. r → r1 /. t → t1 /. μ → μ1 /. dt → t /. S → S1
Plot[Sqrt /@ {v8, v8BS, v10}, {t, 0, Sqrt[1]}]
```

0.0114254




```
0.10450583572185568` - Sqrt[v8 /. t → 1]
```

0.0484233

```
(*Test if equation is solved: *)
VarEq := Simplify[D[#, t] + σ^2 S^2 D[#, {S, 2}] +
    μ S D[#, S] - 2 r # + σ^2 S^2 (D[V[t, S], S] - V^(0,1)[t, S])^2] &
Simplify[VarEq[VarHedgingError[V^(0,1)[t, S]]] /. EW[4] → 3];

var10[V_] :=
 1/2 S^4 σ^4 V^(0,2)[t, S]^2 dt^4 - 1/12 (S^2 σ^2 (S^2 (-32 r^2 + σ^4 + 8 r (3 μ + σ^2)) V^(0,2)[t, S]^2 +
       4 S (-10 r + 6 μ + 5 σ^2) V^(0,2)[t, S] V^(1,1)[t, S] - 8 V^(1,1)[t, S]^2)) dt^6 +
 1/12 S^2 (2 r S^2 (44 r^3 + 2 μ σ^4 + σ^6 - 4 r^2 (16 μ + 5 σ^2) + r (24 μ^2 + 16 μ σ^2 + σ^4))
     V^(0,2)[t, S]^2 - S V^(0,2)[t, S] ((-152 r^3 + 12 μ^2 σ^2 + 16 μ σ^4 + 5 σ^6 +
         4 r^2 (52 μ + 31 σ^2) - 2 r (36 μ^2 + 56 μ σ^2 + 19 σ^4)) V^(1,1)[t, S] +
       S σ^2 (36 r^2 + 12 μ^2 + 20 μ σ^2 + 9 σ^4 - 8 r (5 μ + 4 σ^2)) V^(1,2)[t, S]) +
    2 ((32 r^2 + 12 μ^2 + 28 μ σ^2 + 15 σ^4 - 4 r (10 μ + 9 σ^2)) V^(1,1)[t, S]^2 +
       4 S σ^2 (-3 r + 2 (μ + σ^2)) V^(1,1)[t, S] V^(1,2)[t, S] + S^2 σ^4 V^(1,2)[t, S]^2)) dt^8 +
 1/(720 σ^2) S^2 (S^2 (7104 r^6 - 192 r^5 (95 μ + 177 σ^2) + 4 r σ^8 (197280 μ + 965759 σ^2) +
       σ^10 (345600 μ + 1843201 σ^2) + 96 r^4 (160 μ^2 + 550 μ σ^2 - 137 σ^4) - 96 r^3
        (40 μ^3 + 40 μ^2 σ^2 - 2545 μ σ^4 - 6853 σ^6) - 32 r^2 (15 μ^2 σ^4 - 20235 μ σ^6 - 83336 σ^8))
     V^(0,2)[t, S]^2 - 4 (240 S^2 σ^2 (r^4 + 14 r^3 σ^2 + 71 r^2 σ^4 + 154 r σ^6 + 120 σ^8)
       V^(0,3)[t, S] V^(1,1)[t, S] - 2 (888 r^4 + 180 μ^3 σ^2 + 540 μ^2 σ^4 + 525 μ σ^6 -
         28633 σ^8 - 456 r^3 (5 μ + 11 σ^2) + 30 r^2 (52 μ^2 + 94 μ σ^2 - 527 σ^4) -
         6 r (60 μ^3 + 240 μ^2 σ^2 + 275 μ σ^4 + 6257 σ^6)) V^(1,1)[t, S]^2 -
       6 S^2 σ^4 ((44 r^2 - 60 r μ + 20 μ^2 - 76 r σ^2 + 60 μ σ^2 + 41 σ^4) V^(1,2)[t, S]^2 +
         S σ^2 (-14 r + 10 μ + 13 σ^2) V^(1,2)[t, S] V^(1,3)[t, S] + S^2 σ^4 V^(1,3)[t, S]^2) +
       3 S σ^2 V^(1,1)[t, S] ((552 r^3 - 120 μ^3 - 500 μ^2 σ^2 - 630 μ σ^4 - 253 σ^6 - 4
           r^2 (250 μ + 249 σ^2) + 2 r (300 μ^2 + 700 μ σ^2 + 389 σ^4)) V^(1,2)[t, S] +
         2 S σ^2 ((-48 r^2 + 60 r μ - 20 μ^2 + 78 r σ^2 - 40 μ σ^2 + 9 σ^4) V^(1,3)[t, S] +
           5 σ^2 (2 S (r + 3 σ^2) V^(1,4)[t, S] + S^2 σ^2 V^(1,5)[t, S] + 2 V^(2,3)[t, S])))) +
     2 S V^(0,2)[t, S] (120 S^2 σ^2 (-4 r^2 + 6 r μ + 35 r σ^2 + 12 μ σ^2 + 82 σ^4)
       (r^3 + 12 r^2 σ^2 + 47 r σ^4 + 60 σ^6) V^(0,3)[t, S] +
      120 S^3 σ^4 (r^4 + 18 r^3 σ^2 + 119 r^2 σ^4 + 342 r σ^6 + 360 σ^8) V^(0,4)[t, S] +
      7104 r^5 V^(1,1)[t, S] - 18240 r^4 μ V^(1,1)[t, S] + 13920 r^3 μ^2 V^(1,1)[t, S] -
      3360 r^2 μ^3 V^(1,1)[t, S] - 37056 r^4 σ^2 V^(1,1)[t, S] + 36960 r^3 μ σ^2 V^(1,1)[t, S] -
      8400 r^2 μ^2 σ^2 V^(1,1)[t, S] + 960 r μ^3 σ^2 V^(1,1)[t, S] - 75624 r^3 σ^4 V^(1,1)[t, S] +
      96000 r^2 μ σ^4 V^(1,1)[t, S] + 1920 r μ^2 σ^4 V^(1,1)[t, S] -
      120 μ^3 σ^4 V^(1,1)[t, S] + 77532 r^2 σ^6 V^(1,1)[t, S] + 222960 r μ σ^6 V^(1,1)[t, S] -
      240 μ^2 σ^6 V^(1,1)[t, S] + 665020 r σ^8 V^(1,1)[t, S] + 172650 μ σ^8 V^(1,1)[t, S] +
      719974 σ^10 V^(1,1)[t, S] - 3312 r^4 S σ^2 V^(1,2)[t, S] + 6720 r^3 S μ σ^2 V^(1,2)[t, S] -
      4320 r^2 S μ^2 σ^2 V^(1,2)[t, S] + 960 r S μ^3 σ^2 V^(1,2)[t, S] + 9480 r^3 S σ^4
```

$$V^{(1,2)}[t,S] - 7800\, r^2 S \mu \sigma^4 V^{(1,2)}[t,S] + 3480\, r S \mu^2 \sigma^4 V^{(1,2)}[t,S] - 360\, S \mu^3 \sigma^4 V^{(1,2)}[t,S] + 25140\, r^2 S \sigma^6 V^{(1,2)}[t,S] + 3600\, r S \mu \sigma^6 V^{(1,2)}[t,S] - 840\, S \mu^2 \sigma^6 V^{(1,2)}[t,S] + 83130\, r S \sigma^8 V^{(1,2)}[t,S] - 690\, S \mu \sigma^8 V^{(1,2)}[t,S] + 86142\, S \sigma^{10} V^{(1,2)}[t,S] + 696\, r^3 S^2 \sigma^4 V^{(1,3)}[t,S] - 1080\, r^2 S^2 \mu \sigma^4 V^{(1,3)}[t,S] + 600\, r S^2 \mu^2 \sigma^4 V^{(1,3)}[t,S] - 120\, S^2 \mu^3 \sigma^4 V^{(1,3)}[t,S] - 1248\, r^2 S^2 \sigma^6 V^{(1,3)}[t,S] + 1320\, r S^2 \mu \sigma^6 V^{(1,3)}[t,S] - 300\, S^2 \mu^2 \sigma^6 V^{(1,3)}[t,S] + 474\, r S^2 \sigma^8 V^{(1,3)}[t,S] + 270\, S^2 \mu \sigma^8 V^{(1,3)}[t,S] + 963\, S^2 \sigma^{10} V^{(1,3)}[t,S] - 180\, r^2 S^3 \sigma^6 V^{(1,4)}[t,S] + 180\, r S^3 \mu \sigma^6 V^{(1,4)}[t,S] + 540\, S^3 \mu \sigma^8 V^{(1,4)}[t,S] + 1500\, S^3 \sigma^{10} V^{(1,4)}[t,S] - 30\, r S^4 \sigma^8 V^{(1,5)}[t,S] + 90\, S^4 \mu \sigma^8 V^{(1,5)}[t,S] + 435\, S^4 \sigma^{10} V^{(1,5)}[t,S] + 30\, S^5 \sigma^{10} V^{(1,6)}[t,S] - 120\, r S^2 \sigma^6 V^{(2,3)}[t,S] + 180\, S^2 \mu \sigma^6 V^{(2,3)}[t,S] + 510\, S^2 \sigma^8 V^{(2,3)}[t,S] + 90\, S^3 \sigma^8 V^{(2,4)}[t,S] + 60\, S \sigma^6 V^{(3,2)}[t,S]\Big)\Big)\, dt^{10} + O[dt]^{11}$$

```
1.1 - 55 * (Exp[-0.25] - 1)
```

```
13.266
```