

```

Exit[]

a = 0.0072973525376; M = 510998.910;

En[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k^2 - (Z * a)^2]))^2]);
Table[N[En[i]], {i, 4}]

{13.6059, 3.40148, 1.51176, 0.850365}

M = {{-k / r, En + m + Z * a / r}, {- (En - m + Z * a / r), k / r}};

B = r^2 * D[M, r]

{{k, -a Z}, {a Z, -k}}

S = Eigensystem[B, -2][[2]];
v = Transpose[{S[[2]], S[[1]]}] * a * Z; v // MatrixForm

$$\begin{pmatrix} k + \sqrt{k^2 - a^2 Z^2} & k - \sqrt{k^2 - a^2 Z^2} \\ a Z & a Z \end{pmatrix}$$


A = Transpose[{{Z * a,  $\left(k - \sqrt{k^2 - a^2 Z^2}\right)$ }}, {{ $\left(k - \sqrt{k^2 - a^2 Z^2}\right)$ , Z * a}}]]; A // MatrixForm

$$\begin{pmatrix} a Z & k - \sqrt{k^2 - a^2 Z^2} \\ k - \sqrt{k^2 - a^2 Z^2} & a Z \end{pmatrix}$$


Simplify[v.{u1, u2}]

{k (u1 + u2) + (u1 - u2)  $\sqrt{k^2 - a^2 Z^2}$ , a (u1 + u2) Z}

Simplify[Inverse[v].B.v]

{{ $\sqrt{k^2 - a^2 Z^2}$ , 0}, {0,  $-\sqrt{k^2 - a^2 Z^2}$ }}

Expand[(M.M)[[1, 1]]]


$$-En^2 + m^2 + \frac{k^2}{r^2} - \frac{2 a En Z}{r} - \frac{a^2 Z^2}{r^2}$$


Exit[]

$Assumptions = k^2 > Z^2 * a^2 && Z > 0 && a > 0

k^2 > a^2 Z^2 && Z > 0 && a > 0

M = {{-(m + 1) / r - b * r, En - V[r]}, {-En + V[r], m / r + b * r}}; M // MatrixForm

$$\begin{pmatrix} \frac{-1-m}{r} - b r & En - V[r] \\ -En + V[r] & \frac{m}{r} + b r \end{pmatrix}$$


```

**K1 = Simplify [M.M + D[M, r]]**

$$\left\{ \left\{ b + 2 b m + b^2 r^2 + \frac{2 + 3 m + m^2 - E n^2 r^2}{r^2} + 2 E n V[r] - V[r]^2, -\frac{E n - V[r] + r V'[r]}{r} \right\}, \right. \\ \left. \left\{ \frac{E n - V[r] + r V'[r]}{r}, b - \frac{m}{r^2} + \left( \frac{m}{r} + b r \right)^2 - (E n - V[r])^2 \right\} \right\}$$

**K2 = {{(3 \* m + 2) / r ^ 2, -(D[V[r], r] + (E n - V[r]) / r)},**  
**{(D[V[r], r] + (E n - V[r]) / r), -m / r ^ 2}}**

$$\left\{ \left\{ \frac{2 + 3 m}{r^2}, -\frac{E n - V[r]}{r} - V'[r] \right\}, \left\{ \frac{E n - V[r]}{r} + V'[r], -\frac{m}{r^2} \right\} \right\}$$

**Expand [K1 - K2] // MatrixForm**

$$\begin{pmatrix} b - E n^2 + 2 b m + \frac{m^2}{r^2} + b^2 r^2 + 2 E n V[r] - V[r]^2 & 0 \\ 0 & b - E n^2 + 2 b m + \frac{m^2}{r^2} + b^2 r^2 + 2 E n V[r] - V[r]^2 \end{pmatrix}$$

**V[r\_] := r ^ p;**

**K3 = Simplify [K2 \* r ^ 2]**

$$\{\{2 + 3 m, -r (E n + (-1 + p) r^p)\}, \{r (E n + (-1 + p) r^p), -m\}\}$$

**{va, ve} = Simplify [Eigensystem [K3]]; va**

**ve \* (r (E n + (-1 + p) r ^ p)) // MatrixForm**

$$\left\{ 1 + m - \sqrt{1 + 4 m + 4 m^2 - E n^2 r^2 - 2 E n (-1 + p) r^{2+p} - r^{2+2 p} + 2 p r^{2+2 p} - p^2 r^{2+2 p}}, \right. \\ \left. 1 + m + \sqrt{1 + 4 m + 4 m^2 - E n^2 r^2 - 2 E n (-1 + p) r^{2+p} - r^{2+2 p} + 2 p r^{2+2 p} - p^2 r^{2+2 p}} \right\} \\ \begin{pmatrix} 1 + 2 m - \sqrt{1 + 4 m + 4 m^2 - E n^2 r^2 - 2 E n (-1 + p) r^{2+p} - r^{2+2 p} + 2 p r^{2+2 p} - p^2 r^{2+2 p}} & r (E n + (-1 + p) r^p) \\ 1 + 2 m + \sqrt{1 + 4 m + 4 m^2 - E n^2 r^2 - 2 E n (-1 + p) r^{2+p} - r^{2+2 p} + 2 p r^{2+2 p} - p^2 r^{2+2 p}} & r (E n + (-1 + p) r^p) \end{pmatrix}$$