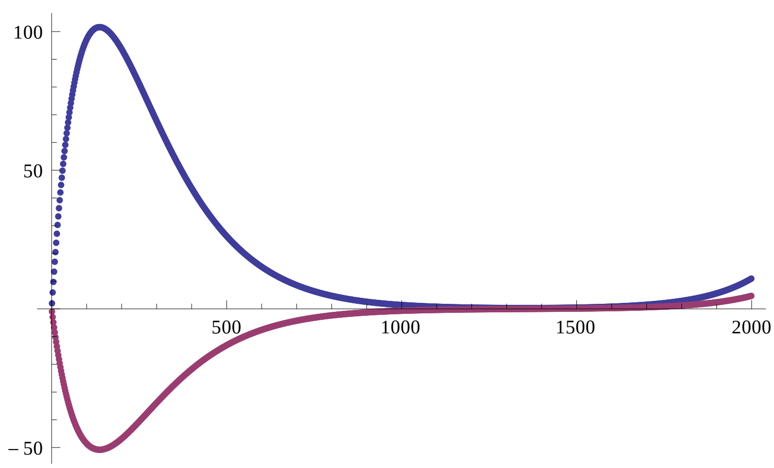
```
Exit[]

a = 7297352537.6 * 10^-12; M = 510998.910; Z = 1; k = -1;
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k^2 - (Z * a)^2]))^2]);
Table[N[Energie[i]], {i, 10}]

{13.6059, 3.40148, 1.51176, 0.850365,
 0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}

f[u_, r_] := Simplify[{(Z * a / r + 2 - Enn) * u[[2]] - k / r * u[[1]],
    k / r * u[[2]] + (Enn - Z * a / r) * u[[1]]}];
k = -1; Z = 1; U =.
n = 1000;
h = 2000 / n;
Enn = 13.6059 / M;
u = {(91.35044102604739` (-3.662751763692355` + Enn) (1.6262886176197724` + Enn)) /
    ((-0.0109728221664999` + Enn) (181.38339842774778` + Enn)), -1};
r = 1; U = {{r, u}};
Do[
 k0 = h * f[u, r]; k1 = h * f[u + k0 / 2, r + h / 2];
 k2 = h * f[u + k1 / 2, r + h / 2]; k3 = h * f[u + k2, r + h];
 u += 1 / 6 * (k0 + 2 * k1 + 2 * k2 + k3); r += h;
 AppendTo[U, {r, u}], {n}]; x =.;
ListPlot[
 Table[{#[[1]], 137^(i - 2) * #[[2, i]]} & /@ U[[1 ;; n]], {i, 2}] // N, PlotRange → All]
```




```
X[1000]
```

$\frac{\pi}{4}$

```
X[1000] / n
```

$\frac{1}{10}$

```
-g[

lambda[n_] := Sqrt[1 - (1 - En[n] / M) ^ 2];
gamma := Sqrt[k ^ 2 - Z ^ 2 * a ^ 2];
f[r_, n_, k_] := -r ^ gamma * Exp[-lambda[n] * r] * (((n - 1 + gamma) / (1 - En[n] / M) - k) *
      Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r] +
    (n - 1) * Hypergeometric1F1[1 - (n - 1), 2 * gamma + 1, 2 * lambda[n] * r]);
g[r_, n_, k_] := Sqrt[2 * M / En[n] - 1] * r ^ gamma * Exp[-lambda[n] * r] *
  (((n - 1 + gamma) / (1 - En[n] / M) - k) *
      Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r] -
    (n - 1) * Hypergeometric1F1[1 - (n - 1), 2 * gamma + 1, 2 * lambda[n] * r]);

g[1, 1, -1] / f[1, 1, -1]

-274.068
```

Randbedingungen

r << 1

```
Exit[]

a = 7297352537.6 * 10 ^ -12; M = 510998.910; k = -1; Z = 1;

Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
Table[N[Energie[i]], {i, 10}]

{13.6059, 3.40148, 1.51176, 0.850365,
 0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}

f[u_, r_] := Simplify[{(Z * a / r + 2 - En) * u[[2]] - k / r * u[[1]],
    k / r * u[[2]] + (En - Z * a / r) * u[[1]]}];

u := r ^ (s + n) * {a[n], b[n]}

Collect[Simplify[(f[u, r] - D[u, r]) / r ^ (s - 1)], r] // MatrixForm
```

$$\begin{pmatrix} -(-2+\text{En})\, r^{1+n}\, b[n] - r^n\, ((k+n+s)\, a[n] - a\, Z\, b[n]) \\ \text{En}\, r^{1+n}\, a[n] + r^n\, (-a\, Z\, a[n] + (k-n-s)\, b[n]) \end{pmatrix}$$

=>

```
13.605873075061169`

s = Sqrt[k ^ 2 - (Z * a) ^ 2];
```

```
En =.;
Simplify[Inverse[{{Z * a / En, (n + s - k) / En}, {(n + s + k) / (2 - En), Z * a / (En - 2)}}]]
```

$$S[n\_, En\_] := \left\{\left\{\frac{a\ En\ Z}{n^2 + 2\ n\ \sqrt{k^2 - a^2\ Z^2}}, -\frac{(-2 + En)\ \left(-k + n + \sqrt{k^2 - a^2\ Z^2}\right)}{n\ \left(n + 2\ \sqrt{k^2 - a^2\ Z^2}\right)}\right\},\right.$$
$$\left.\left\{\frac{En\ \left(k + n + \sqrt{k^2 - a^2\ Z^2}\right)}{n\ \left(n + 2\ \sqrt{k^2 - a^2\ Z^2}\right)}, \frac{a\ (-2 + En)\ Z}{n\ \left(n + 2\ \sqrt{k^2 - a^2\ Z^2}\right)}\right\}\right\};$$

$$DS[n\_, En\_] := \left\{\left\{\frac{a\ Z}{n^2 + 2\ n\ \sqrt{k^2 - a^2\ Z^2}}, -\frac{\left(-k + n + \sqrt{k^2 - a^2\ Z^2}\right)}{n\ \left(n + 2\ \sqrt{k^2 - a^2\ Z^2}\right)}\right\},\right.$$
$$\left.\left\{\frac{\left(k + n + \sqrt{k^2 - a^2\ Z^2}\right)}{n\ \left(n + 2\ \sqrt{k^2 - a^2\ Z^2}\right)}, \frac{a\ Z}{n\ \left(n + 2\ \sqrt{k^2 - a^2\ Z^2}\right)}\right\}\right\};$$

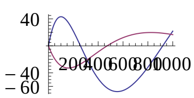
```
S[m, En] // MatrixForm
DS[m, En] // MatrixForm
```

$$\left\{\left\{\frac{a\ En\ Z}{-k^2 + n^2 + 2\ n\ s + s^2 + a^2\ Z^2}, \frac{(-2 + En)\ (k - n - s)}{-k^2 + n^2 + 2\ n\ s + s^2 + a^2\ Z^2}\right\},\right.$$
$$\left.\left\{\frac{En\ (k + n + s)}{-k^2 + n^2 + 2\ n\ s + s^2 + a^2\ Z^2}, \frac{a\ (-2 + En)\ Z}{-k^2 + n^2 + 2\ n\ s + s^2 + a^2\ Z^2}\right\}\right\}$$

$$\begin{pmatrix} \frac{a\ En\ Z}{m^2+2\ m\ \sqrt{k^2-a^2\ Z^2}} & -\frac{(-2+En)\ \left(-k+m+\sqrt{k^2-a^2\ Z^2}\right)}{m\ \left(m+2\ \sqrt{k^2-a^2\ Z^2}\right)} \\ \frac{En\ \left(k+m+\sqrt{k^2-a^2\ Z^2}\right)}{m\ \left(m+2\ \sqrt{k^2-a^2\ Z^2}\right)} & \frac{a\ (-2+En)\ Z}{m\ \left(m+2\ \sqrt{k^2-a^2\ Z^2}\right)} \end{pmatrix}$$

$$\begin{pmatrix} \frac{a\ Z}{m^2+2\ m\ \sqrt{k^2-a^2\ Z^2}} & -\frac{-k+m+\sqrt{k^2-a^2\ Z^2}}{m\ \left(m+2\ \sqrt{k^2-a^2\ Z^2}\right)} \\ \frac{k+m+\sqrt{k^2-a^2\ Z^2}}{m\ \left(m+2\ \sqrt{k^2-a^2\ Z^2}\right)} & \frac{a\ Z}{m\ \left(m+2\ \sqrt{k^2-a^2\ Z^2}\right)} \end{pmatrix}$$

```
UN[R_, N_, En_] := Module[{u = {1, (k + s) / Z / a}, U = {1, (k + s) / Z / a} * R ^ s},
  For[n = 1, n < N, n++,
   u = S[n, En].u;
   U += u * R ^ (s + n);
  ];

  U]

U[r_, g_, En_] :=
 Module[{u = {1, (k + s) / Z / a}, U = {0, 0}, DU = {0, 0}, du = {0, 0}, n = 0},
  Label[begin];
  U += u * r ^ (s + n);
  DU += du * r ^ (s + n);
  n++;
  du = DS[n, En].u + S[n, En].du;
  u = S[n, En].u;
  If[(#[[1]] > g  || #[[2]] > g) &[Abs[u * R ^ (s + n) / U /. r → R]], Goto[begin]];
  {n, U, DU}]

R = 1000; g = 0.01; rU = U[r, g, 1 / M];
Plot[{rU[[2, 1]], rU[[2, 2]] * 137}, {r, 0, R}, PlotRange → All]
```




```
EN[iEn_, g2_] := Module[{rU, fU, n = 0, i, En = iEn, ll},
  Label[begin];
  fU = U[r, g, En];

  rU = fU /. r → R;

  If[rU[[2, 1]] * rU[[2, 2]] > 0,

   En -= (rU[[2, 1]] + rU[[2, 2]]) / (rU[[3, 1]] + rU[[3, 2]]);

   n++
    Goto[begin];
  ];

  {n, En * M, Abs[rU[[2, 1]] - rU[[2, 2]]]}
 ]

R = 3000; g = 0.001; EN[13 / M, 0.1]
```

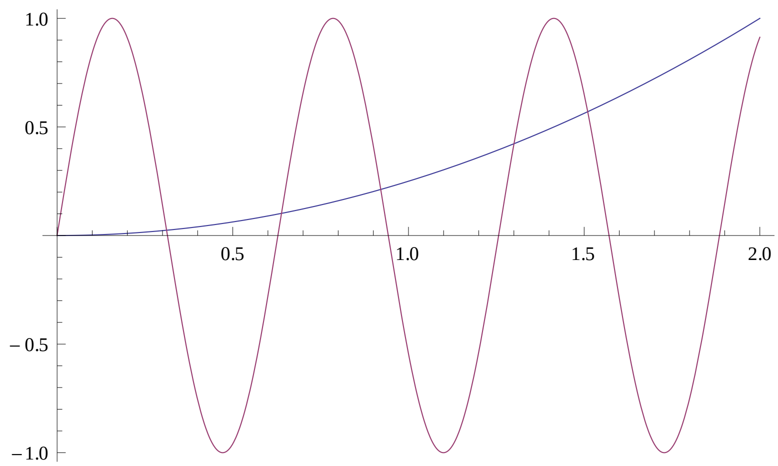
{8, 13.6059, 0.000441718}

```
- {0, 13.605873075061169
```

```
R = 2000; g = 0.001;

plot[{rU[[2, 1]], rU[[3, 1]]}, 100, R]

En = 4 / M; rU[[2, 1]]

129.072

n = 0; x

10

n = 0; While[x = n; x < 10, n++; Print[n]]

1

2

3

4

5

6

7

8

9

10
```

```
plot[liste_, R_] := Module[{nN = 100, table, max, st = {Red, Green, Blue}},
  liste / (Max[Abs[#]] & /@ (Table[# /. r → i * R / nN, {i, 0, nN}] & /@ liste))
 ]
ll = plot[{r ^ 2, Sin[10 * r]}, 2]
Plot[ll, {r, 0, 2}]
```

$\left\{\frac{r^2}{4}, -\text{Csc}[11]\ \text{Sin}[10\ r]\right\}$



r gegen Inifinity

```
Exit[]
```

```
M = 510998.910; a = 7297352537.6 * 10 ^ -12;
```

```
a = 7297352537.6 * 10 ^ -12; M = 510998.910; k = -1; Z = 1;
```

```
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
Table[N[Energie[i]], {i, 10}]
```

```
$Assumptions = 1 > En > 0;
```

```
s = (En - 1) * Z * a / L; L := Sqrt[(2 - En) * En]; L =.; s =.
```

```
f[u_, r_] := Simplify[{(Z * a / r + 2 - En) * u[[2]] - k / r * u[[1]],
    k / r * u[[2]] + (En - Z * a / r) * u[[1]]}];
```

```
u =.
```

```
U = {(hb[r] - ha[r]) * L / En, hb[r] + ha[r]} * Exp[-r * L];
```

```
#[[2]] & /@ Simplify[Solve[Simplify[(f[U, r] - D[U, r]) / Exp[-r * L] * En * r] == 0,
    {ha'[r], hb'[r]}][[1]]] /. ha[r] → u[[1]] /. hb[r] → u[[2]] /. r → rr
```

Part::partd : Part specification u⟦1⟧ is longer than depth of object. ≫

Part::partd : Part specification u⟦2⟧ is longer than depth of object. ≫

$$\left\{\frac{1}{2\,\mathrm{En}\,L\,rr}\left(\left(\mathrm{En}^3\,rr + \mathrm{En}\,L^2\,rr + a\,L^2\,Z - \mathrm{En}^2\,(2\,rr + a\,Z)\right)u[\![1]\!] + \left(\mathrm{En}^3\,rr + \mathrm{En}\,L\,(2\,k + L\,rr) - a\,L^2\,Z - \mathrm{En}^2\,(2\,rr + a\,Z)\right)u[\![2]\!]\right),\right.$$
$$\left.\frac{1}{2\,\mathrm{En}\,L\,rr}\left(\left(-\mathrm{En}^3\,rr + \mathrm{En}\,L\,(2\,k - L\,rr) + a\,L^2\,Z + \mathrm{En}^2\,(2\,rr + a\,Z)\right)u[\![1]\!] + \left(-\mathrm{En}^3\,rr + 3\,\mathrm{En}\,L^2\,rr - a\,L^2\,Z + \mathrm{En}^2\,(2\,rr + a\,Z)\right)u[\![2]\!]\right)\right\}$$

$$\mathbf{F[u\_, rr\_] := Simplify\left[\left\{\frac{-a\,(-1 + En)\,Z\,u[\![1]\!] + \left(\sqrt{-(-2 + En)\,En}\;k - a\,Z\right)u[\![2]\!]}{\sqrt{-(-2 + En)\,En}\;rr},\right.\right.}$$
$$\mathbf{\left.\left.\left(\left(\sqrt{-(-2 + En)\,En}\;k + a\,Z\right)u[\![1]\!] + \left(4\,En\,rr - 2\,En^2\,rr - a\,Z + a\,En\,Z\right)u[\![2]\!]\right)\Big/\left(\sqrt{-(-2 + En)\,En}\;rr\right)\right\}\right]}$$

```
r[x_] := 1 / x; u := {a[n], b[n]} * x ^ (n + s)
```

```
g1 = Collect[Simplify[(F[u, r[x]] * D[r[x], x] - D[u, x]) * x ^ (2 - s)], {a[n], b[n]}]; g1
```

$$\left\{-\left(x^{1+n}\left(a\,(-1 + \mathrm{En})\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}\;Z + L\left(\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}\;n + a\,Z - a\,\mathrm{En}\,Z\right)\right)a[n]\right)\Big/\left(\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}\;L\right) - \frac{x^{1+n}\left(\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}\;k - a\,Z\right)b[n]}{\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}},\right.$$
$$-\frac{x^{1+n}\left(\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}\;k + a\,Z\right)a[n]}{\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}} - \frac{1}{\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}\;L}$$
$$\left.x^n\left(-2\,\mathrm{En}^2\,L + x\left(\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}\;L\,n - a\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}\;Z - a\,L\,Z\right) + \mathrm{En}\left(4\,L + a\sqrt{-(-2 + \mathrm{En})\,\mathrm{En}}\;x\,Z + a\,L\,x\,Z\right)\right)b[n]\right\}$$

$$\mathbf{g2 := \left\{x^n\,(-n)\,a[n] + x^n\,(-k + a\,Z / L)\,b[n],\right.}$$
$$\mathbf{\left.-x^{1+n}\,(k + a\,Z / L)\,a[n] + x^n\,(-n\,x - 2\,s\,x + -2 * L)\,b[n]\right\};\ g2\ //\ MatrixForm}$$

$$\begin{pmatrix} -n\,x^n\,a[n] + x^n\left(-k + \frac{a\,Z}{L}\right)b[n] \\ -x^{1+n}\left(k + \frac{a\,Z}{L}\right)a[n] + x^n\,(-2\,L - n\,x - 2\,s\,x)\,b[n] \end{pmatrix}$$

```
a[n_] := (Z * a / L - k) / n * b[n]
```

```
g2[[2]]
```

$$x^n\,(-2\,L - n\,x - 2\,s\,x)\,b[n] - \frac{x^{1+n}\left(-k + \frac{a\,Z}{L}\right)\left(k + \frac{a\,Z}{L}\right)b[n]}{n}$$

```
M1 = {{L, 2 - En}, {En, L}}; Eigenvalues[M1]; B = Transpose[Eigenvectors[M1]];
```

```
$Assumptions = 1 > En > 0
```

$1 > \mathrm{En} > 0$

=>

```
a[n_] := (Z * a / L - k) / n * b[n]
b[n_] := b0 * Product[((k ^ 2 - a ^ 2 * Z ^ 2 / L ^ 2) / i - (i + 2 * s)) / 2 / L, {i, 1, n}]
b[4]
```

$$\frac{1}{16\,L^4}\,\mathrm{b0}\left(-1+k^2-2\,s-\frac{a^2\,Z^2}{L^2}\right)\left(-4-2\,s+\frac{1}{4}\left(k^2-\frac{a^2\,Z^2}{L^2}\right)\right)\left(-3-2\,s+\frac{1}{3}\left(k^2-\frac{a^2\,Z^2}{L^2}\right)\right)\left(-2-2\,s+\frac{1}{2}\left(k^2-\frac{a^2\,Z^2}{L^2}\right)\right)$$

```
((k ^ 2 - a ^ 2 * Z ^ 2 / L ^ 2) / i - (i + 2 * s)) / 2 / L
Exit[];
```

$$\#[[1,2]]\ \&\ /@\ \mathrm{Solve}\left[\frac{-i-\frac{2\,a\,(-1+\mathrm{En})\,Z}{\sqrt{(2-\mathrm{En})\,\mathrm{En}}}+\frac{k^2-\frac{a^2\,Z^2}{(2-\mathrm{En})\,\mathrm{En}}}{i}}{2\sqrt{(2-\mathrm{En})\,\mathrm{En}}}==0,\ \mathrm{En}\right]$$

```
{Root[-4.803202207768027100482068692830 9 × 10^47 Z^4 +
    Z^2 (3.6079474730614027105151092882086 × 10^52 i^2 + 3.6079474730614027544710557805907 ×
        10^52 k^2 + 2.4016011038840135502410343464154 × 10^47 Z^2) #1 +
    (-6.7753159274232856880568066514557 × 10^56 i^4 + 1.3550631854846571376113613302911 × 10^57
        i^2 k^2 - 6.7753159274232856880568066514557 × 10^56 k^4 -
       1.8039737365307014925688025430557 × 10^53 i^2 Z^2 -
       3.6079474730614027544710557805907 × 10^52 k^2 Z^2) #1^2 +
    (1.0162973891134928532085209977184 × 10^57 i^4 - 2.0325947782269857064170419954367 × 10^57
        i^2 k^2 + 1.0162973891134928532085209977184 × 10^57 k^4 +
       2.2549671706633769000388151535575 × 10^53 i^2 Z^2 +
       9.0198686826535068861776394514 77 × 10^51 k^2 Z^2) #1^3 +
    (-5.0814869455674642660426049885918 × 10^56 i^4 + 1.0162973891134928532085209977184 × 10^57
        i^2 k^2 - 5.0814869455674642660426049885918 × 10^56 k^4 -
       1.0823842419184208840079540605574 × 10^53 i^2 Z^2) #1^4 +
    (8.4691449092791071100710083143 20 × 10^55 i^4 - 1.6938289818558214220142016628639 × 10^56
        i^2 k^2 + 8.4691449092791071100710083143 20 × 10^55 k^4 +
       1.8039737365307013662465412671998 × 10^52 i^2 Z^2) #1^5 &, 1],
 Root[-4.8032022077680271004820686928309 × 10^47 Z^4 + Z^2
     (3.6079474730614027105151092882086 × 10^52 i^2 + 3.6079474730614027544710557805907 × 10^52
        k^2 + 2.4016011038840135502410343464154 × 10^47 Z^2) #1 +
    (-6.7753159274232856880568066514557 × 10^56 i^4 + 1.3550631854846571376113613302911 × 10^57
        i^2 k^2 - 6.7753159274232856880568066514557 × 10^56 k^4 -
       1.8039737365307014925688025430557 × 10^53 i^2 Z^2 -
```

$3.6079474730614027544710557805907 \times 10^{52}\ k^2\ Z^2\big)\ \#1^2 +$
$\big(1.0162973891134928532085209977184 \times 10^{57}\ i^4 - 2.0325947782269857064170419954367 \times 10^{57}$
$i^2\ k^2 + 1.0162973891134928532085209977184 \times 10^{57}\ k^4 +$
$2.2549671706633769000388151535575 \times 10^{53}\ i^2\ Z^2 +$
$9.0198686826535068861776394514477 \times 10^{51}\ k^2\ Z^2\big)\ \#1^3 +$
$\big(-5.0814869455674642660426049885918 \times 10^{56}\ i^4 + 1.0162973891134928532085209977184 \times 10^{57}$
$i^2\ k^2 - 5.0814869455674642660426049885918 \times 10^{56}\ k^4 -$
$1.0823842419184208840079540605574 \times 10^{53}\ i^2\ Z^2\big)\ \#1^4 +$
$\big(8.4691449092791071100710083143320 \times 10^{55}\ i^4 - 1.6938289818558214220142016628639 \times 10^{56}$
$i^2\ k^2 + 8.4691449092791071100710083143320 \times 10^{55}\ k^4 +$
$1.8039737365307013662465412671998 \times 10^{52}\ i^2\ Z^2\big)\ \#1^5\ \&,\ 2\big],$
$\mathrm{Root}\big[-4.8032022077680271004820686928309 \times 10^{47}\ Z^4 + Z^2$
$\big(3.6079474730614027105151092882086 \times 10^{52}\ i^2 + 3.6079474730614027544710557805907 \times 10^{52}$
$k^2 + 2.4016011038840135502410343464154 \times 10^{47}\ Z^2\big)\ \#1 +$
$\big(-6.7753159274232856880568066514557 \times 10^{56}\ i^4 + 1.3550631854846571376113613302911 \times 10^{57}$
$i^2\ k^2 - 6.7753159274232856880568066514557 \times 10^{56}\ k^4 -$
$1.8039737365307014925688025430557 \times 10^{53}\ i^2\ Z^2 -$
$3.6079474730614027544710557805907 \times 10^{52}\ k^2\ Z^2\big)\ \#1^2 +$
$\big(1.0162973891134928532085209977184 \times 10^{57}\ i^4 - 2.0325947782269857064170419954367 \times 10^{57}$
$i^2\ k^2 + 1.0162973891134928532085209977184 \times 10^{57}\ k^4 +$
$2.2549671706633769000388151535575 \times 10^{53}\ i^2\ Z^2 +$
$9.0198686826535068861776394514477 \times 10^{51}\ k^2\ Z^2\big)\ \#1^3 +$
$\big(-5.0814869455674642660426049885918 \times 10^{56}\ i^4 + 1.0162973891134928532085209977184 \times 10^{57}$
$i^2\ k^2 - 5.0814869455674642660426049885918 \times 10^{56}\ k^4 -$
$1.0823842419184208840079540605574 \times 10^{53}\ i^2\ Z^2\big)\ \#1^4 +$
$\big(8.4691449092791071100710083143320 \times 10^{55}\ i^4 - 1.6938289818558214220142016628639 \times 10^{56}$
$i^2\ k^2 + 8.4691449092791071100710083143320 \times 10^{55}\ k^4 +$
$1.8039737365307013662465412671998 \times 10^{52}\ i^2\ Z^2\big)\ \#1^5\ \&,\ 3\big],$
$\mathrm{Root}\big[-4.8032022077680271004820686928309 \times 10^{47}\ Z^4 + Z^2$
$\big(3.6079474730614027105151092882086 \times 10^{52}\ i^2 + 3.6079474730614027544710557805907 \times 10^{52}$
$k^2 + 2.4016011038840135502410343464154 \times 10^{47}\ Z^2\big)\ \#1 +$
$\big(-6.7753159274232856880568066514557 \times 10^{56}\ i^4 + 1.3550631854846571376113613302911 \times 10^{57}$
$i^2\ k^2 - 6.7753159274232856880568066514557 \times 10^{56}\ k^4 -$
$1.8039737365307014925688025430557 \times 10^{53}\ i^2\ Z^2 -$
$3.6079474730614027544710557805907 \times 10^{52}\ k^2\ Z^2\big)\ \#1^2 +$
$\big(1.0162973891134928532085209977184 \times 10^{57}\ i^4 - 2.0325947782269857064170419954367 \times 10^{57}$
$i^2\ k^2 + 1.0162973891134928532085209977184 \times 10^{57}\ k^4 +$
$2.2549671706633769000388151535575 \times 10^{53}\ i^2\ Z^2 +$
$9.0198686826535068861776394514477 \times 10^{51}\ k^2\ Z^2\big)\ \#1^3 +$
$\big(-5.0814869455674642660426049885918 \times 10^{56}\ i^4 + 1.0162973891134928532085209977184 \times 10^{57}$
$i^2\ k^2 - 5.0814869455674642660426049885918 \times 10^{56}\ k^4 -$
$1.0823842419184208840079540605574 \times 10^{53}\ i^2\ Z^2\big)\ \#1^4 +$
$\big(8.4691449092791071100710083143320 \times 10^{55}\ i^4 - 1.6938289818558214220142016628639 \times 10^{56}$
$i^2\ k^2 + 8.4691449092791071100710083143320 \times 10^{55}\ k^4 +$

$$
\begin{aligned}
&\quad 1.8039737365307013662465412671998\times 10^{52}\, i^2\, Z^2\big)\ \#1^5\ \&,\ 4\big],\\
&\mathrm{Root}\big[-4.8032022077680271004820686928309\times 10^{47}\, Z^4 + Z^2\\
&\quad \big(3.6079474730614027105151092882086\times 10^{52}\, i^2 + 3.6079474730614027544710557805907\times 10^{52}\\
&\qquad k^2 + 2.4016011038840135502410343464154\times 10^{47}\, Z^2\big)\ \#1 +\\
&\quad \big(-6.7753159274232856880568066514557\times 10^{56}\, i^4 + 1.3550631854846571376113613302911\times 10^{57}\\
&\qquad i^2\, k^2 - 6.7753159274232856880568066514557\times 10^{56}\, k^4 -\\
&\qquad 1.8039737365307014925688025430557\times 10^{53}\, i^2\, Z^2 -\\
&\qquad 3.6079474730614027544710557805907\times 10^{52}\, k^2\, Z^2\big)\ \#1^2 +\\
&\quad \big(1.0162973891134928532085209977184\times 10^{57}\, i^4 - 2.0325947782269857064170419954367\times 10^{57}\\
&\qquad i^2\, k^2 + 1.0162973891134928532085209977184\times 10^{57}\, k^4 +\\
&\qquad 2.2549671706633769000388151535575\times 10^{53}\, i^2\, Z^2 +\\
&\qquad 9.0198686826535068861776394514 77\times 10^{51}\, k^2\, Z^2\big)\ \#1^3 +\\
&\quad \big(-5.0814869455674642660426049885918\times 10^{56}\, i^4 + 1.0162973891134928532085209977184\times 10^{57}\\
&\qquad i^2\, k^2 - 5.0814869455674642660426049885918\times 10^{56}\, k^4 -\\
&\qquad 1.0823842419184208840079540605574\times 10^{53}\, i^2\, Z^2\big)\ \#1^4 +\\
&\quad \big(8.4691449092791071100710083143 20\times 10^{55}\, i^4 - 1.6938289818558214220142016628639\times 10^{56}\\
&\qquad i^2\, k^2 + 8.4691449092791071100710083143 20\times 10^{55}\, k^4 +\\
&\qquad 1.8039737365307013662465412671998\times 10^{52}\, i^2\, Z^2\big)\ \#1^5\ \&,\ 5\big]\big\}
\end{aligned}
$$

2 M

1.022×10^{6}

```
ET[i_, k_, Z_, a_] :=
 M * {(2 i^4 - 4 i^2 k^2 + 2 k^4 + 8 a^2 i^2 Z^2 - Sqrt[((-2 i^4 + 4 i^2 k^2 - 2 k^4 - 8 a^2 i^2 Z^2)^2 - 4 (i^4 -
        2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2) (a^2 i^2 Z^2 + a^2 k^2 Z^2 - 2 Sqrt[a^4 i^2 k^2 Z^4 - a^6 i^2 Z^6]))]) /
    (2 (i^4 - 2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2)), (2 i^4 - 4 i^2 k^2 + 2 k^4 + 8 a^2 i^2 Z^2 -
     Sqrt[((-2 i^4 + 4 i^2 k^2 - 2 k^4 - 8 a^2 i^2 Z^2)^2 - 4 (i^4 - 2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2)
        (a^2 i^2 Z^2 + a^2 k^2 Z^2 + 2 Sqrt[a^4 i^2 k^2 Z^4 - a^6 i^2 Z^6]))]) /
    (2 (i^4 - 2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2)), 2 - (2 i^4 - 4 i^2 k^2 + 2 k^4 + 8 a^2 i^2 Z^2 +
     Sqrt[((-2 i^4 + 4 i^2 k^2 - 2 k^4 - 8 a^2 i^2 Z^2)^2 - 4 (i^4 - 2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2)
        (a^2 i^2 Z^2 + a^2 k^2 Z^2 - 2 Sqrt[a^4 i^2 k^2 Z^4 - a^6 i^2 Z^6]))]) /
    (2 (i^4 - 2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2)), 2 - (2 i^4 - 4 i^2 k^2 + 2 k^4 + 8 a^2 i^2 Z^2 +
     Sqrt[((-2 i^4 + 4 i^2 k^2 - 2 k^4 - 8 a^2 i^2 Z^2)^2 - 4 (i^4 - 2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2) (a^2 i^2 Z^2 +
          a^2 k^2 Z^2 + 2 Sqrt[a^4 i^2 k^2 Z^4 - a^6 i^2 Z^6]))]) / (2 (i^4 - 2 i^2 k^2 + k^4 + 4 a^2 i^2 Z^2))}
```

```
{#[[1]] - #[[3]], #[[2]] - #[[4]]} & /@ Table[ET[n, -1, 1, 0.001], {n, 0, 10}]
```

$\{\{5.67323\times10^{-11}, 5.67323\times10^{-11}\}, \{0.0000273441, 0.000027344\}, \{1.61363\times10^{-10}, -8.48963\times10^{-11}\}, \{3.80272\times10^{-11}, -1.80558\times10^{-11}\}, \{-8.75485\times10^{-11}, 1.21638\times10^{-10}\}, \{-8.49015\times10^{-11}, -1.50054\times10^{-10}\}, \{-8.9154\times10^{-11}, 3.72012\times10^{-11}\}, \{-3.70403\times10^{-11}, 1.27461\times10^{-10}\}, \{1.89942\times10^{-10}, 1.38865\times10^{-10}\}, \{-9.75765\times10^{-11}, -5.68694\times10^{-11}\}, \{1.33551\times10^{-10}, 3.1188\times10^{-11}\}\}$

```
Table[M * ET[n, -1, 1, a] - Energie[n + 1], {n, 0, 10}]
```

$\{-5.67315\times10^{-11}, -1.9084\times10^{-7}, 5.30991\times10^{-11}, -1.3482\times10^{-10}, 3.24187\times10^{-11}, -5.86517\times10^{-11}, 5.5719\times10^{-11}, 2.10913\times10^{-11}, 1.87879\times10^{-11}, 1.01339\times10^{-10}, -1.46405\times10^{-12}\}$

```
Series[M * ET[n, -1, 1, a] - Energie[n + 1], {n, 0, 5}]
```

$-5.67315\times10^{-11} - 1.26477\times10^{-10}\, n^2 - 2.84217\times10^{-14}\, n^3 - 2.54019\times10^{-10}\, n^4 + 1.42109\times10^{-14}\, n^5 + O[n]^6$

```
M = 510998.910;
```

```
s = (En - 1) * Z * a / L; L := Sqrt[(2 - En) * En];
```

```
a = 7297352537.6 * 10^-12; M = 510998.910; Z = 1; k = -1;
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k^2 - (Z * a)^2]))^2]);
Table[N[Energie[i]], {i, 10}]
```

{13.6059, 3.40148, 1.51176, 0.850365,
0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}

Verhältnis bei r=0

```
a = 7297352537.6 * 10^-12; M = 510998.910; k = -1; Z = 1;
s = Sqrt[k^2 - (Z * a)^2];
```

$$S[n\_] := \left\{\left\{\frac{a\ En\ Z}{n^2 + 2\,n\sqrt{k^2 - a^2 Z^2}},\ -\frac{(-2 + En)\left(-k + n + \sqrt{k^2 - a^2 Z^2}\right)}{n\left(n + 2\sqrt{k^2 - a^2 Z^2}\right)}\right\},\right.$$

$$\left.\left\{\frac{En\left(k + n + \sqrt{k^2 - a^2 Z^2}\right)}{n\left(n + 2\sqrt{k^2 - a^2 Z^2}\right)},\ \frac{a\ (-2 + En)\ Z}{n\left(n + 2\sqrt{k^2 - a^2 Z^2}\right)}\right\}\right\}\ \text{/. En -> Enn;}$$

```
S[
 10]
```

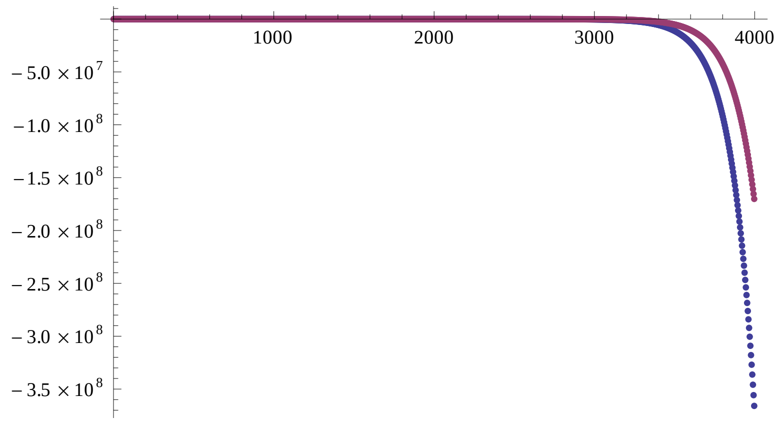
{{0.0000608115 Enn, -0.1 (-2 + Enn)}, {0.0833335 Enn, 0.0000608115 (-2 + Enn)}}

```
Enn =.; u = {1, (k + s) / Z / a}; U = u;
For[n = 1, n < 3, n++,
 u = S[n].u;
 U = Simplify[U + u];
]; n =.;
Simplify[U[[1]] / U[[2]]]
```

$$-\frac{91.3504\ (-3.66275 + Enn)\ (1.62629 + Enn)}{(-0.0109728 + Enn)\ (181.383 + Enn)}$$

Runge von links

```
f[u_, r_] := Simplify[{(Z * a / r + 2 - Enn) * u[[2]] - k / r * u[[1]],
    k / r * u[[2]] + (Enn - Z * a / r) * u[[1]]}];
k = -1; Z = 1; U =.
n = 1000;
h = 4000 / n;
Enn = 13.605 / M;
u = {(91.35044102604739` (-3.662751763692355` + Enn) (1.6262886176197724` + Enn)) /
    ((-0.0109728221664999` + Enn) (181.38339842774778` + Enn)), -1};
r = 1; U = {{r, u}};
Do[
 k0 = h * f[u, r]; k1 = h * f[u + k0 / 2, r + h / 2];
 k2 = h * f[u + k1 / 2, r + h / 2]; k3 = h * f[u + k2, r + h];
 u += 1 / 6 * (k0 + 2 * k1 + 2 * k2 + k3); r += h;
 AppendTo[U, {r, u}], {n}]; x =.;
ListPlot[
 Table[{#[[1]], 137 ^ (i - 2) * #[[2, i]]} & /@ U[[1 ;; n]], {i, 2}] // N, PlotRange → All]
```




```
Sum[A[n] * r ^ n / n !, {n, 0, 10}]
```

$$A[0] + r\,A[1] + \frac{1}{2} r^2 A[2] + \frac{1}{6} r^3 A[3] + \frac{1}{24} r^4 A[4] + \frac{1}{120} r^5 A[5] + \frac{1}{720} r^6 A[6] + \frac{r^7 A[7]}{5040} + \frac{r^8 A[8]}{40\,320} + \frac{r^9 A[9]}{362\,880} + \frac{r^{10} A[10]}{3\,628\,800}$$

```
D[%, {r, 4}]
```

$$A[4] + r\,A[5] + \frac{1}{2} r^2 A[6] + \frac{1}{6} r^3 A[7] + \frac{1}{24} r^4 A[8] + \frac{1}{120} r^5 A[9] + \frac{1}{720} r^6 A[10]$$

```
Sum[A[n + 4] * r ^ n / n !, {n, 0, 10}]
```

$$A[4] + r\,A[5] + \frac{1}{2} r^2 A[6] + \frac{1}{6} r^3 A[7] + \frac{1}{24} r^4 A[8] + \frac{1}{120} r^5 A[9] + \frac{1}{720} r^6 A[10] + \frac{r^7 A[11]}{5040} + \frac{r^8 A[12]}{40\,320} + \frac{r^9 A[13]}{362\,880} + \frac{r^{10} A[14]}{3\,628\,800}$$