```
Exit[]
```

```
n = 4;
```

```
EW[1] = 0; EW[2] = 1; Moments = Table[W ^ n → EW[n], {n, 3 * n, 1, -1}]
```

$\{W^{12} \to EW[12],\ W^{11} \to EW[11],\ W^{10} \to EW[10],\ W^{9} \to EW[9],\ W^{8} \to EW[8],\ W^{7} \to EW[7],\ W^{6} \to EW[6],\ W^{5} \to EW[5],\ W^{4} \to EW[4],\ W^{3} \to EW[3],\ W^{2} \to 1,\ W \to 0\}$

```
ExpValue[a_] := Simplify[a - a + Expand[Normal[a]] /. Moments]
```

```
Cov[a_, b_] := Simplify[ExpValue[a b] - ExpValue[a] ExpValue[b]]
```

```
Var[a_] := Cov[a, a]
```

```
dX = μ dt ^ 2 + σ W dt;
```

```
dS = S (Series[Exp[dX], {dt, 0, n}] - 1);
```

```
dV = Series[V[t + dt ^ 2, S + dS], {dt, 0, n}] - V[t, S];
```

```
dP[Δ_] := dV - Δ dS - (V [t, S] - Δ S) (Exp[dt ^ 2 r] - 1)
```

```
VarHedgingError[Δ_] := Var[dP[Δ]]
```

■ Hedging Ratios:

```
(* Variance minimizing *)
Δ0 = Simplify[Cov[dS, dV] / Var[dS]]
```

$$V^{(0,1)}[t, S] + \frac{1}{2} S\,\sigma\,EW[3]\,V^{(0,2)}[t, S]\,dt + \left(\frac{1}{4} S\left(4\mu + \sigma^2\left(-1 - 2\,EW[3]^2 + 3\,EW[4]\right)\right) V^{(0,2)}[t, S] + \frac{1}{6} S^2\sigma^2\,EW[4]\,V^{(0,3)}[t, S] + V^{(1,1)}[t, S]\right) dt^2 + \frac{1}{24} S\,\sigma\left(\left(12\,\mu\,EW[3] + \sigma^2\left(EW[3] + 12\,EW[3]^3 - 25\,EW[3]\,EW[4] + 15\,EW[5]\right)\right) V^{(0,2)}[t, S] + 2\,S\left(6\,\mu\,EW[3] - \sigma^2\left(EW[3] + 2\,EW[3]\,EW[4] - 4\,EW[5]\right)\right) V^{(0,3)}[t, S] + S^2\sigma^2\,EW[5]\,V^{(0,4)}[t, S] + 12\,EW[3]\,V^{(1,2)}[t, S]\right) dt^3 + O[dt]^4$$

```
(* Wilmott's *)
ΔW = Normal[Simplify[Δ0 /. EW[3] → 0 /. EW[4] → 3] + O[dt] ^ 3]
```

$$V^{(0,1)}[t, S] + dt^2\left(S\left(\mu + 2\sigma^2\right) V^{(0,2)}[t, S] + \frac{1}{2} S^2\sigma^2\,V^{(0,3)}[t, S] + V^{(1,1)}[t, S]\right)$$

```
(* Black Scholes *)
ΔBS = ΔW /. dt → 0
```

$V^{(0,1)}[t, S]$

■ Hedging results:

```
VarHedgingError[Δ]
```

$$S^2\,\sigma^2\,\left(\Delta - V^{(0,1)}[t,S]\right)^2\,dt^2 - S^2\,\sigma^3\,EW[3]\,\left(\Delta - V^{(0,1)}[t,S]\right)\,\left(-\Delta + V^{(0,1)}[t,S] + S\,V^{(0,2)}[t,S]\right)\,dt^3 + \frac{1}{12}\,S^2\,\sigma^2\,\Big(\left(24\,\mu + \sigma^2\,(-3 + 7\,EW[4])\right)\,V^{(0,1)}[t,S]^2 - 6\,S\,\Delta\,\left(4\,\mu + \sigma^2\,(-1 + 3\,EW[4])\right)\,V^{(0,2)}[t,S] + 3\,S^2\,\sigma^2\,(-1 + EW[4])\,V^{(0,2)}[t,S]^2 + \Delta\,\left(24\,\Delta\,\mu - 3\,\Delta\,\sigma^2 + 7\,\Delta\,\sigma^2\,EW[4] - 4\,S^2\,\sigma^2\,EW[4]\,V^{(0,3)}[t,S] - 24\,V^{(1,1)}[t,S]\right) + 2\,V^{(0,1)}[t,S]\,\Big(-24\,\Delta\,\mu + 3\,\Delta\,\sigma^2 - 7\,\Delta\,\sigma^2\,EW[4] + 3\,S\,\left(4\,\mu + \sigma^2\,(-1 + 3\,EW[4])\right)\,V^{(0,2)}[t,S] + 2\,S^2\,\sigma^2\,EW[4]\,V^{(0,3)}[t,S] + 12\,V^{(1,1)}[t,S]\Big)\Big)\,dt^4 - \frac{1}{12}\,\Big(S^2\,\sigma^3\,\Big(-\left(24\,\mu\,EW[3] + \sigma^2\,(-2\,EW[3] + 3\,EW[5])\right)\,V^{(0,1)}[t,S]^2 - 6\,S^2\,\left(2\,\mu\,EW[3] + \sigma^2\,(-EW[3] + EW[5])\right)\,V^{(0,2)}[t,S]^2 - S\,V^{(0,2)}[t,S]\,\Big(\Delta\,\left(-60\,\mu\,EW[3] + \sigma^2\,(8\,EW[3] - 15\,EW[5])\right) + 2\,S^2\,\sigma^2\,(-EW[3] + EW[5])\,V^{(0,3)}[t,S] + 12\,EW[3]\,V^{(1,1)}[t,S]\Big) - \Delta\,\Big(24\,\Delta\,\mu\,EW[3] - 2\,\Delta\,\sigma^2\,EW[3] + 3\,\Delta\,\sigma^2\,EW[5] - 2\,S^2\,\left(6\,\mu\,EW[3] - \sigma^2\,(EW[3] - 4\,EW[5])\right)\,V^{(0,3)}[t,S] - S^3\,\sigma^2\,EW[5]\,V^{(0,4)}[t,S] - 24\,EW[3]\,V^{(1,1)}[t,S] - 12\,S\,EW[3]\,V^{(1,2)}[t,S]\Big) - V^{(0,1)}[t,S]\,\Big(-48\,\Delta\,\mu\,EW[3] + 4\,\Delta\,\sigma^2\,EW[3] - 6\,\Delta\,\sigma^2\,EW[5] + S\,\left(60\,\mu\,EW[3] + \sigma^2\,(-8\,EW[3] + 15\,EW[5])\right)\,V^{(0,2)}[t,S] + 2\,S^2\,\left(6\,\mu\,EW[3] - \sigma^2\,(EW[3] - 4\,EW[5])\right)\,V^{(0,3)}[t,S] + S^3\,\sigma^2\,EW[5]\,V^{(0,4)}[t,S] + 24\,EW[3]\,V^{(1,1)}[t,S] + 12\,S\,EW[3]\,V^{(1,2)}[t,S]\Big)\Big)\Big)\,dt^5 + O[dt]^6$$

```
(* Wilmott's Improvement: *)
Simplify[VarHedgingError[ΔW] - VarHedgingError[ΔBS]]
```

$$-\frac{1}{2}\,\left(S^3\,\sigma^3\,EW[3]\,V^{(0,2)}[t,S]\,\left(2\,S\,\left(\mu + 2\,\sigma^2\right)\,V^{(0,2)}[t,S] + S^2\,\sigma^2\,V^{(0,3)}[t,S] + 2\,V^{(1,1)}[t,S]\right)\right)\,dt^5 - \frac{1}{12}\,\Big(S^2\,\sigma^2\,\left(2\,S\,\left(\mu + 2\,\sigma^2\right)\,V^{(0,2)}[t,S] + S^2\,\sigma^2\,V^{(0,3)}[t,S] + 2\,V^{(1,1)}[t,S]\right)\,\Big(3\,S\,\left(2\,\mu + \sigma^2\,(-5 + 3\,EW[4])\right)\,V^{(0,2)}[t,S] + S^2\,\sigma^2\,(-3 + 2\,EW[4])\,V^{(0,3)}[t,S] + 6\,V^{(1,1)}[t,S]\Big)\Big)\,dt^6 + O[dt]^7$$

```
(*Define WilmottsPricing equation in a
 more readable way and check if it is correct: *)
```

$$\textbf{WilmottPricing} = \left(-r\,V[t,S] + r\,S\,V^{(0,1)}[t,S] + \frac{1}{2}\,S^2\,\sigma^2\,V^{(0,2)}[t,S] + V^{(1,0)}[t,S]\right)\,dt^2 + \Big(r\,\hat{}\,2\,/\,2\,\left(S\,V^{(0,1)}[t,S] - V[t,S]\right) + (\sigma\,\hat{}\,4\,(7\,/\,24\,EW[4] - 1) + (r - \mu\,/\,2)\,(\mu + 2\,\sigma\,\hat{}\,2))\,S\,\hat{}\,2\,V^{(0,2)}[t,S] + (2\,r + \sigma\,\hat{}\,2\,(EW[4] - 1))\,/\,4\,S\,\hat{}\,3\,\sigma\,\hat{}\,2\,V^{(0,3)}[t,S] + S\,\hat{}\,4\,/\,24\,\sigma\,\hat{}\,4\,EW[4]\,V^{(0,4)}[t,S] + r\,S\,V^{(1,1)}[t,S] + \left(S\,\hat{}\,2\,\sigma\,\hat{}\,2\,V^{(1,2)}[t,S] + V^{(2,0)}[t,S]\right)\,/\,2\Big)\,dt\,\hat{}\,4;$$

```
Simplify[ExpValue[dP[ΔW]] - WilmottPricing]
```

$$\frac{1}{6}\,S^2\,\sigma^3\,EW[3]\,\left(3\,V^{(0,2)}[t,S] + S\,V^{(0,3)}[t,S]\right)\,dt^3 + O[dt]^5$$

▪ Iterative Solution

```
ΔW
```

$$V^{(0,1)}[t,S]+dt^2\left(S\left(\mu+2\sigma^2\right)V^{(0,2)}[t,S]+\frac{1}{2}S^2\sigma^2V^{(0,3)}[t,S]+V^{(1,1)}[t,S]\right)$$

```
(* Wilmott's hedging Ratio with higher
 derivatives replaced by their Black Scholes versions: *)
ΔWit = Simplify[ΔW /. V3]
```

$$V^{(0,1)}[t,S]+dt^2\,S\left(-r+\mu+\sigma^2\right)V^{(0,2)}[t,S]$$

```
(* Simplify the higher order term in Wilmott's pricing equation using the Black-
 Scholes equation: *)Simplify[
 WilmottPricing /. μ → μ - σ^2/2 /. EW[3] → 0 /. EW[4] → 3 /. V4 /. V3 /. Vt2 /. Vt1]
```

$$-\frac{1}{2}dt^4S^2\left(r^2+\mu\left(\mu+\sigma^2\right)-r\left(2\mu+\sigma^2\right)\right)V^{(0,2)}[t,S]$$

```
(*Check if it equals the term given in Wilmott's book: *)

Simplify[(μ - r) (r - μ - σ^2) + (r^2 + μ (μ + σ^2) - r (2 μ + σ^2))]
```

0

```
(* Replacement rules for higher derivatives
  using the Black Scholes equation (as done above): *)

V3 = Solve[D[-r V[t, S] + r S V^(0,1)[t, S] + 1/2 S^2 σ^2 V^(0,2)[t, S] + V^(1,0)[t, S] == 0, S],
   D[V[t, S], {S, 3}]][[1, 1]]
```

$$V^{(0,3)}[t,S]\to-\frac{2\left(r\,S\,V^{(0,2)}[t,S]+S\,\sigma^2V^{(0,2)}[t,S]+V^{(1,1)}[t,S]\right)}{S^2\sigma^2}$$

```
V4 = Solve[D[-r V[t, S] + r S V^(0,1)[t, S] + 1/2 S^2 σ^2 V^(0,2)[t, S] + V^(1,0)[t, S] == 0, {S, 2}],
   D[V[t, S], {S, 4}]][[1, 1]]
```

$$V^{(0,4)}[t,S]\to-\frac{1}{S^2\sigma^2}2\left(r\,V^{(0,2)}[t,S]+\sigma^2V^{(0,2)}[t,S]+r\,S\,V^{(0,3)}[t,S]+2S\sigma^2V^{(0,3)}[t,S]+V^{(1,2)}[t,S]\right)$$

```
Vt2 = Solve[D[-r V[t, S] + r S V^(0,1)[t, S] + 1/2 S^2 σ^2 V^(0,2)[t, S] + V^(1,0)[t, S] == 0, t],
   D[V[t, S], {t, 2}]][[1, 1]]
```

$$V^{(2,0)}[t,S]\to\frac{1}{2}\left(2\,r\,V^{(1,0)}[t,S]-2\,r\,S\,V^{(1,1)}[t,S]-S^2\sigma^2V^{(1,2)}[t,S]\right)$$

```
Vt1 = Solve[-r V[t, S] + r S V^(0,1)[t, S] + 1/2 S^2 σ^2 V^(0,2)[t, S] + V^(1,0)[t, S] == 0,
    D[V[t, S], t]][[1, 1]]
```

$$V^{(1,0)}[t, S] \to \frac{1}{2} \left(2\, r\, V[t, S] - 2\, r\, S\, V^{(0,1)}[t, S] - S^2 \sigma^2 V^{(0,2)}[t, S]\right)$$