```
F[t_] := A f[t] + b[t]
```

- ## Exact Solution of f' + A f + b = 0: f[t+h]

```
sol =
 Simplify[Series[Exp[-h A] (f[t] - Integrate[Exp[η A] b[t + η], {η, 0, h}]), {h, 0, 3}]]
```

$$f[t] + (-b[t] - A\, f[t])\, h + \frac{1}{2}\left(A\, b[t] + A^2 f[t] - b'[t]\right) h^2 + \frac{1}{6}\left(-A^2 b[t] - A^3 f[t] + A\, b'[t] - b''[t]\right) h^3 + O[h]^4$$

- ## Crank Nicolson (with arbitrary μ)

```
Collect[f[t + h] - f[t] + h μ F[t] + h (1 - μ) F[t + h], f[t + h]]
```

$$h\,(1-\mu)\, b[h+t] - f[t] + h\,\mu\,(b[t] + A\, f[t]) + (1 + A\, h\,(1-\mu))\, f[h+t]$$

```
approx = Simplify[Series[-(h (1 - μ) b[h + t] - f[t] + h μ (b[t] + A f[t]))/(1 + A h (1 - μ)), {h, 0, 3}]]
```

$$f[t] + (-b[t] - A\, f[t])\, h - (-1+\mu)\left(A\, b[t] + A^2 f[t] - b'[t]\right) h^2 - \frac{1}{2}\left((-1+\mu)\left(2A^2(-1+\mu)\, b[t] + 2A^3(-1+\mu)\, f[t] + 2A\, b'[t] - 2A\,\mu\, b'[t] - b''[t]\right)\right) h^3 + O[h]^4$$

```
Simplify[approx - sol]
```

$$-\frac{1}{2}\left((-1+2\mu)\left(A\, b[t] + A^2 f[t] - b'[t]\right)\right) h^2 + \frac{1}{6}\left(A^2 b[t] + A^3 f[t] - A\, b'[t] - 3(-1+\mu)\left(2A^2(-1+\mu)\, b[t] + 2A^3(-1+\mu)\, f[t] + 2A\, b'[t] - 2A\,\mu\, b'[t] - b''[t]\right) + b''[t]\right) h^3 + O[h]^4$$

```
Simplify[approx - sol] /. μ -> 1/2
```

$$\frac{1}{6}\left(A^2 b[t] + A^3 f[t] - A\, b'[t] + \frac{3}{2}\left(-A^2 b[t] - A^3 f[t] + A\, b'[t] - b''[t]\right) + b''[t]\right) h^3 + O[h]^4$$

- ## Check if two iterations yield correct result

```
ff[h_, t_, f_] := Exp[-h A] (f - Integrate[Exp[η A] b[t + η], {η, 0, h}])
```

```
D[ff[h, f0], h] + A ff[h, f0] + b[t + h]
```

$$0$$

```
Simplify[Exp[h A] Expand[ff[h / 2, h / 2, ff[h / 2, 0, f0]] - ff[h, 0, f0]]]
```

$$-\int_0^{\frac{h}{2}} e^{A\eta}\, b[\eta]\, d\eta + \int_0^{h} e^{A\eta}\, b[\eta]\, d\eta - e^{\frac{A h}{2}} \int_0^{\frac{h}{2}} e^{A\eta}\, b\left[\frac{h}{2}+\eta\right] d\eta$$

```
Simplify[Series[%, {h, 0, 3}]]
```

$$O[h]^4$$

Make two apporximate iterations

```
Series[Integrate[Exp[η A] b[t + η], {η, 0, h}], {h, 0, 3}]
```

$$b[t]\, h + \frac{1}{2}(A\, b[t] + b'[t])\, h^2 + \frac{1}{6}\left(A^2\, b[t] + 2 A\, b'[t] + b''[t]\right) h^3 + O[h]^4$$

$$\left(1 - h\, A / 2 + \frac{h\,{}^\wedge 2}{8}\, A\,{}^\wedge 2\right)\left(f0 - b[t]\, h / 2 - \frac{1}{8}(A\, b[t] + b'[t])\, h^2\right) - ff[h / 2, t, f0] + O[h]\,{}^\wedge 3$$

$$O[h]^3$$

$$\left(\frac{1}{1 + A\,\mu\, h + \frac{1}{2} A^2\,\mu^2\, h^2}\right)\left(\left(1 + (-A + A\,\mu)\, h + \frac{1}{2}(-A + A\,\mu)^2\, h^2\right)\right.$$
$$\left(f0 - b[t]\,(1-\mu)\, h - \frac{1}{2}(A\, b[t] + b'[t])\,((1-\mu)\, h)^2\right) - b[t + (1-\mu)\, h]\,\mu\, h -$$
$$\left.\frac{1}{2}(A\, b[t + (1-\mu)\, h] + b'[t + (1-\mu)\, h])\,(\mu\, h)^2\right) - ff[h, t, f0] + O[h]\,{}^\wedge 3$$

$$\frac{1}{6}\left(A^3\, f0 - 3 A^3\, f0\,\mu + 3 A^3\, f0\,\mu^2 + A^2\, b[t] - 3 A^2\,\mu\, b[t] + 3 A^2\,\mu^2\, b[t] + \right.$$
$$\left. 2 A\, b'[t] - 6 A\,\mu\, b'[t] + 6 A\,\mu^2\, b'[t] + b''[t] - 3\,\mu\, b''[t] + 3\,\mu^2\, b''[t]\right) h^3 + O[h]^4$$

$$\left(\frac{1}{1 + A\,\mu\, h}\right)(1 + (-A + A\,\mu)\, h)\left(f0 - b[t]\, h - \frac{1}{2}(A\, b[t] + b'[t])\, h^2\right) - ff[h, t, f0] + O[h]\,{}^\wedge 3$$

$$\left(-\frac{A^2\, f0}{2} + A^2\, f0\,\mu\right) h^2 + O[h]^3$$