```
Exit[]
```

```
Moments = Table[W^n → Limit[D[Exp[t^2/2], {t, n}], t -> 0], {n, 4, 1, -1}]
```

$$\{W^4 \to 3,\ W^3 \to 0,\ W^2 \to 1,\ W \to 0\}$$

```
dX = (u - s^2/2) dt^2 + s W dt;
```

```
dS = Normal[Series[S ((Normal[Series[Exp[x], {x, 0, 4}]] /. x → dX) - 1), {dt, 0, 4}]]
```

$$dt\ s\ S\ W + dt^2 \left(-\frac{s^2 S}{2} + S\ u + \frac{1}{2} s^2 S\ W^2\right) + dt^3 \left(-\frac{1}{2} s^3 S\ W + s\ S\ u\ W + \frac{1}{6} s^3 S\ W^3\right) +$$
$$\frac{1}{24} dt^4 \left(3 s^4 S - 12 s^2 S\ u + 12 S\ u^2 - 6 s^4 S\ W^2 + 12 s^2 S\ u\ W^2 + s^4 S\ W^4\right)$$

```
dV =
  Normal[Series[Normal[Series[V[a, b], {a, t, 2}, {b, S, 4}] - V[t, S]] /. a → t + dt^2 /.
     b → S + dS, {dt, 0, 4}]];
```

```
P = dV - Δ dS - r (V[t, S] - Δ S) dt^2;
```

```
(*Mean:*)
EP = Collect[P, W] /. Moments;
```

```
EP2 = Collect[Normal[Series[P^2, {dt, 0, 4}]], W] /. Moments;
```

```
Va = Normal[Series[EP2 - EP^2, {dt, 0, 4}]]
```

$$dt^2 \left(s^2 S^2 \triangle^2 - 2 s^2 S^2 \triangle\ V^{(0,1)}[t, S] + s^2 S^2 V^{(0,1)}[t, S]^2\right) +$$
$$\frac{1}{2} dt^4 \left(s^4 S^2 \triangle^2 + 4 s^2 S^2 u \triangle^2 - 2 s^4 S^2 \triangle\ V^{(0,1)}[t, S] - 8 s^2 S^2 u \triangle\ V^{(0,1)}[t, S] + s^4 S^2 V^{(0,1)}[t, S]^2 + 4 s^2 S^2 u\ V^{(0,1)}[t, S]^2 - 6 s^4 S^3 \triangle\ V^{(0,2)}[t, S] - 4 s^2 S^3 u \triangle\ V^{(0,2)}[t, S] + 6 s^4 S^3 V^{(0,1)}[t, S]\ V^{(0,2)}[t, S] + 4 s^2 S^3 u\ V^{(0,1)}[t, S]\ V^{(0,2)}[t, S] + s^4 S^4 V^{(0,2)}[t, S]^2 - 2 s^4 S^4 \triangle\ V^{(0,3)}[t, S] + 2 s^4 S^4 V^{(0,1)}[t, S]\ V^{(0,3)}[t, S] - 4 s^2 S^2 \triangle\ V^{(1,1)}[t, S] + 4 s^2 S^2 V^{(0,1)}[t, S]\ V^{(1,1)}[t, S]\right)$$

```
H = Normal[Series[Solve[D[Va, Δ] == 0, Δ][[1, 1, 2]], {dt, 0, 4}]]
```

$$V^{(0,1)}[t, S] + dt^2 \left(\frac{3}{2} s^2 S\ V^{(0,2)}[t, S] + S\ u\ V^{(0,2)}[t, S] + \frac{1}{2} s^2 S^2 V^{(0,3)}[t, S] + V^{(1,1)}[t, S]\right) +$$
$$\frac{1}{4} dt^4 \left(-3 s^4 S\ V^{(0,2)}[t, S] - 14 s^2 S\ u\ V^{(0,2)}[t, S] - 8 S\ u^2\ V^{(0,2)}[t, S] - s^4 S^2 V^{(0,3)}[t, S] - 4 s^2 S^2 u\ V^{(0,3)}[t, S] - 2 s^2 V^{(1,1)}[t, S] - 8 u\ V^{(1,1)}[t, S]\right)$$

```
Series[Simplify[EP /. Δ → H], {dt, 0, 4}]
```

$$\left(-r\ V[t, S] + r\ S\ V^{(0,1)}[t, S] + \frac{1}{2} s^2 S^2 V^{(0,2)}[t, S] + V^{(1,0)}[t, S]\right) dt^2 +$$
$$\frac{1}{8} \left(12 r s^2 S^2 V^{(0,2)}[t, S] + 2 s^4 S^2 V^{(0,2)}[t, S] + 8 r S^2 u\ V^{(0,2)}[t, S] - 4 s^2 S^2 u\ V^{(0,2)}[t, S] - 4 S^2 u^2 V^{(0,2)}[t, S] + 4 r s^2 S^3 V^{(0,3)}[t, S] + 4 s^4 S^3 V^{(0,3)}[t, S] + s^4 S^4 V^{(0,4)}[t, S] + 8 r S\ V^{(1,1)}[t, S] + 4 s^2 S^2 V^{(1,2)}[t, S] + 4 V^{(2,0)}[t, S]\right) dt^4 + O[dt]^5$$

```
(*Use H up to first order and Solve for V up to first,
then insert into derivative terms in second order part*)
Normal[
 Series[H - dt^2 D[-r V[t, S] + r S V^(0,1)[t, S] + 1/2 s^2 S^2 V^(0,2)[t, S] + V^(1,0)[t, S], S],
  {dt, 0, 2}]]
```

$$V^{(0,1)}[t, S] + dt^2 \left(-r S V^{(0,2)}[t, S] + \frac{1}{2} s^2 S V^{(0,2)}[t, S] + S u V^{(0,2)}[t, S]\right)$$

```
HnextOrder = V^(0,1)[t, S] + dt^2 (-r + 1/2 s^2 + u) S V^(0,2)[t, S]
```

$$V^{(0,1)}[t, S] + dt^2 S \left(-r + \frac{s^2}{2} + u\right) V^{(0,2)}[t, S]$$

```
Series[Simplify[(EP /. Δ → H) - Expand[
      D[-r V[t, S] + r S V^(0,1)[t, S] + 1/2 s^2 S^2 V^(0,2)[t, S] + V^(1,0)[t, S], t] / 2] dt^4 -
    Expand[D[-r V[t, S] + r S V^(0,1)[t, S] + 1/2 s^2 S^2 V^(0,2)[t, S] + V^(1,0)[t, S], S] S r / 2]
     dt^4 - Expand[D[-r V[t, S] + r S V^(0,1)[t, S] + 1/2 s^2 S^2 V^(0,2)[t, S] + V^(1,0)[t, S],
       {S, 2}] s^2 S^2 / 4] dt^4], {dt, 0, 4}]
```

$$\left(-r V[t, S] + r S V^{(0,1)}[t, S] + \frac{1}{2} s^2 S^2 V^{(0,2)}[t, S] + V^{(1,0)}[t, S]\right) dt^2 +$$
$$\frac{1}{4}\left(-2 r^2 S^2 V^{(0,2)}[t, S] + 3 r s^2 S^2 V^{(0,2)}[t, S] + 4 r S^2 u V^{(0,2)}[t, S] -\right.$$
$$\left. 2 s^2 S^2 u V^{(0,2)}[t, S] - 2 S^2 u^2 V^{(0,2)}[t, S] + 2 r V^{(1,0)}[t, S]\right) dt^4 + O[dt]^5$$

```
Collect[Expand[
  1/4 (-2 r^2 S^2 V^(0,2)[t, S] + 3 r s^2 S^2 V^(0,2)[t, S] + 4 r S^2 u V^(0,2)[t, S] - 2 s^2 S^2 u V^(0,2)[t,
       S] - 2 S^2 u^2 V^(0,2)[t, S] + 2 r V^(1,0)[t, S])], {S^2 V^(0,2)[t, S], V^(1,0)[t, S]}]
```

$$S^2 \left(-\frac{r^2}{2} + \frac{3 r s^2}{4} + r u - \frac{s^2 u}{2} - \frac{u^2}{2}\right) V^{(0,2)}[t, S] + \frac{1}{2} r V^{(1,0)}[t, S]$$

```
Expand[(u - r) (r - u - s^2)]
```

$$-r^2 + r s^2 + 2 r u - s^2 u - u^2$$