

```

H[1] = -h; H[2] = 0; H[3] = h;
d[j_, i_] := H[i] ^ j / j!;
d[1, i_] := H[i] - H[i] ^ 3 / 3! b[0] / b[2] + H[i] ^ 4 / 4! b[1] b[0] / b[2] ^ 2;
d[2, i_] :=
  H[i] ^ 2 / 2 - H[i] ^ 3 / 3! b[1] / b[2] + H[i] ^ 4 / 4! (b[1] ^ 2 / b[2] ^ 2 - b[0] / b[2])
d[3, i_] := 0
d[4, i_] := 0

```

```

erV[n_] := Table[b[i] UnitStep[2.9 - i] - Sum[d[i, j] w[j - 2], {j, 1, 3}], {i, n}]

```

```

erV[6]

```

$$\begin{aligned}
& \left\{ b[1] - \left(-h + \frac{h^4 b[0] b[1]}{24 b[2]^2} + \frac{h^3 b[0]}{6 b[2]} \right) w[-1] - \left(h + \frac{h^4 b[0] b[1]}{24 b[2]^2} - \frac{h^3 b[0]}{6 b[2]} \right) w[1], \right. \\
& b[2] - \left(\frac{h^2}{2} + \frac{1}{24} h^4 \left(\frac{b[1]^2}{b[2]^2} - \frac{b[0]}{b[2]} \right) + \frac{h^3 b[1]}{6 b[2]} \right) w[-1] - \\
& \left(\frac{h^2}{2} + \frac{1}{24} h^4 \left(\frac{b[1]^2}{b[2]^2} - \frac{b[0]}{b[2]} \right) - \frac{h^3 b[1]}{6 b[2]} \right) w[1], 0, 0, \\
& \left. \frac{1}{120} h^5 w[-1] - \frac{1}{120} h^5 w[1], -\frac{1}{720} h^6 w[-1] - \frac{1}{720} h^6 w[1] \right\}
\end{aligned}$$

```

er[n_] := Sum[erV[n][[i]] ^ 2, {i, 1, n}]

```

```

er[2]

```

$$\begin{aligned}
& \left(b[1] - \left(-h + \frac{h^4 b[0] b[1]}{24 b[2]^2} + \frac{h^3 b[0]}{6 b[2]} \right) w[-1] - \left(h + \frac{h^4 b[0] b[1]}{24 b[2]^2} - \frac{h^3 b[0]}{6 b[2]} \right) w[1] \right)^2 + \\
& \left(b[2] - \left(\frac{h^2}{2} + \frac{1}{24} h^4 \left(\frac{b[1]^2}{b[2]^2} - \frac{b[0]}{b[2]} \right) + \frac{h^3 b[1]}{6 b[2]} \right) w[-1] - \right. \\
& \left. \left(\frac{h^2}{2} + \frac{1}{24} h^4 \left(\frac{b[1]^2}{b[2]^2} - \frac{b[0]}{b[2]} \right) - \frac{h^3 b[1]}{6 b[2]} \right) w[1] \right)^2
\end{aligned}$$

Unconditional minimization

```

minArgUncon =

```

```

Solve[Simplify[{D[er[2], w[1]] == 0, D[er[2], w[-1]] == 0}], {w[-1], w[1]}][[1]]

```

$$\begin{aligned}
& \{ w[-1] \rightarrow \\
& \left(3 \left(-h^3 b[1]^3 + 2 h^3 b[0] b[1] b[2] + 4 h^2 b[1]^2 b[2] - 4 h^2 b[0] b[2]^2 - 12 h b[1] b[2]^2 + \right. \right. \\
& \left. \left. 24 b[2]^3 \right) \right) / \left(h^2 \left(h^4 b[0]^2 + 6 h^2 b[1]^2 - 18 h^2 b[0] b[2] + 72 b[2]^2 \right) \right), w[1] \rightarrow \\
& \left. - \left(3 \left(-h^3 b[1]^3 + 2 h^3 b[0] b[1] b[2] - 4 h^2 b[1]^2 b[2] + 4 h^2 b[0] b[2]^2 - 12 h b[1] b[2]^2 - \right. \right. \right. \\
& \left. \left. \left. 24 b[2]^3 \right) \right) / \left(h^2 \left(h^4 b[0]^2 + 6 h^2 b[1]^2 - 18 h^2 b[0] b[2] + 72 b[2]^2 \right) \right) \right\}
\end{aligned}$$

```

Series[Sqrt[Simplify[er[6] /. minArgUncon]], {h, 0, 6}]

```

$$\frac{1}{360} \sqrt{9 b[1]^2 + b[2]^2} h^4 + \frac{(9 b[0] b[1]^2 + b[1]^2 b[2] + b[0] b[2]^2) h^6}{4320 b[2] \sqrt{9 b[1]^2 + b[2]^2}} + O[h]^7$$

M-Matrix (!):

```
Simplify[er[2] /. b[0] → -b[2] / 10 /. b[1] → 2 b[2]]
```

$$\frac{1}{57600} \left((-240 b[2] + h^2 ((120 + 80 h + 41 h^2) w[-1] + (120 - 80 h + 41 h^2) w[1]))^2 + 4 (240 b[2] + h ((120 + 2 h^2 + h^3) w[-1] + (-120 - 2 h^2 + h^3) w[1]))^2 \right)$$

```
{minEr, minArg} =
```

```
Refine[Minimize[{Simplify[er[2] /. b[0] → -b[2] / 10 /. b[1] → 2 b[2] /. b[2] → -10],  
w[-1] ≤ 0, w[1] ≤ 0}, {w[-1], w[1]}], {h < 1 / 2, h > 0, b[2] < 0}];
```

```
Series[
```

```
Simplify[erV[6] /. b[0] → -b[2] / 10 /. b[1] → 2 b[2] /. b[2] → -10 /. minArg], {h, 0, 5}]
```

$$\{0, 0, 0, 0, \frac{h^4}{6} + O[h]^6, \frac{h^4}{36} + O[h]^6\}$$

Minimization of first two orders (convection dominant):

```
{minEr, minArg} = Refine[
```

```
Minimize[{Simplify[er[2] /. b[0] → -b[2] / 10 /. b[1] → 2 b[2] / h /. b[2] → -10],  
w[-1] ≤ 0, w[1] ≤ 0}, {w[-1], w[1]}], {h < 1 / 2, h > 0, b[2] < 0}];
```

```
Series[Simplify[erV[6] /. b[0] → -b[2] / 10 /. b[1] → 2 b[2] / h /. b[2] → -10 /. minArg],  
{h, 0, 5}]
```

$$\{0, 0, 0, 0, \frac{h^3}{6} - \frac{h^5}{960} + O[h]^6, \frac{5 h^4}{144} + O[h]^6\}$$

Minimization of first two plus fifth orders (convection dominant):

```
{minEr, minArg} = Refine[
```

```
Minimize[{Simplify[er[5] /. b[0] → -b[2] / 10 /. b[1] → 2 b[2] / h /. b[2] → -10],  
w[-1] ≤ 0, w[1] ≤ 0}, {w[-1], w[1]}], {h < 1 / 2, h > 0, b[2] < 0}];
```

```
Series[Simplify[erV[6] /. b[0] → -b[2] / 10 /. b[1] → 2 b[2] / h /. b[2] → -10 /. minArg],  
{h, 0, 5}]
```

$$\{O[h]^7, O[h]^8, 0, 0, \frac{h^3}{6} - \frac{h^5}{960} + O[h]^6, \frac{5 h^4}{144} + O[h]^6\}$$

Fazit: It is $O(h^4)$, and $O(h^3)$ for convection dominant points!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!