

Exit[]

Moments = Table[W ^ n → Limit[D[Exp[t ^ 2 / 2], {t, n}], t -> 0], {n, 4, 1, -1}]

{W ^ 4 → 3, W ^ 3 → 0, W ^ 2 → 1, W → 0}

dX = (u - s ^ 2 / 2) dt ^ 2 + s W dt;

dS = Normal[Series[S ((Normal[Series[Exp[x], {x, 0, 4}]] /. x → dX) - 1), {dt, 0, 4}]]

dt s S W + dt ^ 2 $\left(-\frac{s^2 S}{2} + S u + \frac{1}{2} s^2 S W^2\right) + dt^3 \left(-\frac{1}{2} s^3 S W + s S u W + \frac{1}{6} s^3 S W^3\right) +$

$\frac{1}{24} dt^4 (3 s^4 S - 12 s^2 S u + 12 S u^2 - 6 s^4 S W^2 + 12 s^2 S u W^2 + s^4 S W^4)$

dV =

Normal[Series[Normal[Series[V[a, b], {a, t, 2}, {b, S, 4}] - V[t, S]] /. a → t + dt ^ 2 /.
b → S + dS, {dt, 0, 4}]];

P = dV - Δ dS - r (V[t, S] - Δ S) dt ^ 2;

(*Mean:*)

EP = Collect[P, W] /. Moments;

EP2 = Collect[Normal[Series[P ^ 2, {dt, 0, 4}]], W] /. Moments;

Va = Normal[Series[EP2 - EP ^ 2, {dt, 0, 4}]]

dt ^ 2 (s ^ 2 S ^ 2 Δ ^ 2 - 2 s ^ 2 S ^ 2 Δ V ^ (0,1) [t, S] + s ^ 2 S ^ 2 V ^ (0,1) [t, S] ^ 2) +

$\frac{1}{2} dt^4 (s^4 S^2 \Delta^2 + 4 s^2 S^2 u \Delta^2 - 2 s^4 S^2 \Delta V^{(0,1)} [t, S] - 8 s^2 S^2 u \Delta V^{(0,1)} [t, S] +$
 $s^4 S^2 V^{(0,1)} [t, S]^2 + 4 s^2 S^2 u V^{(0,1)} [t, S]^2 - 6 s^4 S^3 \Delta V^{(0,2)} [t, S] - 4 s^2 S^3 u \Delta$
 $V^{(0,2)} [t, S] + 6 s^4 S^3 V^{(0,1)} [t, S] V^{(0,2)} [t, S] + 4 s^2 S^3 u V^{(0,1)} [t, S] V^{(0,2)} [t, S] +$
 $s^4 S^4 V^{(0,2)} [t, S]^2 - 2 s^4 S^4 \Delta V^{(0,3)} [t, S] + 2 s^4 S^4 V^{(0,1)} [t, S] V^{(0,3)} [t, S] -$
 $4 s^2 S^2 \Delta V^{(1,1)} [t, S] + 4 s^2 S^2 V^{(0,1)} [t, S] V^{(1,1)} [t, S])$

H = Normal[Series[Solve[D[Va, Δ] == 0, Δ][[1, 1, 2]], {dt, 0, 4}]]

V ^ (0,1) [t, S] + dt ^ 2 $\left(\frac{3}{2} s^2 S V^{(0,2)} [t, S] + S u V^{(0,2)} [t, S] + \frac{1}{2} s^2 S^2 V^{(0,3)} [t, S] + V^{(1,1)} [t, S]\right) +$

$\frac{1}{4} dt^4 (-3 s^4 S V^{(0,2)} [t, S] - 14 s^2 S u V^{(0,2)} [t, S] - 8 S u^2 V^{(0,2)} [t, S] -$
 $s^4 S^2 V^{(0,3)} [t, S] - 4 s^2 S^2 u V^{(0,3)} [t, S] - 2 s^2 V^{(1,1)} [t, S] - 8 u V^{(1,1)} [t, S])$

Series[Simplify[EP /. Δ → H], {dt, 0, 4}]

$\left(-r V[t, S] + r S V^{(0,1)} [t, S] + \frac{1}{2} s^2 S^2 V^{(0,2)} [t, S] + V^{(1,0)} [t, S]\right) dt^2 +$

$\frac{1}{8} (12 r s^2 S^2 V^{(0,2)} [t, S] + 2 s^4 S^2 V^{(0,2)} [t, S] + 8 r S^2 u V^{(0,2)} [t, S] -$
 $4 s^2 S^2 u V^{(0,2)} [t, S] - 4 S^2 u^2 V^{(0,2)} [t, S] + 4 r s^2 S^3 V^{(0,3)} [t, S] + 4 s^4 S^3 V^{(0,3)} [t, S] +$
 $s^4 S^4 V^{(0,4)} [t, S] + 8 r S V^{(1,1)} [t, S] + 4 s^2 S^2 V^{(1,2)} [t, S] + 4 V^{(2,0)} [t, S]) dt^4 + O[dt]^5$

(*Use H up to first order and Solve for V up to first,
then insert into derivative terms in second order part*)

Normal[

$$\text{Series}\left[H - dt^2 D\left[-r V[t, S] + r S V^{(0,1)}[t, S] + \frac{1}{2} S^2 S^2 V^{(0,2)}[t, S] + V^{(1,0)}[t, S], S\right], \{dt, 0, 2\}\right]$$

$$V^{(0,1)}[t, S] + dt^2 \left(-r S V^{(0,2)}[t, S] + \frac{1}{2} S^2 S V^{(0,2)}[t, S] + S u V^{(0,2)}[t, S]\right)$$

$$\text{HnextOrder} = V^{(0,1)}[t, S] + dt^2 \left(-r + \frac{1}{2} S^2 + u\right) S V^{(0,2)}[t, S]$$

$$V^{(0,1)}[t, S] + dt^2 S \left(-r + \frac{S^2}{2} + u\right) V^{(0,2)}[t, S]$$

Series[Simplify[(EP /. Δ → H) - Expand[

$$D\left[-r V[t, S] + r S V^{(0,1)}[t, S] + \frac{1}{2} S^2 S^2 V^{(0,2)}[t, S] + V^{(1,0)}[t, S], t\right] / 2] dt^4 -$$

$$\text{Expand}\left[D\left[-r V[t, S] + r S V^{(0,1)}[t, S] + \frac{1}{2} S^2 S^2 V^{(0,2)}[t, S] + V^{(1,0)}[t, S], S\right] S r / 2\right]$$

$$dt^4 - \text{Expand}\left[D\left[-r V[t, S] + r S V^{(0,1)}[t, S] + \frac{1}{2} S^2 S^2 V^{(0,2)}[t, S] + V^{(1,0)}[t, S], \{S, 2\}\right] S^2 S^2 / 4\right] dt^4, \{dt, 0, 4\}]$$

$$\left(-r V[t, S] + r S V^{(0,1)}[t, S] + \frac{1}{2} S^2 S^2 V^{(0,2)}[t, S] + V^{(1,0)}[t, S]\right) dt^2 +$$

$$\frac{1}{4} \left(-2 r^2 S^2 V^{(0,2)}[t, S] + 3 r S^2 S^2 V^{(0,2)}[t, S] + 4 r S^2 u V^{(0,2)}[t, S] - 2 S^2 S^2 u V^{(0,2)}[t, S] - 2 S^2 u^2 V^{(0,2)}[t, S] + 2 r V^{(1,0)}[t, S]\right) dt^4 + O[dt]^5$$

Collect[Expand[

$$\frac{1}{4} \left(-2 r^2 S^2 V^{(0,2)}[t, S] + 3 r S^2 S^2 V^{(0,2)}[t, S] + 4 r S^2 u V^{(0,2)}[t, S] - 2 S^2 S^2 u V^{(0,2)}[t, S] - 2 S^2 u^2 V^{(0,2)}[t, S] + 2 r V^{(1,0)}[t, S]\right), \{S^2 S^2 V^{(0,2)}[t, S], V^{(1,0)}[t, S]\}]$$

$$S^2 \left(-\frac{r^2}{2} + \frac{3 r S^2}{4} + r u - \frac{S^2 u}{2} - \frac{u^2}{2}\right) V^{(0,2)}[t, S] + \frac{1}{2} r V^{(1,0)}[t, S]$$

Expand[(u - r) (r - u - S^2)]

$$-r^2 + r S^2 + 2 r u - S^2 u - u^2$$