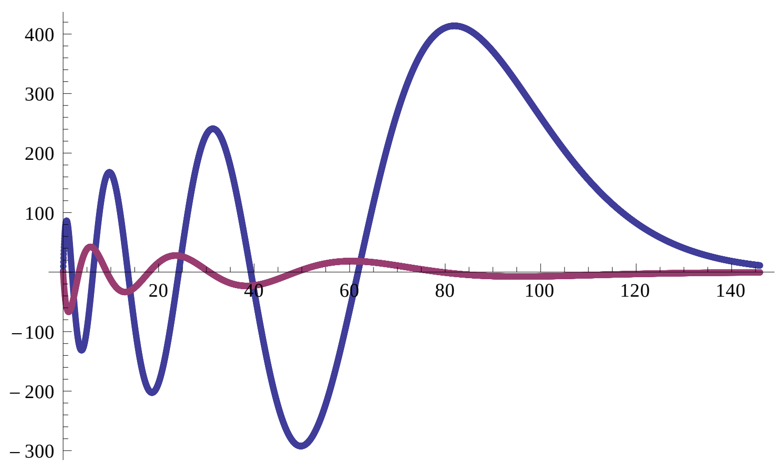
```
Exit[]

a = 7 297 352 537.6 * 10 ^ -12; M = 510 998.910; Z = 1; k = -1;
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
Table[N[Energie[i]], {i, 10}]

{13.6059, 3.40148, 1.51176, 0.850365,
 0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}

'

f[u_, r_] := Simplify[{(Z * a / r + 2 - Enn) * u[[2]] - k / r * u[[1]],
    k / r * u[[2]] + (Enn - Z * a / r) * u[[1]]}];
k = -1; Z = 1; U =.
n = 5000;
h = 20 000 / n;
Enn = 0.27766906844567757` / M;
u = {(91.35044102604739` (-3.662751763692355` + Enn) (1.6262886176197724` + Enn)) /
    ((-0.0109728221664999` + Enn) (181.38339842774778` + Enn)), -1};
r = 1; U = {{r, u}};
Do[
 k0 = h * f[u, r]; k1 = h * f[u + k0 / 2, r + h / 2];
 k2 = h * f[u + k1 / 2, r + h / 2]; k3 = h * f[u + k2, r + h];
 u += 1 / 6 * (k0 + 2 * k1 + 2 * k2 + k3); r += h;
 AppendTo[U, {r, u}], {n}]; x =.;
ListPlot[Table[{#[[1]] * a, (137 ^ (i - 2) * #[[2, i]]) ^ 1} & /@ U[[1 ;; n]], {i, 2}] // N,
 PlotRange → All]
```



```
lambda[n_] := Sqrt[1 - (1 - En[n] / M) ^ 2];
gamma := Sqrt[k ^ 2 - Z ^ 2 * a ^ 2];
f[r_, n_, k_] := -r ^ gamma * Exp[-lambda[n] * r] * (((n - 1 + gamma) / (1 - En[n] / M) - k) *
      Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r] +
     (n - 1) * Hypergeometric1F1[1 - (n - 1), 2 * gamma + 1, 2 * lambda[n] * r]);
g[r_, n_, k_] := Sqrt[2 * M / En[n] - 1] * r ^ gamma * Exp[-lambda[n] * r] *
   (((n - 1 + gamma) / (1 - En[n] / M) - k) *
      Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r] -
     (n - 1) * Hypergeometric1F1[1 - (n - 1), 2 * gamma + 1, 2 * lambda[n] * r]);
```

```
g[1, 1, -1] / f[1, 1, -1]
```

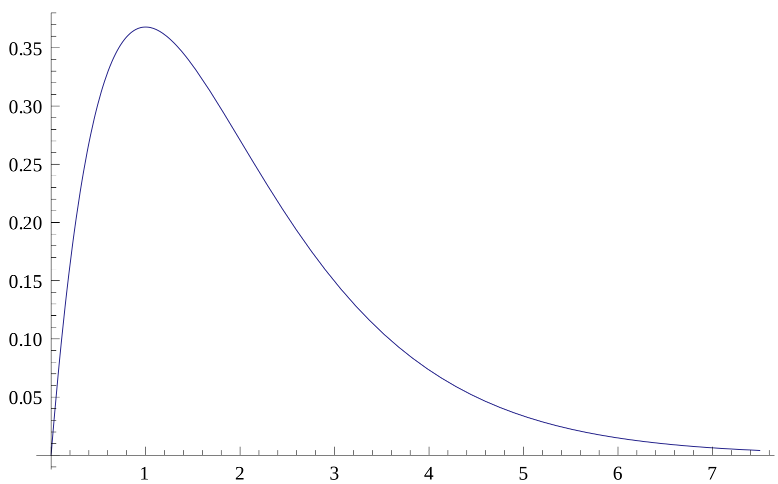
-274.068

```
a = 7297352537.6 * 10 ^ -12; M = 510998.910; Z = 1; k = -1;
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
lambda[n_] := Sqrt[1 - (1 - Energie[n] / M) ^ 2];
gamma := Sqrt[k ^ 2 - Z ^ 2 * a ^ 2];
```

```
Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r / M / a] *
  r ^ gamma * Exp[-lambda[n] * r / M / a] /. n → 1
```

$1.000000000000000\, e^{-1.95695\times 10^{-6}\, r}\, r^{0.999973}$

```
Plot[Hypergeometric1F1[-(n - 1), 2 * gamma + 1, 2 * lambda[n] * r / a] *
   r ^ gamma * Exp[-lambda[n] * r / a] /. n → 1, {r, 0, 7.5}, PlotRange → All]
```




# Randbedingungen

## r << 1

```
Exit[]
```

```
a = 7 297 352 537.6 * 10 ^ -12; M = 510 998.910; k = -1; Z = 1;
```

```
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
Table[N[Energie[i]], {i, 10}]
```

$$\left\{M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(1.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right), M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(2.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right),\right.$$

$$M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(3.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right), M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(4.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right),$$

$$M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(5.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right), M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(6.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right),$$

$$M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(7.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right), M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(8.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right),$$

$$\left.M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(9.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right), M\left(1.-\frac{1.}{\sqrt{1.+\frac{a^2 Z^2}{\left(10.+\sqrt{k^2-1.\,a^2 Z^2}-1.\,\mathrm{Abs}[k]\right)^2}}}\right)\right\}$$

```
e
```

```
f[u_, r_] := Simplify[{(Z * a / r + 2 - En) * u[[2]] - k / r * u[[1]],
    k / r * u[[2]] + (En - Z * a / r) * u[[1]]}];
```

$L = \sqrt{2\,\mathrm{En} - \mathrm{En}^2}$ `; L =.`

```
u := x ^ (s + n) * {a[n], b[n]} * Exp[-x * L];
```

```
g1 = Collect[Simplify[(f[u, x] - D[u, x]) / x^(s - 1) / Exp[-x * L]], x]; g1 // MatrixForm
```

$$\begin{pmatrix} -x^{1+n}\,(-L\,a[n]+(-2+En)\,b[n]) - x^n\,(k\,a[n]+n\,a[n]+s\,a[n]-a\,Z\,b[n]) \\ x^{1+n}\,(En\,a[n]+L\,b[n]) + x^n\,(-a\,Z\,a[n]+k\,b[n]-n\,b[n]-s\,b[n]) \end{pmatrix}$$

```
g2 = Table[Simplify[Sum[D[g1, {x, n2}] / n2!, {n, 0, 5}] /. x → 0], {n2, 0, 5}];
g2 // MatrixForm
```

$$\begin{pmatrix} -k\,a[0]-s\,a[0]+a\,Z\,b[0] & -a\,Z\,a[0]+(k-s)\,b[0] \\ L\,a[0]-(1+k+s)\,a[1]+2\,b[0]-En\,b[0]+a\,Z\,b[1] & En\,a[0]-a\,Z\,a[1]+L\,b[0]-b[1]+k\,b[1 \\ L\,a[1]-(2+k+s)\,a[2]+2\,b[1]-En\,b[1]+a\,Z\,b[2] & En\,a[1]-a\,Z\,a[2]+L\,b[1]-2\,b[2]+k\,b[ \\ L\,a[2]-(3+k+s)\,a[3]+2\,b[2]-En\,b[2]+a\,Z\,b[3] & En\,a[2]-a\,Z\,a[3]+L\,b[2]-3\,b[3]+k\,b[ \\ L\,a[3]-(4+k+s)\,a[4]+2\,b[3]-En\,b[3]+a\,Z\,b[4] & En\,a[3]-a\,Z\,a[4]+L\,b[3]-4\,b[4]+k\,b[ \\ L\,a[4]-(5+k+s)\,a[5]+2\,b[4]-En\,b[4]+a\,Z\,b[5] & En\,a[4]-a\,Z\,a[5]+L\,b[4]-5\,b[5]+k\,b[ \end{pmatrix}$$

```
Det[{{-s - k, a Z}, {-a Z, -s + k}}]
```

$$-k^2+s^2+a^2\,Z^2$$

```
Exit[]
```

```
g4 = Simplify[Inverse[{{(n + s - k), -a}, {a, (n + s + k)}}].{{L, -En + 2}, {En, L}}]
```

$$\left\{\left\{\frac{a\,En+L\,(k+n+s)}{a^2-k^2+(n+s)^2},\ \frac{-(-2+En)\,k+a\,L-(-2+En)\,(n+s)}{a^2-k^2+(n+s)^2}\right\},\ \left\{\frac{-a\,L+En\,(-k+n+s)}{a^2-k^2+(n+s)^2},\ \frac{a\,(-2+En)+L\,(-k+n+s)}{a^2-k^2+(n+s)^2}\right\}\right\}$$

$L = \sqrt{2\,En - En^2}$ `; g5 = Simplify[Eigenvalues[g4]]; g5`

$$\left\{0,\ \frac{2\left(a\,(-1+En)+\sqrt{-(-2+En)\,En}\,\,(n+s)\right)}{a^2-k^2+(n+s)^2}\right\}$$

$$\text{Simplify}\left[\frac{2\left(a\,(-1+En)+\sqrt{-(-2+En)\,En}\,\,(n+s)\right)}{a^2-k^2+(n+s)^2}\ \text{/. s → Sqrt[k^2 - a^2]}\right]$$

$$\frac{2\left(a\,(-1+En)+\sqrt{-(-2+En)\,En}\,\left(\sqrt{-a^2+k^2}\,+n\right)\right)}{n\left(2\sqrt{-a^2+k^2}\,+n\right)}$$

```
g6 = Simplify[Transpose[Eigenvectors[g4]]]
```

$$\left\{\left\{\frac{a\,(-2+En)+\sqrt{-(-2+En)\,En}\,\,(k+n+s)}{a\sqrt{-(-2+En)\,En}\,-En\,(k+n+s)},\ \frac{-a\,En+\sqrt{-(-2+En)\,En}\,\,(k-n-s)}{a\sqrt{-(-2+En)\,En}\,-En\,(k+n+s)}\right\},\ \{1,1\}\right\}$$

```
Inverse[{{T1, T2}, {1, 1}}]
```

$$\left\{\left\{\frac{1}{T1-T2},\ -\frac{T2}{T1-T2}\right\},\ \left\{-\frac{1}{T1-T2},\ \frac{T1}{T1-T2}\right\}\right\}$$

```
Simplify[Inverse[g6]].{0, 1}
```

$$\left\{\frac{a\ En+\sqrt{-(-2+En)\ En}\ (-k+n+s)}{2\left(a\ (-1+En)+\sqrt{-(-2+En)\ En}\ (n+s)\right)},\ \frac{a\ (-2+En)+\sqrt{-(-2+En)\ En}\ (k+n+s)}{2\left(a\ (-1+En)+\sqrt{-(-2+En)\ En}\ (n+s)\right)}\right\}$$

```
Exit[]
```

$$\frac{a\ En+\sqrt{-(-2+En)\ En}\ (-k+n+s)}{2\left(a\ (-1+En)+\sqrt{-(-2+En)\ En}\ (n+s)\right)}\Bigg/\left(\frac{a\ En+\sqrt{-(-2+En)\ En}\ (-k+n+s)}{2\left(a\ (-1+En)+\sqrt{-(-2+En)\ En}\ (n+s)\right)}\ /.\ n\to(n-1)\right)*\frac{2\left(a\ (-1+En)+\sqrt{-(-2+En)\ En}\ (n+s)\right)}{a^2-k^2+(n+s)^2}$$

$$\left(2\left(0.00729735\ (-1+En)+\sqrt{(2-En)\ En}\ (-1+n+s)\right)\left(0.00729735\ En+\sqrt{(2-En)\ En}\ (1+n+s)\right)\right)\Big/\left(\left(0.00729735\ En+\sqrt{(2-En)\ En}\ (n+s)\right)\left(-0.999947+(n+s)^2\right)\right)$$

```
Simplify[Inverse[g6].g4.g6]
```

$$\left\{\{0,0\},\left\{0,\left(2\left(a^2\ (-1+En)^2+2\ a\ (-1+En)\ \sqrt{-(-2+En)\ En}\ (n+s)-(-2+En)\ En\ (n+s)^2\right)\right)\Big/\left(\left(a\ (-1+En)+\sqrt{-(-2+En)\ En}\ (n+s)\right)\left(a^2-k^2+(n+s)^2\right)\right)\right\}\right\}$$

$$\text{Limit}\left[n\ \frac{a\ En+\sqrt{-(-2+En)\ En}\ (-k+n+s)}{2\left(a\ (-1+En)+\sqrt{-(-2+En)\ En}\ (n+s)\right)}\Bigg/\left(\frac{a\ En+\sqrt{-(-2+En)\ En}\ (-k+n+s)}{2\left(a\ (-1+En)+\sqrt{-(-2+En)\ En}\ (n+s)\right)}\ /.\ n\to(n-1)\right)*\frac{2\left(a\ (-1+En)+\sqrt{-(-2+En)\ En}\ (n+s)\right)}{a^2-k^2+(n+s)^2},\ \{n\to\text{Infinity}\}\right]$$

$$\left\{2\sqrt{-(-2+En)\ En}\right\}$$

```
{0.000026626031501830738`, 6.656530030069163`*^-6,
 2.958447944512166`*^-6, 1.6641228148817078`*^-6, 1.0650367227027502`*^-6,
 7.39607883493143`*^-7, 5.433848546676501`*^-7, 4.1602871314605494`*^-7,
 3.2871384492150213`*^-7, 2.6625808169367104`*^-7}
```

**#[[1, 2]] & /@ Solve[**
$$2\left(a(-1+En)+\sqrt{(2-En)\,En}\;(-1+n+s)\right)\left(a(-2+En)+\sqrt{(2-En)\,En}\;(k+n+s)\right)==0,$$
**En] M /. s -> Sqrt[k ^ 2 - (Z * a) ^ 2] /. n → 3**

$\{6.04705,\ 1.022\times10^{6},\ 1.51176,\ 1.022\times10^{6}\}$

**Table[Energie[n], {n, 1, 10 + 5}]**

{13.6053, 3.40141, 1.51174, 0.850356, 0.544228, 0.377936, 0.277667, 0.212589,
0.167972, 0.136057, 0.112444, 0.0944841, 0.0805071, 0.0694168, 0.0604698}

**k = -1;**

**Table[**
$$M\left(\left(k^2-2\sqrt{-a^2+k^2}\;n+n^2-\sqrt{\left(k^4+n^3\left(-4\sqrt{-a^2+k^2}+n\right)+k^2\left(-4\sqrt{-a^2+k^2}\;n+6\,n^2\right)-a^2\left(k^2+n\left(-2\sqrt{-a^2+k^2}+5\,n\right)\right)\right)}\right)\Big/\left(k^2+n\left(-2\sqrt{-a^2+k^2}+n\right)\right)\right),\ \{n, 6, 9+5\}]$$

**Exit[]**

{3.40137, 1.51173, 0.850351, 0.544225, 0.377935, 0.277666, 0.212588, 0.167971, 0.136057}

**$Assumptions = k > 0**

k > 0

**Simplify[Series[Energie[n] / M /. Z → 1, {a, 0, 4}]]**

$$\frac{a^2}{2\,n^2}+\frac{(-3\,k+4\,n)\,a^4}{8\,k\,n^4}+O[a]^5$$

**Simplify[Series[**$1-\sqrt{1-a^{\wedge}2\Big/\left(k^2+n\,2\sqrt{-a^2+k^2}+n^{\wedge}2\right)}$**, {a, 0, 4}] /. n → (n - k)]**

$$\frac{a^2}{2\,n^2}+\frac{(-3\,k+4\,n)\,a^4}{8\,k\,n^4}+O[a]^5$$

**k = -4; Table[M** $\left(1-\sqrt{1-a^{\wedge}2\Big/\left(k^2+n\,2\sqrt{-a^2+k^2}+n^{\wedge}2\right)}\right)$**, {n, 0, 9 + 5}]**

**M (1 − Sqrt[1 − a^2/(k² + n 2 √(−a² + k²) + n^2)])**

$$M\left(1-\sqrt{1-\frac{a^2}{k^2+2\sqrt{-a^2+k^2}\,n+n^2}}\right)$$

**Simplify[Expand[(n + √(k² − a² Z²) − Abs[k])^2]]**

$$k^2+n^2-a^2 Z^2+2\,n\sqrt{k^2-a^2 Z^2}-2\left(n+\sqrt{k^2-a^2 Z^2}\right)\mathrm{Abs}[k]+\mathrm{Abs}[k]^2$$

**Exit[]**

**Simplify[Solve[$\frac{2\left(a\,(-1+En)+\sqrt{-(-2+En)\,En}\,(n+s)\right)}{a^2-k^2+(n+s)^2}$ == 0, En] /. s -> Sqrt[k^2 − a^2]]**

$$\left\{\left\{En\to\left(k^2-2\sqrt{-a^2+k^2}\,n+n^2-\sqrt{\left(k^4+n^3\left(-4\sqrt{-a^2+k^2}+n\right)+k^2\left(-4\sqrt{-a^2+k^2}\,n+6\,n^2\right)-a^2\left(k^2+n\left(-2\sqrt{-a^2+k^2}+5\,n\right)\right)\right)}\right)\Big/\left(k^2+n\left(-2\sqrt{-a^2+k^2}+n\right)\right)\right\},\right.$$
$$\left.\left\{En\to\left(k^2-2\sqrt{-a^2+k^2}\,n+n^2+\sqrt{\left(k^4+n^3\left(-4\sqrt{-a^2+k^2}+n\right)+k^2\left(-4\sqrt{-a^2+k^2}\,n+6\,n^2\right)-a^2\left(k^2+n\left(-2\sqrt{-a^2+k^2}+5\,n\right)\right)\right)}\right)\Big/\left(k^2+n\left(-2\sqrt{-a^2+k^2}+n\right)\right)\right\}\right\}$$

**1 − ($\sqrt{\left(k^4+n^3\left(4\sqrt{-a^2+k^2}+n\right)+2\,k^2\,n\left(2\sqrt{-a^2+k^2}+3\,n\right)-a^2\left(k^2+n\left(2\sqrt{-a^2+k^2}+5\,n\right)\right)\right)}$) / ($k^2+n\left(2\sqrt{-a^2+k^2}+n\right)$)**

$$1-\left(\sqrt{\left(k^4+n^3\left(4\sqrt{-a^2+k^2}+n\right)+2\,k^2\,n\left(2\sqrt{-a^2+k^2}+3\,n\right)-a^2\left(k^2+n\left(2\sqrt{-a^2+k^2}+5\,n\right)\right)\right)}\right)\Big/\left(k^2+n\left(2\sqrt{-a^2+k^2}+n\right)\right)$$

**Expand[$k^4+n^3\left(4\sqrt{-a^2+k^2}+n\right)+2\,k^2\,n\left(2\sqrt{-a^2+k^2}+3\,n\right)-a^2\left(k^2+n\left(2\sqrt{-a^2+k^2}+5\,n\right)\right)$]**

$$-a^2 k^2+k^4-2\,a^2\sqrt{-a^2+k^2}\,n+4\,k^2\sqrt{-a^2+k^2}\,n-5\,a^2 n^2+6\,k^2 n^2+4\sqrt{-a^2+k^2}\,n^3+n^4$$

$$\textbf{Expand}\left[\left(k^2 + n\left(2\sqrt{-a^2+k^2} + n\right)\right)\text{^}2\right]$$

$$k^4 + 4k^2\sqrt{-a^2+k^2}\,n - 4a^2n^2 + 6k^2n^2 + 4\sqrt{-a^2+k^2}\,n^3 + n^4 - \left(-a^2k^2 + k^4 - 2a^2\sqrt{-a^2+k^2}\,n + 4k^2\sqrt{-a^2+k^2}\,n - 5a^2n^2 + 6k^2n^2 + 4\sqrt{-a^2+k^2}\,n^3 + n^4\right)$$

$$a^2k^2 + 2a^2\sqrt{-a^2+k^2}\,n + a^2n^2$$

```
#[[1, 2]] & /@ Solve[
    2 (a (-1 + En) + Sqrt[(2 - En) En] (-1 + n + s)) (a (-2 + En) + Sqrt[(2 - En) En] (k + n + s)) == 0,
    En] * M /. s -> Sqrt[k^2 - (Z * a)^2] /. n → 2
```

$$\Bigg\{2M,\ \frac{2a^2M}{4 + a^2 + 4k + 2k^2 - a^2Z^2 + 4\sqrt{k^2-a^2Z^2} + 2k\sqrt{k^2-a^2Z^2}},$$

$$\left(M\left(2 + 2a^2 + 4\sqrt{k^2-a^2Z^2} + 2\left(k^2-a^2Z^2\right) - \sqrt{\left(-4a^2\left(1 + a^2 + k^2 - a^2Z^2 + 2\sqrt{k^2-a^2Z^2}\right) + \left(-2 - 2a^2 - 4\sqrt{k^2-a^2Z^2} - 2\left(k^2-a^2Z^2\right)\right)^2\right)}\right)\right)\Big/\left(2\left(1 + a^2 + k^2 - a^2Z^2 + 2\sqrt{k^2-a^2Z^2}\right)\right),$$

$$\left(M\left(2 + 2a^2 + 4\sqrt{k^2-a^2Z^2} + 2\left(k^2-a^2Z^2\right) + \sqrt{\left(-4a^2\left(1 + a^2 + k^2 - a^2Z^2 + 2\sqrt{k^2-a^2Z^2}\right) + \left(-2 - 2a^2 - 4\sqrt{k^2-a^2Z^2} - 2\left(k^2-a^2Z^2\right)\right)^2\right)}\right)\right)\Big/\left(2\left(1 + a^2 + k^2 - a^2Z^2 + 2\sqrt{k^2-a^2Z^2}\right)\right)\Bigg\}$$

```
s = Sqrt[k^2 - (Z * a)^2];
```

```
En =.;
Simplify[Inverse[{{Z * a / En, (n + s - k) / En}, {(n + s + k) / (2 - En), Z * a / (En - 2)}}]]
```

$$S[n\_, En\_] := \left\{\left\{\frac{a\,En\,Z}{n^2 + 2\,n\,\sqrt{k^2 - a^2\,Z^2}}, -\frac{(-2 + En)\left(-k + n + \sqrt{k^2 - a^2\,Z^2}\right)}{n\left(n + 2\,\sqrt{k^2 - a^2\,Z^2}\right)}\right\}, \left\{\frac{En\left(k + n + \sqrt{k^2 - a^2\,Z^2}\right)}{n\left(n + 2\,\sqrt{k^2 - a^2\,Z^2}\right)}, \frac{a\,(-2 + En)\,Z}{n\left(n + 2\,\sqrt{k^2 - a^2\,Z^2}\right)}\right\}\right\};$$

$$DS[n\_, En\_] := \left\{\left\{\frac{a\,Z}{n^2 + 2\,n\,\sqrt{k^2 - a^2\,Z^2}}, -\frac{\left(-k + n + \sqrt{k^2 - a^2\,Z^2}\right)}{n\left(n + 2\,\sqrt{k^2 - a^2\,Z^2}\right)}\right\}, \left\{\frac{\left(k + n + \sqrt{k^2 - a^2\,Z^2}\right)}{n\left(n + 2\,\sqrt{k^2 - a^2\,Z^2}\right)}, \frac{a\,Z}{n\left(n + 2\,\sqrt{k^2 - a^2\,Z^2}\right)}\right\}\right\};$$

```
S[m, En] // MatrixForm
DS[m, En] // MatrixForm
```

$$\left\{\left\{\frac{a\,En\,Z}{-k^2 + n^2 + 2\,n\,s + s^2 + a^2\,Z^2}, \frac{(-2 + En)\,(k - n - s)}{-k^2 + n^2 + 2\,n\,s + s^2 + a^2\,Z^2}\right\}, \left\{\frac{En\,(k + n + s)}{-k^2 + n^2 + 2\,n\,s + s^2 + a^2\,Z^2}, \frac{a\,(-2 + En)\,Z}{-k^2 + n^2 + 2\,n\,s + s^2 + a^2\,Z^2}\right\}\right\}$$

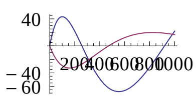
$$\begin{pmatrix} \frac{a\,En\,Z}{m^2 + 2\,m\,\sqrt{k^2 - a^2\,Z^2}} & -\frac{(-2 + En)\left(-k + m + \sqrt{k^2 - a^2\,Z^2}\right)}{m\left(m + 2\,\sqrt{k^2 - a^2\,Z^2}\right)} \\ \frac{En\left(k + m + \sqrt{k^2 - a^2\,Z^2}\right)}{m\left(m + 2\,\sqrt{k^2 - a^2\,Z^2}\right)} & \frac{a\,(-2 + En)\,Z}{m\left(m + 2\,\sqrt{k^2 - a^2\,Z^2}\right)} \end{pmatrix}$$

$$\begin{pmatrix} \frac{a\,Z}{m^2 + 2\,m\,\sqrt{k^2 - a^2\,Z^2}} & -\frac{-k + m + \sqrt{k^2 - a^2\,Z^2}}{m\left(m + 2\,\sqrt{k^2 - a^2\,Z^2}\right)} \\ \frac{k + m + \sqrt{k^2 - a^2\,Z^2}}{m\left(m + 2\,\sqrt{k^2 - a^2\,Z^2}\right)} & \frac{a\,Z}{m\left(m + 2\,\sqrt{k^2 - a^2\,Z^2}\right)} \end{pmatrix}$$

```
S[n, En].u
```

$S[n, En].\{x^{n+s}\,a[n], x^{n+s}\,b[n]\}$

```
UN[R_, N_, En_] := Module[{u = {1, (k + s) / Z / a}, U = {1, (k + s) / Z / a} * R ^ s},
  For[n = 1, n < N, n++,
   u = S[n, En].u;
   U += u * R ^ (s + n);
  ];

  U]

U[r_, g_, En_] :=
 Module[{u = {1, (k + s) / Z / a}, U = {0, 0}, DU = {0, 0}, du = {0, 0}, n = 0},
  Label[begin];
  U += u * r ^ (s + n);
  DU += du * r ^ (s + n);
  n++;
  du = DS[n, En].u + S[n, En].du;
  u = S[n, En].u;
  If[(#[[1]] > g  || #[[2]] > g) &[Abs[u * R ^ (s + n) / U /. r → R]], Goto[begin]];
  {n, U, DU}]

R = 1000; g = 0.01; rU = U[r, g, 1 / M];
Plot[{rU[[2, 1]], rU[[2, 2]] * 137}, {r, 0, R}, PlotRange → All]
```

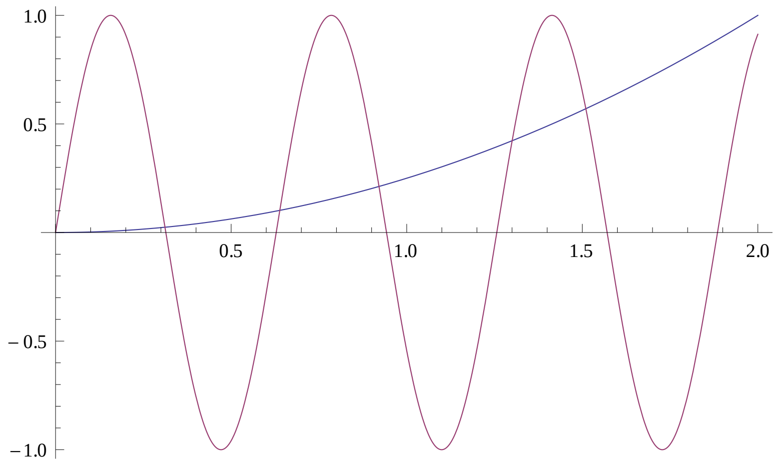



```
EN[iEn_, g2_] := Module[{rU, fU, n = 0, i, En = iEn, ll},
  Label[begin];
  fU = U[r, g, En];

  rU = fU /. r → R;

  If[rU[[2, 1]] * rU[[2, 2]] > 0,

   En -= (rU[[2, 1]] + rU[[2, 2]]) / (rU[[3, 1]] + rU[[3, 2]]);

   n++
    Goto[begin];
  ];

  {n, En * M, Abs[rU[[2, 1]] - rU[[2, 2]]]}
 ]

R = 3000; g = 0.001; EN[13 / M, 0.1]
{8, 13.6059, 0.000441718}

- {0, 13.605873075061169
```

```
R = 2000; g = 0.001;

plot[{rU[[2, 1]], rU[[3, 1]]}, 100, R]

En = 4 / M; rU[[2, 1]]
129.072

n = 0; x
10

n = 0; While[x = n; x < 10, n++; Print[n]]
1
2
3
4
5
6
7
8
9
10
```

```
plot[liste_, R_] := Module[{nN = 100, table, max, st = {Red, Green, Blue}},
  liste / (Max[Abs[#]] & /@ (Table[# /. r → i * R / nN, {i, 0, nN}] & /@ liste))
 ]
ll = plot[{r ^ 2, Sin[10 * r]}, 2]
Plot[ll, {r, 0, 2}]
```

$\left\{\frac{r^2}{4}, -\mathrm{Csc}[11]\ \mathrm{Sin}[10\ r]\right\}$




# r gegen Inifinity

```
Exit[]
```

```
a = 7 297 352 537.6 * 10 ^ -12; M = 510 998.910; k = -1; Z = 1;
```

```
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
Table[N[Energie[i]], {i, 10}]
```

{13.6059, 3.40148, 1.51176, 0.850365,
 0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}

```
$Assumptions = 1 > En > 0;
```

```
s = (En - 1) * Z * a / L; L := Sqrt[(2 - En) * En]; L =.; s =.
```

```
f[u_, r_] := Simplify[{(Z * a / r + 2 - En) * u[[2]] - k / r * u[[1]],
    k / r * u[[2]] + (En - Z * a / r) * u[[1]]}];
```

```
u =.
```

```
U = {(hb[r] - ha[r]) * L / En, hb[r] + ha[r]} * Exp[-r * L];
```

```
#[[2]] & /@ Simplify[Solve[Simplify[(f[U, r] - D[U, r]) / Exp[-r * L] * En * r] == 0,
      {ha'[r], hb'[r]}][[1]]] /. ha[r] → u[[1]] /. hb[r] → u[[2]] /. r → rr
```

Part::partd : Part specification u⟦1⟧ is longer than depth of object. ≫

Part::partd : Part specification u⟦2⟧ is longer than depth of object. ≫

$$\Big\{\frac{1}{2\,\mathrm{En}\,L\,\mathrm{rr}}\Big(\big(\mathrm{En}^3\,\mathrm{rr}+\mathrm{En}\,L^2\,\mathrm{rr}+a\,L^2\,Z-\mathrm{En}^2\,(2\,\mathrm{rr}+a\,Z)\big)\,u[\![1]\!]+\big(\mathrm{En}^3\,\mathrm{rr}+\mathrm{En}\,L\,(2\,k+L\,\mathrm{rr})-a\,L^2\,Z-\mathrm{En}^2\,(2\,\mathrm{rr}+a\,Z)\big)\,u[\![2]\!]\Big),$$
$$\frac{1}{2\,\mathrm{En}\,L\,\mathrm{rr}}\Big(\big(-\mathrm{En}^3\,\mathrm{rr}+\mathrm{En}\,L\,(2\,k-L\,\mathrm{rr})+a\,L^2\,Z+\mathrm{En}^2\,(2\,\mathrm{rr}+a\,Z)\big)\,u[\![1]\!]+\big(-\mathrm{En}^3\,\mathrm{rr}+3\,\mathrm{En}\,L^2\,\mathrm{rr}-a\,L^2\,Z+\mathrm{En}^2\,(2\,\mathrm{rr}+a\,Z)\big)\,u[\![2]\!]\Big)\Big\}$$

$$\mathbf{F[u\_,\ rr\_] := Simplify\Big[\Big\{\frac{-a\,(-1+En)\,Z\,u[\![1]\!]+\left(\sqrt{-(-2+En)\,En}\;k-a\,Z\right)u[\![2]\!]}{\sqrt{-(-2+En)\,En}\;rr},}$$
$$\mathbf{\Big(\Big(\sqrt{-(-2+En)\,En}\;k+a\,Z\Big)\,u[\![1]\!]+\left(4\,En\,rr-2\,En^2\,rr-a\,Z+a\,En\,Z\right)u[\![2]\!]\Big)\Big/\Big(\sqrt{-(-2+En)\,En}\;rr\Big)\Big\}\Big]}$$

```
r[x_] := 1 / x; u := {a[n], b[n]} * x ^ (n + s)
```

```
g1 = Collect[Simplify[(F[u, r[x]] * D[r[x], x] - D[u, x]) * x ^ (2 - s)], {a[n], b[n]}]; g1
```

$$\Big\{-\Big(x^{1+n}\Big(a\,(-1+\mathrm{En})\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}\;Z+L\Big(\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}\;n+a\,Z-a\,\mathrm{En}\,Z\Big)\Big)\,a[n]\Big)\Big/\Big(\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}\;L\Big)-\frac{x^{1+n}\left(\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}\;k-a\,Z\right)b[n]}{\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}},$$
$$-\frac{x^{1+n}\left(\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}\;k+a\,Z\right)a[n]}{\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}}-\frac{1}{\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}\;L}\,x^n\Big(-2\,\mathrm{En}^2\,L+x\Big(\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}\;L\,n-a\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}\;Z-a\,L\,Z\Big)+\mathrm{En}\Big(4\,L+a\sqrt{-(-2+\mathrm{En})\,\mathrm{En}}\;x\,Z+a\,L\,x\,Z\Big)\Big)\,b[n]\Big\}$$

$$\mathbf{g2 := \{x^n\,(-n)\,a[n]+x^n\,(-k+a\,Z/L)\,b[n],\ -x^{1+n}\,(k+a\,Z/L)\,a[n]+x^n\,(-n\,x-2\,s\,x+-2*L)\,b[n]\};\ g2\ //\ MatrixForm}$$

$$\begin{pmatrix} -n\,x^n\,a[n]+x^n\left(-k+\frac{a\,Z}{L}\right)b[n] \\ -x^{1+n}\left(k+\frac{a\,Z}{L}\right)a[n]+x^n\,(-2\,L-n\,x-2\,s\,x)\,b[n] \end{pmatrix}$$

```
a[n_] := (Z * a / L - k) / n * b[n]
```

```
g2[[2]]
```

$$x^n\,(-2\,L-n\,x-2\,s\,x)\,b[n]-\frac{x^{1+n}\left(-k+\frac{a\,Z}{L}\right)\left(k+\frac{a\,Z}{L}\right)b[n]}{n}$$

```
M1 = {{L, 2 - En}, {En, L}}; Eigenvalues[M1]; B = Transpose[Eigenvectors[M1]];
```

```
$Assumptions = 1 > En > 0
```

$1 > En > 0$

# = >

```
Exit[]
a[n_] := (Z * a / L - k) / n * b[n]
b[n_] := b0 * Product[((k ^ 2 - a ^ 2 * Z ^ 2 / L ^ 2) / i - (i + 2 * s)) / 2 / L, {i, 1, n}]
b[4]
```

$$\frac{1}{16\,L^4}\,b0\left(-1+k^2-2\,s-\frac{a^2 Z^2}{L^2}\right)\left(-4-2\,s+\frac{1}{4}\left(k^2-\frac{a^2 Z^2}{L^2}\right)\right)\left(-3-2\,s+\frac{1}{3}\left(k^2-\frac{a^2 Z^2}{L^2}\right)\right)\left(-2-2\,s+\frac{1}{2}\left(k^2-\frac{a^2 Z^2}{L^2}\right)\right)$$

```
((k ^ 2 - a ^ 2 * Z ^ 2 / L ^ 2) / i - (i + 2 * s)) / 2 / L
```

$$\frac{-i-2\,s+\frac{k^2-\frac{a^2 Z^2}{L^2}}{i}}{2\,L}$$

```
Exit[];
En =.;
```

$$\text{Solve}\left[-i*\sqrt{(2-En)\,En}\;-2\,a\,(-1+En)\;+\frac{\sqrt{(2-En)\,En}\;-\frac{a^2}{\sqrt{(2-En)\,En}}}{i}\;==0,\,En\right]\;//\;\text{MatrixForm}$$

$$\begin{pmatrix}
En\to\frac{2-4\,i^2+8\,a^2\,i^2+2\,i^4-\sqrt{\left(-2+4\,i^2-8\,a^2\,i^2-2\,i^4\right)^2-4\left(1-2\,i^2+4\,a^2\,i^2+i^4\right)\left(a^2+a^2\,i^2-2\sqrt{a^4\,i^2-a^6\,i^2}\right)}}{2\left(1-2\,i^2+4\,a^2\,i^2+i^4\right)}\\
En\to\frac{2-4\,i^2+8\,a^2\,i^2+2\,i^4+\sqrt{\left(-2+4\,i^2-8\,a^2\,i^2-2\,i^4\right)^2-4\left(1-2\,i^2+4\,a^2\,i^2+i^4\right)\left(a^2+a^2\,i^2-2\sqrt{a^4\,i^2-a^6\,i^2}\right)}}{2\left(1-2\,i^2+4\,a^2\,i^2+i^4\right)}\\
En\to\frac{2-4\,i^2+8\,a^2\,i^2+2\,i^4-\sqrt{\left(-2+4\,i^2-8\,a^2\,i^2-2\,i^4\right)^2-4\left(1-2\,i^2+4\,a^2\,i^2+i^4\right)\left(a^2+a^2\,i^2+2\sqrt{a^4\,i^2-a^6\,i^2}\right)}}{2\left(1-2\,i^2+4\,a^2\,i^2+i^4\right)}\\
En\to\frac{2-4\,i^2+8\,a^2\,i^2+2\,i^4+\sqrt{\left(-2+4\,i^2-8\,a^2\,i^2-2\,i^4\right)^2-4\left(1-2\,i^2+4\,a^2\,i^2+i^4\right)\left(a^2+a^2\,i^2+2\sqrt{a^4\,i^2-a^6\,i^2}\right)}}{2\left(1-2\,i^2+4\,a^2\,i^2+i^4\right)}
\end{pmatrix}$$

'

$$ET[i\_, k\_, Z\_, a\_] := \left(2\, i^4 - 4\, i^2 k^2 + 2\, k^4 + 8\, a^2 i^2 Z^2 - \sqrt{\left(\left(-2\, i^4 + 4\, i^2 k^2 - 2\, k^4 - 8\, a^2 i^2 Z^2\right)^2 - 4\left(i^4 - 2\, i^2 k^2 + k^4 + 4\, a^2 i^2 Z^2\right)\left(a^2 i^2 Z^2 + a^2 k^2 Z^2 + 2\sqrt{a^4 i^2 k^2 Z^4 - a^6 i^2 Z^6}\right)\right)}\right) \Big/ \left(2\left(i^4 - 2\, i^2 k^2 + k^4 + 4\, a^2 i^2 Z^2\right)\right)$$

```
Table[M * ET[n, -1, 1, a] - Energie[n + 1], {n, 0, 10}]
```

$\{-5.67315\times10^{-11}, -1.9084\times10^{-7}, 5.30991\times10^{-11}, -1.3482\times10^{-10}, 3.24187\times10^{-11}, -5.86517\times10^{-11}, 5.5719\times10^{-11}, 2.10913\times10^{-11}, 1.87879\times10^{-11}, 1.01339\times10^{-10}, -1.46405\times10^{-12}\}$

```
Series[M * ET[n, -1, 1, a] - Energie[n + 1], {n, 0, 5}]
```

$-5.67315\times10^{-11} - 1.26477\times10^{-10}\, n^2 - 2.84217\times10^{-14}\, n^3 - 2.54019\times10^{-10}\, n^4 + 1.42109\times10^{-14}\, n^5 + O[n]^6$

```
M = 510998.910;

s = (En - 1) * Z * a / L; L := Sqrt[(2 - En) * En];

a = 7297352537.6 * 10 ^ -12; M = 510998.910; Z = 1; k = -1;
Energie[n_] := M * (1 - 1 / Sqrt[1 + (Z * a / (n - Abs[k] + Sqrt[k ^ 2 - (Z * a) ^ 2])) ^ 2]);
Table[N[Energie[i]], {i, 10}]
```

{13.6059, 3.40148, 1.51176, 0.850365, 0.544233, 0.377939, 0.277669, 0.21259, 0.167972, 0.136058}

# Verhältnis bei r=0

```
a = 7297352537.6 * 10 ^ -12; M = 510998.910; k = -1; Z = 1;
s = Sqrt[k ^ 2 - (Z * a) ^ 2];
```

$$S[n\_] := \left\{\left\{\frac{a\, En\, Z}{n^2 + 2\, n\sqrt{k^2 - a^2 Z^2}}, -\frac{(-2 + En)\left(-k + n + \sqrt{k^2 - a^2 Z^2}\right)}{n\left(n + 2\sqrt{k^2 - a^2 Z^2}\right)}\right\}, \left\{\frac{En\left(k + n + \sqrt{k^2 - a^2 Z^2}\right)}{n\left(n + 2\sqrt{k^2 - a^2 Z^2}\right)}, \frac{a\,(-2 + En)\, Z}{n\left(n + 2\sqrt{k^2 - a^2 Z^2}\right)}\right\}\right\} /.\ En \to Enn;$$

```
S[
 10]
```

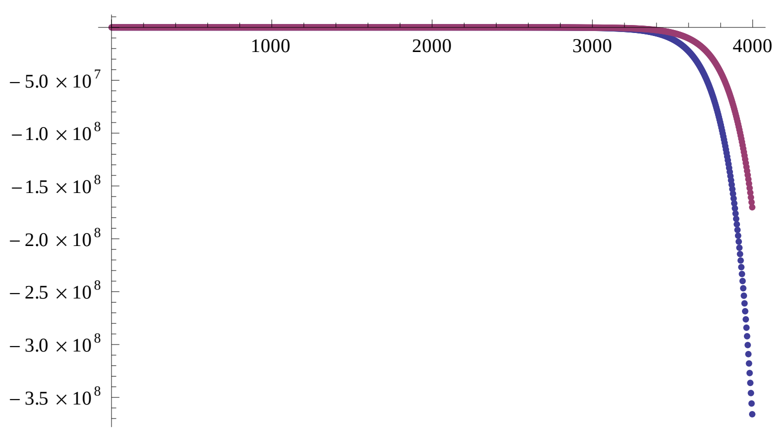
{{0.0000608115 Enn, -0.1 (-2 + Enn)}, {0.0833335 Enn, 0.0000608115 (-2 + Enn)}}

```
Enn =.; u = {1, (k + s) / Z / a}; U = u;
For [n = 1, n < 3, n ++,
 u = S[n].u;
 U = Simplify [U + u];
]; n =.;
Simplify [U[[1]] / U[[2]]]
```

$$-\frac{91.3504\ (-3.66275 + \text{Enn})\ (1.62629 + \text{Enn})}{(-0.0109728 + \text{Enn})\ (181.383 + \text{Enn})}$$

# Runge von links

```
f[u_, r_] := Simplify[{(Z * a / r + 2 - Enn) * u[[2]] - k / r * u[[1]],
    k / r * u[[2]] + (Enn - Z * a / r) * u[[1]]}];
k = -1; Z = 1; U =.
n = 1000;
h = 4000 / n;
Enn = 13.605 / M;
u = {(91.35044102604739` (-3.662751763692355` + Enn) (1.6262886176197724` + Enn)) /
    ((-0.0109728221664999` + Enn) (181.38339842774778` + Enn)), -1};
r = 1; U = {{r, u}};
Do[
 k0 = h * f[u, r]; k1 = h * f[u + k0 / 2, r + h / 2];
 k2 = h * f[u + k1 / 2, r + h / 2]; k3 = h * f[u + k2, r + h];
 u += 1 / 6 * (k0 + 2 * k1 + 2 * k2 + k3); r += h;
 AppendTo[U, {r, u}], {n}]; x =.;
ListPlot[
 Table[{#[[1]], 137 ^ (i - 2) * #[[2, i]]} & /@ U[[1 ;; n]], {i, 2}] // N, PlotRange → All]
```




```
Sum[A[n] * r ^ n / n !, {n, 0, 10}]
```

$$A[0] + r\,A[1] + \frac{1}{2} r^2 A[2] + \frac{1}{6} r^3 A[3] + \frac{1}{24} r^4 A[4] + \frac{1}{120} r^5 A[5] + \frac{1}{720} r^6 A[6] + \frac{r^7 A[7]}{5040} + \frac{r^8 A[8]}{40320} + \frac{r^9 A[9]}{362880} + \frac{r^{10} A[10]}{3628800}$$

```
D[%, {r, 4}]
```

$$A[4] + r\,A[5] + \frac{1}{2}\,r^2\,A[6] + \frac{1}{6}\,r^3\,A[7] + \frac{1}{24}\,r^4\,A[8] + \frac{1}{120}\,r^5\,A[9] + \frac{1}{720}\,r^6\,A[10]$$

```
Sum[A[n + 4] * r ^ n / n !, {n, 0, 10}]
```

$$A[4] + r\,A[5] + \frac{1}{2}\,r^2\,A[6] + \frac{1}{6}\,r^3\,A[7] + \frac{1}{24}\,r^4\,A[8] + \frac{1}{120}\,r^5\,A[9] + \frac{1}{720}\,r^6\,A[10] + \frac{r^7\,A[11]}{5040} + \frac{r^8\,A[12]}{40\,320} + \frac{r^9\,A[13]}{362\,880} + \frac{r^{10}\,A[14]}{3\,628\,800}$$