

$$\text{simplify} \left[\frac{(1 - \mu h A) u_1}{1 + (1 - \mu) h A} - \text{Exp}[-h A] u_1 + O[h]^3 \right]$$

$$\frac{1}{2} A^2 u_1 (1 - 2 \mu) h^2 + O[h]^3$$

$$u[h + t] - u[t] + O[h]^3$$

$$u'[t] h + \frac{1}{2} u''[t] h^2 + O[h]^3$$

$$u[t] - u[t - h] + O[h]^3$$

$$u'[t] h - \frac{1}{2} u''[t] h^2 + O[h]^3$$

$$\text{Series}[u[t + h], \{h, 0, 3\}]$$

$$u[t] + u'[t] h + \frac{1}{2} u''[t] h^2 + \frac{1}{6} u^{(3)}[t] h^3 + O[h]^4$$

$$\text{Series}[u[t - h], \{h, 0, 3\}]$$

$$u[t] - u'[t] h + \frac{1}{2} u''[t] h^2 - \frac{1}{6} u^{(3)}[t] h^3 + O[h]^4$$

$$u[t + h] - u[t] - h ((1 - \mu) (D[u[x], x] /. x \rightarrow t + h) + \mu D[u[x], x] /. x \rightarrow t)$$

$$-u[t] + u[h + t] - h (\mu u'[t] + (1 - \mu) u'[h + t])$$

$$\text{Series}[\%, \{h, 0, 3\}]$$

$$\left(-\frac{1}{2} u''[t] + \mu u''[t] \right) h^2 + \left(-\frac{1}{3} u^{(3)}[t] + \frac{1}{2} \mu u^{(3)}[t] \right) h^3 + O[h]^4$$

$$f[t_]:=D[u[x],x]+A u[x] /. x \rightarrow t$$

$$\text{Series}[\mu f[t + h / 2] + (1 - \mu) f[t - h / 2], \{h, 0, 3\}] /. \mu \rightarrow \frac{1}{2}$$

$$(A u[t] + u'[t]) + \frac{1}{8} (A u''[t] + u^{(3)}[t]) h^2 + O[h]^4$$

$$u(h+t)-u(t) == h u'(h+t) + O[h]^2$$

$$\frac{1}{2} h (u'[t] + u'[h + t]) = u'[t] h + \frac{1}{2} u''[t] h^2 + O[h]^3$$

$$\frac{h}{2} (u'[h + t] + u'[t]) - u[h + t] + u[t] + O[h]^3$$

$$O[h]^3$$

$$-u[t] + u[h + t] - h (\mu u'[t] + (1 - \mu) u'[h + t]) /. \mu \rightarrow \frac{1}{2}$$

$$-u[t] + u[h + t] - h \left(\frac{u'[t]}{2} + \frac{1}{2} u'[h + t] \right)$$

$$\text{CN}[\mathbf{A_}] := \frac{1 - \mathbf{A} / 2}{1 + \mathbf{A} / 2}$$

$$\text{CN}[\mathbf{h} \mathbf{A} / 2] \text{CN}[\mathbf{h} \mathbf{B}] \text{CN}[\mathbf{h} \mathbf{A} / 2] - \text{CN}[\mathbf{h} (\mathbf{A} + \mathbf{B})] + \mathcal{O}[\mathbf{h}]^4$$

$$\frac{1}{16} \left(\mathbf{A}^3 + 4 \mathbf{A}^2 \mathbf{B} + 4 \mathbf{A} \mathbf{B}^2 \right) \mathbf{h}^3 + \mathcal{O}[\mathbf{h}]^4$$

$$\text{Simplify}[\text{MatrixExp}[\{\{0, -1\}, \{0, 0\}\}]]$$

$$\{\{1, -1\}, \{0, 1\}\}$$

$$(1 - \mathbf{h} \mathbf{A} / 2) \mathbf{u}[\mathbf{t} + \mathbf{h}] - (1 + \mathbf{h} \mathbf{A} / 2) \mathbf{u}[\mathbf{t}] - (\mathbf{D}[\mathbf{u}[\mathbf{t}], \mathbf{t}] - \mathbf{A} \mathbf{u}[\mathbf{t}]) / 2 \mathbf{h} -$$

$$((\mathbf{D}[\mathbf{u}[\mathbf{t}], \mathbf{t}] - \mathbf{A} \mathbf{u}[\mathbf{t}]) / 2 /. \mathbf{t} \rightarrow \mathbf{t} + \mathbf{h}) \mathbf{h} + \mathcal{O}[\mathbf{h}]^4$$

$$- \frac{1}{12} \mathbf{u}^{(3)}[\mathbf{t}] \mathbf{h}^3 + \mathcal{O}[\mathbf{h}]^4$$

$$\frac{(1 + \mathbf{h} \mathbf{A} / 2)}{(1 - \mathbf{h} \mathbf{A} / 2)} \mathbf{u}[\mathbf{t}] - \text{Exp}[\mathbf{h} \mathbf{A}] \mathbf{u}[\mathbf{t}] + \mathcal{O}[\mathbf{h}]^4$$

$$\frac{1}{12} \mathbf{A}^3 \mathbf{u}[\mathbf{t}] \mathbf{h}^3 + \mathcal{O}[\mathbf{h}]^4$$

$$\mathbf{A} = \mathbf{A1} + \mathbf{A2};$$

$$\mathbf{a} = \mathbf{u2} - \frac{(1 - \mathbf{h} \mathbf{A} / 2)}{(1 + \mathbf{h} \mathbf{A} / 2)} \mathbf{u1}$$

$$- \frac{\left(1 - \frac{1}{2} (\mathbf{A1} + \mathbf{A2}) \mathbf{h}\right) \mathbf{u1}}{1 + \frac{1}{2} (\mathbf{A1} + \mathbf{A2}) \mathbf{h}} + \mathbf{u2}$$

$$\mathbf{b} = \mathbf{u2} - \frac{(1 - \mathbf{h} \mathbf{A1} / 4) (1 - \mathbf{h} \mathbf{A2} / 2) (1 - \mathbf{h} \mathbf{A1} / 4)}{(1 + \mathbf{h} \mathbf{A1} / 4) (1 + \mathbf{h} \mathbf{A2} / 2) (1 + \mathbf{h} \mathbf{A1} / 4)} \mathbf{u1}$$

$$- \frac{\left(1 - \frac{\mathbf{A1} \mathbf{h}}{4}\right)^2 \left(1 - \frac{\mathbf{A2} \mathbf{h}}{2}\right) \mathbf{u1}}{\left(1 + \frac{\mathbf{A1} \mathbf{h}}{4}\right)^2 \left(1 + \frac{\mathbf{A2} \mathbf{h}}{2}\right)} + \mathbf{u2}$$

$$\mathbf{a} - \mathbf{b} + \mathcal{O}[\mathbf{h}]^3$$

$$\mathcal{O}[\mathbf{h}]^3$$

$$\mathbf{a} = (1 + \mathbf{h} \mathbf{A} / 2) \mathbf{u}[\mathbf{t} + \mathbf{h}] - (1 - \mathbf{h} \mathbf{A} / 2) \mathbf{u}[\mathbf{t}]$$

$$- \left(1 - \frac{1}{2} (\mathbf{A1} + \mathbf{A2}) \mathbf{h}\right) \mathbf{u}[\mathbf{t}] + \left(1 + \frac{1}{2} (\mathbf{A1} + \mathbf{A2}) \mathbf{h}\right) \mathbf{u}[\mathbf{h} + \mathbf{t}]$$

$$\mathbf{b} = (1 + \mathbf{h} \mathbf{A1} / 4) (1 + \mathbf{h} \mathbf{A2} / 2) (1 + \mathbf{h} \mathbf{A1} / 4) \mathbf{u}[\mathbf{t} + \mathbf{h}] -$$

$$(1 - \mathbf{h} \mathbf{A1} / 4) (1 - \mathbf{h} \mathbf{A2} / 2) (1 - \mathbf{h} \mathbf{A1} / 4) \mathbf{u}[\mathbf{t}]$$

$$- \left(1 - \frac{\mathbf{A1} \mathbf{h}}{4}\right)^2 \left(1 - \frac{\mathbf{A2} \mathbf{h}}{2}\right) \mathbf{u}[\mathbf{t}] + \left(1 + \frac{\mathbf{A1} \mathbf{h}}{4}\right)^2 \left(1 + \frac{\mathbf{A2} \mathbf{h}}{2}\right) \mathbf{u}[\mathbf{h} + \mathbf{t}]$$

$$u[t+h] - (1-h A) u[t] - (D[u[t], t] + A u[t]) h + O[h]^3$$

$$\frac{1}{2} u''[t] h^2 + O[h]^3$$

$$u[t+h] - (1-h A1) (1-h A2) u[t] - (D[u[t], t] + A u[t]) h + O[h]^3$$

$$\left(-A1 A2 u[t] + \frac{u''[t]}{2} \right) h^2 + O[h]^3$$

Clear[A]

\$Assumptions =

Element[h, Reals] && Element[A, Reals] && Element[B, Reals] && Element[u, Reals]

h ∈ Reals && A ∈ Reals && B ∈ Reals && u ∈ Reals

u1 = Max[0, (1-h (A+B)) u]

Max[0, (1-(A+B) h) u]

u2 = Max[0, (1-h B) Max[0, (1-h A) u]]

Max[0, (1-B h) Max[0, (1-A h) u]]

Series[u1 - u2, {h, 0, 3}]

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[ u + (-A u - B u) h + A B u h^2 + O[h]^4 (B == 0 && u > 0 && A u == 0) ||
    (B == 0 && u > 0 && A u ≥ 0 && A u - B u > 0) ||
    (B ≤ 0 && u > 0 && A u == 0 && A u + B u < 0) ||
    (B ≥ 0 && u > 0 && A u == 0 && A u - B u < 0) ||
    (B > 0 && u > 0 && A u > 0 && A u - B u ≥ 0) ||
    (B < 0 && u ≠ 0 && A u > 0 && A u + B u ≤ 0) ||
    (B > 0 && u ≠ 0 && A u > 0 && A u - B u ≤ 0) ||
    (B == 0 && u > 0 && A u ≤ 0 && A u + B u < 0 && A u^2 < 0) ||
    (B < 0 && u > 0 && A u > 0 &&
        A u + B u ≥ 0 && A u - B u > 0) ||
    (B > 0 && u > 0 && A u < 0 && A u + B u ≤ 0 && A u^2 < 0) ||
    (B > 0 && u ≠ 0 && A u < 0 && A u + B u ≥ 0 &&
        A u - B u < 0) || (B < 0 && u > 0 && A u < 0 &&
        A u + B u < 0 && A u - B u ≤ 0 && A u^2 < 0) ||
    (B < 0 && u ≠ 0 && A u < 0 && A u + B u < 0 &&
        A u - B u > 0 && A u^2 < 0)
0
u ≤ 0
[ u + (-A u - B u) h + A B u h^2 + O[h]^4 ! (B == 0 && u > 0 && A u == 0) ||
    (B == 0 && u > 0 && A u ≥ 0 && A u - B u > 0) ||
    (B ≤ 0 && u > 0 && A u == 0 && A u + B u < 0) ||
    (B ≥ 0 && u > 0 && A u == 0 && A u - B u < 0) ||
    (B > 0 && u > 0 && A u > 0 && A u - B u ≥ 0) ||
    (B < 0 && u ≠ 0 && A u > 0 && A u + B u ≤ 0) ||
    (B > 0 && u ≠ 0 && A u > 0 && A u - B u ≤ 0) ||
    (B == 0 && u > 0 && A u ≤ 0 && A u + B u < 0 &&
        A u^2 < 0) || (B < 0 && u > 0 && A u > 0 &&
        A u + B u ≥ 0 && A u - B u > 0) || (B > 0 &&
        u > 0 && A u < 0 && A u + B u ≤ 0 && A u^2 < 0) ||
    (B > 0 && u ≠ 0 && A u < 0 && A u + B u ≥ 0 &&
        A u - B u < 0) || (B < 0 && u > 0 && A u < 0 &&
        A u + B u < 0 && A u - B u ≤ 0 && A u^2 < 0) ||
    (B < 0 && u ≠ 0 && A u < 0 && A u + B u < 0 &&
        A u - B u > 0 && A u^2 < 0) ) && u > 0 &&
    -u + A h u < 0 && u - A h u - B h u + A B h^2 u >
0
0
True

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Exit[];

v[h_] := u[0] + h a[u[0]]

u2[h_] := v[h] + h b[v[h]]

gl[v_] := Simplify[D[v[h], h] - a[v[h]] - b[v[h]] + O[h]^2]

gl[u2]

$$(a[u[0]] (-a'[u[0]] + b'[u[0]]) - b[u[0]] (a'[u[0]] + b'[u[0]))) h + O[h]^2$$

a[x_] := A x; b[x_] := B x;

Clear[a, b]

$$U[f_, u0_, h_] := \frac{-2 h f[u0] - 2 u0 + h u0 f'[u0]}{-2 + h f'[u0]}$$

v = U[a, u0, h];

t = U[b, v, h];

D[t, h] - a[t] - b[t] + O[h]^3

$$(-b[u0] a'[u0] + a[u0] b'[u0]) h +$$

$$\frac{1}{4} (a[u0] a'[u0]^2 - 2 b[u0] a'[u0] b'[u0] + 2 a[u0] b'[u0]^2 +$$

$$b[u0] b'[u0]^2 - 2 a[u0]^2 a''[u0] - 4 a[u0] b[u0] a''[u0] - 2 b[u0]^2 a''[u0] +$$

$$4 a[u0]^2 b''[u0] + 2 a[u0] b[u0] b''[u0] - 2 b[u0]^2 b''[u0]) h^2 + O[h]^3$$

$$\frac{1}{4} (A^3 u0 + B^3 u0) h^2 + O[h]^3$$

gl[U[b, v, h] /. h -> # &]

$$\frac{1}{2} \left(a \left[\frac{-2 h a[u0] + u0 (-2 + h a'[u0])}{-2 + h a'[u0]} [h] \right] + \right.$$

$$(-2 (a[u[0]] + b[u[0]]) - 2 ((a'[u[0]] + b'[u[0]]) u'[0]) h +$$

$$(-u'[0]^2 a''[u[0]] - u'[0]^2 b''[u[0]] - (a'[u[0]] + b'[u[0]]) u''[0]) h^2 + O[h]^3) +$$

$$\left(2 + h a' \left[\frac{-2 u0 - 2 h a[u0] + h u0 a'[u0]}{-2 + h a'[u0]} [h] \right] \right)$$

$$\left(\left(\frac{-2 u0 - 2 h a[u0] + h u0 a'[u0]}{-2 + h a'[u0]} \right)' [h] + \frac{4 a[u0]}{(-2 + h a'[u0])^2} [h] \right) \right)$$

$$\text{Solve} \left[u[t+h] - u[t] - f[u[t]] h - \frac{1}{2} f'[u[t]] h (u[t+h] - u[t]) = 0, u[t+h] \right]$$

$$\left\{ \left\{ u[h+t] \rightarrow \frac{-2 h f[u[t]] - 2 u[t] + h u[t] f'[u[t]]}{-2 + h f'[u[t]]} \right\} \right\}$$

CN[u, a, h] + O[h]^3

$$(-a[u[t]] + u'[t]) h + \frac{1}{2} (-a'[u[t]] u'[t] + u''[t]) h^2 + O[h]^3$$

u1[h_] := u[0] + h / 2 a[u[0]]

v[h_] := u1[h] + h b[u1[h]]

u2[h_] := v[h] + h / 2 a[v[h]]

gl[v_] := Simplify[D[u2[h], h] - a[u[h]] - b[u[h]] + O[h]^3]

g1[u2]

$$\left(b[u[0]] a'[u[0]] + a[u[0]] \left(\frac{1}{2} a'[u[0]] + b'[u[0]] \right) - (a'[u[0]] + b'[u[0]]) u'[0] \right) h + \frac{1}{16} \left(12 b[u[0]]^2 a''[u[0]] + 12 a[u[0]] (a'[u[0]] b'[u[0]] + b[u[0]] a''[u[0]]) + 3 a[u[0]]^2 (a''[u[0]] + 2 b''[u[0]]) - 8 (u'[0]^2 a''[u[0]] + u'[0]^2 b''[u[0]] + (a'[u[0]] + b'[u[0]]) u''[0]) \right) h^2 + O[h]^3$$

Series[1 + Inverse[g[x], x],

Inverse[g[x], x]

Inverse::nonopt : Options expected (instead of x) beyond

position 1 in Inverse[g[x], x]. An option must be a rule or a list of rules. >>

Sum[(D[InverseFunction[Identity + h g][x], {h, k}] /. x -> 0) / k! x^k, {k, 0, 3}]

InverseFunction[g h + Identity][0] + x (g InverseFunction'[g h + Identity])[0] +

$$\frac{1}{2} x^2 (g^2 \text{InverseFunction}''[g h + \text{Identity}])[0] +$$

$$\frac{1}{6} x^3 (g^3 \text{InverseFunction}^{(3)}[g h + \text{Identity}])[0]$$

Series[$\frac{1}{1+h}$, {h, 0, 3}]

$$1 - h + h^2 - h^3 + O[h]^4$$