```
Exit[]
```

```
Moments = Table[W^n → Limit[D[Exp[t^2/2], {t, n}], t -> 0], {n, 4, 1, -1}]
```

$$\{W^4 \to 3,\ W^3 \to 0,\ W^2 \to 1,\ W \to 0\}$$

```
Moments = {ϕ^4 → kurt, ϕ^3 → skew, ϕ^2 → 1, ϕ → 0}
```

$$\{\phi^4 \to \text{kurt},\ \phi^3 \to \text{skew},\ \phi^2 \to 1,\ \phi \to 0\}$$

```
n = 3;
```

```
S[dt_] := S Exp[(μ - σ^2/2) dt^2 + σ ϕ dt];
```

```
dΠ = Series[V[t + dt^2, S[dt]] - Δ S[dt], {dt, 0, n}] - (V[t, S] - Δ S)
```

$$\left(-S\,\Delta\,\sigma\,\phi + S\,\sigma\,\phi\,V^{(0,1)}[t, S]\right)\,dt +$$
$$\frac{1}{2}\left(-2\,S\,\Delta\,\mu + S\,\Delta\,\sigma^2 - S\,\Delta\,\sigma^2\,\phi^2 + 2\,S\,\mu\,V^{(0,1)}[t, S] - S\,\sigma^2\,V^{(0,1)}[t, S] + S\,\sigma^2\,\phi^2\,V^{(0,1)}[t, S] + S^2\,\sigma^2\,\phi^2\,V^{(0,2)}[t, S] + 2\,V^{(1,0)}[t, S]\right)\,dt^2 +$$
$$\frac{1}{6}\left(-6\,S\,\Delta\,\mu\,\sigma\,\phi + 3\,S\,\Delta\,\sigma^3\,\phi - S\,\Delta\,\sigma^3\,\phi^3 + 6\,S\,\mu\,\sigma\,\phi\,V^{(0,1)}[t, S] - 3\,S\,\sigma^3\,\phi\,V^{(0,1)}[t, S] + S\,\sigma^3\,\phi^3\,V^{(0,1)}[t, S] + 6\,S^2\,\mu\,\sigma\,\phi\,V^{(0,2)}[t, S] - 3\,S^2\,\sigma^3\,\phi\,V^{(0,2)}[t, S] + 3\,S^2\,\sigma^3\,\phi^3\,V^{(0,2)}[t, S] + S^3\,\sigma^3\,\phi^3\,V^{(0,3)}[t, S] + 6\,S\,\sigma\,\phi\,V^{(1,1)}[t, S]\right)\,dt^3 + O[dt]^4$$

```
A1 = Simplify[SeriesCoefficient[dΠ, 1]]
```

$$S\,\sigma\,\phi\left(-\Delta + V^{(0,1)}[t, S]\right)$$

```
A2 = Simplify[SeriesCoefficient[dΠ, 2]]
```

$$\frac{1}{2}\left(-2\,S\,\Delta\,\mu + S\,\Delta\,\sigma^2 - S\,\Delta\,\sigma^2\,\phi^2 + S\left(2\,\mu + \sigma^2\left(-1 + \phi^2\right)\right)V^{(0,1)}[t, S] + S^2\,\sigma^2\,\phi^2\,V^{(0,2)}[t, S] + 2\,V^{(1,0)}[t, S]\right)$$

```
E2 = Series[Expand[Normal[dΠ^2]] /. Moments, {dt, 0, n + 1}];
```

```
E1 = Expand[Normal[dΠ]] /. Moments;
```

```
Var = Simplify[E2 - E1^2]
```

$$S^2\,\sigma^2\left(\Delta - V^{(0,1)}[t, S]\right)^2\,dt^2 -$$
$$S^2\,\text{skew}\,\sigma^3\left(\Delta - V^{(0,1)}[t, S]\right)\left(-\Delta + V^{(0,1)}[t, S] + S\,V^{(0,2)}[t, S]\right)\,dt^3 +$$
$$\frac{1}{12}S^2\,\sigma^2\left(\left(24\,\mu + (-15 + 7\,\text{kurt})\,\sigma^2\right)V^{(0,1)}[t, S]^2 - 6\,S\,\Delta\left(4\,\mu + 3\,(-1 + \text{kurt})\,\sigma^2\right)V^{(0,2)}[t, S] + 3\,(-1 + \text{kurt})\,S^2\,\sigma^2\,V^{(0,2)}[t, S]^2 + \Delta\left(24\,\Delta\,\mu - 15\,\Delta\,\sigma^2 + 7\,\text{kurt}\,\Delta\,\sigma^2 - 4\,\text{kurt}\,S^2\,\sigma^2\,V^{(0,3)}[t, S] - 24\,V^{(1,1)}[t, S]\right) + 2\,V^{(0,1)}[t, S]\left(-24\,\Delta\,\mu + 15\,\Delta\,\sigma^2 - 7\,\text{kurt}\,\Delta\,\sigma^2 + 3\,S\left(4\,\mu + 3\,(-1 + \text{kurt})\,\sigma^2\right)V^{(0,2)}[t, S] + 2\,\text{kurt}\,S^2\,\sigma^2\,V^{(0,3)}[t, S] + 12\,V^{(1,1)}[t, S]\right)\right)\,dt^4 + O[dt]^5$$

```
dVar = Series[CoefficientList[D[Var, Δ], Δ], {dt, 0, n + 1}]
```

$$\left\{-2\left(S^2\,\sigma^2\,V^{(0,1)}[t,S]\right)dt^2+\left(-2\,S^2\,\text{skew}\,\sigma^3\,V^{(0,1)}[t,S]-S^3\,\text{skew}\,\sigma^3\,V^{(0,2)}[t,S]\right)dt^3+\right.$$
$$\frac{1}{6}\left(-24\,S^2\,\mu\,\sigma^2\,V^{(0,1)}[t,S]+15\,S^2\,\sigma^4\,V^{(0,1)}[t,S]-7\,\text{kurt}\,S^2\,\sigma^4\,V^{(0,1)}[t,S]-\right.$$
$$12\,S^3\,\mu\,\sigma^2\,V^{(0,2)}[t,S]+9\,S^3\,\sigma^4\,V^{(0,2)}[t,S]-9\,\text{kurt}\,S^3\,\sigma^4\,V^{(0,2)}[t,S]-$$
$$\left.2\,\text{kurt}\,S^4\,\sigma^4\,V^{(0,3)}[t,S]-12\,S^2\,\sigma^2\,V^{(1,1)}[t,S]\right)dt^4+O[dt]^5,$$
$$\left.2\,S^2\,\sigma^2\,dt^2+2\,S^2\,\text{skew}\,\sigma^3\,dt^3+\left(4\,S^2\,\mu\,\sigma^2-\frac{5\,S^2\,\sigma^4}{2}+\frac{7}{6}\,\text{kurt}\,S^2\,\sigma^4\right)dt^4+O[dt]^5\right\}$$

```
fr = Normal[-Series[dVar[[1]] / dVar[[2]], {dt, 0, 1}]]
```

$$V^{(0,1)}[t,S]+\frac{1}{2}\,dt\,S\,\text{skew}\,\sigma\,V^{(0,2)}[t,S]$$

```
Simplify[Series[Normal[Var] /. Δ → fr, {dt, 0, n + 1}]]
```

$$\frac{1}{4}\,S^4\left(-1+\text{kurt}-\text{skew}^2\right)\sigma^4\,V^{(0,2)}[t,S]^2\,dt^4+O[dt]^5$$

```
BS = Expand[Simplify[Normal[Series[E1 /. Δ → fr, {dt, 0, n}] -
      Series[(V[t, S] - fr S) (Exp[r dt^2] - 1), {dt, 0, n}]]] / dt^2]
```

$$-r\,V[t,S]+r\,S\,V^{(0,1)}[t,S]+\frac{1}{2}\,dt\,r\,S^2\,\text{skew}\,\sigma\,V^{(0,2)}[t,S]-$$
$$\frac{1}{2}\,dt\,S^2\,\text{skew}\,\mu\,\sigma\,V^{(0,2)}[t,S]+\frac{1}{2}\,S^2\,\sigma^2\,V^{(0,2)}[t,S]+$$
$$\frac{1}{2}\,dt\,S^2\,\text{skew}\,\sigma^3\,V^{(0,2)}[t,S]+\frac{1}{6}\,dt\,S^3\,\text{skew}\,\sigma^3\,V^{(0,3)}[t,S]+V^{(1,0)}[t,S]$$

```
Simplify[2 / S^2 (1/2 dt r S^2 skew σ V^(0,2)[t, S] -
    1/2 dt S^2 skew μ σ V^(0,2)[t, S] + 1/2 S^2 σ^2 V^(0,2)[t, S] + 1/2 dt S^2 skew σ^3 V^(0,2)[t, S])]
```

$$\sigma\left(\sigma+dt\,\text{skew}\left(r-\mu+\sigma^2\right)\right)V^{(0,2)}[t,S]$$