```
Exit[]
```

```
Moments = Table[W^n → Limit[D[Exp[t^2/2], {t, n}], t -> 0], {n, 4, 1, -1}]
```

$$\{W^4 \to 3,\ W^3 \to 0,\ W^2 \to 1,\ W \to 0\}$$

```
Moments = {ϕ^4 → kurt, ϕ^3 → skew, ϕ^2 → 1, ϕ → 0}
```

$$\{\phi^4 \to \text{kurt},\ \phi^3 \to \text{skew},\ \phi^2 \to 1,\ \phi \to 0\}$$

```
n = 4;
```

```
S[dt_] := S Exp[(μ - σ^2/2) dt^2 + σ ϕ dt];
```

```
dΠ = Series[V[t + dt^2, S[dt]] - Δ S[dt], {dt, 0, n}] - (V[t, S] - Δ S)
```

$$\left(-S\,\Delta\,\sigma\,\phi + S\,\sigma\,\phi\,V^{(0,1)}[t, S]\right)\,dt +$$
$$\left(-S\,\Delta\,\mu + \frac{1}{2}S\,\Delta\,\sigma^2 - \frac{1}{2}S\,\Delta\,\sigma^2\,\phi^2 + \left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}S\,\sigma^2\,\phi^2\right)V^{(0,1)}[t, S] + \right.$$
$$\left.\frac{1}{2}S^2\,\sigma^2\,\phi^2\,V^{(0,2)}[t, S] + V^{(1,0)}[t, S]\right)\,dt^2 +$$
$$\left(-S\,\Delta\,\mu\,\sigma\,\phi + \frac{1}{2}S\,\Delta\,\sigma^3\,\phi - \frac{1}{6}S\,\Delta\,\sigma^3\,\phi^3 + \left(S\,\mu\,\sigma\,\phi - \frac{1}{2}S\,\sigma^3\,\phi + \frac{1}{6}S\,\sigma^3\,\phi^3\right)V^{(0,1)}[t, S] + \right.$$
$$\left.S\,\sigma\,\phi\left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}S\,\sigma^2\,\phi^2\right)V^{(0,2)}[t, S] + \frac{1}{6}S^3\,\sigma^3\,\phi^3\,V^{(0,3)}[t, S] + S\,\sigma\,\phi\,V^{(1,1)}[t, S]\right)$$
$$dt^3 + \left(-\frac{1}{2}S\,\Delta\,\mu^2 + \frac{1}{2}S\,\Delta\,\mu\,\sigma^2 - \frac{1}{8}S\,\Delta\,\sigma^4 - \frac{1}{2}S\,\Delta\,\mu\,\sigma^2\,\phi^2 + \frac{1}{4}S\,\Delta\,\sigma^4\,\phi^2 - \frac{1}{24}S\,\Delta\,\sigma^4\,\phi^4 + \right.$$
$$\left(\frac{S\,\mu^2}{2} - \frac{1}{2}S\,\mu\,\sigma^2 + \frac{S\,\sigma^4}{8} + \frac{1}{2}S\,\mu\,\sigma^2\,\phi^2 - \frac{1}{4}S\,\sigma^4\,\phi^2 + \frac{1}{24}S\,\sigma^4\,\phi^4\right)V^{(0,1)}[t, S] +$$
$$\frac{1}{2}\left(\left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}S\,\sigma^2\,\phi^2\right)^2 + 2\,S\,\sigma\,\phi\left(S\,\mu\,\sigma\,\phi - \frac{1}{2}S\,\sigma^3\,\phi + \frac{1}{6}S\,\sigma^3\,\phi^3\right)\right)V^{(0,2)}[t, S] +$$
$$\frac{1}{2}S^2\,\sigma^2\,\phi^2\left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}S\,\sigma^2\,\phi^2\right)V^{(0,3)}[t, S] +$$
$$\frac{1}{24}S^4\,\sigma^4\,\phi^4\,V^{(0,4)}[t, S] + \left(S\,\mu - \frac{S\,\sigma^2}{2} + \frac{1}{2}S\,\sigma^2\,\phi^2\right)V^{(1,1)}[t, S] +$$
$$\left.\frac{1}{2}S^2\,\sigma^2\,\phi^2\,V^{(1,2)}[t, S] + \frac{1}{2}V^{(2,0)}[t, S]\right)\,dt^4 + O[dt]^5$$

```
A1 = Simplify[SeriesCoefficient[dΠ, 1]]
```

$$S\,\sigma\,\phi\left(-\Delta + V^{(0,1)}[t, S]\right)$$

```
A2 = Simplify[SeriesCoefficient[dΠ, 2]]
```

$$\frac{1}{2}\left(-2\,S\,\Delta\,\mu + S\,\Delta\,\sigma^2 - S\,\Delta\,\sigma^2\,\phi^2 + \right.$$
$$\left.S\left(2\,\mu + \sigma^2\left(-1 + \phi^2\right)\right)V^{(0,1)}[t, S] + S^2\,\sigma^2\,\phi^2\,V^{(0,2)}[t, S] + 2\,V^{(1,0)}[t, S]\right)$$

```
E2 = Series[Expand[Normal[dΠ^2]] /. Moments, {dt, 0, n + 1}];

E1 = Expand[Normal[dΠ]] /. Moments;

Var = Simplify[E2 - E1^2]
```

$$S^2 \sigma^2 \left(\Delta - V^{(0,1)}[t, S]\right)^2 dt^2 -$$
$$S^2 \text{skew}\, \sigma^3 \left(\Delta - V^{(0,1)}[t, S]\right) \left(-\Delta + V^{(0,1)}[t, S] + S\, V^{(0,2)}[t, S]\right) dt^3 +$$
$$\frac{1}{12} S^2 \sigma^2 \left(\left(24\mu + (-15 + 7\,\text{kurt})\,\sigma^2\right) V^{(0,1)}[t, S]^2 - 6\, S\, \Delta \left(4\mu + 3\,(-1 + \text{kurt})\,\sigma^2\right) V^{(0,2)}[t, S] + 3\,(-1 + \text{kurt})\, S^2 \sigma^2 V^{(0,2)}[t, S]^2 + \Delta \left(24\,\Delta\,\mu - 15\,\Delta\,\sigma^2 + 7\,\text{kurt}\,\Delta\,\sigma^2 - 4\,\text{kurt}\, S^2 \sigma^2 V^{(0,3)}[t, S] - 24\, V^{(1,1)}[t, S]\right) + 2\, V^{(0,1)}[t, S] \left(-24\,\Delta\,\mu + 15\,\Delta\,\sigma^2 - 7\,\text{kurt}\,\Delta\,\sigma^2 + 3\, S \left(4\mu + 3\,(-1 + \text{kurt})\,\sigma^2\right) V^{(0,2)}[t, S] + 2\,\text{kurt}\, S^2 \sigma^2 V^{(0,3)}[t, S] + 12\, V^{(1,1)}[t, S]\right)\right) dt^4 -$$
$$\frac{1}{6} \left(S^2 \text{skew}\, \sigma^3 \left(\left(-12\mu + 7\sigma^2\right) V^{(0,1)}[t, S]^2 + 6\, S^2 \left(-\mu + \sigma^2\right) V^{(0,2)}[t, S]^2 + S\, V^{(0,2)}[t, S] \left(30\,\Delta\,\mu - 19\,\Delta\,\sigma^2 + S^2 \sigma^2 V^{(0,3)}[t, S] - 6\, V^{(1,1)}[t, S]\right) + \Delta \left(-12\,\Delta\,\mu + 7\,\Delta\,\sigma^2 + 2\, S^2 \left(3\mu - 2\sigma^2\right) V^{(0,3)}[t, S] + 12\, V^{(1,1)}[t, S] + 6\, S\, V^{(1,2)}[t, S]\right) + V^{(0,1)}[t, S] \left(S \left(-30\mu + 19\sigma^2\right) V^{(0,2)}[t, S] - 2 \left(-12\,\Delta\,\mu + 7\,\Delta\,\sigma^2 + S^2 \left(3\mu - 2\sigma^2\right) V^{(0,3)}[t, S] + 6\, V^{(1,1)}[t, S] + 3\, S\, V^{(1,2)}[t, S]\right)\right)\right)\right) dt^5 + O[dt]^6$$

```
dVar = Series[CoefficientList[D[Var, Δ], Δ], {dt, 0, n + 1}]
```

$$\Big\{-2 \left(S^2 \sigma^2 V^{(0,1)}[t, S]\right) dt^2 + \left(-2\, S^2 \text{skew}\, \sigma^3 V^{(0,1)}[t, S] - S^3 \text{skew}\, \sigma^3 V^{(0,2)}[t, S]\right) dt^3 +$$
$$\frac{1}{6} \left(-24\, S^2 \mu \sigma^2 V^{(0,1)}[t, S] + 15\, S^2 \sigma^4 V^{(0,1)}[t, S] - 7\,\text{kurt}\, S^2 \sigma^4 V^{(0,1)}[t, S] - 12\, S^3 \mu \sigma^2 V^{(0,2)}[t, S] + 9\, S^3 \sigma^4 V^{(0,2)}[t, S] - 9\,\text{kurt}\, S^3 \sigma^4 V^{(0,2)}[t, S] - 2\,\text{kurt}\, S^4 \sigma^4 V^{(0,3)}[t, S] - 12\, S^2 \sigma^2 V^{(1,1)}[t, S]\right) dt^4 +$$
$$\frac{1}{6} \left(-24\, S^2 \text{skew}\, \mu \sigma^3 V^{(0,1)}[t, S] + 14\, S^2 \text{skew}\, \sigma^5 V^{(0,1)}[t, S] - 30\, S^3 \text{skew}\, \mu \sigma^3 V^{(0,2)}[t, S] + 19\, S^3 \text{skew}\, \sigma^5 V^{(0,2)}[t, S] - 6\, S^4 \text{skew}\, \mu \sigma^3 V^{(0,3)}[t, S] + 4\, S^4 \text{skew}\, \sigma^5 V^{(0,3)}[t, S] - 12\, S^2 \text{skew}\, \sigma^3 V^{(1,1)}[t, S] - 6\, S^3 \text{skew}\, \sigma^3 V^{(1,2)}[t, S]\right) dt^5 + O[dt]^6,$$
$$2\, S^2 \sigma^2 dt^2 + 2\, S^2 \text{skew}\, \sigma^3 dt^3 + \left(4\, S^2 \mu \sigma^2 - \frac{5\, S^2 \sigma^4}{2} + \frac{7}{6}\,\text{kurt}\, S^2 \sigma^4\right) dt^4 +$$
$$\left(4\, S^2 \text{skew}\, \mu \sigma^3 - \frac{7}{3} S^2 \text{skew}\, \sigma^5\right) dt^5 + O[dt]^6\Big\}$$

```
fr = Simplify[-dVar[[1]] / dVar[[2]]]
```

$$V^{(0,1)}[t, S] + \frac{1}{2} S\,\text{skew}\, \sigma\, V^{(0,2)}[t, S]\, dt +$$
$$\left(\frac{1}{4} S \left(4\mu + \left(-3 + 3\,\text{kurt} - 2\,\text{skew}^2\right)\sigma^2\right) V^{(0,2)}[t, S] + \frac{1}{6}\,\text{kurt}\, S^2 \sigma^2 V^{(0,3)}[t, S] + V^{(1,1)}[t, S]\right) dt^2 + \frac{1}{24} S\,\text{skew}\, \sigma \left(\left(12\mu + \left(-5 - 25\,\text{kurt} + 12\,\text{skew}^2\right)\sigma^2\right) V^{(0,2)}[t, S] + 4 \left(S \left(3\mu - (2 + \text{kurt})\,\sigma^2\right) V^{(0,3)}[t, S] + 3\, V^{(1,2)}[t, S]\right)\right) dt^3 + O[dt]^4$$

```
fra = fr /. skew → 0
```

$$V^{(0,1)}[t, S] + \left(\frac{1}{4} S \left(4 \mu + (-3 + 3\ \text{kurt})\ \sigma^2\right) V^{(0,2)}[t, S] + \frac{1}{6}\ \text{kurt}\ S^2\ \sigma^2\ V^{(0,3)}[t, S] + V^{(1,1)}[t, S]\right) dt^2 + 0[dt]^4$$

```
Series[Normal[Var] /. Δ → fra, {dt, 0, n + 1}]
```

$$\frac{1}{4} S^2\ \sigma^2 \left(-S^2\ \sigma^2\ V^{(0,2)}[t, S]^2 + \text{kurt}\ S^2\ \sigma^2\ V^{(0,2)}[t, S]^2\right) dt^4 + \frac{1}{12} \left(-3\ S^4\ \text{skew}\ \sigma^5\ V^{(0,2)}[t, S]^2 - 9\ \text{kurt}\ S^4\ \text{skew}\ \sigma^5\ V^{(0,2)}[t, S]^2 - 2\ S^5\ \text{skew}\ \sigma^5\ V^{(0,2)}[t, S]\ V^{(0,3)}[t, S] - 2\ \text{kurt}\ S^5\ \text{skew}\ \sigma^5\ V^{(0,2)}[t, S]\ V^{(0,3)}[t, S]\right) dt^5 + 0[dt]^6$$

```
BS = Series[E1 /. Δ → fra, {dt, 0, n}] -
  Series[(V[t, S] - fra S) (Exp[r dt^2] - 1), {dt, 0, n}]
```

$$\left(-r\ V[t, S] + r\ S\ V^{(0,1)}[t, S] + \frac{1}{2} S^2\ \sigma^2\ V^{(0,2)}[t, S] + V^{(1,0)}[t, S]\right) dt^2 + \frac{1}{6} \left(3\ S^2\ \text{skew}\ \sigma^3\ V^{(0,2)}[t, S] + S^3\ \text{skew}\ \sigma^3\ V^{(0,3)}[t, S]\right) dt^3 + \left(\frac{1}{12} \left(-6\ r^2\ V[t, S] + 6\ r^2\ S\ V^{(0,1)}[t, S] + 12\ r\ S^2\ \mu\ V^{(0,2)}[t, S] - 9\ r\ S^2\ \sigma^2\ V^{(0,2)}[t, S] + 9\ \text{kurt}\ r\ S^2\ \sigma^2\ V^{(0,2)}[t, S] + 2\ \text{kurt}\ r\ S^3\ \sigma^2\ V^{(0,3)}[t, S] + 12\ r\ S\ V^{(1,1)}[t, S]\right) + \frac{1}{24} \left(-12\ S^2\ \mu^2\ V^{(0,2)}[t, S] + 42\ S^2\ \mu\ \sigma^2\ V^{(0,2)}[t, S] - 18\ \text{kurt}\ S^2\ \mu\ \sigma^2\ V^{(0,2)}[t, S] - 15\ S^2\ \sigma^4\ V^{(0,2)}[t, S] + 7\ \text{kurt}\ S^2\ \sigma^4\ V^{(0,2)}[t, S] + 12\ S^3\ \mu\ \sigma^2\ V^{(0,3)}[t, S] - 4\ \text{kurt}\ S^3\ \mu\ \sigma^2\ V^{(0,3)}[t, S] - 6\ S^3\ \sigma^4\ V^{(0,3)}[t, S] + 6\ \text{kurt}\ S^3\ \sigma^4\ V^{(0,3)}[t, S] + \text{kurt}\ S^4\ \sigma^4\ V^{(0,4)}[t, S] + 12\ S^2\ \sigma^2\ V^{(1,2)}[t, S] + 12\ V^{(2,0)}[t, S]\right)\right) dt^4 + 0[dt]^5$$

```
BSC = Simplify[CoefficientList[BS / dt^2, dt]]
```

$$\left\{-r\ V[t, S] + r\ S\ V^{(0,1)}[t, S] + \frac{1}{2} S^2\ \sigma^2\ V^{(0,2)}[t, S] + V^{(1,0)}[t, S],\right.$$

$$\frac{1}{6} S^2\ \text{skew}\ \sigma \left(3 \left(r - \mu + \sigma^2\right) V^{(0,2)}[t, S] + S\ \sigma^2\ V^{(0,3)}[t, S]\right),$$

$$-\frac{1}{2} r^2 \left(V[t, S] - S\ V^{(0,1)}[t, S]\right) + r\ S \left(\frac{1}{4} S \left(4 \mu + \left(-3 + 3\ \text{kurt} - 2\ \text{skew}^2\right) \sigma^2\right) V^{(0,2)}[t, S] + \frac{1}{6}\ \text{kurt}\ S^2\ \sigma^2\ V^{(0,3)}[t, S] + V^{(1,1)}[t, S]\right) + \frac{1}{24} \left(S^2 \left(-12 \mu^2 - 6 \left(-7 + 3\ \text{kurt} - 2\ \text{skew}^2\right) \mu\ \sigma^2 + \left(-15 + 7\ \text{kurt} - 2\ \text{skew}^2\right) \sigma^4\right) V^{(0,2)}[t, S] + 2\ S^3\ \sigma^2 \left(-2\ (-3 + \text{kurt})\ \mu + 3\ (-1 + \text{kurt})\ \sigma^2\right) V^{(0,3)}[t, S] + \text{kurt}\ S^4\ \sigma^4\ V^{(0,4)}[t, S] + 12\ S^2\ \sigma^2\ V^{(1,2)}[t, S] + 12\ V^{(2,0)}[t, S]\right)\right\}$$

```
eq = Simplify[BSC /. kurt → 3 /. skew → 0]
```

$$\left\{-r\,V[t,S]+r\,S\,V^{(0,1)}[t,S]+\frac{1}{2}S^2\,\sigma^2\,V^{(0,2)}[t,S]+V^{(1,0)}[t,S],\right.$$
$$0,\ \frac{1}{8}\left(-4\,r^2\left(V[t,S]-S\,V^{(0,1)}[t,S]\right)+\right.$$
$$2\,S^2\left(-2\,\mu^2-2\,\mu\,\sigma^2+\sigma^4\right)V^{(0,2)}[t,S]+4\,S^3\,\sigma^4\,V^{(0,3)}[t,S]+S^4\,\sigma^4\,V^{(0,4)}[t,S]+$$
$$4\,r\,S\left(S\left(2\,\mu+3\,\sigma^2\right)V^{(0,2)}[t,S]+S^2\,\sigma^2\,V^{(0,3)}[t,S]+2\,V^{(1,1)}[t,S]\right)+$$
$$\left.\left.4\,S^2\,\sigma^2\,V^{(1,2)}[t,S]+4\,V^{(2,0)}[t,S]\right)\right\}$$

```
V2 = Solve[eq[[1]] == 0, D[V[t, S], {S, 2}]][[1, 1]]
```

$$V^{(0,2)}[t,S]\to\frac{2\left(r\,V[t,S]-r\,S\,V^{(0,1)}[t,S]-V^{(1,0)}[t,S]\right)}{S^2\,\sigma^2}$$

```
Vt = Solve[eq[[1]] == 0, D[V[t, S], t]][[1, 1]]
```

$$V^{(1,0)}[t,S]\to\frac{1}{2}\left(2\,r\,V[t,S]-2\,r\,S\,V^{(0,1)}[t,S]-S^2\,\sigma^2\,V^{(0,2)}[t,S]\right)$$

```
Simplify[Simplify[eq[[3]] /. D[V2, {S, 2}] /. D[V2, S] /. D[Vt, t] /. V2] / V2[[2]]]
```

$$-\frac{1}{2}S^2\left(r^2+\mu\left(\mu+\sigma^2\right)-r\left(2\,\mu+\sigma^2\right)\right)$$

```
Series[(Exp[r dt^2] - 1), {dt, 0, n}]
```

$$r\,dt^2+\frac{r^2\,dt^4}{2}+O[dt]^5$$

```
err = Simplify[Series[Simplify[Normal[dΠ] /. Δ → fr], {dt, 0, n}] -
    Series[(V[t, S] - fr S) (Exp[r dt^2] - 1), {dt, 0, 2}] /. V2]
```

$$-\left(1+\text{skew}\,\phi-\phi^2\right)\left(r\,V[t,S]-r\,S\,V^{(0,1)}[t,S]-V^{(1,0)}[t,S]\right)dt^2+O[dt]^3$$

```
Expand[(ϕ^2 - 1)^2] /. Moments
```

$$-1+\text{kurt}$$

```
Expand[(ϕ^2 - skew ϕ - 1)^2] /. Moments
```

$$-1+\text{kurt}-\text{skew}^2$$

```
fr
```

$$V^{(0,1)}[t,S]+\frac{1}{2}S\,\text{skew}\,\sigma\,V^{(0,2)}[t,S]\,dt+$$
$$\left(\frac{1}{4}S\left(4\,\mu+\left(-3+3\,\text{kurt}-2\,\text{skew}^2\right)\sigma^2\right)V^{(0,2)}[t,S]+\frac{1}{6}\,\text{kurt}\,S^2\,\sigma^2\,V^{(0,3)}[t,S]+V^{(1,1)}[t,S]\right)$$
$$dt^2+\frac{1}{24}S\,\text{skew}\,\sigma\left(\left(12\,\mu+\left(-5-25\,\text{kurt}+12\,\text{skew}^2\right)\sigma^2\right)V^{(0,2)}[t,S]+\right.$$
$$\left.4\left(S\left(3\,\mu-(2+\text{kurt})\,\sigma^2\right)V^{(0,3)}[t,S]+3\,V^{(1,2)}[t,S]\right)\right)dt^3+O[dt]^4$$

```
err = Simplify[Series[Simplify[Normal[dΠ] /. Δ → (fr /. skew → 0)], {dt, 0, n}] -
    Series[(V[t, S] - (fr /. skew → 0) S) (Exp[r dt^2] - 1), {dt, 0, 2}] /. V2]
```

$$\left(-1+\phi^2\right)\ \left(r\ V[t,S]-r\ S\ V^{(0,1)}[t,S]-V^{(1,0)}[t,S]\right)\ dt^2+O[dt]^3$$