O(h^4) error analysis

```
eq1 = b1 == wm (1 - h ^ 2 b0 / 6 / b2) + wp h ^ 4 b1 b0 / b2 ^ 2 / 4 !
```

$$b1 = \left(1 - \frac{b0\, h^2}{6\, b2}\right) wm + \frac{b0\, b1\, h^4\, wp}{24\, b2^2}$$

```
eq2 = b2 == wm (-h ^ 2 b1 / 3 ! / b2) + wp (h ^ 2 / 2 + h ^ 4 / 4 ! (b1 ^ 2 / b2 ^ 2 - b0 / b1))
```

$$b2 = -\frac{b1\, h^2\, wm}{6\, b2} + \left(\frac{h^2}{2} + \frac{1}{24}\left(-\frac{b0}{b1} + \frac{b1^2}{b2^2}\right) h^4\right) wp$$

```
Solve[{eq1, eq2}, {wm, wp}]
```

$$\left\{\left\{wm \to -\frac{6\, b1 \left(-12\, b1\, b2^2 - b1^3\, h^2 + b0\, b1\, b2\, h^2 + b0\, b2^2\, h^2\right)}{72\, b1\, b2^2 + 6\, b1^3\, h^2 - 12\, b0\, b1\, b2\, h^2 - 6\, b0\, b2^2\, h^2 + b0^2\, b2\, h^4},\right.\right.$$
$$\left.\left. wp \to -\frac{24 \left(-6\, b1\, b2^3 - b1^3\, b2\, h^2 + b0\, b1\, b2^2\, h^2\right)}{h^2 \left(72\, b1\, b2^2 + 6\, b1^3\, h^2 - 12\, b0\, b1\, b2\, h^2 - 6\, b0\, b2^2\, h^2 + b0^2\, b2\, h^4\right)}\right\}\right\}$$

$$\mathbf{Series}\left[-\frac{6\, b1 \left(-12\, b1\, b2^2 - b1^3\, h^2 + b0\, b1\, b2\, h^2 + b0\, b2^2\, h^2\right)}{72\, b1\, b2^2 + 6\, b1^3\, h^2 - 12\, b0\, b1\, b2\, h^2 - 6\, b0\, b2^2\, h^2 + b0^2\, b2\, h^4}, \{h, 0, 10\}\right]$$

$$b1\, h^4 + \frac{b0\, b1\, h^6}{12\, b2} + \frac{\left(-b0\, b1^3 + 2\, b0^2\, b1\, b2 - b0^2\, b2^2\right) h^8}{144\, b2^3} + \frac{1}{1728\, b1\, b2^5}$$
$$\left(b0\, b1^6 - 4\, b0^2\, b1^4\, b2 + 4\, b0^3\, b1^2\, b2^2 - 2\, b0^3\, b1\, b2^3 - b0^3\, b2^4\right) h^{10} + O[h]^{11}$$

O(h^2) with M-Matrix

b[i] in Einheiten von h^i

```
erV[n_] :=
 Table[b[i] UnitStep[2.9 - i] - h ^i (w0 KroneckerDelta[0, i] + wp + (-1) ^i wm), {i, 1, n}]
```

```
er[n_] := Sum[erV[n][[i]] ^ 2, {i, 1, n}]
```

```
er[2]
```

$$(-h\,(-wm + wp) + b[1])^2 + \left(-h^2\,(wm + wp) + b[2]\right)^2$$

Minimization using first two orders (convections dominant):

```
{minEr, minArg} = Refine[Minimize[{er[2] /. b[1] → 2 b[2] / h, wp ≤ 0, wm ≤ 0}, {wp, wm}],
  {h < 1 / 2, h > 0, b[2] < 0}]
```

$$\left\{\frac{b[2]^2}{1+h^2},\ \left\{wp \to \frac{2\,b[2]+h^2\,b[2]}{h^2\,(1+h^2)} - \frac{\sqrt{-b[2]^2+\frac{b[2]^2}{1+h^2}+\frac{h^2\,b[2]^2}{1+h^2}}}{h\,(1+h^2)},\right.\right.$$

$$wm \to \frac{1}{h^2\,(1+h^2)}\left(-2\,b[2]+h^2\,b[2]+h^2\left(\frac{2\,b[2]+h^2\,b[2]}{h^2\,(1+h^2)} - \frac{\sqrt{-b[2]^2+\frac{b[2]^2}{1+h^2}+\frac{h^2\,b[2]^2}{1+h^2}}}{h\,(1+h^2)}\right)-\right.$$

$$\left.h^4\left(\frac{2\,b[2]+h^2\,b[2]}{h^2\,(1+h^2)} - \frac{\sqrt{-b[2]^2+\frac{b[2]^2}{1+h^2}+\frac{h^2\,b[2]^2}{1+h^2}}}{h\,(1+h^2)}\right)\right) - \frac{1}{h\,(1+h^2)}\left(\sqrt{\left(-9\,b[2]^2+\right.}\right.$$

$$\frac{b[2]^2}{1+h^2}+\frac{h^2\,b[2]^2}{1+h^2}+12\,h^2\,b[2]\left(\frac{2\,b[2]+h^2\,b[2]}{h^2\,(1+h^2)} - \frac{\sqrt{-b[2]^2+\frac{b[2]^2}{1+h^2}+\frac{h^2\,b[2]^2}{1+h^2}}}{h\,(1+h^2)}\right)-$$

$$\left.\left.\left.4\,h^4\left(\frac{2\,b[2]+h^2\,b[2]}{h^2\,(1+h^2)} - \frac{\sqrt{-b[2]^2+\frac{b[2]^2}{1+h^2}+\frac{h^2\,b[2]^2}{1+h^2}}}{h\,(1+h^2)}\right)^2\right)\right)\right\}\right\}$$

```
Series[Simplify[erV[5] /. b[1] → 2 b[2] / h /. minArg], {h, 0, 1}]
```

$$\left\{\left(-b[2]-2\sqrt{b[2]^2}\right)h+O[h]^2,\ -b[2]+O[h]^2,\ -2\,b[2]\,h+O[h]^2,\ O[h]^2,\ O[h]^3\right\}$$

Minimization using first three orders(convection dominant):

```
{minEr, minArg} = Refine[Minimize[{er[3] /. b[1] → 2 b[2] / h, wp ≤ 0, wm ≤ 0}, {wp, wm}],
  {h < 1 / 2, h > 0, b[2] < 0}];
minEr
```

$$\frac{b[2]^2+4\,h^2\,b[2]^2+h^4\,b[2]^2}{1+h^2+h^4}$$

```
Series[Simplify[erV[5] /. b[1] → 2 b[2] / h /. minArg], {h, 0, 1}]
```

$$\left\{\left(-b[2]-2\sqrt{b[2]^2}\right)h+O[h]^2,\ -b[2]+O[h]^2,\ -2\,b[2]\,h+O[h]^2,\ O[h]^2,\ O[h]^3\right\}$$

Minimization using first four orders (convection dominant):

```
{minEr, minArg} = Refine[Minimize[{er[4] /. b[1] → 2 b[2] / h, wp ≤ 0, wm ≤ 0}, {wp, wm}],
   {h < 1 / 2, h > 0, b[2] < 0}];
minEr
```

$$\frac{b[2]^2 + 4 h^2 b[2]^2 + 5 h^4 b[2]^2 + h^6 b[2]^2}{1 + h^2 + h^4 + h^6}$$

```
Series[Simplify[erV[5] /. b[1] → 2 b[2] / h /. minArg], {h, 0, 1}]
```

$$\left\{\left(-b[2] - 2\sqrt{b[2]^2}\right) h + O[h]^2, -b[2] + O[h]^2, -2 b[2] h + O[h]^2, O[h]^2, O[h]^3\right\}$$

Minimization using first two orders (not convections dominant):

```
{minEr, minArg} =
 Refine[Minimize[{er[2] /. b[1] → b[2] / 2 / h, wp ≤ 0, wm ≤ 0}, {wp, wm}],
  {h < 1 / 2, h > 0, b[2] < 0}]
```

$$\left\{0, \left\{wp \to \frac{3 b[2]}{4 h^2}, wm \to \frac{\frac{b[2]}{2} + \frac{1}{2} h^2 b[2]}{2 h^2 \left(1 + h^2\right)}\right\}\right\}$$

```
Simplify[erV[5] /. b[1] → b[2] / h / 2 /. minArg]
```

$$\left\{0, 0, -\frac{1}{2} h b[2], -h^2 b[2], -\frac{1}{2} h^3 b[2]\right\}$$

Minimization using first three orders:

```
{minEr, minArg} = Refine[Minimize[{er[3] /. b[1] → 2 b[2], wp ≤ 0, wm ≤ 0}, {wp, wm}],
   {h < 1 / 2, h > 0, b[2] < 0}];
minEr
```

$$\frac{4 h^4 b[2]^2}{1 + h^4}$$

```
Series[Simplify[erV[5] /. b[1] → 2 b[2] /. minArg], {h, 0, 1}]
```

$$\left\{O[h]^4, 0, O[h]^2, O[h]^2, O[h]^4\right\}$$

Minimization using first four orders:

```
{minEr, minArg} = Refine[Minimize[{er[4] /. b[1] → 2 b[2], wp ≤ 0, wm ≤ 0}, {wp, wm}],
   {h < 1 / 2, h > 0, b[2] < 0}];
minEr
```

$$\frac{5 h^4 b[2]^2}{1 + h^4}$$

```
Series[Simplify[erV[5] /. b[1] → 2 b[2] /. minArg], {h, 0, 1}]
```

$$\left\{O[h]^4, 0, O[h]^2, O[h]^2, O[h]^4\right\}$$

Fazit: It stays O(h^2), and O(h) for convection dominant points