Research Statement of Johannes Girsch

1. Background

One of the most celebrated mathematical results of the last century was the proof of Fermat's Last Theorem by Taylor and Wiles. Essentially, they proved a relation between modular forms and elliptic curves, conjectured by Taniyama and Shimura, which was the last missing piece in a proof of Fermat's longstanding conjecture. This correspondence is a special case of a much bigger picture, the so-called Langlands program, which consists of various conjectures involving automorphic and Galois representations.

One part of this program is the local Langlands correspondence, which for GL_n and F a finite extension of $\mathbb{Q}_p$, states that there is a natural bijection between irreducible smooth representations of $\mathrm{GL}_n(F)$ with coefficients in the complex numbers and certain n-dimensional Galois representations. This bijection is characterized by a list of properties; one of which is the preservation of L-functions, ϵ-factors and γ-factors, the so called local constants.

More concretely, one can associate these rational functions to pairs of representations on both sides of the correspondence and matching elements should have the same local constants. In general it can be quite challenging to define the local constants, but their study has proven to be of great importance in the representation theory of reductive p-adic groups. For example a very useful tool are converse theorems, one version due to Jacquet roughly says that if π, π' are two irreducible representations of $\mathrm{GL}_n(F)$ and the γ-factors associated to the pairs $\pi \times \tau$ and $\pi' \times \tau$ are equal for all irreducible representations τ of $\mathrm{GL}_{n-1}(F)$, then π is isomorphic to π'.

1.1. Local Langlands in Families.

Wiles's proof of Fermat's last theorem, actually relied on proving that families of Galois representations related to a particular Galois representation modulo ℓ come from families of modular forms. Motivated by this, it is natural to ask whether the local Langlands correspondence varies well in families. "Local Langlands in families for $\mathrm{GL}_n(F)$", conjectured by Emerton-Helm in [EH] and proven by Helm-Moss in [HM], interpolates the local Langlands correspondence across families of certain Galois representations. The proof by Helm and Moss crucially uses Moss's construction of γ-factors for certain families of smooth representations of $\mathrm{GL}_n(F)$ (i.e. smooth representations of $\mathrm{GL}_n(F)$ with coefficients in some $\mathbb{Z}_\ell$-algebra A, where ℓ is a prime not equal to p) and a converse theorem for these γ-factors.

1.2. γ-Factors of Pairs.

It is then natural to ask if the local Langlands correspondence for other reductive groups also varies well in families. Recently Dat, Helm, Kurinczuk and Moss began to develop this in the case of classical groups (e.g. [DHKM]). If one hopes to prove such a correspondence by following the ideas of the proof for $\mathrm{GL}_n(F)$, one would need a theory of γ-factors and a converse theorem for families of representations of a classical group times a general linear group. There are various different methods to define local constants (e.g. Rankin-Selberg, Langlands-Shahidi), each having different advantages and caveats. One method which is quite uniform for all classical groups G is the "Doubling Method" of Piateski-Shapiro and Rallis which associates local constants to representations of $G \times \mathrm{GL}_1$. Recently there has been an extension of the doubling method by Cai, Friedberg, Ginzburg and Kaplan ([CFGK]) which allows one to define local constants for complex representations of $G \times \mathrm{GL}_m$.

2. Current and Past Research

For a reductive p-adic group G let $\mathrm{Rep}_A(G)$ be the category of smooth A-representations of G, i.e. $A[G]$-modules such that each element has an open stabilizer in G.

2.1. The Doubling Method in Algebraic Families.

In my PhD thesis and subsequent work I developed the doubling method and its generalization in the setting of families (see [G1] and [G2]). Let A be a noetherian $\overline{\mathbb{Z}}_\ell$-algebra and π a smooth admissible $A[G]$-module. Moreover, let ρ be a $\overline{\mathbb{Z}}_\ell$-lattice in an absolutely irreducible generic $\overline{\mathbb{Q}}_\ell$-representation of $\mathrm{GL}_m(F)$. In [G2], I was then able to construct a γ-factor $\gamma(\pi \times \rho)$ associated to the pair $\pi \times \rho$, which lies in the localization of $A[X^{\pm 1}]$ with respect to the multiplicative subset S which consists of Laurent polynomials with leading and trailing coefficient a unit. The Γ-factor constructed above behaves nicely under base change and interpolates the Γ-factors of Cai-Friedberg-Ginzburg-Kaplan over $\mathbb{C}$. More concretely, fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and suppose the residue field $\kappa(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ is isomorphic to the complex numbers. Then the reduction of the γ-factor $\gamma(\pi \times \rho) \in S^{-1} \cdot A[X^{\pm 1}]$ modulo $\mathfrak{p}$ equals the γ-factor $\gamma((\pi \otimes \kappa(\mathfrak{p})) \times \rho)$ constructed by Cai-Friedberg-Ginzburg-Kaplan.

For the envisioned applications of γ-factors of pairs to the local Langlands correspondence in families one also needs to be able to vary ρ in families. The missing bit here is a theory of generalized Speh representations in families. Namely, in their construction, Cai-Friedberg-Ginzburg-Kaplan associate to a generic representation of $\mathrm{GL}_m(F)$ a generalized Speh representation of $\mathrm{GL}_{km}(F)$ for some positive integer k. Together with David Helm, I have started a project which tries to obtain such a construction in the setting of families. Generalized Speh representations are representations which can be realized in certain degenerate Whittaker spaces and our investigations in these representations have resulted in results about $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$, which are very interesting in their own right.

2.2. **A stratification of** $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$**.** In general irreducible representations of $\mathrm{GL}_n(F)$ can be very abstract objects, however by using the theory of Whittaker models one can realize them in certain explicit spaces of functions:

Let U_n be the unipotent radical of the standard Borel in $\mathrm{GL}_n(F)$, i.e. the group of unipotent uppertriangular matrices. For each integer partition λ of n one can define a character $\psi_\lambda \colon U_n \to \mathbb{C}^\times$. Moreover, Zelevinsky proved that to each irreducible representation $\pi \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ one can associate an integer partition of n, the highest derivative partition λ_π of π. Moeglin and Waldspurger then proved that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(F)}\left(\pi, \mathrm{Ind}_{U_n}^{\mathrm{GL}_n(F)}(\psi_{\lambda_\pi})\right) = 1,$$

where $\mathrm{Ind}_{U_n}^{\mathrm{GL}_n(F)}(\psi_{\lambda_\pi})$ is equal to the smooth vectors in the function space

$$\{f\colon \mathrm{GL}_n(F) \to \mathbb{C} \mid f(ug) = \psi_{\lambda_\pi}(u) f(g), u \in U_n, g \in \mathrm{GL}_n(F)\}$$

under right translation.

The notion of the highest derivative partition allows us to define a "stratification" of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ in the following way. Firstly, recall the dominance order on integer partitions, i.e. for two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ of n we say that $\lambda' \preceq \lambda$ if and only if $\sum_{i=1}^{j} \lambda'_i \leq \sum_{i=1}^{j} \lambda_i$ for all $j \geq 1$.

For any partition λ of n, we can then consider the full subcategories $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{\preceq\lambda}$ (respectively $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{\prec\lambda}$) of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$, whose objects are representations for which every irreducible subquotient has highest derivative partition λ' satisfying $\lambda' \preceq \lambda$ (respectively $\lambda' \prec \lambda$). Let $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$ be the Serre quotient of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\preceq\lambda}$ by $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\prec\lambda}$. We study this category by constructing a progenerator for it, i.e. an element $W_\lambda \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$ with endomorphism ring $E_\lambda = \mathrm{End}_{\mathrm{GL}_n(F)}(W_\lambda)$ such that

$$\pi \mapsto \mathrm{Hom}_{\mathrm{GL}_n(F)}(W_\lambda, \pi)$$

yields an equivalence of categories between $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$ and the category of right E_λ-modules. It is then natural to study the rings E_λ and we are able to prove the following.

Theorem 1 (G.-Helm, [GH])**.** *For all integer partitions λ of n the rings E_λ are commutative and reduced.*

Moreover, we are able to give an explicit description of E_λ, similar to the description of the Bernstein center due to Bernstein-Deligne.

In contrast to the representation theory of finite groups, $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ is not semisimple. However, Bernstein was able to prove a so-called block decomposition of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$, i.e. there are certain full subcategories $\mathrm{Rep}_{\mathbb{C}}^{\mathfrak{s}}(\mathrm{GL}_n(F))$ of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ (which we call blocks of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$) such that any smooth representation π can be written as a unique direct sum

$$\pi = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(\mathrm{GL}_n(F))} \pi^{\mathfrak{s}},$$

where $\pi^{\mathfrak{s}} \in \mathrm{Rep}_{\mathbb{C}}^{\mathfrak{s}}(\mathrm{GL}_n(F))$. Moreover, there are no nonzero morphisms between objects in different blocks. Bernstein also proved that his decomposition is the finest possible, but each of these blocks are still rather complicated. What our above result shows is that each blcok has a stratification, where each of the strata categories is equivalent to a module category over an explicit commutative ring.

There is a (provably different) stratification of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ also indexed by integer partitions due to Schneider and Zink (see [SZ]). We conjecture that the stratification of Schneider-Zink is related to the one we construct via the Zelevinsky involution and I hope to investigate this question further in the future. Schneider and Zink also prove that their strata categories are module categories over some rings, however they have much less control over these rings, e.g. they can only prove they are commutative up to nilpotent elements. Hence we hope that our results also lead to a better understanding of their construction.

3. Future Research Plan

3.1. **Stratifications of Categories of Reductive p-adic Groups.** By using the highest derivative partition we can analogously define stratifications of $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(\mathrm{GL}_n(F))$ or even integrally of $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathrm{GL}_n(F))$ and we are also able to construct progenerators. However, their endomorphism rings (which we denote by $E_{\lambda,\overline{\mathbb{F}}_\ell}$ respectively $E_{\lambda,\overline{\mathbb{Z}}_\ell}$) are much harder to control. Nevertheless, we make the following conjecture.

Conjecture. *The rings $E_{\lambda,\overline{\mathbb{Z}}_\ell}$ are reduced, commutative and flat over $\overline{\mathbb{Z}}_\ell$.*

Proving this conjecture would have to include the study of $E_{\lambda,\overline{\mathbb{F}}_\ell}$ and unfortunately many of the techniques of Bernstein-Zelevinksy we use to prove Theorem 1 break down. However, similar issues were resolved by Minguez-Secherre in their classification of irreducible representations of $\mathrm{GL}_n(F)$ over k in [MS].

Objective 1. ***Study the rings $E_{\lambda,\overline{\mathbb{F}}_\ell}$ by using the techniques of Minguez-Secherre.***

Helm and Moss showed in their proof of the local Langlands correspondence in families for $\mathrm{GL}_n(F)$ that the endomorphism ring of the Gelfand-Graev representation $W_{(n)} = \text{c-Ind}_{U_n}^{\mathrm{GL}_n(F)}(\psi_{(n)})$ is isomorphic to the ring of global functions on the stack of Langlands parameters for GL_n over $\overline{\mathbb{Z}}_\ell$. Hence it seems reasonable that the commutative rings E_λ also have interpretations in terms of the moduli space of Langlands paramters.

Objective 2. ***Explore if the stratification we construct corresponds to a natural structure on the moduli space of Langlands parameters and more generally under the emergent categorical Local Langlands conjecture.***

For more general quasisplit reductive groups G, degenerate Whittaker models are indexed by nilpotent orbits $\mathcal{O}$ in the Lie algebra $\mathfrak{g}$ of G. For each such nilpotent orbit $\mathcal{O}$, one can then again construct spaces of degenerate Whittaker models $W_\mathcal{O}$ and one also has an ordering on the set of nilpotent orbits, where $\mathcal{O}' \leq \mathcal{O}$ if $\mathcal{O}'$ is contained in the Zariski closure of $\mathcal{O}$. A stratification on $\mathrm{Rep}_\mathbb{C}(G)$ is then defined by setting $\mathrm{Rep}_\mathbb{C}(G)^{\leq \mathcal{O}}$ (resp. $\mathrm{Rep}_\mathbb{C}(G)^{< \mathcal{O}}$) to be the full subcategories consisting of representations π such that for each irreducible subquotient π' of π, if $\mathrm{Hom}_G(W_{\mathcal{O}'}, \pi') \neq 0$ then $\mathcal{O}' \leq \mathcal{O}$ (resp. $\mathcal{O}' < \mathcal{O}$). We can then consider the Serre quotient $\mathrm{Rep}_\mathbb{C}(G)^{=\mathcal{O}} = \mathrm{Rep}_\mathbb{C}(G)^{\leq \mathcal{O}} / \mathrm{Rep}_\mathbb{C}(G)^{< \mathcal{O}}$.

Objective 3. ***Construct a progenerator for*** $\mathrm{Rep}_\mathbb{C}(G)^{=\mathcal{O}}$ ***and study its endomorphism ring.***

One approach to this question could be to start with the depth zero part $\mathrm{Rep}_\mathbb{C}(G)^0$ and similar to our results in the $\mathrm{GL}_n(F)$-case, use generalized Gelfand-Graev representations of finite groups to construct a progenerator. A different approach would be to try to prove that the spaces of degenerate Whittaker models $W_\mathcal{O}$ are projective. For the regular nilpotent orbit this can be achieved by Rodier approximation (see Appendix A of [Hansen]), which unfortunately does not directly apply to the more general case. However, we believe that a variation of the techniques of Rodier could cover all cases.

3.2. **γ-Factors in Families.** A long term research objective (once a theory of generalized Speh representations in families has been developed) is to explore potential applications of the theory of γ-factors in families coming from the twisted doubling method to the local Langlands correspondence in families for classical groups, similar to what Helm and Moss did for $\mathrm{GL}_n(F)$. One step in that direction would be to prove a converse theorem in the spirit of Helm and Moss [HM].

Objective 4. ***Study possible applications of γ-factors to the local Langlands correspondence in families for classical groups***

Roughly speaking, I hope to prove a result along the following lines. Let W be the Gelfand-Graev representation of a quasi-split classical group G with rank n, which means that $W = \text{c-Ind}_U^G \psi_U$, where U is the unipotent radical of a Borel subgroup and ψ_U a nondegenerate character of U. Moreover, set V_{n-1} to be the "universal co-Whittaker module" of $\mathrm{GL}_{n-1}(F)$. Both of these representations are admissible when viewed as representations with coefficients in their endomorphism rings. By using the construction of γ-factors in families we can consider $\gamma(W \times V_{n-1})$, which gives rise to a Laurent series in $(\mathrm{End}(W) \otimes \mathrm{End}(V_{n-1}))[X]][X^{-1}]$. Helm and Moss constructed a linear map $\theta_{n-1}\colon \mathrm{End}(V_{n-1}) \to \overline{\mathbb{Z}}_\ell$ and the result I hope to prove in the future is the following.

Conjecture. *The ring* $\mathrm{End}(W)$ *is generated by* $(1 \otimes \theta_{n-1})\left((1 \otimes z)a\right)$*, where* a *runs over all coefficients of* $\gamma(W \times V_{n-1})$ *and* $z \in \mathrm{End}(V_{n-1})$*.*

This conjecture would be a key step in a proof of local Langlands in families for classical groups, which is equivalent to constructing a map from $\mathrm{End}(W)$ to the ring of functions R on the moduli space of Langlands parameters satisfying some favorable properties. Basically by local Langlands over the complex numbers one can construct such a map $\Phi\colon \mathrm{End}(W) \to R \otimes \overline{\mathbb{Q}}_\ell$. Now if the above conjecture is true one has generators for $\mathrm{End}(W)$ and can then show that the image actually lies in R which proves local Langlands in families for classical groups.

Unfortunately there is not even yet a converse theorem for the γ-factors coming from the doubling method in the complex coefficient setting. However, in work in progress joint with Elad Zelingher we are working towards such a result. We are developing the doubling method to define the analogues of γ-factors for representations of classical groups of Lie type. This should then allow us to compute γ-factors explicitly for complex depth-zero representations of classical groups over p-adic fields. By using these computations we hope to prove a converse theorem for complex depth zero representations of classical groups.

Objective 5. ***Develop the analogue of the doubling method of Cai-Friedberg-Ginzburg-Kaplan for finite groups and use these results to compute γ-factors explicitly for complex depth zero representations and prove a converse theorem.***

On a different note, Robert Kurinczuk and I have recently began to study the Langlands-Shahidi method, which associates local constants to generic representations of a quasi-split reductive p-adic group. The techniques developed in my PhD thesis also work in this setting and developing the Langlands-Shahidi method in the setting of families seems within reach.

Objective 6. ***Develop the Langlands-Shahidi method in the setting of families and explore applications in collaboration with Robert Kurinczuk.***

References

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