4. Relations and Functions

Binary relations

A n-ary relation on sets $A_1, A_2, \ldots, A_n$ is a subset of the cartesian product $A_1 \times A_2 \times \cdots \times A_n$, i.e., a set of n-tuples $\langle a_1, a_2, \ldots, a_n \rangle$, where $a_i \in A_i$ for all $i \in [1..n]$. Binary relations are just relations with *arity* two.

Notation and terminology

Take $\mathcal{R} \subseteq A \times B$ to be a binary relation between two sets A and B, where A is called the *domain*, and B the *codomain* of $\mathcal{R}$. If $\langle a, b \rangle \in \mathcal{R}$, we say that a stands in relation $\mathcal{R}$ with b, or that a is related to b under $\mathcal{R}$. It is, however, more common to use the infix notation $a\mathcal{R}b$ to express this fact. In the case that $A = B$, the relation is *defined over*, or is a relation *on* the set in question. The notation $a\not\mathcal{R}b$ is a shorthand for $\neg(a\mathcal{R}b)$. Sometimes, the abbreviation $a\mathcal{R}b\mathcal{R}c$ is used, meaning that $a\mathcal{R}b \wedge b\mathcal{R}c$ (e.g., $0 < r < 1$ when $0 < r$ and $r < 1$). Some familiar binary relations are the

- order comparison ($<, >, \leq, \geq$);
- equality ($=$);
- subset ($\subseteq$); and
- divisibility ($|$) relations.

Properties of binary relations

Binary relations can be classified based on which properties they satisfy. This gives rise to various families of relations, like orders, equivalence relations, dependency relations, and functions. Some of the more common of these properties are described next. In the following examples, we assume a relation $\mathcal{R}$ defined over the set S.

✔ Reflexivity

If $\mathcal{R}$ is *reflexive*, then every element of the set S is related to itself under $\mathcal{R}$.

$$\forall a \in S : a\mathcal{R}a$$

Examples: $\leq, \geq, \subseteq, \supseteq$, and $=$. Divisibility is also reflexive, since every integer divides itself.

✔ Irreflexivity

A relation $\mathcal{R}$ is *irreflexive* (or *strict*) when no element of S is related to itself under $\mathcal{R}$.

$$\forall a \in S : \neg(a\mathcal{R}a) \qquad \text{(or think of this as } \langle a, a \rangle \notin \mathcal{R}\text{)}$$

Examples: $<, >$ (*strictly* less or greater than), $\subset$, and $\supset$.

✔ Transitivity

When a relation $\mathcal{R}$ is *transitive*, given three elements a, b, and c in S, the following condition holds: If a is related to b, and b is related to c, then a is also related to c.

$$\forall a, b, c \in S : a\mathcal{R}b \wedge b\mathcal{R}c \implies a\mathcal{R}c \qquad \text{(or more compactly } a\mathcal{R}b\mathcal{R}c \implies a\mathcal{R}c\text{)}$$

Examples: $<$, $>$, $\leq$, $\geq$, $\mid$ and $=$.

✔ Symmetry

A relation $\mathcal{R}$ is *symmetric* when the following condition holds for any two elements a and b in S: If a is related to b, then b is also related to a.

$$\forall a, b \in S : a\mathcal{R}b \iff b\mathcal{R}a$$

Examples: $=$ and $\equiv$.

✔ Asymmmetry

A relation $\mathcal{R}$ is *asymmetric* when the following condition holds for any two elements a and b in S: If a is related to b, then b is *not* related to a.

$$\forall a, b \in S : a\mathcal{R}b \implies \neg(b\mathcal{R}a)$$

Examples: $<, >, \subset$, and $\supset$. A relation which is not symmetric doesn't have to be asymmetric. As an example, $\leq$ is neither symmetric nor asymmetric. The only relation which is both symmetric and asymmetric is the *empty relation* ($\varnothing$).

✔ Antisymmetry

An *antisymmetric* relation satisfies the condition that, for every pair of elements a and b of S, if a is related to b, and b is related to a, then $a = b$.

$$\forall a, b \in S : (a\mathcal{R}b \wedge b\mathcal{R}a) \implies a = b$$

Examples: $\leq, \geq$, and divisibility, when defined over the natural numbers. Divisibility on the integers, however, is not antisymmetric, since $a \mid -a$ and $-a \mid a$ but $a \neq -a$. Every asymmetric relation is antisymmetric, since the condition $aRb \wedge bRa$ is false if $\mathcal{R}$ is asymmetric, and an implication with a false premise is automatically true. Note that a relation can be both symmetric and antisymmetric.

✔ Totality

A relation $\mathcal{R}$ is *total* when the following condition holds for any two elements a and b in S: Either a is related to b, or b is related to a, under $\mathcal{R}$.

$$\forall a, b \in S : a\mathcal{R}b \vee b\mathcal{R}a$$

Examples: $\leq$ and $\geq$. A total relation is always reflexive.