FunkAn - 2

Milos Mathias Koch - skg675

Januar 2021

Problem 1

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space, H.

a

Note that $\langle f_N, h\rangle = \frac{1}{N}\langle\sum_{n=1}^{N^2} e_n, h\rangle = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, h\rangle$. By Parseval's identity, we have $\sum_{n=1}^{\infty}|\langle e_n, h\rangle|^2 < \infty$, for all $h \in H$. Hence, for all $\varepsilon > 0$ there exists $k \in \mathbb{N}$, such that $\sum_{n=k}^{\infty}|\langle e_n, h\rangle|^2 < \varepsilon$. As

$$\frac{1}{N}\sum_{n=1}^{N^2}|\langle e_n, h\rangle| \le \frac{1}{N}\sum_{n=1}^{k-1}|\langle e_n, h\rangle| + \frac{1}{N}\sum_{n=k}^{N^2}|\langle e_n, h\rangle| \ (*),$$

and by the Cauchy-Schwartz inequality for sums, we have

$$\frac{1}{N}\sum_{n=k}^{N^2}|\langle e_n, h\rangle| = \frac{1}{N}\sum_{n=k}^{N^2}1\cdot|\langle e_n, h\rangle| \le \frac{1}{N}\sqrt{N^2-k+1}\sqrt{\sum_{n=k}^{N^2}|\langle e_n, h\rangle|} < \sqrt{\frac{N^2-k+1}{N^2}}\sqrt{\varepsilon} \ (**).$$

Combining $(*)$ and $(**)$, it is easy to see that $\frac{1}{N}\sum_{n=1}^{N^2}|\langle e_n, h\rangle|$ goes to 0 as N goes to ∞. Hence by HW4 and Riesz representation theorem (for Hilbert spaces), we know that $f_N \to 0$ weakly.

By direct computation

$$\begin{aligned}
\|f_N\|^2 &= \langle f_N, f_N \rangle \\
&= \left\langle \frac{1}{N}\sum_{i=1}^{N^2} e_i, \frac{1}{N}\sum_{j=1}^{N^2} e_j \right\rangle \\
&= \frac{1}{N^2}\left\langle \sum_{i=1}^{N^2} e_i, \sum_{j=1}^{N^2} e_j \right\rangle \\
&= \frac{1}{N^2}\sum_{i=1}^{N^2}\sum_{j=1}^{N^2} \langle e_i, e_j \rangle \\
&= \frac{1}{N^2}\sum_{i=1}^{N^2} \langle e_i, e_i \rangle \\
&= \frac{1}{N^2} N^2 = 1.
\end{aligned}$$

b

Consider an element, f, in the convex hull $\mathrm{co}\{f_N : N \in \mathbb{N}\}$. Then $f = \sum_{i=1}^n a_i f_{N_i}$ with positive a_i's and $\sum_{i=1}^n a_i = 1$ and $N_j \in \mathbb{N}$ for all $j \in \mathbb{N}$. Hence $\|f\| = \|\sum_{i=1}^n a_i f_{N_i}\| \leq \sum_{i=1}^n a_i \|f_{N_i}\| = \sum_{i=1}^n a_i = 1$. Therefore, the convex hull is contained in the closed, norm unit ball, and so is the closed convex hull. By the Banach-Alaoglu theorem, the closed, norm unit ball in H^* is weak* compact. By the Riesz representation theorem, H^* is also a Hilbert space, hence the closed, norm unit ball is weakly compact. As Hilbert spaces are isometric isomorphic to its dual, the closed, norm unit ball in H is weakly compact. As weak and norm closures coincides for convex sets, (theorem 5.7), the norm closure of $\mathrm{co}\{f_N : N \in \mathbb{N}\}$ is weakly closed in H. As closed subsets of compact sets are compact, the norm closure of $\mathrm{co}\{f_N : N \in \mathbb{N}\}$ is weakly compact.
As f_N converges weakly to 0, as $N \to \infty$, 0 is the weak limit of elements in $\mathrm{co}\{f_N : N \in \mathbb{N}\}$, and so 0 is in the weak closure of $\mathrm{co}\{f_N : N \in \mathbb{N}\}$. We have just argued that the norm closure and the weak closure of $\mathrm{co}\{f_N : N \in \mathbb{N}\}$ coincides, hence $0 \in K$.

c

Let $N \in \mathbb{N}$, and assume $f_N = ag + (1-a)h$ is a non-trivial convex combination for some $a \in [0,1]$ and $g, h \in K$. Then g and h are limits of convex combinations of f_N's, Hence we have

$$f_N = a \sum_{i=1}^{\infty} b_i f_{N_i} + (1-a) \sum_{j=1}^{\infty} c_j f_{N_j},$$

where $\sum_{i=1}^{\infty} b_i = \sum_{j=1}^{\infty} c_j = 1$, and $b_i, c_j \geq 0$ for all $i, j \in \mathbb{N}$. Hence

$$\frac{1}{N} \sum_{n=1}^{N^2} e_n = a \sum_{i=1}^{\infty} b_i \frac{1}{N_i} \sum_{n=1}^{N_i^2} e_n + (1-a) \sum_{j=1}^{\infty} c_j \frac{1}{N_j} \sum_{n=1}^{N_j^2} e_n.$$

taking inner product with e_{N^2} yields

$$\frac{1}{N} = a \sum_{\{i : N_i^2 \geq N^2\}} \frac{b_i}{N_i} + (1-a) \sum_{\{j : N_j^2 \geq N^2\}} \frac{c_j}{N_j} \ (*).$$

As $\frac{1}{n}(a \sum_{i=1}^{\infty} b_i + (1-a) \sum_{j=1}^{\infty} c_j) = \frac{1}{n}$ for all $n \in \mathbb{N}$, and as the b- and c-sums are increasing, $(*)$ can only hold if $N_i = N_j = N$ for all $i, j \in \mathbb{N}$, hence $f_N = f_{N_i} = f_{N_j}$. Thus f_N is an extreme point for all $N \in \mathbb{N}$.

Now consider a convex combination for 0 of elements in K. Let $0 = ax + (1-a)y$ with $x, y \in K$. As x and y a limits of convex combinations of something ($\{f_N : N \in \mathbb{N}\}$) with

$$\langle f_N, e_n \rangle \geq 0 \text{ for all } n \in \mathbb{N}$$

we, by continuity of inner products, have that $\langle x, e_n \rangle \geq 0$, and $\langle y, e_n \rangle \geq 0$. As $0 = \langle 0, e_n \rangle = \langle x, e_n \rangle + \langle y, e_n \rangle$ for all $n \in \mathbb{N}$ along with our observation implies $0 = \langle x, e_n \rangle = \langle y, e_n \rangle$ for all $n \in \mathbb{N}$. As $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis, this implies that $x = y = 0$, hence 0 is an extreme point of K.

d

There does not. First, a small fact from general topology; Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence, i.e. $x_n \to x$, in a space, in which limits are unique. then $\{x\} \cup (x_n)_{n \in \mathbb{N}}$ is compact.

Proof. Let O be an open covering of $\{x\} \cup (x_n)_{n\in\mathbb{N}}$. Let O_x be an open set containing x. By convergence, there is only finitely many points in the sequence not contained in O_x. For each such point, x_i, choose open sets, O_i, from O, such that $x_i \in O_i$. This forms a finite subcovering of $\{x\} \cup (x_n)_{n\in\mathbb{N}}$. $\square$

As limits in the weak topology are unique, $\{f_N : N \in \mathbb{N}\} \cup \{0\}$ is weakly compact, hence weakly closed. By Milman's theorem, every extreme point of K is contained in $\{f_N : N \in \mathbb{N}\} \cup \{0\}$.

Problem 2

a

By HW4, we know that $x_n \to x$ weakly if and only if $f(x_n) \to f(x)$ for all $f \in X^*$. As T is linear and continuous, $g \circ T \in X^*$ for all $g \in Y^*$, hence $g(Tx_n)$ converges to $g(Tx)$ for all $g \in Y^*$, hence $Tx_n \to Tx$ weakly.

b

By HW4, we know that $(x_n)_{n\in\mathbb{N}}$ is bounded, hence every subsequence is also bounded. By proposition 8.2, every subsequence, x_{n_j} has a further subsequence $x_{n_{j_i}}$ such that $Tx_{n_{j_i}}$ converges in norm. As norm convergence implies weak convergence, each further subsequence has to satisfy that under T it converges to Tx, else it would contradict our assumption of weak convergence. From general metric space theory, we know that if every subsequence $(z_{n_j})_{j\in\mathbb{N}}$ of a sequence, $(z_n)_{n\in\mathbb{N}}$, in a metric, has a further subsequence, $(z_{n_{j_i}})_{i\in\mathbb{N}}$, that converges to the same point z, then $z_n \to z$ as $n \to \infty$. Applying this to $(Tx_n)_{n\in\mathbb{N}}$ shows that $Tx_n \to Tx$ as $n \in \mathbb{N}$ in the metric induced by the norm, in other words, in norm.

c

Assume for a contradiction that T is not compact. Then, by theorem 8.2, $T(\overline{B_H(0,1)})$ is not totally bounded, hence there exists $\delta > 0$, such that $T(\overline{B_H(0,1)})$ cannot be covered by finitely many balls of radius δ. In that spirit, choose $Tx \in T(\overline{B_H(0,1)})$, and for each $n \in \mathbb{N} \setminus \{1\}$, let be Tx_n such that

$$Tx_n \in T(\overline{B_H(0,1)}) \setminus \bigcup_{i=1}^{n-1} B_Y(Tx_{x-1}, \delta).$$

By the lack of total boundedness, this set is always non-empty. It is easy to see that

$$\|Tx_n - Tx_m\| > \delta, \text{ for } n \neq m.$$

Thus, $(x_n)_{n\in\mathbb{N}} \subset B_H(0,1)$ is a sequence in the Hilbert space H. As the weak topology on H^* and the weak-$*$ topology on H^* coincides, the respective topologies coincides on $\overline{B_{H^*}(0,1)}$. As H is seperable, so is H^*, hence $\overline{B_{H^*}(0,1)}$ is by, theorem 5.13 metrizable. As $H \cong H^*$ for all Hilbert spaces $\overline{B_H(0,1)}$ is also metrizable in the weak topology. As H is a Hilbert space, $\overline{B_H(0,1)}$ is by the Banach-Alaoglu weakly compact. Hence $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$. However, as, for $n_j \neq n_i$

$$\delta < \left\|Tx_{n_j} - Tx_{n_i}\right\|,$$

this subsequence is not a norm-convergent sequence under T. Hence we have shown the contrapositive of the desired result.

d

Let $(x_n)_\mathbb{N}$ be a weakly convergent sequence converging to x. Then by problem $2a$, $(Tx_n)_\mathbb{N}$ converges weakly to Tx in $\ell_1(\mathbb{N})$. As sequences in $\ell_1(\mathbb{N})$, by remark 5.3, converges weakly if and only if they converge in norm, $(Tx_n)_\mathbb{N}$ converges in norm to Tx. As $\ell_2(\mathbb{N})$ is a Hilbert space and $\ell_1(\mathbb{N})$ is a Banach space, we have, by problem $2c$, that T is compact.

e

Assume for a contradiction that T is a surjective, compact operator. By the open mapping theorem, T is also an open mapping. By compactness, $T(B_X(0,1))$ has compact closure, and as $T(B_X(0,1))$ is open, it contains the norm closure of $B(0,\varepsilon)$ for some $\varepsilon > 0$. Hence the norm closure of $B(0,\varepsilon)$ is a closed set in the compact set $\overline{T(B_X(0,1))}$, hence it is itself compact. This is a contradiction with problem 3, in the first mandatory assignment.

f

By direct computaion, as t is real-valued

$$\begin{aligned}\langle Mf, g\rangle &= \int_{[0,1]} tf(t)\overline{g(t)}\, dm(t)\\ &= \int_{[0,1]} f(t)\overline{tg(t)}\, dm(t)\\ &= \langle f, Mg\rangle.\end{aligned}$$

By HW6, M has no eigenvalues. As M is self-adjoint, if it was compact, the spectral theorem would provide several eigenvalues. As this is not the case, M cannot be compact

Problem 3

a

As K is continuous, $[0,1]$ is compact and the Lebesgue measure is finite on $[0,1]$, compactness of T follows from theorem 9.6, as T is the kernel operator of K.

✓

T is kernel operator of $\tilde{K}$ with $\tilde{K}(s,t) = K(t,s)$.

b

By direct computation

$$
\begin{aligned}
\langle T(f), g\rangle &= \int \int K(s,t)f(t)\ dm(t)\overline{g(s)}\ dm(s) \\
&= \int \int K(s,t)f(t)\overline{g(s)}\ dm(t)\ dm(s) \\
&\overset{(*)}{=} \int K(s,t)f(t)\overline{g(s)}\ dm \otimes m(s,t) \\
&\overset{(*)}{=} \int \int K(s,t)f(t)\overline{g(s)}\ dm(s)\ dm(t) \\
&= \int \int K(s,t)\overline{g(s)}\ dm(s)\ f(t)dm(t) \\
&\overset{(**)}{=} \int \int \overline{K(s,t)g(s)}\ dm(s)\ f(t)dm(t) \\
&= \int \overline{\int K(s,t)g(s)\ dm(s)}\ f(t)dm(t) \\
&= \int f(t)\overline{\int K(s,t)g(s)\ dm(s)}\ dm(t) \\
&= \langle f, T(g)\rangle,
\end{aligned}
$$

$\leftarrow$ if $K(s,t) = K(t,s)$

where $(*)$ is due to Fubini's theorem, and $(**)$ is due to K being real-valued.

$(\checkmark)$

why is Fubini justified?

Conclusion?

c

Let $f \in L_2([0,1], m)$. For every $s \in [0,1]$, by direct computation, we see

$$
\begin{aligned}
Tf(s) &= \int_{[0,1]} K(s,t)f(t)\ dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)\ dm(t) + \int_{(s,1]} K(s,t)f(t)\ dm(t) \\
&\overset{(*)}{=} \int_{[0,s]} K(s,t)f(t)\ dm(t) + \int_{\{s\}} K(s,t)f(t)\ dm(t) + \int_{(s,1]} K(s,t)f(t)\ dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)\ dm(t) + \int_{[s,1]} K(s,t)f(t)\ dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)\ dm(t) + \int_{[s,1]} (1-t)sf(t)\ dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)\ dm(t) + s\int_{[s,1]} (1-t)f(t)\ dm(t),
\end{aligned}
$$

where $(*)$ is due to the fact that $\{s\}$ is a null-set. At $s = 0$, the first term is an integral over a null-set, hence 0, and the second term is 0 times some number, hence zero. identically, at $s = 1$, the second term is an integral over a null-set, hence 0, and the first term is 0 times some number, hence zero. Therefore $Tf(0) = Tf(1) = 0$. Now, let $(s_n)_{n\in\mathbb{N}} \subseteq [0,1]$ be a convergent sequence to some $s \in [0,1]$. By the Cauchy-Shwartz inequality, we get √

$$\begin{aligned}\left|\int_{[0,s]} tf(t)\ dm(t) - \int_{[0,s_n]} tf(t)\ dm(t)\right| &= \left|\int_{[\min(s,s_n),\max(s,s_n)]} tf(t)\ dm(t)\right| \\ &\leq \left\|1_{[\min s,s_n,\max s,s_n]}\right\|_2 \|tf(t)\|_2 \\ &= \sqrt{|s-s_n|}\|tf(t)\|_2.\end{aligned}$$

why $\|tf(t)\|_2 < \infty$?

And hence we see that the first function given by an integral is Hölder continuous, hence continuous. Note that $\int_{[s,1]}(1-t)f(t)\ dm(t) = \int_{[0,1]}(1-t)f(t)\ dm(t) - \int_{[0,s)}(1-t)f(t)\ dm(t)$, hence the second function given by an integral is also Hölder continuous. As all other components of Tf is trivially continuous, we conclude that Tf is continuous.

What does "all other components" mean? Tf is composition of cont. functions.

Problem 4

a

By HW7P1a $g_k \in \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$. We claim that, for non-negative Schwartz functions, f, the following statement hold:

If $\int_{\mathbb{R}} |x|^k f(x)\ dm(x) < \infty$, then, by proposition, the Fourier transform of f, $\mathcal{F}(f)$, is k times differentiable, and

$$i^k \frac{d^k}{d\xi^k}\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} x^k e^{-i\xi x} f(x)\ dm(x) \text{ for all } \xi \in \mathbb{R}.$$

True, but proof or ref is missing.

We, by proposition 11.4, already know $\mathcal{F}(g_0)(\xi) = g_0(\xi) = e^{-\frac{\xi^2}{2}}$. By our observation, we can compute the rest of the desired Fourier transform by differentiating (and dividing by the appropriate scalar):

$$\begin{aligned}
\mathcal{F}(g_1)(\xi) &= i\frac{d}{d\xi}g_0(\xi) = -i\xi e^{-\frac{\xi^2}{2}} \\
\mathcal{F}(g_2)(\xi) &= -\frac{d^2}{d\xi^2}g_0(\xi) = -(\xi^2-1)e^{-\frac{\xi^2}{2}} = -\xi^2 e^{-\frac{\xi^2}{2}} + e^{-\frac{\xi^2}{2}} \\
\mathcal{F}(g_3)(\xi) &= -i\frac{d^3}{d\xi^3}g_0(\xi) = i\xi(\xi^2-3)e^{-\frac{\xi^2}{2}} = i\xi^3 e^{-\frac{\xi^2}{2}} - i3\xi e^{-\frac{\xi^2}{2}}
\end{aligned}$$

b

By the previous problem, we see that $h_0 := g_0$ works. Again, by the previous problem. We also see that $h_3 := g_1$ works. In the same spirit we see that. By direct computation, we see, for every $\xi \in \mathbb{R}$

$$\begin{aligned}
\mathcal{F}(g_2 - \frac{1}{2}g_0)(\xi) &= -\xi^2 e^{-\frac{\xi^2}{2}} + e^{-\frac{\xi^2}{2}} - \frac{1}{2}e^{-\frac{\xi^2}{2}} = i^2(g_2(\xi) - \frac{1}{2}g_0(\xi)) \\
\mathcal{F}(g_3 - \frac{3}{2}g_1)(\xi) &= i\xi^3 e^{-\frac{\xi^2}{2}} - i3\xi e^{-\frac{\xi^2}{2}} + i\frac{3}{2}\xi e^{-\frac{\xi^2}{2}} = -i\xi^3 e^{-\frac{\xi^2}{2}} - i\frac{3}{2}\xi e^{-\frac{\xi^2}{2}} = i(g_3 - \frac{3}{2}g_1).
\end{aligned}$$

c

Let $\mathcal{F}^*$ denote the inverse Fourier transform. Corollary 12.12 allows us to write $\mathcal{F}(\mathcal{F}^*(f)) = f$ for every Schwartz function, f. This implies that $\mathcal{F}^2(f^\vee) = \mathcal{F}(f)$, where $f^\vee(\alpha) = \int_{\mathbb{R}} f(x)e^{ix\alpha}\, dm(x)$ denotes the inverse fourier transform of f. Computing this yields

$$\mathcal{F}^2(f^\vee)(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\, dm(x) \text{ for all } \xi \in \mathbb{R}.$$

If we compare this to the inverse Fourier transform of f, we see that $f^\vee(-z) = \mathcal{F}^2(f^\vee)(z)$ for all $z \in \mathbb{R}$. So applying the Fourier transform twice yields a sign change in the argument. As the Fourier transform is a automorphism on the Schwartz space $\mathscr{S}(\mathbb{R})$, this implies that $f^\vee(z) = \mathcal{F}^4(f^\vee)(z)$ for all $z \in \mathbb{R}$. Computing the Fourier transform a last time, gives us

$$f(z) = \mathcal{F}^4(f)(z) \text{ for all } z \in \mathbb{R}$$

d

and linearity of Fourier transform

By the last problem, we have that any complex number, λ, satisfying $\mathcal{F}(f) = \lambda f$, must satisfy $\lambda^4 = 1$. The complex numbers satisfying this are exactly $1, -1, i, -i$. As all those are examples of eigenvalues, by problem 4.b, the eigenvalues are exactly those numbers.

Problem 5

From general topology, we know that dense sets intersect every non-empty open set. If that was not the case, then the complement of a disjoint open, non-empty set would contain the dense set, contradicting that the closure of the dense set is the entire space. In other words, every non-empty open set in $[0,1]$ contains at least one $x_i \in (x_n)_{n\in\mathbb{N}}$. As such we have, for every open set, U, that $\frac{1}{2^i} \leq \mu(U)$. Thus the only open set with μ-measure 0, is the empty set. Hence

$$\operatorname{supp}(\mu) = \left(\bigcup_{U \text{ open with } \mu(U)=0} U \right)^c = \emptyset^c = [0,1].$$