Mandatory assignment - FunkAn

Jannat Ahmad - xrg664

December 14, 2020

Problem 1

a)

To show that $\|\cdot\|_0$ is a norm on X, we show that it satisfies all the conditions of definition 1.1 in the lecture notes.

$$\|x+y\|_0 = \|x+y\|_X+\|T(x+y)\|_Y = \|x+y\|_X+\|Tx+Ty)\|_Y \leq \|x\|_X+\|y\|_X+\|Tx\|_Y+\|Ty\|_Y$$

$$= (\|x\|_X + \|Tx\|_Y) + (\|y\|_X + \|Ty\|_Y) = \|x\|_0 + \|y\|_0$$

for $x, y \in X$. The inequality holds because $\|\cdot\|_X$, and $\|\cdot\|_Y$ are norms on X and Y respectively, and since $T : X \to Y$ then we have $Tx, Ty \in Y$, for $x, y \in X$. Now we've shown that the first condition is satisfied.

Now we show that the seccond condition is satisfied:

$$\|\alpha x\|_0 = \|\alpha x\|_X+\|T\alpha x\|_Y = \|\alpha x\|_X+\|\alpha Tx\|_Y = |\alpha|\|x\|_X+|\alpha|\|Tx\|_Y = |\alpha|(\|x\|_X+\|Tx\|_Y) = |\alpha|\|x\|_0$$

For $\alpha \in \mathbb{K}$ and $x \in X$. Now we've shown the seccond condition.

Now we're gonna show that the third condition is satisfied. we assume $0 = \|x\|_0 = \|x\|_X + \|Tx\|_Y \Leftrightarrow \|x\|_X = -\|Tx\|_Y$, and the only way this is possible is if $\|x\|_X = -\|Tx\|_Y = 0$, because $\|x\|_X \geq 0$ per definition, and since we have $\|\cdot\|_X$ is a norm on X, we have $\|x\|_X = 0 \Leftrightarrow x = 0$, so now we've shown $\|x\|_0 = 0 \Leftrightarrow x = 0$. So now we've shown that the third condition is satisfied. So now we'we shown that $\|\cdot\|_0$ is a norm on X.

Now we want to show $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent $\Leftrightarrow$ T is bounded. We start by showing $\Rightarrow$. We assume $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent, so we have from definition 1.4 from the lecture notes that there exists $0 < C_1 < C_2$ such that:

$$C_1\|x\|_X \leq \|x\|_0 \leq C_2\|x\|_X$$

We have

$$\|x\|_0 \leq C_2\|x\|_X \Leftrightarrow \|x\|_X + \|Tx\|_Y \leq C_2\|x\|_X \Leftrightarrow \|Tx\|_Y \leq C_2\|x\|_X - \|x\|_X \leq C_2\|x\|_X$$

So there exists $C = C_2 > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$, so T is bounded. Now we're gonna show the converse statement $\Leftarrow$. We assume T is bounded, and we want to show that the two norms $\|\cdot\|_X$, and $\|\cdot\|_0$ are equivalent, i.e we want to show that there exists $0 < C_1 \leq C_2 < \infty$ such that:

$$C_1\|x\|_X \leq \|x\|_0 \leq C_2\|x\|_X \quad , x \in X$$

Since we have that T is bounded then we have that there exists $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$, so we have:

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq \|x\|_X + C\|x\|_X = (C+1)\|x\|_X$$
$$\|x\|_X = \|x\|_0 - \|Tx\|_Y \leq \|x\|_0$$

We have the inequality because $\|Tx\|_Y \geq 0$ So we have that there exists $0 < C_1 \leq C_2 < \infty$, such that

$$C_1\|x\| \leq \|x\|_0 \leq C_2\|x\|_X$$

Where $C_1 = 1$, and $C_2 = C + 1$, where $C > 0$, so we have from definition 1.4 in the lecture notes that $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent norms on X.

b)

If we have that X is finite dimensional, then we have that any two norms on X are equivalent by theorem 1.8 in the lecture notes, so $\|\cdot\|_X$ and $\|\cdot\|_0$ may be equivalent norms, since they are both norms on X, and then we have from problem 1a, that T is bounded.

c)

We assume that X is infinite, and we want to show that there exists a linear map $T : X \to Y$ which is not bounded. Since we have assumed that X is infinte we have as a consequence of Zorn's lemma that X admits a Hamel basis defined as $B_x = (e_i)_{i \in I}$ where I is an index set and e_i for $i \in I$ are elements in X. We now define a linear map $T : X \to Y$, and shows that it is not bounded. We let every element in X be normalized so we have:

$$T\left(\frac{e_i}{\|e_i\|}\right) = i \cdot y$$

For $0 \neq y \in Y, i \in \mathbb{N}$, where y is fixed. if we have that $i \notin \mathbb{N}$, then we let:

$$T\left(\frac{e_i}{\|e_i\|}\right) = 0$$

This is well-defined because we have that $\left\{\frac{e_i}{\|e_i\|}\right\} \subseteq X$ is linearly independent, which is because $\left\{\frac{e_i}{\|e_i\|}\right\}$ in B_X

$$\left\{\frac{e_i}{\|e_i\|}\right\}_{i\in I} \subseteq \{x \in X : \|x\| \leq 1\} = A$$

So we have:

$$T\left\{\frac{e_i}{\|e_i\|}\right\}_{i\in I} \subseteq TA$$

So we'll have:

$$0 < i\|y\| \leq \sup_{x\in A} \|Tx\|$$

For each $i \in I$, so there exists a linear map $T : X \to Y$ which is not bounded.

d)

We assume again X is infinite, then we have from problem 1c, that there exists a linear map $T : X \to Y$ which is not bounded, and then we have from problem 1a that there exists a norm $\|\cdot\|_0$ on X which is not equivalent to the given norm $\|\cdot\|_X$. We have that this norm $\|\cdot\|_X$ satisfies:

$$\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_0$$

for all $x \in X$. we have the inequality because $\|Tx\|_Y \geq 0$.

We have that $\|\cdot\|_0$, and $\|\cdot\|_X$ are norms on the vector space X such that $\|\cdot\|_X \leq \|\cdot\|_0$. So we have from HW 3 problem 1, that since the two norms aren't equivalent, then X can't be complete with respect to both norms. So if we have that $(X, \|\cdot\|_X)$ is a Banach space, then $(X, \|\cdot\|_X)$ is complete, so we have that $(X, \|\cdot\|_0)$ can't be complete.

e)

We set $X = \ell_1(\mathbb{N})$ equipped with the 2 norms $\|\cdot\|_1$-norm and $\|\cdot\|_\infty$. We start by showing that these two norms are inequivalent. We do this by taking a finite sequence

$(y_n)_{n\in\mathbb{N}} \subset \ell_1(\mathbb{N})$. we then have:

$$\|y\|_1 = \sum_{i=1}^{n} |y_i| \geq \max_{i=1,\dots,n}\{|y_i|\} = \|y\|_\infty$$

We want to show that the two norms are inequivalent, so we want to look at a sequence $(a_n)_{n\in\mathbb{N}}$ for which it holds that there does not exist a $C > 0$ such that $\|a_n\|_1 \leq C\|a_n\|_\infty$. We look at the sequence:

$$(a_n)_{n\in\mathbb{N}} = (a_1, ..., a_k, 0, 0, ..., 0) = (1, 1, .., 1, 0, 0, ..., 0)$$

We then have that :

$$\|a_n\|_1 = \sum_{i=1}^{k} |1| = \sum_{i=1}^{k} 1 = k$$

And we have that:

$$\|a_n\|_\infty = \max_{i\in\mathbb{N}}\{|a_i|\} = 1$$

And we have that we can for every $C > 0$ find a $k > C$, so there does not exist a $C > 0$ such that $\|a_n\|_1 \leq C\|a_n\|_\infty$, hence $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are inequivalent norms.

From the Riesz-Fischer theorem we have that $(\ell_p(\mathbb{N}), \|\cdot\|_p)$ is a Banach space for $1 \leq p < \infty$, so we have $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is a Banach space, and Banach space is complete, so $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is complete.

Now we want to show that $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete, so we find a cauchy sequence which has points in $\ell_1(\mathbb{N})$, but it's limit is not in $\ell_1(\mathbb{N})$. We look at the sequence of sequences, $(y_n(k))_{n\in\mathbb{N}}$. Where we have that $y_n(k) = \frac{1}{k}$ when $1 \leq k \leq n$, and $y_n(k) = 0$ when $k > n$. Since we have that $y_n(k)$ is finite wrt $\|\cdot\|_1$ for all n for each k, then we have that $(y_n(k))_{n\in\mathbb{N}} \subseteq \ell_1(\mathbb{N})$ for all n and k. We claim that $y(k) = \frac{1}{k}$ for all $k \in \mathbb{N}$, and we show this:

$$\|y_n(k) - y(k)\|_\infty = \max_{n\in\mathbb{N}}\{|y_n(k) - y(k)|\} = \left|\frac{1}{n+1}\right| \to 0$$

So we have that $(y_n(k))_{n\in\mathbb{N}}$ is a Cauchy sequence wrt $\|\cdot\|_\infty$-norm. But since we have that $\sum_{n=1}^{\infty} \left|\frac{1}{n+1}\right| \to \infty$, then we have that $y(k) \notin \ell_1(\mathbb{N})$, so we have that $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete.

Problem 2

a)

To show that f is bounded, we want to start by showing that f is linear. We let $\alpha, \beta \in \mathbb{C}, (a_1, b_1, 0, 0, ..., 0), (a_2, b_2, 0, 0, ..., 0) \in M$. Then we have:

$$f(\alpha(a_1, b_1, 0, 0, ..., 0) + \beta(a_2, b_2, 0, 0, ..., 0)) = f((\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, 0, 0, ..., 0))$$

$$= \alpha a_1+\beta a_2+\alpha b_1+\beta b_2 = \alpha(a_1+b_1)+\beta(a_2+b_2) = \alpha f(a_1, b_1, 0, 0, ..., 0)+\beta f(a_2, b_2, 0, 0, ..., 0)$$

So we have that f is linear.

So now to show that f is bounded we show that there exists $C > 0$ such that $\|f(a, b, 0, 0, ...)\| \leq C\|(a, b, 0, 0, ...)\|_p$ for all $(a, b, 0, 0...) \in M$ where $a, b \in \mathbb{C}$. We have that.

$$\|f(a, b, 0, 0, ...)\| = |f(a, b, 0, 0, ...)| = |a + b| \leq |a| + |b| = \|(a, b)\|_1$$

Since we have that $\mathbb{C}^2$ is a finite-dimensional vector space then we have from theorem 1.6 from the lecture notes, that any two norms on $\mathbb{C}^2$ are equivalent. Hence we have that $\|\cdot\|_1$ and $\|\cdot\|_p$ are equivalent. Hence we have from definition 1.4 that there exists $0 < C < \infty$ such that $\|(a, b)\|_1 \leq C\|(a, b)\|_p$, where both $\|(a, b)\|_1$ and $\|(a, b)\|_p$ are norms on $\mathbb{C}^2$, so we have:

$$\|(a, b)\|_1 \leq C\|(a, b)\|_p = C\sqrt[p]{|a|^p + |b|^p + |0|^p + |0|^p + ...} = C\|(a, b, 0, 0, ...)\|_p$$

So now we've shown that there exists $C > 0$ such that $\|f(a, b, 0, 0, ...)\| \leq C\|(a, b, 0, 0, ...)\|_p$ for all $(a, b, 0, 0, ...) \in \mathbb{C}$ hence we've shown that f is bounded.
Now we want to compute $\|f\|$. I claim that $\|f\| = 2^{1-\frac{1}{p}}$, and I prove this, by first proving $\|f\| \geq 2^{1-\frac{1}{p}}$. We let $b = \left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, 0, ...\right)$, then we have:

$$\|b\|_p = \left\|\left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, 0, ...\right)\right\|_p = \left(\left|\frac{1}{2^{\frac{1}{p}}}\right|^p + \left|\frac{1}{2^{\frac{1}{p}}}\right|^p\right)^{\frac{1}{p}} = \left(\frac{1}{2} + \frac{1}{2}\right)^{\frac{1}{p}} = 1$$

And we have

$$\|f\| = \sup\left\{|a + b| \mid \|(a, b, 0, 0, ...)\|_p = 1\right\} \geq \left|\frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}}\right|$$

We have the inequality because $\left|\frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}}\right| \in \left\{|a + b| \mid \|(a, b, 0, 0, ...)\|_p = 1\right\}$

We have $\left|\frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}}\right| = \frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}} = 2\frac{1}{2^{\frac{1}{p}}} = 2^{1-\frac{1}{p}}$

So now we've shown that $\|f\| \geq 2^{1-\frac{1}{p}}$. And now we want to show $\|f\| \leq 2^{1-\frac{1}{p}}$

$|a+b| \leq |a|+|b| = \|(a,b,0,0,...)\|_1 = \|(a\cdot 1, b\cdot 1, 0, 0, ...)\|_1 \leq \|(a,b,0,0,...)\|_p \|\|(1,1,0,0,...)\|_q$

Where we have the seccond inequality from Hölder's inequality where $1 = \frac{1}{p} + \frac{1}{q}$. We let $\|(a,b,0,0,...)\|_p = 1$, so we'll have:

$$|a+b| \leq \|(1,1,0,0,...)\|_q = (|1|^q + |1|^q)^{\frac{1}{q}} = 2^{\frac{1}{q}}$$

But since we have that Hölder's inequality holds for p and q which satisfies $\frac{1}{p} + \frac{1}{q} = 1$, and we fix p, then we have that $q = \frac{p}{p-1}$, so:

$$|a+b| \leq 2^{\frac{1}{q}} = 2^{\frac{p-1}{p}} = 2^{1-\frac{1}{p}}$$

Since we have that this equality holds for all $|a+b|$ for which it holds $\|(a,b,0,0,...)\|_p = 1$, then we have:

$$\|f\| = \sup\left\{|a+b| \mid \|(a,b,0,0,...)\|_p = 1\right\} \leq 2^{1-\frac{1}{p}}$$

Hence we've shown $\|f\| \leq 2^{1-\frac{1}{p}}$, so now we've shown $\|f\| = 2^{1-\frac{1}{p}}$.

b)

We want to show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$. We start by showing the existence of a linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$. So now we start by assuming $1 < p < \infty$.

We've shown in problem 2a that f is linear, and we have that it is bounded hence it is continuous, so we have that $f \in M^*$, so we have from corollary 2.6 in the lecture notes that there must exist $F \in (\ell_p(\mathbb{N}))$, such that $F_{|M} = f$ and $\|F\| = \|f\|$

Now we want to show the uniqueness of a linear functional F on $\ell_p(\mathbb{N})$ extending f, and satisfying $\|F\| = \|f\|$. We start by assuming that we have two linear functionals F and F' on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$, and then we want to show that $F = F'$.
We have from problem 5 in HW 1 that if $\frac{1}{p} + \frac{1}{q} = 1$, then we'll have:

$$(\ell_p(\mathbb{N}))^* \cong \ell_q((N))$$

for $1 < p < \infty$. We define this with the linear function $T : \ell_q(\mathbb{N}) \to (\ell_p(\mathbb{N}))^*$, with:

$$Tx = f(x)$$

Where we have that $f : \ell_p(\mathbb{N}) \to \mathbb{C}$ is given by:

$$fx(y) = \sum_{n\in\mathbb{N}}$$

For a given $x \in \ell_q(\mathbb{N})$, and for any $y \in \ell_p(\mathbb{N})$ we let $F : \ell_p(\mathbb{N}) \to \mathbb{C}$ be given by:

$$F(a_1, a_2, a_3, ...) = a + b$$

This is seen to be a Hahn Banach extension of f. To show uniquness we want to show that there exist another Hahn-Banach extension F' which satisfies $\|F'\| = \|f\|$, for which it holds $F' = F$. We show this by contradiction, so we assume $F' \neq F$. I couldn't finish this up but then the idea was to conclude $\|F\| \neq \|F'\|$. Since both $\|F'\| = \|f\|$, and $\|F\| = \|f\|$, then we should have $\|F\| = \|F'\|$, so since we had $\|F\| \neq \|F'\|$, then we would have a contradiction, and hence $F = F'$. Hence there is a unique functional F on $\ell_p(\mathbb{N}$ extending f and satisfying $\|f\| = \|F\|$.

c)

Problem 3

a)

We show that no linear map $F : X \to \mathbb{K}^n$ is injective by contradiction. So we assume that a linear map $F : X \to \mathbb{K}^n$ is injective. We let $x_1, ..., x_{n+1}$ be linearly independent, and then we'll have that $F(x_1), ..., F(x_{n+1})$ is linearly dependent, because in $\mathbb{K}^n$ we can have at most n linearly independent vectors. Since $F(x_1), ..., F(x_{n+1})$ are linearly dependent we have that there exists $\alpha_1, ..., \alpha_{n+1}$, where at least one of them is non-zero, such that

$$F(\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1}) = \alpha_1 F(x_1) + ... + \alpha_{n+1} F(x_{n+1}) = 0$$

Where the first equality comes from linearity of F. Since F is injective we have $\ker(F) = \{0\}$, so since we have $F(\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1}) = 0$, then:

$$\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1} = 0$$

Since $x_1, ..., x_{n+1}$ are linearly independent, then we have that $\alpha_1, ..., \alpha_{n+1} = 0$, so we have a contradiction, and hence $F : X \to \mathbb{K}^n$ is not injective, so no linear map $F : X \to \mathbb{K}^n$ is injective.

b)

To show that $\bigcap_{j=1}^n \ker(f_j) \neq \{0\}$ we start by considering the linear map $F : X \to \mathbb{K}^n$ given by:

$$F(x) = (f_1(x), ..., f_n(x)), \quad x \in X$$

We have in problem 3a shown that no linear map is so we have that the linear map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), ..., f_n(x)), \quad x \in X$ isn't injective either, hence we have:

$$\ker(F) \neq \{0\}$$

This means that there exists $0 \neq x \in X$, such that $(f_1(x), ..., f_n(x)) = F(x) = 0$. This means that there exists $0 \neq x \in X$ such that $f_j(x) = 0$ for all $j = 1, ..., n$, so we have:

$$\bigcap_{j=1}^n \ker(f_j) = \ker(F) \neq \{0\}$$

Which is what we wanted to show.

c)

We can pick a non-zero $z \in \bigcap_{j=1}^n \ker(f_j)$, and then we define $y = \frac{z}{\|z\|}$, then $y \in \bigcap_{j=1}^n \ker(f_j)$, because:

$$f_j(y) = f_j\left(\frac{z}{\|z\|}\right) = \frac{f_j(z)}{\|z\|} = 0$$

And since z is non-zero y is non-zero as well, and we have $\|y\| = \|\frac{z}{\|z\|}\| = \frac{\|z\|}{\|z\|} = 1$, and we have:

$$\|y - x_j\| = \|f_j\|\|y - x_j\| \geq \|f_j(y - x_j)\| = |f_j(y - x_j)| = |f_j(y) - f_j(x_j)| = |0 - f_j(x_j)| = |-\|x_j\|| = \|x_j\|$$

We have the first equality from theorem 2.7 (b) from the lecture notes, since $f_j \in X^*$ for $j = 1, ..., n$ we have that $\|f_j\| = 1$, and the seccond last equality comes from theorem 2.7 (b) in the notes as well, since $0 \neq x_j \in X$, we then have that $f_j(x_j) = \|x_j\|$. So now we've shown that there exists $y \in \bigcap_{j=1}^n \ker(f_j) \subset X$ such that $\|y\| = 1$, and $\|y - x_j\| \geq \|x_j\|$, which is what we wanted to show.

d)

We start by denoting the finite family of closed balls not containing 0 by $\{B_i\}_{i=1,...,n}$. To show that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls such that none of the balls contains 0, we show $S \not\subset \bigcup_{i=1}^n B_i$. So we have to show that $\exists x \in S$ such that $x \notin \bigcup_{i=1}^n B_i$.

We do this by starting with showing that B_i is convex. If we take $x, y \in B_i$ then we have:

$$\|\alpha x + (1-\alpha)y - p\| = \|\alpha x - \alpha p \ + (1-\alpha)y - p + \alpha p\| = \|\alpha(x-p) + (1-\alpha)y - p(1-\alpha)\|$$

$$= \|\alpha(x-p)+(1-\alpha)(y-p)\| \leq \|\alpha(x-p)\|+\|(1-\alpha)(y-p)\| = |\alpha|\|x-p\|+|1-\alpha|\|y-p\|$$

$$= \alpha\|x - p\| + (1 - \alpha)\|y - p\| \leq \alpha r + (1 - \alpha)r = \alpha r + r - \alpha r = r$$

So now we have shown that for $x, y \in B_i$ we have $\alpha x + (1-\alpha)y \in B_i$, so B_i is convex. so we have from a corollary to the Hahn Banach theorem that if $x \in B_i$ then Re $\lambda_i(x) \geq 1$, where λ_i is a linear functional.

We have that if we take $x \in V = \bigcap_{i=1}^n \ker(\lambda_i)$, then $\lambda_i(x) = 0$ for all $i = 1, ..., n$, but for $x \in B_i$ we have that Re $\lambda_i(x) \geq 1$, so none of the $x \in V$ is in any of the B_i, hence $V \cap B_i = \emptyset$. So now if you take $x \in V \cap S \subset S$, then $x \notin B_i$, because $V \cap S \cap B_i = V \cap (B_i \cap S) = V \cap \emptyset$. So we have shown that $\exists x \in S \Rightarrow x \notin B_i$ for all $i = 1, ..., n$ hence $S \not\subset B_i$ for all $i = 1, ..., n$, hence $S \not\subset \bigcup_{i=1}^n B_i$, so one cannot cover the unit sphere S with a finite family of closed balls such that none of the balls contains 0.

e)

We show that S is non-compact by contradiction. We assume that S is compact, and then we have that every open cover of S has a finite subcover. So if we for any $x \in S$ consider:

$$B_x = \{v \in X | \|x - v\| < \frac{1}{2}\}$$

then we have that if we take $x \in S$, we see that $\|x - x\| = 0 < \frac{1}{2}$, so this $x \in B_x \subset \bigcup_{x \in S} B_x$, hence we have $S \subset \bigcup_{x \in S} B_x$, hence $\{B_x\}_{x \in S}$ is an open cover of S. So we have that $\{B_x\}_{x \in S}$ has to contain a finite subcover $\{B_{x_i}\}_{x_i \in S}$ of S for $i = 1, ..., n$.

Since we have that $\{B_{x_i}\}_{x_i \in S}$ is a finite subcover of S for $i = 1, ..., n$, we have that $S \subset \bigcup_{x_i \in S} B_{x_i}$ for $i = 1, ..., n$. Since we have that $B_{x_i} \subset \overline{B_{x_i}}$ because the closure of B_{x_i} is the smallest set containing B_{x_i}. So we'll have:

$$S \subset \bigcup_{x_i \in S} B_{x_i} \subset \bigcup_{x_i \in S} \overline{B_{x_i}}$$

So we'll have that $\{\overline{B_{x_i}}\}_{x_i \in S}$ is a familiy of closed balls (the closure of an open ball with radius $\frac{1}{2}$ is a closed ball with radius $\frac{1}{2}$) which covers S such that none of them contains 0. The reason why none of these balls contains 0, is because when $x \in S$ we have that $\|x\| = 1$, so $\|x - 0\| = \|x\| = 1 \geq \frac{1}{2}$. This contradicts with problem 3d, so we have that S may be non-compact.

We can from this deduce that the closed unit ball in X is non-compact. We denote the closed unit ball by:

$$B = \{x \in X | \ \|x\| \leq 1\}$$

We have that $S \subset B$. And we have that a closed subset of a compact space is compact, but since S is non-compact, then B is non-compact.

Problem 4

a)

To be able to talk about whether E_n is absorbing or not we first have to show that E_n is convex. To show that E_n is convex we start by taking $f, g \in E_n$, since we have that $f, g \in E_n$ then we have that $\int_{[0,1]} |f|^3 dm \leq n$ and $\int_{[0,1]} |g|^3 dm \leq n$ for $n \geq 1$, so we have that f and g are measurable, and we have $\|f\|_3 < \infty$, and $\|g\|_3 < \infty$, so we have that $f, g \in L_3([0,1], m)$, hence we have from minkowski's inequality:

$$\left(\int_{[0,1]} |\alpha f + (1-\alpha) g|^3 dm\right)^{\frac{1}{3}} \leq \left(\int_{[0,1]} |\alpha f|^3 dm\right)^{\frac{1}{3}} + \left(\int_{[0,1]} |(1-\alpha) g|^3 dm\right)^{\frac{1}{3}}$$
$$= \alpha \left(\int_{[0,1]} |f|^3 dm\right)^{\frac{1}{3}} + (1-\alpha) \left(\int_{[0,1]} |g|^3 dm\right)^{\frac{1}{3}} \leq \alpha n^{\frac{1}{3}} + (1-\alpha) n^{\frac{1}{3}} = n^{\frac{1}{3}}$$

for all $0 \leq \alpha \leq 1$ So we have:

$$\int_{[0,1]} |\alpha f + (1-\alpha)g|^3 dm \leq n$$

Furthermore we have $alphaf+(1-\alpha)g \in L_1([0,1],m)$. Hence we have that $alphaf+(1-\alpha)g \in E_n$, so E_n is convex. Given $n \geq 1$ the set $E_n \subset L_1([0,1],m)$ is not absorbing, because for E_n to be able to be absorbing it has hold that for all $0 \neq f \in L_1([0,1],m)$ there has to exist $t > 0$ such that $t^{-1}f \in E_n$. But this does not hold for all $f \in L_1([0,1],m)$, because if we look at:

$$f(x) = x^{\frac{-1}{3}}$$

Then we have:

$$\|f\|_1 = \int_{[0,1]} x^{\frac{1}{3}} dm = \int_0^1 x^{\frac{1}{3}} dx = \frac{3}{2} < \infty$$

Hence $f \in L_1([0,1],m)$. And we have for any $t > 0$:

$$\int_{[0,1]} |t^{-1}f|^3 dm = t^{-3} \int_0^1 \frac{1}{x} dx \approx \infty$$

so we have that $t^{-1}f \notin E_n$, hence E_n is not absorbing.

b)

To show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$ we show that $\text{Int}(E_n) = \emptyset$ for all $n \geq 1$, but we show this by contraditcion, so we assume $\text{Int}(E_n) \neq \emptyset$ for some $n \geq 1$. If we have that $\text{Int}(E_n) \neq \emptyset$, then we have that there exists $f \in \text{Int}(E_n)$, so we have the open ball:

$$B(f,\varepsilon) = \{g \in L_1([0,1],m) : \|f-g\| < \varepsilon\} \subseteq E_n$$

for some $\varepsilon > 0$. For $0 \neq g \in L_1([0,1],m)$ we let $h = f + \frac{\varepsilon}{2\|g\|_1}g$, and we have :

$$\left\| f - \left(f + \frac{\varepsilon}{2\|g\|_1} g \right) \right\|_1 = \left\| -\frac{\varepsilon}{2\|g\|_1} g \right\|_1 = \left| \frac{\varepsilon}{2\|g\|_1} \right| \|g\|_1 = \frac{\varepsilon}{2\|g\|_1} \|g\|_1 = \frac{\varepsilon}{2} < \varepsilon$$

So we have $h = f + \frac{\varepsilon}{2\|g\|_1} g \in B(f,\varepsilon) \subseteq E_n$, so we have:

$$g = (h - f)\frac{2\|g\|_1}{\varepsilon} \in L_3([0,1], m)$$

Since $h \in E_n$, and since any function in E_n is in $L_3([0,1], m)$ as well we have that $h \in L_3([0,1], m)$, and $f \in L_3([0,1], m)$. so now we've shown that $g \in L_1([0,1], m) \Rightarrow g \in L_3([0,1], m)$, so we have $L_1([0,1], m) \subseteq L_3([0,1], m)$, but we have from HW2 that $L_3([0,1], m) \subsetneq L_1([0,1], m)$,, so we have a contradiction, hence $\text{Int}(E_n) = \emptyset$

c)

To show that E_n is closed we start by taking a sequence $(f_k)_{k \in \mathbb{N}} \subset E_n$ for which it holds $\|f_n - f\|_1 \to 0$. We have from the Bolzano-Weierstrass proppery that there is a subsequence $(f_{n_k})_{n_k \in \mathbb{N}}$ which converges pointwise in E_n, so we have:

$$\int_{[0,1]} |f|^3 dm \leq \liminf_{n_k \to \infty} |f_{n_k}|^3 dm \leq \liminf_{n_k \to \infty} n = n$$

Where we have that the first inequality comes from Fatou's lemma, and we have the seccond inequality because $f_{n_k} \in E_n$, so we have that $f \in E_n$, so we have that E_n is closed in $L_1([0,1], m)$

d)

We have from definition 3.12(ii) from the lecture notes that to show that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$ then we have to show that there exists a sequence $(E_n)_{n \geq 1}$ where E_n for $n \geq 1$ are nowhere dense sets, and:

$$L_3([0,1], m) = \bigcup_{n=1}^{\infty} E_n$$

From 4b we have that $\text{Int}(E_n) = \emptyset$, for all $n \geq 1$ and from 4c we have the E_n is closed for all $n \geq$, so $\overline{E_n} = E_n$ hence $\text{Int}(\overline{E_n}) = \text{Int}(E_n) = \emptyset$, so we have that E_n for $n \geq 1$ are nowhere dense sets. We have:

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm \leq n \right\} = \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm < \infty \right\}$$

$$= \{ f \in L_1([0,1], m) : f \in L_3([0,1], m) \} = L_3([0,1], m)$$

Where we have the last equality because $L_3([0,1], m) \subsetneq L_1([0,1], m)$. Now we have shown that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$

Problem 5

a)

From the reverse triangular inequality we have $\big|\|x\| - \|x_n\|\big| \leq \|x - x_n\|$ Since we have that $x_n \to x$ in norm as $n \to \infty$, then we have for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ for which it holds:

$$\big|\|x\| - \|x_n\|\big| \leq \|x - x_n\| < \varepsilon$$

So we have that $\|x_n\| \to \|x\|$ for $n \to \|x\|$.

b)

We define $H = \ell_2(\mathbb{N})$. Since H is seperable we can consider $(e_n)_{n\geq 1}$, so we let $x_n = e_n$. We have that $(e_n)_{n\geq 1}$ is a normal orthonormal basis for H.
We let $x \in H$, and then from Bessel's inequality we'll have:

$$\sum_{n\in\mathbb{N}} |\langle e_n, x\rangle|^2 \leq \|x\|^2 < \infty$$

Since $\sum_{n\in\mathbb{N}} |\langle e_n, x\rangle|^2 < \infty$, then we have that $\sum_{n\in\mathbb{N}} |\langle e_n, x\rangle|^2$ converges, so we have that the corresponding sequence $|\langle e_n, x\rangle|^2$ converges to $0 = \langle 0, x\rangle$, hence $\langle e_n, x\rangle \to \langle 0, x\rangle$. We have that a Hilbert space is a Banch space as well, then we have that H is a Banach space, since it is a Hilbert space. Furthermore we have that $(e_n)_{n\leq 1}$ is a sequence, then it is a net, since every sequence is a net, so we have from HW 4 problem 2a that since $\langle e_n, x\rangle \to \langle 0, x\rangle$, then we have $e_n \to 0$:

Furtermore we have $\|e_n\| = 1$, since $(e_n)_{n\geq 1}$ is an orthonormal basis. So we have $\|e_n\| = 1 \to 1 \neq 0 = \|0\|$, so if we suppose that $x_n \to x$ weakly, then it doesn't hold that $\|x_n\| \to \|x\|$.

c)

We assume that $\|x_n\| \leq 1$ for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Since we have that $x_n \to x$ weakly, then we have:

$$\|x\| = \langle x, x\rangle = \lim_{n\to\infty} \langle x, x_n\rangle$$

And furthermore we have $\langle x, x_n\rangle \leq \|x_n\|$, so we have:

$$\|x\| = \langle x, x\rangle = \lim_{n\to\infty} \langle x, x_n\rangle \le \liminf_{n\to\infty} \|x_n\|$$

And now since we have that $\|x_n\| \le 1$ for all $n \ge 1$, then we have $\|x\| \le 1$.