FunkAn Mandatory Assignment 2

Rune Thinggaard Hansen
kjb933

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Problem 1

a)

By HW4 we have weak convergence to 0 if and only $\phi(f_N) \to \phi(0) = 0$ for any functional ϕ. By Riez Representation theorem (HW1), we have that any functional can be represented as an inner product with a fixed vector $y \in H$, i.e $\phi = \langle \cdot, y \rangle$. Denote by y_n the n'th coordinate of y w.r.t the basis $(e_n)_{n\in\mathbb{N}}$. Since $\|y\|^2 = \sum_{i=1}^{\infty} |y_n|^2 < \infty$ and since the harmonic series diverge, we know that for chosen ε there exists some $K \in \mathbb{N}$ so that for all $n > K$ we have that $|y_n|^2 < \frac{1}{n} \Rightarrow |y_n| < \frac{1}{\sqrt{n}}$. Therefore we have:

This holds for $\frac{1}{N} \leq \frac{1}{n}$, i.e. for $N \geq n$. But you state the inequality for $n \geq K+1$, and there is no relation between N and K.

$$|\phi_y(f_N)| = |\langle f_N, y \rangle| \leq \frac{1}{N} \sum_{n=1}^{N^2} |\langle e_n, y \rangle| = \frac{1}{N} \sum_{n=1}^{N^2} |y_n|$$

$$< \frac{1}{N} \sum_{n=1}^{K} |y_n| + \frac{1}{N} \sum_{n=K+1}^{N^2} \frac{1}{\sqrt{n}} \leq \frac{1}{N} \sum_{n=1}^{K} |y_n| + \sum_{n=K+1}^{N^2} \frac{1}{n\sqrt{n}}$$

$\leq$

We know that $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges so as K increases its tail is upper bounded by arbitrary ε. Therefore, we can choose K to be large enough such that the second term is smaller than epsilon. Thus we get:

$$|\phi_y(f_N)| < \frac{1}{N} \sum_{n=1}^{K} |y_n| + \varepsilon$$

Which goes to ε as $N \to \infty$. Since ε and ϕ_y were arbitrary, we have that $f_N \to 0$ weakly. (✓)
Now we show that $\|f_N\|^2 = 1$:

$$\|f_N\|^2 = \frac{1}{N^2} \sum_{n=1}^{N^2} \sum_{m=1}^{N^2} \langle e_n, e_m \rangle = \frac{1}{N^2} \sum_{n=1}^{N^2} \langle e_n, e_n \rangle = \frac{1}{N^2} N^2 = 1.$$ ✓

And thus $\|f_N\| = 1$ for all N.

b)

We have that K is included in the set A consisting of functions of the following form:

$$f = \sum_{i=1}^{\infty} \beta_i f_{N_i}, \quad \sum_{i=1}^{\infty} \beta_i \leq 1, \quad \forall i \in \mathbb{N} : \beta_i \geq 0$$

Why?

If $\{0\} \in K$ then $A = K$ by a similar argument as in HW5. We see that:

Then write it down!

$$\|f\| \leq \sum_{i=1}^{\infty} |\beta_i| \|f_{N_i}\| = 1$$

w*-topology only makes sense on dual spaces, so you need to establish/note that H is a dual space.

Thus $K \subset A \subset B_H(0,1)$. We have that $B_H(0,1)$ is weak*-compact by Banach-Alagou and the weak* and weak-topologies are the same since a Hilbert space is reflexive. Therefore $B_H(0,1)$ is weakly compact. We furthermore have by theorem 5.7 of the notes that K is weakly closed since it's closed in norm and the norm closure and weak-closure are identical for convex sets (which a convex hull always is). Therefore K is a weakly closed set in a weakly compact set making it weakly compact.

We had from the previous task that $f_N \to 0$ weakly and therefore 0 is in the weak closure. Since K is weakly closed $0 \in K$.

c)

Choose an $N \in \mathbb{N}$ and assume $f_N = \alpha g_1 + (1-\alpha) g_2$ with $g_1, g_2 \in K$ and $0 < \alpha < 1$. Then by a similar argument as of HW5, we have that the g_i's would be of the form:

Same as before, because it is not immediately applicable here as the f_n's are not an ONB.

$$f_N = \alpha \sum_{i=1}^{\infty} \beta_i f_{N_i} + (1-\alpha) \sum_{j=1}^{\infty} \gamma_j f_{M_j}$$

That is:

$$\frac{1}{N} \sum_{n=1}^{N^2} e_n = \alpha \sum_{i=1}^{\infty} \frac{\beta_i}{N_i} \sum_{n=1}^{N_i^2} e_n + (1-\alpha) \sum_{j=1}^{\infty} \frac{\gamma_j}{M_j} \sum_{n=1}^{M_j^2} e_n$$

$$\sum_{i=1}^{\infty} \beta_i \leq 1, \sum_{j=1}^{\infty} \gamma_j \leq 1, \forall i \in \mathbb{N} : \beta_i \geq 0, \forall j \in \mathbb{N} : \gamma_j \geq 0$$

By taking the inner product on both sides with e_{N^2} and e_{N^2+1} respectively, we get:

$$\frac{1}{N} = \alpha \sum_{\{i | N_i^2 \geq N^2\}} \frac{\beta_i}{N_i} + (1-\alpha) \sum_{\{j | M_j^2 \geq N^2\}} \frac{\gamma_j}{M_j}$$

$$0 = \alpha \sum_{\{i | N_i^2 \geq N^2+1\}} \frac{\beta_i}{N_i} + (1-\alpha) \sum_{\{j | M_j^2 \geq N^2+1\}} \frac{\gamma_j}{M_j}$$

And by subtracting the two, we get:

$$\frac{1}{N} = \alpha \sum_{\{i|N_i^2=N^2\}} \frac{\beta_i}{N} + (1-\alpha) \sum_{\{j|M_j^2=N^2\}} \frac{\gamma_j}{N} \leq \frac{1}{N}\left(\alpha \sum_{i=1}^{\infty} \beta_i + (1-\alpha)\sum_{j=1}^{\infty}\gamma_j\right) \leq \frac{1}{N}$$

Where the two last inequalities holds as equalities if and only if $\sum_{\{i|N_i^2=N^2\}} \beta_i = \sum_{\{j|M_j^2=N^2\}} \gamma_j = 1$ which implies that each non-zero β_i and γ_j correspond to $N_i = M_j = N$. Thus we have:

$$f_N = \alpha \sum_{i=1}^{\infty} \beta_i f_{N_i} + (1-\alpha)\sum_{j=1}^{\infty} \gamma_j f_{M_j} = \alpha f_N + (1-\alpha) f_N$$

(✓)

Which exactly makes f_N an extreme point.

Assume that $0 \in K$ can be written as a linear combination of the f_N's:

$$0 = \alpha \sum_{i=1}^{\infty} \frac{\beta_i}{N_i} \sum_{n=1}^{N_i^2} e_n + (1-\alpha)\sum_{j=1}^{\infty} \frac{\gamma_j}{M_j}\sum_{n=1}^{M_j^2} e_n$$

By taking the inner product with e_k and e_{k+1} for any $k \in \mathbb{N}$ and subtracting these from each other, we respectively get:

$$0 = \alpha \sum_{\{i|N_i^2 \geq k\}} \frac{\beta_i}{N_i} + (1-\alpha) \sum_{\{j|M_j^2 \geq k\}} \frac{\gamma_j}{M_j}$$

$$0 = \alpha \sum_{\{i|N_i^2 \geq k+1\}} \frac{\beta_i}{N_i} + (1-\alpha) \sum_{\{j|M_j^2 \geq k+1\}} \frac{\gamma_j}{M_j}$$

$$0 = \alpha \sum_{\{i|N_i^2 = k\}} \frac{\beta_i}{N_i} + (1-\alpha) \sum_{\{j|M_j^2 = k\}} \frac{\gamma_j}{M_j}$$

Same issue as before.

Since this was for arbitrary $k \in \mathbb{N}$ and all terms are positive, this implies that each β_i and γ_j is 0. This means $g_1 = g_2 = 0$ making it an extreme point.

(✓)

d)

There are no more extreme points.
Proof: K is convex and by task b it is weakly compact. Therefore we have by Krein-Milman:

$$K = \overline{\mathrm{co}(\mathrm{Ext}(K))}^{\tau_w}$$

Let's assume that $\mathrm{Ext}(K) = \{f_N\}_{N\in\mathbb{N}} \cup \{0\} \cup A$ for some non-empty set A with $\{f_N\}_{N\in\mathbb{N}} \cap A = \emptyset$. We remember by theorem 5.7 of the notes that the weak closure coincides with the norm closure. Krein-Milman states:

$$\overline{\mathrm{co}(\{f_N\}_{N\in\mathbb{N}})}^{\tau_w} = \overline{\mathrm{co}(\{f_N\}_{N\in\mathbb{N}} \cup \{0\} \cup A)}^{\tau_w}$$

Choose non-zero $f_a \in A$. The above equation implies that there exists $\beta_i \geq 0$ with $\sum_{i=1}^{\infty} \beta_i \leq 1$ and $\beta_1 \neq 0$ such that:

Same issue.

$$f_a = \sum_{i=1}^{\infty} \beta_i f_{N_i} = \beta_1 f_{N_1} + (1-\beta_1) \sum_{i=2}^{\infty} \frac{\beta_i}{1-\beta_1} f_{N_i}$$

If originally $\beta_1 = 0$ we can permutate the sum so that the first term isn't 0. We can be sure that $\beta_1 \neq 1$ since otherwise $f_a = f_{N_1}$ which it isn't by assumption. But since f_a was an Extreme Point we can conclude from the above equation that:

$$f_a = f_{N_1} = \sum_{i=1} \frac{\beta_i}{1-\beta_1} f_{N_i}$$

(✓)

Which implies $f_a \in \{f_N\}_{N \in \mathbb{N}}$ which is a contradiction.

Problem 2

a)

We will show that $f(Tx_n) \to f(Tx)$ for all $f \in Y^*$ which by HW4 will imply weak convergence. Choose $f \in Y^*$. We have that:

$$f(Tx_n) = (f \circ T)(x_n) = g(x_n)$$

Where $g = f \circ T$. We have that $\|g\| \leq \|f\|\|T\| < \infty$, so $g \in X^*$. By weak convergence of x_n we have that $g(x_n) \to g(x)$ so $f(Tx_n) \to f(Tx)$. Thus $Tx_n \to Tx$ weakly.

✓

b)

By HW4, we have that $R \equiv \sup_{n \in \mathbb{N}}(\|x_n\|) < \infty$ since it converges weakly. This implies that $T((x_n)_{n \in \mathbb{N}}) \subset T(B_X(0,R)) = R \cdot T(B_X(0,1)) \subset R\overline{T(B_X(0,1))}$ which is compact in norm by the definition of compact operators.
Compactness in norm means that any sequence will have a norm convergent subsequence, so there exists $(Tx_{n_k})_{k \in \mathbb{N}}$ which is convergent in norm towards some limit we call y.
By the assumption $x_n \to x$ weakly together with the previous task we have that $Tx_n \to Tx$ weakly. We know norm convergence implies weak convergence so $Tx_{n_k} \to y$ weakly. Since a subsequence of a convergent sequence always converges to the same limit as the sequence it came from, this implies $Tx_n \to y$ weakly. Since $Tx_n \to Tx$ and $Tx_n \to y$ weakly, we must have that $y = Tx$ by the uniqueness of limits.
So Tx is an accumulation point for Tx_n. Assume there was another accumulation point z for which some subsequence $(x_{n_l})_{l \in \mathbb{N}}$ has the property that $(Tx_{n_l})_{l \in \mathbb{N}} \to z$ in norm. Then we'd have that $Tx_{n_l} \to z$ weakly which by the same argument as above implies $Tx = z$. Thus, Tx is the only accumulation point.

Assume that Tx isn't the limit of $(Tx_n)_{n\in\mathbb{N}}$ but is just an accumulation point. Then there should exists some subsequence $(Tx_{n_k})_{k\in\mathbb{N}}$ which doesn't have Tx as an accumulation point. Since they again are in a compact set, there exists some sub-sub-sequence $(Tx_{n_{k_l}})_{l\in\mathbb{N}}$ with a limit $w \neq Tx$. But this would make w an accumulation point of $(Tx_n)_{n\in\mathbb{N}}$ which is a contradiction to Tx being the only accumulation point.
Therefore we must have that $Tx_n \to Tx$ in norm, or that $\|Tx_n - Tx\| \to 0$

c)

Let's follow the hint! Assume that T isn't compact. Then $T(B_H(0,1))$ isn't totally bounded and thus there exists a δ such that $T(B_H(0,1))$ isn't contained in any finite amount of balls of radius δ.
This means we are able to create a sequence $(x_n)_{n\in\mathbb{N}}$ in the unit ball such that $\|Tx_n - Tx_m\| \geq \delta$ for $n \neq m$. If this wasn't the case, there would only be finitely many points $(x_n)_{n\in I}$ with $I = \{1, \ldots, M\}$ such that $\|Tx_n - Tx_m\| \geq \delta$. Then we could cover $T(B_H(0,1))$ with the finitely many balls $\{B_Y(Tx_n, \delta)\}_{n\in I}$ which would be a contradiction.

Note that this is an antilinear isomorphism.

We have that $H^* \cong H$ since it's a Hilbert space. By theorem 5.13 the topological space $(\overline{B_{H^*}(0,1)}, \tau_{w*}) \cong (\overline{B_H 0, 1}, \tau_w)$ is metrizable by some metric $\|\cdot\|_{\tau^w}$. So the above mentioned sequence $(x_n)_{n\in\mathbb{N}}$ in the unit ball is also a sequence within a bounded set of a metric space and therefore it has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which is convergent with respect to $\|\cdot\|_{\tau^w}$. Since $\|\cdot\|_{\tau^w}$ induces the weak topology, this means that $(x_{n_k})_{k\in\mathbb{N}}$ converges weakly and thus $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence.
By the contrapositive of this statement, we have that if every weakly convergent series $(x_n)_{n\in\mathbb{N}}$ has the property that $\|Tx_n - Tx_m\| \to 0$ (which by the completeness of Y is equivalent to the existence of x for which $\|Tx_n - Tx\| \to 0$) then T is compact.

How do you know that the limit of $(Tx_n)_{n\in\mathbb{N}}$ is in $T(X)$?

d)

Remark 5.3 states that a sequence converges weakly in $\ell_1(\mathbb{N})$ if and only if it converges in norm. For an operator $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ we therefore have by task (a) that if a sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly in $\ell_2(\mathbb{N})$ then $(Tx_n)_{n\in\mathbb{N}}$ converges weakly in $\ell_1(\mathbb{N})$ which implies that $(Tx_n)_{n\in\mathbb{N}}$ converges in norm which again implies by c) that $T \in \mathcal{K}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ since $\ell_2(\mathbb{N})$ is a Hilbert Space.

e)

We have by theorem 3.15 of the notes that if T surjective then T is also open. This means that $T(B_X(0,1))$ contains some open ball $B_Y(0,\varepsilon)$ for some ε. This also means that $B_Y(0,1) \subset T(B_X(0, \frac{1}{\varepsilon})) = \frac{1}{\varepsilon}T(B_X(0,1)$ which again means that $\overline{B_Y(0,1)} \subset \frac{1}{\varepsilon}\overline{T(B_X(0,1))}$. By compactness of T we know that $\frac{1}{\varepsilon}\overline{T(B_X(0,1)}$ is compact, which implies that $\overline{B_Y(0,1)}$ is a closed subset of a compact set making it compact. This is known not to be true (the unit ball of an infinite dimensional Banach Space is not compact by the first mandatory assignment) which means T can't be onto. ✓

f)

What are f, g?

Since $\bar{t} = t$, we have that:

$$\langle M^* g, f \rangle = \langle g, Mf \rangle = \int_{[0,1]} g\overline{tf}dm(t) = \int_{[0,1]} tg\overline{f}dm(t) = \langle Mg, f \rangle$$

So M is self-adjoint.
By HW6 we have that M has no eigenvalues. But if M was self-adjoint and compact, there would be an ONB for H consisting of eigenvectors to M by the Spectral Theorem for compact operators. The set of eigenvectors is empty though, and $L_2([0,1], m)$ is not empty. Thus M isn't compact. ✓

Since $L^2([0,1], m)$ separable and infinite dimensional.

Problem 3

a)

By monotonicity of the Lesbegue Integral on non-negative functions, we have:

why is k measurable?

$$\int_{[0,1]\times[0,1]} |K| dm^2 \leq \int_{[0,1]\times[0,1]} 1 dm^2 = 1 < \infty$$

Thus $K \in L_2([0,1] \times [0,1], m^2)$. So T is Hilbert Schmidt by proposition 9.12 of the notes, so T is compact by proposition 9.11 of the notes.

9.12 requires σ-finite. justify that.

← you need to identify $T = T_{\tilde{k}}$, $\tilde{k}(s,t) = k(t,s)$

b)

$K = \overline{K}$ since it's real, so we have:

$$\langle Tf, g \rangle = \int_{[0,1]} (Tf)(s) \cdot \overline{g(s)} dm(s) = \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) dm(t) \cdot \overline{g(s)} dm(s)$$
$$= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \cdot \overline{g(s)} dm(t) dm(s) = \int_{[0,1]} f(t) \int_{[0,1]} \overline{K(s,t) g(s)} dm(s) dm(t)$$
$$= \int_{[0,1]} f(t) \overline{\int_{[0,1]} K(s,t) g(s) dm(s)} dm(t) = \langle f, Tg \rangle$$

requires $k(s,t) = k(t,s)$

Equality 4 holds due to Fubini since $[0,1]$ is σ-finite (it's simply finite) and $K \in C([0,1])$, $f, g \in L_2([0,1], m)$ making the product integrable. So $T = T^*$!

elaborate this. How does it imply $k(s,t)f(t)g(s) \in L_1([0,1]^2)$?

c)

We have that:

$$
\begin{aligned}
(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t) &= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{(s,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)t \cdot f(t)dm(t) + \int_{[s,1]} (1-t)s \cdot f(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)
\end{aligned}
$$

Where we have used that the singleton $\{s\}$ is a $m(t)$-null set.
Now to show continuity. Choose a sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \to s$. Denote $a_n = \min(s_n, s)$ and $b_n = \max(s_n, s)$. Notice that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$. We see that $1_{[a_n,b_n]}tf(t)$ is dominated by $f(t)$ which is in $L_1([0,1],m)$ (Since $L_2([0,1],m) \subset L_1([0,1],m)$ by HW2). So by Lesbegue Dominated Convergence and the fact that singletons are Lesbegue-null sets, we have:

$$
\lim_{n\to\infty} \left| \int_{[0,s_n]} tf(t)dm(t) - \int_{[0,s]} tf(t)dm(t) \right| \leq
$$

$$
\lim_{n\to\infty} \int_{[0,1]} 1_{[a_n,b_n]} |tf(t)| dm(t) = \int_{[0,1]} \lim_{n\to\infty} 1_{[a_n,b_n]} |tf(t)| dm(t) = 0.
$$

So the first integral is sequentially continuous which is the same as continous on first-countable spaces (which [0,1] is).
For the second integral, we do the same. We see that the $1_{[a_n,b_n]}(1-t)f(t)$ again is dominated by $f(t)$ which is integrable, so by Lesbegue Dominated Convergence:

$$
\lim_{n\to\infty} \left| \int_{[s_n,1]} (1-t)f(t)dm(t) - \int_{[s,1]} (1-t)f(t)dm(t) \right| \leq
$$

$$
\lim_{n\to\infty} \int_{[0,1]} 1_{[a_n,b_n]} |(1-t)f(t)| dm(t) = \int_{[0,1]} \lim_{n\to\infty} 1_{[a_n,b_n]} |(1-t)f(t)| dm(t) = 0
$$

So both integrals are continuous. Therefore we have that $(Tf)(s)$ is the product and sum of continuous functions ($1-s$, the first integral, s and the second integral) and is therefore continuous. Furthermore, since singeltons are Lesbegue-null-sets, we have that:

$$
\begin{aligned}
(Tf)(0) &= (1-0)\int_{\{0\}} tf(t)dm(t) + 0 \cdot \int_{[0,1]} (1-t)f(t)dm(t) = 0 \\
(Tf)(1) &= (1-1)\int_{[0,1]} tf(t)dm(t) + 1 \cdot \int_{\{1\}} (1-t)f(t)dm(t) = 0
\end{aligned}
$$

And thus everything is shown.

Problem 4

a)

We have that a polynomial times a gaussian function differentiated produces another polynomial times a gaussian function:

$$\frac{\partial}{\partial x}\left(p(x)e^{-x^2/2}\right) = \left(\frac{\partial}{\partial x}p(x)\right)e^{-x^2/2} - x \cdot p(x)e^{-x^2/2} = q(x)e^{-x^2/2}$$

Here $q(x) = \frac{\partial}{\partial x}p(x) - xp(x)$ is another polynomial since the derivative of a polynomial is another polynomial and the set of polynomials with real coefficients produce a ring. Thus, by doing this recursively, we get that for suitable polynomials p and q that:

$$x^\beta \frac{\partial^\alpha}{\partial x^\alpha} g_k(x) = x^\beta \cdot p(x)e^{-x^2/2} = q(x)e^{-x^2/2}$$

And by recursive application of l'Hoptial's rule (as many as the degree of q), we get that this function goes to 0 for $x \to \pm\infty$. Thus, we also have that $g_k \in \mathcal{S}(\mathbb{R})$ for $k \in \{0, 1, 2, 3\}$.

✓

We calculate their Fourier Transforms soon but first, some arguments:
Lesbegue integrals can be computed as a Riemann integrals: We have that each $e^{-i\xi x}g_k(x)$ is Riemann-integrable on each interval $[-N, N]$ since they are bounded and continuous. An application of Corollary 12.11 in Measures, Integrals and Martingales by Schilling gives us that their Lesbegue integral can be computed as a Riemann integral.

How do you ensure that the equality still hold for $N \to \infty$? (You need DCT for that)

An integral with end points of the form $\pm\infty + i\xi$ can be replaced with $\pm\infty$ for end points. If we integrate the function $e^{-i\xi x}g_k(x)$ over a large rectangle with one side in $\mathbb{R}$ and another side in $\mathbb{R} + i\xi$ and the two last sides connecting these two then we get zero by the Cauchy's residue theorem (due to the fact that the integrand is holomorphic and has no poles). By making the rectangle broad enough, the integral boundaries goes to infinity and since $|g_k|$ goes to 0, we have that the sides of the rectangle will contribute with a factor that becomes arbitrarily small. Thus we get that the two line integrals on $\mathbb{R}$ and $\mathbb{R} + i\xi$ cancel.

✓

Now we actually find the Fourier Transform of our four functions.
We have that the first one is well known by proposition 11.4 in the notes.

$$\mathcal{F}(g_0)(\xi) = e^{-\xi^2/2}$$

So $\mathcal{F}(g_0) = g_0$.
For $\mathcal{F}(g_1)$ we "complete the square", use the substitution $u = x + i\xi$ and remember that the integral of an odd function from minus infinity to infinity

dies:

DCT

$$
\begin{aligned}
\mathcal{F}(g_1)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} x e^{-x^2/2} dm(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2 - i\xi x} dx = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-(x/\sqrt{2}+i\xi/\sqrt{2})^2} dx \\
&= \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \int_{-\infty+i\xi}^{\infty+i\xi} (u - i\xi) e^{-u^2/2} du = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-u^2/2} du - \frac{i\xi e^{-\xi^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du \\
&= -i\xi e^{-\xi^2/2}
\end{aligned}
$$

✓

So $\mathcal{F}(g_1) = -ig_1$.

For the next one, we complete the square, use the same substitution as before and then we split it into mulitple integrals. We use integration by parts on the first term with the two functions u and $u \cdot e^{-u^2/2}$ where the second has an antiderivate that is $\frac{\partial}{\partial u}(-e^{-u^2/2}) = ue^{-u^2/2}$. We furthermore use that the last term is the integration over an odd function from minus infinity to infinity, so it dies:

$$
\begin{aligned}
\mathcal{F}(g_2)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} x^2 e^{-x^2/2} dm(x) = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \int_{-\infty+i\xi}^{\infty+i\xi} (u - i\xi)^2 e^{-u^2/2} du \\
&= \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \left(\int_{-\infty+i\xi}^{\infty+i\xi} u^2 e^{-u^2/2} du - \xi^2 \int_{-\infty+i\xi}^{\infty+i\xi} e^{-u^2/2} du - 2i\xi \int_{-\infty+i\xi}^{\infty+i\xi} u e^{-u^2/2} \right) \\
&= \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \left(-u e^{-u^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-u^2/2} du - \xi^2 \int_{-\infty}^{\infty} e^{-u^2/2} du - 0 \right) \\
&= e^{-\xi^2/2} \left(1 - \xi^2\right)
\end{aligned}
$$

✓

So $\mathcal{F}(g_2) = g_0 - g_2$.
Method for the next one is the same as before: Complete the square, make the same substitution, expand the cube, acknowledge that the odd terms die and use integration by parts on the last term:

$$
\begin{aligned}
\mathcal{F}(g_3)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} x^3 e^{-x^2/2} dm(x) = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \int_{-\infty+i\xi}^{\infty+i\xi} (u - i\xi)^3 e^{-u^2/2} du \\
&= \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} u^3 e^{-u^2/2} du + i\xi^3 \int_{-\infty}^{\infty} e^{-u^2/2} du - 3\xi^2 \int_{-\infty}^{\infty} u e^{-u^2/2} du - 3i\xi \int_{-\infty}^{\infty} u^2 e^{-u^2/2} du \right) \\
&= \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \left(0 + i\xi^3 \int_{-\infty}^{\infty} e^{-u^2/2} du - 0 + 3i\xi u e^{-u^2/2}\Big|_{-\infty}^{\infty} - 3i\xi \int_{-\infty}^{\infty} e^{-u^2/2} du \right) \\
&= i e^{-\xi^2/2} (\xi^3 - 3\xi)
\end{aligned}
$$

✓

So $\mathcal{F}(g_3) = ig_3 - 3ig_1$. We're done!

b)

We can write up the matrix representation A of $\mathcal{F}$ w.r.t the basis defined by $\{g_0, g_1, g_2, g_3\}$ (notice that these are linearly independent, so they actually constitute a four-dimensional space):

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -i & 0 & -3i \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

And then one can find the eigenfunctions corresponding to the four eigenvalues $\{1, i, -1, -i\}$ to get the folllowing:

$$\begin{aligned} &\mathcal{F}(g_0) = g_0. \\ &\mathcal{F}(g_3 - \frac{3}{2}g_1) = ig_3 - 3ig_1 + i\frac{3}{2}g_1 = i(g_3 - \frac{3}{2}g_1) \\ &\mathcal{F}(g_2 - \frac{1}{2}g_0) = \frac{1}{2}g_0 - g_2 = -(g_2 - \frac{1}{2}g_0). \\ &\mathcal{F}(g_1) = -ig_1 \end{aligned}$$

Which gives us $h_0 = g_0$, $h_1 = g_3 - \frac{3}{2}g_1$, $h_2 = g_2 - \frac{1}{2}g_0$ and $h_3 = g_1$ works. Crazy. it really is!

c)

We take the Fourier Transform twice and evaluate at $-\xi$:

Maybe argue →

$$\mathcal{F}(\mathcal{F}(f))(-\xi) = \int_{\mathbb{R}} e^{i\xi\omega} \int_{\mathbb{R}} e^{-i\omega x} f(x) dm(x) dm(\omega) = \mathcal{F}^*(\mathcal{F}(f)(\xi) = f(\xi)$$

that $\mathcal{F}f \in \mathcal{S}(\mathbb{R})$

Where the last equality comes from Corollary 12.12(iii). So we have that $\mathcal{F}(\mathcal{F}(f))(\xi) = f(-\xi)$. Therefore we have:

$$\mathcal{F}^4(f)(\xi) = \mathcal{F}^2(\mathcal{F}^2(f))(\xi) = f(-(-\xi)) = f(\xi)$$

There we have it. ✓

d)

From c) and linearity of the Fourier Transform we have for non-zero f with $\mathcal{F}(f) = \lambda f$ that $f = \mathcal{F}^4(f) = \lambda^4 f$ which implies that $\lambda^4 = 1$. There are 4 solutions for this equation by the fundamental theory of algebra and they are $1, -1, i$ and $-i$. Thus $\lambda \in \{1, i, -1, -i\}$ and we have furthermore found that these four values are eigenvalues in task b), so they are precisely the eigenvalues. ✓

Problem 5

Choose $x \in [0, 1]$. HW8 says that $x \in \text{supp}(\mu)$ if and only if the integral of any continuous function $f : [0, 1] \to [0, 1]$ with $f(x) > 0$ will be positive.

↑ and compact support.

Choose such a continuous f with $f(x) \neq 0$. Then by continuity, there exits a δ' such that $f(x') > \frac{f(x)}{2}$ for $x' \in (x-\delta', x+\delta')$ and thus also for $x' \in [x-\delta, x+\delta]$ for $\delta \equiv \frac{\delta'}{2}$. Therefore we also have $\min_{x' \in [x-\delta, x+\delta]}(f(x')) > \frac{f(x)}{2}$ (the minimum exists since it's the minimum of a contionous function over a compact set in $\mathbb{R}$). We have that $\{x_n\}_{n\in\mathbb{N}}$ is dense in $[0,1]$ so there exists a sequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges towards x, so there exists $l \in \mathbb{N}$ with $x_{n_l} \in [x-\delta, x+\delta]$.
By monotonicity of integrals of positive functions and positivity of measures, we get:

$$\begin{aligned}\int_{[0,1]} f d\mu = \int_{[0,1]} f \sum_{n\in\mathbb{N}} 2^{-n} d\delta_{x_n} \geq \int_{[x-\delta,x+\delta]} f \sum_{n\in\mathbb{N}} 2^{-n} d\delta_{x_n} \\ \geq \int_{[x-\delta,x+\delta]} \min_{x'\in[x-\delta,x+\delta]}(f(x')) \sum_{n\in\mathbb{N}} 2^{-n} d\delta_{x_n} = \min_{x'\in[x-\delta,x+\delta]}(f(x')) \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}([x-\delta, x+\delta]) \\ > \frac{f(x)}{2} \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}([x-\delta, x+\delta]) \geq \frac{f(x)}{2} 2^{-n_l} \delta_{x_{n_l}}([x-\delta, x+\delta]) = \frac{f(x)}{2} 2^{-n_l} > 0\end{aligned}$$

Since this was for any continuous $f : [0,1] \to [0,1]$ with $f(x) > 0$ we have that $x \in \text{supp}(\mu)$ and since x was chosen arbitrarily, we have that $[0,1] \subset \text{supp}(\mu)$. Since we look at a measure on $[0,1]$ we also have that $\text{supp}(\mu) \subset [0,1]$ so $\text{supp}(\mu) = [0,1]$. ✓

Thank you for an amazing course! It's truly been a pleasure and I've learned incredibly much. I hope that it isn't too tedious to get through correcting all of these tasks. And I hope that you most definitely don't click on the following smiley for your own sake :) - 1000000 points for linking to that song.