# Functional Analysis
# Mandatory Assignment 2
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## Problem 1

Let $H$ be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.

**(a)** To show that $f_N \to 0$ weakly, as $N \to \infty$, while $\|f_N\| = 1$, for all $N \geq 1$, we want to show that

$$\langle f_N, y\rangle \to \langle 0, y\rangle = 0, \qquad \forall y \in H$$

We can write $y$ as a linear combination of the basis

$$y = \sum_{i}^{\infty} \alpha_i e_i$$

and now we can calculate $\langle f_N, y\rangle$ by linearity of the first coordinate in the inner product:

$$\begin{aligned}
\langle f_N, y\rangle &= \langle \frac{1}{N}(e_1 + e_2 + ... + e_{N^2}), \alpha_1 e_1 + \alpha_2 e_2 + ...\rangle \\
&= \frac{1}{N}\Big(\langle e_1, \alpha_1 e_1 + \alpha_2 e_2 + ...\rangle + \langle e_2, \alpha_1 e_1 + \alpha_2 e_2 + ...\rangle + ... + \langle e_{N^2}, \alpha_1 e_1 + \alpha_2 e_2 + ...\rangle\Big) \\
&= \frac{1}{N}\Big(\overline{\alpha_1}\langle e_1, e_1\rangle + \overline{\alpha_1}\langle e_2, e_1\rangle + ... + \overline{\alpha_1}\langle e_{N^2}, e_1\rangle + \overline{\alpha_2}\langle e_1, e_2\rangle + ... + \overline{\alpha_2}\langle e_{N^2}, e_2\rangle + ...\Big)
\end{aligned}$$

We know that $\langle e_i, e_j\rangle = 0$ when $i \neq j$ and $\langle e_i, e_j\rangle = 1$ when $i = j$, since $(e_n)_{n\geq 1}$ is a orthonormal basis, which means:

$$\begin{aligned}
\langle f_N, y\rangle &= \frac{1}{N}\Big(\overline{\alpha_1}\langle e_1, e_1\rangle + \overline{\alpha_1}\langle e_2, e_1\rangle + ... + \overline{\alpha_1}\langle e_{N^2}, e_1\rangle + \overline{\alpha_2}\langle e_1, e_2\rangle + ... + \overline{\alpha_2}\langle e_{N^2}, e_2\rangle + ...\Big) \\
&= \frac{1}{N}\Big(\overline{\alpha_1}\langle e_1, e_1\rangle + \overline{\alpha_2}\langle e_2, e_2\rangle + ... + \overline{\alpha_{N^2}}\langle e_{N^2}, e_{N^2}\rangle\Big) = \sum_{i=1}^{N^2} \frac{\overline{\alpha_i}}{N}
\end{aligned}$$

It is clear that $\sum_{i=1}^{N^2} \frac{\overline{\alpha_i}}{N} \to 0$ as $N \to \infty$, since $\frac{\overline{\alpha_i}}{N} \to 0$ as $N \to \infty$ for all $i \in \mathbb{N}$. Therefore, it is shown that $f_N \to 0$ weakly, as $N \to \infty$.
To show $\|f_N\| = 1$ for all $N \geq 1$ we ones again use that the inner product of the bases is 1 or 0.

$$\begin{aligned}
\|f_N\| = \sqrt{\langle f_N, f_N\rangle} &= \sqrt{\langle \frac{1}{N}(e_1 + ... + e_{N^2}), \frac{1}{N}(e_1 + ... + e_{N^2})\rangle} \\
&= \sqrt{\frac{1}{N^2}\langle e_1 + ... + e_{N^2}, e_1 + ... + e_{N^2}\rangle} \\
&= \sqrt{\frac{1}{N^2}(\langle e_1, e_1\rangle + ... + \langle e_{N^2}, e_{N^2}\rangle)} = \sqrt{\frac{1}{N^2}\cdot N^2} = \sqrt{1} = 1
\end{aligned}$$

**(b)** Let $K$ be the norm closure of $\mathrm{co}\{f_N : N \geq 1\}$. To argue that $K$ is weakly compact, and that $0 \in K$, let $M = \mathrm{co}\{F_N : N \geq 1\}$, which means $K = \overline{M}^{\|\cdot\|}$. Since $M$ is convex, according to definition 7.7 (Musats notes), we use theorem 5.7 to conclude $K = \overline{M}^{\|\cdot\|} = \overline{M}^{\mathcal{T}_w}$. Let $x \in M$, which means that $x = \sum_{i=1}^n \alpha_i f_{N_i}$, $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$. The following shows that $K$ is bounded, remember from part (a) that $\|f_N\| = 1$:

$$\|\sum_{i=1}^n \alpha_i f_{N_i}\| = \|\alpha_1 f_{N_1} + ... + \alpha_n f_{N_n}\| \leq \alpha_1 \|f_{N_1}\| + ... + \alpha_n \|f_{N_n}\| = \sum_{i=1}^n \alpha_i = 1$$

which means that if $x \in M$, then $\|x\| \leq 1$, and then by closure $x \in K$ makes $\|x\| \leq 1$, and hence $K$ is bounded. Since $K$ is a bounded, convex set of $H$, which is a reflexsive Banach space (because $H$ is a Hilbert space), $K$ is weakly compact.
Furthermore, we want to show that $0 \in K$. This follows by part (a). Since $f_N \to 0$ weakly, as $N \to \infty$, and all $f_N \in M$ and $K = \overline{M}^{\mathcal{T}_w}$ it must follow that $0 \in K$.

**(c)** To show that 0, as well as each $f_N$, $N \geq 1$, are extreme points in $K$, note that $f_N = N^{-1} \sum_{n=1}^{N^2} e_i$ only have non-negative coordinates. If we look at the convex hull

$$\mathrm{co}\{f_N : N \geq 1\} = \left\{ \sum_{i=1}^n \alpha_i f_{N_i} : \quad \alpha > 0, \quad \sum_{i=1}^n \alpha_i = 1, \quad n \in \mathbb{N} \right\}$$

it is clear that the elements of $\mathrm{co}\{f_N : N \geq 1\}$ only have non-negative coordinates, too. Although we look at the closure of the convex hull, there are still only non-negative coordinates. This means if $x \in K \subset H$, then $x$ only have non-negative coordinates. Let $x = 0 \in K$, and let $0 < \alpha < 1$ such that $\alpha x_1 + (1-\alpha)x_2 = x$. Look at the $i$'th coordinate

$$\alpha x_{1,i} + (1-\alpha)x_{2,i} = x_i = 0$$

Since $\alpha > 0$ and $(1-\alpha) > 0$ one of $x_{1,i}$ or $x_{2,i}$ must be negative or $x_{1,i} = x_{2,i} = 0$. Since $x_{1,i}$ and $x_{2,i}$ can not be negative, it must hold that $x_{1,i} = x_{2,i} = 0$, which according to definition 7.1 (Musats notes) means that 0 is an extreme point.
To show that $f_N$ are extreme points remember from part (b) that if $x \in K$, then $\|x\| \leq 1$. Look at $x = f_N \in K$, where $N \geq 1$, that means $\|x\| = \|f_N\| = 1$. Let $0 < \alpha < 1$ and $x_1, x_2 \in K$, which means $\|x_1\| \leq 1$ and $\|x_2\| \leq 1$. Let $x = \alpha x_1 + (1-\alpha)x_2$, then the following must hold

$$\|x\| = \|\alpha x_1 + (1-\alpha)x_2\| \leq \alpha\|x_1\| + \|x_2\| - \alpha\|x_2\| = \alpha(\|x_1\| - \|x_2\|) + \|x_2\|$$

Since $\|x\| = 1$ and $\alpha > 0$ it is easily seen that $\|x_1\| \geq \|x_2\|$, because otherwise we will get that $1 = \|x\| < 1$. Since $x_1$ and $x_2$ can switch places, we can conclude that $\|x_1\| = \|x_2\|$. This means that

$$1 = \|x\| \leq \|x_2\| \leq 1$$

so $\|x\| = \|x_1\| = \|x_2\| = 1$. This means $x, x_1, x_2 \in \partial B(0,1)$. This means that $x = \alpha x_1 + (1-\alpha)x_2$ only is possible if $x_1 = x_2 = x$.

# Problem 2

Let $X$ and $Y$ be infinite dimensional Banach spaces.

**(a)** Let $T \in \mathcal{L}(X, Y)$. To show that $x : n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$, for a sequence $(x_n)_{n\geq 1}$ in $X$ and $x \in X$, we use that $Tx_n \to Tx$ weakly if and only if $g(Tx_n) \to g(Tx)$ for all $g \in Y^*$. Since $g \in Y^*$ we know that $g$ is a bounded linear function, just like $T$. Hence

$$\begin{aligned}|g(Tx_n) - g(Tx)| &= |g(Tx_n - Tx)| \\ &\leq \|g\|_{Y^*}\|Tx_n - Tx\|_Y \\ &= \|g\|_{Y^*}\|T(x_n - x)\|_Y \\ &\leq \|g\|_Y\|T\|\|x_n - x\|_X\end{aligned}$$

Since $x_n \to x$ weakly, we know that $|f(x_n) - f(x)| \to 0$, $\forall f \in X^*$. Therefore, we can conclude that

$$\|x_n - x\|_X = \sup_{f\in X^*\setminus\{0\}} \left(\frac{|f(x_n - x)|}{\|f\|_{X^*}}\right)$$

Choose $f \in X^*$ such that $\|f\|_{X^*} = 1$. For a given $\varepsilon > 0$, the following must hold

$$\|x_n - x\|_X < \frac{|f(x_n - x)|}{\|f\|_{X^*}} + \frac{\varepsilon}{2\|T\|} = |f(x_n - x)| + \frac{\varepsilon}{2\|T\|}$$

Since $x_n \to x$ weakly, there exists $N \in \mathbb{N}$ such that

$$|f(x_n) - f(x)| = |f(x_n - x)| < \frac{\varepsilon}{2\|T\|} \qquad \text{for} \quad n > N$$

All this gives us:

$$\begin{aligned}|g(Tx_n) - g(Tx)| &\leq \|g\|_{Y^*}\|T\|\|x_n - x\|_X \\ &< \|g\|_Y\|T\|\left(|f(x_n - x)| + \frac{\varepsilon}{2\|T\|}\right) \\ &< \|g\|_Y\|T\|\left(\frac{\varepsilon}{2\|T\|} + \frac{\varepsilon}{2\|T\|}\right) \\ &= \|g\|_{Y^*}\varepsilon\end{aligned}$$

Since $g$ is bounded one could choose $\varepsilon' = \frac{\varepsilon}{\|g\|_{Y^*}}$ and it is clear that $Tx_n \to Tx$ weakly

**(b)** Let $T \in \mathcal{K}(X, Y)$, which means that $T$ is compact. To show that $x_n \to x$ weakly, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, for a sequence $(x_n)_{n\geq 1}$ in $X$ and $x \in X$, let $(x_n)_{n\geq 1}$ be a sequence in $X$ and suppose $x_n \to x$ weakly, as $n \to \infty$. For contradiction suppose that $\|Tx_n - Tx\| \not\to 0$ as $n \to \infty$. This means that for $\varepsilon > 0$ there exists a subsequence $(x_{n_k})_{k\geq 1}$ such that

$$\|Tx_{n_k} - Tx\| > \varepsilon \qquad \text{for all} \quad k \geq 1$$

Because $x_n \to x$ weakly as $n \to \infty$, we know that $x_{n_k} \to x$ weakly as $k \to \infty$. Using proposition 8.2 (Musat's notes) and the compactness of $T$ and the fact that $(x_{n_k})_{k\geq 1}$ is bounded, we know that there exists a subsequence such that $\|Tx_{n_{k_i}} - Tx'\| \to 0$ as $i \to \infty$ for some $x' \in X$. Because $x_{n_k} \to x$ weakly as $k \to \infty$, we know from part (a) that $Tx_{n_k} \to Tx$ weakly as $k \to \infty$, and subsequently $Tx_{n_{k_i}} \to Tx$ weakly as $i \to \infty$.
If a sequence converges by norm to something, it must converges weakly to the same. This is true since if $(y_n)_{n\geq 1}$ is a sequence in $Y$ and $\|y_n - y\| \to 0$ as $n \to \infty$, then let $g \in Y^*$ then

$$|g(y_n) - g(y)| = |g(y_n - y)| \leq \|g\|\|y_n - y\| \to 0 \qquad \text{as} \quad n \to \infty$$

because $\|y_n - y\| \to 0$. So $y_n \to y$ weakly as $n \to \infty$.
So when $\|Tx_{n_{k_i}} - Tx'\| \to 0$ as $i \to \infty$ and $Tx_{n_{k_i}} \to Tx$ weakly as $i \to \infty$ we can conclude that $Tx = Tx'$. This means that $\|Tx_{n_{k_i}} - Tx\| \to 0$ as $i \to \infty$ but this contradicts that fact that $\|Tx_{n_{k_i}} - Tx\| > \varepsilon$ for all $k \geq 1$ and therefore

$$\|Tx_n - Tx\| \to 0 \qquad \text{as} \quad n \to \infty$$

**(c)** Let $H$ be a separable infinite dimensional Hilbert space. To show that if $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in $H$ converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$, we have to prove that $T$ is compact. Suppose $T$ is not compact for contradiction. If $T$ is not compact then $T(\overline{B_H(0,1)})$ is not totally bounded, according to proposition 8.2 (Musat's notes). This means that there exists $\delta > 0$ such that every finite union of open balls with radius $\delta$ does not cover $T(\overline{B_H(0,1)})$.
Now we define a sequence $(x_n)_{n\geq 1}$. Lets start by chosen $x_1 \in \overline{B_H(0,1)}$ at random. Then $B_Y(Tx_1, \delta)$ will not cover $T(\overline{B_H(0,1)})$ because it is not totally bounded. Now choose $x_2 \in \overline{B_H(0,1)}$ such that $Tx_2 \in \big(B_Y(Tx_1, \delta)\big)^c$. Again $\bigcup_{i=1}^2 \big(B_Y(Tx_i, \delta)\big)$ will not cover $T(\overline{B_H(0,1)})$ and so forth. Let $x_{n+1} \in \overline{B_H(0,1)}$ such that $Tx_{n+1} \in \big(\bigcup_{i=1}^n B_y(Tx_i, \delta)\big)^c$. For this constructed sequence $(x_n)_{n\geq 1}$ we know that $\|Tx_n - Tx_m\| \geq \delta$ for $n \neq m$.
Since $H$ is a separable Hilbert space, then so is the dual space $H^*$. By theorem 6.1 (Musat's notes) the closed unit ball $\overline{B_{H^{**}}(0,1)}$ is compact in the $w^*$-topology. By theorem 5.13 (Musat's notes) $\big(\overline{B_{H^{**}}(0,1)}, \tau_{w^*}\big)$ is metrizable. This means $\overline{B_{H^{**}}(0,1)}$ is compact in the $w^*$-topology, and sequences in $\overline{B_{H^{**}}(0,1)}$ will have a converging subsequence. If we consider a sequence $(z_n)_{n\geq 1}$ in $\overline{B_H(0,1)}$, then $(\hat{z})_{n\geq 1}$ will be a corresponding sequence in $\overline{B_{H^{**}}(0,1)}$. $(\hat{z})_{n\geq 1}$ will have a converging subsequence $(\hat{z}_{n_k})_{n\geq 1}$ in $\overline{B_{H^{**}}(0,1)}$ as $k \to \infty$ in the $w^*$-topology.
Let $f \in H^*$ then $f(z_{n_k}) = \hat{z}_{n_k} f \to \hat{z} f = f(z)$ as $k \to \infty$. This means that all sequences in $\overline{B_H(0,1)}$ must converge weakly, including the subsequence of the previously constructed sequence, that is $(x_{n_k})_{k\geq 1}$ as $k \to \infty$. Since $x_{n_k} \to x$ weakly as $k \to \infty$ then $\|Tx_{n_k} - Tx\| \to 0$ by assumption. But since $\|Tx_m - Tx_n\| \geq \delta$ for $m \neq n$ we also know that $\|Tx_{n_k} - Tx\| \not\to 0$ as $k \to \infty$. This contradiction shows that $T$ is compact.

**(d)** To show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact note that $\ell_2(\mathbb{N})$ and $\ell_1(\mathbb{N})$ are Banach spaces, so the reguirements in part (a) are fulfilled. So if we have a sequence $(x_n)_{n\geq 1}$ in $\ell_2(\mathbb{N})$ and $x_n \to x \in \ell_2(\mathbb{N})$ weakly as $n \to \infty$, then $Tx_n \to Tx$ weakly as $n \to \infty$ in $\ell_1(\mathbb{N})$. According to remark 5.3 (Musat's notes) a sequence converges weakly in $\ell_1(\mathbb{N})$ if and only if it converges in norm. This means that

$$\|Tx_n - Tx\| \to 0 \qquad \text{as} \quad n \to \infty$$

Thus by part (c) $T$ is compact, as $\ell_2(\mathbb{N})$ is a (separable) Hilbert space according to page 3 in Musat's notes).

**(e)** To show that no $T \in \mathcal{K}(X, Y)$ is onto, suppose for contradiction that $T$ is onto. According to theorem 3.15 (Musat's notes) then $T$ is open. As $X$ and $Y$ are normed vector spaces and $T$ is open then there exists $r > 0$, such that $B_Y(0, r) \subset T(B_X(0, 1))$, according to page 18 in Musat's notes. Since the closure preserves inclusion $\overline{B_Y(0, r)} \subset \overline{T(B_X(0, 1)}$. We know that $\overline{B_Y(0, r)} = r\overline{B_Y(0, 1)}$. From problem 3e in mandatory assignment 1 we know that a closed unit-ball in a infinite dimensional vector space is not compact. This means that $\overline{B_Y(0, 1)}$ is not compact, and then $r\overline{B_Y(0, 1)}$ is not compact. But at the same time $r\overline{B_Y(0, 1)}$ is a closed subset of $\overline{T(B_X(0, 1))}$ and $\overline{T(B_X(0, 1))}$ is compact because $T$ is a compact operator. This implies that $r\overline{B_Y(0, 1)}$ is compact. This is a contradiction and hence no $T \in \mathcal{K}(X, Y)$ can be onto.

**(f)** Let $H = L_2([0, 1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0, 1]$. To justify that $M$ is self-adjoint, we need to show that $\langle Mf, g\rangle = \langle f, Mg\rangle$ for all $f, g \in H$. Let $t \in [0, 1]$ which means that $t = t$, then

$$\begin{aligned}
\langle Mf, g\rangle &= \int_{[0,1]} Mf(t)\overline{g(t)}dm(t) \\
&= \int_{[0,1]} tf(t)\overline{g(t)}dm(t) \\
&= \int_{[0,1]} f(t)\overline{\overline{t}g(t)}dm(t) \\
&= \int_{[0,1]} f(t)\overline{tg(t)}dm(t) \\
&= \int_{[0,1]} f(t)\overline{Mg(t)}dm(t) \\
&= \langle f, Mg\rangle
\end{aligned}$$

So $M = M^*$ and $M$ is self-adjoint.

To justify that $M$ is not compact, we suppose $M$ is compact for contradiction. From HW4 (problem 4) we know that $H$ is separable. Since $H$ is separable, infinite-dimensional Hilbert space and $M$ is self-adjoint and assumed compact, then by theorem 10.1 (Musat's notes) $H$ has an orthonormal basis consisting of eigenvalues $\lambda_n \in \mathbb{R}$. But in HW6 (problem 3) we showed that $M$ has no eigenvalues. Here is the contradiction, that makes $M$ non compact.

# Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$, where $m$ is the Lebesque measure. Define $K : [0,1] \times [0,1] \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1, \\ (1-t)s, & \text{if } 0 \le s \le t \le 1, \end{cases}$$

and consider $T \in \mathcal{L}(H,H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t), \qquad s \in [0,1], \quad f \in H.$$

**(a)** To justify that $T$ is compact, we just have to use proposition 9.12 (Musat's notes), where $X = Y = [0,1]$ and $\mu = \nu = m$.

**(b)** To show that $T = T^*$ we have to show that $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in H$. Notice that $K(s,t) = K(t,s)$

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} Tf(s)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} f(t)K(t,s)dm(t)\overline{g(s)}dm(t)dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} f(t)\overline{K(t,s)g(s)}dm(t)dm(s) \\
\text{(switch the integrals)} \qquad &= \int_{[0,1]}\int_{[0,1]} f(t)\overline{K(t,s)g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t)\int_{[0,1]} \overline{K(t,s)g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t)\overline{Tg(t)}dm(t) \\
&= \langle f, Tg\rangle
\end{aligned}$$

Hence $T = T^*$.

**(c)** To show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t), \qquad s \in [0,1], \quad f \in H$$

split up the integral

$$\begin{aligned}
Tf(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t)
\end{aligned}$$

To show that $Tf$ is continuous on $[0,1]$ remember that $f \in H$ is continuous. It is clear that $(1-s)$ and $s$ are continuous and so are the integrals.
To show that $(Tf)(0) = (Tf)(1) = 0$ look at the following calculation:

$$\begin{aligned}
(Tf)(0) &= (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]}(1-t)f(t)dm(t) \\
&= \int_{[0,0]} tf(t)dm(t) = 0
\end{aligned}$$

$$\begin{aligned}
(Tf)(1) &= (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]}(1-t)f(t)dm(t) \\
&= \int_{[1,1]} (1-t)f(t)dm(t) = 0
\end{aligned}$$

This holds since we integrate over singletons $[0,0] = \{0\}$ and $[1,1] = \{1\}$.

# Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

**(a)** For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$.
To justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$, we use definition 11.10(Musat's notes). First notice that $g_k \in C^\infty(\mathbb{R})$ for all $k \geq 0$. Secondly we have to show that

$$\lim_{\|x\| \to \infty} x^\beta \partial^\alpha g_k(x) = 0$$

for all multi-indices $\alpha, \beta$. Look at the following calculations, first where $\alpha = 1$:

$$\frac{\partial}{\partial x} g_k(x) = kx^{k-1}e^{-\frac{x^2}{2}} + x^k \cdot (-x)e^{-\frac{x^2}{2}} = (kx^{k-1} - x^{k+1})e^{-\frac{x^2}{2}}$$

It is clear that, if we continue with the differentiations, we will get a polynomial for every $\alpha \in \mathbb{N}$ like this

$$\frac{\partial^\alpha}{\partial x^\alpha} g_k(x) = \text{Pol}(x) \cdot e^{-\frac{x^2}{2}}$$

which means that

$$x^\beta \frac{\partial^\alpha}{\partial x^\alpha} g_k(x) = \text{Pol}(x) \cdot e^{-\frac{x^2}{2}}$$

We know from previous courses that

$$\lim_{\|x\| \to \infty} (\text{Pol}(x) \cdot e^{-\frac{x^2}{2}}) = 0$$

and now it is shown that $g_k \in \mathscr{S}(\mathbb{R})$.

To compute the Fourier transform of $g_0$, $g_1$, $g_2$ and $g_3$ we start with $g_0$. Note that $g_0(x) = e^{-\frac{x^2}{2}}$. By proposition 11.4 (Musat's notes) we get $\mathcal{F}(g_0) = \hat{g_0}(\xi) = e^{-\frac{\xi^2}{2}}$. Looking at $g_1(x) = xe^{-\frac{x^2}{2}}$, and using that $e^{-\frac{x^2}{2}}$ and $\sin(x\xi)$ are even, and $\cos(x\xi)$ and $x$ are odd, we know that

$$\begin{aligned}
\mathcal{F}(g_1) = \hat{g_1}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} xe^{-\frac{x^2}{2}} \cos(x\xi)dx - \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} xe^{-\frac{x^2}{2}} \sin(x\xi)dx \\
&= -\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} xe^{-\frac{x^2}{2}} \sin(x\xi)dx \\
&= -\frac{2i}{\sqrt{2\pi}} \int_0^\infty xe^{-\frac{x^2}{2}} \sin(x\xi)dx \\
&= -\frac{2i}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} \xi e^{-\frac{\xi^2}{2}} \\
&= -i\xi e^{-\frac{\xi^2}{2}}
\end{aligned}$$

Now we do the same form of calculation for the Fourier transform for $g_2$ and $g_3$. Remember that $x^2$ is even and $x^3$ is odd:

$$\begin{aligned}
\mathcal{F}(g_2) = \hat{g_2}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2e^{-\frac{x^2}{2}} \cos(x\xi)dx - \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2e^{-\frac{x^2}{2}} \sin(x\xi)dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2e^{-\frac{x^2}{2}} \cos(x\xi)dx \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2e^{-\frac{x^2}{2}} \cos(x\xi)dx \\
&= \frac{2}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} (1-\xi^2) e^{-\frac{\xi^2}{2}} \\
&= (1-\xi^2) e^{-\frac{\xi^2}{2}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(g_3) = \hat{g_3}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^3e^{-\frac{x^2}{2}} \cos(x\xi)dx - \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x^3e^{-\frac{x^2}{2}} \sin(x\xi)dx \\
&= -\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x^3e^{-\frac{x^2}{2}} \sin(x\xi)dx \\
&= -\frac{2i}{\sqrt{2\pi}} \int_0^\infty x^3e^{-\frac{x^2}{2}} \sin(x\xi)dx \\
&= -\frac{2i}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} (3\xi - \xi^3) e^{-\frac{\xi^2}{2}} \\
&= -i(3\xi - \xi^3) e^{-\frac{\xi^2}{2}}
\end{aligned}$$

**(b)** To find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$ we first have to find a $h_0$, such that $\mathcal{F}(h_0) = h_0$. If we let $h_0 = g_0$ then

$$\mathcal{F}(h_0) = \mathcal{F}(g_0) = e^{-\frac{x^2}{2}} = g_0 = h_0$$

and it is clear this $h_0$ works.
Now we have to find a $h_1$ such that $\mathcal{F}(h_1) = i \cdot h_1$. Remember $\mathcal{F}$ is linear. If we let $h_1 = 2g_3 - 3g_1$ then

$$\begin{aligned}\mathcal{F}(h_1) &= \mathcal{F}(2g_3 - 3g_1) = 2\mathcal{F}(g_3) - 3\mathcal{F}(g_1) = 2(-i(3x - x^3)e^{-\frac{x^2}{2}}) - 3(-i)xe^{-\frac{x^2}{2}} \\ &= i(-6xe^{-\frac{x^2}{2}} + 2x^3e^{-\frac{x^2}{2}} + 3xe^{-\frac{x^2}{2}}) = i(2x^3e^{-\frac{x^2}{2}} + (3-6)xe^{-\frac{x^2}{2}}) \\ &= i(2x^3e^{-\frac{x^2}{2}} - 3xe^{-\frac{x^2}{2}}) = i(2g_3 - 3g_1) = ih_1\end{aligned}$$

and it is clear this $h_1$ works.
Now we have to find a $h_2$ such that $\mathcal{F}(h_2) = -h_2$. If we let $h_2 = g_0 - 2g_2$ then

$$\begin{aligned}\mathcal{F}(h_2) &= \mathcal{F}(g_0 - 2g_2) = \mathcal{F}(g_0) - 2\mathcal{F}(g_2) = e^{-\frac{x^2}{2}} - 2(1 - x^2)e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} - 2e^{-\frac{x^2}{2}} + 2x^2e^{-\frac{x^2}{2}} \\ &= -e^{-\frac{x^2}{2}} + 2x^2e^{-\frac{x^2}{2}} = -(e^{-\frac{x^2}{2}} - 2x^2e^{-\frac{x^2}{2}}) = -(g_0 - 2g_2) = -h_2\end{aligned}$$

and it is clear this $h_2$ works.
Now we have to find a $h_3$ such that $\mathcal{F}(h_3) = -i \cdot h_3$. If we let $h_3 = g_1$ then

$$\mathcal{F}(h_3) = \mathcal{F}(g_1) = -ixe^{-\frac{x^2}{2}} = -ig_1 = -i \cdot h_3$$

and it is clear this $h_2$ works.

**(c)** To show that $\mathcal{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$, we use that

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ix\xi} dx$$

According to corollary 12.14 (Musat's notes) everything below is well-defined as all functions are Schwartz functions.
Futhermore, we know from definition 12.10 (Musat's notes) that

$$\mathcal{F}^*(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{ix\xi} dx$$

which means that

$$\mathcal{F}(f)(\xi) = \mathcal{F}^*(f)(-\xi) \qquad \forall \varepsilon \in \mathbb{R}$$

From this it is easily seen that

$$\mathcal{F}(\mathcal{F}(f)(\xi)) = \mathcal{F}(\mathcal{F}^*(f)(-\xi)) \qquad \forall \varepsilon \in \mathbb{R}$$

$$\mathcal{F}^2(f)(\xi) = f(-\xi) \qquad \forall \varepsilon \in \mathbb{R}$$

and then it is clear that

$$\mathcal{F}^4(f)(\xi) = \mathcal{F}^2(\mathcal{F}^2(f)(\xi)) = \mathcal{F}^2(f)(-\xi) = f(-(-\xi)) = f(\xi) \qquad \forall \varepsilon \in \mathbb{R}$$

and it is shown that $\mathcal{F}^4(f) = f$.

**(d)** To show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$, we use the fact that $\mathcal{F}^4(f) = f$ known from part (c). Combined with $\mathcal{F}(f) = \lambda f$ we get (remember $\mathcal{F}$ is linear)

$$f = \mathcal{F}^4(f) = \mathcal{F}^3(\lambda f) = \mathcal{F}^2(\lambda^2 f) = \mathcal{F}(\lambda^3 f) = \lambda^4 f$$

Now we just have to solve $\lambda^4 = 1$ and in the complex numbers we have exactly that $\lambda \in \{1, i, -1, -i\}$.
To conclude that the eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$ we assume for contradiction that $\mu \notin \{1, i, -1, -i\}$ is an eigenvalue, but then $\mathcal{F}(f) = \mu f$ and we have just shown that $\mu \in \{1, i, -1, -i\}$ which is a contradiction.

# Problem 5

Let $(x_n)_{n\geq 1}$ be a dense subset of $[0, 1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0, 1]$. To show that $\operatorname{supp}(\mu) = [0, 1]$ we use problem 3 from HW8. Let us look at the open sets of $[0, 1]$ with measure 0. But an open set $U \subset [0, 1]$ ($U \neq \emptyset$) will contain an open interval $(a, b)$, where $0 \leq a < b \leq 1$. Because $(x_n)_{n\geq 1}$ is dense in $[0, 1]$ it will have elements in the interval $(a, b)$ and therefore $0 < \mu\big((a, b)\big) \leq \mu(U)$. Hence the only open $\mu$-null set is $\emptyset$. Let $N$ be the union of all open $\mu$-null sets, which is the largest open $\mu$-null set of $[0, 1]$ according to problem 3 from HW8. Then $N = \emptyset$ and $\operatorname{supp}(\mu) = N^c = [0, 1]$.