Functional Analysis
Mandatory Assignment 1
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Problem 1

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $T : X \to Y$ be a linear map. Set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$, for all $x \in X$. To show that $\|\cdot\|_0$ is a norm on X we use definition 1.1 from Musat's notes. First we check the triangle inequality:

$$\|x_1 + x_2\|_0 = \|x_1 + x_2\|_X + \|T(x_1 + x_2)\|_Y = \|x_1 + x_2\|_X + \|Tx_1 + Tx_2\|_Y$$

Note that $T(x_1+x_2) = Tx_1+Tx_2$ since T is linear. The following holds since $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms, and therefore fulfil the triangle inequality:

$$\begin{aligned}\|x_1 + x_2\|_0 = \|x_1 + x_2\|_X + \|Tx_1 + Tx_2\|_Y &\leq \|x_1\|_X + \|x_2\|_X + \|Tx_1\|_Y + \|Tx_2\|_Y \\ &= \|x_1\|_0 + \|x_2\|_0\end{aligned}$$

Now we show that $\|\alpha x\|_0 = |\alpha|\|x\|_0$:

$$\|\alpha x\|_0 = \|\alpha x\|_X + \|T(\alpha x)\|_Y = \|\alpha x\|_X + \|\alpha\ Tx\|_Y$$

Again it holds that $T(\alpha x) = \alpha\ Tx$ due to the linearity of T. Now we use that $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms:

$$\|\alpha x\|_0 = \|\alpha x\|_X + \|\alpha\ Tx\|_Y = |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y = |\alpha|\Big(\|x\|_X + \|Tx\|_Y\Big) = |\alpha|\|x\|_0$$

At last we need to show that $\|x\|_0 = 0$ if and only if $x = 0$. Since $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms, and therefore positive, the following holds

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y = 0 \qquad \Longleftrightarrow \qquad \|x\|_X = 0 \quad \wedge \quad \|Tx\|_Y = 0$$

Since $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms $\|x\|_X = 0$ and $\|Tx\|_Y = 0$ if and only if $x = 0$ and $Tx = 0$. Since T is linear $Tx = 0$ if and only is $x = 0$. And now it is shown that $\|\cdot\|_0$ is a norm on X.

To show that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded we use proposition 1.10 (Musat's notes) which tells us that if T is bounded then there exists $C > 0$ such that $\|Tx\| \leq C\|x\|$, for all $x \in X$. Suppose that T is bounded, which means that there exists such a C. It is clear that

$$\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_0$$

And since $\|Tx\|_Y \leq C\|x\|_X$, for all $x \in X$, it must hold that

$$\|x\|_X \leq \|x\|_0 = \|x\|_X + \|Tx\|_Y \leq \|x\|_X + C\|x\|_X = (1 + C)\|x\|_X$$

According to definition 1.4 (Musat) this means that $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent. To show the other way suppose $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent, which means that there exists C_1 and C_2 such that $0 < C_1 \leq C_2 < \infty$ and

$$C_1\|x\|_X \leq \|x\|_0 = \|x\|_X + \|Tx\|_Y \leq C_2\|x\|_X$$

From this it is clear that

$$\|Tx\|_Y \leq C_2\|x\|_X - \|x\|_X = (C_2 - 1)\|x\|_X$$

Notice that $C_2 \geq 1$ and then it is shown that T is bounded.

(b) To show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional, we use theorem 1.6 (Musat), which says that if X i finite dimensional, then any two norms on X are equivalent. Since $\|\cdot\|_X$ and $\|\cdot\|_0$ are two norms on X, they are equivalent and we have just shown (in part (a)) that this implies that T is bounded.

(c) Suppose that X is infinite dimensional. To show that there exists a linear map $T : X \to Y$, which is not bounded, we use the hint, which says that there exists a Hamel basis $(e_i)_{i \in I}$ for X. Since Y is non-zero, there exists a $y' \in Y$ where $y' \neq 0$. Let $y = \frac{y'}{\|y'\|_Y}$. Then it holds that $\|y\|_Y = 1$. We can now make a family $(y_i)_{i\in I}$, such that $y_i = i \cdot y$. According to the hint, there exists a unique and linear map $T : X \to Y$ such that $T(e_i) = y_i$, for all $i \in I$. Assume for contradiction that T is bounded, which means that

$$\exists C > 0 : \|Tx\|_Y \leq C\|x\|_X$$

This means:

$$\|Te_i\|_Y = \|y_i\|_Y = \|i \cdot y\|_Y = i \cdot \|y\|_Y = i \leq C \cdot \|e_i\|_X = C$$

But this is a contradiction, since we can choose $i > C$.

(d) Suppose again that X is infinite dimensional. Let T be the non-bounded linear map from part (c). Let $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ as in part (a). According to part (a) $\|\cdot\|_0$ and $\|\cdot\|_X$ are not equivalent, since T isn't bounded. Thus, there exists a norm, $\|\cdot\|_0$ that is not equivalent to the given norm $\|\cdot\|_X$. Furthermore, it is clear that $\|x\|_X \leq \|x\|_0$, for all $x \in X$, since

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y$$

which implies

$$\|x\|_X = \|x\|_0 - \|Tx\|_Y \leq \|x\|_0$$

since $\|\cdot\|_Y \geq 0$
To show that $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space we use definition 1.4. Since $\|\cdot\|_X$ and $\|\cdot\|_0$ are not equivalent, there doesn't exists C_1 and C_2 such that:

$$C_1\|x\|_X \leq \|x\|_0 \leq C_2\|x\|_X, \qquad \forall x \in X$$

Since we just show that $\|x\|_X \leq \|x\|_0$ for all $x \in X$, $C_1 = 1$ fits the inequality. Therefore, C_2 can't exists. From this it follows that there exists $x' \in X$ such that $\|x'\|_0 > n\|x'\|_X$ for $n \in \mathbb{N}$. From this it follows that a sequence $(x_n)_{n\geq 1}$ that converges to x' in $(X, \|\cdot\|_X)$ isn't convergent in $(X, \|\cdot\|_0)$, which means that $(X, \|\cdot\|_0)$ isn't complete.

(e) An example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ satisfying $\|x\|' \leq \|x\|$, for all $x \in X$, such that $(X, \|\cdot\|)$ is complete, while $(X, \|\cdot\|')$ is not, could be

$$\ell_1(\mathbb{N}) = \left\{(x_n)_{n\geq 1} \subset \mathbb{K} : \|(x_n)_{n\geq 1}\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty\right\}$$

According to Musats notes (Lecture 1, page 3) $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is a Banach space and complete (according to Musats notes Lecture 1, page 1). Let

$$\|(x_n)_{n\geq 1}\|' = \sum_{n=1}^{\infty} \frac{1}{n}|x_n|$$

This is clearly a norm on $\ell_1(\mathbb{N})$, since the following holds:

$$\begin{aligned}\|(x_n)_{n\geq 1} + (y_n)_{n\geq 1}\|' &= \sum_{n=1}^{\infty} \frac{1}{n}|x_n + y_n| \leq \sum_{n=1}^{\infty} \frac{1}{n}(|x_n| + |y_n|) = \sum_{n=1}^{\infty} \frac{1}{n}|x_n| + \sum_{n=1}^{\infty} \frac{1}{n}|y_n| \\ &= \|(x_n)_{n\geq 1}\|' + \|(y_n)_{n\geq 1}\|'\end{aligned}$$

$$\|\alpha(x_n)_{n\geq 1}\|' = \sum_{n=1}^{\infty} \frac{1}{n}|\alpha x_n| = |\alpha| \sum_{n=1}^{\infty} \frac{1}{n}|x_n| = |\alpha|\|(x_n)_{n\geq 1}\|'$$

$$\|(x_n)_{n\geq 1}\|' = \sum_{n=1}^{\infty} \frac{1}{n}|x_n| = 0 \qquad \Longleftrightarrow \qquad x_n = 0 \qquad \forall n \geq 1$$

It is clear that $\|\cdot\|' \leq \|\cdot\|_1$. According to part (d) $(\ell_1(\mathbb{N}), \|\cdot\|')$ is not complete, because if $(\ell_1(\mathbb{N}), \|\cdot\|')$ was complete then $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ would not be complete, which is a contradiction.

Problem 2

Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), \|\cdot\|_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, ...) : a, b \in \mathbb{C})$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, 0, ...) = a + b$, for all $a, b \in \mathbb{C}$.

(a) To show that f is bounded on $(M, \|\cdot\|_p)$ we use the knowledge about bounded functions, which is that f is bounded if

$$\exists K > 0 : \quad |f(a, b, 0, 0, ...)| \leq K\|(a, b, 0, 0, ...)\|_p$$

or

$$\exists K > 0 : \quad |a + b| \leq K \sqrt[p]{|a|^p + |b|^p}$$

If $a = b = 0$ it is clear that the inequality holds, since it will evaluate to 0 on both sides of the inequality sign. Since $\sqrt[p]{|a|^p + |b|^p} > 0$ in any other situation, it means that f is bounded if

$$\exists K > 0 : \quad \frac{|a + b|}{\sqrt[p]{|a|^p + |b|^p}} \leq K$$

If we look at the situation where $p = 1$ and use the triangle inequality:

$$\frac{|a + b|}{|a| + |b|} \leq \frac{|a| + |b|}{|a| + |b|} = 1$$

and it is clear, that f is bounded by $K = 1$ in this situation. To compute $\|f\|$ we use this:

$$\|f\| = \sup\{|f(a, b, 0, 0, ...)| : \|(a, b, 0, 0, ...)\|_1 \leq 1\} = \sup\{|a + b| : |a| + |b| \leq 1\}$$

Since $|a + b| \leq |a| + |b| \leq 1$, and since an example where we can switch the inequalities with equality, is easily found ($a = b = \frac{1}{2}$), it must hold that $\|f\| = 1$ in the situation $p = 1$.
Look now at the situation $p = 2$ and let $a = x_1 + iy_1$ and $b = x_2 + iy_2$:

$$\frac{|a + b|}{\sqrt{|a|^2 + |b|^2}} = \frac{\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}}{\sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2}} = \frac{\sqrt{x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2}}{\sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}}$$

Since $(x_1 - x_2)^2 \geq 0$ it follows $x_1^2 + x_2^2 - 2x_1x_2 \geq 0$ and from this it follows that $x_1^2 + x_2^2 \geq 2x_1x_2$.

$$\begin{aligned}\frac{|a + b|}{\sqrt{|a|^2 + |b|^2}} &= \frac{\sqrt{x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2}}{\sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}} \\ &\leq \frac{\sqrt{x_1^2 + x_2^2 + x_1^2 + x_2^2 + y_1^2 + y_2^2 + y_1^2 + y_2^2}}{\sqrt{x_1^2 + x_2^2 + y_1^2 + y_2^2}} \\ &= \sqrt{\frac{2(x_1^2 + x_2^2 + y_1^2 + y_2^2)}{x_1^2 + x_2^2 + y_1^2 + y_2^2}} = \sqrt{2}\end{aligned}$$

Which means that f is bounded on $(M, \|\cdot\|_2)$ by $K = \sqrt{2}$. To compute $\|f\|$ we use that we know $|a + b| \leq \sqrt{2}\sqrt{|a|^2 + |b|^2} \leq \sqrt{2}$ since $\sqrt{|a|^2 + |b|^2} \leq 1$, when we try to compute $\|f\|$. If we look at the example $a = b = \frac{\sqrt{2}}{2}$ we see that $\sqrt{(\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2} = 1$ and $|\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}| = \sqrt{2}$, which means $\|f\| = \sqrt{2}$.
Now we have looked at the situations of $p = 1$ and $p = 2$, so lets generalize. First notice that

$$(x + y)^p = t_1x^p + t_2x^{p-1}y + t_3x^{p-2}y^2 + ... + t_{p-2}x^2y^{p-2} + t_{p-1}xy^{p-1} + t_py^p$$

where t_n are the binomial coefficients. Furthermore, notice $x^a \cdot y^b \leq x^p + y^p$ where $a+b=p$. From this the following must hold:

$$(x+y)^p \leq 2^p(x^p+y^p)$$

Since the sum of row number p in Pascal's triangle is 2^p and at hte same time it's the sum of the coefficients $(t_1+t_2+...+t_p)$. Now we can show that f is bounded:

$$\frac{|a+b|}{\sqrt[p]{|a|^p+|b|^p}} \leq \frac{|a|+|b|}{\sqrt[p]{|a|^p+|b|^p}} = \frac{\sqrt[p]{(|a|+|b|)^p}}{\sqrt[p]{|a|^p+|b|^p}} = \sqrt[p]{\frac{(|a|+|b|)^p}{|a|^p+|b|^p}} \leq \sqrt[p]{\frac{(2^p(|a|^p+|b|^p)}{|a|^p+|b|^p}} = 2$$

Which means f is bounded on $(M, \|\cdot\|_p)$. To compute

$$\|f\| = \sup\{|a+b| : \sqrt[p]{|a|^p+|b|^p} \leq 1\}$$

It is enough to look at $\sqrt[p]{|a|^p+|b|^p} = 1$ or $|a|^p+|b|^p = 1$. Let $|a| = |b|$ to get the highest $|a+b|$, which means

$$2|a|^p = 1 \qquad \Longleftrightarrow \qquad |a|^p = \frac{1}{2} \qquad \Longleftrightarrow \qquad |a| = \sqrt[p]{\frac{1}{2}} = \frac{1}{\sqrt[p]{2}}$$

Due to this let $a = b = \frac{1}{\sqrt[p]{2}}$, which means

$$|a+b| = \left|2 \cdot \frac{1}{\sqrt[p]{2}}\right| = \frac{2}{\sqrt[p]{2}} = \frac{\sqrt[p]{2^p}}{\sqrt[p]{2}} = \sqrt[p]{2^{p-1}}$$

Then $\|f\| = \sqrt[p]{2^{p-1}}$.

(b) To show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$ we first show the existence. Let $q(x) = \sqrt[p]{2^{p-1}}\|x\|_p$. This is a norm since q is proportional to a norm. According to the complex Hahn-Banach extension theorem (2.5 (Musat)) it holds that $|f(x)| \leq q(x)$ for $x \in M$. Then there exists a linear functional, F, such that $F|_M = f$ and $|F(x)| \leq q(x)$ for all $x \in \ell_p(\mathbb{N})$. According to corollary 2.6 we can furthermore conclude $\|F\| = \|f\|$ and then the existence is shown.

To show the uniqueness we use problem 5 from HW1. From this problem we know that if $\frac{1}{p}+\frac{1}{q} = 1$ then there exists an isometric isomorphism $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$. From this we can write

$$F(x) = \sum_{n=1}^{\infty}(x_n y_n) \qquad \text{for} \qquad y = (y_n)_{n\geq 1} \in \ell_q(\mathbb{N}) \qquad \text{and} \qquad x = (x_n)_{n\geq 1} \in \ell_p(\mathbb{N})$$

Since $\frac{1}{p}+\frac{1}{q} = 1$ it must hold that $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$. From part (a) and the proof of existence we know that

$$\|F\| = \|f\| = \sqrt[p]{2^{p-1}} = 2^{\frac{p-1}{p}} = 2^{\frac{1}{q}}$$

Since F is represented by $y \in \ell_q(\mathbb{N})$ we know that $\|y\|_q = 2^{\frac{1}{q}}$. We know that

$$F|_M(x) = a+b$$

which means $y = (1, 1, y_3, y_4, ...)$. It's clear that

$$\|y\|_q = \left(\sum_{i=1}^{\infty}|y_i|^q\right)^{\frac{1}{q}} = \left(|1|^q + |1|^q + |y_3|^q + ...\right)^{\frac{1}{q}} = 2^{\frac{1}{q}}$$

Which means that $0 = y_3 = y_4 = ...$ Therefore, it must hold that $y = (1, 1, 0, 0, ...)$ and

$$F(x) = a + b$$

.

Assume now that $F' \in (\ell_p(\mathbb{N}))^*$ is a linear functional with $F'|_M = f$ and $\|F'\| = \|f\|$. With the same arguments as before we can show that $F'(x) = a + b$, which means that $F(x) = F'(x)$ and the uniqueness is shown.

(c) To show that if $p = 1$, then there are infinitely many linear functional F on $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$ we first show the existence in the same way as in part (b). This time we can find infinitely many linear functionals with the two properties. On example could be $F_i : \ell_p(\mathbb{N}) \to \mathbb{C}$. where $F_i(x_1, x_2, x_3, ...) = a + b + x_i$, where $i \geq 3$. F_i is clearly linear and $F_i|_M = f$. Since F_i is a extension of f and since we know from part (a) that $\|f\| = 1$ it follows:

$$\|F_i\| \geq \|f\| = 1$$

If we use the definition of norms on functionals we get:

$$\begin{aligned}\|F_i\|_1 &= \sup\{|F_i| : \|x\|_1 \leq 1\} = \sup\{|a + b + x_i| : \|x\|_1 \leq 1\} \\ &\leq \sup\{|a| + |b| + |x_i| : \|x\|_1 \leq 1\} \leq 1\end{aligned}$$

Therefore, it must hold that $\|F_i\| = 1$. Then it is shown that there exists infinitely many linear functionals with the properties.

Problem 3

Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

(a) Let $n \geq 1$ be an integer. To show that no linear map $F : X \to \mathbb{K}^n$ is injective, lets suppose that F is injective for contradiction. If F is injective then all $x \in X$ are imaged into different values in $\mathbb{K}^n$. Because $\dim(X) = \dim(F(X)) \leq \dim(\mathbb{K}^n) = n$ we have that $\dim(X) \leq n$, which contradicts that X is infinite dimensional.

(b) Let $n \leq 1$ be an integer and let $f_1, f_2, ..., f_n \in X^*$. To show that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$$

let $F : X \to \mathbb{K}^n$ where $F(x) = \big(f_1(x), f_2(x), ..., f_n(x)\big)$ for all $x \in X$. According to part (a) F is not injective. From this it follows that there exists $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $F(x_1) = F(x_2)$, which means $F(x_1) - F(x_2) = 0$, since F is linear the following must hold $F(x_1 - x_2) = 0$. Let $x' = x_1 - x_2$. It is clear that $x' \neq 0$ (since $x_1 \neq x_2$) and $F(x') = 0$. This means that there exists $x' \in X$, where $x' \neq 0$ such that $F(x') = 0$. From this it follows that $f_1(x') = 0$, $f_2(x') = 0$, ... , $f_n(x') = 0$ and therefore $x' \in \ker(f_j)$, $j = 1, 2, ..., n$. This means that $x' \in \bigcap_{j=1}^n \ker(f_j)$ and then $\bigcap_{j=1}^n \ker(f_j) \neq \{0\}$.

(c) Let $x_1, x_2, ..., x_n \in X$. To show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, 2, ..., n$ we use theorem 2.7 (Musat), which says that for $x_i \neq 0$ there exists $f_i \in X^*$ such that $\|f_i\| = 1$ and $f_i(x_i) = \|x_i\|$. In the situation $x_i = 0$ it inequality clearly holds, for any $y \in X$ where $\|y\| = 1$. Let

$$F(x) = \big(f_1(x), f_2(x), ..., f_m(x)\big)$$

There is a little issue about the notation here. f_1 refers to the first f_i where $x_i \neq 0$, and so on, all the way to f_m which refers to the last f_i where $x_i \neq 0$. According to part (b) there exists $y' \in \bigcap_{j=1}^m \ker(f_j)$ where $y' \neq 0$. Let

$$y = \frac{y'}{\|y'\|}$$

and it's clear that $\|y\| = 1$ and $f_i(y) = 0$, for $i = 1, ..., m$. Look at $\|x_i\|$ and use the linearity of f_i:

$$\|x_i\| = f_i(x_i) = f_i(x_i) - f_i(y) = f_i(x_i - y) = f_i\left(\|x_i - y\|\frac{x_i - y}{\|x_i - y\|}\right) = \|x_i - y\| f_i\left(\frac{x_i - y}{\|x_i - y\|}\right)$$

Since $\frac{x_i - y}{\|x_i - y\|}$ is on the unit ball, it must hold that $f_i\big(\frac{x_i - y}{\|x_i - y\|}\big) \leq 1$ (since ${}_i\| = 1$) and

$$\|x_i\| = \|x_i - y\| f\left(\frac{x_i - y}{\|x_i - y\|}\right) \leq \|x_i - y\| = \|y - x_i\|$$

Hereby the required is shown.

(d) To show that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0, assume for contradiction that one can cover S with a finite family of closed balls in X such that none of the balls contains 0. Let $\{x_1, x_2, ..., x_n\}$ be the centers of these finitely many balls. Since all of these balls do not contain 0, the radii of the balls must fulfill that $r_i < \|x_i\|$. Let y be chosen as in part (c), which means that $\|y\| = 1$ (or $y \in S$) and $\|y - x_i\| \geq \|x_i\| > r_i$. This means that none of the balls contains y, but $y \in S$. This is a contradiction, since the balls cover S.

(e) To show that S is non-compact assume for contradiction that S is compact. Let $A = \bigcup_{x \in S} \mathcal{B}(x, \frac{1}{2})$, where $\mathcal{B}(x, \frac{1}{2})$ is the open ball with center in x and radius $\frac{1}{2}$. According to the topology course A is an open covering of S, since all $x \in S$ is the center of one of the balls in A.

If S is compact, A contains a finite subcover. Even if the balls is closed in this subcover, no one of the balls contains 0 and according to part (d) this is impossible and we have a contradiction, which means that S is non-compact.
To show that the unit ball in X in non-compact, assume for contradiction that the unit ball in X is compact. Once again this means that all open coverings contains a finite subcovering. Look at the following two-part open covering:

$$\left(\bigcup_{x\in S}\mathcal{B}\Big(x,\frac{1}{2}\Big)\right)\cup\left(\bigcup_{n=2}^{\infty}\mathcal{B}\Big(0,1-\frac{1}{n}\Big)\right)$$

If the unit ball in X is compact, this covering contains a finite subcovering. Since

$$S\cap\left(\bigcup_{n=2}^{\infty}\mathcal{B}\Big(0,1-\frac{1}{n}\Big)\right)=\emptyset$$

S can be covered by finitely many sets in $\bigcup_{n=2}^{\infty}\mathcal{B}\Big(0,1-\frac{1}{n}\Big)$, but this we have just shown is not possible. Therefore, we have a contradiction and the unit ball is non-compact.

Problem 4

Let $L_1([0,1],m)$ and $L_3([0,1],m)$ be the Lebesgue spaces on $[0,1]$. It is known from Homework 2 that $L_3([0,1],m)\subsetneq L_1([0,1],m)$. For $n\geq 1$ define

$$E_n:=\left\{f\in L_1([0,1],m):\int_{[0,1]}|f|^3dm\leq n\right\}$$

(a) To determine if the set $E_n\subset L_1([0,1],m)$ is absorbing for a given $n\geq 1$ we first need to show that E_n is convex, which it is if for $f_1,f_2\in E_n$ and $0<\alpha<1$ it holds that $\alpha f_1+(1-\alpha)f_2\in E_n$. Therefore, assume that $f_1,f_2\in E_n$ and $0<\alpha<1$. Let $f=\alpha f_1+(1-\alpha)f_2$. It's clear that $f\in L_1([0,1],m)$ due to properties of spaces. We need to show that

$$\int_{[0,1]}|f|^3dm\leq n$$

To show that we use integration calculation know from the course "Analyse 2":

$$\begin{aligned}\int_{[0,1]}|f|^3dm&=\int_{[0,1]}|\alpha f_1+(1-\alpha)f_2|^3dm\\&\leq\int_{[0,1]}|\alpha f_1|^3+|(1-\alpha)f_2|^3dm\\&=|\alpha|^3\int_{[0,1]}|f_1|^3dm+|1-\alpha|^3\int_{[0,1]}|f_2|^3dm\end{aligned}$$

Since $f_1,f_2\in E_n$ it must hold that $\int_{[0,1]}|f_1|^3\leq n$ and $\int_{[0,1]}|f_2|^3\leq n$. And since $0<\alpha<1$ it must hold that $|\alpha| = \alpha$ and $|1-\alpha| = 1-\alpha$.

$$\begin{aligned}\int_{[0,1]} |f|^3 dm &\leq |\alpha|^3 \int_{[0,1]} |f_1|^3 dm + |1-\alpha|^3 \int_{[0,1]} |f_2|^3 dm \\ &\leq \alpha^3 n + (1-\alpha)^3 n = (\alpha^3 + (1-\alpha)^3)n \\ &= (\alpha^3 + 1 - 3\alpha + 3\alpha^2 - \alpha^3)n = (1 - 3\alpha + 3\alpha^2)n\end{aligned}$$

Look at $f(x) = 3x^2 - 3x + 1$. The determinant of this polynomial is $d = (-3)^2 - 4 \cdot 3 \cdot 1 = -3$, which means that the polynomial don't have roots. Since $f(0) = 1$ and $f(1) = 1$, it's clear that $0 \leq f(x) \leq 1$ if $x \in [0,1]$. From this we can conclude the following:

$$\int_{[0,1]} |f|^3 dm \leq (1 - 3\alpha + 3\alpha^2)n \leq n$$

and it's shown that $f \in E_n$, which means E_n is convex. Assume now for contradiction that E_n is absorbing, which means that

$$\forall f \in L_1([0,1]), m) \qquad \exists t > 0 \qquad \text{such that} \qquad t^{-1} f \in E_n$$

This means that the following holds

$$\int_{[0,1]} |t^{-1} f|^3 dm \leq n \qquad \iff \qquad \int_{[0,1]} |f|^3 dm \leq t^3 n$$

Now look at $f(x) = x^{-1/3}$.

$$\int_{[0,1]} |x^{-1/3}| dm = \left[\frac{3}{2} x^{2/3}\right]_0^1 = \frac{3}{2} < \infty$$

This show that $f \in L_1([0,1], m)$, but

$$\int_{[0,1]} |x^{-1/3}|^3 dm = \int_{[0,1]} x^{-1} dm = \lim_{a \to 0^+} [\ln(x)]_a^1 = \lim_{a \to 0^+} (\ln(1) - \ln(a)) = \lim_{a \to 0^+} (-\ln(a))$$

Since $\ln(a) \to -\infty$ when $a \to 0$, we have a contradiction and E_n is not absorbing in $L_1([0,1], m)$.

(b) To show that E_n has empty interior in $L_1([0,1], m)$, for all $n \geq 1$ assume for contradiction that $\text{Int}(E_n) \neq \emptyset$, which means

$$\exists f_0 \in E_n \quad \text{and} \quad \exists r > 0 \quad \text{such that} \quad I = \{g \in L_1([0,1], m) : \|f_0 - g\|_1 < r\} \subset E_n$$

Let $f \in L_1([0,1], m)$ be arbitrary. Look at

$$g = f_0 + \frac{r}{2} \frac{f}{\|f\|_1}$$

Because of the properties of spaces it holds that $g \in L_1([0,1], m)$. Since

$$\|f_0 - g\|_1 = \left\| f_0 - \Big(f_0 + \frac{r}{2}\frac{f}{\|f\|_1}\Big)\right\|_1 = \left\|\frac{r}{2}\frac{f}{\|f\|_1}\right\|_1 = \frac{r}{2}\frac{\|f\|_1}{\|f\|_1} = \frac{r}{2} < r$$

it must hold that $g \in I$. We used that r and $\|f\|_1$ are constants, which means that $\left\|\frac{r}{2}\frac{f}{\|f\|_1}\right\|_1 = \left|\frac{r}{2\|f\|_1}\right|\|f\|_1 = \frac{r}{2}\frac{\|f\|_1}{\|f\|_1}$. Since $g \in I \subset E_n \subset L_3([0,1],m)$ and $f_0 \in L_3([0,1],m)$ it must hold that

$$f = \frac{2\|f\|}{r}(g - f_0) \in L_3([0,1].m)$$

But since f was arbitrary in $L_1([0,1],m)$, it means that

$$L_1([0,1],m) \subseteq L_3([0,1],m)$$

which is a contradiction to $L_3([0,1],m) \subsetneq L1([0,1],m)$. Therefore, it must hold that $\text{Int}(E_n) = \emptyset$.

(c) To show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$ assume (f_k) is a sequence where $f_k \in E_n$ for all k. Let $f_k \to f$ for $k \to \infty$, where $f \in L_1([0,1],m)$. We want to show that $f \in E_n$, since this implies that E_n is closed in $L_1([0,1],m)$. Since $E_n \subset L_1([0,1],m)$ per definition. According to corollary 2.19 (Folland) (a corollary to Fatou's lemma) it must hold that

$$\int_{[0,1]} |f|^3 dm \leq \liminf \int_{[0,1]} |f_k|^3 dm$$

Since $f_k \to f$ implies $|f_k|^3 \to |f|^3$. Furthermore, we know that $\int_{[0,1]} |f_k|^3 dm \leq n$ since all $f_k \in E_n$. From this we can conclude

$$\int_{[0,1]} |f|^3 dm \leq \liminf \int_{[0,1]} |f_k|^3 dm \leq n$$

Therefore, it's shown that $f \in E_n$ and E_n is according to this closed in $L_1([0,1],m)$.

(d) To show that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$ we use definition 3.12 (Musat). This definition says that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$ if $L_3([0,1],m)$ can be written as $\bigcup_{n\geq 1}^{\infty} E_n$ for some E_n, where these E_n are nowhere dense. Let E_n be the same E_n as previously in this problem. To show that

$$\bigcup_{n\geq 1}^{\infty} E_n = L_3([0,1,m)$$

assume first that $f \in \bigcup_{n\geq 1}^{\infty} E_n$, which means $\exists n' \in [0,\infty)$ such that $f \in E_{n'}$. This means

$$\int_{[0,1]} |f|^3 dm \leq n' < \infty$$

and furthermore

$$\|f\|_3^3 = \int_{[0,1]} |f|^3 dm \leq n' \quad \implies \quad \|f\|_3 \leq \sqrt[3]{n'}$$

Which means $f \in L_3([0,1], m)$.
Next assume $f \in L_3([0,1], m)$, which means $\|f\|_3 = n$ where $0 \leq n \leq \infty$. When the following holds

$$\|f\|_3^3 = n^3 \qquad \Longrightarrow \qquad \int_{[0,1]} |f|^3 dm = n^3 \leq n^3 + 1$$

Then it holds that $f \in E_{n^3+1}$ implies $f \in \bigcup_{n\geq 1}^{\infty} E_n$.
According to definition 3.12 E_n is nowhere dense if $\text{Int}(\overline{E_n}) = \emptyset$. According to part (b) $\text{Int}(E_n) = \emptyset$ and according to part (c) $E_n = \overline{E_n}$, which combined gives us $\text{Int}(\overline{E_n}) = \emptyset$, and it's shown that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$.

Problem 5

Let H be an infinite dimensional Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

(a) Suppose that $x_n \to x$ in norm, as $n \to \infty$, which means $\|x_n - x\| = \|x - x_n\| \to 0$, as $n \to \infty$. To show that $\|x_n\| \to \|x\|$, as $n \to \infty$, we use the triangle inequality

$$\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\| \qquad \Longleftrightarrow \qquad \|x_n\| - \|x\| \leq \|x_n - x\| = \|x - x_n\|$$

Furthermore:

$$\|x\| = \|x - x_n + x_n\| \leq \|x - x_n\| + \|x_n\| \qquad \Longleftrightarrow \qquad \|x\| - \|x_n\| \leq \|x - x_n\| = \|x_n - x\|$$

Combined this gives us

$$\Big| \|x\| - \|x_n\| \Big| \leq \|x - x_n\| \to 0 \qquad \text{as} \qquad n \to \infty$$

which means $\|x_n\| \to \|x\|$, as $n \to \infty$.

(b) Suppose that $x_n \to x$ weakly, as $n \to \infty$. It don't follow that $\|x_n\| \to \|x\|$, as $n \to \infty$, and here comes a counterexample.
Let $(e_n)_{n\geq 1}$ be an orthonormal basis in H. Choose $f \in H^*$, such that $f(e_n) = \frac{1}{n}$ for all $n \geq 1$, and expand this by linearity. It is clear that $f(e_n) \to 0 = f(0)$, as $n \to \infty$. According to HW4 problem 2 we have $\forall f \in H^*$ $f(x_n) \to f(x)$ if and only if $x_n \to x$, as $n \to \infty$. This means that $e_n \to 0$, but $\|e_n\| = 1$ for all $n \geq 1$ and $\|0\| = 0$, which opviously contradicts each other.