Mandatory assignment, FunkAn1

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Problem 1

Let $(X, \|\cdot\|_X)$ and $(X, \|\cdot\|_Y)$ be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $T : X \to Y$ be a linear map. Set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$, for all $x \in X$. Show that $\|\cdot\|_0$ is a norm on X. Show next that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded.

First of all lets show that $\|\cdot\|_0$ is a norm on X. I wanna show that the three properties of def. 1.1 of the lecture notes holds.

- Triangle inequality:

  By definition we obtain

  $$\|x + y\|_0 = \|x + y\|_X + \|T(x + y)\|_Y$$

  Furthermore, since T is linear and X, Y are both normed vector spaces we obtain the following

  $$\begin{aligned}\|x + y\|_X + \|T(x + y)\|_Y &\leq (\|x\|_X + \|y\|_X) + (\|Tx\|_Y + \|Ty\|_Y) \\ &= \|x\|_X + \|Tx\|_Y + \|y\|_X + \|y\|_Y \\ &= \|x\|_0 + \|y\|_0\end{aligned}$$

  Which gives the triangle inequality $\|x + y\|_0 \leq \|x\|_0 + \|y\|_0$ for $x, y \in X$.

- Scalar multiplication:

  $$\begin{aligned}\|\alpha x\|_0 &= \|\alpha x\|_X + \|\alpha Tx\|_Y \\ &= |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y \\ &= |\alpha|\|x\|_0\end{aligned}$$

- Non-seminorm:

  First notice that $\|x\|_0 = 0 \Leftrightarrow \|x\|_X = -\|Tx\|_Y$. Furthermore observe that $\|x\|_X \geq 0$ by definition of the norm, why the only solution to the presented equation is that $\|Tx\|_Y = 0$. However this only holds $\Leftrightarrow x = 0$ since $\|\cdot\|_Y$ is a well-defined norm. So now we have obtained that $\|x\|_0 = 0 \Leftrightarrow x = 0$.

This shows that $\|\cdot\|_0$ is a norm on X.

Now lets show that the two norms are equivalent $\Leftrightarrow$ T is bounded.

"$\Rightarrow$": Assume that the two norms are equivalent, hence by def. 1.4 this means that there exists $0 < C_1, C_2 < \infty$ s.t.

$$C_1\|x\|_0 \leq \|x\|_X \leq C_2\|x\|_0$$

for $x \in X$. I wish to show that T is bounded, which means that there exist $C > 0$ s.t.

$$\|Tx\|_Y \leq C\|x\|_X$$

for all $x \in X$. See that

$$\begin{aligned} C_1\|x\|_0 &\leq \|x\|_X \\ C_1(\|x\|_X + \|Tx\|_Y) &\leq \|x\|_X \\ \|x\|_X + \|Tx\|_Y &\leq \frac{1}{C_1}\|x\|_X \end{aligned}$$

Since $0 \leq \|x\|_X, \|Tx\|_Y < \infty$ (from the inequality), we have shown that $\|Tx\| < D\|x\|_X$ for som $D > 0$, hence bounded.

"$\Leftarrow$": Observe that

$$\begin{aligned} \|Tx\|_Y = \|x\|_0 - \|x\|_X \leq C\|x\|_X \\ \|x\|_0 \leq C\|x\|_X + \|x\|_X \\ \|x\|_0 \leq (C+1)\|x\|_X \end{aligned}$$

Now we only need to show that there exists $D > 0$ s.t. $\|x\|_X \leq D\|x\|_0$ which gives that the norms are equivalent. See that

$$\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_0$$

which shows that $\|x\|_X \leq 1 \cdot \|x\|_0$, and since $1 > 0$ it is a valid constant, why the desired has been obtained. $\square$

(b) Show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.

By thm. 1.6 we have that any two norms on X are equivalent when X is a finite dimensional vector space. From (a) $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent on a linear map T, which implies that T is bounded. But T was an arbitrary map, why all linear maps must be bounded with the assumption that dim $X = n < \infty$. $\square$

(c) Suppose that X is infinite dimensional. Show that there exists a linear map $T : X \to Y$, which is not bounded (= not continuous).

Since X is infinite dimensional we choose to take a Hamel basis B_X for X defined as $B_X := \{b_i : i \in I\}$ for some index I. Assume without loss of generality that $I \supseteq \mathbb{N}$. Now lets define a linear map $T : X \to Y$ and show that this is not bounded. Let every $b \in X$ be normalized s.t. we can set

$$T\left(\frac{b_i}{\|b_i\|}\right) = i \cdot y$$

for $y \in Y$ with $y \neq 0$ as a fixed element and $i \in \mathbb{N}$. Set $T\left(\frac{b_i}{\|b_i\|}\right) = 0$ if $i \notin \mathbb{N}$. This is a well-defined and linear map (by its construction) since $\left\{\frac{b_i}{\|b_i\|}\right\}$ is a linear independent

subset of X (it is contained in our Hamel basis).
Furthermore

$$\left\{ \frac{b_i}{\|b_i\|} \right\} \subseteq \{b \in X : \|b\| \leq 1\} := N$$

and

$$\sup_{x \in N} \|Tb\| \geq i\|y\| > 0$$

for each $i \in I \supseteq \mathbb{N}$. This shows that T is not bounded. $\square$

(d) Suppose again that X is infinite dimensional. Argue that there exist a norm $\|\cdot\|_0$ on X, which is *not* equivalent to the given norm $\|\cdot\|_X$, and which satisfies $\|x\|_X \leq \|x\|_0$, for all $x \in X$. Conclude that $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space.

X is again infinite dimensional. Then by (c) we know that T is not bounded, and then by (a) we can derive that the two norms $\|\cdot\|_0$ and $\|\cdot\|_X$ on X are *not* equivalent. Lets set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ which, by removing the positive norm $\|Tx\|_Y$, gives the desired inequality:

$$\|x\|_X \leq \|x\|_0$$

$\forall x \in X$.
Now, using the result found in problem 1 from HW3, we can conclude that since the norms are *not* equivalent, then X is not complete wrt *both* norms.
Now lets assume that $(X, \|\cdot\|_X)$ is a Banach space, hence complete, then $(X, \|\cdot\|_0)$ cannot be complete, or else this would imply that the norms were equivalent.

(e) Give an example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ satisfying $\|x\|' \leq \|x\|$, for all $x \in X$, such that $(X, \|\cdot\|)$ is complete, while $(X, \|\cdot\|')$ is not.

Take $(\ell_1(\mathbb{N}))$ with the $\|\cdot\|_1$-norm and $\|\cdot\|_\infty$-norm. From the lecture notes $(\ell_p(\mathbb{N}), \|\cdot\|_p)$ is complete for $1 \leq p < \infty$, which gives us that $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is complete.

Take an arbitrary sequence $x = (x_1, x_2, ..., x_n) \in \ell_1(\mathbb{N})$. Then

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \geq |x_1 + x_2 + ... + x_n| \geq \max_{i \in 1,...,n}\{|x_i|\} = \|x\|_\infty$$

which shows that $\|\cdot\|_\infty \leq \|\cdot\|_1$.

Now lets show that the norms are inequivalent. Take the sequence $(z_n)_{n\in\mathbb{N}} = (z_1, z_2, ..., z_k, 0, 0, ...)$ where $z_i = 1$ for $i \leq k$, but then $\|z_n\|_1 = k$ while $\|z_n\|_\infty = 1$. Thus there cannot exist C s.t. $k \leq C \cdot 1$, because we can always pick a bigger k, hence the norms are *not* equivalent.

Now all we need to show is that $(\ell_1(\mathbb{N}), \|\cdot\|)$ is not complete. Lets take the sequence of sequences $((y_n)(k))_{n\in\mathbb{N}} = \frac{1}{k}$ for $1 \leq k \leq n$ and $(y_n)(k) = 0$ for $k > n$. Then

$y_n(k) \in \ell_1$ for all n and each k, as they all have finite sum with the $\|\cdot\|_1$-norm. Let $y(k) = \frac{1}{k}$ for all $k \in \mathbb{N}$, and notice that

$$\|y_n(k) - y(k)\|_\infty = \sup\{|y_n(k) - y(k)|\} = |\frac{1}{n+1}| \to 0 \text{ for } n \to \infty$$

So it is Cauchy sequence wrt $\|\cdot\|_\infty$-norm. But $y(k) \notin \ell_1$ since $\sum_{n=1}^\infty |\frac{1}{n+1}| \to \infty$ for $n \to \infty$, hence it is not complete.

Problem 2

Let $1 \le p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), \|\cdot\|_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, ...) : a, b \in \mathbb{C}$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, 0, ...) = a + b$, for all $a, b \in \mathbb{C}$.

(a) Show that f is bounded on $(M, \|\cdot\|_p)$ and compute $\|f\|$.

First of all lets show that f is bounded on $(M, \|\cdot\|_p)$.
Let $x = (x_1, x_2, 0, 0, ...) \in M$. As $\frac{1}{p} + \frac{1}{\frac{p}{p-1}} = 1$ (with $q = \frac{p}{p-1}$) we obtain by Hölders inequality and the triangle inequality:

$$\begin{aligned}
|fx| = |x_1 + x_2| &\le |x_1| + |x_2| \\
&= \sum_{i=1}^{2} |x_i \cdot 1| \\
&\le \left(\sum_{i=1}^{2} |x_i|^{\frac{1}{p}}\right) \left(\sum_{i=1}^{2} |1|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \\
&= \left(\sum_{i=1}^{2} |x_i|^{\frac{1}{p}}\right) \cdot 2^{1-\frac{1}{p}} \\
&= \|x\|_p \cdot 2^{1-\frac{1}{p}}
\end{aligned}$$

Where I have used that $\frac{1}{q} = \frac{1}{\frac{p}{p-1}} = 1 - \frac{1}{p}$. So this shows that f is bounded on $(M, \|\cdot\|_p)$.

Now lets compute $\|f\|$.
We have just shown that for every $1 \le p < \infty$ we have that $|fx| \le 2^{1-\frac{1}{p}}\|x\|_p$ so

$$2^{1-\frac{1}{p}} \in \{C > 0 : |fx| \le C\|x\|_p\}$$

hence

$$\|f\| = \inf\{C > 0 : |fx| \le C\|x\|_p\} \le 2^{1-\frac{1}{p}}$$

Now lets construct a sequence $z \in M$ st. $\|z\|_p = 1$.
Let $z = (\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, 0, ...)$ and see that

$$\|z\|_p = \left(|\frac{1}{2^{\frac{1}{p}}}|^p + |\frac{1}{2^{\frac{1}{p}}}|^p\right)^{\frac{1}{p}} = \left(\frac{1}{2} + \frac{1}{2}\right)^{\frac{1}{p}} = 1$$

And since

$$|fz| = |\frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}}| = 2\frac{1}{2^{\frac{1}{p}}} = 2^{1-\frac{1}{p}}$$

Then $2^{1-\frac{1}{p}} \in \{|fx| : \|x\|_p = 1\}$ and it then follows that

$$2^{1-\frac{1}{p}} \leq \sup\{|fx| : \|x\|_p = 1\} = \|f\|$$

And we can conclude that $\|f\| = 2^{1-\frac{1}{p}}$. $\square$

(b) Show that if $i \leq p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Since f comes from a Banach space it is linear, and it is also bounded, hence continuous, why it follows that $f \in M^*$, so by cor. 2.6 in the lecture notes there exist $F \in (\ell_p(\mathbb{N}))^*$ st. $F|_M = f$ and $\|F\| = \|f\|$.

By problem 5 in HW1, we know if $\frac{1}{p} + \frac{1}{q} = 1$ then we obtain $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$ for $1 < p < \infty$. This means that the map maintains the norm. We can now set $F(x) = \sum_{n=1}^{\infty} x_n y_n$ for $y := (y_n)_{n\geq 1} \in \ell_q(\mathbb{N})$ and $x := (x_n)_{n\geq 1} \in \ell_p(\mathbb{N})$.
By our previous calculations we know that $2^{\frac{1}{q}} = 2^{1-\frac{1}{p}} = \|f\| = \|F\|$, and since F is represented by $y \in \ell_q(\mathbb{N})$ we must have that $\|y\|_q = 2^{\frac{1}{q}}$.
See that $F|_M(x) = f(x) = x_1 + x_2$ so $y = (1, 1, y_3, y_4, ...)$ and we furthermore get that

$$\|y\|_q = \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{\frac{1}{q}} = (|1|^1 + |1|^q + |y_3|^q + ...)^{\frac{1}{q}} = \|F\| = 2^{\frac{1}{q}}$$

so for $\|y\|_q = \|F\|$ to be valid due to the criteria of isometry this forces $y_3, y_4, ... = 0$, and we may conclude that $y = (1, 1, 0, 0, ...)$.
Now lets assume that $F' \in (\ell_p(\mathbb{N}))^*$ is another linear functional st. $F'|_M = f$ and $\|F'\| = \|f\|$. But then we would be able to use same argument as before, since our $y = (1, 1, y_3, y_4, ...)$ was for arbitrary $y_3, y_4, ...$, and get $F'|_M(x) = x_1 + x_2$. Hence $F(x) = F'(x)$ which shows that a linear functional extending f and satisfying $\|F\| = \|f\|$ is unique. $\square$

(c) Show that if $p = 1$, then there are infinitely many linear functional F on $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Let $p = 1$, define $F_i : \ell_1(\mathbb{N}) \to \mathbb{K}$ and let it be given by $(x_1, x_2, x_3, ...) \mapsto x_1 + x_2 + x_i$ for $i > 2$. This is clearly a linear functional on $\ell_1(\mathbb{N})$ and furthermore an extension on $\ell_1(\mathbb{N})$ since $F_i|_M(x) = x_1 + x_2 = f(x)$, for $x \in M$.
Since F_i extends f we must have that

$$\|F_i\| \geq \|f\| = 2^{1-\frac{1}{1}} = 1$$

Now see that

$$\begin{aligned}\|F_i\|_1 &= \sup\{|F_i x| : \|x\|_1 = 1\} \\ &= \sup\{|x_1 + x_2 + x_i| : \|x\|_1 = 1\} \\ &\leq \sup\{|x_1| + |x_2| + |x_i| : \|x\|_1 = 1\} \\ &\leq 1\end{aligned}$$

which follows by definition of $\|\cdot\|_1$.
Now we have that $\|F_i\| = 1 = \|f\|$. So F_i is a linear functional extending f, and since we can define this for any $i > 2$, there is infinitely many linear functionals on $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$. $\square$

Problem 3

Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $n \geq 1$ be an integer. Show that no linear map $F : X \to \mathbb{K}^n$ is injective.

I wanna show this by contradiction, so lets assume that the linear map $F : X \to \mathbb{K}^n$ is injective.
Let $x_1, ..., x_{n+1} \in X$ be linear independent and $F(x_1), ..., F(x_{n+1})$ be linear dependent. Then there exists scalars $\alpha_1, ..., \alpha_{n+1}$ where at least one of them is non-zero s.t.

$$\alpha_1 F(x_1) + ... + \alpha_{n+1} F(x_{n+1}) = 0$$

But by linearity we obtain that

$$\alpha_1 F(x_1) + ... + \alpha_{n+1} F(x_{n+1}) = F(\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1}) = 0$$

And since F is injective it follows that

$$\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1} = 0$$

However all $x_1, ..., x_{n+1}$ was linear independent why it must follow that all the scalars are zero, which is a contradiction, hence no linar map $F : X \to \mathbb{K}^n$ is injective. $\square$

(b) Let $n \geq 1$ be an integer and let $f_1, f_2, ..., f_n \in X^*$. Show that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}.$$

Lets consider the map $F : X \to \mathbb{K}^n$ defined by $F(x) = (f_1(x), f_2(x), ..., f_n(x))$ for $x \in X$. F is linear since it is defined only by linear functions, and we may conclude from (a) that F is not injective. This shows that $\ker(F) \neq \{0\}$, hence

$$\ker((f_1(x), f_2(x), ..., f_n(x)) \neq 0$$

which shows that there exists $x \neq 0$ s.t.

$$F(x) = (f_1(x), f_2(x), ..., f_n(x)) = 0 \Leftrightarrow f_1(x), f_2(x), ..., f_n(x) = 0$$

And therefore we obtain

$$0 \neq \ker(F) = \cap_{j=1}^n \ker(f_j)$$

$\square$

(c) Let $x_1, x_2, ..., x_n \in X$. Show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, 2, ..., n$.

Choose $0 \neq z \in \cap_{j=1}^n \ker(f_j)$. Define $y = \frac{z}{\|z\|}$ and see that we get by linearity:

$$f_j(y) = f_j\left(\frac{z}{\|z\|}\right) = \frac{1}{\|z\|} f_j(z)$$

for all j. But since $z \in \cap_{j=1}^n \ker(f_j)$ we obtain that $f_j(z) = 0$ why

$$f_j(y) = \frac{1}{\|z\|} f_j(z) = 0$$

which implies that $y \in \cap_{j=1}^n \ker(f_j)$. Observe that

$$\|y\| = \|\frac{z}{\|z\|}\| = \frac{\|z\|}{\|z\|} = 1$$

Now lets take $y \in \cap_{j=1}^n \ker(f_j)$ where $\|y\| = 1$. Notice that $\|f_j\| = 1$, from lecture notes 2.7(b), since $f_j \in X^*$ and X is a normed vector space. This shows that

$$\begin{aligned}
\|y - x_j\| &= f_j \cdot \|y - x_j\| \\
&\geq \|f_j(y - x_j)\| \\
&= |f_j(y - x_j)| \\
&= |f_j(y) - f_j(x_j)| \\
&= |0 - \|x_j\|| \\
&= \|x_j\|
\end{aligned}$$

Where we have used the definition of the operator norm, linearity of f_j, 2.7(b) from the lecture notes and that $y \in \cap_{j=1}^n \ker(f_j)$. $\square$

(d) Show that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

We wanna show that $S \nsubseteq \cup_{i=1}^n B_i$, where B_i are closed balls.
Lets take $x \in S$ and show that $x \notin \cup_{i=1}^n B_i$. More specific lets take $x \in \cap_{j=1}^n \ker(f_j) \cap S \subseteq S$. For x to be in B_i for all $i \geq 1$, B_i being convex, then by Hahn-Banach thm. $\mathrm{Re}(f_j(x)) \geq 1$ must hold. First of all lets show that B_i is convex.
For B_i to be convex we must have that $\alpha x + (1 - \alpha)y \in B_i$, $\forall x, y \in B_i$ and for all

$0 \leq \alpha \leq 1$. This holds if $\|\alpha x + (1-\alpha)y - p\| \leq r$, p being the center of the ball and r the radius. Lets show this

$$\begin{aligned}
\|\alpha x + (1-\alpha)y - p\| &= \|\alpha x - \alpha p + (1-\alpha)y - p + \alpha p\| \\
&= \|\alpha(x-p) + (1-\alpha)y - p(1+\alpha)\| \\
&\leq \|\alpha(x-p)\| + \|(1-\alpha)(y-p)\| \\
&= |\alpha|\|x-p\| + |1-\alpha|\|y-p\| \\
&\leq \alpha r + (1-\alpha)r \\
&= r
\end{aligned}$$

Hence B_i is convex.
Back to our x, since $x \in \cap_{j=1}^n \ker(f_j)$ we know that $f_j(x) = 0$ why $\mathrm{Re}(f_j(x)) = 0$, which is not larger or equal to 1 which shows that $x \notin B_i$ for all i. This shows that

$$\cap_{j=1}^n \ker(f_j) \cap B_i = \emptyset \Rightarrow \cap_{j=1}^n \ker(f_j) \cap B_i \cap S = \emptyset$$

And we have obtained that $x \notin \cup_{i=1}^n B_i$ as wanted. $\square$

(e) Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

I wanna show this by contradiction, so lets assume that S is compact.
Lets take an arbitrary $x \in S$ and consider the open ball

$$B_x = \{v \in X \mid \|x - v\| < \frac{1}{2}\}$$

Notice that $B_x \subseteq \cup_{x \in S} B_x$.
So if we look at $x \in S$ then it follows that $\|x - x\| = 0 < \frac{1}{2}$, why $x \in B_x$, hence $S \subseteq \cup_{x \in S} B_x$.
It now follows that $\{B_x\}_{x \in S}$ is an open cover of S, and by definition of compactness it follows that every open cover of S has a finite subcover, lets call it $\{B_{x_i}\}_{x \in S}$ for $1 \leq i \leq n$.
Now notice that $B_{x_i} \subseteq \overline{B_{x_i}}$ for $i = 1, ..., n$ why it follows that $S \subseteq \cup_{x_i \in S} \overline{B_{x_i}}$. Furthermore we know that the closure of a open ball is a closed ball, and since $\|x - 0\| = \|x\| = 1$ which is larger than $\frac{1}{2}$ we also know that $0 \notin \overline{B_{x_i}}$ for all $x_i \in S$.
We have now shown that there exists a finite family of closed balls covering S, where $0 \notin \overline{B_{x_i}}$. This is a contradiction by (d), why S is non-compact.

We have that $S \subseteq B$, with B being the closed unit ball. We just showed that S is non-compact, why B is also, since a closed subset of a compact space is compact hence a closed subset of a non-compact space is non-compact. $\square$

Problem 4

Let $L_1([0,1], m)$ and $L_3([0,1], m)$ be the Lebesgue spaces on $[0,1]$. Recall from HW2 that $L_3([0,1], m) \subsetneq L_1([0,1], m)$. For $n \geq 1$, define

$$E_n := \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm \leq n \right\}.$$

(a) Given $n \geq 1$, is the set $E_n \subset L_1([0,1],m)$ absorbing? Justify.

In order to talk about absorbing the set has to be convex, so lets show this.
We already used it once in problem 3, but lets refresh it. For E_n to be convex $\alpha f + (1-\alpha)g \in E_n \; \forall f, g \in E_n$ and for all $0 \leq \alpha \leq 1$ must hold. In this case this means that we have to show that

$$\int_{[0,1]} |\alpha f + (1-\alpha)g|^3 dm \leq n$$

By Minkowskis inequality we obtain

$$\begin{aligned}
\left(\int_{[0,1]} |\alpha f + (1-\alpha)g|^3 dm\right)^{\frac{1}{3}} &\leq \left(\int_{[0,1]} |\alpha f|^3 dm\right)^{\frac{1}{3}} + \left(\int_{[0,1]} |(1-\alpha)g|^3 dm\right)^{\frac{1}{3}} \\
&= \left(\int_{[0,1]} \alpha |f|^3 dm\right)^{\frac{1}{3}} + \left(\int_{[0,1]} (1-\alpha)|g|^3 dm\right)^{\frac{1}{3}} \\
&= \alpha \left(\int_{[0,1]} |f|^3 dm\right)^{\frac{1}{3}} + (1-\alpha)\left(\int_{[0,1]} |g|^3 dm\right)^{\frac{1}{3}} \\
&\leq \alpha n^{\frac{1}{3}} + (1-\alpha) n^{\frac{1}{3}} \\
&= n^{\frac{1}{3}}
\end{aligned}$$

Which shows that E_n is convex. Now lets return to justify if E_n is absorbing.
To be absorbing the following must hold

$$\forall f \in L_1([0,1],m) \exists t > 0 : t^{-1} f \in E_n$$

Our claim is that E_n isn't absorbing, lets proof this.
Let $f(t) = t^{-\frac{1}{3}}$, see that

$$\begin{aligned}
\|f\|_1 = \int_{[0,1]} |f| dm &= \int_0^1 x^{-\frac{1}{3}} dx \\
&= \frac{3}{2}
\end{aligned}$$

This is obviously finite and since $f(t)$ is measurable we obtain that $f \in L_1([0,1],m)$.
Now take $t > 0$ and see that

$$\int_{[0,1]} |f|^3 dm = \int_0^1 \frac{1}{x} dx \approx \infty$$

This shows that $f \notin L_3([0,1],m)$, why there doesn't exists $t > 0$ st. $t^{-1} f \in E_n$. This furthermore shows that $\int_{[0,1]} |t^{-1} f|^3 dm \approx \infty$ why E_n is *not* absorbing. $\square$

(b) Show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.

I wanna show this by contradiction, so lets assume that $\text{Int}(E_n) \neq \emptyset \; \forall n \geq 1$. Then it follows that there exists $f \in \text{Int}(E_n)$. Furthermore we have an open ball

$$B(f,\epsilon) := \{g \in L_1([0,1],m) : \|f - g\|_1 < \epsilon\} \subseteq E_n$$

for $\epsilon > 0$. For $0 \neq g \in L_1([0,1],m)$ we have that

$$\begin{aligned} \|f-(f+\frac{\epsilon}{2\|g)\|_1}g\|_1 &= \|f-f-\frac{\epsilon}{2\|g\|_1}g\|_1 \\ &= \|-\frac{\epsilon}{2\|g\|_1}g\|_1 \\ &= |-\frac{\epsilon}{2\|g\|_1}|\|g\|_1 \\ &= \frac{\epsilon}{2\|g\|_1}\|g\|_1 \\ &= \frac{\epsilon}{2} < \epsilon \end{aligned}$$

This shows that $k := f + \frac{\epsilon}{2\|g\|_1}g \in B(f,\epsilon)$ by how we defined the ball. Now see that since $k \in B(f,\epsilon) \subseteq E_n$ it follows that $k \in L_3([0,1],m)$. Furthermore, since $f \in E_n$ it also follows that $f \in L_3([0,1],m)$. Notice that $g = (k-f)\frac{2\|g\|_1}{\epsilon}$ why we can conclude that $g \in L_3([0,1],m)$ which shows that $L_1([0,1],m) \subseteq L_3([0,1],m)$ which is a contradiction since we have from HW2 that $L_3([0,1],m) \subsetneq L_1([0,1],m)$. We have now obtained that $\mathrm{Int}(E_n) = \emptyset$ why E_n has empty interior in $L_1([0,1],m)$ for all $n \geq 1$. $\square$

(c) Show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$.

To show that E_n is closed in $L_1([0,1],m)$ we wanna show that for a sequence $(f_k)_{k\in\mathbb{N}} \subseteq E_n$ it also holds that the limit of the sequence is in E_n. Lets proof this.
Take a sequence $(f_k)_{k\in\mathbb{N}} \subseteq E_n$ where $\|f_k - f\| \to 0$ and $f \in L_1([0,1],m)$. From Bolzano-weierstrass we have that there is a subsequence $(f_{n_k})_{n_k\in\mathbb{N}}$ which converges pointwise. This shows, together with Fatou's lemma, that

$$\begin{aligned} \|f\|_3^3 = \int_{[0,1]} |f|^3 dm &\leq \lim_{n_k\to\infty}\inf \int_{[0,1]} |f_{n_k}|^3 dm \\ &\leq \lim_{n_k\to\infty}\inf n \\ &= n \end{aligned}$$

This shows that $f \in E_n$, and since f was the limit of the sequence we have obtained the desired. $\square$

(d) Conclude from (b) and (c) that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

By def. 3.12(ii) in the lecture notes $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$ if there exists a sequence $(E_n)_{n\geq 1}$ of nowhere dense sets st. $L_3([0,1],m) = \cup_{n=1}^{\infty} E_n$.
First lets show that $(E_n)_{n\geq 1}$ $\forall n \geq 1$ is a set that is nowhere dense. By def. 3.12(i) in the lecture notes a subset is nowhere dense if $\mathrm{Int}(\overline{E_n}) = \emptyset$, for $n \geq 1$.
From (b) we know that $\mathrm{Int}(E_n) = \emptyset$ and from (c) that E_n is closed $\forall n \geq 1$, why $E_n = \overline{E_n}$.
This gives us that

$$\mathrm{Int}(E_n) = \mathrm{Int}(\overline{E_n}) = \emptyset$$

Which shows that $(E_n)_{n\geq 1}$ is nowhere dense.

Now I wanna show that $L_3([0,1],m) = \cup_{n=1}^{\infty} E_n$.
Observe that

$$\begin{aligned}\cup_{n=1}^{\infty} E_n &= \cup_{n=1}^{\infty}\{f \in L_1([0,1],m) : \int_{[0,1]} |f|^3 dm \leq n\} \\ &= \{f \in L_1([0,1],m) : \int_{[0,1]} |f|^3 dm \leq \infty\} \\ &= \{f \in L_1([0,1],m) : f \in L_3([0,1],m)\} \\ &= L_3([0,1],m)\end{aligned}$$

Where I have used from HW2 that $L_3([0,1],m) \subsetneq L_1([0,1],m)$. □

Problem 5

Let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

(a) Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$?

Yes, it follows. Notice that

$$\|x\| = \|x - x_n + x_n\| \leq \|x - x_n\| + \|x_n\|$$

and similarly

$$\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\|$$

Gathering these we obtain the reverse triangle inequality

$$|\|x\| - \|x_n\|| \leq \|x - x_n\|$$

Now let $\epsilon > 0$. Since $x_n \to x$ in norm, there exist $N \in \mathbb{N}$ s.t.

$$n \geq N \Rightarrow |\|x\| - \|x_n\|| \leq \|x - x_n\| \leq \epsilon$$

Which proves that $\|x_n\| \Rightarrow \|x\|$ as $n \to \infty$. □

(b) Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$?

No, it doesn't follow. Let $H = \ell_2(\mathbb{N})$ and let $x_n = (e_n)_{n\geq 1}$ be the usual orthonormal basis of H. We can look at this basis since H is separable.
See that

$$\langle e_n, e_m\rangle = \delta_{mn}$$

where $\delta_{mn} = 1$ if $m = n$ and 0 otherwise.
The claim is that $e_n \to 0$ weakly but that $\|e_n\| \to \|0\| = 0$ doesn't hold. Lets proof this.
For $x \in H$ we have

$$\sum_n |\langle e_n, x\rangle|^2 \leq \|x\|^2 \text{ (Bessel's inequality)}$$

Therefore we get that
$$|\langle e_n, x\rangle|^2 \to \langle 0, x\rangle = 0$$
which holds since the series above converges, since $\|x\|^2 < \infty$, why its corresponding sequence must go to zero, and we obtain

$$\langle e_n, x\rangle \to \langle 0, x\rangle$$

hence by HW4 problem 2(a) we obtain that $e_n \to 0$ weakly. We can use this since a Hilbert space is a Banach space and a net is said to be a more generalized case of a sequence. Furthermore the f presented in HW4 can by the top of page 13 in the lecture notes be seen as the inner product why we obtain $e_n \to 0$ weakly $\Leftrightarrow \langle e_n, a\rangle \to \langle 0, a\rangle$.

Now see that $\|e_n\| = 1$ for every n, and since $\|0\| = 0$ and $\|e_n\|$ doesn't converge we obtain that $\|e_n\| \to 0$ isn't true.

(c) Suppose that $\|x_n\| \leq 1$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Is it true that $\|x\| \leq 1$?

Yes, it is true. A property of weak convergence is that the norm is (sequentially) weakly lower-semicontinuous, which means that $\|x\| \leq \lim_{n\to\infty}\inf\|x_n\|$. Lets proof this.

See that since $x_n \to x$ weakly it follows that

$$\|x\| = \langle x, x\rangle = \lim_{n\to\infty}\langle x, x_n\rangle$$

and
$$\langle x, x_n\rangle \leq \|x_n\|$$
why it follows that
$$\lim_{n\to\infty}\langle x, x_n\rangle \leq \lim_{n\to\infty}\inf\|x_n\|$$
So this shows $\|x\| \leq \lim_{n\to\infty}\inf\|x_n\|$ hence that $\|x\| \leq 1$. □