FunkAn Mandatory Assignment 2

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Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$ for all $N \geq 1$.

a)

Show that $f_N \to 0$ weakly, as $N \to \infty$, while $\|f_N\| = 1$ for all $N \geq 1$.

By Homework 4 problem 2, we know that f_N converges weakly to 0 iff $g(f_N)$ converges to $g(0) = 0$ for all $g \in H^*$
Let $g \in H^*$, then by Riesz representation theorem there exists a unique element $y \in H$ such that $g(x) = \langle x, y\rangle$ for all $x \in H$. Hence $g(f_N) = \langle f_n, y\rangle$. So we need to show, that

$$g(f_N) = \langle f_N, y\rangle \to g(0) = \langle 0, y\rangle = 0, \text{ for } N \to \infty$$

Hence we will show that $|g(f_N) - 0| < \varepsilon$ for some $k \geq N_\varepsilon$. This follows from the following calculations

$$\begin{aligned}
|g(f_N)| = |\langle f_N, y\rangle| &= |\langle N^{-1}\sum_{n=1}^{N^2} e_n, \sum_{i=1}^{\infty} \alpha_i e_i\rangle| \\
&= |\langle f_N, \sum_{i=1}^{k} \alpha_i e_i + \sum_{i=k+1}^{\infty} \alpha_i e_i\rangle| \\
&\leq |\langle f_N, \sum_{i=1}^{k} \alpha_i e_i\rangle| + |\langle f_N, \sum_{i=k+1}^{\infty} \alpha_i e_i\rangle|
\end{aligned}$$

We know that $\alpha_i e_i$ converges to zero for $i \to \infty$. Hence $\sum_{i=k+1} \alpha_i e_i < \frac{\varepsilon}{2}$ for $k \geq N_\varepsilon$. Thus we get

$$|\langle f_N, \sum_{i=k+1}^{\infty} \alpha_i e_i\rangle| \leq \|f_N\| \| \sum_{i=k+1}^{\infty} \alpha_i e_i\| \leq \| \sum_{i=k+1}^{\infty} \alpha_i e_i\| < \frac{\varepsilon}{2}$$

Next we have

$$|\langle f_N, \sum_{i=1}^{k} \alpha_i e_i \rangle| = N^{-1} |\langle \sum_{n=1}^{N^2} e_n, \sum_{i=1}^{k} \alpha_i e_i \rangle|$$

$$= N^{-1} \sum_{i=1}^{k} \overline{\alpha_i} |\langle \sum_{n=1}^{N^2} e_n, e_i \rangle| = N^{-1} \sum_{i=1}^{k} \overline{\alpha_i} \|e_i\|$$

Where $\langle \sum_{n=1}^{N^2} e_n, e_i \rangle = \|e_i\|^2$ if $i \in \{1, \ldots, N^2\}$ and is 0 otherwise.
We conclude that

$$|g(f_N)| \leq |\langle f_N, \sum_{i=1}^{k} \alpha_i e_i \rangle| + |\langle f_N, \sum_{i=k+1}^{\infty} \alpha_i e_i \rangle| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence f_N converges weakly to 0 by Homework 4 problem 2.
Next we show that $\|f_N\| = 1$.

$$\|f_N\| = N^{-1} \left\| \sum_{n=1}^{N^2} e_n \right\| = N^{-1} \left(\sum_{n=1}^{N^2} \|e_n\|^2 \right)^{\frac{1}{2}} = N^{-1} (N^2)^{\frac{1}{2}} = 1$$

Where we have used the Pythagorean theorem because e_i is perpendicular to all e_n for $n = 1, \ldots, N^2$ except $n = i$.

b)

Let K be the norm closure of $\text{co}\{f_N : N \geq 1\}$. Argue that K is weakly compact, and that $0 \in K$.

We start by showing that K is weakly compact.
K is a convex set, since by definition the convex hull is convex, and because the closure of a convex set is convex. This follows from the fact that if $(x_n)_{n \geq 1} \subset A$ and $(y_n)_{n \geq 1} \subset A$ with $\lim_{n \to \infty} x_n = x \in \overline{A}$ and $\lim_{n \to \infty} y_n = y \in \overline{A}$. Then

$$\alpha x_n + (1 - \alpha) y_n \in A$$

So

$$\lim_{n \to \infty} (\alpha x_n + (1 - \alpha) y_n) = \alpha \lim_{n \to \infty} x_n + (1 - \alpha) \lim_{n \to \infty} y_n = \alpha x + (1 - \alpha) y \in \overline{A}$$

Hence $\overline{A}$ is convex. Now by theorem 5.7 we have that $\overline{K}^{\|\cdot\|} = \overline{K}^{\tau_w}$ i.e that the norm and weak closures coincide.
Hence

$$K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w}$$

Now let $x \in \text{co}\{f_N : N \geq 1\}$ then

$$\|x\| = \|\sum_{i=1}^{n} \alpha_i f_{N_i}\| \leq \sum_{i=1}^{n} \alpha_i \|f_{N_i}\| \leq \sum_{i=1}^{n} \alpha_i \leq 1$$

Since $\|f_N\| = 1$ for all $N \geq 1$.
This implies that $x \in \overline{B_H(0,1)}$ hence if $x \in K \Rightarrow x \in \overline{\overline{B_h(0,1)}} = \overline{B_H(0,1)}$ so $K \subset \overline{B_H(0,1)}$. By 2.10 H is reflexive because it is a Hilbert space, so by 6.3 $\overline{B_H(0,1)}$ is weakly compact. Now since any closed subset of a compact space is compact, we conclude that K is weakly compact.
Next we show that $0 \in K$
We just showed in a) that $f_N \to 0$
Since each $f_N \in \{f_N : N \geq 1\} \subset \text{co}\{f_N : N \geq 1\}$ by definition, then 0 must be in the closure of $\text{co}\{f_N : N \geq 1\}$, i.e. $0 \in K$.

c)

Show that 0, as well as f_N are extreme points in K.
We will start by showing that 0 is an extreme point.
Recall that b is an extreme point if $b = \alpha x + (1-\alpha)y \Rightarrow x = y = b$.
We know that $K = \overline{\text{co}\{f_N : N \geq 1\}}$, so there exists sequences $(x_n)_{n\geq 1} \subset K$ and $(y_n)_{n\geq 1} \subset K$ with $\lim_{n\to\infty} x_n = x \in \overline{K}$ and $\lim_{n\to\infty} y_n = y \in \overline{K}$.
This gives that

$$\begin{aligned} 0 = \langle 0, e_k\rangle = \langle \alpha x + (1-\alpha)y, e_k\rangle \\ = \langle \alpha x, e_k\rangle + \langle (1-\alpha)y, e_k\rangle = \alpha\langle x, e_k\rangle + (1-\alpha)\langle y, e_k\rangle. \end{aligned}$$

Now if we can show that both $\langle x, e_k\rangle \geq 0$ and $\langle y, e_k\rangle \geq 0$, we are done, since $\alpha \geq 0$ and $(1-\alpha) \geq 0$.

$$\langle x, e_k\rangle = \langle \sum_{i=1}^{n} \alpha_i f_{N_i}, e_k\rangle = \sum_{i=1}^{n} \alpha_i \langle f_{N_i}, e_k\rangle$$

Where

$$\langle f_{N_i}, e_k\rangle = \langle N_i^{-1} \sum_{n=1}^{N_i^2} e_n, e_k\rangle = N_i^{-1}\langle \sum_{n=1}^{N_i^2} e_n, e_k\rangle =\geq 0.$$

Thus $\langle x, e_k\rangle \geq 0$ and a similar argument holds for $\langle y, e_k\rangle$.
We conclude that 0 is an extreme point.
Next we will show that f_N is an extreme point for each $N \geq 1$.
This will be done by showing that if f_N can be written as $f_N = \alpha x + (1-\alpha)y, x, y \in K$ then $f_N = x = y$.
We will start by showing that $\|x\| = \|y\| = 1$. We know from $b)$ that if $x \in K$ then $\|x\| \leq 1$. We note that

$$1 = |\langle|f_N, f_N\rangle| \leq \|f_N\|\|\alpha x + (1-\alpha)y\| = \alpha\|x\| + (1-\alpha)\|y\|$$

Now if $\|x\| < 1$ then $1 \leq \alpha\|x\| + (1-\alpha)\|y\| < \alpha + (1-\alpha) = 1$, which is a contradiction. Hence $\|x\| = 1$, and the exact same argument holds for y.
Now we have that $|\langle f_N, x\rangle \leq \|f_N\|\|x\| = 1$, however

$$1 = |\langle f_N, f_N\rangle| \leq \alpha|\langle x, f_N\rangle| + (1-\alpha)|\langle y, f_N\rangle|$$

so if $|\langle x, f_N\rangle| < 1$ then by the same argument as before we would have a contradiction. Hence $|\langle x, f_N\rangle| = 1$, and of course, the same holds for y.
So now

$$|\langle x, f_N\rangle| = 1 = \|f_N\|\|x\|$$

so by the Cauhcy Schwartz inequality we know that this holds iff $kf_N = x$ and $k'f_N = y$.
So now all we need to show is that $k = k' = 1$.
For this notice that

$$k = k \cdot 1 = k\|x\| = k\|kf_N\| = k|k| = k \Rightarrow k = \pm 1.$$

This also holds for k'.
Now we note that $k, k' = -1$ is not possible, since

$$f_N = \alpha k f_N + (1-\alpha)k' f_N$$

and in each combination of k and k' being negative leads to a contradiction since $0 < \alpha < 1$.
Hence we have showed that for some arbitrary f_N then $f_N = x = y$ for any convex combination of elements from K. Hence each f_N is extreme.

d)

Are the any other extreme points in K?

We want to show that there are no other extreme points. This will be done by showing that $\text{Ext}(K) = \{f_N : N \geq 1\} \cup \{0\} = F \cup \{0\}$. We have just shown one inclusion in c), so we need to show the other inclusion i.e. $Ext(K) \subset F \cup \{0\}$.
We showed in b) that $K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w}$ is a weakly compact subset of (H, τ_w) which is a LCTVS. Hence by theorem 7.9 $Ext(K) \subset \overline{F}^{\tau_w} =$. By definition this is exactly the union of F with all its weak limit points. So if we can show that every weak limit point converges to some element in F or to 0 , then we are done. Assume for contradiction that there exists some $x \in \overline{F}^{\tau_w}$ with $0 \neq x \neq f_N$ for all $N \geq 1$, and remember that f_N converges weakly to 0.
Then there exists some sequence $(f_{N_i})_{i \geq 1}$ in F converging weakly to x. By definition this means that for every neighbourhood U of x then $(f_{N_i})_{i \geq 1}$ is eventually in U.
But f_N is never infinitely many times in a neighbourhood of any $x \neq 0$ since

that would make x and accumulation point, and since τ_w is Hausdorff, a sequence can't have an accumulation point different from its limit.
Hence x can't exist. Therefore the only accumulation point is 0 and we conclude that $\text{Ext}(K) = \{f_N : N \geq 1\} \cup \{0\} = F \cup \{0\}$.

Problem 2

Let X and Y be infinite dimensional Banach spaces.

a)

Let T be a continuous linear map $T : X \to Y$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly as $n \to \infty$, implies that $Tx_n \to Tx$ weakly as $n \to \infty$.

We know from Homework 4 problem 2 that $x_n \to x$ weakly iff $g(x_n) \to g(x)$ for all $g \in X^*$, $g : X \to \mathbb{K}$
Now again by Homework 4 problem 2 we have that $Tx_n \to Tx$ weakly iff $f(Tx_n) \to f(Tx)$ for all $f \in Y^*$, $f : Y \to \mathbb{K}$.
Now $f \circ T \in X^*$ for all $f \in Y^*$, hence

$$f(Tx_n) = f \circ T(x_n) \to f \circ T(x) = f(Tx)$$

Which was what we wanted.

b)

Let $T \in \mathcal{K}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly as $n \to \infty$, implies that $\|Tx_n - Tx\| \xrightarrow{w} 0$ as $n \to \infty$.

Let $T \in \mathcal{K}(X, Y)$ and let $(x_n)_{n\geq} \subset X$ with $x_n \xrightarrow{w} x \in X$ as $n \to \infty$.
Since $T \in \mathcal{K}(X, Y)$ we have from $a)$ that $Tx_n \to Tx$ weakly as $n \to \infty$ and by Homework 4 problem 2 we get that $\sup\{|x_n\| : n \geq 1\} < \infty$ i.e. $(x_n)_{n\geq 1}$ is bounded. In particular every subsequence $(x_{n_k})_{k\geq 1}$ is bounded. Thus we get from 8.2 that there exists a subsequence $(x_{n_{k_l}})_{l\geq 1}$ such that $(Tx_{n_{k_l}})_{l\geq 1}$ converges in norm to some element in Y.
Now since $Tx_n \xrightarrow{w} Tx$ we must have that $Tx_{n_{k_l}} \xrightarrow{w} Tx$ for $n \to \infty$ for each subsequence $T(x_{n_{k_l}})_{l\geq 1}$.
We assert that this means that $\|Tx_{n_{k_l}} - Tx\| \to 0$ as $l \to \infty$. So assume for contradiction that $Tx_{n_{k_l}} \xrightarrow{w} Tx$ as $l \to \infty$ but $\|Tx_{n_{k_l}} - y\| \to 0$ for some $Tx \neq y \in Y$.
Now since norm convergence implies weak convergence we have that $Tx_{n_{k_l}} \xrightarrow{w} y$ for $l \to \infty$, but since τ_w is Hausdorff, the limit is unique and we have a contradiction.
Thus every subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ contains a subsequence $x_{n_{k_l}}$ such

that $(Tx_{n_{k_l}})_{l\geq 1}$ converges to Tx in norm.
This implies that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$ since if not, that means that $\|Tx_n - Tx\| \nrightarrow 0$ as $n \to \infty$ which is equivalent to saying that there exists some $\varepsilon > 0$ and $k \in \mathbb{N}$ so for all $n_k > k$ then $\|Tx_{n_k} - Tx\| \geq \varepsilon$. But then $(Tx_{n_k})_k \geq 1$ cant contain a subsequence converging to Tx, which contradicts our statement. Hence we are done.

c)

Let H be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H,Y)$ satisfies that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H,Y)$.

We will prove this by contraposition i.e. assume that T is not compact, then we want to show that whenever there exists a sequence $(x_n)_{n\geq 1}$ which converges weakly to $x \in H$ it implies that $< |Tx_n - Tx_m\| \geq \varepsilon$ for all $n \neq m$.
We want to construct this sequence $(x_n)_{n\geq 1}$.
Since T is not compact we know from 8.2 that $T(\overline{B_H(0,1)})$ is not totally bounded. Hence we cant cover it with a finite union of $\varepsilon-$balls.
Now let $x_1 \in \overline{B_H(0,1)}$ then $B_Y(Tx_1, \varepsilon)$ does not cover $T(\overline{B_H(0,1)})$.
Next let $Tx_2 \in T(\overline{B_(0,1)})$ such that $Tx_2 \cap T(B_H(0,1)) = \emptyset$, and let x_2 be one of the elements being mapped to Tx_2 under T.
Now recursively we let $Tx_n \in T(B_H(0,1))$ such that $Tx_n \cap (cup_{i=1}^{n-1} B_Y(Tx_i, \varepsilon)) = \emptyset$ and x_n be one of the elements being mapped to Tx_n under T.
Then $\|Tx_n - Tx_m\| \geq \varepsilon$ for all $n \neq m$.
I unfortunately couldnt manage to get farther than this.

d)

Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

Let $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ and let $(x_n)_{n\geq 1}$ converge weakly to some $x \in \ell_2(\mathbb{N})$. Then a) tells us that $Tx_n \to Tx$ weakly, and by c) if $\|Tx_n - Tx\| \to 0$ as $n \to \infty$ for all such $(x_n)_{n\geq 1}$, then T will be compact.
Now since $(Tx_n)_{n\geq 1} \in \ell_1$ it will also converge in norm, by remark 5.3 hence we are done.

e)

Show that no $T \in \mathcal{K}(X,Y)$ is onto.

Assume for contradiction that $T \in \mathcal{K}(X,Y)$ is onto. The open mapping theorem then tells us that T is open. This tells us that $T(B_X(0,1))$ is open since $B_X(0,1)$ is open in X. By page 18 in the notes, we have that there exists some $r > 0$ such that

$$B_Y(0,r) \subset T(B_X(0,1))$$

Hence

$$\overline{B_Y(0,r)} \subset \overline{T(B_X(0,1))}$$

since closures preserve inclusion.
Recall that T is compact, hence $\overline{T(B_X(0,1))}$ is compact while $\overline{B_Y(0,r)}$ is compact, since it is a closed subset of a compact set.
We now consider different values of r and see if we can find a contradiction in each case.
For $r = 1$ we have that $\overline{B_Y(0,r)} = \overline{B_Y(0,1)}$ which is never compact.
For $r > 1$ we have $\overline{B_Y(0,1)} \subset \overline{B_Y(0,r)}$ which would make $\overline{B_Y(0,1)}$ compact. However this is never compact by Mandatory 1 Problem 3 e).
For $r < 1$ consider the map $f : Y \to Y$ given by $f(x) = \frac{x}{r}$, which is continuous. We claim that we can scale the open unit ball by some $r > 0$.

$$rB(0,1) = B(0,r)$$

Assume that $x \in rB(0,1)$ then there exists $x' \in B(0,1)$ such that $x = rx'$ hence

$$\|x\| = \|rx'\| < r$$

Thus $x \in B(0,r)$.
For the other inclusion note that if $x \in B(0,r)$ then $x = r\frac{x}{r}$ and

$$\left\|\frac{x}{r}\right\| < \frac{r}{r} = 1$$

so $\frac{x}{r} \in B(0,1)$ hence $x \in rB(0,1)$. So now

$$f(\overline{B_Y(0,r)}) = \frac{1}{r}\overline{B_Y(0,1)} = \overline{\frac{1}{r}B_Y(0,1)}$$

which is compact since f is continuous and $\overline{B_Y(0,r)}$ is compact. However, by the same argument as before, this is not compact.
We conclude that no $T \in \mathcal{K}(X,Y)$ is onto.

f)

Let $H = L_2([0,1], m)$ and consider the operator $M \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ given by $Mf(t) = tf(t)$ for $f \in H$ and $t \in [0,1]$. Justify that M is self-adjoint but not compact.

The following calculation shows that M is self-adjoint.

$$\langle Mf(t), g(t)\rangle = \langle tf(t), g(t)\rangle = t\langle f(t), g(t)\rangle = \langle f(t), tg(t)\rangle = \langle f(t), Mg(t)\rangle$$

Now assume for contradiction that M is compact.
Since L_2 is separable by Homework 4 problem 4 we can use the spectral theorem 10.1. This tells us that H has an ONB that consists of eigenvectors for M, but by Homework 6, we know that M has no eigenvalues, therefore it has no eigenvectors.
We conclude that M is not compact.

Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$. Define $K : [0,1] \times [0,1] \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1 \\ (1-t)s, & \text{if } 0 \le s \le t \le 1 \end{cases}$$

and consider $T \in \mathcal{L}(H, H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t) f(t) dm(t), \quad s \in [0,1],\ f \in H$$

a)

Justify that T is compact.

We know that $[0,1]$ is compact and Hausdorff, and that m is a finite borel-measure. It then follows from theorem 9.6 that T is compact.

b)

Show that $T = T^*$

This follows from the following calculation, using that $K(s,t) = K(t,s)$

$$\begin{aligned}
\langle f, Tg \rangle &= \int_{[0,1]} f(s)(Tg)(s) dm(s) \\
&= \int_{[0,1]} f(s) \left(\int_{[0,1]} K(t,s) g(t) dm(t) \right) dm(s) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(t,s) g(t) f(s) dm(t) \right) dm(s) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(s,t) g(t) f(s) dm(s) \right) dm(t) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(s,t) f(s) dm(s) \right) g(t) dm(t) \\
&= \int_{[0,1]} (Tf)(s) g(t) dm(t) \\
&= \langle Tf, g \rangle
\end{aligned}$$

where we used the Fubini-Tonelli theorem. This is possible since

$$\begin{aligned}
\int_{[0,1]\times[0,1]} |K(s,t)g(t)f(s)|d(s,t) &= \int_{[0,1]} \left(\int_{[0,1]} |K(s,t)g(t)f(s)|dm(s) \right) dm(t) \\
&= \int_{[0,1]} \left(\int_{[0,1]} |K(s,t)||g(t)||f(s)|dm(s) \right) dm(t) \\
&\leq \int_{[0,1]} \left(\int_{[0,1]} |g(t)||f(s)|dm(s) \right) dm(t) \\
&= \int_{[0,1]} |g(t)| \left(\int_{[0,1]} |f(s)|dm(s) \right) dm(t) \\
&\leq \int_{[0,1]} |g(t)|K dm(t) \leq KK' < \infty
\end{aligned}$$

Where we used that $|K(s,t) \leq 1|$ and that $f, g \in L_2([0,1], m) \subset L_1([0,1], m)$.

c)

Show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t), \quad s \in [0,1], f \in H$$

Use this to show that Tf is continuous on $[0,1]$ and that $(Tf)(0) = (Tf)(1) = 0$.

By using the definition of $K(s,t)$ we get that

$$\begin{aligned}
(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)
\end{aligned}$$

since the first term is exactly when $0 \leq t \leq s$ and the second term is when $s \leq t \leq 1$.
It then follows that Tf is bounded since $L_2 \subset L_1$

$$\begin{aligned}
(Tf)(s) &= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t) \\
&\leq \int_{[0,s]} f(t)dm(t) + \int_{[s,1]} f(t)dm(t) \\
&= \int_{[0,1]} f(t)dm(t) = \|f\|_1 < \infty
\end{aligned}$$

Finally we have that

$$\begin{aligned}
(Tf)(0) &= \int_{[0,0]} (1-0)tf(t)dm(t) + \int_{[0,1]} (1-t)\cdot 0 \cdot f(t)dm(t) \\
&= \int_{[0,1]} (1-1)tf(t)dm(t) + \int_{[1,1]} (1-t)\cdot 1 \cdot f(t)dm(t) \\
&= (Tf)(1) \\
&= 0+0=0
\end{aligned}$$

Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

a)

We start by justifying that $g_k \in \mathscr{S}(\mathbb{R})$.
First of all $g_k \in C^\infty(\mathbb{R})$ for every $k = 0,1,2,3$ since it is composed of infinitely differentiable functions. Next we check the definition of being a Schwartz function.

$$x^\beta \partial^\alpha (x^k e^{-\frac{1}{2}x^2}) = x^\beta (e^{-\frac{1}{2}x^2} \cdot Pol_{|k|}(x)) = e^{-\frac{1}{2}x^2} \cdot Pol_{|k|+|\beta|} \to 0 \text{ for } \|x\| \to \infty$$

where $Pol_{|k|}$ denotes a polynomial of degree k.
Next we compute $\mathcal{F}(g_k)$ for $k = 0,1,2,3$

$$\mathcal{F}(g_0) = \mathcal{F}(e^{-\frac{1}{2}x^2}) = \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} e^{-ix\xi} dm(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} e^{-ix\xi} d(x) = e^{-\frac{1}{2}\xi^2}$$

Where the final equality is a calculation done in the proof of 11.4 in the notes. Now in order to find $\mathcal{F}(g_k)$ for $k > 0$ we need to use proposition 11.13 d), which states that the Fourier transform $\mathcal{F}(x^k f) = i^k(\partial \hat{f})$. This is possible since each g_k is a Schwartz function, and each $x^k \in C^\infty(\mathbb{R})$. Hence the Fourier tranforms are given as

$$\begin{aligned}
&\mathcal{F}(g_1) = \mathcal{F}(xe^{-\frac{1}{2}x^2}) = i(-\xi e^{-\frac{1}{2}\xi^2}) = -i\xi e^{-\frac{1}{2}\xi^2} \\
&\mathcal{F}(g_2) = \mathcal{F}(x^2 e^{-\frac{1}{2}x^2}) = i(-ie^{-\frac{1}{2}\xi^2} + i\xi^2 e^{-\frac{1}{2}\xi^2}) = (1-\xi^2)e^{-\frac{1}{2}\xi^2} \\
&\mathcal{F}(g_3) = \mathcal{F}(x^3 e^{-\frac{1}{2}x^2}) = i\left((\xi^2-1)\xi e^{-\frac{1}{2}\xi^2} + 2\xi(-e^{-\frac{1}{2}\xi^2})\right) = i(\xi^3 - 3\xi)e^{-\frac{1}{2}\xi^2}
\end{aligned}$$

b)

Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$ for $k = 0, 1, 2, 3$.
First we need to find h_0 such that $\mathcal{F}(h_0) = h_0$
Let $h_0 = g_0 = e^{-\frac{1}{2}x^2}$ then

$$\mathcal{F}(h_0) = \mathcal{F}(g_0) = e^{-\frac{1}{2}\xi^2} = h_0$$

Next we need to find h_1 such that $\mathcal{F}(h_1) = ih_1$
Let $h_1 = 2g_3 - 3g_1 = (2x^3 - 3x)e^{-\frac{1}{2}x^2}$, then

$$\begin{aligned}
\mathcal{F}(h_1) &= \mathcal{F}(2g_3 - 3g_1) \\
&= 2\mathcal{F}(g_3) - 3\mathcal{F}(g_1) \\
&= 2i(x^3 - 3x)e^{-\frac{1}{2}x^2} - 3(-ixe^{-\frac{1}{2}x^2}) \\
&= i(2x^3 - 6x)e^{-\frac{1}{2}x^2} + 3ixe^{-\frac{1}{2}x^2} \\
&= i(2x^3 - 3x)e^{-\frac{1}{2}x^2} = ih_1
\end{aligned}$$

Next we need to find h_2 such that $\mathcal{F}(h_2) = -h_2$
Let $h_2 = 2g_2 - g_0 = (2x^2 - 1)e^{-\frac{1}{2}x^2}$, then

$$\begin{aligned}
\mathcal{F}(h_2) &= \mathcal{F}(2g_2 - g_0) \\
&= 2\mathcal{F}(g_2) - \mathcal{F}(g_0) \\
&= 2(1 - x^2)e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}x^2} \\
&= -(2x^2 - 1)e^{\frac{1}{2}x^2} \\
&= -h_2
\end{aligned}$$

Lastly we need to find h_3 such that $\mathcal{F}(h_3) = -ih_3$
Let $h_3 = g_1 = xe^{-\frac{1}{2}x^2}$, then

$$\mathcal{F}(h_3) = \mathcal{F}(g_1) = -ixe^{-\frac{1}{2}x^2} = -ih_3$$

c)

Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$.

Denote by $\check{f}$ the inverse Fourier transform as given in the notes. Then

$$\begin{aligned}
\mathcal{F}^2(f) &= \mathcal{F}(\mathcal{F}(f)) \\
&= \mathcal{F}(\hat{f}) \\
&= \int_{\mathbb{R}} \hat{f}(y) e^{-ixy} dm(y) \\
&= \check{\hat{f}}(-x) \\
&= f(-x)
\end{aligned}$$

Since

$$\check{f}(-x) = \int_{\mathbb{R}} f(y) e^{-ixy} dm(y).$$

and

$$\check{\hat{f}}(-x) = f(-x)$$

by 12.12, since $f \in \mathscr{S}(\mathbb{R})$.

d)

Show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, -1, i, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ are precisely $\{1, -1, i, -i\}$.
Let $f \in \mathscr{S}(\mathbb{R})$ non-zero and $\mathcal{F}(f) = \lambda f$. Then

$$\mathcal{F}(\mathcal{F}(f)) = \mathcal{F}(\lambda f) = \lambda \mathcal{F}(f) = \lambda^2 f \Rightarrow F^4(f) = \lambda^4 f = \mathcal{F}(f) = \lambda f \Rightarrow \lambda^4 = \lambda$$

The only $\lambda \in \mathbb{C}$ that fullfill this are $\lambda = \{1, -1, i, -i\}$.
Remember that $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{F}$ if $\mathcal{F}(f) = \lambda f$. But if λ is an eigenvalue, then $\mathcal{F}(f) = \lambda f = \mathcal{F}^4(f) = \lambda^4 f$, hence as we just argued, λ has to be in the set $\{1, -1, i, -i\}$.

Problem 5

Let $(x_n)_{n \geq 1}$ be a dense subset of $[0, 1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}$ on $[0, 1]$. Show that supp(μ)=[0, 1].

By Homework 8 problem 3, the support of μ is defined to be the union of all subset $U \subset [0, 1]$ such that $\mu(U) = 0$.
We notice that since $(x_n)_{n \geq 1}$ is dense in $[0, 1]$ we have that $\mu(U) = 0$ for no $U \in [0, 1]$, hence $N = \emptyset$. But that means that

$$\text{supp}(\mu) = N^{\mathsf{C}} = \emptyset^{\mathsf{C}} = [0, 1]$$