Mandatory Assignment 1, Functional Analysis

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Problem 1

Part (a)

We first show that $\|\cdot\|_0$ satisfies Lecture notes **definition 1.1** for being a norm, using that $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms and that $T : X \longrightarrow Y$ is a linear map. Using **Definition 1.1(a)** for $\|\cdot\|_X$ and $\|\cdot\|_Y$ we see for all $x, y \in X$ that

$$\begin{aligned}\|x+y\|_0 &= \|x+y\|_X + \|T(x+y)\|_Y = \|x+y\|_X + \|T(x)+T(y))\|_Y \\ &\leq \|x\|_X + \|y\|_X + \|T(x)\|_Y + \|T(y)\|_Y \leq \|x\|_0 + \|y\|_0\end{aligned}$$

Again using linearity of T and **Definition 1.1(b)** for the two norms, we see that for all $\alpha \in \mathbb{K}$ and all $x \in X$ we have that

$$\|\alpha x\|_0 = \|\alpha x\|_x + \|T(\alpha x)\|_Y = \|\alpha x\|_x + \|\alpha T(x)\|_Y = |\alpha|\,\|x\|_X + |\alpha|\,\|T(x)\|_Y = |\alpha|\,\|x\|_0$$

Lastly using linearity of T, so $T(0) = 0$, and **Definition 1.1(c)** for the two norms, we find that

$$\|0\|_0 = \|0\|_X + \|T(0)\|_Y = \|0\|_X + \|0\|_Y = 0$$

If $\|x\|_0 = 0$ we see that since $\|x\|_X > 0$ if and only if $x \neq 0$ and $\|T(x)\| \geq 0$ for all $x \in X$ we can therefore conclude $x = 0$ and then we have proved that $\|x\|_0 = 0$ if and only if $x = 0$. We have now showed that $\|\cdot\|_0$ is a norm.
Assume that $\|\cdot\|_X$ and $\|\cdot\|_0$ are equvalent. Then there per **Definition 1.4** exit $0 < C_1 \leq C_2 < \infty$ such that $C_1\|x\|_X \leq \|x\|_0 \leq C_2\|x\|_X$, holds for all $x \in X$. We can therefore see that

$$\begin{aligned}\|x\|_X + \|T(x)\|_Y = \|x\|_0 \leq C_2\|x\|_X \Rightarrow \\ \|T(x)\|_Y \leq (C_2 - 1)\|x\|_X\end{aligned}$$

Which means that T satisfies **Proposition 1.10(3)**, so that means T is bounded. Assume now that T is bounded, then per **Proposition 1.10(c)** we have that there exit a $C > 0$ such that $\|T(x)\|_Y \leq C\|x\|_X$ for all $x \in X$. Therefore we get that

$$\|x\|_X \leq \|x\|_X + \|T(x)\|_Y \leq \|x\|_X + C\|x\|_X = (C+1)\|x\|_X$$

So we have that $\|\cdot\|_X$ and $\|\cdot\|_0$ er equivalent.

Part (b)

For a given linear map, $T : X \longrightarrow Y$, we have that since X is finite dimensional we can use **Theorem 1.6** to say that any two norm on X, in particular $\|\cdot\|_X$ and $\|\cdot\|_0$, are equivalent. Then it follows from **Mandatory problem 1(a)** that T is bounded.

I has infinitely many elements

Part (c)

For a X that is infinite dimensional we know that there exits a normalized Hamel basis, $(e_i)_{i\in I}$ with $\|e_i\|_X = 1$ for all $i \in I$. We also know that I has infinite elements, which means that $\text{card}(I) \geq \text{card}(\mathbb{N})$. Hence there exits a surjektiv funciton, $f : I \longrightarrow \mathbb{N}$. Since Y is a non-zero normed vector space, choose $0 \neq y \in Y$ and let $y_i = f(i)y$ for all $i \in I$. We now have (from the hint) that there exits precisely one linear map $T : X \longrightarrow Y$ such that $T(e_i) = y_i$ for all $i \in I$. For a given $C > 0$ let $\lceil \frac{C+1}{\|y\|} \rceil = N_C \in \mathbb{N}$ then since f is surjective there exits a $i_0 \in I$ such that $f(i_0) = N_C$. We now have that

$$C \|e_{i_0}\|_X = C < N_C \cdot \|y\|_Y = \|N_C y\|_Y = \|f(i_0)y\| = \|T(e_{i_0})\|$$

Hence the linear map $T : X \longrightarrow Y$ cannot be bounded by any constant $C > 0$.

Part (d)

let $Y = \mathbb{K}$ (so either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let $\|\cdot\|_Y = |\cdot|$(modulus). Then we have a non-zero normed vector space over $\mathbb{K}$. Define $T : X \longrightarrow \mathbb{K}$ using **Mandatory problem 1(c)** so T is a linear map that is not bounded. Let $\|x\|_0 = \|x\|_X + \|T(x)\|_Y$ (a norm per Mandatory problem 1(a)). We have trivally that $\|x\|_X \leq \|x\|_0$ for all $x \in X$. Since T is not bounded it follows that for all $C > 0$ there exits a $x \in X$ such that $\|T(x)\|_Y > C \|x\|_X$, so therefore we have that the two norms cannot be equivalent. If $(X, \|\cdot\|_X)$ is a Banach space then it follows from contraposition of **Homework week 3, problem 1** that $(X, \|\cdot\|_0)$ cannot be complete.

Part (e)

Take $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|_1)$ and let $\|\cdot\|' = \|\cdot\|_\infty = \sup\{|x(k)| : k \geq 1\}$. Then we know from Riesz-fischer completeness theorem (**Schilling, Theorem 13.7**) that $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is complete. We also note that $\|x\|_\infty \leq \|x\|_1$ for all $x \in \ell_1(\mathbb{N})$ and therefore we also have that $\|x\|_\infty < \infty$ for all $x \in (\ell_1(\mathbb{N})$, so it is a norm on $(\ell_1(\mathbb{N})$. Consider the sequence in $(x_n)_{n\in\mathbb{N}} \subset (\ell_1(\mathbb{N})$ given by $x_n = \left(\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots\right)$. Then for all $\epsilon > 0$ we can choose $N_\epsilon = \lceil \frac{1}{\epsilon} \rceil$ such that for all $n, m \geq N_\epsilon$ and

$$\|x_n - x_m\|_\infty = \sup\{|x_n(k) - x_m(k)| : k \geq 1\} = \frac{1}{\min\{m, n\} + 1} \leq \frac{1}{N_\epsilon + 1} < \epsilon$$

So we have that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\|\cdot\|_\infty$. But we also know that $x_n \to x$ where $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ and [in $\|\cdot\|_\infty$

where does x live?

$$\|x\|_1 = \sum_{k=1}^{\infty} |x(k)| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

So we have that $x \notin \ell_1(\mathbb{N})$ and the sequence can therefore not converges in $\ell_1(\mathbb{N})$. We can now conclude that $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete.

Problem 2

Part (a)

We first note that $f : M \longrightarrow \mathbb{C}$ is a linear map and the norm on $\mathbb{C}$ is modulus. We have that since $|\cdot|$ satisfies the triangle inequality we have for all $m = (a, b, 0, 0, \ldots) \in M$

$$|f(m)|^p = |a + b|^p \leq (|a| + |b|)^p \leq (2\max\{|a|, |b|\})^p \leq 2^p (|a|^p + |b|^p) = 2^p \|m\|_p^p$$

So $|f(m)| \leq 2\|m\|_p$ for all $m \in M$ and therefore f is bounded. Let $m = (a, b, 0, 0, \ldots) \in M \subset \ell_p(\mathbb{N})$ and let $y = (1, 1, 0, 0, \ldots) \in \ell_q(\mathbb{N})$ where $q = \frac{p-1}{p}$ for $p > 1$ and $q = \infty$ for $p = 1$. Then $q = \frac{p}{p-1}$. we have that $f_y(m) = \sum_{k=1}^{\infty} y(k)m(k) = m(1) + m(2) = a + b = f(m)$ for all $m \in M$. Using Hölders inequality (**Schilling, Theorem 13.2**) we have that

$$|f(m)| = |f_y(m)| = |\sum_{k=1}^{\infty} y(k)m(k)| \leq \|m\|_p \|y\|_q$$

Be more explicit in that the calculations only hold for $p>1$, but the formula holds for $p\geq 1$

Where we have that $\|y\|_q = (\sum_{n=1}^{\infty} |y_n|^q)^{\frac{1}{q}} = (1+1)^{\frac{1}{q}} = 2^{\frac{p-1}{p}}$, which holds for all $p \in [1,\infty)$, since for $p=1$ we have that $\|y\|_\infty = 1 = 2^0$. We now see per **Remark 1.11** that

$$\|f\| = \sup\left\{|f(m)| : \|m\|_p \leq 1\right\} \leq \sup\left\{\|m\|_p \|y\|_q \,:\, \|m\|_p \leq 1\right\} \leq \|y\|_q = 2^{\frac{p-1}{p}}$$

For all $p \in [1,\infty)$ we have that if $m_1 = \left((\frac{1}{2})^{\frac{1}{p}}, (\frac{1}{2})^{\frac{1}{p}}, 0, 0, \dots\right)$ then we see that $\|m_1\|_p = \left(\left((\frac{1}{2})^{\frac{1}{p}}\right)^p + \left((\frac{1}{2})^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}} = (\frac{1}{2}+\frac{1}{2})^{\frac{1}{p}} = 1$. We calculate that $|f(m_1)| = |(\frac{1}{2})^{\frac{1}{p}} + (\frac{1}{2})^{\frac{1}{p}}| = 2(\frac{1}{2})^{\frac{1}{p}} = 2^{\frac{p-1}{p}}$. Since $\|f\| = \sup\left\{|f(m)|\,;\, \|m\|_p \leq 1\right\} \geq |f(m_1)|$ we can therefore conclude that $\|f\| \geq 2^{\frac{p-1}{p}}$. Combining this with the other inequality we have that $\|f\| = 2^{\frac{p-1}{p}}$ for all $p \in [1,\infty)$. ✓

Part (b)

Using **Homework week 1, problem 5** we have that for all $p \in (1,\infty)$ that $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$ where $q = \frac{p}{p-1}$. We also have from **Homework week 1, problem 5** that all linear functionals $F \in (\ell_p(\mathbb{N}))^*$ can be given by $F(y) = f_x(y) = \sum_{k=1}^{\infty} x(k)y(k)$ for all $y \in \ell_p(\mathbb{N})$, where $x \in \ell_q(\mathbb{N})$. with $\|F\|=\|x\|_q$.

We want to show that $f_x|_M = f$ if and only if $x(1)=1$ and $x(2)=1$. If $x(1)=1$ and $x(2)=1$ we have that for all $m = (a,b,0,0,\dots) \in M$ that $f_x(m) = \sum_{k=1}^{\infty} x(k)m(k) = x(1)m(1) + x(2)m(2) = a+b = f(m)$ and therefore $f_x|_M = f$. If $x(1) \neq 1$ choose $m_1 = (1,0,0,\dots) \in M$, then we have that $f_x(m_1) = \sum_{k=1}^{\infty} x(k)m_1(k) = x(1) \neq 1 = f(m_1)$ and hence $f_x|M \neq f$. The argument for $x(2) \neq 1$ follows with the same idea where $m_2 = (0,1,0,0\dots) = M$ and then $f_x(m_2) = x(2) \neq 1 = f(m_2)$.
If we look at $x_1 = (1,1,0,0,\dots) \in \ell_q(\mathbb{N})$ we must have that $f_{x_1}|_M = f$ and that $\|x_1\|_q = (|1|^q + |1|^q)^{1/q} = 2^{\frac{p-1}{p}}$. Since $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$ we have that $\|f_{x_1}\| = \|x_1\|_q = 2^{\frac{p-1}{p}} = \|f\|$. Hence the linear functional f_{x_1} on $\ell_p(\mathbb{N})$ is a extension of f and satisfies $\|f_{x_1}\| = \|f\|$. Let $x_2 \in \ell_q(\mathbb{N})$ such that $f_{x_2}|_M = f$ and assume that $x_2 \neq x_1$. Then we have that $x_2(1) = 1$ and $x_2(2) = 1$ but there must exits a $n \in \mathbb{N}$ such that $x_2(n) \neq 0 = x_1(n)$, otherwise $x_1 = x_2$. Hence we have that

$$\|f\| = 2^{\frac{p-1}{p}} = 2^{\frac{1}{q}} < (1+1+|x(n)|^q)^{\frac{1}{q}} \leq \left(\sum_{k=1}^{\infty} |x_2(k)|^q\right)^{\frac{1}{q}} = \|x\|_q = \|f_{x_2}\|$$

Therefore any linear functional F on $\ell_p(\mathbb{N})$ extending f that is different form f_{x_1} has $\|F\| > \|f\|$ and we now conclude that f_{x_1} must be the unique extension where $\|f_{x_1}\| = \|f\|$. ✓

Part (c)

and, again, $\|F\|=\|x\|_q$.

Again using **Homework week 1, problem 5** we have that $(\ell_1(\mathbb{N}))^* \cong \ell_\infty(\mathbb{N})$ and again we use that all linear functionals $F \in (\ell_1(\mathbb{N}))^*$ can be given by $F(y) = f_x(y) = \sum_{k=1}^{\infty} x(k)y(k)$ for all $y \in \ell_1(\mathbb{N})$, where $x \in \ell_\infty(\mathbb{N})$. Let $x_n \in \ell_\infty(\mathbb{N})$ be given by $x_n = (1,1,\dots,1,0,0,\dots)$ where the first n places are ones and the rest are zero. Then we have that $\|x_n\|_\infty = 1$ for all $n \in \mathbb{N}$. For $n \geq 2$ we see that for any $m = (a,b,0,0,\dots) \in M$ we get $f_{x_n}(m) = \sum_{k=1}^{\infty} x_n(k)m(k) = a+b = f(m)$. Since we have that $(\ell_1(\mathbb{N}))^* \cong \ell_\infty(\mathbb{N})$ we get that for all $n \geq 2$ we find that $\|f_{x_n}\| = \|x_n\|_\infty = 1 = \|f\|$ and since $f_{x_n}|_M = f$ we have infinitely many linear functionals f_{x_n} on $\ell_1(\mathbb{N})$ extending f and satisfying $\|f_{x_n}\| = \|f\|$. ✓

Problem 3

Part (a)

messy Notation

Let $(e_i)_{i\in I}$ be a hamel basis for X. We have that $\text{card}(\{1,2,\ldots,n+1\}) \leq \text{card}(I)$ and hence there exits a injective map $F : \{1,2,\ldots,n+1\} \longrightarrow I$. Now define the subset $\{e_1, e_2, \ldots, e_{n+1}\} \subset (e_i)_{i\in I}$ by $e_k = e_{f(k)}$ for $k = 1,2,\ldots,n+1$. We now set $\text{span}\{e_1, e_2, \ldots, e_n, e_{n+1}\} = X_{n+1} \subset X$ so let $F_{n+1} : X_{n+1} \longrightarrow \mathbb{K}^n$ be the restriction of F to the set X_{n+1}. Then F_{n+1} is also a linear map and it holds that $\ker(F_{n+1}) \subset \ker(F)$. Using results from basic linear algebra we have that $n+1 = \dim(X_{n+1})) = \dim(\ker(F_{n+1})) + \dim(F_{n+1}(X_{n+1}))$. Since $\dim(F_{n+1}(X_{n+1})) \leq \dim(\mathbb{K}^n) = n$ this means that $\dim(\ker(Fn+1)) \geq 1$ so especially $\ker(F_{n+1}) \neq \{0\}$ and therefore $\ker(F) \neq \{0\}$. We know that a linear map, F, is injective if and only if $\ker(F) = \{0\}$, which means that F cannot be injective.

Part (b)

For a given $n \in \mathbb{N}$ define $F : X \longrightarrow \mathbb{K}^n$ by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ for all $x \in X$. We have that F is linear since we can use linearify of the f_i's to get that for all $\alpha, \beta \in \mathbb{K}$ and all $x, y \in X$:

$$\begin{aligned} F(\alpha x + \beta y) &= (f_1(\alpha x + \beta y), \ldots, f_n(\alpha x + \beta y)) = (\alpha f_1(x) + \beta f_1(y), \ldots, \alpha f_n(x) + \beta f_n(x)) \\ &= \alpha\,(f_1(x), \ldots, f_n(x)) + \beta\,(f_1(y), \ldots, f_n(y)) = \alpha F(x) + \beta F(y) \end{aligned}$$

It now follows from **Mandatory problem 3(a)** that F cannot be injective which for a linear map is equivalent to $\ker(F) \neq \{0\}$. Since $F(x) = 0$ if and only if $f_i(x) = 0$ for all $i = 1, \ldots, n$ we have that $\cap_{j=1}^n \ker(f_j) = \ker(F)$ and therefore $\cap_{j=1}^n \ker(f_j) \neq \{0\}$.

Part (c)

If $x_i = 0$ then it is trivial that $= \|y - x_i\| = \|y\| \geq 0 = \|x_i\|$, so assume that $x_1, \ldots, x_n \in X$ are all different from 0. Then it follows from **Theorem 2.7(b)** that for all $x_j, j = 1,2,\ldots,n$ there exits $f_j \in X^*$ such that $\|f_j\| = 1$ and $f_j(x_j) = \|x_j\|$. Since per **Mandatory problem 3(b)** $\cap_{j=1}^n \ker(f_j) \neq \{0\}$, we can find $0 \neq y' \in \cap_{j=1}^n \ker(f_j) = \ker(F)$. Since $\ker(F)$ is a subspace we can normalize y' and still be in $\ker(F)$, so let $y = \frac{y'}{\|y'\|} \in \cap_{j=1}^n \ker(f_j) = \ker(F)$ with $\|y\| = 1$ We now have for all $j = 1,2,\ldots,n$

$$\|x_j\| = f_j(x_j) = f_j(x_j) - 0 = f_j(x_j) - f_j(y) = f_j(x_j - y) \leq \|f_j\| \cdot \|x_j - y\| = \|y - x_j\|$$

Where we used linearity of f_j, that $y \in \ker(f_j)$ and the properties of f_j that $f_j(x_j) = \|x_j\|$ and $\|f_j\| = 1$

Part(d)

Given a finite familiy of closed ball $\overline{B(x_j, r_j)}$ index form $j = 1,2,\ldots,n$ with center x_j and radius r_j and where none of them contain 0. This means that $r_j < \|x_j\|$ because otherwise $\|x_j - 0\| = \|x_j\| \leq r_j$ and then $0 \in \overline{B(x_j, r_j)}$. Since x_j cannot be equal to 0 for any $j = 1, \ldots, n$, let y be from **Mandatory problem 3(c)**. Then we have that $\|y\| = 1$ so $y \in S = \{x \in X : \|x\| = 1\}$, but for all $j = 1,2,\ldots,n$ we have that $r_j < \|x_j\| \leq \|y - x_j\|$ so $y \notin \overline{B(x_j, r_j)}$ and therefore $S = \{x \in X : \|x\| = 1\} \not\subset \cup_{j=1}^n \overline{B(x_j, r_j)}$

Part (e)

Assume for contradiction that S is compact. Let $B(x, r = \frac{\|x\|}{2})$ be the ball centered at x with radius $\frac{\|x\|}{2}$, then $0 \notin \overline{B(x, \frac{\|x\|}{2})}$ if $x \neq 0$. We have that $S \subset \cup_{x \in S} B(x, \frac{\|x\|}{2})$ so from assumption of compactness we have that there exits a finite set, $A \subset S$, such that $S \subset \cup_{x \in A} B(x, \frac{\|x\|}{2})$ (**Folland, Section 4.4**). Then since $B(x, \frac{\|x\|}{2}) \subset \overline{B(x, \frac{\|x\|}{2})}$ we have that $S \subset \cup_{x \in A} B(x, \frac{\|x\|}{2}) \subset \cup_{x \in A} \overline{B(x, \frac{\|x\|}{2})}$. But we have that $0 \notin \overline{B(x, r = \frac{\|x\|}{2})}$ for all $x \in S$ so especially for all $x \in A \subset S$. Then we have a contradiction with **Mandatory problem 3(d)** and therefore S is non-compact. Since S is a closed set of the closed unit ball in X it follows from contraposition of **Folling, proposition 4.22** that the closed unit ball cannot be compact.

Problem 4

Part (a)

why?

For a given $n \in \mathbb{N}$ let $f(x) = x^{\frac{-1}{3}} 1_{(0,1]}$. Then we have that $f \in L_1([0,1], m)$ since $\int_{[0,1]} |f| dm =$ justify $\int_0^1 x^{\frac{-1}{3}} dx = \frac{3}{2} < \infty$ but for all $t > 0$ we have that $\int_{[0,1]} |t^{-1} f|^3 dm = t^{-3} \int_0^1 x^{-1} d(x) = \infty$ and Lebesgue → improper Riemann

hence $t^{-1} f \notin E_n$ for all $n \in \mathbb{N}$. It now follows from the notes that E_n cannot be absorbing even if it is convex.

Part(b)

We are going to proof it by contradiction. So for all $n \in \mathbb{N}$ assume that $(E_n)^\circ \neq \emptyset$ then there exits a $f \in (E_n)^\circ$ so we can construct a open ball around f with radius $\epsilon > 0$ contanied in E_n, so $B(f, \epsilon) = \{g \in L_1([0,1], m) : \|f - g\|_1 < \epsilon\} \subset E_n$. For a given $g \in L_1([0,1], m)$ different from 0 we have that $f + \frac{g}{\|g\|_1} \frac{\epsilon}{2} \in B(f, \epsilon)$ since $\left\| f - \left(f + \frac{g}{\|g\|_1} \frac{\epsilon}{2} \right) \right\|_1 = \frac{\epsilon}{2} \left\| \frac{g}{\|g\|_1} \right\|_1 = \frac{\epsilon}{2} < \epsilon$. So since $B(f, \epsilon) \subset E_n \subset L_3([0,1], m)$ we can use linearity of $L_3([0,1], m)$ to get that $f + \frac{g}{\|g\|_1} \frac{\epsilon}{2} - f = \frac{g}{\|g\|_1} \frac{\epsilon}{2} \in L_3([0,1], m)$ and then $\frac{g}{\|g\|_1} \frac{\epsilon}{2} \cdot \frac{2}{\epsilon} \|g\|_1 = g \in L_3([0,1], m)$ so we get that $L_1([0,1], m) \subset L_3([0,1], m)$ which is a contradiction with **Homework week 2, problem 2(b)** and therefore $(E_n)^\circ = \emptyset$ for all $n \in \mathbb{N}$.

Part (c)

For at given $n \in \mathbb{N}$ let $(f_k)_{k \in \mathbb{N}}$ be a given convergent sequence in E_n, $f_k \to f$ we have that since E_n is a subset of $L_1([0,1], m)$ this means that $f_k \xrightarrow{L_1} f$. We now know (**Schilling, Corollary 13.8**) that there exits a subsequence $\left(f_{k_p}\right)_{p \in \mathbb{N}}$ such that $f_{k_p}(x) \to f(x)$ m-almost everywhere. Then we also have that $\lim_{p \to \infty} |f_{k_p}(x)|^3 = \limsup_{p \to \infty} |f_{k_p}(x)|3 = |f(x)|^3$ m-almost everywhere. So we get that

$$\int_{[0,1]} |f|^3 dm = \int_{[0,1]} \liminf_{p \to \infty} |f_{k_p}|^3 dm \leq \liminf_{p \to \infty} \int_{[0,1]} |f_{k_p}|^3 dm \leq \liminf_{p \to \infty} n = n$$

Where we used Fatous lemma (**Schilling, Theorem 9.11**) and lastly used that $f_{k_p} \in E_n$. Then we see that $f \in E_n$ for all convergent sequences $f_n \to f$ and hence E_n is closed.

Part (d)

We have trivially that for all $n \in \mathbb{N}$ it holds that $E_n \subset L_3([0,1], m)$, since $n < \infty$. Therefore we get $\cup_{n=1}^{\infty} E_i \subset L_3([0,1])$. For a given $f \in L_3([0,1], m)$ there exits a $C > 0$ such that $\int_{[0,1]} |f|^3 dm \leq C$. Let $N_C = \lceil C \rceil$, then we have that all $f \in E_{N_C}$ and therefore also that

$f \in \cup_{n=1}^{\infty} E_n$. Hence we have that $L_3([0,1],m) \subset \cup_{n=1}^{\infty} E_n$ and therefore $L_3([0,1],m) = \cup_{n=1}^{\infty} E_n$. It now follows from **Definition 3.12(ii)** that since E_n is a sequence of nowhere dense and closed such that $L_3([0,1],m) = \cup_{n=1}^{\infty} E_n$, then $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$

Problem 5

Part (a)

If $x_n \to x$ in norm, as $n \to \infty$ then it holds that $\|x_n - x\| \to 0$. Using the reverse triangle inequality we get that for all $n \in \mathbb{N}$ we have that

$$0 \leq |\, \|x_n\| - \|x\| \,| \leq \|x_n - x\|$$

Since $\|x_n - x\| \to 0$ we get that $|\|x_n\| - \|x\|| \to 0$ which is convergens in $\mathbb{R}$. So we have that $\|x_n\| \to \|x\|$ for $n \to \infty$

Part (b)

A counterexample is to look at an ortohormal basis $(e_n)_{n\in\mathbb{N}}$ for H. We have from **Homework week 2, problem 1** that for all $f \in H^*$ there exits a $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x \in H$. For a given $y \in H$ we have from Bessel's inequality (**Schilling, Theorem 26.19(iii)**) that $\sum_{n=1}^{\infty} |\langle y, e_n\rangle|^2 \leq \|y\|^2 < \infty$. Since it is a convergent sum we must have that $|\langle y, e_n\rangle|^2 \to 0$ for $n \to \infty$, which only happens if $\langle y, e_n\rangle \to 0$, which again is equivalent to $\langle e_n, y\rangle = \overline{\langle y, e_n\rangle} \to 0$. Since it holds for all $y \in H$ that $\langle e_n, y\rangle$ goes to 0, then it holds that $f(e_n)$ converges to $0 = f(0)$ for all $f \in H^*$. We have from **Homework week 4, problem 2(a)** that a sequence x_n converges weakly to x if and only if $f(x_n)$ converges to $f(x)$ for all $f \in H^*$. So we see that e_n convergens weakly to 0, but it does not hold for its norm. Since it is a orthonormal basis we have that $\|e_n\| = 1$ for all $n \in \mathbb{N}$. Which means that $\|e_n\| \to 1 \neq 0 = \|0\|$ concluding the counterexample.

Part (c)

Assume that $x \neq 0$ (since $\|x\| = 0 \leq 1$ if $x = 0$) then it follows from **Theorem 2.7(b)** that there exits $f \in H^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Since we have that x_n converges weakly to x for $n \to \infty$ we know from **Homework week 4, problem 2(a)** that then $f(x_n)$ converges to $f(x)$, so it also holds that $|f(x_n)| \to |f(x)|$. We now see that

$$\|x\| = |\, \|x\| \,| = |f(x)| = \lim_{n\to\infty} |f(x_n)| \leq \lim_{n\to\infty} 1 = 1$$

Where we use that $|f(x_n)| \leq \|f\|\,\|x_n\| \leq 1 \cdot 1 = 1$ for all $n \in \mathbb{N}$