Solutions for Mandatory Assignment 1 for FunkAn 2020

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Problem 1

(a) Show that $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ is a norm on X. Show that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded.

Proof First we show that $\|x\|_0$ is a norm on X.

$$\begin{aligned} \|x+y\|_0 &= \|x+y\|_X + \|T(x+y)\|_Y = \|x+y\|_X + \|Tx+Ty\|_Y \\ &\le \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y = \|x\|_0 + \|y\|_0, \quad x, y \in X \end{aligned} \tag{1}$$

$$\begin{aligned} \|\alpha x\|_0 &= \|\alpha x\|_X + \|T(\alpha x)\|_Y = |\alpha|\|x\|_X + \|\alpha Tx\|_Y \\ &= |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y = |\alpha|\|x\|_0, \quad \alpha \in \mathbb{K}, x \in X \end{aligned} \tag{2}$$

$$\begin{aligned} &\|x\|_0 = 0 \quad \text{if and only if} \quad x = 0 \\ &\text{i.e.} \quad \|0\|_0 = \|0\|_X + \|T0\|_Y = 0 + 0 = 0 \end{aligned} \tag{3}$$

$\|x\|_0$ satisfies all three criteria above. Hence, $\|x\|_0$ is a norm on X.
Next we show that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded.
"$\Rightarrow$": Assume that T is bounded. Then there exists $C > 0, C \in \mathbb{K}$ such that $\|Tx\|_Y \le C\|x\|_X$, for all $x \in X$. We have

$$\|x\|_X \le \|x\|_0 = \|x\|_X + \|Tx\|_Y \le \|x\|_X + C\|x\|_X = |1+C|\|x\|_X$$

Take $C_1 = 1$ and $C_2 = |1+C|$, by definition, the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent since $C_1\|x\|_X \le \|x\|_0 \le C_2\|x\|_X$.
"$\Leftarrow$": Assume that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent. Then there exists $0 < C_1 \le C_2 < \infty, C_1, C_2 \in \mathbb{K}$ such that $C_1\|x\|_X \le \|x\|_0 \le C_2\|x\|_X$, for all $x \in X$. Take $C = C_2$, by definition, T is bounded since $\|Tx\|_Y \le C\|x\|_X$, for all $x \in X$. □

(b) Show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.

Proof Choose a basis $\{e_1, e_2, \ldots, e_n\}$ in X which may be taken to be unit vectors. Then,

$$Tx = \sum_{i=1}^{n} x_i Te_i,$$

and so by the triangle inequality,

$$\|Tx\| = \|\sum_{i=1}^{n} x_i Te_i\| \le \sum_{i=1}^{n} |x_i|\|Te_i\|$$

Letting

$$M = \sup_i \{\|Te_i\|\}$$

and using the fact that

$$\sum_{i=1}^{n} |x_i| \leq C\|x\|$$

for some $C > 0$ which follows from the fact that any two norms on a finite-dimensional space are equivalent, one finds

$$\|Tx\| \leq (\sum_{i=1}^{n} |x_i|)M \leq CM\|x\|$$

Thus, T is a bounded linear operator. $\square$

(c) Suppose that X is infinite dimensional. Show that there exists a linear map $T : X \to Y$, which is not bounded (= not continuous).

Proof Consider a sequence $(a_j)_1^n$ of linearly independent vectors in X. Define a linear functional f:

$$f(a_j) = j\|a_j\|$$

for each $j = 1, 2, \ldots,$ n. Complete this sequence of linearly independent vectors to a vector space basis of X (i.e. a Hamel basis $(e_i)_{i\in I}$ for X.), and define f at the other vectors in the basis to be zero. That is

$$f(e_i) = \begin{cases} i\|e_i\| & \text{if} \quad i \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

f so defined will extend uniquely to a linear map on X and it is not bounded since there exists no C such that $\|fx\| \leq C\|x\|$ for all $x \in X$. Then, define a linear map T:

$$\begin{aligned} T : & X \to Y \\ & x \mapsto f(x)y \end{aligned}$$

where y is an arbitrary nonzero vector in Y. T so defined is not bounded since f is not bounded. $\square$

Another solution (just a discussion, cannot make sure it is right):

Proof Choose an infinite linearly independent set $\{x_n | n \in \mathbb{N}\}$ such that $\|x_n\| = 1$. An infinite linearly independent set exists, since X is infinite-dimensional. Normalizing the vectors does not influence the linear independence. There is a Hamel basis B containing this set.
Then there is a linear map $T : X \to Y$(here $Y = \mathbb{R}$) such that $Tx_n = n$ and $Tb = 0$ for $b \in B\backslash\{x_n | n \in \mathbb{N}\}$. Let $\{r_n\}_n$ be any sequence of rationals which converges to b. Then $\lim_n Tr_n = b$, but $Tb = 0$. We can see that T is unbounded (= not continuous). $\square$

(d) Suppose that X is infinite dimensional. Argue that there exists a norm $\|\cdot\|_0$ on X, which is not equivalent to the given norm $\|\cdot\|_X$, and which satisfies $\|x\|_X \leq \|x\|_0$, for all $x \in X$. Conclude that $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space.

Proof There are many ways to construct such a norm $\|\cdot\|_0$. Here is an example:
Let $(X, \|\cdot\|_X)$ be an infinite dimensional normed space and $T : X \to \mathbb{R}$ be a linear non bounded map for the norm $\|\cdot\|_X$. Define $\|\cdot\|_0 := \|x\|_X + |Tx|$, for all $x \in X$. Then $\|\cdot\|_0$ is a norm which is not equivalent to $\|\cdot\|_X$ since T is not bounded, which we have proved in problem (a). Also, the norm $\|\cdot\|_0$ satisfies $\|x\|_X \leq \|x\|_X + |Tx| = \|x\|_0$, for all $x \in X$.
Next we show that $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space.
Assume that $(X, \|\cdot\|_0)$ were complete. Consider the identity map $I : (X, \|\cdot\|_0) \to (X, \|\cdot\|_X)$. Since $\|Ix\|_X = \|x\|_X \leq \|x\|_0$, we have that I is a continuous linear map. Since $(X, \|\cdot\|_X)$ is a Banach space, I is also surjective. By the open mapping theorem, I is open. Whence, $I^{-1} : (X, \|\cdot\|_X) \to (X, \|\cdot\|_0)$ is continuous. Since I^{-1} is linear, it is also bounded:

$$\|x\|_0 = \|I^{-1}X\|_0 \leq C\|x\|_X$$

for some constant $C > 0$.
Therefore $\frac{1}{C}\|x\|_0 \leq \|x\|_X \leq \|x\|_0$. By definition, the two norms $\|\cdot\|_0$ and $\|\cdot\|_X$ would be equivalent. But this is a contradiction with the fact that they are not. Hence, $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space. □

An Example Here is another example to construct a norm $\|\cdot\|_0$.
Define $T : x_i \mapsto iy \in Y$ for all $\|x_i\| = 1$. Define $\|\cdot\|_0 := \|x\|_X + \|Tx\|_Y$, for all $x \in X$.

(e) Give an example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ satisfying $\|x\|' \leq \|x\|$, for all $x \in X$, such that $(X, \|\cdot\|)$ is complete, while $(X, \|\cdot\|')$ is not.
An Example Take $(X, \|\cdot\|) = (L_2([0,1], m), \|\cdot\|_2)$. Define $\|x\|' := \|\cdot\|_1$. For $f \in L_2([0,1], m)$, by Cauchy-Schwarz:

$$\int_{[0,1]} |f(x)| \mathrm{d}m(x) \leq \sqrt{\int_{[0,1]} |f(x)|^2 \mathrm{d}m(x)}.$$

Note that $(L_2([0,1], m), \|\cdot\|_2)$ is complete. Next we show that $(L_2([0,1], m)$ is not complete under $\|\cdot\|_1$.
Assume that $L_2([0,1], m)$ is complete under $\|\cdot\|_1$. Then by the Banach isomorphism theorem, there would be a constant C such that for any $f \in L_2([0,1], m)$, $\|f\|_2 \leq C\|f\|_1$. In particular, if $f = \chi_A$ (the function that is equal to 1 on A and 0 otherwise) for $A \subset [0,1]$ measurable, we would get $m(A)^{\frac{1}{2}-1} \leq C$, hence

$$\inf_{A : m(A) > 0} m(A) > 0.$$

This condition implies that, together with finiteness of the measure space, that $L_2([0,1],m)$ is finite dimensional. However, $L_2([0,1],m)$ is infinite dimensional. Hence, $(L_2([0,1],m), \|\cdot\|_1)$ is not complete.

Problem 2

(a) Show that f is bounded on $(M, \|\cdot\|_p)$ and compute $\|f\|$.

Proof By Hölder's inequality (which mentioned in HW1_FunkAn20-21.pdf Problem 5),

$$\|fx\| \leq |a| + |b| \leq (|a|^p + |b|^p)^{\frac{1}{p}}(1+1)^{1-\frac{1}{p}} = 2^{(1-\frac{1}{p})}\|x\|_p$$

for $x = (a, b, 0, 0, \ldots)$ and $p \in [1, \infty)$. Here we have $C = 2$ such that:

$$\|fx\| \leq 2^{(1-\frac{1}{p})}\|x\|_p \leq C\|x\|_p$$

for all $x \in M$. Hence f is bounded on $(M, \|\cdot\|_p)$.
Since equality can be attained for $x = (1, 0, 0, 0, \ldots)$ we have

$$\|f\|_p = 2^{(1-\frac{1}{p})}$$

for $p \in [1, \infty)$. And it is trivial that $\|f\|_1 = 1$. $\square$

Another solution (probably the same solution):

Proof By a consequence of Hölder's inequality, in general, for vectors in $\mathbb{C}^n$ where $0 < r < p$:

$$\|x\|_p \leq \|x\|_r \leq n^{(\frac{1}{r}-\frac{1}{p})}\|x\|_p$$

Here we have $C = 2^{(1-\frac{1}{p})}$ such that:

$$\|fx\| = \|x\|_1 \leq C\|x\|_p$$

for all $x \in M$. Hence f is bounded on $(M, \|\cdot\|_p)$.
By Remark 1.11. from Lecture1_FunkAn20-21.pdf, $\|f\| = \sup\{\|fx\| : \|x\| = 1\}$. Let $x = (a, b, 0, 0, \ldots)$ where $a = b = \frac{1}{\sqrt[p]{2}}$ such that $\|x\|_p = 1$. Then

$$\|f\| = \sup_{\|x\|_p=1} |fx| \geq \frac{2}{\sqrt[p]{2}} = 2^{(1-\frac{1}{p})},$$

for $p \in [1, \infty)$. And it is trivial that $\|f\|_1 = 1$. $\square$

(b) Show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Proof Given a generic vector

$$x = (a, b, x_3, x_4, \ldots) = ae_1 + be_2 + x_3e_3 + x_4e_4 + \cdots .$$

From (a), we have $fx \leq 2^{(1-\frac{1}{p})}\|x\|_p$. If $1 < p < \infty$, by Theorem 2.3 from Lecture2_FunkAn20-21.pdf, suppose $F : \ell_p(\mathbb{N}) \to \mathbb{K}$ is a bounded extension of f with

$$\|F\|_p = 2^{1-\frac{1}{p}}.$$

Let $Fe_j = \alpha_j$. We can see that for all $j \geq 3$, by Hölder's inequality, it holds that

$$\begin{aligned}
\|F(ae_1 + be_2 + x_je_j)\| &= |a + b + x_j\alpha_j| \\
&\leq (|a|^p + |b|^p + |x_j|^p)^{\frac{1}{p}}(1 + 1 + |\alpha_j|^{\frac{p}{p-1}})^{1-\frac{1}{p}} \\
&= (2 + |\alpha_j|^{\frac{p}{p-1}})^{1-\frac{1}{p}}\|x\|_p
\end{aligned}$$

Since equality can be attained, we have

$$(2 + |\alpha_j|^{\frac{p}{p-1}})^{1-\frac{1}{p}} \leq \|F\|_p = 2^{1-\frac{1}{p}}, \quad \forall j \geq 3.$$

This shows $\alpha_j = 0$ for all $j \geq 3$, and it follows that

$$Fx = a + b$$

for all $x \in \ell_p(\mathbb{N})$. Thus there is a unique linear functional F as desired. □

(c) Show that if $p = 1$, then there are infinitely many linear functional F on $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Proof Given a generic vector

$$x = (a, b, x_3, x_4, \ldots) = ae_1 + be_2 + x_3e_3 + x_4e_4 + \cdots .$$

If $p = 1$, note that for any $\{\alpha_j\}_{j\geq 3}$ with $\sup_{j\geq 3}|\alpha_j| \leq 1$, $Fe_j = \alpha_j$,

$$Fx = a + b + \sum_{j=3}^{\infty}\alpha_jx_j$$

is a bounded extension of f with

$$\|F\|_1 = 1 = \|f\|_1.$$

This shows that there are infinitely many linear functional F on $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$ for $p = 1$. $\square$

Problem 3

(a) Let $n \geq 1$ be an integer. Show that no linear map $F : X \to \mathbb{K}^n$ is injective.

Proof Since X is an infinite dimensional normed vector space, we can take a $n+1$-dimensional normed vector space Y in X. Then we have $F|_Y : Y \to \mathbb{K}^n$. By Rank-Nullity Theorem,

$$\dim(Y) = \dim(\operatorname{Im} F|_Y) + \dim(\operatorname{Ker} F|_Y).$$

Since $\dim(\mathbb{K}^n) = n$, it is clear that $\dim(\operatorname{Im} F|_Y) \leq n$. Since Y is a $n+1$-dimensional normed vector space over $\mathbb{K}$, it deduces that $\dim(\operatorname{Ker} F|_Y) \geq 1$. Hence, $F|_Y$ is not injective and F is not injective. $\square$

(b) Let $n \geq 1$ be an integer and let $f_1, f_2, \ldots, f_n \in X^*$. Show that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}.$$

Proof Consider the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), x \in X$. Suppose that when $x = 0$ we have $F(x) = (0, 0, \ldots, 0)$. According to the previous problem, we have at least one more different vector $x' \neq 0$ such that $F(x') = (0, 0, \ldots, 0)$. Namely,

$$f_i(0) = f_i(x') = 0,$$

for all $i = 1, 2, \ldots, n$. Hence,

$$\{0, x'\} \subseteq \bigcap_{j=1}^{n} \ker(f_j)$$

Now we can conclude that $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$. $\square$

(c) Let $x_1, x_2, \ldots, x_n \in X$. Show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, 2, \ldots, n$.

Proof For $1 \leq j \leq n$ let $f_j \in X^*$ be a bounded functional such that $f_j(x_j) = \|x_j\|$ and $\|f_j\| = 1$. Then the intersection of kernels

$$\bigcap_{j=1}^{n} \ker(f_j)$$

is a nontrivial subspace of X (which we have proved in the previous problem). Pick $y \in \bigcap_{j=1}^{n} \ker(f_j)$ such that $\|y\| = 1$ and notice that

$$\|y - x_j\| = \|f_j\|\|y - x_j\| \geq |f_j(y - x_j)| = |f_j(y) - f_j(x_j)| = |0 - \|x_j\|| = \|x_j\|$$

which proves the claim. (Note that by definition of the operator norm we have $\|f_j\|\|y-x_j\| \geq |f_j(y-x_j)|$) $\square$

(d) Show that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

Proof Let $B_j := \{B(x_j, r_j) : r_j < \|x_j\|\}$ be closed balls and these closed balls not containing 0 since $r_j < \|x_j\|$, $j = 1, 2, \ldots$.
Assume that we have a finite family of closed balls B_j covering S, $j = 1, 2, \ldots, n$. These balls are closed convex sets since any closed ball in a normed vector space is convex. ($\|tx+(1-t)y-p\| \leq t\|x-p\| + (1-t)\|y-p\| \leq r$, for all $x, y \in B(p, r), 0 \leq t \leq 1$.) Then, by Theorem 3.6 (Lecture3_FunkAn20-21.pdf), we can find continuous functionals g_j such that $\mathrm{Re} g_j(x) \geq 1$ for all $x \in B_j$. (If $x \in \bigcap_{j=1}^n \ker(g_j)$, then $g_j(x) = 0$ for all j. But $x \in B_j$ implies $\mathrm{Re} g_j(x) \geq 1$.)
Since the parallel hyperplane (given by the Hahn-Banach separation theorems) through the origin has codimension one and is disjoint from the ball, we have a subspace $\bigcap_{j=1}^n \ker(g_j)$ of finite codimension in X that is disjoint from the given finite set of closed balls. Namely, this vector space $\bigcap_{j=1}^n \ker(g_j)$ does not intersect any of the B_j.
Note that $\bigcap_{j=1}^n \ker(g_j) \neq \{0\}$ (problem (b)). Hence, we can always find an $x \in (\cap_{j=1}^n \ker(g_j)) \cap S$, i.e. $x \notin \cup_j B_j$ but $x \in S$, which is a contradiction. Therefore, no finite number of closed balls not containing 0 can cover S. $\square$

Another solution:

Proof Let $B_j := \{B(x_j, r_j) : r_j < \|x_j\|\}$ be closed balls and these closed balls not containing 0 since $r_j < \|x_j\|$, $j = 1, 2, \ldots$. Assume that we have a finite family of closed balls B_j covering S, $j = 1, 2, \ldots, n$. According to (c), there exist $y \in X$ with $\|y\| = 1$, which means $y \in S$, such that $\|y - x_i\| \geq \|x_i\| > r_i$. Hence, $y \notin B(x_i, r_i) \cap S$ and then $y \notin \cup_i (B(x_i, r_i)) \cap S$. Therefore, no finite number of closed balls not containing 0 can cover S. $\square$

(e) Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

Proof The previous problem implies that S is non-compact. Now we prove it. Assume that S were compact and for any $x \in S$ consider $B_x = \{v \in X | \|x - v\| < \frac{1}{2}\}$. Then $\{B_x\}_{x \in S}$ is an open cover of S and by compactness it has to contain a finite subcover $\{B_{x_1}, B_{x_2}, \ldots, B_{x_n}\}$. However, $\{\overline{B_{x_1}}, \overline{B_{x_2}}, \ldots, \overline{B_{x_n}}\}$ is a finite family of closed balls that can cover S, where $\overline{B_{x_i}} = \{v \in X | \|x_i - v\| \leq \frac{1}{2}\}$, $i = 1, 2, \ldots, n$. And none of $\overline{B_{x_i}}$ contains 0. Because given an $\|x\| = 1$ as an element of S and so $\|x - 0\| = \|x\| = 1 > \frac{1}{2}$. This contradicts what we have proved in problem (d). Thus, S is non-compact.

For the second part. Denote the closed unit ball by $\overline{B_u}$. Assume that $\overline{B_u}$ were compact. Since S is a closed subset of $\overline{B_u}$, S would be compact, which deduces that X would be finite dimensional. However X is infinite dimensional. Hence the closed unit ball in X is non-compact.
Another way to show that the closed unit ball in X is non-compact:
S is a closed subset of the closed unit ball. Since a closed subset of a compact space is compact, S is compact. But S is not compact. So the closed unit ball in X is non-compact. $\square$

Problem 4

(a) Given $x \geq 1$, is the set $E_n \subset L_1([0,1],m)$ absorbing? Justify.

Proof Since $L_3([0,1],m) \subsetneq L_1([0,1],m)$ we can take $f \in L_1([0,1],m) - L_3([0,1],m)$, i.e., $f \in L_1([0,1],m)$ and $f \notin L_3([0,1],m)$. Then, $\|f\|_1 < \infty$ and $\|f\|_3 = \infty$. Assume that E_n is absorbing. Then there exists $t > 0$ such that $t^{-1}f \in E_n$. That is,

$$t^{-1}\|f\|_3 = \|t^{-1}f\|_3 \leq n$$

which means

$$\|f\|_3 \leq tn < \infty$$

Thus $f \in L_3([0,1],m)$ and it is a contradiction. Therefore, the set $E_n \subset L_1([0,1],m)$ is not absorbing. $\square$

(b) Show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.

Proof Assume that E_n contains a non-empty open set U for some $n \geq 1$. Then there exists $f \in U$. So we have the open ball

$$B(f,\varepsilon) := \{g \in L_1([0,1],m) : \|f-g\|_1 < \varepsilon\} \subseteq E_n$$

for some $\varepsilon > 0$. For $0 \neq g \in L_1([0,1],m)$, we have $h := f + \frac{\varepsilon}{2\|g\|_1}g \in B(f,\varepsilon)$, and so it belongs to $L_1([0,1],m)$. Note that h is in the ball, and the ball is in E_n by assumption. Also note that $\int_{[0,1]}|f|^3 \mathrm{d}m \leq n$ means that $f \in L_3([0,1],m)$, so $E_n \subset L_3([0,1],m)$ by the definition of E_n. Hence, $h \in L_3([0,1],m)$. Then,

$$g = \frac{2\|g\|_1}{\varepsilon}(h-f) \in L_3([0,1],m),$$

from which we conclude that $L_3([0,1],m) = L_1([0,1],m)$. This is a contradiction. Hence, E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$. $\square$

Another solution:

Proof Denote $E_n \subset V := L_1([0,1],m) \cap L_3([0,1],m)$. Note that $\int_{[0,1]} |f|^3 \mathrm{d}m \leq n$ means that $f \in L_3([0,1],m)$, so $E_n \subset L_3([0,1],m)$ by the definition of E_n. If V contains a non-empty open set U. Let $u \in U$ and $f \in L_1([0,1],m)$. Take $t > 0$ such that $u_t := (1-t)u + tf \in U$. Since

$$\|u_t - u\| = |t|\|f - u\|,$$

we have

$$\lim_{t \to 0^+} u_t = u.$$

It follows that $u_t \in V$ and $u \in V$. Hence, $f = \frac{1}{t}(u_t - (1-t)u) \in V$. Then, $V = L_1([0,1],m)$ and E_n does not contain a non-empty open set. Hence, E_n has empty interior in $L_1([0,1],m)$. $\square$

(c) Show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$.

Proof Take a sequence $(f_k) \in E_n$ such that $(f_k) \to f \in L_1([0,1],m)$, as $k \to \infty$. Note that there is a subsequence (f_{n_k}) which converges pointwise almost everywhere. It follows by Fatou's Lemma:

$$\int_{[0,1]} \liminf |f_{n_k}|^3 \mathrm{d}m = \int_{[0,1]} |f|^3 \mathrm{d}m = \|f\|_3^3 \leq \liminf \int_{[0,1]} |f_{n_k}|^3 \mathrm{d}m \leq n$$

Then, we have $f \in E_n$. Whence, every sequence $(f_k) \in E_n$ converges to $f \in E_n$. In other words, E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$. $\square$

(d) Conclude from (b) and (c) that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

Proof From (b) we show that E_n is nowhere dense in $L_1([0,1],m)$. Simply show it again:
Let $f \in E_n$ and $g \in L_1([0,1],m) - L_3([0,1],m)$, then $f + \frac{1}{k}g \to f$ in $L_1([0,1],m)$ but $f + \frac{1}{k}g \notin E_n$ for all k. Hence E_n does not contain any interior points. Namely, E_n is nowhere dense in $L_1([0,1],m)$.
On the other hand, from (c) we show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$.
As $L_3([0,1],m) = \bigcup_n E_n$ is a union of nowhere dense sets, it is of first category in $L_1([0,1],m)$. $\square$

Problem 5

(a) Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$? Give a proof or a counterexample.

Proof $x_n \to x$ in norm, as $n \to \infty \iff \forall \epsilon > 0$ there exists a n_ϵ such that if $n > n_\epsilon$ then $\|x_n - x\| < \epsilon$. In particular, by the triangle inequality, for $n > n_\epsilon$ we have

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| < \epsilon.$$

Hence $\|x_n\| - \|x\|$ as desired. Namely, $\|x_n\| \to \|x\|$, as $n \to \infty$. $\square$

(b) Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$? Give a proof or a counterexample.

A Counterexample Consider an orthonormal basis $(e_n)_{n\geq 1}$ in H. Then (note that $\langle \cdot, \cdot \rangle$ denotes the inner product in H),

$$\langle e_n, e_m \rangle = \delta_{mn}$$

where

$$\delta_{mn} = \begin{cases} 1 & \text{if} \quad m = n; \\ 0 & \text{otherwise.} \end{cases}$$

Since H is an infinite dimensional Hilbert space, then the basis $(e_n)_{n\geq 1}$ is infinite and it converges weakly to 0. But $\|e_n\| = 1 \to 1 \neq 0$. Simply prove this:
For $x \in H$, by Bessel's inequality, we have

$$\sum_n |\langle e_n, x \rangle|^2 \leq \|x\|^2$$

Therefore $|\langle e_n, x \rangle|^2 \to 0$. Hence, $|\langle e_n, x \rangle| \to 0$, i.e. $(e_n)_{n\geq 1}$ converges weakly to 0.
However, $\|e_n\| = 1 \to 1$. Clearly,

$$\|e_n\| \to 1 \neq 0 = \|0\|.$$

Whence, if $x_n \to x$ weakly, as $n \to \infty$, then it does not follow that $\|x_n\| \to \|x\|$, as $n \to \infty$.

(c) Suppose that $\|x_n\| \leq 1$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Is it true that $\|x\| \leq 1$? Give a proof or a counterexample.

Proof By Theorem 2.7 (Lecture2_FunkAn20-21.pdf), there exists a $f \in H^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. By weak convergence,

$$\|x\| = f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} |f(x_n)| \leq \sup_{n\to\infty} \|x_n\| \leq 1.$$

So it is true that $\|x\| \leq 1$. $\square$