Mandatory assignment, FunkAn 2

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Problem 1

Let H be an infinite dimensional seperable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$ for all $N \geq 1$.

(a) Show that $f_N \to 0$ weakly, as $N \to \infty$ while $\|f_N\| = 1$ for all $N \geq 1$.

Since e_n is a basis for H it follows that $f_N \in H$ for all $N \geq 1$.
Now let $F_n : H \to \mathbb{C}$ be any linear bounded functional. By Riesz' representation thm. there exist $h = \sum_{n=1}^{\infty} \alpha_n e_n \in H$ s.t. $F_n(x) = \langle x, h\rangle$. Lets consider this

$$
\begin{aligned}
F_n(f_N) &= \langle N^{-1}\sum_{n=1}^{N^2} e_n, \sum_{n=1}^{\infty} \alpha_n e_n\rangle \\
&= N^{-1}\sum_{n=1}^{N^2}\langle e_n, \sum_{n=1}^{\infty} \alpha_n e_n\rangle \\
&= N^{-1}\sum_{n=1}^{N^2} \alpha_n
\end{aligned}
$$

By def. of weak convergence we want to show that $\frac{1}{\sqrt{N}}\sum_{n=1}^{N} \alpha_n \to 0$ as $n \to \infty$.
Now, by using both the triangle inequality and Cauchy-Schwarz' inequality we obtain that

$$
\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\alpha_n\right)^2 \leq \left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}|\alpha_n|\right)^2 \leq \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{N}}\right)^2\sum_{n=1}^{N}|\alpha_n|^2 = \sum_{n=1}^{N}|\alpha_n|^2
$$

Since $(\alpha_n)_{n\geq 1} \in \ell_2(\mathbb{N})$ by Riesz' representation thm. we now obtain, by def. of $\ell_2(\mathbb{N})$ that

$$
\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\alpha_n\right| \leq \left(\sum_{n=1}^{N}|\alpha_n|^2\right)^{1/2} < \infty \quad \text{for all } N \geq 1
$$

Since $\sum_{n=1}^{N}|\alpha_n|^2 < \infty$ there exist a $C \in \mathbb{C}$ s.t. $\sum_{n=1}^{N}|\alpha_n|^2 \to C$ when $n \to \infty$.
For all $\varepsilon > 0$ there exist m s.t. $\sum_{n=m+1}^{\infty}|\alpha_n|^2 < \varepsilon$. This shows that for any constant $K \geq 1$ $\sum_{n=m+1}^{K+m}|\alpha_n|^2 < \varepsilon$ holds. Now for $N \geq \frac{C^2}{\varepsilon^2}$ we have that

$$
\frac{1}{\sqrt{N}}\sum_{n=1}^{m}|\alpha_n| \leq \frac{\varepsilon}{C}\cdot C = \varepsilon
$$

Now we can use Cauchy Schwarz' inequality and obtain

$$\begin{aligned}
\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\alpha_n\right| &\leq \frac{1}{\sqrt{N}}\sum_{n=1}^{N}|\alpha_n| \\
&= \frac{1}{\sqrt{N}}\sum_{n=1}^{m}|\alpha_n| + \frac{1}{\sqrt{N}}\sum_{n=m+1}^{N}|\alpha_n| \\
&\leq \varepsilon + \frac{1}{\sqrt{N}}\sum_{n=m+1}^{N+m}|\alpha_n| \\
&\leq \varepsilon + \sqrt{\left(\sum_{n=m+1}^{N+m}\frac{1}{N}\right)\left(\sum_{n=m+1}^{N+m}|\alpha_n|^2\right)} \\
&= \varepsilon + \sqrt{1\cdot\left(\sum_{n=m+1}^{N+m}|\alpha_n|^2\right)} \\
&< \varepsilon + \sqrt{\varepsilon}
\end{aligned}$$

This shows that $\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\alpha_n\right| \to 0$ as $N \to \infty$ which implies that $\left|\frac{1}{N}\sum_{n=1}^{N^2}\alpha_n\right| \to 0$ as $N \to \infty$. We now obtain that $\lim_{n\to\infty}\frac{1}{N}\sum_{n=1}^{N^2}\alpha_n = 0$, but $\lim_{n\to\infty}\frac{1}{N}\sum_{n=1}^{N^2}\alpha_n = \lim_{n\to\infty}F_n(f_N)$. Since F is bounded, hence continuous we have now obtained the desired, that $f_N \to 0$ weakly as $N \to \infty$.

Now lets compute $\|f_N\|$.

$$\begin{aligned}
\|f_N\|^2 = \|N^{-1}\sum_{n=1}^{N^2}e_n\|^2 &= |N^{-1}|^2\|\sum_{n=1}^{N^2}e_n\|^2 \\
&= N^{-2}\|\sum_{n=1}^{N^2}e_n\|^2 = N^{-2}\sum_{n=1}^{N^2}\|e_n\|^2 \\
&= N^{-2}\sum_{n=1}^{N^2}1^2 = N^{-2}N^2 \\
&= 1
\end{aligned}$$

This shows that $\|f_N\| = 1$ for all $N \leq 1$. $\square$

Let K be the norm closure of co$\{f_n : N \geq 1\}$.

(b) Argue that K is weakly compact, and that $0 \in K$.

We have that $K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|}$, and since $\text{co}\{f_N : N \geq 1\}$ is convex by definition of the convex hull we obtain, by thm. 5.7, that

$$K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w}$$

i.e. that the norm and the weak closure coincide. This shows that K is weakly closed. Since K is weakly closed, and since we showed in (a) that $f_N \to 0$ weakly as $N \to \infty$, then $0 \in K$.

Now lets consider the unit ball $\overline{B_{H^*}(0,1)} \subset H^*$.
By Alaoglu's thm. we know that $\overline{B_{H^*}(0,1)}$ is compact in the w^*-topology. Since H is a Hilbert space it follows by prop. 2.10 that it is a reflexive Banach space. By thm. 5.9 and the topologies on H^* we obtain that $\tau_w = \tau_{w^*}$ and thereby we get that $\overline{B_{H^*}(0,1)}$ is weakly compact.
By Riesz' representation thm. we have that for every $y \in H$ every element in H^* is given by $F_y = \langle \cdot, y \rangle$. This shows that we have an isomorphism from H^* to H, which sends F_y to y. Then we have an isomorphism between $\overline{B_{H^*}(0,1)}$ and $\overline{B_H(0,1)}$, why $\overline{B_H(0,1)}$ also is weakly compact. Since $K \subseteq \overline{B_H(0,1)}$ we now obtain that K, the weakly closed set, is a subset of a weakly compact set, hence K is weakly compact. $\square$

(c) Show that 0, as well as each f_N, $N \geq 1$ are extreme points in K.

By def. 7.1 we obtain that

$$\text{Ext}(K) = \{x \in K \mid x = \alpha x_1 + (1-\alpha)x_2 \text{ implies } x_1 = x_2 = x,\ x_1, x_2 \in K,\ 0 < \alpha < 1\}$$

Lets first show that $0 \in \text{Ext}(K)$.
Note that by def. $K \subseteq H$ is a non-empty convex compact subset. Lets consider the continuous linear functional $G_n = \langle \cdot, -e_n \rangle \in H^*$ for any $n \in \mathbb{N}$. Note that $G_n(K) \subseteq \mathbb{R}$.
Now let

$$C = \sup_n \{\langle x, -e_n \rangle \mid x \in K\} = \sup_n \{-\langle x, e_n \rangle \mid x \in K\}$$

Since $x \in K$ we know that $x \geq 0$, and we furthermore have that $0 \in K$, why we obtain that $-\langle x, e_n \rangle \leq 0$ for $x \in K$. We can now use lemma 7.5, why we get that $F_n := \{x \in K \mid \text{Re}\langle x, -e_n \rangle = 0\} \neq \emptyset$ is a compact face of K for all $n \in \mathbb{N}$.
We have that $0 \in F_n$ for all $n \in \mathbb{N}$ why $0 \in \cap_{n=1}^{\infty} F_n \neq \emptyset$. Since the only element which is orthogonal on all elements e_n is zero we obtain

$$\bigcap_{n=1}^{\infty} F_n = \{x \in K \mid \text{Re}\langle x, -e_n \rangle = 0,\ \forall n \in \mathbb{N}\} = \{0\}$$

Now we can use remark 7.4(3) to say that $\cap_{n=1}^{\infty} F_n = \{0\}$ is a face of K and by applying remark 7.4(1) we have now reached that $0 \in \text{Ext}(K)$ as desired.

Now lets show that $f_N \in \text{Ext}(K)$.
Lets fix $N \geq 1$ and suppose that $f_N = \alpha x_1 + (1-\alpha)x_2$ for $x_1, x_2 \in K$ and $0 < \alpha < 1$. We know that $1 = \|f_N\|^2 = f_N, f_N\rangle$. Now consider

$$\begin{aligned} 1 = \langle f_N, f_N \rangle &= \langle \alpha x_1 + (1-\alpha)x_2, f_N \rangle \\ &= \alpha \langle x_1, f_N \rangle + (1-\alpha)\langle x_2, f_N \rangle \end{aligned}$$

this implies that

$$\begin{aligned} 0 &= \alpha \langle x_1, f_N \rangle + (1-\alpha)\langle x_2, f_N \rangle - 1 \\ &= \alpha \langle x_1, f_N \rangle + (1-\alpha)\langle x_2, f_N \rangle - (\alpha + (1-\alpha)) \\ &= \alpha(\langle x_1, f_N \rangle - 1) + (1-\alpha)(\langle x_2, f_N \rangle - 1) \end{aligned}$$

since $0 < \alpha < 1$ and $\langle x_1, f_N \rangle$, $\langle x_2, f_N \rangle \geq 0$ we can see that $0 \leq \langle x_i, f_N \rangle \leq 1$ for $i = 1, 2$. But by what we just found this shows that $\langle x_1, f_N \rangle = 1 = \langle x_1, f_N \rangle$.
Now we wanna show that $x_1 = x_2 = f_N$, since it would then follow that $f_N \in \text{Ext}(K)$. That $x_1 = f_N$ and that $x_2 = f_N$ is found with the same approach, why I will only show that $x_1 = f_N$.
See that

$$1 = \|\langle x_1, f_N \rangle\| \leq \|x_1\| \|f_N\| = \|x_1\|$$

by Cauchy-Schwarz. Since $x_1 \in K \subseteq \overline{B_H(0,1)}$, then $\|x_1\| \leq 1$. This shows that

$$1 = \|\langle x_1, f_N \rangle\| = \|x_1\| \|f_N\| = \|x_1\|$$

Then F_N and x_1 are linealy dependent, why $x_1 = \lambda f_N$ for a scalar λ. Then it follows that

$$1 = \langle \lambda f_N, f_N \rangle = \lambda \langle f_N, f_N \rangle = \lambda \|f_N\|^2 = \lambda$$

which shows that $x_1 = f_N$ why $f_N \in \text{Ext}(K)$ for all $N \geq 1$. □

(d) Are there any other extreme points in K?

See that $K = \overline{\text{co}\{f_N \mid N \geq 1}^{\tau_w}$ is a non-empty convex subset for H. By Milmans thm. we get that $\text{Ext}(K) \subseteq \overline{\{f_N \mid N \geq 1\}}^{\tau_w}$.
By (c) we now obtain that $\{f_N \mid N \geq 1\} \cup \{0\} \subseteq \overline{\{f_N \mid N \geq 1\}}^{\tau_w}$.
Since H is a normed space it is metrizable and then $\{f_N \mid N \geq 1\}$ is also metrizable. This shows that $\{f_N \mid N \geq 1\}$ is first countable and it is then enough to consider sequences in $\{f_N \mid N \geq 1\}$ instead of nets.
Now lets assume that $(x_n)_{n \geq 1}$ is a sequence in $\{f_N \mid N \geq 1\}$ which converges weakly to $x \in \overline{\{f_N \mid N \geq 1\}}^{\tau_w}$. It then follows that each $x_i = f_N$ for some $N \geq 1$, why x is equal to some F_N or to zero. We then obtain that

$$\text{Ext}(K) \subseteq \overline{\{f_N \mid N \geq 1\}}^{\tau_w} = \{f_N \mid N \geq 1\} \cup \{0\}$$

And since we by (c) have that

$$\{f_N \mid N \geq 1\} \cup \{0\} \subseteq \text{Ext}(K)$$

we can conclude that $\text{Ext}(K) = \{f_N \mid N \geq 1\} \cup \{0\}$ why there are no other extreme points in K. □

Problem 2

Let X and Y be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n \geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

Assume that $x_n \to x$ weakly as $n \to \infty$ for $x \in X$. From HW 4 problem 2 we know that this holds if and only if $Fx_n \to Fx$ for all $F \in X^*$. I can use this problem since a net is said to be a more general case of a sequence.

Now lets take $G \in Y^*$, then we obtain that the decomposition $G \circ T \in X^*$, why $(G \circ T)(x_n) \to (G \circ T)(x)$ as $n \to \infty$ for all $G \in Y^*$. But this means exactly what we wanted to show, that $Tx_n \to Tx$ weakly as $n \to \infty$. $\square$

(b) Let $T \in \mathcal{K}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$.

Assume that $x_n \to x$ weakly as $n \to \infty$ for $x \in X$. Lets assume by contradiction that $\|Tx_n - Tx\| \nrightarrow 0$ as $n \to \infty$. Then there exist a subsequence $(x_{n_i})_{i\geq 1}$ and $\varepsilon > 0$ s.t. $\|Tx_{n_i} - Tx\| > \varepsilon$ for all $i \geq 1$.
Since $x_n \to x$ weakly as $n \to \infty$, we get that $x_{n_i} \to x$ weakly as $n \to \infty$ as well. We obtain that $(x_{n_i})_{i\geq 1}$ is bounded, which means that it has a subsequence $(x_{n_{i_k}})_{k\geq 1}$ which fulfills that $\|Tx_{n_{i_k}} - Tx'\| \to 0$ as $k \to \infty$ for some $x' \in X$. We can now use (a) to say that $Tx_{n_i} \to Tx$ weakly as $i \to \infty$ since $x_{n_i} \to x$ weakly as $i \to \infty$, but then it also holds that $Tx_{n_{i_k}} \to Tx$ weakly as $k \to \infty$. If something converges by norm to something, then it will also converge weakly to the same, why we must obtain that $Tx' = Tx$ which shows that $\|Tx_{n_{i_k}} - Tx\| \to 0$ as $k \to \infty$. However this is a contradiction to what we found earlier, that $\|Tx_{n_i} - Tx\| > \varepsilon$ for all $i \geq 1$, why we have reached a contradiction and can conclude that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. $\square$

(c) Let H be a separable infinite dimensional Hilbert Space. If $T \in \mathcal{L}(H,Y)$ satisfies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H,Y)$.

Lets assume by contradiction that T is *not* compact (i.e. $T \notin K(H,Y)$), but by prop. 8.2 this holds if and only if the closed unit ball $T(\bar{B}_H(0,1))$ is *not* totally bounded, and by def. this means that there exist $\delta > 0$ s.t. every finite union of open balls with radius δ does not cover $T(\bar{B}_H(0,1))$.
Now lets take an $x_1 \in \bar{B}_H(0,1)$ where $x_1 \in (x_n)_{n\geq 1} \subset \bar{B}_H(0,1)$. Assume that $x_2, x_3, ..., x_n$ are satisfying that $\|Tx_q - Tx_r\| \geq \delta$ for all $1 < q, r \leq n$ and $q \neq r$. Now lets define the set

$$M := T(\bar{B}_H(0,1) \cap (\cup_{i=1}^n B_Y(Tx_i, \delta))^C$$

Observe that $M \neq \emptyset$, since $T(\bar{B}_H(0,1))$ is *not* totally bounded, why we obtain that $T(\bar{B}_H(0,1)) \subset (\cup_{i=1}^n B_Y(Tx_i, \delta))^C$.
Now lets take $x_{n+1} \in \bar{B}_H(0,1)$ s.t. we obtain $Tx_{n+1} \in M$, thereby we also get that $Tx_{n+1} \in (\cup_{i=1}^n B_Y(Tx_i, \delta))^C$ and following this also that $Tx_{n+1} \notin B_Y(Tx_i, \delta)$ for any i. This shows that $\|Tx_{n+1} - Tx_i\| \geq \delta$ for all $i \leq n$. We can continue this process, thereby obtaining a sequence $(x_n)_{n\geq 1}$ s.t. $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$.
By prop. 2.10 H is reflexive, why $\bar{B}_H(0,1)$ is weakly compact by thm. 6.3. This shows that every sequence has a weakly convergent subsequence $(x_{n_k})_{k\geq 1}$. Since we found that $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$ we will then obtain that $\|Tx_{n_k} - Tx\| \geq \delta$, hence that $\|Tx_{n_k} - Tx\| \nrightarrow 0$ as $k \to \infty$, since we assumed that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. This is a contradiction, why T must be compact, i.e. $T \in K(H,Y)$. $\square$

(d) Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

Take $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ and let $(x_n)_{n\geq 1} \in \ell_2(\mathbb{N})$. Suppose further that $x_n \to x$ weakly as $n \to \infty$. By (a) this implies that $Tx_n \to Tx$ weakly in $\ell_1(\mathbb{N})$ as $n \to \infty$. Using remark 5.3 this holds if and only if $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. Now we can use (c) (since $\ell_2(\mathbb{N})$ by def. is a infinite dimensional Hilbert space, and by HW4 problem 4 also separable) to conclude that $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact. $\square$

(e) Show that <u>no</u> $T \in \mathcal{K}(X, Y)$ is onto.

Suppose that $T \in \mathcal{L}(X, Y)$ is compact and onto, thereby surjective and by the Open mapping thm. also open. Since X, Y are normed vector spaces and T is open we get (by p. 18 of the lecture notes) that there exist $r > 0$ s.t. $B_Y(0, r) \subset T(B_X(0, 1))$, hence also that $\overline{B_Y(0, r)} \subset \overline{T(B_X(0, 1))}$ (since closure preserves inclusion). Since T is a compact operator $\overline{T(B_X(0, 1))}$ is compact and it also follows that $\overline{B_Y(0, r)}$ is compact. Now lets consider different values of r.

- $r = 1$
  Then it follows that $\overline{B_Y(0, r)} = \overline{B_Y(0, 1)}$, and since $\overline{B_Y(0, r)}$ is compact so is $\overline{B_Y(0, 1)}$. But since Y is an infinite-dimensional normed space it follows from Riesz's lemma that $\overline{B_Y(0, 1)}$ cannot be compact, why we have reached a contradiction.

- $r > 1$
  Then $\overline{B_Y(0, 1)}$ is a closed set of the compact set $\overline{B_Y(0, r)}$, hence compact as well, but with the same argument as before this is a contradiction.

- $r < 1$
  Lets consider the map $g : Y \to Y$ given by $x \mapsto \frac{1}{r}x$, which is continuous. We know that the image under a continuous function of a compact set is compact, why we obtain that $g(\overline{B_Y(0, 1)}) = \overline{B_Y(0, 1)}$ is compact, which again is a contradiction.

So we have now showed that $\overline{B_Y(0, r)}$ is not compact for any r, which is a contradiction, hence no $T \in \mathcal{K}(X, Y)$ is onto. $\square$

(f) Let $H = L_2([0, 1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0, 1]$. Justify that M is self-adjoint, but not compact.

First lets show that M is self-adjoint.
Observe that $t = \bar{t}$ since t only has real values. Now lets consider the inner product on

H.

$$\begin{aligned}
\langle Mf, g\rangle &= \int_0^1 Mf(t)g(\bar{t})dm(t) \\
&= \int_0^1 tf(t)g(\bar{t})dm(t) \\
&= \int_0^1 f(t)tg(\bar{t})dm(t) \\
&= \int_0^1 f(t)tg(t)dm(t) \\
&= \int_0^1 f(t)Mg(t)dm(t) \\
&= \langle f, Mg\rangle
\end{aligned}$$

Where I have used p. 56 of the lecture notes.
This shows that $M = M^*$ and by def. that it is self-adjoint.

Now lets justify that M is not compact.
Lets assume by contradiction that M is compact. We have furthermore just showed that M is self-adjoint. H is by HW 4 problem 4 seperable and we also know that it is infinite-dimensional, so thm. 10.1 implies that H has an othonormal basis consisting of eigenvectors for M with corresponding eigenvalues. In HW 6 problem 3 we proved that M has no eigenvalues, why we have reached a contradiction, which shows that M is compact. $\square$

Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$, where m is the Lebesgue measure. Define $K : [0,1] \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t, & \textbf{if } 0 \leq t \leq s \leq 1, \\ (1-t)s, & \textbf{if } 0 \leq s \leq t \leq 1, \end{cases}$$

and consider $T \in \mathcal{L}(H, h)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t), \quad s \in [0,1], \; f \in H$$

(a) Justify that T is compact.

Note that $[0,1]$ is in $\mathbb{R}$ hence a compact Hausdorff topological space. Furthermore K is, by how it is defined, continuous on $[0,1] \times [0,1]$, hence $K \in C([0,1] \times [0,1])$. At last, see that since m is the Lebesgue measure it is a finite Borel measure on $[0,1]$. Now we can use thm. 9.6 to conclude tha T is compact. $\square$

(b) Show that $T = T^*$.

Observe that $K(s,t) = K(t,s)$ always. Now lets consider the inner product on H.

$$
\begin{aligned}
\langle Tf, g \rangle &= \int_{[0,1]} Tf(s)\overline{g(s)}\mathrm{dm}(s) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(s,t)f(t)\mathrm{dm}(t) \right) \overline{g(s)}\mathrm{dm}(s) \\
&= \int_{[0,1]\times[0,1]} K(s,t)f(t)\overline{g(s)}\mathrm{dm}(s,t) \\
&= \int_{[0,1]\times[0,1]} K(t,s)f(t)\overline{g(s)}\mathrm{dm}(s,t) \\
&= \int_{[0,1]\times[0,1]} K(t,s)\overline{g(s)}f(t)\mathrm{dm}(s,t) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(t,s)\overline{g(s)}\mathrm{dm}(s) \right) f(t)\mathrm{dm}(t) \\
&= \int_{[0,1]} \overline{Tg(t)}f(t)\mathrm{dm}(t) \\
&= \langle f, Tg \rangle
\end{aligned}
$$

Where I have used p. 56 of the lecture notes and Fubini-Tonelli's thm. twice. This shows that $T = T^*$, hence self-adjoint. □

(c) Show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t), \quad s \in [0,1], \ f \in H.$$

Use this to show that Tf is continuous on $[0,1]$, and that $(Tf)(0) = (Tf)(1) = 0$.

First lets look at $(Tf)(s)$

$$
\begin{aligned}
(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)
\end{aligned}
$$

This follows by linearity of integrals and furthermore that $s \in [0,1]$.

Lets use this to show that Tf is continuous.
By prop. 1.10 Tf is continuous if it is bounded. Lets show this by looking at each integral

separately.
By def. of $L_2([0,1],m)$ we obtain that

$$\left(\int_{[0,1]} |f(t)|^2 dm(t)\right)^{1/2} < \infty.$$

Since $s \in [0,1]$ this also shows that

$$(1-s)\left(\int_{[0,s]} t|f(t)|^2 dm(t)\right)^{1/2} < \infty$$

and at last that

$$(1-s)\int_{[0,s]} tf(t)dm(t) < \infty.$$

The exact same can be done for the other part of (Tf)(s) why we could obtain

$$s\int_{[s,1]} (1-t)f(t)dm(t) < \infty$$

which shows that Tf is bounded on $[0,1]$, hence continuous.

Now lets show that $(Tf)(0) = (Tf)(1) = 0$.
First notice that

$$\begin{aligned}
(Tf)(0) &= (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]} (1-t)f(t)dm(t) \\
&= \int_{[0,0]} tf(t)dm(t) \\
&= 0
\end{aligned}$$

And now that

$$\begin{aligned}
(Tf)(1) &= (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]} (1-t)f(t)dm(t) \\
&= \int_{[1,1]} (1-t)f(t)dm(t) \\
&= 0
\end{aligned}$$

Hence $(Tf)(0) = (Tf)(1) = 0$. $\square$

Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

(a) For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$.
Justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$.
Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

First lets justify that $g_k \in \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$.
By HW 7 problem 1 we obtain that $e^{-x^2} \in \mathscr{S}(\mathbb{R})$, and then for $a = \sqrt{2} \in \mathbb{R} \setminus \{0\}$ that $S_{\sqrt{2}} e^{-x^2} \in \mathscr{S}(\mathbb{R})$. By p. 62 in the lecture notes we obtain $S_{\sqrt{2}} e^{-x^2} = e^{-x^2/2} \in \mathscr{S}(\mathbb{R})$.
By applying HW 7 problem 1 again we have obtained $g_k \in \mathscr{S}(\mathbb{R})$ as desired.

Now lets compute $\mathcal{F}(g_k)$ for $k = 0, 1, 2, 3$.
Let $\varphi(x) := e^{-x^2/2}$ and note that this is integrable. See also that $x^k e^{-x^2/2}$ is integrable.
Note that $\varphi(x) = \hat{\varphi}(x)$ by prop. 11.4 for $n = 1$. Using this and prop. 11.3 we obtain that

$$\begin{aligned}
\mathcal{F}(g_k)(\xi) &= \hat{g}_k(\xi) \\
&= (g_k)^\wedge(\xi) \\
&= (x^k \varphi)^\wedge \xi \\
&= i^k (\partial^k \hat{\varphi})(\xi) \\
&= i^k (\partial^k \varphi)(\xi)
\end{aligned}$$

And we obtain:

<u>$k = 0$</u>.
$\mathcal{F}(g_0)(\xi) = i^0 (\partial^0 \varphi)(\xi) = e^{-\xi^2/2}$

<u>$k = 1$</u>.
$\mathcal{F}(g_1)(\xi) = i^1 (\partial^1 \varphi)(\xi) = -i\xi e^{-\xi^2/2}$

<u>$k = 2$</u>.
$\mathrm{F}(\mathrm{g}_2)(\xi) = i^2 (\partial^2 \varphi)(\xi) = i^2 e^{-\xi^2/2}(\xi^2 - 1) = e^{-\xi^2/2} - \xi^2 e^{-\xi^2/2}$

<u>$k = 3$</u>.
$\mathcal{F}(g_3)(\xi) = i^2 (\partial^3 \varphi)(\xi) = i^3 \xi e^{-\xi^2/2}(3 - \xi^2) = i\xi^3 e^{-\xi^2/2} - 3i\xi e^{-\xi^2/2}$ $\square$

(b) Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

For non-zero $h_0 \in \mathscr{S}(\mathbb{R})$ we wanna show that $\mathcal{F}(h_0) = i^0 h_0 = h_0$.
Lets compute $\mathcal{F}(g_0(\xi))$.

$$\mathcal{F}(g_0(\xi)) = e^{-\xi^2/2} = g_0(\xi)$$

So for $h_0 = g_0$ we obtain $\mathcal{F}(h_0) = h_0$ as desired.

For non-zero $h_1 \in \mathscr{S}(\mathbb{R})$ we wanna show that $\mathcal{F}(h_1) = i^1 h_1 = ih_1$.
Notice that

$$\mathcal{F}(g_3)(\xi) = i\xi^3 e^{-\xi^2/2} - 3i\xi e^{-\xi^2/2} = i(g_3(\xi) - 3g_1(\xi))$$

Now lets compute $\mathcal{F}(g_3(\xi) - \frac{3}{2} g_1(\xi))$.

$$\begin{aligned}
\mathcal{F}(g_3(\xi) - \frac{3}{2} g_1(\xi)) &= \mathcal{F}(g_3(\xi)) - \frac{3}{2} \mathcal{F}(g_1(\xi)) \\
&= i(g_3(\xi) - 3g_1(\xi)) + \frac{3}{2} i\xi e^{-\xi^2/2} \\
&= i(g_3(\xi) - \frac{3}{2} g_1(\xi))
\end{aligned}$$

Why we obtain $\mathcal{F}(h_1) = ih_1$ for $h_1 = g_3 - \frac{3}{2}g_1$.

For non-zero $h_2 \in \mathscr{S}(\mathbb{R})$ we want to show that $\mathcal{F}(h_2) = i^2 h_2 = -h_2$.
First notice that

$$\mathcal{F}(g_2)(\xi) = e^{-\xi^2/2} - \xi^2 e^{-\xi^2/2} = g_0(\xi) - g_2(\xi)$$

Lets compute $\mathcal{F}(g_2(\xi) - \frac{1}{2}g_0(\xi))$.

$$\begin{aligned}
\mathcal{F}(g_2(\xi) - \frac{1}{2}g_0(\xi)) &= \mathcal{F}(g_2(\xi)) - \frac{1}{2}\mathcal{F}(g_0(\xi)) \\
&= g_0(\xi) - g_2(\xi) - \frac{1}{2}g_0(\xi) \\
&= -g_2(\xi) + \frac{1}{2}g_0(\xi) \\
&= -(g_2(\xi) - \frac{1}{2}g_0(\xi))
\end{aligned}$$

Which shows that $\mathcal{F}(h_2) = -h_2$ for $h_1 = g_2 - \frac{1}{2}g_0$.

For non-zero $h_3 \in \mathscr{S}(\mathbb{R})$ we want to show that $\mathcal{F}(h_3) = i^3 h_3 = -ih_2$.
Lets notice that

$$\mathcal{F}(g_1)(\xi) = -i^{-\xi^2/2} = -ig_1(\xi)$$

Why we have obtained that $\mathcal{F}(h_3) = -ih_3$ when $h_3 = g_1$. $\square$

(c) Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$.

Lets compute $\mathcal{F}^2(f)$

$$\begin{aligned}
\mathcal{F}^2(f(\xi)) &= \mathcal{F}(\mathcal{F}(f(\xi)) = \mathcal{F}(\hat{f}(\xi)) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(x) e^{-ix\xi} \mathrm{dx}
\end{aligned}$$

Where I have used def. 11.1, which I can since HW 7 problem 1 states that $\mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$ why $f \in L_1(\mathbb{R})$.
Now lets define $T(f) = S_{-1}(f)$ which by Hw 7 problem 1 is in $\mathscr{S}(\mathbb{R})$ since $f \in \mathscr{S}(\mathbb{R})$.
Now observe that

$$T^2 f(x) = T(Tf(x)) = T(S_{-1}f(x)) = (Tf(-x)) = S_{-1}f(-x) = f(x)$$

Where we have used p. 62 in the lecture notes. This shows that $T^2 = Id$.
Furthermore see that

$$\begin{aligned}
Tf(\xi) &= f(-\xi) \\
&= \mathcal{F}^*(\mathcal{F}(f(-\xi))) \\
&= \mathcal{F}^*(\hat{f}(-\xi)) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(x) e^{-ix\xi} \mathrm{dx} \\
&= \mathcal{F}^2(f(\xi))
\end{aligned}$$

So now we have obtained the desired since

$$\mathcal{F}^4(f) = \mathcal{F}^2(\mathcal{F}^2(f)) = T^2(f) = f.$$

$\square$

(d) Use (c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.

Assume $f \in \mathscr{S}(\mathbb{R})$ is non-zero. To show that $\lambda \in \{1, i, -1, -i\}$ it suffices to show that $\lambda^4 = 1$.
Let $\mathcal{F}(f) = \lambda f$. This would imply that $\lambda^4 f^4 = \mathcal{F}^4(f) = f$ (by (c)), and moreover that $\lambda^4 = \frac{f}{f^4}$.
By (c) we furthermore obtain that

$$f^2 = \mathcal{F}^8(f) = \mathcal{F}^4(\mathcal{F}^4(f)) = \mathcal{F}^4(f) = f$$

why

$$f^4 = (f^2)^2 = f^2 = f$$

Then we obtain

$$\lambda^4 = \frac{f}{f^4} = \frac{f}{f} = 1$$

And we have obtained the desired that $\lambda \in \{1, i, -i - i\}$.
Since these values for λ are the only that satisfy $\mathcal{F}(f) = \lambda(f)$, the eigenvalues of $\mathcal{F}$ are precisely $\{1, i, -1, -i\}$. $\square$

Problem 5

Let $(x_n)_{n \geq 1}$ be a dense subset of $[0, 1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}$ on $[0, 1]$. Show that supp$(\mu) = [0, 1]$.

Using HW 8 problem 3 we have to show that $\mu([0, 1]^C) = 0$.
First lets look at the Dirac mass:

$$\delta_{x_n}([0, 1]^C) = \begin{cases} 0 & , x_n \in [0, 1] \\ 1 & , x_n \notin [0, 1] \end{cases}$$

So we obtain

$$\mu([0, 1]^C) = \sum_{n=1}^{\infty} 2^{-1} \delta_{x_n}([0, 1]^C) = 0$$

since μ is defined on $[0, 1]$ where δ_{x_n} is exactly 0. Now we have obtained, by HW 8 problem 3, that

$$\text{supp}(\mu) = [0, 1]$$

as desired. $\square$