FunkAn Assignment 2

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Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.

a)

Show that $f_N \to 0$ weakly, as $N \to \infty$, while $||f_N|| = 1$, for all $N \geq 1$.

$$||f_N||^2 = \langle N^{-1}\sum_{n=1}^{N^2} e_n, N^{-1}\sum_{n=1}^{N^2} e_n\rangle = N^{-2}\sum_{j,k=1}^{N^2}\langle e_j, e_k\rangle = N^{-2}\sum_{k=1}^{N^2}\langle e_k, e_k\rangle = N^{-2}\sum_{k=1}^{N^2}||e_n||^2 = \frac{N^2}{N^2} = 1$$

Where we used that $(e_n)_{n\geq 1}$ is an orthonormal basis so $\langle e_j, e_k\rangle = 0$ for $j \neq k$ ✓
Now i show that $f_N \to 0$ weakly.
By HMW 4 Pb 2 (or by definition in Folland, i will refer to this result as "definition" of weak convergence) we know that $f_N \to 0$ weakly $\Leftrightarrow F(f_N) \to F(0)$ for all $F \in H^*$. We also know that $F(0) = 0$ for all elements in the dual. By Theorem 5.25 Folland we can write $F(f_n) = \langle f_N, y\rangle$ where y is an unique element of H. Since (e_n) is an ONB we can write $y = \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i$ and as $||y|| < \infty$ for any ϵ there exists a K such that $||\sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i|| < \epsilon$.
Thus $|F(f_N)| = |\langle f_N, y\rangle| = |\langle f_N, \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i\rangle| = |\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle + \langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle|$. Which by the triangle inequality we get:

$$|\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i + \langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle| \leq |\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i| + |\langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle|$$

Firstly we bound the 2nd expression using Cauchy Schwartz as H is a Hilbert space.

$$|\langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle| \leq ||f_N|| \cdot ||\sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle|| < 1 \cdot \epsilon$$

Now to bound the 1st expression:

$$|\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i| = N^{-1}|\sum_{n=1}^{N^2}\langle e_n, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle| = N^{-1}\left|\sum_{n=1}^{N^2}\overline{\sum_{i=1}^{K}\langle y, e_i\rangle\langle e_n, e_i\rangle}\right| \leq N^{-1}\left|\sum_{n=1}^{K}\overline{\langle y, e_i\rangle}\right| < \epsilon \text{ for } N \to \infty$$

Where for the last inequalities we used e_n ONB and $\left|\sum_{n=1}^{K}\overline{\langle y, e_i\rangle}\right|$ being finite.
This shows that for all $F \in H^*$, $F(f_N) \to 0 = F(0)$ for $N \to \infty$ which shows that $f_N \to 0$ weakly. ✓

b)

Let K be the norm closure of $co\{f_N : N \geq 1\}$. Argue that K is weakly compact, and that $0 \in K$.

Firstly we note that K, being the norm closure of a convex set, is convex so by Theorem 5.7 in the notes the norm and weak closures coincide. Thus we have (Why? we omit the $N \geq 1$) $K = \overline{co\{f_n\}}^{||\cdot||} = \overline{co\{f_n\}}$.
We know all Hilbert spaces are reflexive so by Theorem 6.3 in the notes $\overline{B_H(0,1)}$ is compact with

How do you extend from $\|f_n\| = 1$ to $co\{f_n \mid n \in \mathbb{N}\} \subseteq B_H(0,1)$?

respect to the weak topology. As the convex hull is the smallest set containing all convex combinations and all $||f_N|| = 1$ we have that $co\{f_n\} \subset B_H(0,1)$ as the closed unit ball is a convex set contaning all convex combinations of f_N. And as the closed unit ball is closed then K must be contained in it too. Thus K is a weakly closed subset of a weakly compact set and is thus weakly compact.

But these are the same sets?

The sequence $(f_N)_{N\geq 1}$ lies in K as each f_N lies inside it. This sequence converges weakly to 0 thus it its in the weak closure of $co\{f_N : N \geq 1\}$ and hence in the norm closure, K.

✓

c)

Show that 0, as well as each f_N, $N \geq 1$, are extreme points in K.

Show how this extends to $co\{f_n \mid n \in \mathbb{N}\}$

We will first show 0 is an extreme point.
Note that every element in $co\{f_N|N \geq 1\}$ will have a positive inner product with e_n as $\langle f_N, e_n\rangle$ is positive. Let $(x_n)_{n\geq 1}$ be a sequence in $co\{f_N|N \geq 1\}$ converging to x. Let $g_n \in H^*$ be given by $g_n(x) = \langle x, e_n\rangle$, these are continuous function so $\langle x_n, e_n\rangle \to \langle x, e_n\rangle$ for all n. Thus as all $\langle x_n, e_n\rangle \geq 0$ we must have that $\langle x, e_n\rangle \geq 0$. Therefore we have shown that each element in $\overline{co\{f_N|N \geq 1\}}$ will still have positive inner product with e_n.
Let 0 be given as a convex combination $0 = \alpha x + (1-\alpha)y$. Specifically we would also have $0 = \alpha\langle x, e_n\rangle + (1-\alpha)\langle y, e_n\rangle$ for all $n \geq 1$. But 0 is an extreme point of the positive real line thus for each n we have $\langle x, e_n\rangle = \langle y, e_n\rangle = 0$. But by Theorem 5.27(a) Folland we must have that $x = y = 0$. Hence we conclude that 0 is an extreme point of $\overline{co\{f_N|N \geq 1\}} = K$. ✓

Now for the ugly part. Let $f_N = \alpha x + (1-\alpha)y$ be a convex combination in K. Where x is a limit point of $(x_n)_{n\geq 1}$ and y is a limit point of $(y_n)_{n\geq 1}$ $((x_n),(y_n) \in co\{f_N|N \geq 1\})$. Thus we have that $\alpha(x_n)_{n\geq 1} + (1-\alpha)(y_n)_{n\geq 1} \to f_N$. As before note $g_{N^2}(x) = \langle x, e_{N^2}\rangle$. We can apply g_{N^2} (a continuous function) and get.

$$g_{N^2}(\alpha(x_n) + (1-\alpha)(y_n)) = \alpha g_{N^2}(x_n) + (1-\alpha)g_{N^2}(y_n) \to g_{N^2}(f_N) = \frac{1}{N}$$

We will now show that $g_{N^2}(x_n) \leq \frac{1}{N}$:
Note that if $j < N$ $g_{N^2}(f_j) = 0$ and if $j \geq N$ then $g_{N^2}(f_j) = \frac{1}{j} \leq \frac{1}{N}$ For simplicity we note the elements $x_n \in K$ as their convex combination $x_n = \sum_{k=1}^{\infty} \alpha_{n_k} f_k$ where we remember that the sum of the α_{n_k} is 1 and hence there is only a finite set of which they are non-zero thus can also be written as $x_n = \sum_{k=1}^{W_n} \alpha_{n_k} f_k$.

$x_n \in co\{f_n \mid n \in \mathbb{N}\}$

$$g_{N^2}(x_n) = \sum_{k=1}^{W_n} \alpha_{n_k} g_{N^2}(f_k) \leq \sum_{k=1}^{W_n} \alpha_{n_k}\frac{1}{N} = \frac{1}{N}$$

The exact same argument can be made for (y_n).
Therefore the only way for $\alpha g_{N^2}(x_n) + (1-\alpha)g_{N^2}(y_n) \to \frac{1}{N}$ to hold we must have that $g_{N^2}(x_n) \to \frac{1}{N}$ and $g_{N^2}(y_n) \to \frac{1}{N}$.
We know that $(x_n)_{n\geq 1}$ converges to a specific f_j if the sequence (α_{n_j}) converges to 1 ((α_{n_j}) is the sequence of j'th coefficent of the elements in the sequence $(x_n)_{n\geq 1}$).

Why?

We will show that if $g_{N^2}(x_n) \to \frac{1}{N}$ (and respectively for y_n) then $(x_n)_{n\geq 1}$ converges to f_N by showing that (α_{n_j}) converges to 1.
Assume that (α_{n_j}) does not converge to 1, therefore there must exist an $\epsilon > 0$ such that for every L there exist $n > L$ where $|1 - \alpha_{n_j}| > \epsilon$. As $\alpha_{n_j} \leq 1$ we have $r_n = 1 - \alpha_{n_j} > \epsilon$. Now we want to show the contradiction by showing $g_{N^2}(x_n) \not\to \frac{1}{N}$:

$$\left|\frac{1}{N} - g_{N^2}(\alpha(x_n) + (1-\alpha)(y_n))\right| = \frac{1}{N} - \alpha g_{N^2}(x_n) - (1-\alpha)g_{N^2}(y_n)$$

$$\geq \frac{1}{N} - \left(\alpha g_{N^2}(x_n) + (1-\alpha)\frac{1}{N}\right) \geq \alpha\frac{1}{N} - \left(\alpha\sum_{k=1}^{W_n}\alpha_{n_k} g_{N^2}(f_k)\right) = \alpha\frac{1}{N}(1-\alpha_{n_N}) - \left(\alpha\sum_{k=1, k\neq N}^{W_n}\alpha_{n_k} g_{N^2}(f_k)\right)$$

In the last equality we pulled out the N'th element of the sum. Now we use that $\sum_{i=1,i\neq N}^{W_n} \alpha_{n_k} = 1-\alpha_{n_N} = r_n$ (by definition of convex combination coefficents) and that for $k \neq N$ we have $g_{N^2}(f_k) \leq \frac{1}{N+1}$

$$\geq \alpha\left(\frac{r_n}{N} - \frac{r_n}{N+1}\right) \geq \epsilon \cdot \alpha\left(\frac{1}{N} - \frac{1}{N+1}\right)$$

Which contradicts the assumption of $g_{N^2}(x_n) \to \frac{1}{N}$. The exact same argument can be made for $(y_n)_{n\geq 1}$.
Thus we know that (α_{n_j}) converges to 1 and as said before this implies that $(x_n)_{n\geq 1} \to x = f_N$ and (by the same argument) $(y_n) \to x = f_N$.
We finally conclude that for any convex combination in K such that $f_N = \alpha x + (1-\alpha)y$ we must have that $x = y = f_N$ making f_N an extreme point in K.

✓

d)

Are there any other extreme points in K? Justify your answer. (An answer without justification will not be given any credit.)
We have that $K = \overline{co\{f_N\}}^{||\cdot||} = \overline{co\{f_N\}}^{w}$ and H with the weak topology is $LCTVS$ (top of page 27 lecture notes) thus by Milman (Theorem 7.9)

$$Ext(K) \subset \overline{\{f_N\}}^{w} =^{?} \{f_N, N \geq 1\} \cup \{0\}$$

Thus all the extreme points of K are contained in the set of f_N and 0, but we have shown that these points are extreme points. Therefore there are no more extreme points of K.

?: We have not shown the equality $\overline{\{f_N\}}^{w} = \{f_N, N \geq 1\} \cup \{0\}$, we will show it now.
As 0 is a weak limit point of f_n we have that $\overline{\{f_N\}}^{w} \supseteq \{f_N, N \geq 1\} \cup \{0\}$. To show the other way we will show that no sequence in $\{f_n\}$ has other weak limits. Assume that x is the weak limit of such a sequence then by "definition" of weak limit we must have that $\forall g \in H^*$ $g(x)$ is the limit of some sequence in $\{f_N\}$. Specifically we can use $H^* \ni g_1(x) := \langle x, e_1\rangle$. We note that $g_1(\{f_N\}) = \{N^{-1} | \forall N \in \mathbb{N}\}$ which is a set whose only accumulation points are 0 and N^{-1}.
If N^{-1} is an accumulation point: By "definition" of weak convergence and $\{N^{-1}\}_{N\in\mathbb{N}}$ being discrete, any sequence $(f_{N_j})_{j\in\mathbb{N}} \in \{f_N\}$ where $g_1(f_{N_j}) = N_j^{-1} \to N^{-1}$ as $j \to \infty$ will weakly converge to f_N.
If 0 is an accumulation point $((f_{N_j})_{j\in\mathbb{N}} \in \{f_N\}$ and $g_1(f_{N_j}) = N_j^{-1} \to 0)$ then N_j goes to infinity as $j \to \infty$. Thus (f_{N_j}) must have a subsequence where each $N_{j_k} < N_{j_l}$ for $k < l$. This subsequence is also a subsequence of (f_N) so it must converge weakly to 0. Therefore (f_{N_j}) must also converge weakly to 0. Thus we have shown that any sequence in (f_N) must have weak limit points in the set $\{f_N, N \geq 1\} \cup \{0\}$ ✓

Problem 2

Let X and Y be infinite dimensional Banach spaces.

a)

Let $T \in \mathcal{L}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

We again use HMW 4 Pb2 for the "definition" of weak convergence. And Theorem 7.13 for the existence of the Banach space adjoint which we denote T^*, where we note that all $T^*g(x_n)$ are elements in X^*
This is typically used for Hilbert space adjoint ✓

$$x_n \to x \text{ weakly } \Leftrightarrow f(x_n) \to f(x), \forall f \in X^* \Rightarrow T^*g(x_n) \to T^*g(x) \Leftrightarrow g(Tx_n) \to g(Tx) \Leftrightarrow T(x_n) \to T(x) \text{ weakly}$$

b)

Let $T \in \mathcal{K}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $||Tx_n - Tx|| \to 0$, as $n \to \infty$.

By Pb2 HMW 4 we know that $sup\{||x_n||\} < \infty$. So $\{x_1, x_2, ...\}$ is a bounded set, therefore T being compact implies $\overline{T(\{x_1, x_2, ..\})}$ is compact. I will state a result from Analysis 1 regarding norm convergence: If all subsequences of a sequence have convergent subsequence then the original sequence is convergent.

Let (Ty_l) be a subsequence of the sequence $(T(y_n))_{n\geq 1} = (T(x_n - x))_{n\geq 1}$, as T is compact one has that $\overline{T(\{y_1, y_2, ..\})}$ is compact (as (Ty_l) bounded) and thus there exists a converging subsequence (Ty_{l_j}) of (Ty_l) with $Ty_{l_j} \to \gamma$ for some γ. Where does γ live?

Now to show that $\gamma = 0$: From (a) we know that $g(T(x_n)) \to g(T(x))$ for all $g \in Y^*$ thus specifically $g(T(y_{l_j})) = g(T(x_{l_j} - x)) \to g(T(x - x)) = 0$ showing $\gamma = 0$.

Thus we have that $Tx_{n_{l_j}}$ (subsequence of the subsequence Tx_{n_l}) converges to Tx therefore every subsequence of Tx_n has a convergent subsequence converging to Tx and thus the sequence itself must be convergent to Tx, showing $||Tx_n - Tx|| \to 0$. ✓

c)

Let H be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H, Y)$ satisfies that $||Tx_n - Tx|| \to 0$ as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Assume that $||Tx_n - Tx|| \to 0$ as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$ and that T is not compact. T not being compact means that $T(\overline{B_H(0,1)}$ is not totally bounded (Def 8.1 and text below it). Thus there exists, by Proposition 8.2.(4), a sequence $(y_n)_{n\geq 1}$ in $T(\overline{B_H(0,1)})$ that has no convergent subsequences. But being in the image under T of the closed unit ball for each y_n we can pick a x_n such that $Tx_n = y_n$ for each n. Thus we have a sequence $(x_n)_{n\geq 1}$ inside the closed unit ball in H.

How does Prop. 8.2 give you this? Be explicit!

By theorem 6.3 in the notes $\overline{B_H(0,1)}$ is weakly compact thus $(x_n)_{n\geq 1}$ must have a weakly converging subsequence (x_{n_j}) and by b) we know that $T(x_{n_j})$ is a strongly converging sequence (i.e. $||Tx_{n_j} - Tx|| \to 0$) but this sequence is a converging subsequence of $(y_n)_{n\geq 1}$ which had no converging subsequences, thus we reach a contradiction and T must be a compact operator. (✓)

Generally this would be a subnet, not a subseq.

d)

Show that each $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ is compact.

Let $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$, we know that $l_2(\mathbb{N})$ is a separable Hilbert space. Thus we want to use c). So if we show its statement: "If $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ satisfies that $||Tx_n - Tx|| \to 0$ as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in $l_2(\mathbb{N})$ converging weakly to $x \in l_2(\mathbb{N})$, then $T \in \mathcal{K}(l_2(\mathbb{N}), l_1(\mathbb{N}))$". Let $(x_n)_{n\geq 1}$ be a weakly converging sequence in $l_2(\mathbb{N})$ converging to x. By a) we know for any $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ that $Tx_n \to Tx$ weakly, by Remark 5.3 in the notes (the text below the remark) this implies that $||Tx_n - Tx|| \to 0$. Thus we have shown exactly the prerequisites for c) for any $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$. Thus by c) all $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ are compact. ✓

e)

Show that no $T \in \mathcal{K}(X, Y)$ is surjective.

Assume per contradiction that K is a surjective compact map. By Theorem 5.10 Folland, T is an open map. And we know that T is open if and only if $T(B_X(0,1))$ contains a ball centered around 0_Y. Thus $B_Y(0,r) \subset T(B_X(0,1))$. Taking closure on both sides we get $\overline{B_Y(0,r)} \subset \overline{T(B_X(0,1))}$ as T is compact, the right hand side is compact and as $\overline{B_Y(0,r)}$ is a closed subset of a compact set it must be compact. But it is a contradiction with assignment 1 Pb3 e) where we showed that the unit ball in Y is not compact, thus any ball centered around 0 of some radius will not be compact. Hence our assumption that T was injective must be wrong. ✓

Elaborate on this

f)

Let $H = L_2([0,1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. Justify that M is self adjoint, but not compact

As H is a Hilbert space we check: What is fig?

$$\langle Mf, g\rangle = \int_{[0,1]} Mf \cdot \overline{g} dm = \int_{[0,1]} t \cdot f \cdot \overline{g} dm = \int_{[0,1]} f \cdot \overline{t \cdot g} dm = \int_{[0,1]} f \cdot \overline{Mg} dm = \langle f, Mg\rangle$$

Using $L^2([0,1],m)$ separable and infinite-dimensional

Thus M is self-adjoint.
Assume M is compact, then by the spectral theorem (Theorem 10.1 notes) H has an orthonormal basis of eigenvectors of M. But by problem 3a) in HMW 6 we know it has no eigenvalues thus we reach a contradiction and therefore M is not compact. (✓)

Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$ where m is the Lebesgue measure. Define K: $[0,1] \times [0,1] \to \mathbb{R}$ by (see the assignment text). And consider $T \in \mathcal{L}(H,H)$ defined by (see the assignment text).

a)

Justify that T is compact

$[0,1]$ is a compact Hausdorff space and m is a finite measure on it. K is piecewise continuous thus $K \in C([0,1] \times [0,1])$. Then by Theorem 9.6 in the notes, "the associated operator" T_k which is exactly T, is compact.

Piecewise cont. is not necessarilly continous

$T = T_{\tilde{k}}$, $\tilde{k}(s,t) = k(t,s)$.

b)

Show that $T^* = T$.

We firstly note that if x is real: ?

$$\overline{\int_{\mathbb{R}} f(x)dx} = \overline{\int_{\mathbb{R}} \alpha(x)dx + i\int_{\mathbb{R}} \beta(x)dx} = \int_{\mathbb{R}} \alpha(x)dx - i\int_{\mathbb{R}} \beta(x)dx = \int_{\mathbb{R}} \alpha(x) - i\beta(x)dx = \int_{\mathbb{R}} \overline{\alpha(x) + i\beta(x)}dx = \int_{\mathbb{R}} \overline{f(x)}dx$$

Let $f, g \in H$. (That the integrals are finite is shown in the lecture notes in p.46 of lecture 9. So we can use Furbini)

This is shown for $K \in L_2(X \times Y, \mu \otimes \nu)$ which has not been shown here.

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} \overline{g(s)} \int_{[0,1]} K(s,t)f(t)dm(t)dm(s) = \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(s)dm(t) = \int_{[0,1]} f(t)\int_{[0,1]} K(s,t)\overline{g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t)\int_{[0,1]} \overline{K(s,t)g(s)}dm(s)dm(t) = \int_{[0,1]} f(t)\overline{\int_{[0,1]} K(s,t)g(s)dm(s)}dm(t) = \langle f, Tg\rangle
\end{aligned}$$

(✓)

(where we used (s,t) are real) s, t are not fixed?
Which shows that T is self-adjoint

c)

Show that (see the assignment text). Use this to show that Tf is continuous on $[0,1]$ and that $(Tf)(0) = (Tf)(1) = 0$.

As the point s is of measure 0, we know from MI we can split the integral in the following way:

$$\begin{aligned}
Tf(s) = \int_{[0,1]} K(s,t)f(t)dm(t) &= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)
\end{aligned}$$

Use this to show that Tf is continuous:
Firstly we put it back together

$$(1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t) = \int_{[0,1]} K(s,t)f(t)dm(t)$$

Then we use continuity lemma (Lemma 12.4 Schilling) where we note that exactly the same proof can be given for a closed set (like $[0,1]$) instead of an open one like $(0,1)$. Even further, we note that the lemma is only for functions into $\mathbb{R}$ but can be used for function into $\mathbb{C}$ when $(f(x) = a(x)+ib(x))$ a, b are real valued function:
$[0,1]$ is nondegenerate closed. $u : [0,1]\times[0,1]$ where $u(s,t) = K(s,t)f(t)$.
(a) $t \to u(s,t)$ is in $L_1([0,1], m)$ for every fixed $s \in [0,1]$ as its integrable (shown in the lecture notes in p.46 of lecture 9).
(b) $s \to u(s,t)$ is continuous for every fixed $t \in [0,1]$.
(c) $|u(s,t)| = |K(s,t)f(t)| \leq w(t) = |f(t)|$ for all $(s,t) \in [0,1]\times[0,1]$ (where $|f(t)| \in L_1$ by HMW2 Problem 2b).

Idea is fine. But there are many claims which are not proven, or at least further argued for.

Thus we conclude that $\int u(s,t)dm = \int k(s,t)f(t)dm$ is continuous on $[0,1]$.

$$Tf(0) = (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]}(1-t)f(t)dm(t) = 0+0 = 0$$

$$Tf(1) = (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]}(1-t)f(t)dm(t) = 0+0 = 0$$

✓

Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$

a)

For each integer $k \geq 0$, set $g_k(x) = x^k e^{\frac{-x^2}{2}}$ for $x \in \mathbb{R}$. Justify that $g_k \in: \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$. Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

The function $x \to x^k e^{-\frac{x^2}{2}}$ is $C^\infty(\mathbb{R})$. Next notice that $\partial^\beta x^k e^{-\frac{x^2}{2}} = \frac{\partial^\beta}{\partial x^\beta} x^k e^{-\frac{x^2}{2}} = Pol(x)e^{-\frac{x^2}{2}}$ Where $Pol(x)$ is some polynomial in x. Therefore we get $x^\alpha \partial^\beta e^{-\frac{x^2}{2}} = Pol_2(x)e^{-\frac{x^2}{2}}$ where $Pol_2(x)$ is a gain some polynomial in x. But we know from MatIntro that $Pol_2(x)e^{-\frac{x^2}{2}} \to 0$ as $x \to \infty$ as the exponential goes faster to 0 than any polynomial. Thus we conclude that $g_k \in \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$. ✓
Now to computing g_k for $k = 0, 1, 2, 3$. By proposition 11.12 (b) $g_k \in L_p(\mathbb{R})$ for all $1 \leq p < \infty$, specifically we must have that $g_k \in L_1(\mathbb{R})$. $\mathcal{F}(g_0)$ is calculated exactly on page 57 of the notes under Solution 1. As its a matter of just copy pasting what is written, i will omit all the justifications as its 100% exactly what is shown there. The conclusion is $\mathcal{F}(g_0)(\xi) = e^{-\frac{\xi^2}{2}}$.
For g_1 and so on we can use Proposition 11.13 c) and d). As all the partial derivatives of g_0 are in $L_1(\mathbb{R})$

$$\mathcal{F}(g_1) = \mathcal{F}(g_0(x)x) = i\frac{d\hat{g}_0(\xi)}{d\xi} = -i\xi e^{-\frac{\xi}{2}}$$

Which we use to calculate g_k for $k = 2, 3$:

$$\mathcal{F}(g_2) = \mathcal{F}(g_0(x)x^2) = \overset{i^2}{\cancel{i}}\frac{d^2\hat{g}_0(\xi)}{d\xi^2} = (1-\xi^2)e^{-\frac{\xi}{2}}$$

$$\mathcal{F}(g_3) = \mathcal{F}(g_0(x)x^3) = \overset{i^3}{\cancel{i}}\frac{d^3\hat{g}_0(\xi)}{d\xi^3} = i(\xi^3 - 3\xi)e^{-\frac{\xi}{2}}$$

(✓)

b)

Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for k=0,1,2,3.

$$\mathcal{F}(h_0) = \mathcal{F}(g_0) = \mathcal{F}(e^{\frac{-x^2}{2}}) = e^{\frac{-\xi^2}{2}} = i^0 h_0$$

$$\mathcal{F}(h_1) = \mathcal{F}(g_3 - \frac{3}{2}g_1) = \mathcal{F}(e^{\frac{-x^2}{2}}(x^3 - \frac{3}{2}x)) = ie^{\frac{-\xi^2}{2}}(\xi^3 - \frac{3}{2}\xi) = i^1 h_1$$

$$\mathcal{F}(h_2) = \mathcal{F}(g_2 - \frac{1}{2}g_0) = \mathcal{F}(e^{\frac{-x^2}{2}}(x^2 - \frac{1}{2})) = e^{\frac{-\xi^2}{2}}((1-\xi^2) - \frac{1}{2}) = i^2 h_2$$

$$\mathcal{F}(h_3) = \mathcal{F}(g_1) = \mathcal{F}(xe^{\frac{-x^2}{2}}) = -i\xi e^{\frac{-\xi^2}{2}} = i^3 h_3$$

c)

Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$

By the definition of Fourier transform:

$$\hat{f}(\xi) = \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$$

$$\mathcal{F}^2(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{-i\xi x} dx$$

As $f \in \mathscr{S}(\mathbb{R})$ by definition 12.10 and corollary 12.12(iii) in the notes we know.

$$f(x) = \check{f}(x) = \mathcal{F}^*(\hat{f}(\xi)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} dx$$

By comparing the last two equations we see that $\mathcal{F}^2(f(x)) = f(-x)$. Thus $\mathcal{F}^4(f(x)) = \mathcal{F}^2(f(-x)) = f(x)$ for all $f \in \mathscr{S}(\mathbb{R})$

d)

Use (c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{\pm 1, \pm i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\lambda \in \{\pm 1, \pm i\}$.

From (c) we know that $\mathcal{F}^4(f) = f(x) = \lambda^4 f$ thus $\lambda^4 = 1$. As $\lambda \in \mathbb{C}$ the solutions are $\lambda = \{\pm 1, \pm i\}$. By the definition of eigenvalue ($\mathcal{F}f = \lambda f$) and by the fundamental theorem of algebra we know these 4 values are all the eigenvalues of $\mathcal{F}$.

The fundamental thm of algebra does not guarantee that all four solutions of $\lambda^4 = 1$ are eigenvalues. This follows from 4.6.

Problem 5

Let $(x_n)_{n\geq 1}$ be a dense subset of $[0,1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0,1]$. Show that $\operatorname{supp}(\mu) = [0,1]$.

Let N be the union of all open subsets U of $[0,1]$ such that $\mu(U) = 0$. By Problem 3 HMW 8 we know $supp(\mu) = N^c$. To show that $N^c = supp(\mu) = [0,1]$ we must show that $N = \emptyset$. To show that we must show that if an open set U has measure 0 then it must be the empty-set.
Assume U is an non-empty open set with $\mu(U) = 0$, by the definition of μ we must have that $x_n \notin U$ for any n. As U is non-empty and per the definition of open there must exist an open ball of radius ϵ around an element $x \in U$ of which all elements are contained in U. But by the defintion of dense in $[0,1]$, $B(x,\epsilon)$ must contain an element x_k of $(x_n)_{n\geq 1}$ which contradicts the assumption that $\mu(U) = 0$ as $\mu(U) \geq 2^{-k} > 0$. Thus the assumption of U being non-empty was wrong and we conclude that if we have an open set in $[0,1]$ such that $\mu(U) = 0$ it must be empty. Thus $N = \emptyset \Rightarrow N^c = supp(\mu) = [0,1]$.