FunkAn, Mandatory 1

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- **Problem 1** Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_y)$ be (non-zero) normed vector spaces over $\mathbb{K}$.

  (a) Let $T : X \longrightarrow Y$ be a linear map. Set $||x||_0 = ||x||_X + ||Tx||_Y$, for all $x \in X$. Show that $||\cdot||_0$ is a norm on X. Show next that the two norms $||\cdot||_x$ and $||\cdot||_0$ are equivalent if and only if T is bounded.

  Firstly we check that $||x + y||_0 \leq ||x||_0 + ||y||_0$. But this follows from

  $$||x + y||_0 = ||x + y||_X + ||Tx + Ty||_Y \leq ||x||_X + ||y||_X + ||Tx||_Y + ||Ty||_Y = ||x||_0 + ||y||_0$$

  Now to check that for all $\alpha \in \mathbb{K}$, $||\alpha x||_0 = |\alpha| \cdot ||x||_0$ we see that

  $$||\alpha x||_0 = ||\alpha x||_X + ||T\alpha x||_Y = |\alpha| \cdot ||x||_X + |\alpha| \cdot ||Tx||_Y = |\alpha|(||x||_X + ||Tx||_Y) = |\alpha| \cdot ||x||_0$$

  and finally we see that

  $$||x||_0 = ||x||_X + ||Tx||_Y = 0 \Leftrightarrow ||x||_X = 0 \text{ and } ||Tx||_Y = 0 \Leftrightarrow x = 0$$

  hence $||\cdot||_0$ is a norm.

  Now suppose that T is bounded. Then there exists $C > 0$ such that $||Tx||_Y \leq C||x||_X$ for all $x \in X$, so we get

  $$||x||_X \leq ||x||_X + ||Tx||_Y \leq ||x||_X + C||x||_X = (1 + C)||x||_X$$

  hence $||x||_X \leq ||x||_0 \leq (1 + C)||x||_X$, hence $||\cdot||_X$ and $||\cdot||_0$ are equivalent norms.

  Now suppose T is not bounded. Then for all $C > 0$ there exists $x \in X$ such that $||Tx||_Y > C||x||_X$. So we suppose that $||\cdot||_X$ and $||\cdot||_0$ are equivalent. Then there exists $c, C > 0$ such that $c||x||_X \leq ||x||_0 \leq C||x||_X$ for all $x \in X$. But this is a contradiction, as we know for all $C > 0$ there exists $x \in X$ such that $C||x||_X < ||Tx||_Y \leq ||x||_X + ||Tx||_Y = ||x||_0$. So in this case $||\cdot||_X$ and $||\cdot||_0$ are not equivalent norms.

  (b) Show that any linear map $T : X \longrightarrow Y$ is bounded, if X is finite dimensional.

  We want to show that there exists $K > 0$ such that $||Tx||_Y \leq K||x||_X$ for all $x \in X$. Since $\dim(X) < \infty$, we can consider a basis for X, $B = \{x_1, ..., x_n\}$, so we can write each $x \in X$ as $x = \sum_{i=1}^n \alpha_i x_i$ for some $\alpha_1, ..., \alpha_n \in \mathbb{K}$. Also since X is finite dimensional, we know that $||\cdot||_X$ and $||\cdot||_\infty$ are equivalent norms, where $||\sum_{i=1}^n \alpha_i x_i||_\infty = \max(|\alpha_1|, ..., |\alpha_n|)$. Since they are equivalent there are $c, C > 0$ such that $c||x||_X \leq ||x||_\infty \leq C||x||_X$. So we consider

  $$||Tx||_Y = ||T\sum_{i=1}^n \alpha_i x_i||_Y = ||\sum_{i=1}^n \alpha_i Tx_i||_Y \leq \sum_{i=1}^n |\alpha_i| \cdot ||Tx_i||_Y \leq \sum_{i=1}^n ||x||_\infty ||Tx_i||_Y$$

  $$= \left(\sum_{i=1}^n ||Tx_i||_Y\right) ||x||_\infty \leq \left(\sum_{i=1}^n ||Tx_i||_Y\right) C||x||_X$$

  hence for all $x \in X$ we have that $||Tx||_Y \leq K||x||_X$ with $K = \left(\sum_{i=1}^n ||Tx_i||_Y\right) C$

(c) Suppose that X is infinite dimensional. Show that there exists a linear map $T : X \longrightarrow Y$, which is not bounded.

Let $(x_i)_{i\in I} \subset X$ be a Hamel basis. Then it is easily seen that $(e_i)_{i\in I} = \left(\frac{x_i}{||x_i||_X}\right)_{i\in I}$ is also a Hamel basis, since each $x \in X$ can be written as $x = \sum_{i\in I} \lambda_i x_i = \sum_{i\in I}(\lambda_i||x_i||_X)e_i$, and we will still have finitely many $\lambda_i||x_i||_X \neq 0$. We then get that $||e_i||_X = 1$ for each $i \in I$.

Now let $(y_i)_{i\in I} \subset Y$ be a family such that $||y_i||_Y \longrightarrow \infty$ as $i \longrightarrow \infty$. Then there is a unique linear map $T : X \longrightarrow Y$ such that $T(e_i) = y_i$. Now since $||y_i||_Y \longrightarrow \infty$ as $i \longrightarrow \infty$, we have that for each $K > 0$ there exists $i \in I$ such that $||T(e_i)||_Y = ||y_i||_Y > K = K||e_i||_X$, hence T is not bounded.

(d) Suppose again that X is infinite dimensional. Argue that there exists a norm $||\cdot||_0$ on X, which is not equivalent to the given norm $||\cdot||_X$, and which satisfies $||x||_X \leq ||x||_0$, for all $x \in X$. Conclude that $(X, ||\cdot||_0)$ is not complete if $(X, ||\cdot||_X)$ is a Banach space.

From (c) we know there exists a linear map which is not bounded. Let $T : X \longrightarrow Y$ be such a map, and define $||x||_0 = ||x||_X + ||Tx||_Y$ as in (a). We have shown in (a) that this is a norm, and that it is equivalent to $||\cdot||_X$ if and only if T is bounded, hence as we have chosen T to not be bounded, we get that these are not equivalent norms. It is also clear that since $||Tx||_Y \geq 0$ we have $||x||_0 = ||x||_X + ||Tx||_Y \geq ||x||_X$ for all $x \in X$.

We suppose that $(X, ||\cdot||_X)$ is a Banach space, adn consider the identity map $\mathrm{id}_X : (X, ||\cdot||_X) \longrightarrow (X, ||\cdot||_0)$ which is clearly a surjective map. Now if $(X, ||\cdot||_0)$ is a Banach space, the open mapping theorem says that the identity map must be an open map, but this is true if and only if the two norms induce the same topology which they do if and only if they are equivalent norms. But $||\cdot||_0$ is constructed such that they are not equivalent norms, hence $(X, ||\cdot||_0)$ cannot be Banach and is therefor not complete.

(e) Give an example of a vector space X equipped with two inequivalent norms $||\cdot||$ and $||\cdot||'$ satisfying $||x||' \leq ||x||$, for all $x \in X$, such that $(X, ||\cdot||)$ is complete, while $(X, ||\cdot||')$ is not.

We consider the space $(\ell_l(\mathbb{N}), ||\cdot||_1)$ and let $||\cdot||' = ||\cdot||_\infty$. Then

$$||(x_n)_{n\geq 1}||_\infty = \max_{n\geq 1}(|x_n|) \leq \sum_{n=1}^{\infty} |x_n| = ||(x_n)_{n\geq 1}||_1$$

for all $(x_n)_{n\geq 1} \in \ell_1(\mathbb{N})$. Now we $(a_n)_{n\geq 1}$ be the sequence with n 1's and all zeros after. Then $||(a_n)_{n\geq 1}||_\infty = 1$ for all $n \in \mathbb{N}$ and $||(a_n)_{n\geq 1}||_1 = n$. But this means there cannot exist a $C > 0$ such that $||(a_n)_{n\geq 1}||_1 \leq C||(a_n)_{n\geq 1}||_\infty = C$ for all $n \in N$, since you can always find an $n \in \mathbb{N}$ such that $n > C$. In particular this means there is no $C > 0$ such that $||(x_n)_{n\geq 1}||_1 \leq C||(x_n)_{n\geq 1}||_\infty$ for all $(x_n)_{n\geq 1} \in \ell_1(\mathbb{N})$, hence the norms are not equivalent, and by the same argument as in (d) we get that $(\ell_1(\mathbb{N}), ||\cdot||_\infty)$ is not complete.

- **Problem 2** Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), ||\cdot||_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, 0, ...) : a, b \in \mathbb{C}\}$$

Let $f : M \longrightarrow \mathbb{C}$ be give by $f(a, b, 0, 0, 0, ...) = a + b$, for all $a, b \in \mathbb{C}$.

(a) Show that f is bounded on $(M, ||\cdot||_p)$ and compute $||f||$.

We see that $|a| = (|a|^p)^{1/p} \leq (|a|^p + |b|^p)^{1/p}$ for all $a, b \in \mathbb{C}$, so we get that

$$|f(a, b, 0, ...)| = |a + b| \leq |a| + |b| \leq 2(|a|^p + |b|^p)^{1/p} = 2||(a, b, 0, ...)||_p$$

hence f is bounded.

Now we want to calculate $||f|| = \sup\{|f(a, b, 0, ...)| : ||(a, b, 0, ...)||_p = 1\}$. We see that

$\sup\{|f(a, b, 0, ...)| : ||(a, b, 0, ...)||_p = 1\} = \sup\{|a+b| : |a|^p+|b|^p = 1\} \leq \sup\{|a|+|b| : |a|^p+|b|^p = 1\}$

Now both $|a| + |b|$ and $|a|^p + |b|^p$ can be considered as functions on $\mathbb{R}^2_{>0} \longrightarrow \mathbb{R}$, so for $p > 1$ we can use the Lagrange method and find that this is maximized when $|a| = |b|$, and thus get that

$$2|a|^p = 1 \Rightarrow |a| = \frac{1}{2^{1/p}}$$

and we then get that

$$||f|| = \sup\{|f(a, b, 0, ...)| : ||(a, b, 0, ...)||_p = 1\} \leq \sup\{|a| + |b| : |a|^p + |b|^p = 1\} = \frac{2}{2^{1/p}}$$

Now we see that $\frac{2}{2^{1/p}} \in \{|f(a, b, 0, ...)| : ||(a, b, 0, ...)||_p = 1\}$ since $|f(\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}}, 0, ...)| = \frac{2}{2^{1/p}}$, so we have that

$$\frac{2}{2^{1/p}} \leq ||f|| \leq \frac{2}{2^{1/p}}$$

hence $||f|| = \frac{2}{2^{1/p}}$. For $p = 1$ we clearly get that $\sup\{|a| + |b| : |a|^p + |b|^p = 1\} = 1$, and $1 = |f(\frac{1}{2}, \frac{1}{2}, 0, ...)| \in \{|f(a, b, 0, ...)| : ||(a, b, 0, ...)||_p = 1\}$, so $||f|| = 1$.

(b) Show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $||F|| = ||f||$.

We know that $\ell_q(\mathbb{N}) \cong (\ell_p(\mathbb{N}))^*$ isometrically when $\frac{1}{q} + \frac{1}{p} = 1$ or equivalently $q = \frac{p}{p-1}$. We have that all $F : \ell_p(\mathbb{N}) \longrightarrow \mathbb{C}$ are given as $F_{(y_n)_{n\geq1}}((x_n)_{n\geq1}) = \sum_{n=1}^{\infty} x_n y_n$ for some $(y_n)_{n\geq1} \in \ell_q(\mathbb{N})$, where $||F_{(y_n)_{n\geq1}}|| = ||(y_n)_{n\geq1}||_q$.

Now let $(a_n)_{n\geq1}$ be the sequence with 1 in the i'th place and zeros everywhere else, and let $F_{(y_n)_{n\geq1}}$ be an extension of f with $||F_{(y_n)_{n\geq1}}|| = ||f||$ which exists by corollary 2.6. We then have

$$1 = f(a_1) = f(a_2) = F(a_1) = F(a_2) = y_1 = y_2$$

Suppose now that $y_n \neq 0$ for some $n > 2$. Then

$$||F_{(y_n)_{n\geq1}}|| = ||(y_n)_{n\geq1}||_q = \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{1/q} = \left(\sum_{n=1}^{\infty} |y_n|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$= \left(1 + 1 + \sum_{n=3}^{\infty} |y_n|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} > 2^{\frac{p-1}{p}} = \frac{2}{2^{1/p}} = ||f||$$

hence we must have $y_n = 0$ for all $n > 2$. Thus $F_{(y_n)_{n\geq1}} \in (\ell_p(\mathbb{N}))^*$ is the map determined by $(y_n)_{n\geq1} = (1, 1, 0, 0, 0, ...)$ which is unique, as $(y_n)_{n\geq1} \longmapsto F_{(y_n)_{n\geq1}}$ is an isomorphism.

(c) Show that if $p = 1$, then there are infinitely many linear functionals F on $\ell_1(\mathbb{N})$ extending f and satisfying $||F|| = ||f||$.

For each $2 < i \in \mathbb{N}$ consider $F_i : \ell_p(\mathbb{N}) \longrightarrow \mathbb{C}$, given by $F_i(a_1, a_2, a_3, ...) = a_1 + a_2 + a_i$. Clearly this is an extension of f, as $F(a, b, 0, 0, 0, ...) = a + b = f(a, b, 0, 0, 0, ...)$. We see now that

$$||F_i|| = \sup\{|a_1 + a_2 + a_i| : \sum_{n=1}^{\infty} |a_n| = 1\} \leq \sup\{|a_1| + |a_2| + |a_i| : \sum_{n=1}^{\infty} |a_n| = 1\} \leq 1$$

but $1 \in \{|a_1 + a_2 + a_i| : \sum_{n=1}^{\infty} |a_n| = 1\}$, as you can take $(a_n)_{n\geq 1}$ with $a_1 = a_2 = a_i = \frac{1}{3}$ and zeros everywhere else. So we get that $||F_i|| = 1 = ||f||$ for all $i > 2$, hence there are infinitely many extensions F of f such that $||F|| = ||f||$.

- **Problem 3** Let X be an infinite dimensional normed vector space over $\mathbb{K}$.

  (a) Let $n \geq 1$ be an integer. Show that no linear map $F : X \longrightarrow \mathbb{K}^n$ is injective.

  Let $\{x_m\}_{m\in\mathbb{N}} \subset X$ be an infinite linearly independent set, and suppose that $F : X \longrightarrow \mathbb{K}^n$ is an injective linear map. Then $\{F(x_m)\}_{m\in\mathbb{N}} \subset \mathbb{K}^n$ is linearly independent, since if there exists $0 \neq \alpha \in \mathbb{K}$ such that $F(x_i) = \alpha F(x_j) = F(\alpha x_j)$ then F is not injective, as $x_i \neq \alpha x_j$ since $\{x_m\}_{m\in\mathbb{N}}$ is linearly independent. It is also clear that $\{F(x_m)\}_{m\in\mathbb{N}}$ is an infinite set, since F is injective, so $F(x_i) \neq F(x_j)$ for $i \neq j$. But then $\{F(x_m)\}_{m\in\mathbb{N}} \subset \mathbb{K}^n$ cannot be linearly independent, as a linearly independent set in $\mathbb{K}^n$ has at most $n = \dim(\mathbb{K}^n)$ elements. Hence there cannot exist an injective linear map $F : X \longrightarrow \mathbb{K}^n$.

  (b) Let $n \geq 1$ be an integer and let $f_1, f_2, ..., f_n \in X^*$. Show that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq 0$$

  We consider the map $F : X \longrightarrow \mathbb{K}^n$ given by $F(x) = (f_1(x), ..., f_n(x))$. By the linearity of the f_i's we get that F is linear since

$$\begin{aligned} F(\alpha x + \beta y) &= (f_1(\alpha x + \beta y), ..., f_n(\alpha x + \beta y)) \\ &= (\alpha f_1(x) + \beta f_1(y), ..., \alpha f_n(x) + \beta f_n(y)) \\ &= (\alpha f_1(x), ..., \alpha f_n(x)) + (\beta f_1(y), ..., \beta f_n(y)) \\ &= \alpha(f_1(x), ..., f_n(x)) + \beta(f_1(y), ..., f_n(y)) = \alpha F(x) + \beta F(y) \end{aligned}$$

  Now we want to show that

$$\ker(F) = \bigcap_{j=1}^{n} \ker(f_j)$$

  Let $x \in \ker(F)$. Then $(0, ..., 0) = F(x) = (f_1(x), ..., f_n(x))$ hence $f_i(x) = 0$ for all $i = 1, ..., n$, so $x \in \bigcap_{j=1}^{n} \ker(f_j)$, so $\ker(F) \subset \bigcap_{j=1}^{n} \ker(f_j)$.

  Now let $x \in \bigcap_{j=1}^{n} \ker(f_j)$. Then $F(x) = (f_1(x), ..., f_n(x)) = (0, ..., 0)$ so $x \in \ker(F)$, so $\bigcap_{j=1}^{n} \ker(f_j) \subset \ker(F)$. This means

$$\ker(F) = \bigcap_{j=1}^{n} \ker(f_j)$$

  Now we know that a linear map F is injective if and only if $\ker(F) = 0$, but by (a) F cannot be injective, hence

$$\bigcap_{j=1}^{n} \ker(f_j) = \ker(F) \neq 0$$

  as we wanted to show.

(c) Let $x_1, x_2, ..., x_n \in X$. Show that there exists $y \in X$ such that $||y|| = 1$ and $||y - x_j|| \geq ||x_j||$ for all $j = 1, 2, ..., n$.

If $x_j = 0$ for any $j = 1, ..., n$, clearly $||y - x_j|| = ||y|| \geq ||x_j|| = 0$, so we may assume $x_1, ..., x_n \neq 0$. Then by theorem 2.7 b) we have for each $j = 1, ..., n$ there exist $f_j \in X^*$ such that $f_j(x_j) = ||x_j||$ and $||f_j|| = 1$.

By (b) we have that $\bigcap_{j=1}^n \ker(f_j) \neq \{0\}$, so we consider a $0 \neq y \in \bigcap_{j=1}^n \ker(f_j)$. It is clear that if $y \in \bigcap_{j=1}^n \ker(f_j)$, then $\frac{y}{||y||} \in \bigcap_{j=1}^n \ker(f_j)$, since if $f_j(y) \neq 0$ for all $j = 1, ..., n$ then $f_j(\frac{y}{||y||}) = \frac{1}{||y||} f_j(y) \neq 0$ for all $j = 1, ..., n$, so we may assume that $||y|| = 1$.

Now we just need to show that $||y - x_j|| \geq ||x_j||$ for all $j = 1, ..., n$. For arbitrary $j = 1, ..., n$ we have the following:

- Since $||f_j|| = 1$, we get that $|f(y - x_j)| \leq ||y - x_j|| \cdot ||f_j|| = ||y - x_j||$.
- Since f_j is linear $f_j(y - x_j) = f_j(y) - f_j(x_j)$.
- Since $y \in \bigcap_{j=1}^n \ker(f_j)$, we have that $f_j(y) = 0$ for each $j = 1, ..., n$, hence $|f_j(y) - f_j(x_j)| = |f_j(x_j)|$.

Combining this we get that

$$||y - x_j|| \geq |f_j(y - x_j)| = |f_j(y) - f(x_j)| = |f(x_j)| = ||x_j||$$

which is what we wanted to show.

(d) Show that one cannot cover the unit sphere $S = \{x \in X : ||x|| = 1\}$ wit a finite family of closed balls in X such that none of the balls contain 0.

Assume for contradiction that S can be covered by a finite family of closed balls not containing 0. Then

$$S = \bigcup_{i=1}^n \overline{B_{r_j}(x_j)}$$

for some $x_1, ..., x_n \in X$, such that $0 \notin \overline{B_{r_j}(x_j)}$ for any $j = 1, ..., n$. But then we must have $r_j < ||x_j||$ for each $j = 1, ..., n$ since otherwise $||x_j - 0|| = ||x_j|| \leq r_j$, so $0 \in \overline{B_{r_j}(x_j)}$.

Now by (c) there exist $y \in X$ with $||y|| = 1$, hence $y \in S$, and such that $||y - x_j|| \geq ||x_j|| > r_j$ for all $j = 1, ..., n$, which means $y \not\overline{B_{r_j}(x_j)}$ for any $j = 1, ..., n$ so $y \notin \bigcup_{i=1}^n \overline{B_{r_j}(x_j)} = S$ which is a contradiction, hence S cannot be covered by a finite family of closed balls not containing 0.

(e) Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

Assume for contradiction that S is compact, and consider the open cover

$$\left(\bigcup_{x \in S} B_{1/2}(x) \right) \cap S = S$$

Now since S is assumed to be compact, there exist a finite subcover, so there are $x_1, ..., x_n \in S$ such that

$$\left(\bigcup_{j=1}^n B_{1/2}(x_j) \right) \cap S = S$$

But $B_{1/2}(x_j) \subset \overline{B_{1/2}(x_j)}$ for all $j = 1, ..., n$, so $\bigcup_{j=1}^n B_{1/2}(x_j) \subset \bigcup_{j=1}^n \overline{B_{1/2}(x_j)}$ hence we get

$$\left(\bigcup_{j=1}^n \overline{B_{1/2}(x_j)} \right) \cap S = S \Rightarrow S \subset \bigcup_{j=1}^n \overline{B_{1/2}(x_j)}$$

We now note that $0 \notin \overline{B_{1/2}(x_j)}$ for any $j = 1, ..., n$ since $||x_j - 0|| = ||x_j|| = 1 > \frac{1}{2}$, but then by (d) there exist $y \in S$ such that $y \notin \bigcup_{j=1}^n \overline{B_{1/2}(x_j)}$ which contradicts $S \subset \bigcup_{j=1}^n \overline{B_{1/2}(x_j)}$, so S cannot be compact.

We see that the closed unit ball $\overline{B_1(0)} = B_1(0) \cup S$, and we note that $B_1(0) \cap S = \emptyset$. So any open cover of S together with $B_1(0)$ will be an open cover of $\overline{B_1(0)}$, but we have just shown there exists an open cover of S with no finite subcover, so since $B_1(0) \cap S = \emptyset$ any open cover of S with no finite subcover, together with $B_1(0)$ will give an open cover of $\overline{B_1(0)}$ with no finite subcover, hence $\overline{B_1(0)}$ is not compact.

- **Problem 4** Let $L_1([0,1], m)$ and $L_3([0,1], m)$ be the Lebesgue spaces on $[0,1]$. Recall that $L_3([0,1], m) \subsetneq L_1([0,1], m)$. For $n \geq 1$, define

$$E_n := \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm \leq n \right\}$$

(a) Given $n \geq 1$, is the set $E_n \subset L_1([0,1], m)$ absorbing?

We consider $f : [0,1] \longrightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x^{1/3}}$. Then $f \in L_1([0,1], m)$ since

$$\int_{[0,1]} |f| dm = \int_0^1 \frac{1}{x^{1/3}} dx = \left[\frac{3}{2} x^{2/3} \right]_0^1 = \frac{3}{2} < \infty$$

but for all $t > 0$ we have that

$$\int_{[0,1]} |tf|^3 dm = t^3 \int_0^1 \frac{1}{x} dx = t^3 \left[\log(x) \right]_0^1 = \infty > n$$

hence there is no $t > 0$ such that $tf \in E_n$ for any $n \geq 1$, so E_n is not absorbing for any $n \geq 1$.

(b) Show that E_n has empty interior in $L_1([0,1], m)$, for all $n \geq 1$.

Suppose there exists $f \in \text{int}(E_n)$. Then there exists $r > 0$ such that $B_r(f) \subset E_n$. We define $g : [0,1] \longrightarrow \mathbb{R}$ given by $g(x) = \frac{1}{x^{1/3}}$, and consider $f - \frac{r}{2} g$. This is in $B_r(f)$ since by the calculations we did in (a)

$$||f - (f - \frac{r}{2} g)||_1 = ||\frac{r}{2} g||_1 = \frac{r}{2} ||g||_1 = \frac{3r}{4} < r$$

But $f - \frac{r}{2} g \notin E_n$ since we know $||f||_3 = c$ for some $c \leq n$, so by the use of the reverse triangle inequality we get

$$||f - \frac{r}{2} g||_3 \geq \Big| ||f||_3 - ||\frac{r}{2} g||_3 \Big| = |c - \infty| = \infty > n$$

But this is a contradiction hence $\text{int}(E_n) = \emptyset$.

(c) Show that E_n is closed in $L_1([0,1], m)$, for all $n \geq 1$.

Let $(f_m)_{m \in \mathbb{N}} \subset E_n$ be a convergent sequence with $|f - f_m| \longrightarrow 0$ as $m \longrightarrow \infty$ for some $f \in L_1([0,1], m)$. We now know that there is a convergent subsequence $(f_{m_l})_{l \in \mathbb{N}} \subset (f_m)_{m \in \mathbb{N}}$ such that

$\lim_{l\Rightarrow\infty} f_{m_l} = f$ almost everywhere. But then since norms and exponents are continuous we get that $\lim_{l\to\infty} |f_{m_l}|^3 = |f|^3$ almost everywhere. Then we use Fatou's lemma, and get

$$\int_{[0,1]} |f|^3 = \liminf_{l\to\infty} \int_{[} 0,1]|f_{m_l}|^3 \leq n$$

hence $f \in E_n$, so every convergent sequence in E_n has its limit in E_n, hence E_n is closed.

(d) Conclude from (b) and (c) that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

From (c) we get that $E_n = \bar{E_n}$, and then we get from (b) that $\text{int}(\bar{E_n}) = \emptyset$, hence E_n is nowhere dense for each $n \geq 1$. Now we want to show that $L_3([0,1],m) = \bigcup_{n=1}^{\infty} E_n$.

Let $f \in \bigcup_{n=1}^{\infty} E_n$. Then $f \in E_n$ for some n, hence $\int_{[0,1]} |f|^3 dm \leq n$, hence $f \in L_3([0,1],m)$, so $\bigcup_{n=1}^{\infty} E_n \subset L_3([0,1],m)$. Now let $f \in L_3([0,1],m)$. Then $int_{[0,1]}|f|^3 dm < c$ for some $0 < c < \infty$. But then there exists $c < n \in \mathbb{N}$, which means $f \in E_n$ so $f \in \bigcup_{n=1}^{\infty} E_n$, hence $L_3([0,1],m) \subset \bigcup_{n=1}^{\infty} E_n$, so they must be equal.

So $L_3([0,1],m) = \bigcup_{n=1}^{\infty} E_n$ which is the union of a sequence of nowhere dense sets in $L_1([0,1],m)$ hence it is of the first category.

- **Problem 5** Let H be an infinite dimensional Hilbert space with associated norm $||\cdot||$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

(a) Suppose that $x_n \longrightarrow x$ in norm, as $n \longrightarrow \infty$. Does it follow that $||x_n|| \longrightarrow ||x||$, as $n \longrightarrow \infty$? Give a proof or a counterexample.

By assumption $\lim_{n\to\infty} ||x_n - x|| = 0$, so we use the reverse triangle inequality on $\lim_{n\to\infty} \big| ||x_n|| - ||x|| \big|$, and get

$$\lim_{n\to\infty} \big| ||x_n|| - ||x|| \big| \leq \lim_{n\to\infty} ||x_n - x|| \leq 0$$

hence $\lim_{n\to\infty} \big| ||x_n|| - ||x|| \big| = 0$, so $||x_n|| \longrightarrow ||x||$ as $n \longrightarrow \infty$ as we wanted to show.

(b) Suppose that $x_n \longrightarrow x$ weakly, as $n \longrightarrow \infty$. Does it follow that $||x_n|| \longrightarrow ||x||$, as $n \longrightarrow \infty$? Give a proof or a counterexample.

Let $(e_n)_{n\geq 1} \subset H$ be an orthonormal basis. By the Riesz representation theorem, we know that for each $f \in H^*$ there exist $x_f \in H$ such that $f(x) = \langle x, x_f \rangle$ for all $x \in H$. Now using Bessels inequality we get that for each $f \in H^*$

$$\sum_{n=1}^{\infty} |f(e_n)|^2 = \sum_{n=1}^{\infty} \langle e_n, x_f \rangle|^2 \leq ||x_f||^2 < \infty$$

hence the tail of the sequence $(f(e_n))_{n\geq 1}$ must tend towards 0, i.e. $\lim_{n\to\infty} |f(e_n) - f(0)| = \lim_{n\to\infty} |f(e_n)| = 0 = f(0)$ for every $f \in H^*$. Now by HW4 problem 2 a) we get that $(e_n)_{n\geq 1}$ converges weakly to 0. But $\lim_{n\to\infty} ||e_n|| = \lim_{n\to\infty} 1 = 1 \neq 0 = ||0||$, so $(e_n)_{n\geq 1}$ converges weakly to 0 but converges to 1 in norm, so this is a counterexample.

(c) Suppose that $||x_n|| \leq 1$, for all $n \geq 1$, and that $x_n \longrightarrow x$ weakly, as $n \longrightarrow \infty$. Is it true that $||x|| \leq 1$? Give a proof or a counterexample.

By theorem 2.7 b) there exist $f \in X^*$ such that $f(x) = ||x||$ and $||f|| = 1$. Since $x_n \longrightarrow x$ weakly, we know that $\lim_{n\to\infty} |g(x) - g(x_n)| = 0$ for all $g \in X^*$ and in particular for f.

Suppose $||x|| > 1$. Then there exists $\varepsilon > 0$ such that $||x|| \geq 1 + \varepsilon$. We also know that $|f(x_n)| \leq ||x_n|| \cdot ||f|| = ||x_n|| \leq 1$. We then get that

$$|f(x) - f(x_n)| = \big| ||x|| - f(x_n) \big| \geq \big| ||x|| - |f(x_n)| \big| \geq |1 + \varepsilon - 1| \geq \varepsilon$$

for all $n \geq 1$. But then

$$\lim_{n\to\infty} |f(x) - f(x_n)| \geq \varepsilon > 0$$

which is a contradiction, hence we must have $||x|| \leq 1$.