PHD SCHOOL OF SCIENCE
UNIVERSITY OF COPENHAGEN

3H. Final assessment of the PhD thesis

If the final assessment is not ready immediately after the PhD defense, the PhD student must be notified whether the PhD degree is awarded or not.
The final assessment must be submitted by the assessment committee no later than **one week** after the date of the PhD defense.
Please note: The PhD student will receive the final assessment report and typically use it for his/her future career

1.General information

Name: Johannes Agerskov Schiødt

Danish CPR number: 050395-1215 Department: Department of Mathematical Sciences

2. Supervisor(s)

Principal Supervisor
Title & Name: Professor Jan Philip Solovej E-mail: solovej@math.ku.dk

Co-supervisor
Title & Name: E-mail:
Place of employment:

Co-supervisor
Title & Name: E-mail:
Place of employment:

3. PhD thesis

Title of PhD thesis: One-dimensional Dilute Quantum Gases and Their Ground State Energies

Date of defense: 06/06 2023 Can the PhD degree be awarded? [X] **Yes** [ ] **No**

4. Assessment of the PhD thesis

The assessment should include a discussion of the aim and hypothesis, results, methods and conclusions. The different parts of the thesis should be evaluated as well as the mutual relationship between the different parts. The assessment should also evaluate if the state-of-the-art research is presented and understood in the context of the overall conclusion.

Please note: it is possible to attach 2-3 A4 pages instead of filling in section 4.

Short overview of the structure of the thesis

The thesis of Johannes Agerskov, discusses the ground-state properties of one-dimensional quantum systems, of bosonic, fermionic and also anyonic type.

Chapter 1 contains a concise introduction to the subject of the thesis, and to relevant recent results in the literature. The introduction gives a brief summary of the results presented in the manuscript, and of the ideas used in their proof.

Chapter 2 introduces the necessary mathematical concepts needed to formulate the problems studied in the thesis. In particular: many-body wave functions of identical particles; the spectral theorem; the energy functional and the ground state energy; delta interactions, appearing in various exactly solvable models discussed later; the even and the odd scattering length. Then, the dilute regime is introduced, and rigorous results for the ground state energy asymptotics are reviewed, both for fermions and for bosons. Most of the available results in the literature deal with dimension greater or equal than two. A notable feature of the one-dimensional case is the existence of exactly solvable models, for special choices of the interaction potential. The thesis reviews the exact solution of the Lieb-Liniger model, for spinless bosons, via the Bethe ansatz. Then, the thesis also discusses the more complicated case of the Yang-Gaudin model, describing fermions with spin.

Chapter 3 contains a recent preprint of Agerskov, Reuvers and Solovej, about the asymptotics of the ground state energy of dilute, spinless Bose gases in one dimension. The type of interaction potential considered is quite general, and it also allows for a hard-core component. The main result of the chapter is a theorem providing a two-term asymptotics in the density for the ground state energy of the system. The two explicit terms appearing in the result display universality: the first term corresponds to the ground state energy of a non-interacting, 1d Fermi gas, while the second term only depends on the scattering length of the potential (as for the higher dimensional counterparts of such result). The physical significance of the result is drastically different from the two and three dimensional case, as it is consistent with the absence of Bose-Einstein condensation, as is expected in one dimension. Interestingly, a corollary of the result is an analogous two-term asymptiotics for the ground state energy of dilute particles of anyonic type (including fermions), due to the fact that the energy of the 1d anyon gas agrees with the energy of a Bose gas modified by a suitable two-body delta interaction, depending on the statistics parameter.

The strategy of the proof is based on matching upper and lower bounds for the ground state energy of the system, that agree at next-to-leading order in the density. The upper bound is based on the choice of a suitable trial state: namely, the absolute value of the free Fermi gas with Dirichlet boundary conditions, suitably weighted by the solution of the scattering equation. The proof proceeds by restricting the trial state into boxes containing not too many particles, and relies of a precise control of the density matrices of the free Fermi gas in each box. Such control is made particularly non-trivial due to the Dirichlet boundary conditions. The lower bound is based on the comparison with the Lieb-Liniger model with Neumann boundary conditions, which appears as a reference model for the ground state energy. In order to make the comparison rigorous, a crucial role is played by Dyson's lemma. Then, a precise control of the ground state energy of the Lieb-Liniger model allows to obtain the desired expansion for the energy, for small values of the number of particles N. Larger values of N can be considered using an argument based on Legendre transform, and superadditivity of the energy. Finally, the chapter is concluded with the extension of the result to anyons, via a unitary transformation.

Chapter 4 focuses on the ground state energy of spin-1/2 dilute fermions. The main result contained in the chapter is an upper bound for the ground state energy of the system. The expression obtained exhibits universality, however it differs qualitatively from the analogous expressions obtained in dimension higher than one. Specifically, the second order here depends on both the even and the odd scattering lengths. The chapter conjectures that the upper bound is actually sharp, as a next-to-leading order asymptotics for the ground state energy. This is consistent with few explicit cases, namely the hard-core Fermi gas, and the Yang-Gaudin model. Also, the chapter contains a nice check of the conjecture using perturbation theory, via the Feynman-Hellman argument. The proof of the rigorous upper bound is based on a judicious choice of a trial state, that now comes with a non-trivial spin dependence. The most interesting aspect of the proof is, in our view, the fact that the energy of the trial state turns out to be related to the ground state energy of the antiferromagnetic Heisenberg chain, another exactly solvable 1d system. The result has several corollaries, among which an improvement of an old upper bound for the ground state energy of the Lieb-Liniger-Heisenberg (LLH) model due to Girardeau. The proof of the lower bound remains open. The thesis proposes to follow the strategy adopted in Chapter 3: in the present case, Dyson's lemma would allow to bound below the ground state energy using the Lieb-Liniger-Heisenberg model. However, the main problem in concluding the proof is that the Lieb-Liniger-Heisenberg is not exactly solvable via Bethe ansatz. Thus, this comparison, in general, does not allow yet to get an explicit expansion for the ground state energy as function of the density. The chapter is concluded with the discussion of a few special cases, in which the LLH ground state energy can be boundedbelow using the Lieb-Liniger energy, where a more explicit control on the energy can be achieved.

Finally, Chapter 5 nicely summarizes the content of the thesis, and discusses the open problems that can be studied via an extension of the methods introduced in the thesis.

Objectives of the thesis' research
Are the objectives clearly expressed in the thesis, and is the state- of- the- art research well expressed with relevant review of recently published results

The objectives of the research are clearly laid out in the introduction. Further details and state-of-the-art are well-presented in the overview in Chapter 2.

Summary and quality of main findings and scientific achievements

The thesis presents interesting new results on the ground state properties of many-body systems. In the last years there has been a lot of activity in dimensions two and three, but relatively little has been done in dimension one. Therefore, the thesis of Johannes Agerskov introduces substantial progress in the field by both establishing the two-terms asymptotics of the ground state energy in the spinless case and by breaking ground towards the spin ½-case.
The one-dimensional world has special features, among which the existence of several exactly solvable systems. These are of course non-generic situations, however in physics it is often argued that they are the representative of universality classes: systems belonging to the same universality class share the same qualitative features, despite being very different at the microscopic level. The present thesis introduces a strategy that allows to make rigorous use of exactly solvable models, to obtain predictions about nonsolvable one-dimensional systems, in a nonperturbative way. This insight is particularly interesting and noteworthy.
The writing of the thesis, is generally very clear and shows considerable overview and mastery of the technical details and fundamental ideas.

Quality of all papers (unsubmitted, submitted as well as published)

The thesis contains a preprint as Chapter 3. This preprint is clearly publishable in a high quality journal of mathematical physics as it contains the main result on the energy of the 1-dimensional gas.

Other relevant factors

Assessment of oral defence
Comments on the oral presentation and evaluation of the ability to conduct a scientific discussion

Agerskov gave a very good overview of the research of the thesis where the most important points were clearly presented. He answered all questions of the jury with a mastery of the subject and broad understanding of the research field. In the discussion, also the open problem of the lower bound for the spin-1/2 case was treated in detail and Agerskov presented very recent developments with confidence. This demonstrates an excellent level of competence in scientific discussion.

Overall conclusion

Agerskov has written a clear thesis with important new results on the 1-d Bose gas – a very timely subject in a competitive research field. The results are interesting and original and connect different mathematical aspects of quantum mechanics. He has presented the results with high competence and giving a broad perspective.
In conclusion, the jury strongly recommends that he be awarded the Ph.D degree.

5. Members of the assessment committee

Member 1 (SCIENCE, chairman)

Title and Name:	Professor, Søren Fournais
Department	Department of Mathematical Sciences, University of Copenhagen
Date: 6/6 2023	Signature: Søren Fournais

Member 2 (External)

Title and Name:	Professor, Marcello Porta
Place of employment:	Sissa (International School for Advanced Studies), Trieste
Date: 6/6 2023	Signature: Marcello Porta

Member 3 (External)

Title and Name:	Professor, Simona Rota Nodari
Place of employment:	Université Côte d'Azur
Date: 6/6 2023	Signature: Simona Rota Nodari

Once you have completed the form, please sign (all members) and hand it to the PhD secretary at your department. You can find your PhD secretary here: www.science.ku.dk/phd/