Problem 1: Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.

(a) Show that $f_N \to 0$ weakly, as $N \to \infty$, while $\|f_N\| = 1$ for all $N \geq 1$.

First we compute $\|f_N\|$ as follows

$$\|f_N\| = \frac{1}{N}\|\sum_{n=1}^{N^2} e_n\| = \frac{1}{N}(\sum_{n=1}^{N^2}\|e_n\|^2)^{\frac{1}{2}} = \frac{1}{N}(N^2)^{\frac{1}{2}} = 1.$$

This shows that $\|f_N\| = 1$ for all $N \geq 1$. Now given $\epsilon > 0$ and some $g \in H$ we need to show that there exists N_ϵ s.t. $|\langle f_N, g\rangle| < \epsilon$ for all $N \geq N_\epsilon$. By the triangle inequality we get

$$|\langle f_N, g\rangle| = |\langle f_N, \sum_{i=1}^{\infty}\langle g, e_i\rangle e_i\rangle| \leq |\langle f_N, \sum_{i=1}^{M}\langle g, e_i\rangle e_i\rangle| + |\langle f_N, \sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\rangle|$$

For some $M \geq 1$ (using the orthonormal expansion of g). Using that $\|f_N\| = 1$, the orhtonormality of $(e_n)_{n\geq 1}$ and Cauchy-Schwartz inequality, we get

$$\begin{aligned}
|\langle f_N, \sum_{i=1}^{M}\langle g, e_i\rangle e_i\rangle| + |\langle f_N, \sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\rangle| \leq \\
|\langle f_N, \sum_{i=1}^{M}\langle g, e_i\rangle e_i\rangle| + \|f_N\|\cdot\|\sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\| = \\
|\langle \frac{1}{N}\sum_{n=1}^{N^2} e_n, \sum_{i=1}^{M}\langle g, e_i\rangle e_i\rangle| + \|\sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\| = \\
|\frac{1}{N}\sum_{i=1}^{M}\sum_{n=1}^{N^2}\langle e_n, \langle g, e_i\rangle e_i\rangle| + \|\sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\| = \\
|\frac{1}{N}\sum_{i=1}^{M}\sum_{n=1}^{N^2}\overline{\langle g, e_i\rangle}\langle e_n, e_i\rangle| + \|\sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\| \leq \\
\frac{1}{N}|\sum_{i=1}^{M}\overline{\langle g, e_i\rangle}| + \|\sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\|
\end{aligned}$$

Now since $\|\sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\|$ is convergent, for a suitable M we have that $\|\sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\| < \frac{\epsilon}{2}$. Furthermore we see that $|\sum_{i=1}^{M}\overline{\langle g, e_i\rangle}| \leq \sum_{i=1}^{M}|\langle e_i, g\rangle| \leq \sum_{i=1}^{M}\|e_i\|\|g\| = M\cdot g$ so for $N_\epsilon > \frac{2\cdot M\cdot g}{\epsilon}$ we have that $|\langle f_N, g\rangle| \leq \frac{1}{N}|\sum_{i=1}^{M}\overline{\langle g, e_i\rangle}| + \|\sum_{i=M+1}^{\infty}\langle g, e_i\rangle e_i\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $N \geq N_\epsilon$, showing that $f_N \to 0$ weakly, as $n \to \infty$.

Let K be the norm closure of $\text{co}\{f_N : N \geq 1\}$.
(b) Argue that K is weakly compact, and that $0 \in K$.

Since $\text{co}\{f_N : N \geq 1\}$ is convex, we know from Theorem 5.7 that it's norm closure and weak closure coincide, hence K is weakly closed. Since we have shown in (a) that $f_N \to 0$ weakly, as $N \to \infty$, zero must be contained in the weak closure i.e. $0 \in K$. Furthermore we have shown that $\|f_N\| = 1$ for all $N \geq 1$ so we see now that $K \subseteq \overline{B}_H(0,1)$. As H is a Hilbert space, and hence a reflexive Banach space, it follows that the closed unit ball $\overline{B}_H(0,1)$ is weakly compact by Theorem 6.3. Since we have now shown that K is a weakly closed subset of a weakly compact set, it follows that K is itself weakly compact.

(c) Show that 0 as well as each f_N, $N \geq 1$, are extreme points in K.

First assume that $0 \notin \text{Ext}(K)$. This means that there exist $x, y \in K$ with $x, y \neq 0$ and some $0 < \alpha < 1$ s.t. $0 = \alpha x + (1-\alpha)y$. Since K is the weak closure of $\text{co}\{f_N : N \geq 1\}$ there exists $(x_n)_{n\geq 1}, (y_n)_{n \geq 1} \subseteq \text{co}\{f_N : N \geq 1\}$ converging weakly to x and y respectively. Note that x_n can be written as $x_n = \sum_{i=1}^n \beta_i f_i$ for $f_i \in \{f_N : N \geq 1\}$, $\beta_i > 0$, $\sum_{i=1}^n \beta_i = 1$. Now consider $\langle x_n, e_N \rangle = \langle \sum_{i=1}^n \beta_i f_i, e_N \rangle = \sum_{i=1}^n \beta_i \langle f_i, e_N \rangle$, this is greater than or equal to zero, since all the β_i's are strictly positive and $\langle f_i, e_N \rangle = \frac{1}{i} \sum_{j=1}^{i^2} \langle e_j, e_N \rangle$ is either equal to 0 or $\frac{1}{i}$ (Just in case: not to be confused with the imaginary number). This goes for all $n \geq 1$, $N \geq 1$. We now have $\langle x, e_N \rangle = lim_{n\to\infty} \langle x_n, e_N \rangle \geq 0$ for all $N \geq 1$ by continuity of the inner product. Similarly, it can be shown that $\langle y, e_N \rangle \geq 0$. Our assumption that 0 is not an extreme point in K can be expressed as $0 = \langle 0, e_N \rangle = \alpha \langle x, e_N \rangle + (1-\alpha)\langle y, e_N \rangle$ leaving only the possibility that $\langle x, e_N \rangle = \langle y, e_N \rangle = 0$ (since α is strictly positive). Since e_N is an element of an orthonormal basis, it now follows that $x = y = 0$, showing that 0 is indeed an extreme point in K.

Next assume that $f_N \notin \text{Ext}(K)$. Then there exist $x, y \in K$ with $x, y \neq f_N$ and some $0 < \alpha < 1$ s.t. $f_N = \alpha x + (1-\alpha)y$, where x, y are both limits of weakly convergent sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n \geq 1}$ in $\text{co}\{f_N : N \geq 1\}$. Since $(x_n)_{n \geq 1}$ is a sequence in the convex hull, we can write $x_n = \beta_1 f_N + \sum_{i=2}^n \beta_i f_i$ where $f_N \neq f_i$ for all i and where $\sum_{i=2}^n \beta_i = 1$ and the coefficient β_1 is possibly 0 but definitely strictly less than 1 (if it was 1 then x_n would be equal to f_N). Now consider

$$\langle x_n, e_N \rangle = \beta_1 \langle f_N, e_N \rangle + \sum_{i=2}^n \beta_i \langle f_i, e_N \rangle < \frac{\beta_1}{N} + \frac{1-\beta_1}{N} = \frac{1}{N}$$

for all n, since $\langle f_i, e_N \rangle = \frac{1}{i} < \frac{1}{N}$ if $N < i$ and 0 otherwise (we can't have $N = i$), and since $\beta_i + \beta_1 \leq 1$ we must have $\beta_i \leq 1 - \beta_1$. Since this is the case fore all n, this shows that $\langle x, e_N \rangle < \frac{1}{N}$ and similarly it can be shown that $\langle y, e_N \rangle < \frac{1}{N}$. This means that

$$\frac{1}{N} = \langle f_N, e_N \rangle = \alpha \langle x, e_N \rangle + (1-\alpha)\langle y, e_N \rangle < \alpha \frac{1}{N} + (1-\alpha)\frac{1}{N} = \frac{1}{N}$$

which is a contradiction. Hence we must have $x = y = f_N$ (if one of them equals f_N so must the other), showing that each f_N is an extreme point of K.

(d) Are there any other extreme points in K? Justify your answer.

For all $x, y \in K$ there exist $(x_n)_{n\geq 1}, (y_m)_{m\geq 1} \subseteq \text{co}\{f_N : N \geq 1\}$ converging weakly to x and y respectively. Now since $\text{co}\{f_N : N \geq 1\}$ is convex, it follows that $\alpha x_i + (1-\alpha)y_i \in \text{co}\{f_N : N \geq 1\}$ for all $i = 1, .., \max\{n, m\}$ and $0 \leq \alpha \leq 1$. Hence the limit must be contained in the weak closure, i.e. $\alpha x + (1-\alpha)y \in K$, showing that K is convex. Now since K is both the closure of $\text{co}\{f_N : N \geq 1\}$ w.r.t. the weak topology and convex, we can use Theorem 7.9 which states that the set of extreme points in K is a subset of the weak closure of $\{f_N : N \geq 1\}$. Note that any element of $\{f_N : N \geq 1\}$ will eventually either be constant or approach 0, hence $\text{Ext}(K) \subseteq \{f_N : N \geq 1\} \cup \{0\}$ meaning that there are no other extreme points in K.

Problem 2: Let X and Y be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $T(x_n) \to T(x)$ weakly, as $n \to \infty$.

First let $F \in Y^*$, then $F \circ T \in X^*$. Since X is a Banach space and since a sequence is also a net we can use Problem 2(a) HW4, which states that if $(x_n)_{n\geq 1}$ converges weakly to x, then $(f(x_n))_{n\geq 1}$ converges to $f(x)$ for every $f \in X^*$. Hence we see that

$$F(T(x_n)) \to F(T(x))$$

Which is equivalent to $T(x_n) \to T(x)$ weakly.

(b) Let $T \in \mathcal{K}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $\|T(x_n) - T(x)\| \to 0$, as $n \to \infty$.

First note that by the weak convergence of $(x_n)_{n\geq 1}$, the sequence is bounded. Now since T is compact it follows that every bounded sequence $(x_n)_{n\geq 1}$ in X contains a subsequence $(x_{n_k})_{n\geq 1}$ s.t. $(T(x_{n_k}))_{k\geq 1}$ converges in Y by Theorem 8.2. Since $(T(x_{n_k}))_{k\geq 1}$ is a subsequence of $(T(x_n))_{n\geq 1}$ we know from (a) that it converges weakly to $T(x)$, so by uniqueness of the limit, we now have $\|(T(x_{n_k}))_{k\geq 1} - T(x)\| \to 0$ for $n \to \infty$. Now if $T(x_n)$ does not converge to $T(x)$ in norm, then for some $\epsilon > 0$, $(x_n)_{n\geq 1}$ contains a subsequence $(x_{n_m})_{m\geq 1}$ such that $\|T(x_{n_m}) - T(x)\| > \epsilon$ for all m. But since $(x_n)_{n\geq 1}$ is bounded, so are all it's subsequences, meaning (again by 8.2, (a) and uniqueness of the limit) that $(x_{n_m})_{m\geq 1}$ would contain a subsequence $(x_{n_{m_l}})_{l\geq 1}$ such that $(T(x_{n_{m_l}}))_{l\geq 1}$ converges to $T(x)$ in norm, contradicting that $\|T(x_{n_m}) - T(x)\| > \epsilon$ for all m. We conclude that

$\|T(x_n) - T(x)\| \to 0$, as $n \to \infty$.

(c) Let H be a separable infinite dimensional Hilbert space. Show that if $T \in \mathcal{L}(H, Y)$ satisfies that $\|T(x_n) - T(x)\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Following the hint, we assume that T is not compact. By Theorem 8.2 it follows that $T(\overline{B_H(0,1)})$ is not totally bounded, meaning that for all $\delta > 0$ it cannot be covered by a union of finitely many open balls with radius δ. Let $\delta > 0$ be given and let $x_1 \in \overline{B_H(0,1)}$. Since $B_Y(T(x_1), \delta)$ does not cover $T(\overline{B_H(0,1)})$, it means that we can find $x_2 \in \overline{B_H(0,1)}$ such that $T(x_2) \notin B_Y(T(x_1), \delta)$. Similarly we know that $B_Y(T(x_1), \delta) \cup B_Y(T(x_2), \delta)$ does not cover $T(\overline{B_H(0,1)})$ so we can repeat this recursively, obtaining a sequence $(x_n)_{n\geq 1}$ in the closed unit ball of H satisfying $\|T(x_n) - T(x_m)\| \geq \delta$ for all $n \neq m$. As we have argued in Problem 1 (b), $\overline{B_H(0,1)}$ is weakly compact and by Theorem 5.13 it is metrizable, hence it is sequentially compact. Since $(x_n)_{n\geq 1} \in \overline{B_H(0,1)}$, it must contain a subsequence $(x_{n_k})_{k\geq 1}$ converging weakly to some $x \in \overline{B_H(0,1)}$. By assumption $\|T(x_{n_k}) - T(x)\| \to 0$, as $n \to \infty$, but for any two terms with $k \neq m$ we have $\|T(x_{n_k}) - T(x_{n_m})\| \geq \delta$, which is in contradiction to $(T(x_{n_k}))_{k\geq 1}$ converging in norm, hence T must be compact.

(d) Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

First note that since $\ell_1(\mathbb{N}), \ell_2(N)$ are both infinite dimensional Banach spaces, we know from (a) that for any sequence $(x_n)_{n\geq 1}$ in $\ell_2(\mathbb{N})$ converging weakly to some $x \in \ell_2(\mathbb{N})$ we have $T(x_n) \to T(x)$ weakly in $\ell_1(\mathbb{N})$, as $n \to \infty$. We know that a sequence converges weakly $\ell_1(\mathbb{N})$ if and only if it converges in norm, hence we have $\|T(x_n) - T(x)\| \to 0$, as $n \to \infty$. As $\ell_2(\mathbb{N})$ is an infinite dimensional separable Hilbert space, (c) now gives us that $T \in \mathcal{K}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$. Actually each $T \in \mathcal{L}(H, \ell_1(\mathbb{N}))$ is compact, for any separable Hilbert space H.

(e) Show that <u>no</u> $T \in \mathcal{K}(X, Y)$ is onto.

Assume that $T \in \mathcal{K}(X, Y)$ is onto. By the open mapping theorem T is open, meaning that $T(B_X(0,1))$ is open in Y. Therefore there exists $r > 0$ such that $B_Y(0,r) \subseteq T(B_X(0,1))$. It follows that $\overline{B_Y(0,r)} \subseteq \overline{T(B_X(0,1))}$ and since $\overline{T(B_X(0,1))}$ is compact by definition of T being compact, so is $\overline{B_Y(0,r)}$ as it is a closed subset of a compact set. This means that $\frac{1}{r}\overline{B_Y(0,r)} = \overline{B_Y(0,1)}$ must also be compact, since it is just a scaling of the points in $\overline{B_Y(0,r)}$. But as we have shown in the first mandatory assignment, the closed unit ball in an infinite dimensional vector space is non-compact, and since Y by assumption is an infinite dimensional Banach space, this is a contradiction. Thus we conclude that no $T \in \mathcal{K}(X, Y)$ is onto.

(f) Let $H = L_2([0,1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. Justify that M is self-adjoint, but not compact.

First let $f, g \in H$. Then

$$\langle Mf, g\rangle = \int_{[0,1]} tf(t)\overline{g(t)}dm(t) = \int_{[0,1]} f(t)\overline{tg(t)}dm(t) = \langle f, Mg\rangle$$

since t is real, showing that $M = M^*$ and hence is self-adjoint. Now assume that M is compact. Since H is an infinite dimensional separable Hilbert space, we can use Theorem 10.1 stating that H has an orthonormal basis consisting of eigenvectors for M. But from Problem 3 HW6(a) we know that M has no eigenvalues, so this leads to a contradiction, hence M cannot be compact.

Problem 3: Consider the Hilbert space $L_2([0,1], m)$, where m is the Lebesgue measure. Define $K : [0,1] \times [0,1] \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1, \\ (1-t)s, & \text{if } 0 \le s \le t \le 1, \end{cases}$$

And consider $T \in \mathcal{L}(H, H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t), \quad s \in [0,1], \quad f \in H.$$

(a) Justify that T is compact.

It is easily seen that $K(s,t)$ is continuous on $[0,1]$, hence $K \in C([0,1] \times [0,1])$. We know that $[0,1]$ with the Lebesgue measure is a compact Hausdorff topological space. We recognize T as the associated operator $T_K : L_2([0,1], m) \to L_2([0,1], m)$, which is compact by Theorem 9.6.

(b) Show that $T = T^*$.

Let $f, g \in H$ then

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} Tf(s)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} f(t)K(s,t)\overline{g(s)}dm(t)dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} f(t)\overline{K(s,t)g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t)\overline{\int_{[0,1]} K(s,t)g(s)dm(s)}dm(t) = \langle f, Tg\rangle
\end{aligned}$$

By Fubini's theorem and since $K(s,t)$ is real. This shows that T i self-adjoint i.e. $T = T^*$.

(c) Show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t), \quad s \in [0,1], \quad f \in H.$$

Use this to show that Tf is continuous on $[0,1]$ and that $(Tf)(0) = (Tf)(1) = 0$.

We're integrating with respect to t and the value of t is dependent on s, so given $s \in [0,1]$ we can define $K_{s1} : [0,s] \to \mathbb{R}$ and $K_{s2} : [s,1] \to \mathbb{R}$ as $K_{s1}(t) = (1-s)t$ and $K_{s2}(t) = (1-t)s$ Then $K(s,t) = K_{1s}(t),$ if $t \in [0,s]$ and $K(s,t) = K_{2s}(t),$ if $t \in [s,1]$. Therefore we can write

$$\begin{aligned}
(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} K_{s1}(t)f(t)dm(t) + \int_{[s,1]} K_{s2}(t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t)
\end{aligned}$$

We want to show that Tf is continuous on $[0,1]$.

Define $F(s) = \int_{[0,s]} tf(t)dm(t)$ and $G(s) = \int_{[s,1]}(1-t)f(t)dm(t)$ for $s, t \in [0,1]$.
We know that f is Lebesgue integrable and since $t \mapsto t$ and $t \mapsto 1-t$ are continuous, they are also Lebesgue integrable. This means that given $\epsilon > 0$ and $s_1, s_2 \in [0,1]$ with $s_1 \leq s_2$, we can find $\delta > 0$ s.t.

$$|F(s_2) - F(s_1)| = |\int_{[0,s_2]} tf(t)dm(t) - \int_{[0,s_1]} tf(t)dm(t)| = |\int_{[s_1,s_2]} tf(t)dm(t)| < \epsilon$$

and

$$|G(s_2) - G(s_1)| = |\int_{[0,s_2]} (1-t)f(t)dm(t) - \int_{[0,s_1]} (1-t)f(t)dm(t)| = |\int_{[s_1,s_2]} (1-t)f(t)dm(t)| < \epsilon$$

whenever $|s_2 - s_1| < \delta$, showing that both F and G are continuous. Since $s \mapsto s$ and $s \mapsto 1-s$ are both continuous, so are the products $(1-s)F(s)$ and $sG(s)$. We see now that Tf is the sum of two continuous functions on $[0,1]$, which means that it is itself continuous on $[0,1]$.
It is now fairly easily seen that $(Tf)(0) = (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]}(1-t)f(t)dm(t) = 0$ and $(Tf)(1) = (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]}(1-t)f(t)dm(t) = 0$.

Problem 4: Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$
(a) for each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$.
Justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$.
Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

From Problem 1 HW7 we know that the function $x \in \mathbb{R}^n \mapsto e^{-\|x\|^2} \in \mathscr{S}(\mathbb{R}^n)$. For $x \in \mathbb{R}$ we have $-\|x\|^2 = -|x|^2 = -x^2$ meaning that $\lim_{|x|\to\infty} x^\beta \partial^\alpha e^{-x^2} = 0$ for all non-negative integers α, β. Obviously $e^{-\frac{1}{2}x^2} \in C^\infty(\mathbb{R})$ and since dividing $-x^2$ by 2 won't change the limit, we see that $x \in \mathbb{R} \mapsto e^{\frac{-x^2}{2}} \in \mathscr{S}(\mathbb{R})$. By Problem 1$(a)$ HW7 we know that if $f \in \mathscr{S}(\mathbb{R}^n)$ then $x^\alpha f \in \mathscr{S}(\mathbb{R}^n)$ for all multiple-indices, which means that $x^k e^{\frac{-x^2}{2}} \in \mathscr{S}(\mathbb{R})$ for all non-negative integers $k \geq 0$. Now we want to compute the Fourier transform of $g_k(x)$ for $k = 0, 1, 2, 3$. The Fourier transform is given by the integral

$$\begin{aligned}
\hat{g_k}(\xi) &= \int_{\mathbb{R}} g_k(x) e^{-i\langle x,\xi\rangle} dm(x) \\
&= \int_{\mathbb{R}} x^k e^{-\frac{1}{2}x^2} e^{-ix\xi} dm(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^k e^{-\frac{1}{2}x(x+2i\xi)} dx
\end{aligned}$$

First we see that $g_0(x) = e^{-\frac{1}{2}x^2}$, so by proposition 11.4 we have $\hat{g}_0(\xi) = e^{-\frac{1}{2}\xi^2}$.

Since $g_k, x^k g_k \in \mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$ for all non-negative integers k, Proposition 11.13 gives us that $\hat{g}_1(\xi) = i(\frac{\partial}{\partial\xi}\hat{g}_0)(\xi) = i(-\xi e^{-\frac{1}{2}\xi^2}) = -i\xi e^{-\frac{1}{2}\xi^2}$. Following the same method, we see that $\hat{g}_2(\xi) = i(\frac{\partial}{\partial\xi}\hat{g}_1)(\xi) = (1-\xi^2)e^{-\frac{1}{2}\xi^2}$ and $\hat{g}_3(\xi) = i(\frac{\partial}{\partial\xi}\hat{g}_2)(\xi) = i\xi^3 e^{-\frac{1}{2}\xi^2} - 3i\xi e^{-\frac{1}{2}\xi^2}$

(b) Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

First note that $i^0 = 1$ so the Fourier transform of h_0 is itself, hence by (a) we know that $h_0(x) = g_1(x) = e^{-\frac{1}{1}x^2}$ and hence $\hat{h}_0(\xi) = e^{-\frac{1}{2}\xi^2}$. Since $i^3 = -i$ it is easily seen (again by (a)) that if we let $h_3(x) = g_1(x) = xe^{-\frac{1}{2}x^2}$ then $\hat{h}_3(\xi) = -i\xi e^{-\frac{1}{2}x^2} = i^3 h_3(\xi)$.

Let $h_2(x) = g_0(x) - 2g_2(x) = e^{-\frac{1}{2}x^2} - 2x^2e^{-\frac{1}{2}x^2}$ then $\hat{h}(\xi) = e^{-\frac{1}{2}\xi^2} - 2(e^{-\frac{1}{2}\xi^2} - \xi^2 e^{-\frac{1}{2}\xi^2}) = 2\xi^2 e^{-\frac{1}{2}\xi^2} - e^{-\frac{1}{2}\xi^2} = i^2 h_2(\xi)$ since

$$\int_{\mathbb{R}} (g_0(x) - 2g_2(x))e^{-ix\xi}dx = \int_{\mathbb{R}} g_0(x)e^{-ix\xi}dx - 2\int_{\mathbb{R}} g_2(x)e^{-ix\xi}dx.$$

Finally let $h_1(x) = 2g_3(x) - 3g_1(x) = 2x^3e^{-\frac{1}{2}x^2} - 3xe^{-\frac{1}{2}x^2}$ then $\hat{h}_1(\xi) = 2(\xi^3 e^{-\frac{1}{2}\xi^2} - 3\xi e^{-\frac{1}{2}\xi^2}) - 3(-\xi e^{-\frac{1}{2}\xi^2}) = 2\xi^3 e^{-\frac{1}{2}\xi^2} - 3\xi e^{-\frac{1}{2}\xi^2} = ih_1(\xi)$

(c) Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$.

We know that $\mathcal{F}(f(x)) = \hat{f}(\xi)$ so $\mathcal{F}^2(f(x)) = \mathcal{F}(\hat{f}(\xi))$ is given by

$$\int_{\hat{\mathbb{R}}} \hat{f}(\xi)e^{-ix\xi}dm(\xi).$$

We recognize this as the inverse Fourier transform of $\hat{f}(-x)$ (see Definition 12.10) so $\mathcal{F}^2(f(x)) = \mathcal{F}^*(\hat{f}(-x)) = (\hat{f})^{\vee}(-x)$. We know from Proposition 11.13 that since f is a Schwartz function, so is $\hat{f}$ and since $\mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$ we have $f \in L_1(\mathbb{R})$ and $\hat{f} \in L_1(\hat{\mathbb{R}})$. Furthermore, f is clearly continuous since it belongs to $C^\infty(\mathbb{R})$, therefore we can use Theorem 12.11 which states that $f = (\hat{f})^{\vee}$. Hence we now have $\mathcal{F}^2(f(x)) = (\hat{f})^{\vee}(-x) = f(-x)$. Now it is clear that $\mathcal{F}^4(f(x)) = \mathcal{F}^2(\mathcal{F}^2(f(x))) = \mathcal{F}^2(f(-x)) = f(-(-x)) = f(x)$, showing that $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$.

(d) use (c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F} = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.

Problem 5: Let $(x_n)_{n\geq 1}$ be a dense subset of $[0,1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0,1]$. Show that $\text{supp}(\mu) = [0,1]$

First note that $[0,1]$ is a compact Hausdorff space and therefore it is especially locally compact. From Problem 3 HW8 we know that $\text{supp}(\mu) = [0,1] \setminus N$ where N denotes the union of all open subsets U satisfying $\mu(U) = 0$. Note first that $2^{-n}\delta_{x_n} \geq 0$ for all $n \geq 1$ so every term is positive. Let U be any open subset in $[0,1]$. If $U \cup (x_n)_{n\geq 1} \neq \emptyset$ then for atleast one $1 \leq k \leq n$ we would have $\delta_{x_k}(U) = 1$ and hence $\mu(U) \neq 0$, so let $U \cup (x_n)_{n\geq 1} = \emptyset$. Since $(x_n)_{n\geq 1}$ is dense in $[0,1]$ we know that for $x \in U$ we have $x_n \in (x_n)_{n\geq 1}$ s.t. $x_n \to x$. But then $\mu(\{x_n\}) \to \mu(\{x\})$ meaning that

$\delta_{x_n}(\{x_n\}) \to \delta_{x_n}(\{x\})$ but $\delta_{x_n}(\{x_n\})$ is constantly 1 hence $\delta_{x_n}(\{x\}) = 1$ and therefore $\mu(U) \neq 0$. We conclude that $N = \emptyset$ and hence $\operatorname{supp}(\mu) = [0,1] \setminus \emptyset = [0,1]$.