CoCo - Assignment 1

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1.12)

Define the language,

$$D = \{\omega | \omega \text{ contains an even } \# \text{ of } a\text{s and an odd } \# \text{ of } b\text{s and does not contain the substring } ab\}. \tag{0.1}$$

Notice that $D = b(b^2)^*(a^2)^*$ since "does not contain ab as a substring" implies $\omega \in D$ is of the form $\omega = b^i a^j$ for some $i \in 2\mathbb{N}+1$ and some $j \in 2\mathbb{N}$

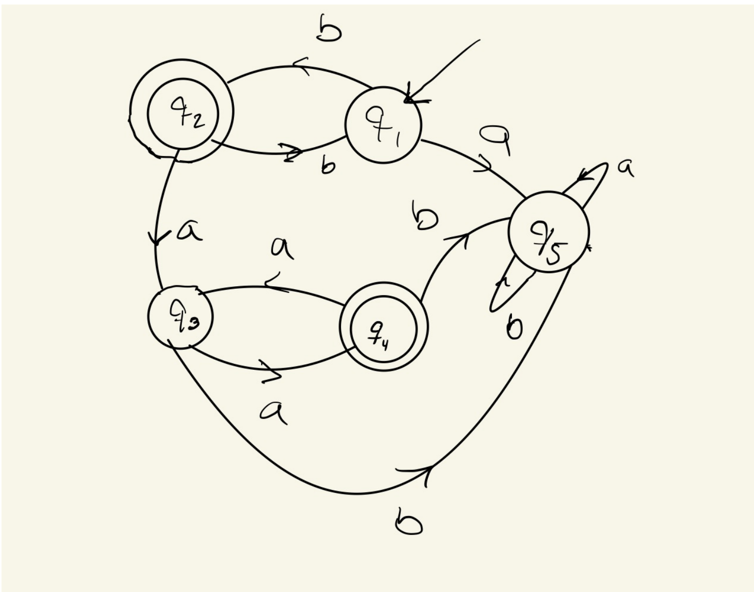

Figure 1: DFA that recognizes D

1.16b)

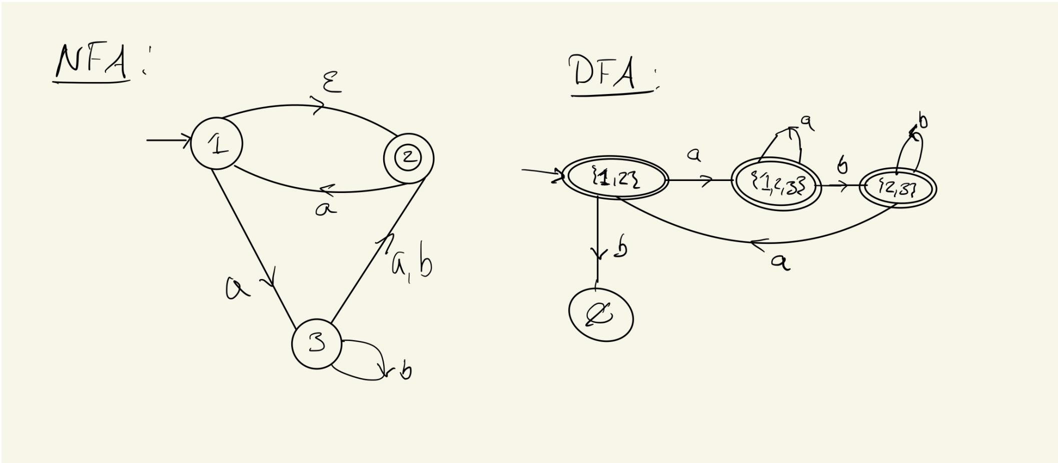

Figure 2: DFA that reckognizes the same language as the NFA given in the problem.

1.29)

Show that the following languages are not regular.

a)

$$A_1 = \{0^n 1^n 2^n \mid n \geq 0\} \tag{0.2}$$

Proof. Assume that A_1 is regular, and let p be the pumping length of A_1. Then by the pumping lemma we known that $0^p1^p2^p$ can be split in $0^p1^p2^p = xyz$ with $|y| \leq |xy| \leq p$ and $|y| > 0$ and such that $xy^iz \in A_1$ for all $i \geq 1$. Clearly, $y = 0^m$ for some $0 < m \leq p$ and thus $xy^2z = 0^{p+m}1^p2^p \notin A_1$. Which contradict the assumption that A_1 is regular. □

b)

$$A_2 = \{\omega\omega\omega \mid \omega \in \{a, b\}^*\}$$

Proof. Assume for contradiction A_2 is regular with pumping length p, and let $\omega = a^pb$. By the pumping lemma there exists x, y, z such that $\omega^3 = xyz$, where $0 < |y| \leq |xy| \leq p$. Evidently, as $|xy| \leq p$, we have $y = a^l$ for some $l \geq 1$, and so $xy^2z = a^{p+l}ba^pba^pb \notin A_2$, which contradicts the pumping lemma. □

c)

$$A_3 = \{a^{2^n} \mid n \geq 0\}$$

Proof. Assume for contradiction A_3 is regular with pumping length p, and pick some $n \in \mathbb{N}$ such that $p < 2^n$. By the pumping lemma there exists x, y, z such that $a^{2^n} = xyz$, where $0 < |y| \leq |xy| \leq p$. But then

$$2^n < \left|xy^2z\right| = 2^n + |y| \leq 2^n + p < 2^{n+1},$$

which shows $xy^2z \notin A_3$. This contradicts the pumping lemma. □

1.44)

We construct a language that is reckognized by a DFA with k states, but not any DFA with $k-1$ states. Consider the following DFA This DFA reckognizes the language

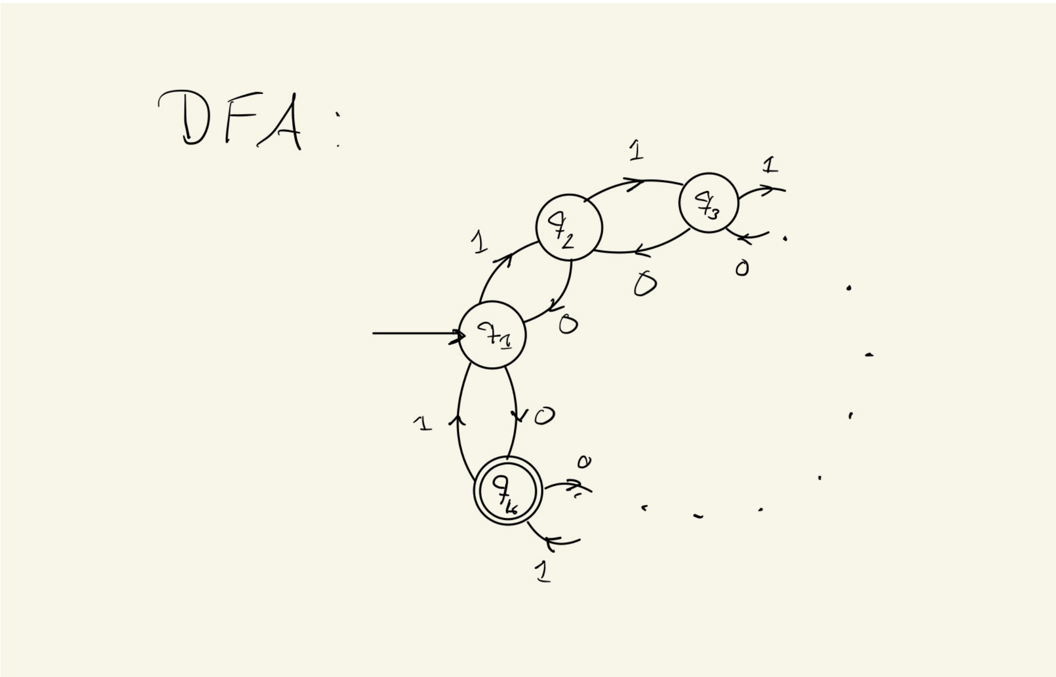

Figure 3: DFA with k states.

$$L = \{w \in \{0,1\}^* \mid \#1(w) - \#0(w) = k - 1 \bmod k\}$$

where $\#1(w)$ denotes the number of 1s in w and $\#0(w)$ denotes the number of 0s in w. This can easily be seen by the fact that 1s move the state clockwise and 0s move the state counter-clockwise. We claim that L is not reckognized by any DFA with $k-1$ states.

Proof. Assume for contradiction that there exists a DFA with $k-1$ states that reckognizes L. Then we might, by the pumping lemma and the proof thereof, take the pumping length to be $k-1$. Now $1^{k-1} \in L$, and splitting this string according to the pumping lemma, $1^{k-1} = xyz$, we immediately conclude that $y = 1^m$ for some $0 < m \leq k-1$. Now clearly $xy^2z = 1^{k+m-1}$ and we thus have

$$\#1(xy^2z) - \#0(xy^2z) = k+m-1 = m-1 \bmod k < k-1 \bmod k.$$

Thereby, $xy^2z \notin L$. This contradict the fact that the pumping length can be chosen to be $k-1$, and we conclude that L is not reckognized by any DFA with $k-1$ states. □

1.50)

We find the minimum pumping lengths of the following languages.

a)

0001^*.
We claim that $p_{min} = 4$.

Proof. Notice that 000 cannot be pumped, since no string contain neither more of less than three 0s. However, any string of length 4 or more is of the form 0001^k for some $k \geq 1$. And clearly choosing $x = 000$, $y = 1$, and $z =$"the rest", we find that $xy^iz \in 0001^*$ for any $i \geq 0$. Thus any string of length 4 or more can be pumped. □

b)

0^*1^*.
We claim that $p_{min} = 1$.

Proof. Notice first that $p_{min} = 0$ is not an option by the pumping lemma point (2), since it states that $|y| > 0$. However, let w be any string in 0^*1^* with length 1 or more. Then we know that w is either of the form $w = 0w_0$ for some $w_0 \in 0^*1^*$ *or* $w = w_11$ for some $w_1 \in 0^*1^*$. In the first case we choose $x = \varepsilon$, $y = 0$, and $z = w_0$. Then $xy^iz \in 0^*1^*$ for any $i \geq 0$. In the second, case we choose $x = w_1$, $y = 1$, and $z = \varepsilon$. Then $xy^iz \in 0^*1^*$ for any $i \geq 0$. Thus any string of length one or more, can be pumped. □

c)

$L = 001 \cup 0^*1^*$.
We claim that $p_{min} = 1$.

Proof. Notice that $001 \in 0^*1^*$, so $A = 001 \cup 0^*1^* = 0^*1^*$, and the rest follows from b). □

d)

$A = 0^*1^+0^+1^* \cup 10^*1$. We claim that $p_{min} = 3$.

Proof. Notice first that $11 \in A$ cannot be pumped since for example $111 \notin A$, showing that $p_{min} > 2$. But if $w \in A$ with $|w| \geq 3$ then either $w = 0...$ *or* $w = 11...$ *or* $w = 101...$ *or* $w = 100...$, where "..." refers to "the rest of the string". In the first case we choose $x = \varepsilon$, $y = 0$, and $z = ...$ then we clearly have $xy^iz \in A$ for any $i \geq 0$. In the second case we choose $x = \varepsilon$, $y = 1$, and $z = 1...$ then we clearly have $xy^iz \in A$ for any $i \geq 0$. In the third case we choose $x = 10$, $y = 1$ and $z = ...$ then we clearly have $xy^iz \in A$ for any $i \geq 0$. Lastly, in the fourth case we choose $x = 1$, $y = 0$, and $z = 0...$ then we clearly have $xy^iz \in A$ for any $i \geq 0$. This proves that any string of length 3 or more can be pumped, and thus $p_{min} \leq 3$. Combing the two bounds on p_{min} shows $p_{min} = 3$. □

e)

$A = (01)^*$. We claim that $p_{min} = 1$.

Proof. Notice first that there are no string in A with length 1, as all strings in A have even length. Now as argued in b) the pumping length cannot be 0. So we migth start out by considering string of length 2 or more. Let $w \in A$ with $|w| \geq 2$, then $w = 01w_0$ for some $w_0 \in A$. Thus we may choose $x = \varepsilon$, $y = 01$, and $z = w_0$, then clearly $xy^iz \in A$ for any $i \geq 0$. This shows that any string of length 2 or more may be pumped. Now since there are no strings of length 1. It is also true that any string in A of length 1 or more can be pumped, and we conclude that $p_{min} = 1$. □

f)

$A = \varepsilon$. We claim that $p_{min} = 1$.

Proof. Since p_{min} cannot be 0 by arguments as above in b), and since it trivially true that all strings in A of length 1 or more (there are none) can be pumped, we conclude that $p_{min} = 1$. Notice that this is a simple consquence of the fact that False $\implies S$ is True for any Boolean statement S. □

g)

$A = 1^*01^*01^*$. We calin that $p_{min} = 3$.

Proof. Notice first that 00 cannot be pumped, showing that $p_{min} > 2$. Now let $w \in A$ with $|w| \geq 3$ then either $w = 1w_1$ for some $w_1 \in A$ *or* $w = 01w_0$ for some $w_0 \in 1^*01^*$ *or* $w = w_21$ for some $w_2 \in A$. In the first case, we may choose $x = \varepsilon$, $y = 1$, and $z = w_1$, then we clearly have $xy^iz \in A$ for any $i \geq 0$. In the second case, we may choose $x = 0$, $y = 1$, and $z = w_0$, then we clearly have $xy^iz \in A$ for any $i \geq 0$. In the third case, we may choose $x = w_2$, $y = 1$, and

$z = \varepsilon$, then we clearly have $xy^iz \in A$ for any $i \geq 0$. Thus any string of length 3 or more can be pumped, and $p_{min} \leq 3$, proving that $p_{min} = 3$. □

h)

$A = 10(11^*0)^*0$ We claim that $p_{min} = 4$.

Proof. Notice that $100 \in A$ cannot be pumped, showing that $p_{min} > 3$. This can be seen since, any downpump of 100 will reduce the length, i.e. $xyz = 100$ with $|y| > 0$ implies that $xy^0z = xz$ has length smaller than 3. But 100 is clearly the shortest string in A so this is not possible. However, any string in A with length 5 or more may be written as xyz with $x = 10$, $y \in (11^*0)^*$, and $z = 0$. But then $y^i \in\in (11^*0)^*$ for any $i \geq 0$, showing that $xy^iz \in A$ for any $i \geq 0$. Thus any string with length greater than or equal to 5 can be pumped. However, notice that $\{w \in A \mid |w| = 4\} = \emptyset$ so in fact any string of length 4 or more may be pumped, showing that $p_{min} = 4$. □

i)

$A = 1011$. We claim that $p_{min} = 5$.

Proof. Notice first that 1011 cannot be pumped. Showing that $p_{min} > 4$. However, no string in A has length 5 or more, so it is true that any string in A with lenght 5 or more can be pumped. Thus $p_{min} \leq 5$, showing that $p_{min} = 5$. □

j)

$A = \Sigma^*$. We claim that $p_{min} = 1$.

Proof. As before, the minimum pumping length satisfies $p_{min} \geq 1$. However, for any string $w \in A$ with $|w| \geq 1$ we have $w = w_1s$ where w_1 is just the first element in w and s is the rest of the string, which might be the empty string. Clearly by the definition of Kleene star operation we have $w_1^is \in A$. So w can be pumped, showing that $p_{min} = 1$. □

1.53)

Let $\Sigma = \{0, 1\}$ and let $D = \{w| \ \#01(w) = \#10(w)\}$, where $\#s(w)$ denotes the number of occurances of the substring s in w. We show that D is a regular language

Proof. We notice that D can be written as a regular expression. To see this, note that the occurance of a substring 01 is equivalent to the string switching from 0s to 1s and the occurance of the substring 10 is equivalent to the string switching from 1s to 0s. Thus having an equal number of occurances of the two substrings means that the string starts and ends at the same letter. Therefore, $D = 1\Sigma^*1 \cup 0\Sigma^*0$, which is a regular expression. Thus by Theorem 1.54 in M. Sipser the language is regular. □

1.71)

a)

Let $B = \{1^k y \mid y \in \{0,1\}^* \text{ and } y \text{ contains at least k 1s, for } k \geq 1\}$. We show that B is regular.

Proof. Notice that if $w = 1^k y$ for some $y \in \{0,1\}^*$ containing at least k 1s for some $k \geq 1$. Then $w = 1z$ for some $z \in \{0,1\}^*$ containing at least one 1, namely $z = 1^{k-1}y$ and the contrary is triavially true. Thus, if we define $\Sigma = \{0,1\}$, we clearly have $B = 1\Sigma^*1\Sigma^*$, which is a regular expression. Thus by Theorem 1.54 in M. Sipser, the language is regular. $\square$

b)

Let $C = \{1^k y \mid y \in \{0,1\}^* \text{ and } y \text{ contains at most k 1s, for } k \geq 1\}$. We show that C is not regular.

Proof. Assume for contradiction that C is regular, and let p be the pumping length. Then $1^p0^p1^p \in C$, and for any splitting, as described by the pumping lemma, $xyz = 1^p0^p1^p$, with $|xy| \leq p$ and $|y| > 0$ we have $y = 1^m$ for some $0 < m \leq p$. But then clearly $xy^0z = 1^{p-m}0^p1^p$ which is *not* in C. Since if $1^{p-m}0^p1^p = 1^k w$, implies $k \leq p - m$, but clearly w most contain at least p 1s then. Thus we have a contradiction with the pumping lemma, showing that C is not regular. $\square$