FunkAn - Mandatory 1

Ida Gottlieb Fjeldhagen, pfc343

December 14th 2020

Problem 1

(a) We want to show that $\|\cdot\|_0$ is a norm on X.

First of all, the map $\|\cdot\|_0$ takes values in $[0,\infty)$ by the formula $\|x\|_0 = \|x\|_X + \|Tx\|_Y$, $x \in X$, since both $\|\cdot\|_X, \|\cdot\|_Y$ are norms. The following three conditions, which a norm by definition needs to satisfy, follows from the facts that $\|\cdot\|_X, \|\cdot\|_Y$ are norms (hence satisfy the same three conditions) and that T is linear, i.e. $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$ for all $\alpha, \beta \in \mathbb{K}$, $x, y \in X$.

(i) Triangle inequality. For all $x, y \in X$ we have

$$\|x+y\|_0 = \|x+y\|_X + \|T(x+y)\|_Y \leq \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y = \|x\|_0 + \|y\|_0.$$

(ii) Absolutely homogeneous. Let $\alpha \in \mathbb{K}$ and $x \in X$. Then

$$\|\alpha x\|_0 = \|\alpha x\|_X + \|T(\alpha x)\|_Y = \|\alpha x\|_X + \|\alpha Tx\|_Y = |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y = |\alpha|\|x\|_0.$$

(iii) Positive definite. We have that $x = 0$ if and only if both $\|x\|_X = 0$ and $\|Tx\|_Y = 0$ (since $T(0) = 0$) if and only if $\|x\|_0 = 0$.

We want to show that the two norms $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent if and only if T is bounded.

Assume T is bounded. Observe first that $\|x\|_X \leq \|x\|_0$ for all $x \in X$ by definition of $\|\cdot\|_0$. By Prop. 1.10 in the notes, since T is assumed bounded, there exists $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ $(*)$ for all $x \in X$. Then for $x \in X$

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq \|x\|_X + C\|x\|_X = (1+C)\|x\|_X \quad (**).$$

Combining the two inequalities $(*)$ and $(**)$ yields

$$\frac{1}{1+C}\|x\|_0 \leq \|x\|_X \leq \|x\|_0 \quad \text{for all } x \in X.$$

Hence the two norms are equivalent.

Assume that the two norms are equivalent, i.e. that there exists $0 < C \leq D < \infty$ such that for all $x \in X$

$$C\|x\|_X \leq \|x\|_0 \leq D\|x\|_X.$$

Then $\|Tx\|_Y = \|x\|_0 - \|x\|_X \leq \|x\|_0 \leq D\|x\|_X$ for all $x \in X$. Hence, again by Prop. 1.10, T is bounded.

(b) We want to show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.
Let $T : X \to Y$ be a linear map. By Theorem 1.6 any two norms on X are equivalent, when X is finite dimensional. Hence in particular the norms $\|\cdot\|_0$ and $\|\cdot\|_X$ on X are equivalent, so by Problem 1 (a) T is bounded.

(c) We want to show that if X is infinite dimensional, there exists a linear map $T : X \to Y$ which is not bounded.

Let $(e_i)_{i\in I}$ be a Hamel basis for X, i.e for every $x \in X$ there is a unique family $(\lambda_i)_{i\in I}$ in $\mathbb{K}$ for which the set $\{i \in I : \lambda_i \neq 0\}$ is finite and $x = \sum_{i\in I} \lambda_i e_i$. Then $(e_i/\|e_i\|_X)_{i\in I}$ is also a Hamel basis, since $x = \sum_{i\in I}(\lambda_i\|e_i\|_X)e_i/\|e_i\|_X$ and $\{i \in I : \lambda_i\|e_i\|_X \neq 0\}$ is finite. So we can choose the Hamel basis $(e_i)_{i\in I}$ such that $\|e_i\|_X = 1$ for every $i \in I$.

Now let $(y_i)_{i\in I}$ be a family in Y satisfying that $\|y_i\|_Y \to \infty$ as $i \to \infty$ (such a family does exist; choose

This only makes sense if I is a subset of $\mathbb{K}$.

e.g. the family $(i \cdot y_i/\|y_i\|_Y)_{i\in I}$. Then it follows from the fact that X is infinite dimensional, hence I contains infinitely many elements). There exists a unique linear map $T : X \to Y$ such that $T(e_i) = y_i$. If T is bounded, there exists $C > 0$ such that

$$\|Tx\|_Y \leq C\|x\|_X \quad \text{for all } x \in X.$$

But since $(y_i)_{i\in I}$ was chosen such that $\|y_i\|_Y \to \infty$ as $i \to \infty$, there exists $i_0 \in I$ such that

$$\|y_i\|_Y > C \quad \text{for all } i \geq i_0.$$

But then

$$\|T(e_i)\|_Y = \|y_i\|_Y > C = C\|e_i\|_X \quad \text{for all } i \geq i_0.$$

This proves that T cannot be bounded.

(d) Suppose X is infinite dimensional. Let $T : X \to Y$ be a linear map, which is not bounded – such a map exists by Problem 1 (c). Let $\|\cdot\|_0$ be the norm associated to T as in Problem 1 (a). By the same problem, the two norms $\|\cdot\|_0$ and $\|\cdot\|_X$ cannot be equivalent, since T is not bounded. Furthermore, we have that for all $x \in X$

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y \geq \|x\|_X.$$

If $(X, \|\cdot\|_X)$ is a Banach space, then $(X, \|\cdot\|_0)$ is not complete. Indeed, assume by contradiction that $(X, \|\cdot\|_0)$ is complete. The identity map $\mathrm{id} : (X, \|\cdot\|_0) \to (X, \|\cdot\|_X)$ sending $x \mapsto x$ is bijective, so by Corollary 3.17 to The Open Mapping Theorem, the inverse identity map $\mathrm{id}^{-1} = \mathrm{id}$ is bounded, i.e. there exists $C > 0$ such that

$$\|x\|_X = \|\mathrm{id}(x)\|_X \geq C\|x\|_0$$

for all $x \in X$. Hence we have that

$$C\|x\|_0 \leq \|x\|_X \leq \|x\|_0$$

for all $x \in X$. But this contradicts the fact, that the two norms are inequivalent. Hence X is not complete with respect to $\|\cdot\|_0$.

(e) We want to give an example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ such that $\|x\|' \leq \|x\|$ for all $x \in X$. Consider the normed vector space $(X, \|\cdot\|) = (l_1(\mathbb{N}), \|\cdot\|_1)$. This space is complete (Remark 1.8). Consider also the norm $\|\cdot\|' = \|\cdot\|_\infty$ on $l_1(\mathbb{N})$. The two norms satisfy

$$\|x\|_\infty = \sup\{|x_n| \,:\, n \in \mathbb{N}\} \leq \sum_{n=1}^{\infty} |x_n| = \|x\|_1$$

for all $x = (x_n)_{n\geq 1} \in l_1(\mathbb{N})$. Furthermore, the two norms are not equivalent: consider the sequence $(x_i)_{i\geq 1} \subset l_1(\mathbb{N})$ where $x_{i_n} = \begin{cases} 1, & n \leq i \\ 0, & n > i \end{cases}$, for each $i \geq 1$. Then

$$\|x_i\|_1 = \sum_{n=1}^{\infty} |x_{i_n}| = \sum_{n=1}^{i} 1 = i,$$

and

$$\|x_i\|_\infty = \sup\{|x_{i_n}| \,:\, n \in \mathbb{N}\} = 1.$$

But there can exist no $C > 0$ such that

$$C\|x_i\|_1 = C \cdot i \leq 1 = \|x_i\|_\infty \quad \text{for all } i \geq 1.$$

Hence the two norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are inequivalent, and by Problem 1 (d) the normed vector space $(l_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete.

Problem 2

(a) We want to show that f is bounded on $(M, \|\cdot\|_p)$ and compute $\|f\|$.
The operator norm $\|f\|$ is defined as

$$\begin{aligned}\|f\| &= \sup\{|f((a,b,0,\dots))| \,:\, \|(a,b,0,\dots)\|_p = 1,\ (a,b,0,\dots) \in M\}\\ &= \sup\{|a+b| \,:\, (|a|^p + |b|^p)^{1/p} = 1,\ a,b \in \mathbb{C}\}\end{aligned}$$

By Hölder's inequality we have that for $a, b \in \mathbb{C}$

$$\begin{aligned}|a+b| \le |a| + |b| &= \|(a,b,0,\dots)\cdot(1,1,0,\dots)\|_1\\ &\le \|(a,b,0,\dots)\|_p \cdot \|(1,1,0,\dots)\|_q,\end{aligned}$$

where $1/p + 1/q = 1$. So when $\|(a,b,0\dots)\|_p = 1$, we have

What if $p=1$?

$$\begin{aligned}|a+b| &\le \|(a,b,0,\dots)\|_p \cdot \|(1,1,0,\dots)\|_q\\ &= (1^q + 1^q)^{1/q}\\ &= 2^{1/q}\\ &= 2^{1-1/p}.\end{aligned}$$

This proves that

$$\|f\| \le 2^{1-1/p}.$$

Furthermore, we have $|a+b| = 2^{1-1/p}$ when $a = b = 1/2^{1/p}$:

$$|a+b| = \frac{1}{2^{1/p}} + \frac{1}{2^{1/p}} = 2^{1-1/p}.$$

And the sequence $(1/2^{1/p}, 1/2^{1/p}, 0, \dots) \in l_1(\mathbb{N})$ belongs to the set over which we take the supremum in $\|f\|$, since it has 1-norm

$$\left(\frac{1}{2^{1/p}}^p + \frac{1}{2^{1/p}}^p\right)^{1/p} = \left(\frac{1}{2} + \frac{1}{2}\right)^{1/p} = 1.$$

This proves that in fact $\|f\| = 2^{1-1/p}$, $1 \le p < \infty$, and furthermore, by Remark 1.11

$$|f((a,b,0,\dots))| \le \|f\| \|(a,b,0,\dots)\|_p = 2^{1-1/p}\|(a,b,0,\dots)\|_p,$$

for all $(a,b,0,\dots) \in M$. So f is bounded on $(M, \|\cdot\|_p)$.

(b) We want to show that if $1 < p < \infty$, then there is a unique linear functional F on $l_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Since we in Problem 2 (a) proved that $f \in M^*$, Corollary 2.6 ensures the existence of $F \in X^*$ satisfying $F|_M = f$ and $\|F\| = \|f\|$. In Problem 5 HW1 we proved that $(l_p(\mathbb{N}))^*$ is isometrically isomorphic to $l_q(\mathbb{N})$, where $1/p + 1/q = 1$, and that there exists $y = (y_n)_{n\ge1} \in l_q(\mathbb{N})$ such that

$$F(x) = \sum_{n=1}^{\infty} x_n y_n, \quad \text{for all } x = (x_n)_{n\ge1} \in l_p(\mathbb{N}).$$

We see that

$$\begin{aligned}y_1 &= F((1,0,0,\dots)) = f((1,0,0,\dots)) = 1,\\ y_2 &= F((0,1,0,\dots)) = f((0,1,0,\dots)) = 1,\end{aligned}$$

since $(1,0,0,\dots), (0,1,0,\dots) \in M$. Furthermore, by Problem 2 (a) we have that $\|F\| = \|f\| = 2^{1-1/p}$, so since $1/p + 1/q = 1$ and the isomorphism is isometric, we have that

$$\|y\|_q = \|F\| = 2^{1-1/p} = 2^{1/q}.$$

I.e.

$$2 = \|y\|_q^q = 2 + \sum_{n=3}^{\infty} |y_n|^q.$$

So $\sum_{n=3}^{\infty} |y_n|^q = 0$ and hence we must have $y_n = 0$ for all $n \geq 3$. We therefore see that $y = (1, 1, 0, 0, \dots) \in l_q(\mathbb{N})$ is the unique corresponding element to F, and hence $F \in (l_p(\mathbb{N}))^*$ is also unique with respect to the relevant properties.

(c) We want to prove that if $p = 1$ there exists infinitely many linear functionals F on $l_1(\mathbb{N})$ extending f satisfying $\|F\| = \|f\|$.

As in Problem 2 (b) the existence is ensured by Corollary 2.6. Now, for $k \geq 2$ define maps $F_k : l_1(\mathbb{N}) \to \mathbb{K}$ by $F_k(x) = \sum_{n=1}^{k} x_n$, for $x = (x_n)_{n\geq 1} \in l_1(\mathbb{N})$. The maps F_k are clearly linear. The operator norm satisfies

$$\begin{aligned}\|F_k\| &= \sup\left\{|F_k(x)| \,:\, \|x\|_1 \leq 1,\, x = (x_n)_{n\geq 1} \in l_1(\mathbb{N})\right\} \\ &= \sup\left\{\left|\sum_{n=1}^{k} x_n\right| \,:\, \sum_{n=1}^{\infty} |x_n| \leq 1\right\} \\ &= 1.\end{aligned}$$

Indeed, for every $k \geq 2$ and every $x = (x_n)_{n\geq 1} \in l_1(\mathbb{N})$ with $\|x\|_1 \leq 1$, we have

$$\left|\sum_{n=1}^{k} x_n\right| \leq \sum_{n=1}^{k} |x_n| \leq \sum_{n=1}^{\infty} |x_n| \leq 1.$$

So $\|F_k\| \leq 1$, and we have equality, since $x = (1, 0, 0, \dots) \in l_1(\mathbb{N})$ satisfies $\sum_{n=1}^{k} x_n = \|x\|_1 = 1$. By Remark 1.11, then $\|F_k(x)\| \leq \|F_k\|\|x\|_1 = \|x\|_1$ for all $x \in l_1(\mathbb{N})$, so $F_k \in (l_1(\mathbb{N}))^*$. Furthermore, F_k extends f and $\|F_k\| = \|f\|$ for every $k \geq 2$, since

$$F_k((a, b, 0, \dots)) = a + b = f((a, b, 0, \dots)) \quad \text{for all } (a, b, 0, \dots) \in M,$$

and by Problem 2 (a),

$$\|f\| = 2^{1-1/1} = 1 = \|F_k\|.$$

This proves that there are infinitely many linear functionals F_k on $l_1(\mathbb{N})$ extending f and satisfying $\|F_k\| = \|f\|$.

Problem 3

(a) We want to prove that no linear map $F : X \to \mathbb{K}^n$, $n \geq 1$, where X is infinite dimensional, is injective.

Let $(u_i)_{i\in I}$ be an infinite linearly independent set in X. Assume by contradiction that there exists a linear map $F : X \to \mathbb{K}^n$ which is injective. We know from the theory of advanced vector spaces, that the image of a linearly independent set under an injective linear map is linearly independent, i.e. $(Fu_i)_{i\in I}$ is a linearly independent set in $\mathbb{K}^n$. The set consists of infinitely many elements, since F is injective, but this is impossible, since $\mathbb{K}^n$ is finitely dimensional and can only contain linearly independent sets of at most n elements. Hence no linear map $F : X \to \mathbb{K}^n$ is injective. ref or proof? ✓

(b) We want to show that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$$

for $n \geq 1$ and $f_1, \dots, f_n \in X^*$.

Consider the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), \dots, f_n(x))$, $x \in X$. The map is linear, since every coordinate f_j, $j = 1, \dots, n$, is linear. So by Problem 2 (a), F cannot be injective, i.e. $\ker F \neq \{0\}$. But the kernel of F is exactly the intersection $\bigcap_{j=1}^{n} \ker(f_j)$, so this proves the claim. show this.

(c) Let $x_1, \dots, x_n \in X$. We want to show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, \dots, n$.

If $x_j = 0$ for every j, then the claim is trivial. Just choose $y \in X$ with $\|y\| = 1$ (this is always possible:

for arbitrary $y \in X$, the element $y/\|y\|$ is in X and has norm 1). Then $\|y - x_j\| = \|y\| = 1 \geq 0 = \|x_j\|$ for every $j = 1, \ldots, n$.

We can therefore assume that x_j is non-zero for every j (if $x_j = 0$ for some j, just cast them away and adjust the value of n in accordance with the number of non-zero elements x_j). Theorem 2.7 (b) then says that there exists $f_1, \ldots, f_n \in X^*$ such that $\|f_j\| = 1$ and $f_j(x_j) = \|x_j\|$ for every $j = 1, \ldots, n$. By Problem 2 (b) there exists $0 \neq y' \in X$ such that $f_j(y') = 0$ for all j. Set $y = y'/\|y'\| \neq 0$. This element also satisfies that $f_j(y) = f_j(y')/\|y'\| = 0$ for all j. Furthermore, $\|y\| = \|y'\|/\|y'\| = 1$.

Now, since the f_j's are bounded linear functionals on X, there exists $c_1, \ldots, c_n > 0$ such that $|f_j(x)| \leq c_j\|x\|$ for all $x \in X$ (Prop. 1.10). Then for every $j = 1, \ldots, n$,

$$c_j\|y - x_j\| \geq |f_j(y - x_j)| = |f_j(y) - f_j(x_j)| = |f_j(x_j)| = \|x_j\|.$$

By Remark 1.11, $\|f_j\| = \inf\{C > 0 \,:\, |f_j(x)| \leq C\|x\|, x \in X\}$, so since $\|f_j\| = 1$, we can choose $c_j = 1$ for every j. Hence we proved the existence of a $y \in X$ with $\|y\| = 1$ such that $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, \ldots, n$.

(d) We want to show that one cannot cover the unit sphere $S = \{x \in X \,:\, \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

Let $x_1, \ldots, x_n \in X$ be finitely many elements in X and assume that the closed balls $\overline{B}_X(x_i, r_i)$, $r_i \geq 0$, $i = 1, \ldots, n$, cover S. We want to prove that at least one of the balls contains 0. By Problem 3 (c) there exists $y \in X$ such that $\|y\| = 1$, i.e. $y \in S$, and $\|y - x_i\| \geq \|x_i\|$ for all i. Since the balls cover S, there exists $i_0 \in \{1, \ldots, n\}$ such that $y \in \overline{B}_X(x_{i_0}, r_{i_0})$, i.e. $\|y - x_{i_0}\| \leq r_{i_0}$. Then it is impossible that $r_{i_0} < \|x_{i_0}\|$, since otherwise

$$\|y - x_{i_0}\| \geq \|x_{i_0}\| > r_{i_0} \geq \|y - x_{i_0}\|.$$

Hence $r_{i_0} \geq \|x_{i_0}\|$. But this means that $0 \in \overline{B}_X(x_{i_0}, r_{i_0})$, since $\|x_{i_0} - 0\| = \|x_{i_0}\| \leq r_{i_0}$. This proves the statement.

(e) We want to show that S is non-compact and deduce that the closed unit ball in X is non-compact. Assume by contradiction that S is compact and consider the open cover $\mathcal{U} = \{B_X(x, 1/2) \,:\, x \in S\}$ consisting of open balls of radius $1/2$ around every point $x \in S$. Then there exists a finite subcover of $\mathcal{U}$, i.e. there exists $N \in \mathbb{N}$ such that $\{B_X(x_i, 1/2) \,:\, x_i \in S, i = 1, \ldots, N\}$ covers S. Then also the set of closed balls $\mathcal{V} = \{\overline{B}_X(x_i, 1/2) \,:\, x_i \in S, i = 1, \ldots, N\}$ covers S. But this contradicts the fact proven in Problem 3 (d), since none of the closed balls in $\mathcal{V}$ contains 0 (for $i = 1, \ldots, N$, $\|x_i - 0\| = 1 > 1/2$). Hence S cannot be compact.

Let $D = \{x \in X \,:\, \|x\| \leq 1\}$ be the closed unit ball in X. Then the unit circle S is a closed subset of D, since the complement $D \setminus S$ is open. Indeed, given $\varepsilon > 0$, then for every $x \in D \setminus S$, the open ball $B(x, \delta)$ where $\delta = \text{dist}(x, S) = \inf\{\|x - y\| \,:\, y \in S\}$ is contained in $D \setminus S$. If D was compact, then the closed subset S of D would also be compact, but this is not the case. So D is also non-compact.

Problem 4

(a) Given $n \geq 1$, the set $E_n = \left\{f \in L_1([0,1], m) \,:\, \int_{[0,1]} |f|^3 dm \leq n\right\} \subseteq L_1([0,1], m)$ is not absorbing. By Problem 2 HW2, $L_3([0,1], m)$ is a proper subspace of $L_1([0,1], m)$, so we can pick $f \in L_1([0,1], m)$ such that the integral

$$\left(\int_{[0,1]} |f|^3 dm\right)^{1/3}$$

is divergent. If there exists $t > 0$ such that $tf \in E_n$, i.e. such that

$$\int_{[0,1]} |tf|^3 dm \leq n,$$

then

$$\left(\int_{[0,1]} |f|^3 dm\right)^{1/3} = \left(t^{-3} \int_{[0,1]} |tf|^3 dm\right)^{1/3} \leq (nt^{-3})^{1/3},$$

contradicting the fact, that the integral was not convergent. Hence there exists no such t, and E_n is not absorbing.

(b) We want to show that E_n has empty interior in $L_1([0,1],m)$ for all $n \geq 1$.
Assume by contradiction that the interior of E_n is non-empty and let $f \in \text{Int}(E_n)$. The interior is defined as the union of all open sets in $L_1([0,1],m)$ containing E_n, so there is an open set containing f. $L_1([0,1],m)$ is a metric space, so there exists $\varepsilon > 0$ such that

contained in.

$$f \in B(f,\varepsilon) = \{g \in L_1([0,1],m) : \|f-g\|_1 < \varepsilon\} \subseteq E_n.$$

Now let $0 \neq g \in L_1([0,1],m)$. Then $g' = f + \frac{\varepsilon}{2\|g\|_1} g \in B(f,\varepsilon) \subseteq E_n$, since

$$\|g'-f\|_1 = \left\| f + \frac{\varepsilon}{2\|g\|_1} g - f \right\|_1 = \frac{\varepsilon}{2\|g\|_1}\|g\|_1 = \frac{\varepsilon}{2} < \varepsilon.$$

E_n is contained in $L_3([0,1],m)$, so $f, g' \in L_3([0,1],m)$. Hence also $g = \frac{2\|g\|_1}{\varepsilon}(g'-f) \in L_3([0,1],m)$. So $L_1([0,1],m) \subseteq L_3([0,1],m)$, which contradicts the fact that $L_3([0,1],m)$ is a proper subspace of $L_1([0,1],m)$ (Problem 2 HW2). Hence E_n has empty interior.

(c) We want to show that E_n is closed in $L_1([0,1],m)$ for all $n \geq 1$.
Let $(f_i)_{i\geq 1}$ be a sequence in E_n and assume that $\lim_{i\to\infty} \|f_i - f\|_1 = 0$ for some $f \in L_1([0,1],m)$. Then, by Fatou's Lemma, we have

$$\int_{[0,1]} |f|^3 dm = \int_{[0,1]} \lim_{i\to\infty} |f_i|^3 dm \leq \lim_{i\to\infty} \int_{[0,1]} |f_i|^3 dm \leq n.$$

L_1-convergence does not imply pointwise convergence

So $f \in E_n$. This proves that E_n is closed in $L_1([0,1],m)$.

(d) From Problem 4 (b) and (c) we get that $\text{Int}(E_n) = \text{Int}(\overline{E_n}) = \emptyset$, which means that E_n is nowhere dense, $n \geq 1$. Observe also that $L_3([0,1],m) = \bigcup_{n\geq 1} E_n$, so $L_3([0,1],m)$ can be written as a union of nowhere dense sets. This means exactly that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$. Show this.

Problem 5

(a) Suppose $x_n \to x$ in norm as $n \to \infty$. We want to prove that $\|x_n\| \to \|x\|$ as $n \to \infty$. Given $\varepsilon > 0$, by assumption there exists $N \in \mathbb{N}$ such that

$$\|x_n - x\| < \varepsilon \quad \text{for all } n > N.$$

By the triangle inequality we have that

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| < \varepsilon \quad \text{for all } n > N.$$

Hence $\|x_n\| \to \|x\|$ as $n \to \infty$.

(b) We want to show that if $x_n \to x$ as $n \to \infty$, it does not necessarily imply that $\|x_n\| \to \|x\|$ as $n \to \infty$, by proposing a counterexample.

Let the sequence be given by a countable orthonormal basis $(e_n)_{n\geq 1}$ for H (which do exist by Prop. 5.29 in Folland, since X is separable). Let $f \in H^*$. By the Riesz Representation Theorem (Problem 1 HW2) there exists $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x \in H$. Bessel's inequality tells us, since $(e_n)_{n\geq 1}$ is an orthonormal set, that

$$\sum_{n=1}^{\infty} |\langle e_n, y\rangle|^2 = \sum_{n=1}^{\infty} |\langle y, e_n\rangle|^2 \leq \|y\|^2,$$

so the sum is convergent, because y is fixed. Thus we must have that the terms tend to zero, i.e. $|\langle e_n, y\rangle|^2 \to 0$ as $n \to \infty$, so also $f(e_n) = \langle e_n, y\rangle \to 0 = f(0)$ as $n \to \infty$. Since $f \in H^*$ was chosen arbitrarily, Problem 2 (a) HW4 implies that $e_n \to 0$ weakly as $n \to \infty$. But $\|e_n\| = 1$ for every $n \geq 1$, so $\|e_n\| \to 1$ as $n \to \infty$. So $\|e_n\| \not\to \|0\| = 0$, which proves the claim.

(c) We want to prove that if $\|x_n\| \leq 1$ for all $n \geq 1$ and $x_n \to x$ weakly as $n \to \infty$, then $\|x\| \leq 1$. Note first that if $x = 0$, then $\|x\| = 0 \leq 1$, so we can assume that $x \neq 0$. Then by Theorem 2.7 (b) there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. By Remark 1.11 we have that

$$|f(x_n)| \leq \|f\|\|x_n\| = \|x_n\| \leq 1.$$

By Problem 2 HW4 we have that $f(x_n) \to f(x)$ as $n \to \infty$. So

$$\|x\| = |f(x)| = \lim_{n\to\infty} |f(x_n)| \leq \lim_{n\to\infty} \|x_n\| \leq 1.$$

Hence $\|x\| \leq 1$.