Mandatory Assignment 2
Functional Analysis

Name: Derya Diana Cosan
KU-id: wxz639
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Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. I set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$ for all $N \geq 1$

(a)

I will in this section show that $f_N \to 0$ weakly as $N \to \infty$. Furthermore I will show that $||f_N|| = 1$ for all $N \geq 1$.
First, I will show that $f_N \to 0$ weakly as $N \to \infty$.
I have set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$ for all $N \geq 1$. Since $(e_n)_{n\geq 1}$ is a orthonormal basis in H it applies that $f_N = N^{-1}\sum_{n=1}^{N^2} e_n \in H$ i.e. $f_N \in H$.
Furthermore, I let $F_y : H \to \mathbb{C}$ be any linear bounded linear functional. From Riesz' representation theorem it applies that there exists $y = \sum_{n=1}^{\infty} y_n e_n \in H$ such that $F_y(x) = \langle x, y\rangle$.
Now look at

$$F_y(f_N) = \langle f_N, y\rangle = \langle N^{-1}\sum_{n=1}^{N^2} e_n, \sum_{n=1}^{\infty} y_n e_n\rangle$$
$$= N^{-1}\sum_{n=1}^{N^2}\langle e_n, \sum_{n=1}^{\infty} y_n e_n\rangle$$
$$= N^{-1}\sum_{n=1}^{N^2} y_n < \infty$$

Needs some $|\cdot|$, because in general $y_n \in \mathbb{C}$ and this is not well-defined.

This $N^{-1}\sum_{n=1}^{N^2} y_n < \infty$ applies because F_y is bounded.

I will now show that $\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n \to 0$ as $n \to 0$. $N \to 0$.
From the triangle inequality and Cauchy-Schwartz inequality it applies that

?

$$\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n\right)^2 \leq \left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} |y_n|\right)^2 = \left(\sum_{n=1}^{N}\frac{1}{\sqrt{N}}|y_n|\right)^2 \leq \sum_{n=1}^{N}\left(\frac{1}{\sqrt{N}}\right)^2\sum_{n=1}^{N}|y_n|^2 = \sum_{n=1}^{N}|y_n|^2$$

Not well-defined.

These inequalities gives

$$\Big|\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n\Big| \leq \Big(\sum_{n=1}^{N}|y_n|^2\Big)^{\frac{1}{2}} < \infty$$

This applies because $(y_n)_{n\geq 1} \in \ell_2(\mathbb{N})$. Because $\sum_{n=1}^{N}|y_n|^2 < \infty$, there exists a constant $C \in \mathbb{C}$ such that $\sum_{n=1}^{N}|y_n|^2 \to C$ for $n \to \infty$. No, this requires that $\sum_{n=1}^{\infty}|y_n|^2 < \infty$, which you have stated without proof.
Hence $\forall \varepsilon > 0\ \exists M$ such that

$$\sum_{n=M+1}^{\infty}|y_n|^2 < \varepsilon$$

Hence for any constant $K \geq 1$ it applies that $\sum_{n=M+1}^{K+M}|y_n|^2 < \varepsilon$. For $N \geq \frac{C^2}{\varepsilon^2}$ we then have

Where did the square go?

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{M}|y_n| \leq \frac{\varepsilon}{C}\cdot C = \varepsilon$$

By using Triangle inequality and Cauchy-Schwarz inequality we then have

$$\begin{aligned}
\Big|\frac{1}{\sqrt{N}}\cdot\sum_{n=1}^{N} y_n\Big| &\leq \frac{1}{\sqrt{N}}\sum_{n=1}^{N}|y_n| \\
&= \frac{1}{\sqrt{N}}\cdot\sum_{n=1}^{M}|y_n| + \frac{1}{\sqrt{N}}\cdot\sum_{n=M+1}^{N}|y_n| \\
&\leq \varepsilon + \frac{1}{\sqrt{N}}\sum_{n=M+1}^{N+M}|y_n| \\
&= \varepsilon + \sum_{n=M+1}^{N+M}\frac{1}{\sqrt{N}}\cdot|y_n| \\
&= \varepsilon + \sqrt{\Big(\sum_{n=M+1}^{N+M}\frac{1}{\sqrt{N}}\cdot|y_n|\Big)^2} \\
&\leq \varepsilon + \sqrt{\Big(\sum_{n=M+1}^{N+M}\frac{1}{N}\Big)\cdot\Big(\sum_{n=M+1}^{N+M}|y_n|^2\Big)} \\
&= \varepsilon + \sqrt{1\cdot\sum_{n=M+1}^{N+M}|y_n|^2} \\
&< \varepsilon + \sqrt{\varepsilon}
\end{aligned}$$

Thus $\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n\right| \to 0$ for $N \to \infty$. This gives that $\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N^2} y_n\right| \to 0$ for $N \to \infty$. Hence

$$\lim_{N\to\infty} F_y(f_N) = \lim_{N\to\infty} N^{-1}\sum_{n=1}^{N^2} y_n = 0$$

Note that F_y is an arbitrary bounded function

Thus, since F_y is bounded i.e. continuous we can now conclude that $f_N \to 0$ weakly as $N \to \infty$ (✓)

I will now show that $||f_N|| = 1$. For that I will compute $||f_N||^2$. Notice that $||e_n|| = 1$ since $(e_n)_{n\geq 1}$ is an orthonormal basis.

$$\begin{aligned}
||f_N||^2 &= ||N^{-1}\sum_{n=1}^{N^2} e_n||^2 \\
&= |N^{-1}|^2 \cdot ||\sum_{n=1}^{N^2} e_n||^2 \\
&= N^{-2}||\sum_{n=1}^{N^2} e_n||^2 \\
&= N^{-2}\sum_{n=1}^{N^2} ||e_n||^2 \\
&= N^{-2}\sum_{n=1}^{N^2} 1^2 \\
&= N^{-2}N^2 \\
&= 1
\end{aligned}$$

What do you use here?

Hence $||f_N|| = 1$. (✓)

(b)

I will in this section argue that K is weakly compact and that $0 \in K$.
I let K be the norm closure of $co\{f_N | N \geq 1\}$ i.e. $K = \overline{co\{f_N | N \geq 1\}}^{||\cdot||}$.

By definition 7.7 (lecture notes) it applies that $co\{f_N | N \geq 1\}$ is convex so by theorem 5.7 (lecture notes) the norm and the weak closure of $co\{f_N | N \geq 1\}$ coincide. This gives that $K = \overline{co\{f_N | N \geq 1\}}^{\tau_w}$, hence K is weakly closed.
So since K is weakly closed and since $f_N \to 0$ as $N \to \infty$ by problem 1a, then $0 \in K$. ✓

I will now show that K is weakly compact. Now look at the unit ball $\overline{B}_{H^*}(0,1) \subset H^*$. Since H is a Hilbert space, hence a normed vector space, then by Alaoglu theorem it applies that $\overline{B}_{H^*}(0,1)$ is compact in the w^*-topology. Furthermore since H is a Hilbert space, H is reflexive. Hence for H^* it applies that $\tau_w = \tau_{w^*}$. Thus $\overline{B}_{H^*(0,1)}$ is weakly compact.
Now by using Riesz' Representation theorem, it applies that every element in H^* can be written in the form $F_y = \langle \cdot, y \rangle$ where $y \in H$. Hence there exists an isomorphism from H^* to H, which sends F_y to y. Thus we get an isomorphism between $\overline{B}_{H^*(0,1)}$ and $\overline{B}_H(0,1)$. Hence $K \subseteq \overline{B}_H(0,1)$ is a weakly closed subset of a weakly compact space, which gives that K is weakly compact.

Careful, this is an antilinear isomorphism,

(✓)

(c)

I will in this section show that 0, as well as each f_N, $N \geq 1$, are extreme points in K.

By definition 7.1 notice

$$Ext(K) = \{x \in K | x = \alpha x_1 + (1-\alpha)x_2 \text{ implies } x_1 = x_2 = x, x_1, x_2 \in K, 0 < \alpha < 1\}$$

I will show that $0 \in Ext(K)$. Notice that $K \subseteq H$ is non-empty convex compact subset. Now for $n \in \mathbb{N}$ look at $h_n = \langle \cdot, -e_n \rangle \in H^*$ where h_n is a continuous linear functional. Notice $h_n(K) \subseteq \mathbb{R}$ and now I let

Why is this true?

$$C = \sup_{n \in \mathbb{N}} \{\langle x, -e_n \rangle | x \in K\} = \sup_{n \in \mathbb{N}} \{-\langle x, e_n \rangle | x \in K\}$$

Because we have that $x \in K$ then $x \geq 0$ and $0 \in K$ and this gives that $C = \sup_{n \in \mathbb{N}} \{-\langle x, e_n \rangle | x \in K\} \leq 0$. Thus by using lemma 7.5 we then have that

$$F_n := \{x \in K | Re\langle x, -e_n \rangle = 0\} \cancel{\neq} \emptyset$$

is a compact face of K for all $n \in \mathbb{N}$.
So now, since $0 \in F_n$ $\forall n \in \mathbb{N}$ thus $0 \in \cap_{n=1}^{\infty} F_n \neq \emptyset$. This gives that

$$\cap_{n=1}^{\infty} F_n = \{x \in K | \langle x, -e_n \rangle = 0 \quad \forall n \in \mathbb{N}\} = \{0\}$$

since the only element which is orthogonal on all e_n is 0. By using lemma 7.4(3) (lecture notes), I can now notice that $\cap_{n=1}^{\infty} F_n = \{0\}$ is a face of K and by using 7.4(1)(lecture notes) it now applies that $0 \in Ext(K)$. ✓

I will now show that $f_N \in Ext(K)$. I start by fixing $N \geq 1$ and assume that

$$f_N = \alpha x_1 + (1-\alpha)x_2 \quad \text{for} \quad x_1, x_2 \in K \quad \text{and} \quad 0 < \alpha < 1$$

Notice furthermore that $1 = ||f_N||^2 = \langle f_N, f_N \rangle$.
Now consider

$$1 = \langle f_N, f_N \rangle = \langle \alpha x_1 + (1-\alpha)x_2, f_N \rangle = \alpha \langle x_1, f_N \rangle + (1-\alpha)\langle x_2, f_N \rangle$$

which gives that

$$\begin{aligned} 0 &= \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle - 1 \\ &= \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle - (\alpha + (1-\alpha)) \\ &= \alpha(\langle x_1, f_N\rangle - 1) + (1-\alpha)(\langle x_2, f_N\rangle - 1) \end{aligned}$$

How do you know this?

Why is this true?

and because $0 < \alpha < 1$ and $\langle x_1, f_N\rangle, \langle x_2, f_N\rangle \geq 0$ it applies that $0 \leq \langle x_i, f_N\rangle \leq 1$ for $i = 1, 2$. Thus we get that

$$\langle x_1, f_N \rangle = 1$$
$$\langle x_2, f_N \rangle = 1$$

I will now show that $x_1 = x_2 = f_N$.
Now look at

$$1 = ||\langle x_1, f_N\rangle|| \leq ||x_1|| \cdot ||f_N|| = ||x_1||$$

where Cauchy Schwartz inequality is used.
Because $x_1 \in K \subseteq \overline{B}_H(0,1)$ it applies that $||x_1|| \leq 1$.
So now, since $||x_1|| \leq 1$ and $1 \leq ||x_1||$, it gives that $||x_1|| = 1$. Thus

$$1 = ||\langle x_1, f_N\rangle|| = ||x_1|| \cdot ||f_N|| = ||x_1||$$

Thus x_1 and f_N are linearly dependent and hence $x_1 = \lambda f_N$ for a scalar λ. Hence

$$1 = \langle \lambda f_N, f_N\rangle = \lambda\langle f_N, f_N\rangle = \lambda||f_N||^2 = \lambda$$

Thus $\lambda = 1$.
So $x_1 = \lambda f_N = 1 \cdot f_N = f_N$. Hence $x_1 = f_N$.For x_2 the same can be done in the same way. Then $x_1 = x_2 = f_N$. Thus $f_N \in Ext(K)\ \forall N \geq 1$. (✓)

(d)

I will in this part argue whether there are any other extreme points in K or not. First, notice that $K = \overline{co\{f_N | N \geq 1\}}^{\tau_w}$ is a non-empty compact convex subset for H. By using Milman (theorem 7.9 lecture notes) it gives

$$Ext(K) \subseteq \overline{\{f_N | N \geq 1\}}^{\tau_w}$$

and furthermore by using problem 1c it gives

$$\{f_N | N \geq 1\} \cup \{0\} \subseteq Ext(K) \subseteq \overline{\{f_N | N \geq 1\}}^{\tau_w}$$

Hence

$$\{f_N | N \geq 1\} \cup \{0\} \subseteq \overline{\{f_N | N \geq 1\}}^{\tau_w}$$

Since H is metrizable it gives that $\{f_N | N \geq 1\}$ is metrizable. Thus $\{f_N | N \geq 1\}$ is first countable. Hence it is not necessary to consider nets and instead it is enough to consider sequences which is in $\{f_N | N \geq 1\}$.
Now suppose that $(x_n)_{n \geq 1}$ is a sequence in $\{f_N | N \geq 1\}$ where $(x_n)_{n \geq 1}$ converges weakly to $x \in \overline{\{f_N | N \geq 1\}}^{\tau_w}$. Thus each $x_i = f_N$ for some $N \geq 1$, hence x is equal to some f_N or 0. We then have ?

Why should the sequence necessarily be constant?

$$Ext(K) \subseteq \overline{\{f_N | N \geq 1\}}^{\tau_w} = \{f_N | N \geq 1\} \cup \{0\}$$

From problem 1.c it applies that

$$\{f_N | N \geq 1\} \cup \{0\} \subseteq Ext(K)$$

and now since

$$Ext(K) \subseteq \{f_N | N \geq 1\} \cup \{0\}$$

It is possible to conclude that

$$Ext(K) = \{f_N | N \geq 1\} \cup \{0\}$$

Hence there are not any other extreme points in K. (✓)

Problem 2

Let X and Y be infinite dimensional Banach spaces.

(a)

Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n \geq 1}$ in X and $x \in X$, I will in this section show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.
So I assume that $x_n \to x$ weakly for $n \to \infty$, and I then want to show that $Tx_n \to Tx$ weakly, as $n \to \infty$.
From HW4 problem 2a we have that $x_n \to x$ weakly for $n \to \infty$ if and only if $f(x_n) \to f(x)$ for all $f \in X^*$. We can use this result from this problem since x_n is a sequence and a sequence is a part of a net, where a net is a generalization.

No, but sequences are nets.

I now take a $g \in Y^*$ which implies that $g \circ T \in X^*$. This gives that

$$(g \circ T)(x_n) \to (g \circ T)(x)$$

for $n \to \infty$ and for all $g \in Y^*$.
Hence by using HW4 problem 2a, it applies that

$$(g \circ T)(x_n) \to (g \circ T)(x) \text{ for } n \to \infty \text{ and for all } g \in Y^*$$

is the same as $Tx_n \to Tx$ weakly as $n \to \infty$. Hence, I can conclude that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$. ✓

(b)

Let $T \in \mathcal{K}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, I will in this section show that $x_n \to x$ weakly, as $n \to \infty$, implies that $||Tx_n - Tx|| \to 0$ as $n \to \infty$.
I will show this by contradiction.
So I suppose by contradiction that $||Tx_n - Tx|| \nrightarrow 0$ for $n \to \infty$. Thus there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $||Tx_n - Tx|| > \varepsilon$ for all $k \in \mathbb{N}$. It applies that $x_n \to x$ weakly for $n \to \infty$ which implies that $x_{n_k} \to x$ weakly for $k \to \infty$.
Thus $(x_{n_k})_{k\in\mathbb{N}}$ is bounded. This gives that $(x_{n_k})_{k\in\mathbb{N}}$ has a convergent subsequence $(x_{n_{k_i}})_{i\in\mathbb{N}}$ such that $||Tx_{n_{k_i}} - Tx'|| \to 0$ for some $x' \in X$, because X and Y are Banach spaces. Because $x_{n_k} \to x$ for some $k \to \infty$ weakly,then by 2a it applies that $Tx_{n_k} \to Tx$ weakly and hence $Tx_{n_{k_i}} \to Tx$ weakly for $i \in \mathbb{N}$. But if this applies then $||Tx_{n_{k_i}} - Tx|| \to 0$ for $i \to \infty$, because we are in Banach spaces. Why do we obtain norm-conv. specifically?
$||Tx_{n_{k_i}} - Tx|| \to 0$ for $i \to \infty$ contradicts with $||Tx_{n_k} - Tx|| > \varepsilon$ for all $k \in \mathbb{N}$. Hence $||Tx_n - Tx|| \to 0$ for $n \to \infty$.

Why? Bounded sequences need not have convergent subsequences.

Why should $(Tx_{n_{k_i}})$ conv. in $T(X)$?

(c)

I let H be a separable infinite dimensional Hilbert space. I will show that, if $T \in \mathcal{L}(H,Y)$ satisfies that $||Tx_n - Tx|| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H,Y)$.
I will show that $T \in \mathcal{K}(H,Y)$ by contradiction. So I assume by contradiction that T is not compact. Then proposition 8.2 (lecture notes) gives that $T(\overline{B}_H(0,1))$ is not totally bounded. By definition of totally bounded this means that $\exists \varepsilon > 0$ such that there are union of finitely many open balls with radius ε which does not covering $T(\overline{B}_H(0,1))$.
I will now show that there exists a sequence $(x_n)_{n\geq 1}$ in the closed unit ball of H such that $||Tx_n - Tx_m|| \geq \varepsilon$ for all $n \neq m$.
Then now take $x_1 \in \overline{B}_H(0,1)$ where it applies that $x_1 \in (x_n)_{n\geq 1} \subset \overline{B}_H(0,1)$. Assume that $x_2, x_3, ..., x_n$ satisfying that $||Tx_q - Tx_r|| \geq \varepsilon$ for all $q, r \leq n$, $q, r > 1$ and $q \neq r$.
$q,r \geq 1$?
Now look at

$$P := T(\overline{B}_H(0,1)) \cap (\cup_{i=1}^{n} B_Y(Tx_i, \varepsilon))^C$$

Notice that $T(\overline{B}_H(0,1)) \not\subset (\cup_{i=1}^{n} B_Y(Tx_i, \varepsilon))$ since $T(\overline{B}_H(0,1))$ is not totally bounded, thus $P \neq \emptyset$.
Now take $x_{n+1} \in \overline{B}_H(0,1)$ such that $Tx_{n+1} \in P$. Especially $Tx_{n+1} \in (\cup_{i=1}^{n} B_Y(Tx_i, \varepsilon))^C$, thus $Tx_{n+1} \notin B_Y(Tx_i, \varepsilon)$ for any i.
Thus $||Tx_{n+1} - Tx_i|| \geq \varepsilon$ for all $i \leq n$, which gives what I want to show, because if I continue this proces I get that $||Tx_n - Tx_m|| \geq \varepsilon$ for all $n \neq m$. ✓
Furthermore H is a reflexive space, since H is a Hilbert space, so by theorem 6.3(lecture notes) it applies that $\overline{B}_H(0,1)$ is weakly compact. Hence every sequence has a weakly

Generally, it will only have a weakly convergent subnet, not a subsequence.

convergent subsequence. Since $\overline{B}_H(0,1)$ is weakly compact, $(x_{n_k})_{k\geq 1}$ can be the weakly convergent subsequence of $(x_n)_{n\geq 1}$. Since $||Tx_n - Tx_m|| \geq \varepsilon$ for all $n \neq m$ it means that $||Tx_{n_k} - Tx|| \nrightarrow 0$ for $k \to \infty$ which is a contradiction for our assumption. So T is compact. (✓)

Why is this the case?

(d)

I will show that $T \in \mathcal{L}(\ell_2,(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact. Now let $(x_n)_{n\geq 1} \in H$ and assume that $x_n \to x$ weakly for $n \to \infty$ so by problem 2a $Tx_n \to Tx$ weakly, as $n \to \infty$, in $\ell_1(\mathbb{N})$.
Since $Tx_n \to Tx$ weakly then by remark 5.3 it applies that $||Tx_n - Tx|| \to 0$.
From HW4 problem 4 we have that $\ell_2(\mathbb{N})$ is separable. Furtermore $\ell_2(\mathbb{N})$ is an infinite dimensional Hilbert space. Then by problem 2c I can conclude that T is compact. ✓

(e)

I will in this section show that no $T \in \mathcal{K}(X,Y)$ is onto. This will be done by contradiction. Let X and Y be infinite dimensional Banach spaces. So now I assume that T is onto, hence open (by open mapping theorem).
Because X, Y are normed vector spaces, since they are Banach spaces, and because T is open then there exists $r > 0$ such that $B_Y(0,r) \subset T(B_x(0,1))$.
It also applies that $\overline{B_Y(0,r)} \subset \overline{T(B_x(0,1))}$, since taking closure preverses inclusion.
Because T is a compact operator, it gives that $\overline{T(B_x(0,1))}$ is compact, then a closed subset of a compact $\overline{T(B_x(0,1))}$ is compact and since $\overline{B_Y(0,r)}$ is a closed subset of $\overline{T(B_x(0,1))}$ then $\overline{B_Y(0,r)}$ is compact.
I will now look at the cases where $r = 1$, $r > 1$ and $r < 1$.
When $r = 1$ then it applies that $\overline{B_Y(0,r)} = \overline{B_Y(0,1)}$ and since $\overline{B_Y(0,r)}$ is compact, it gives that $\overline{B_Y(0,1)}$ is compact, which is a contradiction because we have that since Y is infinite dimensional then by Riezs lemma we have that $\overline{B_Y(0,1)}$ is not compact.
When $r > 1$ then it applies that $\overline{B_Y(0,1)}$ is a closed subset of the compact subset $\overline{B_Y(0,r)}$, which means that $\overline{B_Y(0,1)}$ is also compact, which is again a contradiction, since Y is infinite dimensional hence $\overline{B_Y(0,1)}$ is not compact.
When $r < 1$ I look at $f : Y \to Y$ by $x \mapsto \frac{1}{r}x$, which is continuous. Since it applies that the image under a continuous function of a compact set is compact, then it applies that since Y is infinite dimensional then $f(\overline{B_Y(0,1)}) = \overline{B_Y(0,1)}$ is compact which is again a contradiction, with same argument as earlier. ✓
These contradictions gives that no $T \in \mathcal{K}(X,Y)$ is onto.

(f)

Let $H = L_2([0,1], m)$ and consider the operator $M \in \mathcal{L}(H,H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. I will show that M is self-adjoint, but not compact.

I start by showing that M is self-adjoint i.e. I will show that $M = M^*$.
Notice first that $t = \bar{t}$ since t has only real values. Notice furthermore that $g \in H$. I now look at the inner product on H and we deduced the following

$$\begin{aligned}\langle Mf, g\rangle &= \int_{[0,1]} Mf(t)\overline{g(t)}dm(t) \\ &= \int_{[0,1]} tf(t)\overline{g(t)}dm(t) \\ &= \int_{[0,1]} f(t)\bar{t}\overline{g(t)}dm(t) \\ &= \int_{[0,1]} f(t)\overline{tg(t)}dm(t) \\ &= \int_{[0,1]} f(t)\overline{Mg(t)}dm(t) \\ &= \langle f, Mg\rangle\end{aligned}$$

Hence I have shown that $M = M^*$, which gives that M is self-adjoint.
I will now show that M is not compact. I assume by contradiction that M is compact. From earlier notice that M is self-adjoint. Furtermore $H = L_2([0,1], m)$ is infinite dimensional and from HW4 problem 4 we have that $H = L_2([0,1], m)$ is separable.
Hence by the Spectral Theorem for self-adjoint compact operators (theorem 10.1 in lecture notes), we have that H has an ONB consisting of eigenvectors for M with corresponding eigenvalues.
But from HW6 problem 3a we have that M has no eigenvalues. This gives the contradiction and we can conclude that M is not compact.

Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$ where m is the Lebesgue measure. Define $K : [0,1] \times [0,1] \to \mathbb{R}$ by:

$$K(s,t) = \begin{cases}(1-s)t \text{ if } 0 \le t \le s \le 1 \\ (1-t)s \text{ if } 0 \le s < t \le 1\end{cases}$$

.

and consider $T \in \mathcal{L}(H, H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t), \qquad s \in [0,1],\ f \in H$$

(a)

I will in this section justify that T is compact. First notice that $[0,1]$ is a compact Hausdorff topological space and m is the Lebesgue measure on $[0,1]$. Lebesgue measure on $[0,1]$ is a measure on Borel σ-algebra so m is a finite Borel measure on $[0,1]$. Furthermore K is continuous on $[0,1]\times[0,1]$ so $K \in C([0,1]\times[0,1])$. Hence $T : H \to H$ is a compact operator by theorem 9.6 (lecture notes).

A little wied frasing

This requires $T = T_k$ in fact $T = T_{\tilde{k}}$ with $\tilde{k}(s,t) = k(t,s)$

Check at least $s \to t$.

(b)

I will in this section show that $T = T^*$. For showing this I will show that $\langle Tf, g\rangle = \langle f, Tg\rangle$.
So I consider

$$
\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} Tf(s)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(t,s)f(t)\overline{g(s)}dm(t)dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} \overline{K(s,t)}f(t)\overline{g(s)}dm(t)dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} \overline{K(s,t)}f(t)\overline{g(s)}dm(s)dm(t) \\
&= \int_{[0,1]}\int_{[0,1]} \overline{K(s,t)g(s)}dm(s)f(t)dm(t) \\
&= \int_{[0,1]} \overline{Tg(t)}f(t)dm(t) \\
&= \langle f, Tg\rangle
\end{aligned}
$$

why justified?

where I have used Tonelli-Fubini theorem and the fact that $K(t,s) = K(s,t)$. Hence I have showed that $\langle Tf, g\rangle = \langle f, Tg\rangle$ which gives that $T = T^*$.

(c)

In the following I will show that

$$
(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t) \qquad s \in [0,1], f \in H
$$

so I compute $(Tf)(s)$. Notice that the second equality follows by the linearity of Lebesgue integrals

$$\begin{aligned}(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\ &= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\ &= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\ &= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)\end{aligned}$$

and hence I have shown that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t) \qquad s \in [0,1], f \in H$$

I will now argue that Tf is continuous on $[0,1]$. For showing that Tf is continuous I will show that Tf is bounded. Since $f \in L_2([0,1], m)$ it gives that $||f||_2 < \infty$. Furthermore

not continuity.

$$\left(\int_{[0,1]} |f|^2 dm(t)\right)^{\frac{1}{2}} < \infty$$

Since $\left(\int_{[0,1]} |f|^2 dm(t)\right)^{\frac{1}{2}} < \infty$ and since $0 \leq t \leq 1$ and $0 \leq s \leq 1$ it gives that

$$(1-s)\int_{[0,s]} tf(t)dm(t) < \infty$$

$$s\int_{[s,1]} (1-t)f(t)dm(t) < \infty$$

Hence $(Tf)(s) < \infty$ so Tf is bounded and hence from proposition 1.10 Tf is continuous on $[0,1]$.

P.1.10 is for linear operators

Tf is just a function not necessarilly linear.

Finally I will show that $(Tf)(0) = (Tf)(1) = 0$.

$$\begin{aligned}
(Tf)(0) &= (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]}(1-t)f(t)dm(t) \\
&= \int_{[0,0]} tf(t)dm(t) \\
&= 0
\end{aligned}$$

so $(Tf)(0) = 0$.

$$\begin{aligned}
(Tf)(1) &= (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]}(1-t)f(t)dm(t) \\
&= 0 + \int_{[1,1]}(1-t)f(t)dm(t) \\
&= \int_{[1,1]}(1-t)f(t)dm(t) \\
&= 0
\end{aligned}$$

so $(Tf)(1) = 0$.
Hence I have shown that $(Tf)(0) = (Tf)(1) = 0$.

Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$

(a)

For each integer $k \geq 0$, I set $g_k(x) = x^k e^{\frac{-x^2}{2}}$ for x$\in \mathbb{R}$. I will in this section justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$. Furthermore I will compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$

I first notice that $e^{-||x||^2} = e^{-x^2}$. From HW7 I know that $e^{-||x||^2} \in \mathbb{R}$ belongs to $\mathscr{S}(\mathbb{R})$. Hence $e^{-x^2} \in \mathscr{S}(\mathbb{R})$. Notice furthermore that $S_{\sqrt{2}}(e^{-x^2}) = e^{-\frac{x^2}{2}}$. From HW7 problem 1d it applies that $S_{\sqrt{2}}(e^{-x^2}) \in \mathscr{S}(\mathbb{R})$ (since $\sqrt{2} \in \mathbb{R} \setminus \{0\}$ and $e^{-x^2} \in \mathscr{S}(\mathbb{R})$) so $e^{\frac{-x^2}{2}} \in \mathscr{S}(\mathbb{R})$. Hence by problem 1a in HW7 I can now conclude that $x^k e^{-\frac{x^2}{2}}$.

I will now compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.
I start by fix k and I set $\phi := e^{\frac{-x^2}{2}}$, which is integrable. Furthermore notice that $x^k e^{\frac{-x^2}{2}}$

is also integrable. So since $g_k \in \mathscr{S}(\mathbb{R})$, HW7 1.c gives that $g_k \in L_1(\mathbb{R})$.
Furthermore by proposition 11.4 (lecture notes) for $n = 1$ it applies that $\phi(x) = \hat{\phi}(x)$.
So now consider

$$\begin{aligned}
\mathcal{F}(g_k)(\xi) &= \hat{g_k}(\xi) \\
&= (g_k)\hat{\ }(\xi) \\
&= (x^k\phi)\hat{\ }(\xi) \\
&= i^k(\partial^k\hat{\phi})(\xi) \\
&= i^k(\partial^k\phi)(\xi)
\end{aligned}$$

where I have used definition 11.1(lecture notes) since $g_k \in L_1(\mathbb{R})$ and I have used proposition 11.13(d)(lecture notes), which is possible since $x^k e^{\frac{-x^2}{2}} \in L_1(\mathbb{R})$ and $e^{\frac{-x^2}{2}} \in L_1(\mathbb{R})$.

I will now compute $\mathcal{F}(g_k)$ for $k = 0, 1, 2, 3$.
Then for $k = 0$ I obtain

$$\begin{aligned}
\mathcal{F}(g_0)(\xi) &= i^0(\partial^0\phi)(\xi) \\
&= 1 \cdot e^{\frac{-\xi^2}{2}} \\
&= e^{\frac{-\xi^2}{2}} = g_0
\end{aligned}$$

Thus $\mathcal{F}(g_0)(\xi) = e^{\frac{-\xi^2}{2}}$

For $k = 1$ I obtain

$$\begin{aligned}
\mathcal{F}(g_1)(\xi) &= i^1(\partial^1\phi)(\xi) \\
&= i\partial\phi)(\xi) \\
&= i(e^{\frac{-\xi^2}{2}} \cdot (-\xi)) \\
&= -i\xi e^{\frac{-\xi^2}{2}}
\end{aligned}$$

Thus $\mathcal{F}(g_1)(\xi) = -i\xi e^{\frac{-\xi^2}{2}}$

For $k = 2$ I obtain

$$\begin{aligned}
\mathcal{F}(g_2)(\xi) &= i^2(\partial^2\phi)(\xi) \\
&= i^2(-e^{\frac{-\xi^2}{2}} + \xi^2 e^{\frac{-\xi^2}{2}}) \\
&= e^{\frac{-\xi^2}{2}} - \xi^2 e^{\frac{-\xi^2}{2}}
\end{aligned}$$

Thus $\mathcal{F}(g_2)(\xi) = e^{\frac{-\xi^2}{2}} - \xi^2 e^{\frac{-\xi^2}{2}}$

For $k = 3$ I obtain

$$\begin{aligned}\mathcal{F}(g_3)(\xi) &= i^3(\partial^3\phi)(\xi) \\ &= i^3(\xi e^{\frac{-\xi^2}{2}} + 2\xi e^{\frac{-\xi^2}{2}} - \xi^3 e^{\frac{-\xi^2}{2}}) \\ &= i^3(3\xi e^{\frac{-\xi^2}{2}} - \xi^3 e^{\frac{-\xi^2}{2}}) \\ &= -i(3\xi e^{\frac{-\xi^2}{2}} - \xi^3 e^{\frac{-\xi^2}{2}})\end{aligned}$$

Thus $\mathcal{F}(g_3)(\xi) = -i(3\xi e^{\frac{-\xi^2}{2}} - \xi^3 e^{\frac{-\xi^2}{2}})$.

(b)

I will in this part find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$ for $k = 0, 1, 2, 3$. I will use the computations from problem 4a.

I will find a non-zero function $h_0 \in \mathscr{S}(\mathbb{R})$ for which it applies that $F(h_0) = i^0 h_0$. Consider

$$F(g_0) = i^0 e^{\frac{-\xi^2}{2}} = g_0(\xi)$$

so $h_0 = g_0$ such that $\mathcal{F}(h_0) = i^0 h_0$

Now, I will find a a non-zero function $h_1 \in \mathscr{S}(\mathbb{R})$ for which it applies that $F(h_1) = ih_1$. I compute

$$\mathcal{F}(g_3)(\xi) = i(\xi^3 e^{\frac{-\xi^2}{2}} - 3\xi e^{\frac{-\xi^2}{2}}) = i(g_3(\xi) - 3g_1(\xi))$$

Thus

$$\begin{aligned}\mathcal{F}(g_3 - \frac{3}{2}g_1)(\xi) &= \mathcal{F}(g_3)(\xi) - \frac{3}{2}\mathcal{F}(g_1)(\xi) \\ &= i(g_3(\xi) - 3g_1(\xi)) - \frac{3}{2}(-i\xi e^{\frac{-\xi^2}{2}}) \\ &= i(g_3(\xi) - 3g_1(\xi)) + \frac{3}{2}i\xi e^{\frac{-\xi^2}{2}} \\ &= i(g_3(\xi) - 3g_1(\xi) + \frac{3}{2}\xi e^{\frac{-\xi^2}{2}}) \\ &= i(g_3(\xi) - \frac{3}{2}g_1(\xi))\end{aligned}$$

Hence $h_1 = (g_3 - \frac{3}{2}g_1)$ such that $\mathcal{F}(h_1) = ih_1$

Now, I will find a non-zero function $h_2 \in \mathscr{S}(\mathbb{R})$ for which it applies that $F(h_2) = i^2 h_2 = -h_2$.
I compute

$$\mathcal{F}(g_2)(\xi) = e^{\frac{-\xi^2}{2}} - \xi^2 e^{\frac{-\xi^2}{2}} = g_0(\xi) - g_2(\xi) = -(g_2(\xi) - g_0(\xi))$$

Thus

$$\begin{aligned}
\mathcal{F}(g_2 - \frac{1}{2}g_0)(\xi) &= \mathcal{F}(g_2)(\xi) - \frac{1}{2}\mathcal{F}(g_0)(\xi) \\
&= -(g_2(\xi) - g_0(\xi)) - \frac{1}{2}g_0(\xi) \\
&= -g_2(\xi) + g_0(\xi) - \frac{1}{2}g_0(\xi) \\
&= -g_2(\xi) + \frac{1}{2}g_0(\xi) \\
&= -(g_2(\xi) - \frac{1}{2}g_0(\xi))
\end{aligned}$$

Thus $h_2 = (g_2 - \frac{1}{2}g_0)$ such that $\mathcal{F}(h_2) = i^2 h_2 = -h_2$

Now, I will find a non-zero function $h_3 \in \mathscr{S}(\mathbb{R})$ for which it applies that $F(h_3) = i^3 h_3 = -ih_3$.
I compute

$$\mathcal{F}(g_1)(\xi) = -i\xi e^{\frac{-\xi^2}{2}} = -ig_1(\xi)$$

Thus $h_3 = g_1$ such that $\mathcal{F}(h_3) = i^3 h_3 = -ih_3$

(c)

In this section I will show that $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$

Notice from HW7 1.c that $\mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$ and since $f \in \mathscr{S}(\mathbb{R})$ it gives that $f, \hat{f} \in L_1(\mathbb{R})$.
Notice furthermore that, since $f \in \mathscr{S}(\mathbb{R})$ we have that $\mathcal{F}^*(\mathcal{F}(f)) = \mathcal{F}(\mathcal{F}^*(f)) = f$ (from corollary 12.12(iii) in lecture notes). So now consider

$$\begin{aligned}
\mathcal{F}^2(f)(\xi) &= \mathcal{F}(\mathcal{F}(f))(\xi) \\
&= \mathcal{F}(\hat{f}(\xi)) \\
&= \int_{\mathbb{R}} e^{-ix\xi}\hat{f}(x)dm(x)
\end{aligned}$$

Furthermore it applies that

$$
\begin{aligned}
(S_{-1}f)(\xi) &= f\Big(\frac{\xi}{-1}\Big) \\
&= f(-\xi) \\
&= \mathcal{F}^*(\mathcal{F}(f))(-\xi) \\
&= \mathcal{F}^*(\hat{f})(-\xi) \\
&= \int_{\mathbb{R}} e^{-ix\xi}\hat{f}(x)dm(x) \\
&= \mathcal{F}^2(f)(\xi)
\end{aligned}
$$

Finally I can look at the following and obtaing what I want to show:

$$
\begin{aligned}
(\mathcal{F}^4 f)(x) &= \mathcal{F}^2(\mathcal{F}^2 f)(x) \\
&= \mathcal{F}^2(S_{-1}f)(x) \\
&= \mathcal{F}^2(f)(-x) \\
&= (S_{-1}f)(-x) \\
&= f(x)
\end{aligned}
$$

Hence $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$ ✓

(d)

In this problem, I will show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$ then $\lambda \in \{1, i, -1, -i\}$. I will furthermore conclude that the eigenvalues of $\mathcal{F}$ are precisely $\{1, i, -1, -i\}$

Suppose that $f \in \mathscr{S}(\mathbb{R})$ is non-zero and and $\mathcal{F}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$ then I want to show that $\lambda \in \{1, i, -1, -i\}$.To show this, it is enough to show that $\lambda^4 = 1$. Notice that, $\lambda f = \mathcal{F}(f)$ gives that $\lambda^4 f^4 = \mathcal{F}^4(f)$. ← where is this from ?
From problem 4c I know that $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$. This gives that

$$\lambda^4 f^4 = \mathcal{F}^4(f) = f$$

$\mathcal{F}^4(f) = \lambda^4 f$

so

$$\lambda^4 = \frac{f}{f^4}$$

← f need not be non-zero everywhere !

Furthermore by using $\mathcal{F}^4(f) = f$ from problem 4c we obtain

$$f^2 = \mathcal{F}^8(f) = \mathcal{F}^4(\mathcal{F}^4(f)) = \mathcal{F}^4(f) = f$$

$\mathcal{F}^8(f) = \mathcal{F}^4(\mathcal{F}^4(f))$
$= \mathcal{F}^4(f) = f$

so

$$f^4 = (f^2)^2 = f^2 = f$$

so

$$\lambda^4 = \frac{f}{f^4} = \frac{f}{f} = 1$$

Hence $\lambda^4 = 1$ so $\lambda \in \{1, i, -1, -i\}$.

Since $\{1, i, -1, -i\}$ are the only values for λ which satisfy $\mathcal{F}(f) = \lambda f$, then the eigenvalues for $\mathcal{F}$ are $\{1, i, -1, -i\}$ 0/0

Problem 5

Let $(x_n)_{n \geq 1}$ be a dense subet on $[0, 1]$ and I consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}$ on $[0, 1]$. I want to show that $\text{supp}(\mu) = [0, 1]$
I then want to show that $\text{supp}(\mu) = \text{supp}(\sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}) = [0, 1]$.

Notice that μ is a Radon measure on a locally compact Hausdorff topological space $[0, 1]$. From HW8 problem 3a we then know that it is enough to show that $\mu([0, 1]^c) = 0$. NO.
Notice that

$$\delta_{x_n}([0, 1]^c) = \begin{cases} 0 & \text{if } x_n \notin [0, 1]^c \\ 1 & \text{if } x_n \in [0, 1]^c \end{cases}$$

This would only show $[0, 1]^c = \emptyset \subseteq N$, which is trivial as $\mu(\emptyset) = 0$ axiomatically (N largest open set w/ $\mu(N) = 0$).

Now consider

$$\mu([0, 1]^c) = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}([0, 1]^c) = 0$$

so $\mu([0, 1]^c) = 0$ which applies since $\delta_{x_n}([0, 1]^c) = 0$ for $x_n \notin [0, 1]^c$. Therefore, since μ is a measure on $[0, 1]$ and since $x_n \in [0, 1]$ $\forall n \geq 1$, it applies that $\delta_{x_n}([0, 1]^c) = 0$. Hence $\text{supp}(\mu) = [0, 1]$. 0/0