Assigment 2

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Problem 1 *Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.*

a) Show that $f_N \to 0$ weakly, as $N \to \infty$, while $\|f_N\| = 1$, for all $N \geq 1$.

By *HW4 2a)* $f_N \to 0$ weakly, as $N \to \infty$, if and only if for any linear functional $\phi \in H^*$, $\phi(f_N) \to 0$ in $\mathbb{K}$. But as H is a Hilbert space, by the Riesz representation theorem any linear functional is given by $\phi_z(x) = \langle x, z\rangle$ for some unique $z \in H$. So we need to show that $\langle f_N, z\rangle \to 0$ in $\mathbb{K}$ as $N \to \infty$:

$$|f_N, z| = \left|\langle N^{-1}\sum_{n=1}^{N^2} e_n, z\rangle\right| =$$
$$= N^{-1}\sum_{n=1}^{N^2} |\langle e_n, z\rangle| \leq N^{-1}\sum_{n=1}^{\infty} |\langle e_n, z\rangle| \leq$$
$$\leq^* N^{-1}\|z\|^2 \to 0$$

✓

where in (*) we have used Bessel's inequality: $\sum_{n=1}^{\infty} |\langle e_n, z\rangle| \leq \|z\|^2$.

And by orthonormality of $(e_n)_{n\geq 1}$: $\|f_n\|^2 = \frac{\overbrace{1+\cdots+1}^{N^2}}{N^2} = 1$. (✓)

Elaborate on this equation.

Let K be the norm closure of $\operatorname{co}\{f_N : N \geq 1\}$.

b) Argue that K is weakly compact, and that $0 \in K$.

We know that $\operatorname{co}(\{f_N\})$ is convex, so by *Theorem 5.7* $\overline{\operatorname{co}(\{f_n\})}^{\|\cdot\|} = \overline{\operatorname{co}(\{f_N\})}^{\tau_\omega}$ in H

$\operatorname{co}(\{f_N \mid N \in \mathbb{N}\})$

On the other hand, H is reflexive by *Proposition 2.10*, so by *Theorem 5.9* $\tau_\omega = \tau_{\omega^*}$ in H (τ_{ω^*} as H can be seen as the dual of H^*). As $\|f_N\| = 1$ and $B(0,1)$ is convex in H, $\overline{\operatorname{co}(\{f_N\})} \subset \bar{B}(0,1)$. By Alouglu's theorem, $\bar{B}(0,1)$ is compact in ω^*- topology, so K is compact in τ_{ω^*} (cosed subset of a compact subspace), and hence compact in τ_ω. ✓

By *HW5 Pb 1)*, as $f_N \to 0$ weakly there exists a sequence $(y_n)_{n\geq 1} \subset \operatorname{co}(\{f_N\})$ such that $(y_n) \to 0$ in norm. i.e, $0 \in \overline{\operatorname{co}(\{f_n\})}^{\|\cdot\|} = K$. ✓

c) Show that 0 , as well as each $f_N, N \geq 1$, are extreme points in K.

The elements of $\operatorname{co}(\{f_N\})$ are of the form $\sum_{N=1}^{M} \alpha_N f_N$ with $0 \leq \alpha_N$ and $\sum_{N=1}^{M} \alpha_N = 1$ for some $M \in \mathbb{N}$. Making some computations we can rewrite them as $\sum_{N=0}^{M}(\sum_{j=N+1}^{M^2} \frac{\alpha_j}{j})(e_{N^2+1} + \cdots e_{(N+1)^2})$. In particular, the elements

Do them explicitly!

Something seems off about this...

of $\text{co}(\{f_N\})$ have non-negative coordinates in the base (e_n), so the elements of K, being limits of sequences of them, have also non negative coordinates.

With that remark, we notice 0 is an extreme point: if we have $x, y \in K$ and $0 < \alpha < 1$ such that $0 = \alpha x + (1-\alpha)y$, then coordinate-wise we also have that equality, but $\alpha x, (1-\alpha)y \geq 0$ implies $x_i = y_i = 0$ for each coordinate, so $x = y = 0$.

With respect to f_N, we notice that $\|f_N\| = 1$ and $K \subset \bar{B}(0,1)$, so if we have $x, y \in K$ and $0 < \alpha < 1$ such that $f_N = \alpha x + (1-\alpha)y$ then

$$1 = \|f_N\| = \|\alpha x + (1-\alpha)y\| \leq^* \alpha\|x\| + (1-\alpha)\|y\| \leq^{**} \alpha 1 + (1-\alpha)1 = 1$$

so all the inequalities turn to equalities. The equality in (**) proves that $\|x\| = \|y\| = 1$, and the equality in (*) proves that αx and $(1-\alpha)y$ are proportional, so you have for some $a > 0$ (as x and y have non-negative coordinates, a must be also non negative):

$$\begin{aligned}(1-\alpha)y &= a\alpha x,\\ (1-\alpha)\|y\| &= a\alpha\|x\|,\\ (1-\alpha) &= a\alpha\\ a &= \frac{(1-\alpha)}{\alpha}.\end{aligned}$$

Hence,

$$(1-\alpha)y = \frac{(1-\alpha)}{\alpha}\alpha x = (1-\alpha)x$$

so $x = y$ and then f_N is extreme point.

d) Are there any other extreme points in K? Justify your answer. (An answer without justification will not be given any credit.)

By applying *Theorem 7.9*, as K is weakly compact and $\{f_N\}$ is a subset of K verifying $K = \overline{\text{co}(\{f_N\})}^{\tau_\omega}$ we get $\text{Ext}(K) \subset \overline{(\{f_N\})}^{\tau_\omega}$, and $\overline{(\{f_N\})}^{\tau_\omega} = \{f_N\} \cup \{0\}$ as 0 is the weak limit of (f_N). So $\{f_N\} \cup \{0\}$ are the only extreme points.

How do you know there are not more (weak) accumulation points of $\{f_n\}_{n\in\mathbb{N}}$?

(✓)

Problem 2 *Let X and Y be infinite dimensional Banach spaces.*

a) Let $T \in \mathcal{L}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

We have $Tx_n \to T_x$ weakly if and only if for all $\phi \in Y^*$ $\phi(Tx_x) \to \phi(Tx)$. So let $\phi \in Y^*$. As $T : X \to Y$ is continuous $\phi T \in X^*$. So by continuity $\phi T x_n \to \phi T x$.

Careful, you are not proving this, you are using this fact.

b) *Let $T \in \mathcal{K}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$.*

We will prove $Tx_n \to Tx$ in norm if and only if every subsequence of (Tx_n) has a further subsequence which converges to Tx in norm. So let's take a subsequence of Tx_n (we will refer to it as (Tx_n)). ← This only makes it harder to read...

Typically, we would denote the subsequence by (z_{n_k}) here.

By *Proposition 8.2* T is a compact operator if and only if every bounded sequence (z_n) has a subsequence (z_{k_n}) such that Tz_{k_n} converges in Y. So given our subsequence Tx_n let's consider the sequence x_n. As $x_n \to x$ weakly, by *HW4 P2b)* (x_n) is bounded. By compacity of T there exists a further subsequence (x_{k_n}) such that Tx_{k_n} converges to some $y \in Y$ in norm.

On the other hand, as $Tx_n \to T_x$ weakly (as T compact implies T bounded), then $Tx_{k_n} \to Tx$ weakly. So for any $\phi \in Y^*$ $\phi Tx_{k_n} \to \phi Tx$, but we also have $\phi Tx_{k_n} \to \phi y$, as Tx_{k_n} converges to y in norm. Then $\phi Tx = \phi y$ for every $\phi \in Y^*$, then $y = Tx$. So we have proven that any subsequence of Tx_n has a further subsequence which converges to Tx in norm, then $Tx_n \to Tx$ in norm. ✓

c) *Let H be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H,Y)$ satisfies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$. then $T \in \mathcal{K}(H.Y)$. [Hint: Suppose that T is not compact. Show that there exists $\delta > 0$ and a sequence $(x_n)_{n\geq 1}$ in the closed unit ball of H such that $\|Tx_n - Tx_m\| \geq \delta$, for all $n \neq m$ Show next that $(x_n)_{n\geq 1}$ has a weakly convergent subsequence.]*

Let's suppose T is not compact. By *Proposition 8.2* then there exists some bounded sequence $(x_n)_{n\geq 1}$ (we may assume $(x_n) \subset \bar{B}(0,1)$) such that no subsequence of (Tx_n) converges in norm in Y. By completeness of Y, that's equivalent to no subsequence of (Tx_n) being Cauchy in Y, i.e., for every subsequence (Tx_{k_n}) there exists some $\delta > 0$ such that for all N exists some $m(N), n(N) > N$ such that

$$\|Tx_{k_{m(N)}} - Tx_{k_{n(N)}}\| \geq \delta.$$

Using this property, we can construct a new sequence (Ty_n) out of that one such that $\|Ty_n - Ty_m\| \geq \delta$, and consider the sequence $(y_n) \subset X$. As $(x_n) \subset \bar{B}(0,1)$, $(y_n) \subset \bar{B}(0,1)$. However, H is a Hilbert space, so it's reflexive and by *Theorem 5.9*, $\tau_\omega = \tau_{\omega^*}$ in H (it makes sense as H can be seen as de dual of H^*). By Alouglu's theorem $\bar{B}(0,1)$ is ω^*-compact, so is compact in the weak topology.

On the other hand, H is separable (and H^* likewise, as given a countable orthonormal basis of H, we can consider the standard basis in the dual and it's also a countable orthonormal basis), so by *Theorem 5.13* $(\bar{B}(0,1), \tau_{\omega^*}$ is metrizable, hence $\bar{B}(0,1)$ is sequentally compact.

Then $(y_n) \in \bar{B}(0,1)$ has a weakly convergent *subsequence* and that give us our desired contradiction. ✓

d) Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact. [Hint: Use (a) and (c) and the characterisation of weak convergence of sequences in $\ell_1(\mathbb{N})$, cf. Remark 5.3, Lecture 5.] A reference to Pitt's Theorem, cf. the note in HW6, is not sufficient.

$\ell_2(\mathbb{N})$ is a separable ∞-dimensional Hilbert space, so by *2c)* $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact if and only if for all $(x_n) \subset \ell_2(\mathbb{N})$ which converges weakly to x implies $\|Tx_n - Tx\| \to 0$ in $\ell_1(\mathbb{N})$.

Let $(x_n) \subset \ell_2(\mathbb{N})$ which converges weakly to x. By *2a)*, Tx_n converges weakly to Tx. Then, by *Remark 5.3*, Tx_n converges then to Tx in norm.

e) Show that no $T \in \mathcal{K}(X, Y)$ is onto.

Supose $T \in \mathcal{K}(X, Y)$ and it's onto. Then, by the open mapping theorem (as X, Y are Banach and $T \in \mathcal{L}(X, Y)$), T is open. Hence $T(B_X(0,1)$ is open in Y and $0 \in T(B_X(0,1)$, so there exists some open ball centered at 0 contained in $T(B_X(0,1)$, i.e., for some $c > 0$:

$$cB_Y(0,1) \subset T(B_X(0,1)$$

and taking closures (in norm):

$$c\overline{B_Y(0,1)} \subset \overline{T(B_X(0,1)}$$

T is a compact, so $\overline{T(B_X(0,1)}$ is compact by definition. Then $c\overline{B_Y(0,1)}$ is a closed subspace of a compact set, so it's compact, therefore $\overline{B_Y(0,1)}$ is compact (in norm). However, in the *Mandatory Assigment 1* we saw that that's incompatibe with an infinite dimensional vector space. So we have our contradiction.

why?

✓

f) Let $H = L_2([0,1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. Justify that M is self-adjoint, but not compact.

We have to check $\langle f, Mg \rangle = \langle Mf, g \rangle$ for $f, g \in L_2([0,1], m)$. Indeed:

$$\langle f, Mg \rangle = \int_{[0,1]} \overline{f(t)} t g(t) dm(t) =$$
$$= \int_{[0,1]} \overline{tf(t)} g(t) dm(t) = \langle Mf, g \rangle$$

✓

However, M can't be compact, as by the Spectral Theorem for self-adjoint compact operators *Theorem 10.1*, we would have that there exists an ONB for $\ell_2(\mathbb{N})$ consisting of eigenvectors, but we checked on *HW6 P3* that M doesn't have any eigenvalues.

$L^2([0,1],m)$

(✓)

Remember to note $L^2([0,1],m)$ is separable and infinite-dimensional, so you can apply Thm 10.1

Problem 3 *Consider the Hilbert space $H = L_2([0,1], m)$, where m is the Lebesgue measure. Define $K : [0,1] \times [0,1] \to \mathbb{R}$ by*

$$K(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1 \\ (1-t)s, & \text{if } 0 \le s < t \le 1 \end{cases}$$

and consider $T \in \mathcal{L}(H,H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t) f(t) dm(t), \quad s \in [0,1], \quad f \in H$$

a) Justify that T is compact.

Then you need to show $T = T_{\tilde{K}}$ for some $\tilde{K}$ satisfying

We see we are on the conditions of *Theorem 9.6*, as $X = Y = [0,1]$ are compact Haussdorf topological spaces, m is a finite measure on them and $K \in C([0,1] \times [0,1])$: it's clearly continuous on the interior of $U_1 = \{(t,s) \in [0,1] \times [0,1]; t < s\}$ and $U_2 = \{(t,s) \in [0,1] \times [0,1]; s \le t\}$, and its both expressions coincide in $U_1 \cap U_2 = \{(t,s) \in [0,1] \times [0,1]; t = s\}$.

($\tilde{K}(s,t) = K(t,s)$ in this case)

b) Show that $T = T^*$.

We have to check that $\langle f, Tg \rangle = \langle Tf, g \rangle$ for $f, g \in H$.

$$\begin{aligned} \langle f, Tg \rangle &= \int_{[0,1]} \overline{f(s)} Tg(s) dm(s) = \\ &= \int_{[0,1]} \overline{f(s)} \int_{[0,1]} K(s,t) g(t) dm(t) dm(s) =^* \\ &= \int_{[0,1]} \int_{[0,1]} \overline{f(s)} K(s,t) g(t) dm(s) dm(t) = \\ &= \int_{[0,1]} g(t) \left(\int_{[0,1]} \overline{f(s) K(s,t)} dm(s) \right) dm(t) = \\ &= \int_{[0,1]} g(t) \overline{Tf(t)} dm(t) = \langle Tf, g \rangle. \end{aligned}$$

Only if $K(s,t) = K(t,s)$

And the equality in $*$ in justified as we can apply Fubini. Indeed, as $K \in C([0,1] \times [0,1])$, it's bounded for some $M > 0$. Then, by Tonelli:

$$\begin{aligned} &\int_{[0,1]\times[0,1]} |K(s,t) f(t) g(s)| \, dm(t) dm(s) \le \\ &\le M \int_{[0,1]\times[0,1]} |f(t)| \, |g(s)| \, dm(t) dm(s) = \\ &= M \int_{[0,1]} |f(t)| \, dm(t) \int_{[0,1]} |g(s)| \, dm(s) = \\ &= M \|f\|_1 \|g\|_1 < \infty \end{aligned}$$

c) Show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t), \quad s \in [0,1], \quad f \in H.$$

Use this to show that Tf is continuous on $[0,1]$, and that $(Tf)(0) = (Tf)(1) = 0$.

$$\begin{aligned}
Tf(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) = \\
&= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) = \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) = \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t).
\end{aligned}$$

✓

This expression of Tf allows us to show that it's continuous. First, we will need to prove the general fact that if $g \in L_1([0,1],m)$ then the indefinite integral G defined as:

$$G(s) = \int_{[0,s]} g(t)dm(t),$$

is continuous, i.e, for any $s_0 \in [0,1]$:

$$\lim_{s\to s_0} \int_{[0,s]} g(t)dm(t) = \int_{[0,s_0]} g(t)dm(t).$$

If we write $\int_{[0,s]} g(t)dm(t) = \int_{[0,1]} g(t)\chi_{[0,s]}dm(t)$, (notice $\lim_{s\to s_0} \chi_{[0,s]} = \chi_{[0,s_0]}$) then we will have to show:

$$\lim_{s\to s_0} \int_{[0,1]} g(t)\chi_{[0,s]}dm(t) = \int_{[0,1]} (\lim_{s\to s_0} g(t)\chi_{[0,s]})dm(t).$$

For that, we will only need to prove that the Lebesgue dominated convergence theorem holds. Indeed:

$$\left|g(t)\chi_{[0,s]}\right| \leq |g(t)|;$$

and $\int_{[0,1]} |g(t)|\, dm(t) < \infty$ as $g \in L_1([0,1],m)$.

With that fact, we then notice that the function $s \to \int_{[0,s]} tf(t)dm(t)$ is continuous as $\phi(t) = tf(t)$ is integrable (as t is bounded in $[0,1]$). Likewise

$\uparrow$

$f \in L_2([0,1])$

why $f \in L_1([0,1])$?

$s \to \int_{[s,1]}(1-t)f(t)dm(t) = -\int_{[1,s]}(1-t)f(t)dm(t)$. Then now Tf is clearly continuous as is the sum of continuous functions.

Finally:

$$Tf(0) = (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]}(1-t)f(t)dm(t) = 0;$$
$$Tf(1) = (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]}(1-t)f(t)dm(t) = 0.$$

Problem 4 *Consider the Schwartz space $\mathcal{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.*

a) *For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$. Justify that $g_k \in \mathcal{S}(\mathbb{R})$, for all integers $k \geq 0$. Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$*

By *HW7 Pb1*, $g_0 \in \mathcal{S}(\mathbb{R})$, as $e^{-x^2} \in \mathcal{S}(\mathbb{R})$ and $g_0 = S_{\sqrt{2}}(e^{-x^2})$. By *Example 11.3*, $\mathcal{F}(g_0) = g_0$.

By *HW7 Pb1*, $g_i \in \mathcal{S}(R)$ because is the product of g_0 with a power of x. Then, by *Proposition 11.13 ii)*:

$$\mathcal{F}(g_1)(\xi) = \mathcal{F}(xg_0(x))(\xi) = i\frac{\partial \mathcal{F}(g_0)}{\partial x}(\xi) = -i\xi e^{\frac{-\xi^2}{2}}.$$

Analogously:

$$\mathcal{F}(g_2)(\xi) = \mathcal{F}(xg_1(x))(\xi) = i\frac{\partial \mathcal{F}(g_1)}{\partial x}(\xi) = -1(-e^{\frac{-\xi^2}{2}} + \xi^2 e^{\frac{-\xi^2}{2}}) =$$
$$= e^{\frac{-\xi^2}{2}}(1-\xi^2).$$
$$\mathcal{F}(g_3)(\xi) = \mathcal{F}(xg_2(x))(\xi) = i\frac{\partial \mathcal{F}(g_2)}{\partial x}(\xi) = i((-\xi)e^{\frac{-\xi^2}{2}}(1-\xi^2) + e^{\frac{-\xi^2}{2}}(-2\xi)) =$$
$$= ie^{\frac{-\xi^2}{2}}(\xi^3 - 3\xi).$$

b) *Find non-zero functions $h_k \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$. [Hint: Try suitable linear combinations of the functions g_0, g_1, g_2, g_3 from (a).*

We first notice that we can rewrite the results in *a)* as:

$$\mathcal{F}(g_0) = g_0,$$
$$\mathcal{F}(g_1) = -ig_1,$$
$$\mathcal{F}(g_2) = g_0 - g_2,$$
$$\mathcal{F}(g_3) = i(g_3 - 3g_1),$$

Then for $\mathbf{h_0}$ and $\mathbf{h_3}$ we can set $h_0 = g_0$ and $h_3 = g_1$. For $\mathbf{h_1}$ we set the system of equations:

$$\begin{aligned}
\mathcal{F}(ag_0 + bg_1 + cg_2 + dg_3) &= i(ag_0 + bg_1 + cg_2 + dg_3) \\
ag_0 - ibg_1 + cg_0 - cg_2 + idg_3 - i3dg_1 &= i(ag_0 + bg_1 + cg_2 + dg_3) \\
(a + c)g_0 + (-ib - i3d)g_1 - cg_2 + idg_3 &= iag_0 + ibg_1 + icg_2 + idg_3.
\end{aligned}$$

and making equal the coefficients for each g_i we get:

$$\begin{aligned}
a + c &= ia, \\
-ib - i3d &= ib, \\
-c &= ic, \\
id &= id;
\end{aligned}$$

which has as solution $a = 0, b = 3, c = 0, d = -2$, so we can set $h_1 = 3g_1 - 2g_3$.

Analogously with $\mathbf{h_2}$:

$$\begin{aligned}
\mathcal{F}(ag_0 + bg_1 + cg_2 + dg_3) &= -1(ag_0 + bg_1 + cg_2 + dg_3) \\
ag_0 - ibg_1 + cg_0 - cg_2 + idg_3 - i3dg_1 &= -(ag_0 + bg_1 + cg_2 + dg_3) \\
(a + c)g_0 + (-ib - i3d)g_1 - cg_2 + idg_3 &= -ag_0 - bg_1 - cg_2 - idg_3.
\end{aligned}$$

and making equal the coefficients for each g_i we get:

$$\begin{aligned}
a + c &= -a, \\
-ib - i3d &= -b, \\
-c &= -c, \\
id &= -d;
\end{aligned}$$

which has as solution $a = -1, b = 0, c = 2, d = 0$, so we can set $h_2 = -g_0 + 2g_2$.

c) Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathcal{S}(\mathbb{R})$. [Hint: First compute $\mathcal{F}^2(f)$, which by definition is equal to $\mathcal{F}(\mathcal{F}(f))$, and then compute $\mathcal{F}^4(f)$ which is equal to $\left(\mathcal{F}^2\left(\mathcal{F}^2(f)\right)\right.$.

By *Corolary 12.12 ii)* we know $\mathcal{F}^*(\mathcal{F}(f)) = f, f \in \mathcal{S}(F)$.Then:

$$\begin{aligned}
\mathcal{F}^2(f)(x) = \mathcal{F}(\mathcal{F}(f))(x) = \int_{\mathbb{R}} e^{-ix\xi}\mathcal{F}(f)(\xi)dm(\xi) = \\
= \int_{\mathbb{R}} e^{i(-x)\xi}\mathcal{F}(f)(\xi)dm(\xi) = \mathcal{F}^*(\mathcal{F}(f))(-x) = f(-x),
\end{aligned}$$

so $\mathcal{F}^2(f)(x) = f(-x)$, then $\mathcal{F}^4(f)(x) = \mathcal{F}^2(\mathcal{F}^2(x)) = \mathcal{F}^2(f(-x)) = f(x)$.

d) Use (c) to show that if $f \in \mathcal{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.

Suppose $\mathcal{F}(f) = \lambda f$, then $f = \mathcal{F}^4(f) = \lambda^4 f$ and $f \not\equiv 0$ implies $1 = \lambda^4$. Then $\lambda \in \{1, i, -1, -i\}$, and the conclusion holds, as $\mathcal{F}(f) = \lambda f$ is the eigenvalue equation and in *b)* we found eigenvectors for each $\{1, i, -1, -i\}$.

✓

Problem 5 *Let $(x_n)_{n\geq 1}$ be a dense subset of [0,1] and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0,1]$. Show that $\operatorname{supp}(\mu) = [0,1]$. [Hint: See Problem 3 HW8].*

By *HW8 P3b)* we know that $x \in \operatorname{supp}(\mu)$ if and only if $\int f d\mu > 0$ for any $f : [0,1] \to [0,1]$ continuous with compact support such that $f(x) > 0$. So we show that each $x_n \in \operatorname{supp}(\mu)$. As $\mu(\{x_n\}) = 2^{-n} > 0$, then, for any $f : [0,1] \to [0,1]$ continuous with compact support such that $f(x_n) > 0 :$

$$\int_{[0,1]} f d\mu \geq \int_{\{x_n\}} f d\mu = f(x_n)\mu(\{x_n\}) > 0.$$

Then $(x_n) \subset \operatorname{supp}(\mu)$. Taking closures and using that $\operatorname{supp}(\mu)$ is closed we get to $[0,1] = \operatorname{supp}(\mu)$.

and using that $(x_n)_{n\in\mathbb{N}}$ is dense in $[0,1]$.

✓