Assigment 1

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Exercise 1 (24 *points*) . *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$*

- [5p]. *Let $T : X \to Y$ be a linear map. Set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$, for all $x \in X$. Show that $\|\cdot\|_0$ is a norm on X. Show next that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded.*

Let's check the axioms of norm: for any $\lambda \in \mathbb{K}$, $x, y \in X$

1. $\|\lambda x\|_0 = |\lambda|\,\|x\|_0$.

$$\begin{aligned} \|\lambda x\|_0 = \|\lambda x\|_X + \|T(\lambda x)\|_Y = |\lambda|\,\|x\|_X + \|\lambda T(x)\|_Y = \\ = |\lambda|\,(\|x\|_X + \|T(x)\|_Y) = |\lambda|\,\|x\|_0. \end{aligned}$$

2. $\|x\|_0 \geq 0$ for any $x \in X$ and $\|x\|_0 = 0 \iff x = 0$. The first part is obvious, and for the second part:

$$\begin{aligned} 0 &= \|x\|_0; \\ 0 &= \|x\|_X + \|T(x)\|_Y; \end{aligned}$$

as the two terms are non negative numbers:

$$0 = \|x\|_X = \|T(x)\|_Y,$$

so as $0 = \|x\|_X$ then $x = 0$.

3. $\|x + y\|_0 \leq \|x\|_0 + \|y\|_0$. That will deduce from $\|\cdot\|_X$ and $\|\cdot\|_Y$ being norms and linearity of T.

Let's prove now that $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded.

- $\Longleftarrow$ If T is bounded then there exists $\|T\| = \sup\{\|Tx\|_Y, \|x\|_X \leq 1\}$ and $\|Tx\|_Y \leq \|T\|\|x\|_X$ for any $x \in X$. We have to prove there exist $c, C > 0$ such that $c\|x\|_X \leq \|x\|_0 \leq C\|x\|_X$ for any $x \in X$. First, as $\|Tx\|_Y \geq 0$, $\|x\|_0 \geq \|x\|_X$ for any $x \in X$ (so we take $c = 1$). On the other hand:

$$\|x\|_0 = \|x\|_X + \|T(x)\|_Y \leq \|x\|_X + \|T\|\|x\|_X = (1 + \|T\|)\|x\|_X.$$

So taking $C = 1 + \|T\|$, the two norms are equivalent.

- $\Longrightarrow$ If there exists some $C > 0$ such that $\|x\|_0 \leq C\|x\|_X$, then:

$$\begin{aligned} \|x\|_X + \|T(x)\|_Y &\leq C\|x\|_X; \\ \|T(x)\|_Y &\leq (C - 1)\|x\|_X \end{aligned}$$

and that inequality implies T is bounded.

- [4p]. *Show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.*

For the first part, we now T is bounded if and only if the two norms considered are equivalent on X, and for *Theorem 1.6* any 2 norms on X are equivalent when X is finite-dimensional.

- [5p]. *Suppose that X is infinite dimensional. Show that there exists a linear map $T : X \to Y$, which is not bounded (= not continuous). [Hint: Take a Hamel basis for X.] Let X be a vector space over $\mathbb{K}$. An algebraic basis for X is a family $(e_i)_{i\in I}$ of elements in X with the following property: For each vector space Y over $\mathbb{K}$, and each family $(y_i)_{i\in I}$ in Y there exists precisely one linear map $T : X \to Y$ satisfying $T(e_i) = y_i$, for all $i \in I$. One can show that this condition is equivalent to the more usual definition of an algebraic basis: for each $x \in X$, there is a unique family $(\lambda_i)_{i\in I}$ in $\mathbb{K}$ for which the set $\{i \in I : \lambda_i \neq 0\}$ is finite and $x = \sum_{i\in I} \lambda_i e_i$. When X is infinite dimensional, an algebraic basis is also called a Hamel basis. It is a consequence of Zorn's lemma that each infinite dimensional vector space admits a Hamel basis. You are free to use these facts without further justifications.*

We take a Hamel basis $(e_i) \subset X$ and define a norm on X such that this basis is orthogonal with respect to it, i.e. for $x \in X$, if $x = \sum \lambda_i e_i$ then we define $\|x\| := \sqrt{\sum |\lambda_i|^2}$.

Now we choose $Y = \mathbb{K}$ and a countable subset of the Hamel basis $(e_n)_{n\geq 0} \subset (e_i)$. Also we consider the unique linear map such that $T(e_n) = n$ and $T(e_i) = 0$ otherwise.

Then T is linear and is not bounded, as:

$$\sup\{|Tx|\,, \|x\| = 1\} \geq \sup\{|T(e_i)|\} \geq \sup_{n\geq 0} n.$$

- [5p]. *Suppose again that X is infinite dimensional. Argue that there exists a norm $\|\cdot\|_0$ on X, which is not equivalent to the given norm $\|\cdot\|_X$, and which satisfies $\|x\|_X \leq \|x\|_0$ for all $x \in X$. Conclude that $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space.*

If we take the not bounded linear map T for the previous exercise, the first part show us that the norm $\|x\|_0 := \|x\|_X + \|Tx\|_Y$ is not equivalent to $\|x\|_X$. And also, by constuction $\|\cdot\|_0 \geq \|\cdot\|_X$. Now, for *HW3* Problem 1, if $(X, \|\cdot\|_X)$ and $(X, \|\cdot\|_0)$ were both complete, as $\|\cdot\|_0 \geq \|\cdot\|_X$, then the two norms would be equivalent and they are not, so $(X, \|\cdot\|_0)$ cannot be complete.

- [5p]. *Give an example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ satisfying $\|x\|' \leq \|x\|$, for all $x \in X$, such that $(X, \|\cdot\|)$ is complete, while $(X, \|\cdot\|')$ is not. [Hint: Take $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|_1)$ with a suitable choice of $\|\cdot\|'$; or take $(X, \|\cdot\|) = (L_2([0,1], m), \|\cdot\|_2)$ with a suitable choice of $\|\cdot\|'$, where m is the Lebesgue measure.]*

We take $X = \ell_1(\mathbb{N}) = \{(x_n) \subset \mathbb{K}, \sum_{n=1}^{\infty} |x_n| < \infty\}$, and with norm $\|\cdot\|_1$. By *HW1* Problem 5, $(X, \|\cdot\|_1)$ is a Banach space. In addition, we can also consider the infinity norm $\|(x_n)\|_\infty = \sup_{n\geq 1}\{|x_n|\}$, as it's well-defined on X.

It's clear that $\|\cdot\|_\infty \leq \|\cdot\|_1$, as for each element of the sequence $|x_n| \leq \sum_{n=1}^{\infty} |x_n| = \|(x_n)\|_1$, so $\|(x_n)\|_\infty \leq \|(x_n)\|_1$. However, it can't exist $C \geq 0$ such that $\|\cdot\|_1 \leq C\|\cdot\|_\infty$ for each $(x_n) \in X$. Indeed, if that were the case, if we take $N = [C] + 1$, we can consider the sequence $(x_n) = (\underbrace{1, 1, \ldots, 1}_{N}, 0, \ldots)$.
Then $(x_n) \in X$ and $\|(x_n)\|_\infty = 1$, but $\|(x_n)\|_1 = [C] + 1 > C$, so it's not true that $\|\cdot\|_1 \leq C\|\cdot\|_\infty$. Then the two norms can't be equivalent.

$(X, \|\cdot\|_\infty)$ can't be a Banach space by the same argument of the previous part.

Exercise 2 (20*points*) . *Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), \|\cdot\|_p)$, considered as a vector space over $\mathbb{C}$, given by*

$$M = \{(a, b, 0, 0, \ldots) : a, b \in \mathbb{C}\}$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, 0, \ldots) = a + b$, for all $a, b \in \mathbb{C}$

- [8p]. *Show that f is bounded on $(M, \|\cdot\|_p)$ and compute $\|f\|$. (Answer depends on p.)*

It's easy to show that f is bounded: for any $x = (a, b, 0, \ldots) \in M$ we have:

$$\begin{aligned} |f(x)| &= |a + b| \leq |a| + |b| = |a|^{p\,1/p} + |b|^{p\,1/p} \leq \\ &\leq \|x\|_p + \|x\|_p = 2\|x\|_p \end{aligned}$$

That shows that f is bounded. To compute the exact norm of f we will have to study

$$\sup_{\|x\|_p = 1} \{|f(x)|\} = \sup_{|a|^p + |b|^p = 1} \{|a + b|\} \leq \sup_{|a|^p + |b|^p = 1} \{|a| + |b|\}$$

So we are going to study the maxima of the function $h(t) = t + (1 - t^p)^{1/p}$ with $t \in [0, 1]$.

Computing its critical points and solving $h'(t) = 0$ we get to the equation

$$0 = 1 - (1 - t^p)^{\frac{1}{p} - 1} t^{p-1}$$

which has solution $t_0 = (\frac{1}{2})^{1/p}$. Analyzing the sign of h' we get to the conclusion that h reaches its maximum on t_0, which is the value $h(t_0) = (\frac{1}{2})^{1/p} + (1 - \frac{1}{2}^{1/p} = \frac{2}{2^{1/p}}$.

So $\sup_{\|x\|_p=1}\{|f(x)|\} \leq \frac{2}{2^{1/p}}$, and this value is actually attained for the vector $x = ((\frac{1}{2})^{1/p}, (\frac{1}{2})^{1/p}, 0, \ldots)$. So, $\|f\| = \frac{2}{2^{1/p}}$.

- [7p]. *Show that if* $1 < p < \infty$, *then there is a unique linear functional* F *on* $\ell_p(\mathbb{N})$ *extending* f *and satisfying* $\|F\| = \|f\|$.

We have a trivial extension of f given by $F(a_1, a_2, a_3, \ldots) = a_1 + a_2$. It's lineal and an extension of f, and verifies the norm condition, as we already have $\|F\| \geq \|f\|$ and:

$$\begin{aligned}\|F\| &= \sup_{\sum_{n=1}^{\infty}|a_n|^p=1}\{|F(a_1, a_2, a_3, \ldots)|\} = \\ &= \sup_{\sum_{n=1}^{\infty}|a_n|^p=1}\{|a_1 + a_2|\} \leq^* \sup_{\sum_{n=1}^{\infty}|a_n|^p=1, a_i=0, i\geq 3}\{|a_1| + |a_2|\} = \|f\|\end{aligned}$$

where in (*) the inequality holds because we maximize $|a_1 + a_2|$ if we maximize ${a_1}^p$ and ${a_2}^p$, making $a_i = 0, i \geq 3$. So $\|f\| = \|F\|$.

On the other hand let's prove uniqueness of F, i.e, that $F(0, 0, a_3, a_4, \ldots) = 0$ for any (a_n) with $a_1 = a_2 = 0$. Then let's suppose for some (a_n) that $F(0, 0, a_3, a_4, \ldots) \neq 0$. Then, as $F(0, 0, a_3, a_4, \ldots) = \sum_i F(0, 0, 0 \ldots, a_i, 0, \ldots)$, for some a_i we have $F(0, 0, 0 \ldots, a_i, 0, \ldots) \neq 0$. If we multiply a_i by $\overline{F(0, 0, 0 \ldots, a_i, 0, \ldots)}$ and then normalize it, we can make $|a_i| = 1$ and $\alpha := F(0, 0, 0 \ldots, a_i, 0, \ldots) > 0$.

Now, for any $0 < r < 1$ we define:

$$(x_n) = ((1 - r^p)^{1/p}\frac{1}{2^{1/p}}, (1 - r^p)^{1/p}\frac{1}{2^{1/p}}, 0, \ldots, ra_i, 0, \ldots).$$

It verifies $\sum_{n=1}^{\infty}|x_n|^p = 1$, so $|F(x_n)| \leq \frac{2}{2^{1/p}}$.

On the other hand:

$$\begin{aligned}|F(x_n)| = F(x_n) &= (1 - r^p)^{1/p}F(\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}}, 0, \ldots) + r\alpha = \\ &= (1 - r^p)^{1/p}\frac{2}{2^{1/p}} + r\alpha\end{aligned}$$

Therefore, for any $0 < r < 1$:

$$\begin{aligned}(1 - r^p)^{1/p}\frac{2}{2^{1/p}} + r\alpha &\leq \frac{2}{2^{1/p}}, \\ r\alpha &\leq (1 - (1 - r^p)^{1/p})\frac{2}{2^{1/p}}, \\ \alpha &\leq \underbrace{\frac{(1 - (1 - r^p)^{1/p})}{r}}_{(\triangle)}\frac{2}{2^{1/p}}\end{aligned}$$

If $p > 1$, we can compute the limit $\lim_{r\to 0}(\triangle)$ (using L'Hôpital's rule) and conclude that it tends to 0. That makes $\alpha = 0$ and we get our contradiction. We notice that for $p = 1$ the argument doesn't work as $\triangle = 1$

- [5p]. *Show that if $p = 1$, then there are infinitely many linear functional F on $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.*

We can extend the functional to $F(a_1, a_2, a_3, \ldots) = a_1 + a_2$. It's clearly linear, extends f and its norm (which verifies $\|F\| \geq \|f\|$ due to being an extension of f):

$$\begin{aligned}\|F\| &= \sup_{\sum_{n=1}^{\infty}|a_n|=1} \{|F(a_1, a_2, a_3, \ldots)|\} = \\ &= \sup_{\sum_{n=1}^{\infty}|a_n|=1} \{|a_1 + a_2|\} \leq \sup_{\sum_{n=1}^{\infty}|a_n|=1} \{|a_1| + |a_2|\} \leq 1 = \|f\|\end{aligned}$$

because $\sum_{n=1}^{\infty} |a_n| = 1$ implies $|a_1| + |a_2| \leq 1$.

On the other hand, we have another candidates for extending f:
$F_k(a_1, a_2, a_3, \ldots) = \sum_{i=1}^{k} a_i$ for any $k \geq 3$. Linearity and extension of F_k are clear, and the equality of the norms works by the same argument as before: verifies $\|F_k\| \geq \|f\|$ due to being an extension of f and:

$$\begin{aligned}\|F_k\| &= \sup_{\sum_{n=1}^{\infty}|a_n|=1} \{|F(a_1, a_2, a_3, \ldots)|\} = \\ &= \sup_{\sum_{n=1}^{\infty}|a_n|=1} \{\left|\sum_{i=1}^{k} a_i\right|\} \leq \sup_{\sum_{n=1}^{\infty}|a_n|=1} \{\sum_{i=1}^{k} |a_i|\} \leq 1 = \|f\|\end{aligned}$$

because $\sum_{n=1}^{\infty} |a_n| = 1$ implies $\sum_{i=1}^{k} |a_i| \leq 1$.

(We could also even consider an extension $\bar{F}(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i$, which will be well defined as that series converge absolutely and $\mathbb{K}$ is complete, and will be an extension by the same arguments as above).

Exercise 3 (25 points) . *Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

- [5p]. *Let $n \geq 1$ be an integer. Show that no linear map $F : X \to \mathbb{K}^n$ is injective.*

If F were injective then X would be isomorphic to $F(X)$ which would be a subspace of $\mathbb{K}^n$, so $F(X)$ would be finite dimensional with X infinite-dimensional.

- [5p]. *Let $n \geq 1$ be an integer and let $f_1, f_2, \ldots, f_n \in X^*$. Show that*

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$$

[Hint: Consider the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), x \in X$.].

We consider $F : X \to \mathbb{K}^n$ defined by $F(x) = (f_1(x), f_2(x), \dots, f_n(x)), x \in X$. By the first part, it can't be injective, then $\ker F \neq \{0\}$. On the other hand $\ker F = \{x \in X, (f_1(x), f_2(x), \dots, f_n(x)) = 0\} = \{x \in X, f_i(x) = 0, i = 1, \dots, n\} = \cap_{i=1}^n \ker f_i$.

- [5p]. *Let $x_1, x_2, \dots, x_n \in X$. Show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, 2, \dots n$. [Hint: Use Theorem 2.7 (b) from lectures to get started.]*

Theorem 2.7b) In the lectures says that given X a normed space, $0 \neq x \in X$ then there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$. First we notice that if some $x_j = 0$ then there's nothing to prove for x_j, so we may assume all x_j are non-zero. Let's consider then $f_i \in X^*$ for each x_i given by the theorem. By the second part in this exercise, $\cap_{i=1}^n \ker f_i \neq \{0\}$. We take $0 \neq y \in \cap_{i=1}^n \ker f_i \neq \{0\}$, which can be taken such that $\|y\| = 1$, by normalizing y. We have in consequence:

$$f(x_j - y) = f(x_j) - f(y) = f(x_j) = \|x_j\|;$$
$$|f(x_j - y)| \leq \|f\| \|x_j - y\| = \|x_j - y\|$$

So we get $\|x_j\| \leq \|x_j - y\|$.

- [5p]. *Show that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains* 0.

Assume by contradiction that there exists $x_1, \dots x_n \in X$, $r_1, \dots, r_n > 0$ such that $S \subset \cup_{i=1}^n \bar{B}(x_i, r_i))$ such that $0 \notin \bar{B}(x_i, r_i))$, i.e. $|x_i| > r_i$. By the third part of this exercise there exists $y \in S$ with $\|y - x_j\| \geq \|x_j\| > r_j$. So $y \notin \bar{B}(x_j, r_j)$ for all $j = 1, \dots, n$.

- [5p]. *Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.*

We'll be done when we give an open recover of S by open balls such that the closed balls don't contain 0. Then, if S were compact, there would be a finite recover $S \subset \cup_{i=1}^n B(x_i, r_i)$. But then applying closures (S is closed) $S \subset \overline{\cup_{i=1}^n B(x_i, r_i)} = \cup_{i=1}^n \bar{B}(x_i, r_i)$, and we will have a contradiction. We take the open cover by $\{B(x, \frac{1}{2})\}_{x \in S}$. Then $S \subset \cup_{x \in S} B(x, \frac{1}{2})$ and $0 \notin \bar{B}(x, \frac{1}{2})$.

For the closed unit ball B, it's a closed space in X, same as S (as they are preimages of a continuous function in X, the norm), and then S is closed in B. So if B were compact, then S will be a closed subspace of a compact subspace, and therefore compact. So B can't be compact.

Exercise 4 (20 points) . *Let $L_1([0,1], m)$ and $L_3([0,1], m)$ be the Lebesgue spaces on [0,1] Recall from HW2 that $L_3([0,1], m) \subsetneq L_1([0,1], m)$. For $n \geq 1$, define:*

$$E_n := \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm \leq n \right\}$$

- [6p]. *Given $n \geq 1$, is the set $E_n \subset L_1([0,1],m)$ absorbing? Justify.*

It's not absorbing: if we consider $h(x) = \frac{1}{x^{2/3}}$, then $h \in L_1([0,1],m)$. However, for any $0 < k \in \mathbb{K}$:

$$\int_{[0,1]} \left| k \frac{1}{x^{2/3}} \right|^3 dx = k^3 \int_{[0,1]} \frac{1}{x^2} dx = \infty$$

So for any $0 < k \in \mathbb{K}$, $kh \notin E_n$.

On the other hand E_n is actually convex: if $f, g \in E_n$ and $0 < t < 1$ then:

$$\int_{[0,1]} |tf + (1-t)g|^3 \, dm \leq \int_{[0,1]} t\,|f|^3 + (1-t)\,|g|^3 \, dm \leq tn + (1-t)n = n$$

So $tf + (1-t)g \in E_n$.

- [7p]. *Show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.*

We are going to show that given $f \in E_n$, for any $r > 0$, the open balls: $B_1(f,r) = \{g \in L_1([0,1]) \int_{[0,1]} |f-g| \, dm < r\}$ verify that $B_1(f,r) \not\subset E_n$. In other words, for any E_n, any $f \in E_n$ and any $r > 0$, we are going to give $g_r \in B_1(f,r)$ with $g_r \notin E_n$.

If $f = 0$, then we define $h(x) := \frac{1}{x^{2/3}}$ and consider $g_r(x) = \frac{r}{2}h(x)$. As $\int_{[0,1]} \frac{1}{x^{2/3}} dx = 1$, then $\int_{[0,1]} |g_r - 0| \, dm < r$. However, $g_r \notin E_n$ by our reasoning above on the part that the E_n are not absorbing.

In the general case, we consider $g_r(x) := f(x) + \frac{r}{2}h(x)v(x)$, with $v : [0,1] \to \mathbb{K}$ such that $|v(x)| = 1$ and has the same angle as $f(x)$ (i.e. $v(x) = \frac{f(x)}{|f(x)|}$ if $f(x) \neq 0$, 1 otherwise), so that $|g_r(x)| = |f(x)| + \frac{r}{2}\,|h(x)|$.

Then $g_r \in B_1(f,r)$ and:

$$\int_{[0,1]} |g_r| \, dm = \int_{[0,1]} \left| f + \frac{r}{2}h(x)v(x) \right|^3 dm \geq^* \int_{[0,1]} \frac{r^3}{8} h^3(x) dx = \infty.$$

The inequality (*) we'll come from expanding the binomial. Then $g_r \notin E_n$.

- [8p]. *Show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$.*

We'll prove that for any sequence $\{f_j\} \subset E_n$ such that there exists $f \in L_1[0,1]$ with $\lim_{j\to\infty} \|f_j - f\|_1 = 0$, then $f \in E_n$.

We'll use the result that if $f_j \to f$ in L_1, then there exists a subsequence f_{j_k} such that $f_{j_k} \to f$ almost everywhere. That implies that $|f_{j_k}|^3 \to |f|^3$ almost everywhere.

Now we'll use the Fatou's lemma: given a sequence of non-negative measurable functions $\{g_n\}$ then $\liminf g_n$ is also measurable and:

$$\int \liminf g_n dm \leq \liminf \int g_n dm.$$

In our case with $|f_{j_k}|^3$, we have that $\liminf |f_{j_k}|^3 = |f|^3$ so:

$$\int_{[0,1]} |f|^3 \, dm \leq \liminf \int_{[0,1]} |f_{j_k}|^3 \leq n$$

as $f_{j_k} \in E_n$. Then $f \in E_n$.

- [4p]. *Conclude from (b) and (c) that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$*

We want to show that $L_3([0,1], m)$ is a countable union of nowhere dense sets. The sets E_n verify $Int(\bar{E}_n) = Int(E_n) = \emptyset$, so they are nowhere dense. Finally, $L_3([0,1], m) = \cup_{n=1}^{\infty} E_n$ as if $f \in L_3([0,1], m)$ then $\int_{[0,1]} |f|^3 \, dm < \infty$, so there exists some $n \in \mathbb{N}$ such that $f \in E_n$.

Exercise 5 (11 points) . *Let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.*

- [2p]. *Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$? Give a proof or a counterexample.*

$x_n \to x$ in norm means $\|x_n - x\| \to 0$. We want to show that $\|x_n\| \to \|x\|$, which is equivalent to $|\|x_n\| - \|x\|| \to 0$. We notice that:

$$\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\|,$$

therefore $\|x_n\| - \|x\| \leq \|x_n - x\|$. Analogously, it can be shown that $\|x\| - \|x_n\| \leq \|x_n - x\|$, so $|\|x_n\| - \|x\|| \leq \|x_n - x\|$. So:

$$0 \leq |\|x_n\| - \|x\|| \leq \|x_n - x\|$$

then $|\|x_n\| - \|x\|| \to 0$, so $\|x_n\| \to \|x\|$.

- [5p]. *Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$? Give a proof or a counterexample. [Hint: Consider an orthonormal basis $(e_n)_n \geq 1$ in H, and use* HW4.]

As it has been said in the lectures, any separable Hilbert space admits countable orthonormal basis $(e_n)_{n\geq 1}$ in H. By *HW P2a)* $x_n \to x$ weakly if and only if $f(x_n) \to f(x)$ for any $f \in H^*$. By the Riesz representation theorem we know that any $f \in H^*$ is of the form f_y whith $f_y(x) = \langle x, y\rangle$ for some unique $y \in H$. On the other hand, as $(e_n)_{n\geq 1}$ is orthonormal basis, we get that for any $y \in H$, $y = \sum_n y_n e_n$ with $\{y_n \neq 0\}$ finite and $y_n = \langle y, e_n\rangle$.

Then for any $y \in H$, $f_y(e_n) = \langle e_n, y\rangle = \bar{y_n}$. As the set $\{y_n \neq 0\}$ is finite, $f_y(e_n) \to 0 = f_y(0)$. So for any $f \in H^*$ we have $f(e_n) \to f(0)$, so $e_n \to 0$ weakly. However, $\|e_n\| = 1$ so $\|e_n\| \not\to \|0\|$.

- [4p]. *Suppose that* $\|x_n\| \leq 1$, *for all* $n \geq 1$, *and that* $x_n \to x$ *weakly, as* $n \to \infty$. *Is it true that* $\|x\| \leq 1$? *Give a proof or a counterexample.*

$x_n \to x$ weakly is equivalent to $f(x_n) \to f(x)$ for any $f \in H^*$, i.e, $\langle x_n, y\rangle \to \langle x, y\rangle$ for any $y \in H$. In particular for x, we have $\langle x_n, x\rangle \to \|x\|^2$, so we have $|\langle x_n, x\rangle| \to \|x\|^2$.

On the other hand, we can apply Cauchy-Schwartz inequality:

$$|\langle x_n, x\rangle|^2 \leq \|x_n\|\|x\| \leq \|x\|$$

so

$$|\langle x_n, x\rangle| \leq \|x\|^{1/2}$$

So $\lim |\langle x_n, x\rangle| \leq \|x\|^{1/2}$. Therefore $\|x\|^2 \leq \|x\|^{1/2}$, and that relation holds if and only if $\|x\| \leq 1$.