Assigment 1 -Diffun

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Ex 1

Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $u \in \mathcal{D}'(\mathbb{R})$. We show that $f(x) = \langle u, \varphi(x, \cdot)\rangle$ defines a function in $C_0^\infty(\mathbb{R})$ with $f'(x) = \langle u, \partial_x \varphi(x, \cdot)\rangle$.

Proof. We first of all notice that in the product topology the projection maps $\pi_1 : \mathbb{R}^2 \ni (x, y) \mapsto x \in \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \ni (x, y) \mapsto y \in \mathbb{R}$ are continuous. Therefore, $\pi_1(\text{supp}(\varphi)) \subset \mathbb{R}$ and $\pi_2(\text{supp}(\varphi)) \subset \mathbb{R}$ are compact sets, as they are images of compact sets under continuous maps. Now clearly, we have $\text{supp}(\varphi(x, \cdot)) \subset \pi_2(\text{supp}(\varphi))$ for all $x \in \mathbb{R}$, so $\text{supp}(\varphi(x, \cdot))$ is closed subset of a compact set, and therefore $\varphi(x, \cdot)$ has compact support for all $x \in \mathbb{R}$. Furthermore, $\varphi(x, \cdot)$ is a C^∞ function, since all derivatives are equal to some partial derivative of φ which is continuous by assumtion. Hence $\varphi(x, \cdot) \in C_0^\infty(\mathbb{R})$ for all $x \in \mathbb{R}$ and, f is well-defined. Now by a similar argument we have that $\text{supp}(\varphi(\cdot, y)) \subset \pi_1\text{supp}(\varphi)$ for all $y \in R$ and therefore $\varphi(x, \cdot) \neq 0$ only if $x \in \pi_1\text{supp}(\varphi)$. Therefore, we may conclude that $\text{supp}(f(x)) \subset \pi_1(\text{supp}(\varphi))$. Thus $\text{supp}(f(x))$ is a closed subset of a compact set, hence it is compact.
Thus we know that $f(x)$ is well-defined and have compact support. to show that f is a C^∞ function. We compute the difference coefficient for f

$$\frac{f(x+h) - f(x)}{h} = \left\langle u, \frac{\varphi(x+h, \cdot) - \varphi(x, \cdot)}{h} \right\rangle, \tag{0.1}$$

where we used linearity of $\langle u, \cdot\rangle$. Now we show that for any sequence h_n, such that $h_n \to 0$, we have $\frac{\varphi(x+h_n,\cdot)-\varphi(x,\cdot)}{h_n} \to \partial_x\varphi(x, \cdot)$ in $C_0^\infty(\mathbb{R})$. This is seen by the mean value theorem, since $\frac{\varphi(x+h_n,\cdot)-\varphi(x,\cdot)}{h_n} = \partial_x\varphi(x + \xi_n(x, h_n, \cdot), \cdot)$ for some $0 \leq \xi(x, h_n, \cdot) \leq h_n$. However, since we by the above argument have that $\text{supp}(\varphi(x, \cdot)) \in \pi_2\text{supp}(\varphi) \subset\subset \mathbb{R}$ for all $x \in \mathbb{R}$, we see that $\text{supp}\left(\frac{\varphi(x+h_n,\cdot)-\varphi(x,\cdot)}{h_n}\right) \subset \pi_2\text{supp}(\varphi)$ for all $n \geq 1$. Thus there exist a $j \geq 1$ such that $\text{supp}\left(\frac{\varphi(x+h_n,\cdot)-\varphi(x,\cdot)}{h_n}\right) \in K_j$ for all $n \geq 1$, where K_j is the increasing sequence of compact sets defined in lemma 2.2 in the book. Furthermore, since $\partial_x\varphi(\cdot, \cdot)$ is continuous with compact support, it a well known result that it is uniformly continuous. But then it is clear that $\frac{\varphi(x+h_n,\cdot)-\varphi(x,\cdot)}{h_n} = \partial_x\varphi(x+\xi_n(x, h_n, \cdot), \cdot) \to \partial_x\varphi(x, \cdot)$ uniformly for all $x \in \mathbb{R}$. The same holds for all the derivatives, $\partial_y^m\varphi(x, \cdot)$, by the same argument applied to $\partial_y^m\varphi(x, \cdot)$. Thus we have shown that there exist a $j \geq 1$ such that $\frac{\varphi(x+h_n,\cdot)-\varphi(x,\cdot)}{h_n} \in C^\infty_{K_j}(\mathbb{R})$ for all $n \geq 1$ and

$$\sup\left\{\left|\partial_y^m\left(\frac{\varphi(x+h_n,y)-\varphi(x,y)}{h_n} - \partial_x\varphi(x,y)\right)\right| \;:\; y \in K_j,\; m \leq \alpha\right\} \to 0 \text{ as } n \to \infty \tag{0.2}$$

for all $\alpha \geq 0$. Hence by theorem 2.5(a) we have that $\frac{\varphi(x+h_n,\cdot)-\varphi(x,\cdot)}{h_n} \to \partial_x\varphi(x,\cdot)$ in $C_0^\infty(\mathbb{R})$. It then follows from continuity of $u : C_0^\infty(\mathbb{R}) \to \mathbb{C}$ that $\frac{f(x+h_n)-f(x)}{h_n} \to \langle u, \partial_x\varphi(x,\cdot)\rangle$ as $n \to \infty$, but since this was for any sequence, h_n, converging to 0, we then may conclude that $\frac{f(x+h)-f(x)}{h} \to \langle u, \partial_x\varphi(x,\cdot)\rangle$ as $h \to 0$, such that $f'(x) = \langle u, \partial_x\varphi(x,\cdot)\rangle$. It then follows that f is continuous, since it is differentiable. Now to see that f is C^∞, we simply proceed by induction. Iterating the argument with φ replaced by $\partial_x^m\varphi$, shows that f is $m+1$ times differentiable with $f^{(m+1)}(x) = \langle u, \partial_x^{m+1}\varphi(x,\cdot)\rangle$, thus $f \in C_0^m(\mathbb{R})$. Therefore, by induction, $f \in C_0^k(\mathbb{R})$ for all $k \geq 0$ such that $f \in C_0^\infty(\mathbb{R})$, which completes the proof. □

Ex 2

Consider the function $u : \mathbb{R} \to \mathbb{C}$ given by $u(x) = \exp(-|x|)$, $x \in \mathbb{R}$.

1) We show that $u \in L^1(\mathbb{R})$ and that in the sense of distributions we have

$$\left(1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) u = 2\delta_0. \tag{0.3}$$

where δ_0 is the δ-distribution at 0. That $u \in L^1(\mathbb{R})$ is easily verified: u is measurable, since it is continuous. Futhermore,

$$\begin{aligned}\int_{\mathbb{R}} |u(x)|\,\mathrm{d}x &= 2\int_{[0,\infty)} \exp(-x)\,\mathrm{d}x = 2\lim_{N\to\infty}\int_{[0,N]} \exp(-x)\,\mathrm{d}x \\ &= 2\lim\int_0^N \exp(-x)\,\mathrm{d}x = 2\lim_{N\to\infty}\left[-\exp(-x)\right]_0^N = 2 < \infty.\end{aligned} \tag{0.4}$$

So $u \in L^1(\mathbb{R})$. To verify (0.3), notice that u is C^∞ on $\mathbb{R}_+$ and $\mathbb{R}_-$, and that u is continuous on $\mathbb{R}$. Therefore, by lemma 3.6 in G. Grubb, we have that

$$\frac{\mathrm{d}}{\mathrm{d}x}u = \begin{cases} -\exp(-x), & x > 0 \\ \exp(x), & x < 0. \end{cases} \tag{0.5}$$

which is again an $L^1(\mathbb{R})$ function. Now notice that $\frac{\mathrm{d}}{\mathrm{d}x}u + 2H$ is extendible to a continuous function on $\mathbb{R}$, where H is the Heaviside step function $H = \mathbb{1}_{(0,\infty)}$. Thus lemma 3.6 again applies to this function, giving us

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}u + 2H\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}u + 2\delta_0 = \begin{cases} \exp(-x), & x > 0, \\ \exp(x), & x < 0. \end{cases} = u \tag{0.6}$$

which is equivalent to

$$\left(1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) u = 2\delta_0. \tag{0.7}$$

as desired.

2) We show that if $\phi \in C_0^\infty(\mathbb{R})$ then $u * \phi \in C^\infty(\mathbb{R})$ and

$$\left(1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) u * \phi = 2\phi. \tag{0.8}$$

That $u * \phi \in C^\infty(\mathbb{R})$ follows from theorem 3.16 and by noticing that $u * \phi = \phi * u$ (by definition as the adjoint operation). Now (0.8) follows from Eq (3.42) in G. Grubb. Using this relation and linearity of the convolution we may calculate

$$\left(1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) u * \phi = \left(\left(1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) u\right) * \phi = 2\delta_0 * \phi = 2\left\langle \delta_0, \phi(x - \cdot)\right\rangle = 2\phi(x-0) = 2\phi(x). \tag{0.9}$$

where we also used theorem 3.16 in the third equality again.

Ex 3

a)

Let $f(x) = x^{-3/2}H(x)$, where H is the Heaviside step function. We show that $f|_{\mathbb{R}_+} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, but f is not in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Proof. Notice first that $f_n = \mathbb{1}_{(1/n,\infty)} f$ is a non-negative increasing sequence of functions such that $f_n \uparrow f$ pointwise. f_n are measurable, since

$$\{f_n > a\} = \begin{cases} (1/n, a^{-2/3}) & 0 < a < n^{3/2}, \\ (1/n, \infty) & a = 0, \\ \emptyset & a \geq n^{3/2}, \\ \mathbb{R} & a < 0 \end{cases} \tag{0.10}$$

which are all open sets, *i.e.* $\{f_n > a\} \in \mathcal{B}(\mathbb{R})$ for all $a \in \mathbb{R}$. Thus by the monotone convergence theorem we have f is measurable. Now for any compact set $K \in \mathbb{R}_+$ we have that there exist $a, b > 0$ such that $K \in [a, b]$. Thus we estimate

$$\int_K |f(x)| \,\mathrm{d}x = \int_K f(x) \,\mathrm{d}x \leq \int_{[a,b]} f(x) \,\mathrm{d}x \tag{0.11}$$

since f is continuous on the interval (a, b) we may rewrite this integral as a Riemann integral

$$\int_K |f(x)| \,\mathrm{d}x \leq \int_a^b f(x) \,\mathrm{d}x = \int_a^b x^{-3/2} \,\mathrm{d}x = \left[-2x^{-1/2}\right]_a^b = 2(a^{-1/2} - b^{-1/2}) < \infty. \tag{0.12}$$

Thus $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. On the other hand, $[0, 1]$ is clearly a compact set in $\mathbb{R}$, and by the monotone convergence theorem we have

$$\int_{[0,1]} |f(x)| \, \mathrm{d}x = \int_{[0,1]} f(x) \, \mathrm{d}x = \lim_{n\to\infty} \int_{[0,1]} f_n(x) \, \mathrm{d}x = \lim_{n\to\infty} \int_{(1/n,1]} x^{-3/2} \, \mathrm{d}x \tag{0.13}$$

again since $x^{-3/2}$ is continuous on $(1/n, 1)$ we may rewrite in terms of Riemann integrals

$$\int_{[0,1]} |f(x)| \, \mathrm{d}x = \lim_{n\to\infty} \int_{1/n}^{1} x^{-3/2} \, \mathrm{d}x = \lim_{n\to\infty} \left[-2x^{-1/2}\right]_{1/n}^{1} = 2 \lim_{n\to\infty} \left(n^{1/2} - 1\right) = \infty, \tag{0.14}$$

from which it follows that $f \notin L^1_{\mathrm{loc}}(\mathbb{R})$. □

We now show that $\langle \Lambda, \varphi \rangle = \int_{(0,\infty)} x^{-3/2} \left(\varphi(x) - \varphi(0)\right)$ defines a distribution in $\mathcal{D}'(\mathbb{R})$, which is equal to f on $\mathbb{R}_+$ and on $\mathbb{R}_-$.

Proof. We have already shown that $x^{-3/2}\mathbb{1}_{[0,\infty)}$ is measurable so $\langle \Lambda, \varphi \rangle$ is well defined. That $\langle \Lambda, \cdot \rangle$ is a linear functional is obvious from linearity of the integral. $\langle \Lambda, \varphi \rangle \neq \infty$ will follows from the proof of continuity below. We thus need to show, that $\langle \Lambda, \cdot \rangle$ is also continuous on $C_0^\infty(\mathbb{R})$. To see this, let $a > 0$ and let $\varphi \in C^\infty_{K_j}(\mathbb{R})$ and notice that

$$\begin{aligned} |\langle \Lambda, \varphi \rangle| &= \left| \int_{(0,\infty)} x^{-3/2} \left(\varphi(x) - \varphi(0)\right) \mathrm{d}x \right| \\ &\leq \left| \int_{(0,a]} x^{-3/2} \left(\varphi(x) - \varphi(0)\right) \mathrm{d}x \right| + \left| \int_{(a,\infty)} x^{-3/2} \left(\varphi(x) - \varphi(0)\right) \mathrm{d}x \right|. \end{aligned} \tag{0.15}$$

Now by the mean value theorem $|\langle \Lambda, \varphi \rangle| \left(\varphi(x) - \varphi(0)\right) = \varphi'(\xi(x))x$ where $0 \leq \langle \Lambda, \varphi \rangle = \xi(x) \leq x$. Thus we have

$$\begin{aligned} |\langle \Lambda, \varphi \rangle| &\leq \left| \int_{(0,a]} x^{-1/2} \varphi'(\xi(x)) \, \mathrm{d}x \right| + \left| \int_{(a,\infty)} x^{-3/2} \left(\varphi(x) - \varphi(0)\right) \mathrm{d}x \right| \\ &\leq \max_{x\in\mathbb{R}}(|\varphi'(x)|) \int_{(0,a]} x^{-1/2} \, \mathrm{d}x + 2 \max_{x\in\mathbb{R}}(|\varphi(x)|) \int_{(a,\infty)} x^{-3/2} \, \mathrm{d}x. \end{aligned} \tag{0.16}$$

where the maxima $\max_{x\in\mathbb{R}}(|\varphi'(x)|) = \max_{x\in K_j}(|\varphi'(x)|)$ and $\max_{x\in\mathbb{R}}(|\varphi(x)|) = \max_{x\in K_j}(|\varphi(x)|)$ exist since, $\varphi \in C^\infty_{K_j}(\mathbb{R})$. By the usual conversion of Lebsgue integrals to Riemann integrals, via *e.g.* monotone convergence theorem, we get

$$\begin{aligned} &\max_{x\in K_j}(|\varphi'(x)|) \int_{(0,a]} x^{-1/2} \, \mathrm{d}x + 2 \max_{x\in K_j}(|\varphi(x)|) \int_{(a,\infty)} x^{-3/2} \, \mathrm{d}x \\ &= 2 \max_{x\in K_j}(|\varphi'(x)|) a^{1/2} + 4 \max_{x\in K_j}(|\varphi(x)|) a^{-1/2} \leq C \sup\left\{ \left|\varphi^{(m)}(x)\right| : x \in K_j, \ m \leq 1 \right\} \end{aligned} \tag{0.17}$$

where C might be choosen to be *e.g.* $C = 6$, which is easily seen by setting $a = 1$. Thereby we have shown for any $j \in \mathbb{N}$ that

$$\langle \Lambda, \varphi \rangle \leq C \sup\left\{ \left|\varphi^{(m)}(x)\right| : x \in K_j, \ m \leq 1 \right\}, \tag{0.18}$$

for all $\varphi \in C^{\infty}_{K_j}(\mathbb{R})$. Thus by theorem 2.5(d) we see that $\langle \Lambda, \cdot\rangle$ defines a distribution in $\mathcal{D}'(\mathbb{R})$. That $\Lambda = \Lambda_f$ on $\mathbb{R}_+$ is easily seen: Let $\varphi \in C_0^\infty(\mathbb{R}_+)$, then

$$(\Lambda - \Lambda_f)(\varphi) = \int_{(0,\infty)} x^{-3/2} \left(\varphi(x) - \underbrace{\varphi(0)}_{=0}\right) \mathrm{d}x - \int_{(0,\infty)} x^{-3/2}\varphi(x)\,\mathrm{d}x = 0. \tag{0.19}$$

where we used that $f \in L^1_{\text{loc}}(\mathbb{R})$ in the first equality and that $\text{supp}(\varphi) \subset (0, \infty)$ implies that $\varphi(0) = 0$ and linearity of the integral in the second equality. Thus we have shown that $\Lambda|_{\mathbb{R}_+} - \Lambda_f|_{\mathbb{R}_+} = 0$ which by definition means that $\Lambda = \Lambda_f$ $(= f)$ on $\mathbb{R}_+$. On $\mathbb{R}_-$ both distributions are trivially zero, so $\Lambda = \Lambda_f$ $(= f)$ on $\mathbb{R}_-$ as well. □

b)

Let $g(x) = -2x^{-1/2}H(x)$. We show that $g \in L^1_{\text{loc}}(\mathbb{R})$ and that $g' = \Lambda$.

Proof. Define $g_n = \mathbb{1}_{(1/n,\infty)}g$, then $-g_n$ is an increasing sequence of non-negative functions such that $-g_n \uparrow -g$. g_n are measurable, by a similar argument to one made in (a), or by noticing that g_n may be written as a product of a continuous function $\tilde{g}(x) = \begin{cases} g(x) & x > 1/n \\ -2xn^{3/2} & x \leq 1/n \end{cases}$, and the measurable function $\mathbb{1}_{(1/n,\infty)}$. Thus, $-g_n$ are measurable and by the monotone convergence theorem $-g$ is measurable, from which it follows that g is measurable. Now let K be a compact subset of $\mathbb{R}$, then there exist $a > 0$ such that $K \in (-a, a)$ therefore, we estimate

$$\int_K |g(x)|\,\mathrm{d}x \leq \int_{[-a,a]} |g(x)|\,\mathrm{d}x = \int_{[0,a]} 2x^{-1/2} = \lim_{n\to\infty} \int_{(1/n,a]} 2x^{-1/2}\,\mathrm{d}x, \tag{0.20}$$

where we used the monotonce convergence theorem in the last equality. The last integrals may be rewritten as Riemann integrals and thus we have

$$\int_K |g(x)|\,\mathrm{d}x \leq \lim_{n\to\infty} \int_{1/n}^a 2x^{-1/2}\,\mathrm{d}x = 2\lim_{n\to\infty} \left[2x^{1/2}\right]_{1/n}^a = 4a^{1/2} < \infty. \tag{0.21}$$

Thus it follows that $g \in L^1_{\text{loc}}(\mathbb{R})$. It therefore makes sense to compute the distributional derivative, g'. This can be computed directly from definition, let $\varphi \in C_0^\infty(\mathbb{R})$

$$\langle g', \varphi\rangle = -\langle g, \varphi'\rangle = \int_{(0,\infty)} 2x^{-1/2}\varphi'(x)\,\mathrm{d}x = \int_{(0,\infty)} 2x^{-1/2} \left(\varphi(x) - \varphi(0)\right)'\mathrm{d}x, \tag{0.22}$$

where we used that $(\varphi(x) - \varphi(0))' = \varphi'(x)$ in the last equality. Noticing that $\left|-2x^{1/2}\varphi'(x)\right| \in L^1(\mathbb{R}_+)$, since $\varphi' \in C_0^\infty(\mathbb{R})$, it follows from the dominated convergence theorem that

$$\langle g', \varphi\rangle = \lim_{n\to\infty} \int_{(1/n,n)} 2x^{-1/2} \left(\varphi(x) - \varphi(0)\right)'\mathrm{d}x \tag{0.23}$$

By rewriting in terms of Riemann integrals we have

$$\begin{aligned}\langle g', \varphi\rangle &= \lim_{n\to\infty} \int_{1/n}^{n} 2x^{-1/2} \left(\varphi(x) - \varphi(0)\right)' \mathrm{d}x \\ &= \lim_{n\to\infty} \left(\left[2x^{-1/2} \left(\varphi(x) - \varphi(0)\right)\right]_{1/n}^{n} + \int_{(1/n,n)} x^{-3/2} \left(\varphi(x) - \varphi(0)\right) \mathrm{d}x \right)\end{aligned} \tag{0.24}$$

Now we use that

$$\lim_{n\to\infty} \left(\left[2x^{-1/2} \left(\varphi(x) - \varphi(0)\right]_{1/n}^{n} \right) = 2 \lim_{n\to\infty} \left[n^{-1/2}(\varphi(n) - \varphi(0)) - n^{1/2}(\varphi(1/n) - \varphi(0))\right] = 0, \tag{0.25}$$

which can be seen from the fact that $\varphi(n)$ is bounded, and $\varphi(1/n) - \varphi(0) = \varphi'(\xi_n)/n \leq C/n$ for some $C > 0$ by the mean value theorem. Now notice also that $\left|x^{-3/2} \left(\varphi(x) - \varphi(0)\right)\right| \in L^1(\mathbb{R}_+)$ since, as was also used in part a), we have

$$\left|x^{-3/2} \left(\varphi(x) - \varphi(0)\right)\right| \leq \begin{cases} \max(|\varphi'|)x^{-1/2} & 0 < x < 1 \\ 2\max(|\varphi|)x^{-3/2} & x \geq 1 \end{cases} \tag{0.26}$$

where as usual the top estimate follows from the mean value theorem and the bottom one is straightforward. Clearly, as seen by above in part a), this shows that $\left|x^{-3/2} \left(\varphi(x) - \varphi(0)\right)\right| \in L^1(\mathbb{R}_+)$. But then notice that by the dominated convergence theorem it follows that

$$\int_{(1/n,n)} x^{-3/2} \left(\varphi(x) - \varphi(0)\right) \mathrm{d}x \to \int_{(0,\infty)} x^{-3/2} \left(\varphi(x) - \varphi(0)\right) \mathrm{d}x, \text{ as } n \to \infty. \tag{0.27}$$

We have thereby shown that $\langle g', \phi\rangle = \int_{(0,\infty)} x^{-3/2} \left(\varphi(x) - \varphi(0)\right) \mathrm{d}x = \langle \Lambda, \varphi\rangle$, for all $\varphi \in C_0^\infty(\mathbb{R})$, such that $g' = \Lambda$. $\square$