1 Problem 1

Let $(X, \|\cdot\|_x)$ and $(Y, \|\cdot\|_y)$ be non-zero normed vector spaces over field $\mathbb{K}$where $\mathbb{K} = \mathbb{C} \vee \mathbb{R}$.

1.1 Part a

Let $T : X \to Y$ be a linear map, and let $\|x\|_0 = \|x\|_x + \|Tx\|_y$. First I will show that $\|x\|_0$ is a norm by adhere the properties by defination 1.1. First the triangle inequality:

$$\begin{aligned} \|x+\tilde{x}\|_0 &= \|x+\tilde{x}\|_x + \|T(x+\tilde{x})\|_y \\ &= \|x+\tilde{x}\|_x + \|Tx+T\tilde{x}\|_y && \text{Linar map properties} \\ &\leq \|x\|_x + \|\tilde{x}\|_x + \|Tx\|_y + \|T\tilde{x}\|_y \\ &= \|x\|_0 + \|\tilde{x}\|_0 \end{aligned}$$

Next the scalar properties:

$$\begin{aligned} \|\alpha x\|_0 &= \|\alpha x\|_x + \|T\alpha x\|_y \\ &= \|\alpha x\|_x + \|\alpha T\|_y \\ &= |\alpha|\|x\|_x + |\alpha|\|Tx\|_y \\ &= |\alpha|(\|x\|_x + \|Tx\|_y) = |\alpha|\|x\|_0 \end{aligned}$$

Last the zero properties:

$$\begin{aligned} \|0\|_0 &= \|0\|_x + \|T0\|_y \\ &= 0+0=0 \\ \|x\|_0 &\leq \|x\|_x + \|Tx\|_y, && x \neq 0 \\ 0 < \|x\|_0 &\leq \|x\|_x + \|Tx\|_y \end{aligned}$$

Hence $\|x\|_0$ is a norm. Next we will show that if the norms $\|x\|_0$ and $\|x\|_0$ are equivalent if and only if T is bounded. Two norms are equivalent if:

$$c_1\|x\|_0 \leq \|x\|_x \leq c_2\|x\|_0$$

Suppose that T is bounded. Then there exists a $C > 0$ so $\|Tx\| \leq C\|x\|$. If $C < 1$, then let $C = 1$. So let $c_1 = \dfrac{1}{2C}$

$$\begin{aligned} \frac{1}{2C}\|Tx\|_y &\leq \frac{1}{2C}C\|x\|_x \leq \frac{1}{2}\|x\|_x \\ \frac{1}{2C}\|x\|_x &\leq \frac{1}{2}\|x\|_x \end{aligned}$$

We can use this inequality to show that $\frac{1}{2C}\|x\|_0 \leq \|x\|_x \leq \|x\|_0$:

$$\frac{1}{2C}\|x\|_x + \frac{1}{2C}\|Tx\|_y \leq \frac{1}{2}\|x\|_x + \frac{1}{2}\|x\|_x = \|x\|_x \leq \|x\|_x + \|Tx\|_y$$

Sonversely suppose that $\|x\|_0$ and $\|x\|_x$ are equivalent. Then we have $\|x\|_0 \geq C\|x\|_0$

$$\begin{aligned} \|x\|_x + \|Tx\|_y &\leq C\|x\|_x, \qquad C > 1 \\ \|Tx\|_y &\leq (C-1)\|x\|_x \end{aligned}$$

Hence T is bounded.

1.2 Part b

Suppose that X is finite, then by theorem 1.6, that every two norms are equivalent on finite dimensional vector space. Then $\|x\|_0 = \|x\|_x + \|Tx\|_y$ and $\|x\|_x$ are equivalent, and by problem 1 part a, we have that T is bounded.

1.3 Part c

Let $(e_i)_{i\in\mathbb{N}}$ be a Hamel basis for X, and let $(y_i)_{i\in\mathbb{N}} = (ie_i)_{i\in\mathbb{N}}$, then there exists precisely one linear map with $T(e_i) = i$, and for any C there exsists a N

$$\|Tx_i\| \nleq C\|x_i\|, \qquad i > N \tag{1}$$

Hence T is not bounded.

1.4 Part d

Take the norm $\|x\|_0 \leq \|x\|_x + \|Tx\|_y$, we have showed in problem 1 part a, that it is a norm and in part c, that there exists a T so they are not equivalent.

$$\|x\|_0 \leq \|x\|_x + \|Tx\|_y$$

Let $(X, \|x\|_x)$ be a Banach space. Suppose for contradiction that $(X, \|x\|_0)$ is complete. Then for every cauchy sequence $(x_n)_{n\geq 1}$.

$$\forall \varepsilon > 0\, \exists n_\varepsilon > 0 : \|x_m - x_n\|_0 < \varepsilon, \forall n, m \geq n_\varepsilon$$

We can show that T is continuous at 0, with:

$$\|Tx_m - Tx_n\|_y \leq \|x_m - x_n\|_x + \|T(x_m - x_n)\|_y < \varepsilon, \qquad \|x_m - x_n\| < \epsilon$$

This shows us that T is continuous at 0 ($(x - x_n)_{n\geq 1}$ is a cauchy sequence). By proposition 1.10 is equivalent with T is bounded, and those is a contradiction.

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1.5 Part e

Let $(l_1(\mathbb{N}), \|\cdot\|_1)$ over $\mathbb{C}$ and let $|x|_\infty$, these two norms are inequivalent. Since for any $C \in \mathbb{N}$, we can let $|x_n| = \dfrac{1}{c}$ for n satisfying $C+1 \geq n \geq 1$ for $n \geq C+1$ let $x_n = 0$.

$1 + \dfrac{1}{c} = \|(x_n)_{n\geq 1}\|_1 \not\leq C\|(x_n)_{n\geq 1}\|_\infty = 1$

Now let a sequnce of 1 equal to n so $x_1 = (1, 0, 0, ..)$ and $x_2 = (1, 1, 0, ..)$, we now have that $\|x_n\|_\infty = 1$ for all n but $\|x_n\|_1 = n$. So x_n is a cauchy sequence in $\|\cdot\|'$ but are not in $l_1(\mathbb{N})$. And we have that $(l_1(\mathbb{N}), \|\cdot\|_1)$ is complete but $(l_1(\mathbb{N}), \|\cdot\|_\infty)$ is not.

2 Problem 2

Let $1 \leq p < \infty$ be fixed and the subspace M of the Banach space $(l_p(N), \|\cdot\|_p)$, let M be a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, ...) \colon a, b \in C\}$$

2.1 a

Then

$$\|f\| = \sup_{|x|\leq 1}(|f(x)|), \qquad \|x\|_p = \sqrt[p]{|a|^p + |b|^p}$$

I will first compute $\|f\|$, since we have that $\|f(x)\| \leq \|f\|\|x\|_p$.

$$\begin{aligned}
\sup_{\|x\|_p\leq 1}(|f(x)|) &= \sup_{|x|_p\leq 1}(|a+b|) \\
\|x\|_p &= (|a|^p + |b|^p)^{1/p} \\
&= (|a|^p + |a|^p)^{1/p} && \text{Let } |a| = |b| \\
&= (2|a|^p)^{1/p} \\
&= 2^{1/p}|a| = 1, \\
|a| &= 2^{-1/p}, \\
\sup_{\|x\|_p\leq 1}(|f(x)|) &= |2^{-1/p}| + |2^{-1/p}| = 2^{\frac{p-1}{p}}
\end{aligned}$$

Now we have that f is bounded with $C = 2^{\frac{p-1}{p}}$

$$\begin{aligned}
&\|f(x)\| \leq \|f\|\|x\|_p, \\
&|a+b| \leq 2^{\frac{p-1}{p}}(|a|^p + |b|^p)^{\frac{1}{p}}, && \text{for } 1 \leq p < \infty
\end{aligned}$$

2.2 b

Let

$$l_p(\mathbb{N}) = \left\{ (x_n)_{n\geq 1} \subset \mathbb{K} : \|(x_n)_{n\geq 1}\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 < p < \infty$$

We like to extending f to $l_p(\mathbb{N})$ with F satisfying $\|F\| = \|f\|$.

$$\begin{aligned} \|f\| = \|F\| &= \sup_{\|x\|_p \leq 1} (\|F(x)\|), \\ 2^{\frac{p-1}{p}} &= \sup_{\|x\|_p \leq 1} (|x_1 + x_2 + \sum_{i \in I} \lambda_i x_i|), \qquad x_i \in \mathbb{C} \end{aligned}$$

We can assume that λ_i and x_i is in $\mathbb{R}_+$, with without loss of generality, hence $(x_n)_{n\geq 1} \in l_p(\mathbb{N}) \Rightarrow (|x_n|)_{n\geq 1} \in l_p(\mathbb{N})$. Suppose that $\lambda_3 \neq 0$,then we have the inequality

$$\sup_{\|x\|_p \leq 1} (|x_1 + x_2|) < \sup_{\|x\|_p \leq 1} (|x_1 + x_2 + \lambda_3 x_3|), \qquad x_i \in \mathbb{C}, 0 < \lambda_3$$

This gives us that all $\lambda_i = 0$, hence there is only one unique F that extends f to $l_p(\mathbb{N})$ $\|F\|$.

2.3 c

Let $p = 1$, then we have that F_I with $i \in I$ where I is finite and that $(\lambda_i)_{i\in I}$ so $0 \leq \lambda_i \leq 1$.

$$\sup_{\|x\| \leq 1} (\|F_I(x)\|) = \sup_{\|x\| \leq 1} (|x_1 + x_2 + \sum_{i \in I} \lambda_i x_i|),$$

Then we have that

$$1 = \sup_{\|x\| \leq 1} (|x_1| + |x_2|) \leq \sup_{\|x\| \leq 1} (\|F_I(x)\|) \leq \sup_{\|x\| \leq 1} (|x_1| + |x_2| + \sum_{i \in I} |\lambda_i x_i|) = 1,$$

This show us that extending f, that there do not exists a unique F_I satisfying $\|F_I\| = \|f\|$.

3 3

Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

3.1 a

Let $n \geq 1$ be an interger and let $F \colon X \to \mathbb{K}^n$ be a linear map. Since X is a infinite dimensional normed vector space over $\mathbb{K}$ we have there exsist a $V \subsetneq X$ normed vector space over $\mathbb{K}$ with dimension $n+1$, and we have $F|_V$ by advec is not injectiv, hence F is not injective.

3.2 b

Let $n \geq 1$ be an integer and let $f_1, f_2, ..., f_n \in X^*$ and let $F \colon X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), ..., f_n(x)), x \in X$. If F is injective $F(x) = 0$ only if $x = 0$. From problem 3 part a we have that F is not injective, therefore there $\exists x \in X/\{0\}$ so $F(x) = 0$. we now have that $f_j(x) = 0$, for all $j \leq n$

$$\begin{aligned} \exists x \in X/\{0\} \text{ so } F(x) &= 0, \\ f_j(x) = 0 \Rightarrow x &\in \ker\{f_j\}, && \forall j \leq n \\ x &\in \bigcap_{j=1}^{n} \ker(f_j), \end{aligned}$$

This means that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$$

3.3 c

Let $0 \neq x_1, x_2, ..., x_n \in X$ and by theorem 2.7 (b) there exists $f \in X^*$, such that $\|f\| = 1$ and $f_j(x) = \|x\|$. And from problem 3 part a we have that there $\exists \tilde{y}$ so $0 \neq \tilde{y} \in \bigcap_{j=1}^{n} \ker(f_j)$, and so let $y = \frac{\tilde{y}}{\|\tilde{y}\|}$ so $\|y\| = 1$,

$$\begin{aligned} \|f_j(x_j - y)\| &\leq \|f_j\| \|y - x_j\|, && \text{Inequality for linear maps} \\ \|f_j(x) - f_j(y)\| &\leq 1 \|y - x_j\|, && \text{Since } \|f_j\| = 1, \\ \|x\| &\leq \|y - x_j\| && \text{Since } f_j(y) = 0, \end{aligned}$$

3.4 d

Let $S = \{x \in X \colon \|x\| = 1\}$ be the unit sphere, suppose for contradiction that there exists $x_1, x_2, ...x_n$, such that $|x_j|$ and $\cup b(x_j, r_j)$ cover S, where $r \leq \|x\|$. Then by problem 3 part c, we have that there $\exists y$ so $\|x_j - y\| \geq \|x_j\| > r_j$. This means that y is not in any of the closed balls and $\|y\| = 1$ so $y \in S$, hence there is no finite family of closed balls cover the unit sphere.

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3.5 e

S is compact if for all open cover of S there is a finite subcover. Now let $p \in S$ and $B := \{x \in X| : \|x - p\| < \frac{1}{2}\}$ this is a open cover of S. But proven in problem 3 part d, there is no finite subcover of S in B.

All closed unit balls in X has a center in X, let it be c then the open cover of the closed unit ball given by $B_c := \{x \in X|p \in S : \|x + c - p\| < \frac{1}{2}\} \cup \{x \in X : \|x - c\| < \frac{2}{3}\}$. Since $\{x \in X : \|x - c\| < \frac{2}{3}\}$ do not cover the sphere of the closed unit ball, we can use the same arguments, then there is no finite cover. And the unit ball in X is non-compact.

4 4

Let $L_1([0,1], m)$ and $L_3([0,1], m)$ be the Lebesgue spaces, that is

$$L_p(X, \mu) := \left\{ f := x \to \mathbb{K} \text{ measurable} : \|f\|_p := \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} < \infty \right\}$$

For $n \geq 1$, let

$$E_n := \left\{ f \in L_1([0,1], m) : \int_X |f(x)|^3 dm < n \right\}$$

4.1 a

Given $n \geq 1$, if the set $E_n \subset L_1([0,1], m)$ is absorbing, then E_n needs both be a convex set and $\forall 0 \neq x \in X$, there exists $t > 0$ such that $x \in tA$, or equivalently, $t^{-1}x \in A$. We have that $L_3([0,1], m) \subsetneq L_1([0,1], m)$ and $f \in L_1([0,1], m) - L_3([0,1], m) \neq \emptyset$. I will not show that E_n is a convex set, i will just show that there do not $\exists 0 < t < \infty$ so $t^{-1}f \in E_n$. We can asumme that $\int_{[0,1]} |f|^3 \, dm \geq n$ else $f \in L_3([0,1], m)$.

$$\begin{aligned}
\int_{[0,1]} |t^{-1}f|^3 \, dm &= |t^{-3}| \int_{[0,1]} |f|^3 \, dm, \\
|t^{-3}| \int_{[0,1]} |f|^3 \, dm &\geq |t^{-3}| \left(\int_{[0,1]} |f|^3 \right)^{1/3} && \text{use that } f \notin L_3([0,1], m) \\
&= |t^{-3}| \infty = \infty
\end{aligned}$$

Hence there do not exists a t so $\forall x$ so $tx \in E_n$.

4.2 b

Let $f_1 \in L_1([0,1,m]) - L_3([0,1,m])$ and we can let $0 < \|f_1\| < \delta$, depend on delta since $\|\delta f\| = |\delta|\|f\|$. Now i will shot that $0 \in E_n$ is not an interior E_n, this shows that there no interior point since $f_2 = f_1 + f_e \in L_1([0,1],m)$ for $f_e \in E_n$

$$\begin{aligned} \|f_1 - 0\| = \|f_1\| &< \delta, \\ \|f_1\| &< \varepsilon && \text{Let } \delta = \varepsilon \\ \|f_e - f_2\| = \|f_e - f_1 - f_e\| = \|f_1\| &< \varepsilon && \text{Just to show } \forall f_e \in E_n \text{ is true} \end{aligned}$$

Hence E_n has empty interior.

4.3 c

Let $f \in \overline{E_n}$ and $(f_m)_{m\geq 1} \in (E_n)$ be a sequence converging to uniformly to f. Then $\forall \varepsilon$ there exists a M so for all $M < m$, $\|f_m - f\| < \varepsilon$.

$$\begin{aligned} \int_{[0,1]} |f|^3 dm &\leq \int_{[0,1]} (|f - f_m| + |f_m|)^3 dm, && \text{By triangle inequality} \\ &\leq \int_{[0,1]} (\varepsilon + |f_m|)^3 dm && \text{Use that} \|f_m - f\| < \varepsilon \\ &= n + \varepsilon(3 \int_{[0,1]} |f_n|^2 + \varepsilon |f_n| + \varepsilon^2 \, dm), \\ \infty &> (3 \int_{[0,1]} |f_n|^2 + \varepsilon |f_n| + \varepsilon^2 \, dm), && \text{Since } E_n \subseteq L_3([0,1]) \subsetneq L_2([0,1]) \subsetneq L_1([0,1]) \end{aligned}$$

Hence we have that for $\varepsilon \to 0$, that $m \to \infty$ that $\int_{[0,1]} |f_m|^3 \to \int_{[0,1]} |f|^3 \leq n$. This shows that $f \in E_n$ and that E_n is closed.

4.4 d

Let $(E_n)_{n\geq 1}$ be a sequence with

$$E_n := \left\{ f \in L_1([0,1],m) : \int_X |f(x)|^3 dm \leq n \right\}$$

and we have that $L_1([0,1],m)$ is a topological space. From problem 4 part b and c that E_n is closed and with $Int(\overline{E_n}) = \emptyset$, this is by definition 3.12, that E_n is of nowhere dense.

$$
\begin{aligned}
f_3 &\in L_3([0,1],m) \\
\|f_3\|_3 &= c < \infty \\
\|f_3\|_3^3 &= c^3 < \infty \\
f_3 &\in E_{n>c^3}, \\
\cup_{n\geq 1}(E_n) &= L_3([0,1],m)
\end{aligned}
$$

Since we have a sequence $(En)_{n\leq 1}$ of nowhere dense sets such that $\cup_{n\geq 1}(E_n) = L_3([0,1],m)$. We have by Defination 3.12 (ii) that $L_3([0,1],m)$ is of first category of $L_3([0,1],m)$.

5 5

Let H be an infinite dimensional Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

5.1 a

Suppose that $x_n \to x$ in norm, as $n \to \infty$, mening that.

$$\|x_n - x\| \to 0, n \to \infty$$

Then we have that

$$
\begin{aligned}
&\|x + x_n - x_n\|, && \\
\|x_n\| - \|x_n - x\| \leq \|x\| &\leq \|x_n\| + \|x_n - x\|, && \text{Triangle inequality} \\
\|x_n\| - 0 \leq \|x\| &\leq \|x_n\| + 0, && \text{Let } n \to \infty \\
\|x_n\| &\to \|x\|. &&
\end{aligned}
$$

So it does true that $\|x_n\| \to \|x\|$

5.2 b

Let $(e_n)_{m\leq 1}$ be a orthonormal basis in H, by defination of weakly convergens we have that $e_n \to x$ for all $h \in H$,

$$
\begin{aligned}
\langle e_n, h\rangle &\to \langle h, x\rangle \\
\langle h, x\rangle &= 0 = \langle 0, x\rangle \\
e_n &\to 0,
\end{aligned}
$$

but we have that

$$\langle x_n, x_n \rangle = \|x_n\| \to 1 \neq \|0\| = 0$$

5.3 e

Suppose that $\|x_n\| \leq 1$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Since $(x_n)_{n\geq 1}$ is a bound sequence we have that $\|x\|$ is bound by the limit of the sequence. Hence $\|x\| \leq 1$ It is true.