# FunkAn Mandatory 2

Line Lindholm Jensen, zmj832

January 2021

## Problem 1

*Let $H$ be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$.*
*Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.*

### (a)

*Show that $f_N \to 0$ weakly, as $N \to \infty$, while $||f_N|| = 1$, for all $N \geq 1$.*

First notice that $||f_N|| = 1$, since by Pythagoras identity we have

$$||f_N|| = \Big|\Big|N^{-1}\sum_{n=1}^{N^2} e_n\Big|\Big| = N^{-1}\Big|\Big|\sum_{n=1}^{N^2} e_n\Big|\Big| = N^{-1}\left(\sum_{n=1}^{N^2} ||e_n||^2\right)^{1/2} = \frac{1}{N}\sqrt{N^2} = 1.$$

We wish to show that the sequence $(f_N)_{N\geq 1}$ converges weakly to 0, as $n \to \infty$. From Homework 4, problem 2 we know that the sequence $(f_N)_{N\geq 1}$ in $H$ converges to 0 in the weak topology $\tau_\omega$ on $H$ if and only if the net $(g(f_N))_{N\geq 1}$ converges to $g(0)$ when $N \to \infty$, for every $g \in H^*$.
Now since $H$ is a Hilbert space, then by Riesz representation theorem every $g \in H^*$ is on the form $g(y) = \langle y, x\rangle$, for every $x, y \in H$. So we want to show that the inner product $\langle f_N, x\rangle$ converges to $\langle 0, x\rangle = 0$ for all $x \in H$.

First note we can write $x = \sum_{i=1}^{\infty}\langle x, e_i\rangle e_i$, for all $x \in H$. Then by Parcevals identity we have

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 = ||x||^2 < \infty,$$

meaning there exists $M_1 \geq 1$ such that for $\varepsilon > 0$, then

$$\sum_{i=M_1+1}^{\infty} |\langle x, e_i\rangle|^2 < \left(\frac{\varepsilon}{2}\right)^2$$

Now we see that

$$\begin{aligned}
|\langle f_N, x\rangle| &= |\langle f_N, \sum_{i=1}^{\infty}\langle x, e_i\rangle e_i\rangle| \\
&\leq |\langle f_N, \sum_{i=1}^{M}\langle x, e_i\rangle e_i\rangle| + |\langle f_N, \sum_{i=M+1}^{\infty}\langle x, e_i\rangle e_i\rangle| \\
&= |\langle N^{-1}\sum_{n=1}^{N^2} e_n, \sum_{i=1}^{M}\langle x, e_i\rangle e_i\rangle| + |\langle f_N, \sum_{i=M+1}^{\infty}\langle x, e_i\rangle e_i\rangle| \\
&= |N^{-1}\sum_{n=1}^{N^2}\sum_{i=1}^{M}\overline{\langle x, e_i\rangle}\langle e_n, e_i\rangle| + |\langle f_N, \sum_{i=M+1}^{\infty}\langle x, e_i\rangle e_i\rangle| \\
&\leq N^{-1}\sum_{i=1}^{M}|\overline{\langle x, e_i\rangle}|\sum_{n=1}^{N^2}|\langle e_n, e_i\rangle| + ||f_N||\Big|\Big|\sum_{i=M+1}^{\infty}\langle x, e_i\rangle e_i\Big|\Big| \\
&\leq N^{-1}\sum_{i=1}^{M}|\overline{\langle x, e_i\rangle}| + \left(\sum_{i=M+1}^{\infty}|\langle x, e_i\rangle|^2\right)^{1/2}
\end{aligned}$$

where we have used Cauchy-Schwarz , Pythagoras identity and that $||f_N|| = 1$. Now notice that $N^{-1} \to 0$ when $N \to \infty$. Thus for every $\varepsilon > 0$ we can choose $N_1 \geq 1$ large enough such that

$$N^{-1}\sum_{i=1}^{M}|\overline{\langle x, e_i\rangle}| < \frac{\varepsilon}{2}.$$

Then for all $N, M \geq N_1, M_1$ we have that

$$|\langle f_N, x\rangle| \leq N^{-1}\sum_{i=1}^{M}|\overline{\langle x, e_i\rangle}| + \left(\sum_{i=M+1}^{\infty}|\langle x, e_i\rangle|^2\right)^{1/2} < \frac{\varepsilon}{2} + \sqrt{\left(\frac{\varepsilon}{2}\right)^2} = \varepsilon.$$

So in conclusion when $N \to \infty$, then $|\langle f_N, x\rangle| \to 0$, for all $x \in H$. Hence $f_N \to 0$ weakly, as $N \to \infty$.

## (b)

*Let $K$ be the norm closure of $\mathrm{co}\{f_N : N \geq 1\}$. Argue that $K$ is weakly compact, and that $0 \in K$.*

Since the convex hole by definition is a convex subset of $H$, we get by theorem 5.7 that

$$K = \overline{\mathrm{co}\{f_N : N \geq 1\}}^{||\cdot||} = \overline{\mathrm{co}\{f_N : N \geq 1\}}^{\tau_w},$$

so $K$ is the weak closure of $\mathrm{co}\{f_N : N \geq 1\}$. Note that $H$ is a Hilbert space, so it is reflexive by proposition 2.10. Now from Theorem 6.3 we get that it is reflexive if and only if $\overline{B}_H(0,1)$ is compact with respect to the weak topology $\tau_w$ on $H$. So we want to show that $K$ is in fact a subset of the closed unit ball, i.e. $K \subseteq \overline{B}_H(0,1)$, since closed subsets of compact sets are again compact. Recall that we showed in $(a)$ that $f_N$ has unit lenght for all $N \geq 1$, i.e. $||f_N|| = 1$, hence $f_N \in \overline{B}_H(0,1)$, $\forall N \geq 1$. Now take an arbitrary element of the convex hull $\sum_{i=1}^{n}\alpha_i x_i$, where $x_i \in \{f_N : N \geq 1\}$, $\alpha_i > 0$, $\sum_{i=1}^{n}\alpha_i = 1$, $n \in \mathbb{N}$. Then

$$\Big|\Big|\sum_{i=1}^{n}\alpha_i x_i\Big|\Big| \leq \sum_{i=1}^{n}|\alpha_i|||x_i|| = \sum_{i=1}^{n}|\alpha_i| = 1,$$

so $\mathrm{co}\{f_N : N \geq 1\} \subseteq \overline{B}_H(0,1)$. Hence $K = \overline{\mathrm{co}\{f_N : N \geq 1\}}^{\tau_w} \subseteq \overline{\overline{B}_H(0,1)} = \overline{B}_H(0,1)$. Thus since $\overline{B}_H(0,1)$ is weakly compact and $K$ is a closed subset in the weak topology, then $K$ is also weakly compact.

From $(a)$ $f_N \to 0$ weakly as $n \to \infty$, so 0 is contained in the weakly cloasure, i.e. $0 \in \overline{\{f_N : N \geq 1\}}^{\tau_w}$. By definition the convex hull is the smallest convex set containing $\{f_N : N \geq 1\}$, so clearly we have that $\{f_N : N \geq 1\} \subseteq \mathrm{co}\{f_N : N \geq 1\}$. Hence $0 \in \overline{\{f_N : N \geq 1\}}^{\tau_w} \subseteq \overline{\mathrm{co}\{f_N : N \geq 1\}}^{\tau_w} = K$.

## (c)

*Show that* $0$*, as well as each* $f_N$*,* $N \geq 1$*, are extreme points in* $K$*.*

First we wish to show that $0 \in \mathrm{Ext}(K)$. Suppose that 0 can be written as a convex combination, $0 = \alpha x + (1-\alpha)y$ for $0 < \alpha < 1$ and some $x, y \in K$, where $K$ is the norm closure of $\mathrm{co}\{f_N : N \geq 1\}$. Then we know there exists sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subseteq \mathrm{co}\{f_N : N \geq 1\}$, such that $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ converges, respectively, to $x$ and $y$ in norm, when $n \to \infty$. Now let $e_m$ be an arbitrary element from the orthonormal basis $(e_n)_{n\geq 1}$, then we notice that

$$0 = \langle 0, e_m\rangle = \langle \alpha x + (1-\alpha)y, e_m\rangle = \alpha\langle x, e_m\rangle + (1-\alpha)\langle y, e_m\rangle. \tag{1}$$

We wish to show that $\langle x, e_m\rangle$,$\langle y, e_m\rangle \geq 0$, since this implies $\langle x, e_m\rangle = \langle y, e_m\rangle = 0$, and we would be done. So consider $\langle x, e_m\rangle$, then

$$\langle x, e_m\rangle = \langle \lim_{n\to\infty} x_n, e_m\rangle = \lim_{n\to\infty}\langle x_n, e_m\rangle,$$

by continuity of the inner product. Now note that every element $x_n$ in $\mathrm{co}\{f_N : N \geq 1\}$ is of the form $x_n = \sum_{i=1}^k \alpha_i f_{N_i}$, where $\alpha_i > 0$, and $\sum_{i=1}^k \alpha_i = 1$. Consider $\langle x_n, e_m\rangle$, then

$$\langle x_n, e_m\rangle = \langle \sum_{i=1}^k \alpha_i f_{N_i}, e_m\rangle = \sum_{i=1}^k \alpha_i\langle f_{N_i}, e_m\rangle.$$

Now notice that

$$\langle f_{N_i}, e_m\rangle = \langle N_i^{-1}\sum_{n=1}^{N_i^2} e_n, e_m\rangle = N_i^{-1}\sum_{n=1}^{N_i^2}\langle e_n, e_m\rangle = \begin{cases} 0 & m > N_i^2 \\ N_i^{-1} & m \leq N_i^2 \end{cases}$$

So in particular $\langle f_{N_i}, e_m\rangle \geq 0$, which implies $\langle x_n, e_m\rangle \geq 0$, hence $\langle x, e_m\rangle = \lim_{n\to\infty}\langle x_n, e_m\rangle \geq 0$. So $\langle x, e_m\rangle \geq 0$ and in a similar way we obtain that $\langle y, e_m\rangle \geq 0$. Hence from (1) we have that

$$0 = \alpha\langle x, e_m\rangle + (1-\alpha)\langle y, e_m\rangle$$

and since $\alpha$, $(1-\alpha) > 0$ and both $\langle x, e_m\rangle, \langle y, e_m\rangle \geq 0$, we must have $\langle x, e_m\rangle = \langle y, e_m\rangle = 0$, hence $x = y = 0$ by completeness (5.27 Folland). So 0 is not a proper convex combination of two other points in $K$, i.e. 0 is an extreme point in $K$.

Now we will show that $f_N \in \mathrm{Ext}(K)$ for every $N \geq 1$. Fix $N$ and suppose that $f_N$ can be written as a convex combination, $f_N = \alpha x + (1-\alpha)y$ for $0 < \alpha < 1$ and some $x, y \in K$. Note that $||x|| \leq 1$ since $x \in K \subseteq \overline{B_H(0,1)}$, as we showed in $(b)$. Hence $|\langle x, f_N\rangle| \leq ||x|| \leq 1$, by Cauchy-Schwarz. In the same way we obtain $||y|| \leq 1$ and $|\langle y, f_N\rangle| \leq 1$.
Then by the triangle inequality an linearity of the inner product,

$$\begin{aligned} 1 = |\langle f_N, f_N\rangle| &= |\langle \alpha x + (1-\alpha)y, f_N\rangle| \\ &\leq \alpha|\langle x, f_N\rangle| + (1-\alpha)|\langle y, f_N\rangle| \end{aligned}$$

implies $|\langle x, f_N\rangle| = |\langle y, f_N\rangle| = 1$. Now note since $||f_N|| = 1$ and $||x|| \leq 1$, then

$$||x||||f_N|| = ||x|| \leq |\langle x, f_N\rangle|.$$

Hence we have equality in Cauchy-Schwarz, thus

$$||x||||f_N|| = |\langle x, f_N\rangle| = 1,$$

so $1 = ||x|| = ||f_N||$.
Now by Lemma 26.3 in Schilling, equality in Cauchy-Schwarz implies there exists some $\lambda > 0$ such that $f_N = \lambda x$. But then $||f_N|| = |\lambda|||x||$, so $\lambda = 1$, hence in particular $f_N = x$. In a similar way we get that $f_N = y$. Thus $f_N = x = y$, so we conclude that $f_N$ for every $N \geq 1$ are extreme points in $K$.

### (d)

*Are there any other extreme points in $K$?*

Denote $F = \{f_N : N \geq 1\} \subseteq K$, such that $K = \overline{\text{co}(F)}^{\tau_w}$.
Notice that from $(c)$ we know $F \cup \{0\} \subseteq \text{Ext}(K)$, so we will show that $\text{Ext}(K) \subseteq F \cup \{0\}$.

We claim that the closure of a convex set $X$ is convex. Suppose $X$ is convex, then for all $x, y \in \overline{X}$, there exists sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subseteq X$, such that $x_n \to x$ and $y_n \to y$, when $n \to \infty$. Then since $X$ is convex, $\alpha x_n + (1-\alpha)y_n \in X$, for all $n \geq 1$ and every $0 < \alpha < 1$. Now by linearity of limits

$$\alpha x_n + (1-\alpha)y_n \to \alpha x + (1-\alpha)y,$$

when $n \to \infty$, which implies $\alpha x + (1-\alpha)y \in \overline{X}$. So $\overline{X}$ is convex, which proofs the claim.

Notice from the above and from $(b)$ that $K$ is a non-empty, convex, weakly compact subset of $H$ and $K = \overline{\text{co}(F)}^{\tau_w}$. Further $(H, \tau_w)$ is a Hilbert space with the weak topology, so it is $LCTVS$.
Hence Theorem 7.9 yields $\text{Ext}(K) \subseteq \overline{F}^{\tau_w}$. Now we claim that $\overline{F}^{\tau_w}$ is equal to $F \cup \{0\}$. Notice that $\overline{F}^{\tau_w}$ is the union of $F$ and all the weak limit points in $F$. By $(a)$ $f_N \to 0$ weakly as $N \to \infty$, so since the weak topology is hausdorff and since there are infinitely many elements in $F$ that lies close to 0, then 0 must be the only limit of $(f_N)_{N\geq 1}$. So in particular 0 is the only accumulation point of $(f_N)_{N\geq 1}$, hence every weakly convergent sequence in $F$ must either converge to 0 or an element in $F$. Hence $\text{Ext}(K) \subseteq \overline{F}^{\tau_w} = F \cup \{0\}$.

In conclusion there are no other extreme points in $K$, besides 0 and $f_N, \forall N \geq 1$.

## Problem 2

*Let $X$ and $Y$ be infinite dimensional Banach spaces.*

### (a)

*Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in $X$ and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.*

Suppose $T \in \mathcal{L}(X, Y)$ and $(x_n)_{n\geq 1}$ is a sequence in $X$, where $x \in X$, such that $x_n \to x$ weakly, as $n \to \infty$.

From Homework 4, problem 2 we know that $x_n \to x$ in the weak topology $\tau_\omega$ if and only if $f(x_n)$ converges weakly to $f(x)$ when $n \to \infty$, for every $f \in X^*$. Notice that $y \circ T \in X^*$, for every $y \in Y^*$. Then we see that

$$y(Tx_n) = y \circ T(x_n) \to y \circ T(x) = y(Tx)$$

weakly, as $n \to \infty$. Hence $Tx_n \to Tx$ weakly when $n \to \infty$ by Homework 4, problem 2.

## (b)

*Let $T \in \mathcal{K}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in $X$ and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $||Tx_n - Tx|| \to 0$, as $n \to \infty$.*

Suppose $T \in \mathcal{K}(X,Y)$ and $(x_n)_{n\geq 1}$ is a sequence in $X$ for $x \in X$ such that $x_n \to x$ weakly, as $n \to \infty$. By Homework 4, problem 2 then $\sup\{||x_n|| : n \geq 1\} < \infty$, so $(x_n)_{n\geq 1}$ is bounded, and specially every subsequence $(x_{n_k})_{k\geq 1}$ is bounded. By proposition 8.2 since $T$ is compact and $(x_{n_k})_{k\geq 1}$ is bounded, then there exists a subsequence $(x_{n_{k_j}})_{j\geq 1}$ such that $||Tx_{n_{k_j}} - y|| \to 0$ as $j \to \infty$, for some $y \in Y$. Since norm convergence implies weak convergence, we have that $Tx_{n_{k_j}} \to y$ weakly when $j \to \infty$.
Notice by $(a)$ since $T \in \mathcal{K}(X,Y)$ implies $T \in \mathcal{L}(X,Y)$, then $Tx_n \to Tx$ weakly as $n \to \infty$. But then we must also have that $Tx_{n_{k_j}} \to Tx$ weakly when $j \to \infty$, for every (sub)subsequence $(x_{n_{k_j}})_{j\geq 1}$.
Now since the weak topology $\tau_w$ is Hausdorff, then by uniqueness of limits $y = Tx$, so $||Tx_{n_{k_j}} - Tx|| \to 0$ as $j \to \infty$. Hence every subsequence $(Tx_{n_k})_{k\geq 1}$ of $(Tx_n)_{n\geq 1}$ has a convergent subsequence $(Tx_{n_{k_j}})_{j\geq 1}$ such that $||Tx_{n_{k_j}} - Tx|| \to 0$ when $j \to \infty$.
Now assume for contradiction that $||Tx_n - Tx|| \not\to 0$, when $n \to \infty$. Then there exists $\varepsilon > 0$ such that for every $K \in \mathbb{N}$ we can choose some $n_K > K$ such that $||Tx_{n_K} - Tx|| \geq \varepsilon$. Now for every $K$ choose the smallest $n_K$ such that this is satisfied, then we can construct a sequence $(x_{n_K})_{K\geq 1}$, such that $||Tx_{n_K} - Tx|| \geq \varepsilon$ for all $K \geq 1$. Note that $(x_{n_K})_{K\geq 1}$ is a subsequence of $(x_n)_{n\geq 1}$, so $(Tx_{n_K})_{K\geq 1}$ is in particular a subsequence of $(Tx_n)_{n\geq 1}$, hence $(Tx_{n_K})_{K\geq 1}$ contains a convergent subsequence, but this is a contradiction since $||Tx_{n_K} - Tx|| \geq \varepsilon$ for all $K \geq 1$. Hence, $||Tx_n - Tx|| \to 0$, as $n \to \infty$.

## (c)

*Let $H$ be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H,Y)$ satisfies that $||Tx_n - Tx|| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in $H$ converging weakly to $x \in H$, then $T \in \mathcal{K}(H,Y)$.*

Suppose for contraposition that $T$ is not compact. By proposition 8.2 then $T(\overline{B_H(0,1)})$ is not totally bounded, meaning there exists $\delta > 0$ such that $\overline{B_H(0,1)}$ is not contained in a finite union of open balls with radius $\delta$. We wish to construct a sequence $(x_n)_{n\geq 1}$ in $\overline{B_H(0,1)}$ recursively. First let $x_1 \in \overline{B_H(0,1)}$. Then we may assume for some $n \in \mathbb{N}$, that we have found $x_2, \dots, x_n$ such that $||Tx_j - Tx_k|| \geq \delta$, for all $j \neq k \leq n$. Then define the set $S = T(\overline{B_H(0,1)}) \setminus (\cup_{i=1}^n B_Y(Tx_i, \delta))$, for $x_i \in \overline{B_H(0,1)}$. Notice that $S$ is non-empty, since otherwise $T(\overline{B_H(0,1)}) \subseteq \cup_{i=1}^n B_Y(Tx_i, \delta)$, but this contradicts $T$ not being totally bounded. Now choose $x_{n+1} \in \overline{B_H(0,1)}$ such that $Tx_{n+1} \in S$. Then $Tx_{n+1} \in (\cup_{i=1}^n B_Y(Tx_i, \delta))^{\complement} = \cap_{i=1}^n (B_Y(Tx_i, \delta))^{\complement}$, so $Tx_{n+1} \notin B_Y(Tx_i, \delta)$ for any $i \leq n$. Thus $x_{n+1}$ satisfies that $||Tx_{n+1} - Tx_i|| \geq \delta$ for every $i \leq n$. In this way we can obtain a sequence $(x_n)_{n\geq 1}$ in $\overline{B_H(0,1)}$ satisfying $||Tx_n - Tx_m|| \geq \delta$ for all $n \neq m$.

Now since $H$ is a Hilbert space, then $H$ is reflexive by proposition 2.10 and by Theorem 6.3 it follows that $\overline{B_H(0,1)}$ is compact with respect to the weak topology. Note that since isometries preserves metric properties and $y \mapsto F_y$ is an isometry for $y \in H, F_y \in H^*$, then $H$ and $H^*$ has similar metric properties. Hence $H$ being seperable implies that $H^*$ is seperable. Now by theorem 5.13 then $(\overline{B_H(0,1)}, \tau_w)$ is metrizable, since there is an isometric isomorphism between $H$ and its double dual $H^*$. Notice that

$(\overline{B_H(0,1)}, \tau_w)$ is compact if and only if its sequentially compact, so $(\overline{B_H(0,1)}, \tau_w)$ is weakly sequentially compact, meaning every sequence in $\overline{B_H(0,1)}$ has a subsequence that converges in $\overline{B_H(0,1)}$. Thus we have a subsequence $(x_{n_k})_{k\geq 1} \subseteq (x_n)_{n\geq 1} \subseteq \overline{B_H(0,1)}$ such that $||Tx_{n_k} - Tx|| \to 0$ as $k \to \infty$. But $||Tx_n - Tx_m|| \geq \delta$ for every $n \neq m$, so $||Tx_{n_k} - Tx|| \nrightarrow 0$ as $k \to \infty$. Hence we reach a contradiction, so if $x_n \to x$ as $n \to \infty$ implies $||Tx_n - Tx|| \to 0$ as $n \to \infty$, then $T$ is compact.

## (d)

*Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.*

Let $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ and let $(x_n)_{n\geq 1}$ be a sequence in $\ell_2(\mathbb{N})$ such that $x_n \to x$ weakly when $n \to \infty$. Then by $(a)$ we have $Tx_n \to Tx$ weakly, as $n \to \infty$. Now by remark 5.3 weakly convergence coincides with norm convergence in $\ell_1(\mathbb{N})$, hence $||Tx_n - Tx|| \to 0$, as $n \to \infty$. So we obtain that $x_n \to x$ weakly as $n \to \infty$ implies $Tx_n \to Tx$ in norm, as $n \to \infty$. Hence by $(c)$ $T$ is compact.

## (e)

*Show that no $T \in \mathcal{K}(X, Y)$ is onto.*

Suppose to reach a contradition that $T$ is onto, i.e. surjective. By the open mapping theorem (3.15), then $T$ is open. Now by the note on page 18 we know this means there exists some $r > 0$, such that $B_Y(0,r) \subset T(B_X(0,1))$. Thus in particular $\overline{B_Y(0,r)} \subset \overline{T(B_X(0,1))}$. Now we know that $\overline{T(B_X(0,1))}$ is compact, by the definition of $T$ being compact. So since closed subset of compact sets are again compact, then in particular $\overline{B_Y(0,r)}$ is compact.

We claim that it is possible to scale the unit ball by some constant $r > 0$, such that $r\overline{B(0,1)} = \overline{B(0,r)}$, for all $r > 0$. Let $x \in r\overline{B(0,1)}$, then there exists $x' \in \overline{B(0,1)}$ such that $x = rx'$. Thus $||x|| = r||x'|| \leq r$, so $x \in \overline{B(0,r)}$. For the other way, take $x \in \overline{B(0,r)}$, then $x = \frac{x}{r}r$, so $||\frac{x}{r}|| = \frac{||x||}{r} \leq 1$, since $||x|| \leq r$. Hence $\frac{x}{r} \in \overline{B(0,1)}$, so $x \in r\overline{B(0,1)}$. This proves the claim.

Now for all $r > 0$ we consider the function $f : Y \to Y$, given by $x \mapsto \frac{1}{r}x$, which is definitely continuous. Since $B_Y(0,r)$ is compact we get that $f(\overline{B_Y(0,r)}) = \frac{1}{r}\overline{B_Y(0,r)} = \overline{B_Y(0,1)}$ is compact, since the image of a compact set under a continuous function is compact. Now from Mandatory assignment 1, problem $3(e)$ we know that $\overline{B_Y(0,1)}$ is non-compact, hence we reach a contradiction.

In conclusion no $T \in \mathcal{K}(X, Y)$ is onto.

## (f)

*Let $H = L_2([0,1], m)$, and consider the operator $M \in L(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. Justify that $M$ is self-adjoint, but not compact.*

First note that $t = \bar{t}$, since $t \in [0,1]$. Hence for $f, g \in H$ we have

$$\langle Mf(t), g(t)\rangle = \langle tf(t), g(t)\rangle = \langle f(t), \bar{t}g(t)\rangle = \langle f(t), tg(t)\rangle = \langle f(t), Mg(t)\rangle$$

Thus $M$ is self-adjoint, i.e. $M = M^*$.

Now assume to reach a contradiction that $T$ is compact. Note that $H = L_2([0,1], m)$ is an infinite dimensional Hilbert space, which is indeed separable by Homework 4, problem 4. Since $M$ is a self-adjoint, compact operator on $H$, then the spectral theorem for self-adjoint compact operators yields $H$ has an orthonormal basis consisting of eigenvectors of M with corresponding eigenvalues. But by Homework 6, problem 3 then $M$ has no eigenvectors, so there's a contradiction. Hence $M$ is not compact.

## Problem 3

*Consider the Hilbert space $H = L_2([0,1], m)$, where $m$ is the lebesgue measure. Define $K : [0,1]\times[0,1] \to \mathbb{R}$ by*

$$K(s,t) = \begin{cases} (1-s)t, & if \ \ 0 \le t \le s \le 1 \\ (1-t)s, & if \ \ 0 \le s < t \le 1 \end{cases}$$

*and consider $T \in \mathcal{L}(H,H)$ defined by*

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t), \quad s \in [0,1], \quad f \in H.$$

### (a)

*Justify that $T$ is compact.*

Notice that $[0,1]$ is a compact Hausdorff topological space, since it is a closed and bounded interval in $\mathbb{R}$. Furthermore since $m([0,1]) = 1 < \infty$, then $m$ is a finite Borel measure on $[0,1]$. $K$ is clearly a continuous function, since $(1-s)t$ is continuous for $0 \le t \le s \le 1$, and $(1-t)s$ is continuous for $0 \le s \le t \le 1$. Notice for $(s,t) \in [0,1]\times[0,1]$, then $|K(s,t)| \le 1$ and we have

what about $s \to t$ ? Piecewise cont. does not imply cont.

$$\int_{[0,1]\times[0,1]} |K(s,t)| d(m\otimes m)(s,t) \le \int_{[0,1]}\int_{[0,1]} 1\cdot dm(t)dm(s) = 1 < \infty,$$

$L_1$ ?

where we have used Tonellis Theorem. Hence $K \in L_2([0,1]\times[0,1], m\otimes m)$. Then $T$ is the associated operator $T_k$ as we defined on page 46. Hence $T$ is compact by Theorem 9.6.

Actually $T = T_{\tilde{k}}$, $\tilde{k}(s,t) = k(t,s)$, so this requires that you show $k(s,t) = k(t,s)$

### (b)

*Show that $T = T^*$.*

First we will argue that we can use Fubinis theorem on the integral:

$$\int_{[0,1]\times[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s), \quad f,g \in H.$$

Note that $|K(s,t)| \le 1$. Then by using Tonellis theorem and the fact that $f,g \in L_2([0,1],m)$ implies $f,g \in L_1([0,1],m)$ as we showed in Homework 2, we get that

$$\begin{aligned}\int_{[0,1]\times[0,1]} \left|K(s,t)f(t)\overline{g(s)}\right| dm(t)dm(s) &= \int_{[0,1]}\int_{[0,1]} |K(s,t)|\,|f(t)||\overline{g(s)}|dm(t)dm(s) \\ &\le \int_{[0,1]}\int_{[0,1]} |f(t)||\overline{g(s)}|dm(t)dm(s) \\ &= \int_{[0,1]} |f(t)|dm(t) \int_{[0,1]} |\overline{g(s)}|dm(s) < \infty,\end{aligned}$$

since $f, g \in L_1([0,1], m)$.
Now let $f, g \in H$. We wish to show that $\langle Tf, g\rangle = \langle f, Tg\rangle$. We see that

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} Tf(s)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t)\int_{[0,1]} \overline{K(s,t)}\,\overline{g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t)\int_{[0,1]} \overline{K(t,s)g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t)\overline{Tg(t)}dm(t) \\
&= \langle f, Tg\rangle,
\end{aligned}$$

where we have used that since $K$ is real we have $K(s,t) = \overline{K(s,t)}$ and from the definition of $K$ we see that $K(s,t) = K(t,s)$. ✓

## (c)

*Show that*

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t), \qquad s \in [0,1], \quad f \in H.$$

*Use this to show that $Tf$ is continuous on $[0,1]$, and that $(Tf)(0) = (Tf)(1) = 0$.*

Notice that by linearity of integrals we obtain the wanted equation,

$$\begin{aligned}
(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t).
\end{aligned}$$

Now to show that $Tf$ is continuous we will show by proposition 1.10 that there exists a constant $c > 0$ such that $||Tf|| \leq c\,||f||$, for all $f \in H$. Note that $|t| \leq 1$ and $|s| \leq 1$, so we can bound these. In particular

$|1-s| \leq 1$ and $|1-t| \leq 1$, thus we have

$$\begin{aligned}
||Tf|| = |Tf(s)| &= \left|(1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t)\right| \\
&\leq |1-s|\int_{[0,s]}|t||f(t)|dm(t) + |s|\int_{[s,1]}|(1-t)||f(t)|dm(t) \\
&\leq \int_{[0,s]}|f(t)|dm(t) + \int_{[s,1]}|f(t)|dm(t) \\
&= \int_{[0,1]}|f(t)|dm(t) < \infty,
\end{aligned}$$

by linearity of integrals and since $f \in L_2([0,1],m) \subset L_1([0,1],m)$ (HW.2). Thus there exists some constant $c > 0$ such that $||Tf|| \leq c$, so specially $||Tf|| \leq c||f||$.

NO

$C = ||f||_1$.

This shows T cont

Not Tf.

Note that integrals over the areas $[0,0]$ and $[1,1]$ gives 0, hence

$$\begin{aligned}
(Tf)(0) &= \int_{[0,0]} tf(t)dm(t) + 0 \cdot \int_{[0,1]}(1-t)f(t)dm(t) = 0 \\
(Tf)(1) &= 0 \cdot \int_{[0,1]} tf(t)dm(t) + \int_{[1,1]}(1-t)f(t)dm(t) = 0
\end{aligned}$$

Thus $(Tf)(0) = (Tf)(1) = 0$.

## Problem 4

*Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.*

### (a)

*For each integer $k \geq 0$, set $g_k(x) = x^k e^{-\frac{1}{2}x^2}$, for $x \in \mathbb{R}$. Justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$. Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.*

Notice that $g_k(x) \in C^\infty(\mathbb{R}), \forall x \geq 0$. First we will show that $e^{-\frac{1}{2}x^2} \in \mathscr{S}(\mathbb{R})$.

? $g_k \in C^\infty(\mathbb{R})$ $\forall k \geq 0$ ?

We claim that $\partial^\alpha e^{-\frac{1}{2}x^2} = \mathrm{Pol}_\alpha(x) \cdot e^{-\frac{1}{2}x^2}$, for $x \in \mathbb{R}$, where $\mathrm{Pol}_\alpha(x)$ is some polynomium depending on $\alpha \in \mathbb{N}$. We proof the claim by induction.
When $\alpha = 0$, then

$$\partial^0 e^{-\frac{1}{2}x^2} = e^{-\frac{1}{2}x^2},$$

so the base case holds. Assume the claim holds up to $\alpha = n-1$, then we have that

$$\partial^n e^{-\frac{1}{2}x^2} = \partial\,\mathrm{Pol}_{n-1}(x)e^{-\frac{1}{2}x^2} = \mathrm{Pol}_{\underset{n-2\,?}{n}}(x)e^{-\frac{1}{2}x^2} + \mathrm{Pol}_1(x)\mathrm{Pol}_{n-1}(x)e^{-\frac{1}{2}x^2} = \mathrm{Pol}_n(x)e^{-\frac{1}{2}x^2}.$$

So the claims holds for all $\alpha \in \mathbb{N}$.

Now notice that $\partial^\alpha e^{-\frac{1}{2}x^2} = \mathrm{Pol}_\alpha(x)e^{-\frac{1}{2}x^2} \to 0$ when $|x| \to \infty$, since $e^{-\frac{1}{2}x^2}$ goes to zero faster than any polynomium grows. Further $x^\beta \partial^\alpha e^{-\frac{1}{2}x^2} \to 0$ when $|x| \to \infty$, for all non-negative integers $\alpha, \beta$, since we simply multiply with a polynomium. Hence $e^{-x^2/2} \in \mathscr{S}(\mathbb{R})$. Now by Homework 7, exercise 1 we get that $x^k e^{-x^2/2} \in \mathscr{S}(\mathbb{R})$, for all $k \geq 0$.

Next we will compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$. From proposition 11.4 we get that $\mathcal{F}(g_0) = \hat{g}_0(\xi) = e^{-\frac{1}{2}\xi^2}$, for $\xi \in \mathbb{R}$. Now notice that $e^{-\frac{1}{2}x^2} \in L_1(\mathbb{R})$ and $g_k(x) = x^k e^{-\frac{1}{2}x^2} \in L_1(\mathbb{R})$, since $\mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$. Then by proposition 11.13 we have that $\partial^k \hat{g}_0$ exists and we can compute $\mathcal{F}(g_k)$, for $k = 1, 2, 3$:

$$
\begin{aligned}
\mathcal{F}(g_1) &= \hat{g}_1(\xi) = i(\partial \hat{g}_0)(\xi) = -i\xi e^{-\frac{1}{2}\xi^2} \\
\mathcal{F}(g_2) &= \hat{g}_2(\xi) = i^2(\partial^2 \hat{g}_0)(\xi) = i(\partial \hat{g}_1)(\xi) = (1 - \xi^2)e^{-\frac{1}{2}\xi^2} \\
\mathcal{F}(g_3) &= \hat{g}_3(\xi) = i^3(\partial^3 \hat{g}_0)(\xi) = i(\partial \hat{g}_2)(\xi) = (i\xi^3 - 3i\xi)e^{-\frac{1}{2}\xi^2}
\end{aligned}
$$

## (b)

*Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.*

Let $h_k$ be of the form $h_k = \alpha_0 g_0 + \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3$. Note that $h_k \in \mathscr{S}(\mathbb{R})$, since linear combinations of Schwartz functions are again Schwartz functions, by the continuity of limits.

Then since the Fourier transformation is linear we have that

$$
\begin{aligned}
\mathcal{F}(h_k) &= \alpha_0 \hat{g}_0 + \alpha_1 \hat{g}_1 + \alpha_2 \hat{g}_2 + \alpha_3 \hat{g}_3 \\
&= e^{-\frac{1}{2}x^2}\left(\alpha_0 - ix\alpha_1 + (1 - x^2)\alpha_2 + (ix^3 - 3ix)\alpha_3\right).
\end{aligned}
$$

Further we see that

$$i^k h_k = e^{-\frac{1}{2}x^2}(i^k\alpha_0 + i^k\alpha_1 x + i^k\alpha_2 x^2 + i^k\alpha_3 x^3).$$

Now solving $\mathcal{F}(h_k) = i^k h_k$ for $k = 1$ we have

$$\alpha_0 - ix\alpha_1 + (1 - x^2)\alpha_2 + (ix^3 - 3ix)\alpha_3 = i^k\alpha_0 + i^k\alpha_1 x + i^k\alpha_2 x^2 + i^k\alpha_3 x^3,$$

hence

$$(1 - i)\alpha_0 - 2ix\alpha_1 + (1 - x^2 - ix^2)\alpha_2 - 3ix\alpha_3 = 0.$$

Now choose $\alpha_0 = \alpha_2 = 0$, $\alpha_1 = 3$ and $\alpha_3 = -2$ such that the above is satisfied.

For $k = 2$ we solve

$$2\alpha_0 + (x - ix)\alpha_1 + \alpha_2 + (ix^3 - 3ix + x^3)\alpha_3 = 0$$

Then choose $\alpha_1 = \alpha_3 = 0$, $\alpha_0 = 1$ and $\alpha_2 = -2$.

For $k = 3$ we have

$$(1 + i)\alpha_0 + (1 - x^2 + ix^2)\alpha_2 + (-x^3 - 3ix)\alpha_3 = 0$$

Then choose $\alpha_0 = \alpha_2 = \alpha_3 = 0$ and $\alpha_1 = 1$.

In conclusion we have found the following non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$

$$
\begin{aligned}
h_0 &= g_0 \\
h_1 &= 3g_1 - 2g_3 \\
h_2 &= g_0 - 2g_2 \\
h_3 &= g_1
\end{aligned}
$$

### (c)

*Show that* $\mathcal{F}^4(f) = f$*, for all* $f \in \mathscr{S}(\mathbb{R})$

Consider the Fourier transform of a function $f \in \mathscr{S}(\mathbb{R}) \subseteq L_1(\mathbb{R})$,

$$\mathcal{F}(f) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dm(x).$$

Now since $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$, then $\hat{f} \in L_1(\mathbb{R})$, so we can find the fourier transformation of $\hat{f}$, thus

$$\mathcal{F}\big(\mathcal{F}(f)\big) = \hat{\hat{f}}(\xi) = \int_{\mathbb{R}} \hat{f}(x)e^{-i\xi x}dm(x)$$

Again since $\hat{\hat{f}}(\xi) \in L_1(\mathbb{R})$, we may note that $\mathcal{F}\big(\mathcal{F}(f)\big) = \hat{\hat{f}}(\xi) = \check{\hat{f}}(-\xi)$. Now since $f \in \mathscr{S}(\mathbb{R})$, then $f$ is in particular $C^\infty$, so in fact continuous. Then by Theorem 12.11 we have that $\check{\hat{f}}(-\xi) = f(-\xi)$. Thus $\mathcal{F}^2(f(\xi)) = f(-\xi)$. So we get that

$$\mathcal{F}^4(f(\xi)) = \mathcal{F}^2(\mathcal{F}^2(f(\xi))) = \mathcal{F}^2(f(-\xi)) = f(\xi),$$

hence we obtain that $\mathcal{F}^4(f) = f$.

### (d)

*Use (c) to show that if* $f \in \mathscr{S}(\mathbb{R})$ *is non-zero and* $\mathcal{F}(f) = \lambda f$*, for some* $\lambda \in \mathbb{C}$*, then* $\lambda \in \{1, i, -1, -i\}$*. Conclude that the eigenvalues of* $\mathcal{F}$ *precisely are* $\{1, i, -1, -i\}$*.*

Assume $\mathcal{F}(f) = \lambda f$, for $f \in \mathscr{S}(\mathbb{R})$ non-zero and some $\lambda \in \mathbb{C}$. Notice since the Fourier transformation is linear we can always pull constants out. Hence using $(c)$ we get that

$$f = \mathcal{F}^4(f) = \mathcal{F}^3(\mathcal{F}(f)) = \mathcal{F}^3(\lambda f) = \lambda\mathcal{F}^3(f) = \lambda^2\mathcal{F}^2(f) = \lambda^3\mathcal{F}(f) = \lambda^4 f$$

Hence $\lambda^4 = 1$, and $\lambda - 1$ has 4 roots for $\lambda \in \mathbb{C}$, so there are the following possibilities $\lambda \in \{1, -1, i, -i\}$.

Since the eigenvalues $\lambda$ of $\mathcal{F}$ must satisfy $\mathcal{F}(f) = \lambda f$, it is clear from the above that $\{1, -1, i, -i\}$ are the only possible eigenvalues. Now note from $(b)$ that $\{1, -1, i, -i\}$ are in fact eigenvalues of $\mathcal{F}$, since $\{i^k : k = 1, 2, 3, 4\} = \{1, -1, i, -i\}$. Hence the eigenvalues of $\mathcal{F}$ are precisely $\{1, i, -1, -i\}$.

## Problem 5

*Let* $(x_n)_{n\geq 1}$ *be a dense subset of* $[0, 1]$ *and consider the Radon measure* $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ *on* $[0, 1]$*. Show that* $\operatorname{supp}(\mu) = [0, 1]$*.*

Note that $(x_n)_{n\geq 1}$ being dense in $[0, 1]$ means $\overline{\{x_n\}_{n\geq 1}} = [0, 1]$, or equivalent that all non-empty open sets in $[0, 1]$ will intersect the set $\{x_n\}_{n\geq 1}$, where $\{x_n\}_{n\geq 1}$ is the set consisting of elements from the sequence $(x_n)_{n\geq 1}$.
Let $N$ be the union of all open subsets $U \subseteq [0, 1]$ that satisfies $\mu(U) = 0$. Note that since all open sets $U \subseteq [0, 1]$ contains at least one element from $(x_n)_{n\geq 1}$, then as $\delta_{x_n} = 1$ when $x_n \in U$, we must have $\mu(U) > 0$ for every $U \subseteq [0, 1]$. Hence $N$ must be the empty set.
Now since $[0, 1]$ is indeed a locally compact Hausdorff topological space we get from Homework 8, problem $3(a)$ that the support of $\mu$ is the complement to $N$. So since $N = \emptyset$ we get $\operatorname{supp}(\mu) = [0, 1]$, which is what we wanted.