# Funk.An. Mandatory assignment 1.

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## Problem 1 [24 points]

Let (X, $\|\cdot\|_X$) and (Y, $\|\cdot\|_Y$) be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

a) [5 p]. Let T: X → Y be a linear map. Set $\| x \|_0 = \| x \|_X + \| Tx \|_Y$, for all $x \in X$. Show that $\|\cdot\|_0$ is a norm on X. Show next that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded.

b) [4 p]. Show that any linear map T: X → Y is bounded, if X is finite dimensional.

c) [5 p]. Suppose that X is infinite dimensional. Show that there exists a linear map T: X → Y , which is not bounded (= not continuous). [Hint: Take a Hamel basis for X (see below).]

d) [5 p]. Suppose again that X is infinite dimensional. Argue that there exists a norm $\|\cdot\|_0$ on X, which is not equivalent to the given norm $\|\cdot\|_X$, and which satisfies $\| x \|_X \le \| x \|_0$, for all $x \in X$. Conclude that (X,$\|\cdot\|_0$) is not complete if (X,$\|\cdot\|_X$) is a Banach space.

e) [5 p]. Give an example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ satisfying $\| x \|' \le \| x \|$, for all $x \in X$, such that (X, $\|\cdot\|$) is complete, while (X, $\|\cdot\|'$) is not. [Hint: Take (X, $\|\cdot\|$) = ($\ell_1(\mathbb{N})$, $\|\cdot\|_1$) with a suitable choice of $\|\cdot\|'$; or take (X, $\|\cdot\| = (L_2([0,1], m), \|\cdot\|_2)$ with a suitable choice of $\|\cdot\|'$, where m is the Lebesgue measure.]

### Answers

**a)**

Use align-environment, very dense to read.

$[0,\infty)$

We have that, since $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms that $\|\cdot\|_0$: X → $(0,\infty)$ by definition, and then we check the first condition from definition 1.1 of the lecture notes, $\| x + x' \|_0 = \| x + x' \|_X + \| Tx + Tx' \|_Y \le \| x \|_X + \| x' \|_X + \| Tx \|_Y + \| Tx' \|_Y = \| x \|_0 + \| x' \|_0 \ \forall x, x' \in X$, since $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms. ✓ Then we chech the second condition, $\| \alpha x \|_0 = \| \alpha x \|_X + \| T\alpha x \|_Y = \| \alpha x \|_X + \| \alpha Tx \|_Y = |\alpha| \, \| x \|_X + |\alpha| \, \| Tx \|_Y = |\alpha| \, (\| x \|_X + \| Tx \|_Y) = |\alpha| \, \| x \|_0$, since T is a linear map and $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms. ✓
Now we check the third and last condition of the definition, $\| x \|_0 = 0 \Leftrightarrow \| x \|_X + \| Tx \|_Y = 0$, and $\| x \|_X = 0 \Leftrightarrow x = 0$, and $\| Tx \|_Y = 0 \Leftrightarrow Tx = 0 \Leftrightarrow x = 0$, since T is linear and since $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms, so by definition of $\| x \|_0$ we have that $\| x \|_0 = 0 \Leftrightarrow x = 0$. No. (✓)
So $\|\cdot\|_0$ is a norm.

Now we need to show that $\| x \|_0$ and $\| x \|_X$ are equivalent ⇔ T is bounded. First lets assume that $\| x \|_0$ and $\| x \|_X$ are equivalent. This means that $C_0 \| x \|_0 \le \| x \|_X \le C_X \| x \|_0$, for $0 < C_0 \le C_X < \infty$ by definition 1.4 in the lecture notes. So we have that $C_0 \| x \|_X + C_0 \| Tx \|_Y \le \| x \|_X \le C_X \| x \|_0 \implies \| x \|_X + \| Tx \|_Y \le \frac{1}{C_0} \| x \|_x \le \frac{C_X}{C_0} \| x \|_0 \implies \| Tx \|_Y \le \frac{1}{C_0} \le \frac{C_X}{C_0} \| x \|_0 - \| x \|_X \le \frac{C_X}{C_0} \| x \|_0 \ \forall x \in X$, since $0 < C_0 \le C_X < \infty$ and since ✓ the norms are non-zero by assumption. So T is bounded by Proposition 1.10 (3) from the lecture notes. ✓ Then let's assume that T is bounded, this means that there exists $C > 0$ such that $\boxed{\| Tx \|_Y \le C \| x \|_0}$, for all $x \in X$. So since $\| x \|_0 = \| x \|_X + \| Tx \|_Y$ we have that $\| x \|_0 \le \| x \|_X + C \| x \|_0 \implies \| x \|_0 - C \| x \|_0 \le \| x \|_X \implies C_0 \| x \|_0 \le \| x \|_X$ for $C_0 = 1 - C$ for $0 < C \le 1$. And since $\| x \|_X \le \| x \|_0$ by definition of $\| x \|_0$, then we can pick a $0 < C_0 \le C_X < \infty$ such that $C_0 \| x \|_0 \le \| x \|_X \le C_X \| x \|_0$. So $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent.

Something is wrong here?

Not $\|x\|_0$, but $\|x\|_X$

(✓)

**b)**

?

By theorem 1.6 we have that if X is a finite dimensional vector space, then any two norms are equivalent. Which by (a) means that T is bounded. Or more generally, we have that if X is finite dimensional we can find a minimal distance min($\| x - y \|$) = D, $\forall x, y \in X$ Then if we take some point $x_0 \in X$ and a $\epsilon > 0$, we can let $\delta = \frac{D}{2}$. Then if $\| x - x_0 \| < \delta \implies \| Tx - Tx_0 \| < \epsilon$. So T is continuous when X is finite dimensional, which by proporsition 1.10 in the lecture notes means that T is bounded for all linear maps T: X $\rightarrow$ Y, when X is finite dimensional. 0/

**c)**

Lets suppose that X is infinite dimensional. Then by Zorn's lemma X admits a Hamel basis, which means that $(e_i)_{i \in I}$ of elements in X for with the property that for each vector space Y over $\mathbb{K}$, and each family $(y_i)_{i \in I}$ in Y, there exists precisely one linear map $T : X \rightarrow Y$ satisfying $T(e_i) = y_i$ for all $i \in I$, or equivalently that for each $x \in X$, there is a unique family $(\lambda_i)_{i \in I}$ in $\mathbb{K}$ for which the set $\{i \in I : \lambda_i \neq 0\}$ is finite and $x = \sum_{i \in I} \lambda_i e_i$. But $\exists$ bounded linear maps $X \to Y$ as well?

T is not linear

The existence of a linear map is clear from the definition of an algebraic basis, so we only need to show that it has to be not bounded (not continuous). Since X is infinite dimensional we must have that some of the $\lambda_i' s$ for all $i \in I$ has to be zero since we have finitely many $\lambda_i' s$ which are non-zero for $i \in I$ and since the family of $(\lambda_i)_{i \in I}$ are unique, So we can for example look at the function $T : X \rightarrow Y$ defined by $T(x) = \frac{1}{\|0-x\|}$, where we define $T(x) = 0$ for $x = 0$. This map is obviously discontinuous in 0, so T is therefore not bounded. 0/0

**d)**

Since we by problem (a) showed that $\| \cdot \|_0$ was a norm on X so it exists, and that the norms $\| \cdot \|_X$ and $\| \cdot \|_0$ only are equivalent if and only if T was bounded, and by problem (b) we had that any linear map T was bounded if X was finite dimensional and problem (c) tells us that there exists a linear map which is not bounded when X is infinite dimensional. This means that since we can find a linear map T which is not bounded, so not every linear map is bounded when X is infinite dimensional. So since we can find such a linear map T which isn't bounded we have that the two norms can not be eqivalent by problem (a). And by definition of $\| x \|_0$ and the fact that (X, $\| \cdot \|_X$) and (Y, $\| \cdot \|_Y$) are (non-zero) normed vector space over $\mathbb{K}$, we have that $0 \leq \| Tx \|_Y$ for all $x \in X$, so obviously $\| x \|_X \leq \| x \|_0$, for all $x \in X$. ✓

If (X,$\| \cdot \|_X$) is a Banach space, we can find a Cauchy sequence $(x_n)_{n \geq 1}$ with respect to the metric d i.e., $\forall \epsilon > 0 \exists n_\epsilon \geq 1$ such that $\forall m, n \geq n_\epsilon$, $d(x_n, x_m) = \| x_n - x_m \|_X < \epsilon$, then there exists $x \in X$ such that $lim_{n \rightarrow \infty} \| x_n - x \|_X = 0$. And since T is unbounded and the two norms are not equivalent, then we wouldn't be able to find such a limit for a cauchy sequence with respect to the norm $\| \cdot \|_0$ since the limit wouldn't exist. We can for example look at the map I mentioned in problem (c) which was discontinuous at 0. 0/0

Why?

That just means it does not preserve limits, it is not a divergent Cauchy sequence..?

**e)**

Let us look at the vector space in the hint, i.e. the vector space $(X, \| \cdot \|) = (\ell_1(\mathbb{N}), \| \cdot \|_1)$. So I need to find a norm such that $\| x \|_1 \geq \| x \|_n$, where $\| x \|_n$ and $\| x \|_1$ are inequivalent and where $\| x \|_n$ makes the normed vector space $(\ell_1(\mathbb{N}), \| \cdot \|_n)$ not complete. We know that $\| x \|_2 \leq \| x \|_1$, so by taking the two norm $\| \cdot \|_2$ we would get that the normed vector space $(\ell_1(\mathbb{N}), \| \cdot \|_2)$ would not be complete since we could find a cauchy sequence in $\ell_1(\mathbb{N})$ with no limit inside $\ell_1(\mathbb{N})$ with respect to the two normn since the completion of $(\ell_1(\mathbb{N}), \| \cdot \|_2)$ with respect to $\| \cdot \|_2)$ is $\ell_2(\mathbb{N})$, where $\ell_1(\mathbb{N}) \subset \ell_2(\mathbb{N})$. And these two norms are inequivalent with respect to $\ell_1(\mathbb{N})$, since any two p norms are not equivalent on $\ell_1(\mathbb{N})$ for different p. So we have what we wanted Why?

Why?

(✓)

Idea is correct, but not argued sufficiently.

## Problem 2 [20 points]

Let $1 \leq p < \infty$ be fixed, and consider the subspace $M$ of the Banach space $(\ell_p(\mathbb{N}), || \cdot ||_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, \dots) : a, b \in \mathbb{C}\}.$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, 0, \dots) = a + b$, for all $a, b \in \mathbb{C}$.

a) [8 p]. Show that $f$ is bounded on $(M, || \cdot ||_p)$ and compute $|| f ||$. (answer depends on p.)

b) [7 p]. Show that if $1 < p < \infty$, then there is a unique linear functional $F$ on $\ell_p(\mathbb{N})$ extending $f$ and satisfying $|| F || = || f ||$.

c) [5 p]. Show that if $p = 1$, then there are infinitely many linear functional $F$ on $\ell_p(\mathbb{N})$ extending $f$ and satisfying $|| F || = || f ||$.

That's (a wrong version of) continuity, not linearity!

### Answers

**a)**

$f$ is obviously linear, since we can find $| x - x_0 | < \delta$ for $\delta > 0$ such that $| f(x) - f(x_0) | < \epsilon$ for all $\epsilon > 0$, since every x in $\ell_p(\mathbb{N})$ is bounded and so is the sum by definition. So in particular $| (a, b, 0, 0, \dots) - (a_0, b_0, 0, 0, \dots) | < \delta$ for $\delta > 0$ such that $| f(a, b, 0, 0, \dots) - f(a_0, b_0, 0, 0, \dots) | = | a + b - (a_0 + b_0) | = | a - a_0 + b - b_0 | < \epsilon$ for all $\epsilon > 0$ by definition of $f$. So $f$ is bounded on $(M, || \cdot ||_p)$. Then we compute $|| f || = \sup\{|| fx || : || x || \leq 1\}$ $= \inf\{C > 0 : || fx || \leq C || x ||, x \in \ell_p(\mathbb{N})\}$. So we have that $|| (a, b, 0, 0, \dots) ||_p = (| a |^p + | b |^p + | 0 |^p + \dots)^{\frac{1}{p}} = (| a |^p + | b |^p)^{\frac{1}{p}}$, and $|| f(a, b, 0, 0, \dots) ||_p = || a + b ||_p = (| a + b |^p)^{\frac{1}{p}} = | a + b |$. 0/0

Wrong norm

So for p = 1 we have that $|| f || = \inf\{C > 0 : | a + b | \leq C(| a | + | b |), a, b \in \mathbb{C}\}$,
and for 1 < p < $\infty$ we have that $|| f || = \inf\{C > 0 : | a + b | \leq C(| a |^p + | b |^p)^{\frac{1}{p}}, a, b \in \mathbb{C}\}$. Which is..?

**b)**

Since $f$ is bounded and hence continuous we have that $f \in M^* = \mathcal{L}(M, \mathbb{C})$ by definition of $f$, then by corollary 2.6 in the lecture notes we have that there exists $F \in (\ell_p(\mathbb{N}), || \cdot ||_p)^* = \mathcal{L}((\ell_p(\mathbb{N}), || \cdot ||_p)), \mathbb{C})$ such that $F |_M = f$ and $|| F || = || f ||$. So we only need to show the uniqeness of $F$ on $\ell_p(\mathbb{N})$ for 1 < p < $\infty$. By example 2.11 in the lecture notes we have that $L_p(X, \mu)$ is reflexive for $1 < p < \infty$, so the same is the case for $(\ell_p(\mathbb{N}), || \cdot ||_p)$. Where do you use reflexivity of $\ell^p(\mathbb{N})$? One of these should be the dual What is q?

We know that there is an isometric isomorphism between $\ell_p(\mathbb{N})$ and $\ell_q(\mathbb{N})$ for every $1 < p < \infty$ by HW.1 problem 5. And we know that $F$ exists by corollary 2.6 in the lecture notes, so by isometry there exists a $y \in \ell_q(\mathbb{N})$ such that $F(x) = \sum_{n=1}^{\infty} x_n y_n$, for all $x \in \ell_p(\mathbb{N})$. Where $y$ is such that $|| Fx || = || x ||$ and $|| f || = || F ||$ and $F |_M = f$. This does not prove uniqueness. 0/0

**c)**

Since $f$ is bounded and hence continuous we have that $f \in M^* = \mathcal{L}(M, \mathbb{C})$ by definition of $f$, then by corollary 2.6 in the lecture notes we have that there exists $F \in (\ell_p(\mathbb{N}), || \cdot ||_p)^* = \mathcal{L}((\ell_p(\mathbb{N}), || \cdot ||_p), \mathbb{C})$ such that $F |_M = f$ and $|| F || = || f ||$. So we only need to show that there are infinitely many $F$ on $\ell_1(\mathbb{N})$ such that this is the case for p = 1. By example 2.11 in the lecture notes we have that $L_p(X, \mu)$ is not reflexive for p = 1, so the same is the case for $(\ell_p(\mathbb{N}), || \cdot ||_p)$ for p = 1. Needs a *.

Don't understand this?

So for $F$ being the continuous extension on $\ell_1(\mathbb{N})$, i.e. $F \in \ell_1(\mathbb{N}) \cong \ell_\infty(\mathbb{N})$ we have that the duality gives us some $u \in \ell_\infty(\mathbb{N})$ such that $\forall\ x = (x_n) \in \ell_1(\mathbb{N})$ being a sequence, we have that $F_k(x) = \sum_{i=1}^{k} x_i$. These $F_k(x)$ are obviously linear by construction, since $F_k(\alpha x + \beta y) = \sum_{i=1}^{k} \alpha(x_i) + \beta(y_i) = \sum_{i=1}^{k} \alpha(x_i) + \sum_{i=1}^{k} \beta(y_i) = \alpha F_k(x) + \beta F_k(y)$. And since the $x = (x_n) \in \ell_1(\mathbb{N})$ the $F_k$ are extentsion of $f$ with the same norm as $f$. So we have infinitely many extensions $F$ of $f$ in this case. ? (✓)

Correct extensions, but you have not argued that $||F_k|| = ||f||$.

## Problem 3 [25 points]

Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. a) [5 p]. Let n $\geq 1$ be an integer. Show that no linear map $F : X \to \mathbb{K}^n$ is injective.

b) [5 p]. Let n $\geq 1$ be an integer and let $f_1, f_2, \ldots, f_n \in X^*$. Show that $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$. [Hint: Consider the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), x \in X$.]

c) [5 p]. Let $x_1, x_2, \ldots, x_n \in X$. Show that there exists $y \in X$ such that $|| \ y \ || = 1$ and $|| \ y - x_j \ || \geq || \ x_j \ ||$ for all $j = 1, 2, \ldots, n$. [Hint: Use Theorem 2.7 (b) from lectures to get started.]

d) [5 p]. Show that one cannot cover the unit sphere $S = \{x \in X : \ || \ x \ || \ = 1\}$ with a finite family of closed balls in $X$ such that none of the balls contains 0.

e) [5 p]. Show that $S$ is non-compact and deduce further that the closed unit ball in $X$ is non-compact.

### Answers

#### a)

We proof this by contradiction.
Let's suppose that $F : X \to \mathbb{K}^n$ is injective. Then we can take $x_1, \ldots, x_{n+1} \in X$, where $x_1, \ldots, x_{n+1}$ are linear independent in X, since X is infinite dimensional and we have that $F(x_1), \ldots, F(x_{n+1})$ is linear dependent in $\mathbb{K}^{n+1}$, since we have that $n+1$ vectors in a $n+1$ dimensional vector space are linear dependent. 0/0
Then $\exists \ \alpha_1, \ldots, \alpha_{n+1} \in \mathbb{K}^{n+1}$ not all being 0 such that $\sum_{i=1}^{n+1} \alpha_i F(x_i) = \alpha_1 F(x_1) + \cdots + \alpha_{n+1} F(x_{n+1}) = F(\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}) = 0$ by linear dependence and since $F$ is a linear map. Then since $F$ was assumed injective we deduce that $\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0$, but $\alpha_i = 0$ for some $i \in \mathbb{N}$, since $x_1, \ldots, x_{n+1}$ is linear independent, which is a contradiction, so there is no linear map $F : X \to \mathbb{K}^n$ which is injective. all

(✓)

#### b)

Let's look at the opposite of what we want. For $\bigcap_{j=1}^{n} \ker(f_j) = \{0\}$, means that the only $x \in X$ making $f_j(x) = 0$ for $1 \leq j \leq n$ where $j, n \in \mathbb{N}$ would be $x = \{0\}$, by definition of the kernel and intersection. If we look at the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), x \in X$ as in the hint, we get that F is a linear map since it consists of linear maps by definition of the dual space which says that $X^* = \mathcal{L}(X, \mathbb{K})$. So by these facts we actually have that $F$ isn't injective. This means that $f_j$ aren't injective either $\forall \ j$. for all

So lets assume that $f_j(x) = 0 \ \forall \ j$ for $x = 0$ since $f_j$ are linear maps $\forall \ j$, so in particular we have that $F(\{0\}) = (f_1(0), f_2(0), \ldots, f_n(0)) = \{0\}$ then by the non-injectivety we have that $\exists x_i \in X$ such that $f_j(x_i) = 0$ $\forall \ j$ and for some $i$, so in particular we have that $\exists x_i \in X$ such that $F(x_i) = (f_1(x_i), f_2(x_i), \ldots, f_n(x_i)) = \{0\}$. always true
This means that $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$, since there is another point in $X$ where $F(x) = 0$ by the injectivity of $F$. why i?

?

Very messy.

#### c)

We have by Theorem 2.7 in the lecture notes, that if $0 \neq x \in X$, then there exists $f \in X^*$ such that $|| \ f \ || = 1$ and $f(x) = || \ x \ ||$, so since $X$ is infinite dimensional we can find a $0 \neq y \in X$ so we get that $\exists$ $f \in X^* = \mathcal{L}(X, \mathbb{K})$ such that $|| \ f \ || = 1$ and $f(y) = || \ y \ ||$. And by remark 1.11 from the lecture notes we also have that $|| \ f \ || = \sup\{|| \ f(y) \ || \ : \ || \ y \ || \leq 1\}$, which should be equal to 1 when we combine these two. So this means that $\sup\{|| \ || \ y \ || \ || \ : \ || \ y \ || = 1\} = \{|| \ y \ || \ : \ || \ y \ || \leq 1\} = 1$, which means that $|| \ y \ || = 1$. And by the previous results we have that there is finitely many $0 \neq x_j \in X$ for $1 \leq j \leq n$ since we can find a Hamel basis. This means that $|| \ x_j \ || \leq 1$ by theorem 2.7 (b) in the lecture notes. Then we use remark 1.2 from the notes, which gives that, $|| \ y - x_j \ || \geq || \ y \ ||$ - $|| \ x_j \ || \geq 1 - || \ x_j \ || \geq 1 \geq || \ x_j \ ||$. NO. ??

$\not\leq$

Makes no sense

**d)**

By the note below remark 5.3 in the lecture notes we have that $S$ is weakly dense in the closed unit ball $\overline{B_X(0,1)} = \{x \in X : \ || \, x \, || \ \leq 1\}$ of $X$.

$S$ is dense in $\overline{B_X(0,1)}$ in the weak topology means that the closure of $S$ in this particular topology is equal to $\overline{B_X(0,1)}$. This, by basics of point-set topology, means that every point in $\overline{B_X(0,1)}$ is a limit (in the weak topology) of a net of points in $S$.

If we let $B_i$ for $i = 1, \ldots, n$ be closed balls not containing 0, which are closed convex sets, since any closed ball in a normed vector space is convex. In particular $|| \, tx + (1-t)y - x_0 \, || \leq t \, || \, x - x_0 \, || + (1-t) \, || \, y - x_0 \, || \leq r$ for $x, y \in B(x_0, r), 0 \leq t \leq 1$. Hence we can find continuous functionals $\lambda_i$, such that Re $\lambda_i(x) \geq 1$ for $x \in B_i$. The vector space $V = \bigcap_{i=1}^{n} \ker(\lambda_i)$ does not intersect any of the $B_i$, since if $x \in V$, then $\lambda_i(x) = 0$, for all $i$. But $x \in B_i$ implies that Re $\lambda_i(x) \geq 1$. But $V \neq 0$, because X is infinite-dimensional. So we find an $x \in V \cap S$.

yes, but why?

this is good

And in particular we have by subproblem (c) that there exists $y \in B_i$ $\in X$ such that $|| \, y \, || = 1$ and $|| \, y - x_j \, || \geq || \, x_j \, ||$ for all $j = 1, 2, \ldots, n$, where $|| \, y - x_j \, || = 0$ means that $|| \, x_j \, || = 0$ which can only be the case if $x_j = 0$. Therefore, no finite number of closed balls can cover $S$ without one of them containing 0.

**e)**

We have that $S$ is a subset of the closed unit ball $S \subset \overline{B_X(0,1)} = \{x \in X : \ || \, x \, || \ \leq 1\}$ of $X$.

For $S$ being compact means that every infinite subset of $S$ has a complete accumulation point, but since $S$ is dense in $\overline{B_X(0,1)}$ in the weak topology, this cann't be true, so $S$ is non-compact. ?

By Riesz's lemma which says that for $X$ being a normed space and $S$ being a closed proper subspace of $X$ and $a$ be a real number with $0 < a < 1$, then there exists an $x \in X$ with $|| \, x \, || = 1$ such that $|| \, x - y \, || \geq a$ for all $y \in S$. So we have that since $X$ is an infinite dimensional normed vectorspace, the closed unit ball $\overline{B_X(0,1)}$ of $X$ is non-compact, since we can take an element $x_1 \in S$, and pick an element $x_n \in S$ such that $\mathrm{d}(x_n, S_{n-1}) > a$ for a constant $0 < a < 1$ where $S_{n-1}$ is the linear span of $\{x_1, \ldots, x_{n-1}\}$ and $\mathrm{d}(x_n, S) = \inf_{y \in S} | \, x_n - y \, |$. We easily see that $\{x_n\}$ contains no convergent subsequence, since $S$ is non-compact, which means that the closed unit ball in $X$ is non-compact.

Idea is fine, but we have not proven Riez's lemma in this course. Althoug it should have been realized that Problem c) can be used instead.

## Problem 4 [20 points]

Let $L_1([0,1],m)$ and $L_3([0,1],m)$ be the Lebesgue spaces on $[0,1]$. Recall from HW2 that $L_3([0,1],m) \subsetneq L_1([0,1],m)$. For $n \geq 1$, define

$$E_n := \{f \in L_1([0,1],m) : \int_{[0,1]} \mid f \mid^3 dm \leq n\}.$$

a) [5 p]. Given $n \geq 1$, is the set $E_n \subset L_1([0,1],m)$ absorbing? Justify.

b) [5 p]. Show that $E_n$ has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.

c) [7 p]. Show that $E_n$ is closed in $L_1([0,1],m)$, for all $n \geq 1$.

d) [3 p]. Conclude from (b) and (c) that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

### Answers

#### a)

First we check that $E_n$ is convex. We see that $\forall\ f_1, f_2 \in E_n$ and $\forall\ 0 \leq \alpha \leq 1$, $\alpha f_1 + (1-\alpha) f_2 \in E_n$, since $\int_{[0,1]} \mid \alpha f_1 + (1-\alpha) f_2 \mid^3 dm \leq \int_{[0,1]} \mid \alpha f_1 \mid^3 + \mid (1-\alpha) f_2 \mid^3 dm \leq \int_{[0,1]} \mid \alpha f_1 \mid^3 dm + \int_{[0,1]} \mid (1-\alpha) f_2 \mid^3 dm \leq \int_{[0,1]} \mid \alpha \mid^3 \mid f_1 \mid^3 dm + \int_{[0,1]} \mid (1-\alpha) \mid^3 \mid f_2 \mid^3 dm \leq \mid \alpha \mid^3 \int_{[0,1]} \mid f_1 \mid^3 dm + \mid (1-\alpha) \mid^3 \int_{[0,1]} \mid f_2 \mid^3 dm \leq \alpha^3 n + (1-\alpha)^3 n \leq \alpha n + (1-\alpha) n = n$, since $0 \leq \alpha \leq 1$ for all $\alpha$. So $E_n$ is convex.

$E_n$ is absorbing if and only if $\forall\ 0 \neq f \in L_1([0,1],m)$, $\exists t > 0$ such that $f \in tE_n$, equivalently $t^{-1} f \in E_n$. To show this we can take $f \in L_1([0,1],m)$, then $\int_{[0,1]} \mid f \mid dm < \infty$ and then $\int_{[o,1]} \mid \frac{1}{t} f \mid^3 dm = \int_{[o,1]} \mid \frac{1}{t^3} \mid \mid f \mid^3 dm = \mid \frac{1}{t^3} \mid \int_{[0,1]} \mid f \mid^3 dm \leq$ n, for t large enough where $0 < 1 \leq t$, since that $\frac{1}{t} \int_{[0,1]} \mid f \mid dm < \infty$ for $t \geq 1$ by assumption.

c/v

#### b)

Firstly we notice that $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n$, and we can find an open subset of $E_n$ for every $n \geq 1$. The subset $U_1 \subset E_1$, where $U_1 = \{f \in L_1([0,1],m) : \int_{[0,1]} \mid f \mid^3 dm < n\}$. By definition of an interior point we have that if $f$ is an interior point of $E_n$, then $E_n$ is a neighbourhood of $f$, i.e. $f \in U_n \subset E_n$. So we easily see that $U_1 \subset E_1$ where $U_1$ also is an absorbing set since $E_1$ is absorbing in $L_1([0,1],m)$ by (a).
Then lemma 3.5 in the lecture notes gives us that $f \in U_1 \Leftrightarrow p_{U_1}(f) < 1$, where $p_{U_1}(f) = \inf\{t > 0 : f \in tU_1\} = \inf\{t > 0 : t^{-1} f \in U_1\}$. Then by the same calculations as in problem (a) we can get that $t^{-1} f \in U_1 \Rightarrow \mid \frac{1}{t^3} \mid \int_{[0,1]} \mid f \mid^3 dm < 1$, but this is only true for $t \geq 1$ and $t$ large enough, so this means that $p_{U_1}(f) \geq 1 \Leftrightarrow f \notin U_1$. So $E_n$ has empty interior in $L_1([0,1],m)$ for all $n \geq 1$.

c/v

#### c)

For $E_n$ to be closed in $L_1([0,1],m)$ for all $n \geq 1$, we need to have that any cauchy sequence in $E_n$ has limit in $E_n$. We take $(f_n)$ to be any cauchy sequence of funtions where each $f_n \in E_n$. Then there exists $f$ such that $\lim(f_n) = f$ and there exists $n \geq 1$ such that $(f_n)$ converges uniformly to $f$ since $f_n \in E_n$ and by definition of $E_n$ and $f$ is continuous by definition, since $f \in L_1([0,1],m)$.
Then we can let $\mid f(x) - f_n(x) \mid < \frac{\epsilon}{2}$ and $\mid f_n(x) - n \mid < \frac{\epsilon}{2}$, for $\epsilon > 0$. So we have that $\mid f(x) - n \mid = \mid f(x) - f_n(x) + f_n(x) - n \mid \leq \mid f(x) - f_n(x) \mid + \mid f_n(x) - n \mid < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, for $\epsilon > 0$. So $f \in E_n$ which means that $E_n$ is closed in $L_1([0,1],m)$ for all $n \geq 1$.

?

$|f(x)-n| < \varepsilon$ ?

**d)**

By definition 3.12 in the lecture notes, we need to show that there exists a sequence $(E_n)_{n\geq 1}$ of nowhere dense sets such that $L_3([0,1],m) = \cup_{n\geq 1}^{\infty} E_n$.
If we combine the result from (b) and (c) we get that $E_n = \bar{E_n}$, since $E_n$ is closed in $L_1([0,1],m)$ for all $n \geq 1$ and that $\mathrm{Int}(E_n) = \mathrm{Int}(\bar{E_n}) = \emptyset$ for all $n \geq 1$, which means that $E_n \subset L_1([0,1],m)$ is nowhere dense for all $n \geq 1$ by definition 3.12 (i) in the lecture notes.
And we have that $\cup_{n\geq 1}^{\infty} E_n = \cup_{n\geq 1}^{\infty} \{f \in L_1([0,1],m) : \int_{[0,1]} \mid f \mid^3 dm \leq n\} = \{f : [0,1] \to \mathbb{K} \text{ measureable } : \underline{\mid\mid f \mid\mid_1 :=} (\int_{[0,1]} \underline{\mid f(x) \mid^3 dm}) < \infty\} = \{f : [0,1] \to \mathbb{K} \text{ measureable } : \mid\mid f \mid\mid_3 := (\int_{[0,1]} \mid f(x) \mid^3 dm)^{\frac{1}{3}} < \infty\} = L_3([0,1],m)$. So $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$ by definition 3.12 (ii) in the lecture notes.

## Problem 5 [11 points]

Let $H$ be an infinite dimensional separable Hilbert space with associated norm $|| \cdot ||$, let $(x_n)_{n\geq 1}$ be a sequence in $H$, and let $x \in H$.

a) [2 p]. Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $|| x_n || \to || x ||$, as $n \to \infty$? Give a proof or a counterexample.

b) [5 p]. Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $|| x_n || \to || x ||$, as $n \to \infty$? Give a proof or a counterexample. [Hint: Consider an orthonormal basis $(e_n)_{n\geq 1}$ in $H$, and use HW4.]

c) [4 p]. Suppose that $|| x_n || \leq$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Is it true that $|| x || \leq 1$? Give a proof or a counterexample.

### Answers

**a)**

Since $x_n \to x$ in norm, as $n \to \infty$, then $\lim_{n\to\infty} || x_n - x || = 0$. And we have that $|| x_n ||$ - $|| x || \leq || x_n - x ||$, so by the squeeze lemma, we get that $|| x_n || \to || x ||$ as $n \to \infty$. (✓)

Need $|\cdot|$

**b)**

This is not the definition

But why is this equivalent?

By proposition 5.28 and 5.29 in Folland we have that any Hilbert space has an orthonormal basis where any orthonomal basis countable when $H$ is separable. So we can find an countable basis $(e_n)_{n\geq 1}$ in $H$.
And we have by definition of weak convergense that $x_n \to x$ weakly, as $n \to \infty$ means that $< x_n, y > \to < x, y > \forall\ y \in H$. Then if we consider an orthonormal basis $(e_n)_{n\geq 1}$ in $H$ such that $< e_n, e_m >= 1$ if $n = m$ and 0 otherwise. Then for $x \in H$ we have that $\sum_{n\geq 1} |< e_n, x >|^2 \leq || x ||^2$, with equality when $e_n$ is a basis for a Hilbert space as it is in our case. So we have that $|< e_n, x >|^2 \to 0$, i.e. $< e_n, x > \to 0$. Which means that since $H$ is an infinite dimensional separable Hilbert space we have that $x_n \to 0$ as $n \to \infty$. ?

Why do you say that?

Then by HW4 problem 4 we have that the Hilbert space $\ell_2(\mathbb{N})$ is separable. And by HW4 problem 3 (a) we have that the sequence $(x_n)_{n\geq 1}$ is bounded in $|| \cdot ||_2$, which means that there is a constant $K > 0$ such that $|| x_n ||_2 \leq K$, for all $n \geq 1$. So we have that $|| x_n || \to || 0 ||$ as $n \to \infty$, since $|| 0 ||_2 = 0 \lneq K$ for $K > 0$.
So the statement that $|| x_n || \to || x ||$, as $n \to \infty$ as $x_n \to x$ weakly, as $n \to \infty$ is true.

0/0

Do not see what K is?

**c)**

This is also true by calculations and arguments in problem (b), since we can choose $K > 0$ where $K = 1$ such that $|| x_n || \leq 1$ for all $n \geq 1$, since we are in the same situation as in problem (b) since we again assume that $x_n \to x$ weakly, as $n \to \infty$.

If (b) was true, this argument would work. But (b) is false.

0/0