Homework for week 3, FunkAn, Fall 2018

Problem 1 [Problem 32, p.164, Folland]: Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the vector space X such that $\|\cdot\|_1 \leq \|\cdot\|_2$. If X is complete with respect to both norms, then the norms are equivalent.

Problem 2 [Problem 37, p.164, Folland]: Let X and Y be Banach spaces. If $T\colon X \to Y$ is a linear map such that $f \circ T \in X^*$, for every $f \in Y^*$, then T is bounded.

Problem 3: Let $X = C([0,1])$ be the space of (complex-valued) continuous functions on $[0,1]$. For every $f \in C([0,1])$, set $\|f\|\colon = \int_0^1 |f(t)|dt$. Note that $(C([0,1]), \|\cdot\|)$ is a normed space, but it is not complete. Define a functional $T\colon X \to \mathbb{C}$ by $T(f) = f(1/2)$, for every $f \in X$. Show that T is linear, but not bounded. Is T a closed map?

Problem 4 [Problem 37, p.164, Folland]:
Let $Y = l_1(\mathbb{N})$, and let $X\colon = \{f \in Y\colon \sum_{n=1}^{\infty} n|f(n)| < \infty\}$, equipped with the $l_1(\mathbb{N})$ norm.

(a) Show that X is a proper dense subspace of Y; hence X is not complete.

(b) Define $T\colon X \to Y$ by $Tf(n) = nf(n)$, $f \in X, n \in \mathbb{N}$. Then T is closed, but not bounded.

(c) Let $S = T^{-1}$. Then $S\colon Y \to X$ is bounded and surjective, but not open.