Notes on 1D bosons

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We consider the dilute bose gas in one dimension, where we seek to prove the formula for the ground state energy

$$\frac{E}{L} = \frac{\pi^2}{3}\rho^3\left(1 + 2\rho a + \mathcal{O}\left((\rho a)^2\right)\right). \tag{0.1}$$

We assume that the interaction potential v has compact support, say in the ball of radius b, B_b.

1 Upper bound

We provide the upper bound for (0.1), by using the variational principle with a suitable trial state. We assume for simplicity periodic boundary conditions to begin with. Consider the trial state

$$\Psi(x) = \begin{cases} \omega(\mathcal{R}(x))\frac{\tilde{\Psi}_F(x)}{\sin\left(\frac{\pi}{L}\mathcal{R}(x)\right)} & \text{if } \mathcal{R}(x) < b, \\ \tilde{\Psi}_F(x) & \text{if } \mathcal{R}(x) \geq b, \end{cases} \tag{1.1}$$

where ω is the suitably normalized solution to the two-body scattering equation, *i.e.* $\omega(x) = f(x)\frac{\sin\left(\frac{\pi}{L}b\right)}{f(b)}$ where f is any solution of the two-body scattering equation. $\tilde{\Psi}_F(x) = \mathcal{N}^{1/2}\prod_{i<j}^N \sin\left(\frac{\pi}{L}\left|x_i - x_j\right|\right)$ is the absolute value of the free fermionic ground state, and $\mathcal{R}(x) = \min_{i<j}(|x_i - x_j|)$ is uniquely defined a.e.

The energy of this trial state is then

$$\mathcal{E}(\Psi) = \int \sum_{i=1}^N \left|\nabla_i \Psi\right|^2 + \sum_{i<j}^N v_{ij}\left|\Psi\right|^2, \tag{1.2}$$

where $v_{ij}(x) = v(x_i - x_j)$. Since v is supported in B_b and $\Psi = \tilde{\Psi}_F$ except in the region $B = \{x \in \mathbb{R}^N | \mathcal{R}(x) < b\}$, we may rewrite this as

$$\mathcal{E}(\Psi) = E_0 + \int_B \sum_{i=1}^N \left|\nabla_i \Psi\right|^2 + \sum_{i<j}^N v_{ij}\left|\Psi\right|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2, \tag{1.3}$$

where $E_0 = N\frac{\pi^2}{3}\rho^2$ is the ground state energy of the free Fermi gas. Using that $v \geq 0$, symmetry of exchange of particles, and defining the set $B_{12} = \{x \in \mathbb{R}^N | \mathcal{R}(x) < b,\ \mathcal{R}(x) = |x_1 - x_2|\} \subset A_{12} = \{x \in \mathbb{R}^N | \, |x_1 - x_2| < b\}$ which up to a set of measure zero is the intersection of B and

{1 and 2 are closest}, we find

$$\begin{aligned}
\mathcal{E}(\Psi) &= E_0 + \binom{N}{2}\int_{B_{12}} \sum_{i=1}^N |\nabla_i \Psi|^2 + \sum_{i<j}^N v_{ij}|\Psi|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 \\
&= E_0 + \binom{N}{2}\int_{B_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}\right|^2 + \sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 \\
&= E_0 + \binom{N}{2}\int_{A_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}\right|^2 + \sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 \\
&\quad - \binom{N}{2}\int_{A_{12}\setminus B_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}\right|^2 + \sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 \\
&\leq E_0 + E_1 + \binom{N}{2}\int_{A_{12}\setminus B_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2
\end{aligned} \tag{1.4}$$

where we have defined

$$\tilde{\Psi} = \begin{cases} \omega(x_1-x_2)\frac{\tilde{\Psi}_F(x)}{\sin\left(\frac{\pi}{L}|x_1-x_2|\right)} & \text{if } |x_1-x_2|<b, \\ \tilde{\Psi}_F(x) & \text{if } |x_1-x_2|\geq b, \end{cases}$$

and $E_1 = \binom{N}{2}\int_{A_{12}}\sum_{i=1}^N \left|\nabla_i\tilde{\Psi}\right|^2 + \sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 - \sum_{i=1}^N\left|\nabla_i\tilde{\Psi}_F\right|^2$.
We may estimate

$$\begin{aligned}
\binom{N}{2}\int_{A_{12}\setminus B_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 &= \binom{N}{2}\Bigg(2N\left[\int_{A_{12}\cap A_{13}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 - \int_{B_{12}\cap A_{13}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2\right] \\
&\quad + \binom{N-2}{2}\left[\int_{A_{12}\cap A_{34}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 - \int_{B_{12}\cap A_{34}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2\right]\Bigg) \\
&\leq \binom{N}{2}\left[2N\int_{A_{12}\cap A_{13}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 + \binom{N-2}{2}\int_{A_{12}\cap A_{34}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2\right]
\end{aligned} \tag{1.5}$$

Thus we find

$$\mathcal{E}(\Psi) \leq E_0 + E_1 + E_2^{(1)} + E_2^{(2)} \tag{1.6}$$

with $E_2^{(1)} = \binom{N}{2}2N\int_{A_{12}\cap A_{13}}\sum_{i=1}^N\left|\nabla_i\tilde{\Psi}_F\right|^2$ and $E_2^{(2)} = \binom{N}{2}\binom{N-2}{2}\int_{A_{12}\cap A_{34}}\sum_{i=1}^N\left|\nabla_i\tilde{\Psi}_F\right|^2$.
We notice that since $\tilde{\Psi}_F = |\Psi_F|$ so by the diamagnetic inequality we have $\left|\nabla_i\tilde{\Psi}_F\right|^2 \leq |\nabla_i\Psi_F|^2$, which implies that $\tilde{\Psi}_F$ is in $H^1(\mathbb{R}^N)$. Furthermore, Ψ_F is $C^1(\mathbb{R}^N)$ with a zero set $\{\Psi_F = 0\}$ of measure zero, $\left|\nabla_i\tilde{\Psi}_F\right|^2$ and $|\nabla_i\Psi_F|^2$ are equal a.e. But then $\tilde{\Psi}_F = \Psi_F$ as $H^1(\mathbb{R}^N)$ functions. Hence we may replace $\tilde{\Psi}_F$ with Ψ_F in all integrals above.

1.1 Reduced density matrices

We recall briefly the definition of the reduced density matrices, as we will use some fact about these frequently in the subsequent calculations. For an N-(identical)particle state Ψ, the n-particle reduced density matrix is defined by

$$\gamma^{(n)}(x_1,...,x_n;y_1,...,y_n)=\frac{N!}{(N-n)!}\int\overline{\Psi(x_1,...,x_N)}\Psi(y_1,...,y_N)\,\mathrm{d}x_{n+1}...\,\mathrm{d}x_N. \tag{1.7}$$

For a determinant state $\Psi=\det\left(u_i(x_j)\right)$ with u_i orthonormal states, the one-particle reduced density matrix is given by

$$\gamma^{(1)}(x,y)=\sum_{i=1}^{N}\overline{u_i(x)}u_i(y). \tag{1.8}$$

The n-particle reduced density matrix may be expressed in terms of creation and annihilation operators as

$$\gamma^{(n)}(x_1,...,x_n;y_1,...,y_n)=\left\langle a^\dagger_{x_1}...a^\dagger_{x_n}a_{y_n}...a_{y_1}\right\rangle. \tag{1.9}$$

For the groundstate of a free Hamiltonian (or any quasi free state), Wick's theorem applies and n-particle reduced density matrix of the Fermi groundstate may be computed recusively by

$$\left\langle c^\dagger_{x_1}...c^\dagger_{x_n}c_{y_n}...c_{y_1}\right\rangle=\sum_{i=1}^{n}(-1)^{i-1}\left\langle c^\dagger_{x_1}c_{y_i}\right\rangle\left\langle c^\dagger_{x_2}...c^\dagger_{x_n}c_{y_n}...c_{y_{i+1}}c_{y_{i-1}}...c_{y_1}\right\rangle. \tag{1.10}$$

For the Fermi ground state with periodic boundary conditions, we also have

$$\begin{aligned}\gamma^{(1)}(x,y)=\left\langle c^\dagger_x c_y\right\rangle&=\frac{1}{L}\sum_{j=-(N-1)/2}^{(N-1)/2}\mathrm{e}^{i2\pi(x-y)j/L}=\frac{1}{L}\mathrm{e}^{-i\pi(x-y)(\rho-1/L)}\sum_{j=0}^{N-1}\left(\mathrm{e}^{i2\pi(x-y)/L}\right)j\\&=\frac{1}{L}\mathrm{e}^{-i\pi(x-y)(N-1)/L}\frac{1-\mathrm{e}^{2\pi(x-y)\rho}}{1-\mathrm{e}^{2\pi(x-y)/L}}=\frac{1}{L}\frac{\mathrm{e}^{-i\pi\rho(x-y)}-\mathrm{e}^{i\pi\rho(x-y)}}{\mathrm{e}^{-i\pi(x-y)/L}-\mathrm{e}^{i\pi(x-y)/L}}=\frac{1}{L}\frac{\sin(\pi\rho(x-y))}{\sin\left(\frac{\pi}{L}(x-y)\right)}.\end{aligned} \tag{1.11}$$

For $x-y\ll\rho^{-1}$ we may use the relation

$$\gamma^{(1)}(x,y)=\rho+\frac{\pi^2}{6}\left(\frac{\rho}{L^2}-\rho^3\right)(x-y)^2+\mathcal{O}((x-y)^3). \tag{1.12}$$

1.2 Calculating E_1

Recall the definition

$$E_1=\binom{N}{2}\int_{A_{12}}\sum_{i=1}^{N}\left|\nabla_i\tilde{\Psi}\right|^2+\sum_{i<j}^{N}v_{ij}\left|\tilde{\Psi}\right|^2-\sum_{i=1}^{N}\left|\nabla_i\tilde{\Psi}_F\right|^2 \tag{1.13}$$

We estimate E_1 by splitting it in three terms. First we have

$$\begin{aligned} E_1^{(1)} &= 2\binom{N}{2}\int_{A_{12}} \left|\nabla_1 \tilde{\Psi}\right|^2 \\ &= 2\binom{N}{2}\int_{A_{12}} \overline{\tilde{\Psi}}\left(-\Delta_1 \tilde{\Psi}\right) + 2\binom{N}{2}\int \left[\overline{\tilde{\Psi}}\nabla_1\tilde{\Psi}\right]_{x_1=x_2-b}^{x_1=x_2+b}. \end{aligned} \tag{1.14}$$

The boundary term can be explicitly calculated, and to lowest order in b we find

$$\begin{aligned} 2\binom{N}{2}\int \left[\overline{\tilde{\Psi}}\nabla_1\tilde{\Psi}\right]_{x_1=x_2-b}^{x_1=x_2+b} &= L\left[\frac{\omega(x)}{\sin(\pi x/L)}\partial_x\left(\frac{\omega(x)}{\sin(\pi x/L)}\right)\gamma^{(2)}(x,0)\right]_{-b}^{b} \\ &\quad + L\left[\left(\frac{\omega(x)}{\sin(\pi x/L)}\right)^2 \partial_x\left(\gamma^{(2)}(x,0;y,0)\right)\Big|_{y=x}\right]_{-b}^{b}. \end{aligned} \tag{1.15}$$

Since the continuous function $\frac{\omega(x)}{\sin(\pi x/L)} = \frac{x-a}{b-a}\frac{\sin(\pi b/L)}{\sin(\pi x/L)}$ for $|x|>b$, we see that

$$\partial_x\left(\frac{\omega(x)}{\sin(\pi x/L)}\right)\Big|_{x=\pm b} \approx \pm\pi b/L\frac{\frac{1}{b-a}-1}{(\pi b/L)} = \pm\frac{a}{b^2} \tag{1.16}$$

and we know that $\gamma^{(2)}(x,0) = \frac{\pi^2}{6}\rho^4 x^2$. Furthermore, by Wick's theorem it is straightforward to show that

$$\partial_x\left(\gamma^{(2)}(x,0;y,0)\right)\Big|_{y=x} = \frac{\pi^2}{3}N\rho^3 x + \rho^2 o(\rho x) \tag{1.17}$$

Thus we have

$$E_1^{(1)} = \frac{\pi^2}{3}N\rho^3(a+2b) + 2\binom{N}{2}\int_{A_{12}}\overline{\tilde{\Psi}}(-\Delta_1\tilde{\Psi}) \tag{1.18}$$

Another contribution to E_1 is

$$\begin{aligned} E_1^{(2)} &= -\binom{N}{2}\int_{A_{12}} 2\left|\nabla_1\Psi_F\right|^2 + \sum_{i=3}^{N}\left|\nabla_i\Psi_F\right|^2 = \\ &- \binom{N}{2}\int_{A_{12}}\sum_{i=1}^{N}\overline{\Psi_F}(-\Delta_i\Psi_F) - 2\binom{N}{2}\int\left[\overline{\Psi_F}\nabla_1\Psi_F\right]_{x_1=x_2-b}^{x_1=x_2+b} \\ &= -E_0\binom{N}{2}\int_{A_{12}}\left|\Psi_F\right|^2 - L\left[\partial_y\gamma^{(2)}(x,0;y,0)|_{y=x}\right]_{-b}^{b} \end{aligned} \tag{1.19}$$

Again using (1.17) and $\gamma^{(2)}$ we find

$$E_1^{(2)} = -E_0\frac{1}{2}\frac{\pi^2}{9}N\rho^3 b^3 - \frac{\pi^2}{3}N\rho^3(2b). \tag{1.20}$$

The last contributions are $E_1^{(3)} = \binom{N}{2}\int_{A_{12}}\sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 = \binom{N}{2}\int_{A_{12}} v_{12}\left|\tilde{\Psi}\right|^2 + \binom{N}{2}\int_{A_{12}}\sum_{2\leq i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2$ and $E_1^{(4)} = \int_{A_{12}}\sum_{i=3}^N\left|\nabla_i\tilde{\Psi}\right|^2$. First we notice that

$$
\begin{aligned}
&\binom{N}{2}\int_{A_{12}}\sum_{2\leq i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 \\
&\quad\leq C_1'\int_{A_{12}\cap\mathrm{supp}(v_{34})}\gamma^{(4)}(x_1,x_2,x_3,x_4) + C_2'C\int_{A_{12}\cap\mathrm{supp}(v_{23})}\gamma^{(3)}(x_1,x_2,x_3).
\end{aligned}
\tag{1.21}
$$

To leading order in L, $|x_3-x_4|$ and $|x_1-x_2|$ we find that

$$
\gamma^{(4)}(x_1,x_2,x_3,x_4) = \frac{\pi^4}{9}\rho^8(x_1-x_2)^2(x_3-x_4)^2 \tag{1.22}
$$

and to leading order in L, $|x_1-x_2|$ and $|x_2-x_3|$ we find

$$
\gamma^{(3)}(x_1,x_2,x_3) = \frac{\pi^6}{135}\rho^9\underbrace{(x_1-x_3)^2}_{=[(x_1-x_2)+(x_2-x_3)]^2}(x_1-x_2)^2(x_2-x_3)^2. \tag{1.23}
$$

Therefore we have

$$
\begin{aligned}
&\binom{N}{2}\int_{A_{12}}\sum_{2\leq i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 \\
&\quad\leq C'\Bigg(N^2(\rho b)^3\rho^3\int x^2v(x)\,\mathrm{d}x + N(\rho b)^3\rho^5\int x^4v(x)\,\mathrm{d}x + N(\rho b)^4\rho^4\int x^3v(x)\,\mathrm{d}x \\
&\qquad\qquad + N(\rho b)^5\rho^3\int x^2v(x)\,\mathrm{d}x\Bigg) \\
&\leq C'N^2(\rho b)^5\rho\int v = \text{konst. } E_0N(\rho b)^3\left(b\int v\right)
\end{aligned}
\tag{1.24}
$$

and then we find that

$$
\begin{aligned}
E_1 &= E_1^{(1)} + E_1^{(2)} + E_1^{(3)} + E_1^{(4)} \\
&\leq \frac{\pi^2}{3}N\rho^3a + 2\binom{N}{2}\int_{A_{12}}\left(\overline{\tilde{\Psi}}(-\Delta_1)\tilde{\Psi} + \frac{1}{2}\sum_{i=3}^N\left|\nabla_i\tilde{\Psi}\right|^2 + \frac{1}{2}v_{12}\left|\tilde{\Psi}\right|^2\right) - E_0\frac{1}{2}\frac{\pi^2}{9}N\rho^3b^3 + \frac{1}{2}\rho\int v
\end{aligned}
\tag{1.25}
$$

Using the two body scattering equation this implies

$$
\begin{aligned}
E_1 \leq{}& \frac{\pi^2}{3}N\rho^3a + 2\binom{N}{2}\int_{A_{12}}\overline{\tilde{\Psi}}\omega(-\Delta_1)\frac{\Psi_F}{\sin(\pi(x_1-x_2)/L)} \\
&+ 2\binom{N}{2}\int_{A_{12}}\overline{\tilde{\Psi}}(\nabla_1\omega)\nabla_1\frac{\Psi_F}{\sin(\pi(x_1-x_2)/L)} \\
&+ \binom{N}{2}\int_{A_{12}}\sum_{i=3}^N\overline{\tilde{\Psi}}\frac{\omega}{\sin(\pi(x_1-x_2)/L)}(-\Delta_i)\Psi_F \\
&- E_0\frac{1}{2}\frac{\pi^2}{9}N\rho^3b^3 + \frac{1}{2}\rho\int v
\end{aligned}
\tag{1.26}
$$

Now using that

$$\begin{aligned}&\binom{N}{2}\int_{A_{12}}\sum_{i=3}^{N}\bar{\tilde{\Psi}}\frac{\omega}{\sin(\pi(x_1-x_2)/L)}(-\Delta_i)\Psi_F\\&=E_0\binom{N}{2}\int_{A_{12}}\left|\frac{\omega}{\sin(\pi(x_1-x_2)/L)}\tilde{\Psi}\right|^2-2\binom{N}{2}\int_{A_{12}}\bar{\tilde{\Psi}}\frac{\omega}{\sin(\pi(x_1-x_2)/L)}(-\Delta_1)\Psi_F,\end{aligned} \tag{1.27}$$

$$\binom{N}{2}\int_{A_{12}}\left|\frac{\omega}{\sin(\pi(x_1-x_2)/L)}\tilde{\Psi}\right|^2\leq C_1\left(\frac{b}{L}\right)^2\pi^2\rho^4\left(\frac{L}{\pi}^2\right)Lb=C_1N\rho^3b^3 \tag{1.28}$$

and that

$$\begin{aligned}2\binom{N}{2}\int_{A_{12}}\bar{\tilde{\Psi}}\frac{\omega}{\sin(\pi(x_1-x_2)/L)}(-\Delta_1)\Psi_F&\leq C_2E_0 2\binom{N}{2}\int_{A_{12}}|\Psi_F|^2\\&=C_2E_0\frac{\pi^2}{9}N\rho^3b^3\end{aligned} \tag{1.29}$$

we find that

$$\binom{N}{2}\int_{A_{12}}\sum_{i=3}^{N}\bar{\tilde{\Psi}}\frac{\omega}{\sin(\pi(x_1-x_2)/L)}(-\Delta_i)\Psi_F\leq CE_0N(\rho b)^3. \tag{1.30}$$

Furhtermore we find to leading order in N and ρb that

$$2\binom{N}{2}\int_{A_{12}}\bar{\tilde{\Psi}}\omega(-\Delta_1)\frac{\Psi_F}{\sin(\pi(x_1-x_2)/L)}=\frac{\pi^2}{15}N\rho^2(\rho b)^3, \tag{1.31}$$

and that

$$2\binom{N}{2}\int_{A_{12}}\bar{\tilde{\Psi}}(\nabla_1\omega)\nabla_1\frac{\Psi_F}{\sin(\pi(x_1-x_2)/L)}=\frac{\pi^2}{45}N\rho^2(\rho b)^3. \tag{1.32}$$

Combining everything we find

$$E_1\leq E_0\left(\rho a+\text{konst. }(\rho b)^3\right)+\frac{1}{2}\rho\int v \tag{1.33}$$

1.3 Calculating E_2

Recall that $E_2=E_2^{(1)}+E_2^{(2)}$ with

$$\begin{aligned}E_2^{(1)}&=\binom{N}{2}2N\int_{A_{12}\cap A_{13}}\sum_{i=1}^{N}|\nabla_i\Psi_F|^2\\E_2^{(2)}&=\binom{N}{2}\binom{N-2}{2}\int_{A_{12}\cap A_{34}}\sum_{i=1}^{N}|\nabla_i\Psi_F|^2\end{aligned} \tag{1.34}$$

To estimate these, we first split them in two terms each and use partial integration. Consider first $E_2^{(1)}$:

$$
\begin{aligned}
E_2^{(1)} &= \binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=1}^{N} |\nabla_i \Psi_F|^2 \\
&= \binom{N}{2} 8N \int_{A_{12}\cap A_{13}} |\nabla_1 \Psi_F|^2 + \binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=5}^{N} |\nabla_i \Psi_F|^2 \\
&= \binom{N}{2} 2N \left([bla] \int_{A_{12}\cap A_{13}} \sum_{i=1}^{N} \overline{\Psi_F} (-\Delta_i \Psi_F) \right)
\end{aligned}
\tag{1.35}
$$

For the second term, we can perform partial integration directly, in order to obtain

$$
\begin{aligned}
E_2^{(1,1)} &= \binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=5}^{N} |\nabla_i \Psi_F|^2 = \binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=5}^{N} \overline{\Psi_F} (-\Delta_i \Psi_F) \\
&\leq E_0 N^3 \int_{A_{12}\cap A_{23}} |\Psi_F|^2 \\
&\leq 2E_0 \int_{[0,L]} \int_{[x_2-b, x_2+b]} \int_{x_2-b, x_2+b} \gamma^{(3)}(x_1, x_2, x_3)\, \mathrm{d}x_3\, \mathrm{d}x_1\, \mathrm{d}x_2
\end{aligned}
\tag{1.36}
$$

Changing variable $y_1 = x_1 - x_2$, $y_3 = x_3 - x_2$ and using translational invariance, we find

$$
\begin{aligned}
E_2^{(1,1)} &\leq 2E_0 L \int_{[-b,b]} \int_{[-b,b]} \gamma^{(3)}(y_1, 0, y_3)\, \mathrm{d}y_1\, \mathrm{d}y_3 \\
&\approx 2E_0 L \frac{\pi^6}{135} \rho^9 \int_{[-b,b]} \int_{[-b,b]} y_1^4 y_3^2\, \mathrm{d}y_1\, \mathrm{d}y_3 \\
&= E_0 N \frac{2\pi^6}{15 \cdot 135} (b\rho)^8.
\end{aligned}
\tag{1.37}
$$