Rigorous Stability Theory of Quantum Mechanics

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1 Preliminaries: Hilbert Spaces and Operators

The basic mathematical objects in quantum mechanics are Hilbert spaces and operators defined on them. In order to fix notations we briefly review the definitions.

1.1 DEFINITION (Hilbert Space)**.** A Hilbert Space $\mathcal{H}$ is a vector space endowed with a sesquilinear map $(\cdot,\cdot):\mathcal{H}\times\mathcal{H}\to\mathbb{C}$ (i.e., a map which is conjugate linear in the first variable and linear in the second[1]) such that $\|\phi\| = (\phi,\phi)^{1/2}$ defines a norm on $\mathcal{H}$ which makes $\mathcal{H}$ into a complete metric space.

1.2 REMARK. We shall mainly use the following two properties of Hilbert spaces.

(a) To any closed subspace $V\subset\mathcal{H}$ there corresponds the orthogonal complement $V^\perp$ such that $V\oplus V^\perp = \mathcal{H}$.

(b) Riesz representation Theorem: To any continuous linear functional $\Lambda:\mathcal{H}\to\mathbb{C}$ there is a unique $\psi\in\mathcal{H}$ such that $\Lambda(\phi) = (\psi,\phi)$ for all $\phi\in\mathcal{H}$.

We shall always assume that our Hilbert spaces are separable and therefore that they have countable orthonormal bases.

1.3 DEFINITION (Operators on Hilbert spaces)**.** By an operator (or more precisely densely defined operator) A on a Hilbert space $\mathcal{H}$ we mean a linear map $A: D(A)\to\mathcal{H}$ defined on a *dense* subspace $D(A)\subset\mathcal{H}$. Dense refers to the fact that the norm closure $\overline{D(A)} = \mathcal{H}$.

[1] This is the convention in physics. In mathematics the opposite convention is used.

Note that the domain is part of the definition of the operator. In defining operators one often starts with a domain which turns out to be too small and which one then later extends.

1.4 EXAMPLE. The Hilbert space describing a one-dimensional particle without internal degrees of freedom is $L^2(\mathbb{R})$, the space of square (Lebesgue) integrable functions defined modulo sets of measure zero. The inner product on $L^2(\mathbb{R})$ is given by

$$(g, f) = \int_{\mathbb{R}} \overline{g(x)} f(x)\, dx.$$

The operator describing the kinetic energy is the second derivative operator. $A = -\frac{d^2}{dx^2}$ defined originally on the subspace

$$D(A) = C_0^2(\mathbb{R}) = \big\{ f \in C^2(\mathbb{R}) : f \text{ vanishes outside a compact set } \big\}.$$

Here $C^2(\mathbb{R})$ refers to the twice continuously differentiable functions on the real line. The subscript 0 refers to the compact support.

1.5 DEFINITION (Bounded operators)**.** An operator A is said to be *bounded* on the Hilbert Space $\mathcal{H}$ if $D(A) = \mathcal{H}$ and A is continuous, which by linearity is equivalent to

$$\|A\| = \sup_{\phi,\ \|\phi\|=1} \|A\phi\| < \infty.$$

The number $\|A\|$ is called the norm of the operator A. An operator is said to be *unbounded* if it is not bounded.

1.6 PROBLEM. *(a) Show that if an operator A with dense domain $D(A)$ satisfies $\|A\phi\| \leq M\|\phi\|$ for all $\phi \in D(A)$ for some $0 \leq M < \infty$ then A can be uniquely extended to a bounded operator.*

(b) Show that the kinetic energy operator A from Example 1.4 cannot be extended to a bounded operator on $L^2(\mathbb{R})$.

1.7 EXAMPLE (Hydrogen atom). One of the most basic examples in quantum mechanics is the hydrogen atom. In this case the Hilbert space is $L^2(\mathbb{R}^3; \mathbb{C}^2)$, i.e., the square integrable functions on $\mathbb{R}^3$ with values in $\mathbb{C}^2$. Here $\mathbb{C}^2$ represents the internal spin degrees of freedom. The inner product is

$$(g, f) = \int_{\mathbb{R}^3} g(x)^* f(x)\, dx,$$

The total energy operator is[2]

$$H = -\Delta - \frac{1}{|x|}, \tag{1}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$, is the Laplacian. The domain of H is

$$\begin{aligned} D(H) &= C_0^2\left(\mathbb{R}^3; \mathbb{C}^2\right) \qquad (2) \\ &= \left\{f \in C^2(\mathbb{R}^3; \mathbb{C}^2) : f \text{ vanishes outside a compact subset of } \mathbb{R}^3\right\}. \end{aligned}$$

It is easy to see that if $\phi \in D(H)$ then $H\phi \in L^2(\mathbb{R}^3; \mathbb{C}^2)$.

2 The Principles of Quantum Mechanics

We shall here briefly review the principles of quantum mechanics. The reader with little or no experience in quantum mechanics is advised to also consult standard textbooks in physics.

In quantum mechanics a *pure* state of a physical system is described by a unit vector ψ_0 in a Hilbert space $\mathcal{H}$. The measurable quantities correspond to 'expectation values'

$$\langle A \rangle_{\psi_0} = (\psi_0, A\psi_0),$$

of operators A on $\mathcal{H}$. Of course, in order for this to make sense we must have $\psi_0 \in D(A)$.

The physical interpretation of the quantity $\langle A \rangle_{\psi_0}$ is that it is the average value of 'many' measurements of the observable described by the operator A in the state ψ_0.

As an example $\psi_0 \in C_0^2\left(\mathbb{R}^3; \mathbb{C}^2\right)$ with $\int |\psi_0|^2 = 1$ may represent a state of a hydrogen atom (see Example 1.7). The average value of many measurements of the energy of the atom in this state will be

$$\begin{aligned} \left(\psi_0, (-\Delta - \frac{1}{|x|})\psi_0\right) &= \int_{\mathbb{R}^3} \psi_0(x)^*(-\Delta - \frac{1}{|x|})\psi_0(x)\, dx \\ &= \int_{\mathbb{R}^3} |\nabla \psi_0(x)|^2 - \frac{1}{|x|}|\psi_0(x)|^2\, dx, \end{aligned}$$

[2] We use units in which Planck's constant $\hbar$, twice the electron mass $2m_{\mathrm{e}}$, and the electron charge e are all equal to unity

where the last equality follows by integration by parts.

The general quantum mechanical state, which is not necessarily pure is a statistical average of pure states, i.e, expectations are of the form

$$\langle A \rangle = \sum_{n=1}^{\infty} \lambda_n (\psi_n, A\psi_n),$$

where $0 \leq \lambda_n \leq 1$ with $\sum_n \lambda_n = 1$ and ψ_n is a family of unit vectors.

Of particular interest are the *equilibrium states*, either zero (absolute) temperature or positive temperature states. The zero temperature state is usually a pure state, i.e., given by one vector, whereas the positive temperature states (the Gibbs states) are non-pure. Both the zero temperature states and the positive temperature states are described in terms of the energy operator, the Hamiltonian. We shall here mainly deal with the zero temperature equilibrium states, the *ground states.*

2.1 DEFINITION (Ground States)**.** Consider a physical system described by a Hamiltonian, i.e., energy operator, H on a Hilbert space $\mathcal{H}$. A ground state for the system is a unit vector $\psi_0 \in D(H)$ such that

$$(\psi_0, H\psi_0) = \inf_{\phi \in D(H),\ \|\phi\|=1} (\phi, H\phi)\,.$$

Thus a ground state is characterized by minimizing the energy expectation.

A physical system for which

$$\inf_{\phi \in D(H),\ \|\phi\|=1} (\phi, H\phi) > -\infty$$

is said to be stable (or stable of the first kind).

Likewise, the positive temperature states can be characterized as minimizing the *free* energy, not simply the energy. The free energy is the energy minus the temperature times the entropy. We shall not here discuss the definition of the entropy of a state.

We shall see later that if ψ_0 is a ground state with $(\psi_0, H\psi_0) = \lambda$ then $H\psi_0 = \lambda\psi_0$, i.e., ψ_0 is an eigenvector of H with eigenvalue λ.

2.2 PROBLEM. *Show that the free 1-dimensional particle described in Example 1.4 is stable, but does not have a ground state.*

We shall see in Section 4 that the Hamiltonian H for hydrogen, given in (1) and (2), is stable. It also does not have a ground state, but in this case, however, this is simply because the domain is too small. There is a natural extension of the domain on which the Hamiltonian does have a ground state. We shall discuss these natural extensions later in Sect. 7.

In the next section we discuss in some generality operators and quadratic forms. We shall only be concerned with the eigenvalues of the operators and not with the continuous part of the spectrum. We therefore do not need to understand the spectral theorem in its full generality and we shall not discuss it here. We therefore do not need to understand the more complex questions concerning self-adjointness. We mainly consider semi bounded operators and the corresponding quadratic forms.

The notion of quadratic forms is very essential in quantum mechanics. As we have seen the measurable quantities corresponding to an observable, represented by an operator A are the expectation values which are of the form $(A\psi, \psi)$. In applications to quantum mechanics it is therefore relevant to try to build the general theory as much as possible on knowledge of these expectation values. The map $\psi \mapsto (\psi, A\psi)$ is a special case of a quadratic form.

Finally, we must discuss one of the most important issues of quantum mechanics. The question of statistics of identical particles. If a particle is described by the Hilbert space $\mathcal{H}$ then N particles of the same kind are described by the tensor product Hilbert space $\bigotimes^N \mathcal{H}$. In quantum mechanics identical particles are however indistinguishable which means that all expectation values are independent of whether we interchange two particles. This means that we have a state either on the fully symmetric or fully antisymmetric subspace of $\bigotimes^N \mathcal{H}$. Particles with fully symmetric states are called *Bosons*, whereas particles with fully antisymmetric states are called *Fermions*. We shall denote the fully antisymmetric subspace of $\bigotimes^N \mathcal{H}$ by $\bigwedge^N \mathcal{H}$.

3 Semi-bounded operators and quadratic forms

3.1 DEFINITION (Positive Operators)**.** An operator[3] $A : D(A) \to \mathcal{H}$ defined on a subspace $D(A)$ of $\mathcal{H}$ is said to be positive (or positive definite) if $(\psi, A\psi) > 0$ for all non-zero $\psi \in D(A)$. It is said to be positive semi-definite if $(\psi, A\psi) \geq 0$ for all $\psi \in D(A)$.

If A is a positive definite or semi-definite operator it follows that for all $\phi, \psi \in D(A)$

$$(\psi, A\phi) = (A\psi, \phi). \tag{3}$$

One says that A is a *symmetric* operator.

3.2 PROBLEM. *Prove (3).*

The notion of positivity induces a partial ordering among operators.

3.3 DEFINITION (Operator ordering)**.** If A and B are two operators with $D(A) = D(B)$ then we say that A is (strictly) less than B and write $A < B$ if the operator $B - A$ (which is defined on $D(B - A) = D(A) = D(B)$ is a positive definite operator. We write $A \leq B$ if $B - A$ is positive semi-definite.

3.4 DEFINITION (Semi bounded operators)**.** An operator A is said to be bounded below if $A \geq -cI$ for some scalar c. Here I denotes the identity operator on $\mathcal{H}$. Likewise an operator A is said to be bounded above if $A \leq cI$.

3.5 DEFINITION (Quadratic forms)**.** A quadratic form Q is a mapping $Q : D(Q) \times D(Q) \to \mathbb{C}$ (where $D(Q)$ is a (dense) subspace of $\mathcal{H}$), which is sesquilinear (conjugate linear in the first variable and linear in the second):

$$\begin{aligned} Q(\alpha_1\phi_1 + \alpha_2\phi_2, \psi) &= \overline{\alpha_1}Q(\phi_1, \psi) + \overline{\alpha_2}Q(\phi_2, \psi) \\ Q(\phi, \alpha_1\psi_1 + \alpha_2\psi_2) &= \alpha_1 Q(\phi, \psi_1) + \alpha_2 Q(\phi, \psi_2). \end{aligned}$$

We shall often make a slight abuse of notation and denote $Q(\phi, \phi)$ by $Q(\phi)$. A quadratic form Q is said to be positive definite if $Q(\phi) > 0$ for all $\phi \neq 0$ and positive semi-definite if $Q(\phi) \geq 0$. It is said to be bounded below if $Q(\phi) \geq -c\|\phi\|^2$ (and above if $Q(\phi) \leq c\|\phi\|^2$).

[3] By an operator we simply mean a linear map between vector spaces

3.6 PROBLEM (Cauchy-Schwarz inequality)**.** *Show that if Q is a positive semi-definite quadratic form it satisfies the Cauchy-Schwarz inequality*

$$|Q(\phi,\psi)| \le Q(\phi)^{1/2}Q(\psi)^{1/2}. \tag{4}$$

3.7 DEFINITION (Bounded quadratic forms)**.** A quadratic form Q is said to be bounded if there exists $0 \le M < \infty$ such that

$$|Q(\phi)| \le M\|\phi\|^2,$$

for all $\phi \in D(Q)$.

Note that quadratic forms that are bounded above and below are bounded, but the converse is not true since bounded quadratic forms are not necessarily real.

As for operators we have that if Q is positive semi-definite (or even just bounded above or below) then it is symmetric, meaning

$$Q(\phi,\psi) = \overline{Q(\psi,\phi)}. \tag{5}$$

The proof is the same as in Problem 3.2.

3.8 PROBLEM. *Show that if Q is a bounded quadratic form then it extends to a unique bounded quadratic form on all of $\mathcal{H}$ (compare Problem 1.6).*

3.9 PROBLEM. *Show that if Q is a quadratic form then it is enough to know, $Q(\phi) = Q(\phi,\phi)$ for $\phi \in D(Q)$, in order to determine $Q(\psi_1,\psi_2)$ for all $\psi_1,\psi_2 \in D(Q)$.*

One may ask what property a map $Q(\phi)$ must satisfy in order that $Q(\phi) = Q(\phi,\phi)$ extends to a quadratic form $Q(\psi,\phi)$. The answer is that it should satisfy the parallelogram identity

$$Q(\phi_1+\phi_2) + Q(\phi_1-\phi_2) = 2Q(\phi_1) + 2Q(\phi_2).$$

This is somewhat lengthy to prove. It is, however, not very important for what we shall do here so we leave it as an exercise to the interested reader.

A more important issue is to associate an operator to a quadratic form. It is clear that if A is an operator then $D(A) \ni \phi \mapsto (\phi, A\phi)$ defines a quadratic form. The converse is in a certain sense also true as we discuss in the next problem.

3.10 PROBLEM (Operators corresponding to quadratic forms)**.** *Show that corresponding to a quadratic form there exists a unique operator $A : D(A) \to \mathcal{H}$, with*

$$D(A) = \left\{ \phi \in D(Q) : \sup_{\psi \in D(Q)\setminus\{0\}} \frac{|Q(\psi,\phi)|}{\|\psi\|} < \infty \right\}$$

such that $Q(\phi) = (\phi, A\phi)$ for all $\phi \in D(A)$. Note, that we may have that $D(A)$ is a strict subspace of $D(Q)$. Can you give an example of this? In fact, in general $D(A)$ need not even be dense. See Example 4.4.

When a quadratic form is bounded below it is useful to extend its domain to the whole of $\mathcal{H}$, which can often be done if one allows the value $+\infty$. In order to do this we introduce the concept of norm lower semi-continuous quadratic forms.

3.11 DEFINITION (Norm lower semi-continuity)**.** A function $Q : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ on a (separable) Hilbert space is said to be norm lower semi-continuous if for all $\phi \in \mathcal{H}$ and all sequences $\{\phi_n\} \in \mathcal{H}$ we have

$$\lim_n \|\phi_n - \phi\| = 0 \Rightarrow \liminf_n Q(\phi_n) \geq Q(\phi).$$

3.12 REMARK. We warn the reader that there is another very useful notion of lower semi-continuity, namely *weak lower semi-continuity*. Despite the name, it is actually a stronger assumption, about a function on a Hilbert space, to say that it is weakly lower semi-continuous, than to say that it is norm lower semi-continuous.

3.13 DEFINITION (Weak lower semi-continuity)**.** A function $Q : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ on a (separable) Hilbert space is said to be weakly lower semi-continuous if for all $\phi \in \mathcal{H}$ and all sequences $\{\phi_n\} \in \mathcal{H}$ we have

$$\lim_n (\psi, \phi_n - \phi) = 0 \text{ for all } \psi \in \mathcal{H} \Rightarrow \liminf_n Q(\phi_n) \geq Q(\phi).$$

3.14 THEOREM (Mazur's Theorem for quadratic forms)**.** *If $Q : \mathcal{H} \to \mathbb{R}\cup\{\infty\}$ is a positive semi definite norm continuous, i.e., bounded quadratic form on a Hilbert space $\mathcal{H}$ then Q is weakly lower semi-continuous.*

Proof. Given $\phi, \in \mathcal{H}$ and a sequence ϕ_n such that $(\psi, \phi_n) \to (\psi, \phi)$ for all $\psi \in \mathcal{H}$ we must show that $\liminf_n Q(\phi_n) \geq Q(\phi)$. Using the semi-definiteness of Q we have

$$\begin{aligned} Q(\phi_n) &= Q(\phi_n - \phi) + Q(\phi, \phi_n) + Q(\phi_n, \phi) - Q(\phi, \phi) \\ &\geq Q(\phi, \phi_n) + Q(\phi_n, \phi) - Q(\phi, \phi) \end{aligned}$$

Since Q is bounded we have from the Cauchy-Schwarz inequality (4) that for all ϕ' $|Q(\phi, \phi')| \leq Q(\phi)^{1/2} Q(\phi')^{1/2} \leq CQ(\phi)^{1/2}\|\phi'\|$. Thus Riesz representation Theorem implies that there is a ψ such that $Q(\phi, \phi') = (\psi, \phi')$. Therefore we have

$$Q(\phi, \phi_n) = (\psi, \phi_n) \to (\psi, \phi) = Q(\phi)$$

and we conclude that

$$\liminf_n Q(\phi_n) \geq Q(\phi).$$

$\square$

It is our goal in Sect. 7 below. to show that, under a reasonable assumption called closability, a quadratic form which is bounded below extends uniquely to a norm lower semi-continuous quadratic form.

4 Schrödinger operators

The Schrödinger operator is a generalization of the hydrogen energy operator given in Example 1.7. The generalization amounts to having a general potential V.

4.1 DEFINITION (Schrödinger operator on $C_0^2(\mathbb{R}^n)$)**.** The Schrödinger operator for a particle without internal degrees of freedom moving in a potential $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is

$$H = -\Delta - V$$

with domain $D(H) = C_0^2(\mathbb{R}^n)$.

We also have the Schrödinger quadratic form.

4.2 DEFINITION (Schrödinger quadratic form on $C_0^1(\mathbb{R}^n)$)**.** The Schrödinger quadratic form for a particle without internal degrees of freedom moving in a potential $V \in L^1_{\rm loc}(\mathbb{R}^n)$ is

$$Q(\phi) = \int_{\mathbb{R}^n} |\nabla\phi|^2 - \int_{\mathbb{R}^n} V|\phi|^2$$

with domain $D(Q) = C^1_0(\mathbb{R}^n)$.

Note that in order to define the quadratic form on $C^1_0(\mathbb{R}^n)$ we need only assume that $V \in L^1_{\rm loc}$ whereas for the operator we need $V \in L^2_{\rm loc}$.

4.3 PROBLEM. *If $V \in L^2_{\rm loc}$ show that the operator defined as explained in Problem 3.10 from the Schrödinger quadratic form Q with $D(Q) = C^1_0(\mathbb{R}^n)$ is indeed an extension of the Schrödinger operator $H = -\Delta - V$ to a domain which includes $C^2_0(\mathbb{R}^n)$. If $V \in L^1_{\rm loc} \setminus L^2_{\rm loc}$ then this need not be the case as explained in the next example.*

4.4 EXAMPLE. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = \begin{cases} |x|^{-n/2}, & \text{if } |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then f is in $L^1(\mathbb{R}^n)$ but not in $L^2(\mathbb{R}^n)$. Let $q_1, q_2, \ldots$ be an enumeration of the rational points in $\mathbb{R}^n$ and define $V(x) = \sum_i i^{-2} f(x - q_i)$. Then $V \in L^1_{\rm loc}(\mathbb{R}^n)$ but for all $\psi \in C^1_0(\mathbb{R}^n)$ we have $V\psi \notin L^2$. This follows easily since $|V(x)|^2|\psi(x)|^2 \geq i^{-2}|x - q_i|^{-n}|\psi(x)|^2$ for all i. Therefore the domain of the operator A defined from the Schrödinger quadratic form Q with $D(Q) = C^1_0(\mathbb{R}^n)$ is $D(A) = \{0\}$.

We shall now discuss ways of proving that the Schrödinger quadratic form is bounded from below.

We begin with the Perron-Frobenius Theorem for the Schrödinger operator. Namely, the fact that if we have found a non-negative eigenfunction for the Schrödinger operator then the corresponding eigenvalue is the lowest possible expectation for the Schrödinger quadratic form.

4.5 THEOREM (Perron Frobenius for Schrödinger)**.** *Let $V \in L^1_{\rm loc}(\mathbb{R}^n)$. Assume that $0 < \psi \in C^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and that $(-\Delta - V)\psi(x) = \lambda\psi(x)$ for all x in some set Ω. Then for all $\phi \in C^1_0(\mathbb{R}^n)$ with support in Ω we get*

$$Q(\phi) = \int_{\mathbb{R}^n} |\nabla\phi|^2 - \int_{\mathbb{R}^n} V|\phi|^2 \geq \lambda \int_{\mathbb{R}^n} |\phi|^2.$$

Proof. Given $\phi \in C_0^1(\mathbb{R}^n)$ we can write $\phi = f\psi$, where $f \in C_0^1(\mathbb{R}^n)$. Then

$$\begin{aligned} Q(\phi) &= \int_{\mathbb{R}^n} \left[\psi^2|\nabla f|^2 + |f|^2|\nabla\psi|^2 + \left(\overline{f}\nabla f + f\overline{\nabla f}\right)\psi\nabla\psi\right] - \int_{\mathbb{R}^n} V|f\psi|^2 \\ &\geq \int_{\mathbb{R}^n} \left[|f|^2|\nabla\psi|^2 + \left(\overline{f}\nabla f + f\overline{\nabla f}\right)\psi\nabla\psi\right] - \int_{\mathbb{R}^n} V|f\psi|^2 \\ &= \int_{\mathbb{R}^n} \left[|f|^2\psi(-\Delta - V)\psi\right], \end{aligned}$$

where the last identity follows by integration by parts. Hence $Q(\phi) \geq \lambda \int |f|^2\psi^2 = \lambda \int |\phi|^2$. □

4.6 COROLLARY (Lower bound on hydrogen)**.** *For all $\phi \in C_0^1(\mathbb{R}^3)$ we have*

$$\int |\nabla\phi(x)|^2 dx - \int Z|x|^{-1}|\phi(x)|^2 dx \geq -\frac{Z^2}{4}\int |\phi(x)|^2 dx.$$

Proof. Consider the function $\psi(x) = e^{-Z|x|/2}$. Then for all $x \neq 0$ we have $(-\Delta - Z|x|^{-1})\psi(x) = -\frac{Z^2}{4}\psi(x)$. The statement therefore immediately follows for all $\phi \in C_0^1(\mathbb{R}^3)$ with support away from 0 from the previous theorem. We leave it to the reader to show that all $\phi \in C_0^1(\mathbb{R}^3)$ can be approximated by functions $\phi_n \in C_0^1(\mathbb{R}^3)$ with support away from 0 in such away that

$$\int |\nabla\phi_n(x)|^2 dx - \int Z|x|^{-1}|\phi_n(x)|^2 dx \to \int |\nabla\phi(x)|^2 dx - \int Z|x|^{-1}|\phi(x)|^2 dx.$$

□

It is rarely possible to find positive eigenfunctions. A much more general approach to proving lower bounds on Schrödinger quadratic forms is to use the Sobolev inequality. In a certain sense this inequality is an expression of the celebrated uncertainty principle.

4.7 THEOREM (Sobolev Inequality)**.** *For all $\phi \in C_0^1(\mathbb{R}^n)$ with $n \geq 3$ we have the Sobolev inequality*

$$\|\phi\|_{\frac{2n}{n-2}} \leq \frac{2(n-1)}{n-2}\|\nabla\phi\|_2$$

Proof. Let $u \in C_0^1(\mathbb{R}^n)$ then we have

$$u(x) = \int_{-\infty}^{x_i} \partial_i u(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) dx_i'.$$

Hence

$$|u(x)|^{\frac{n}{n-1}} \le \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} |\partial_i u| dx_i' \right)^{\frac{1}{n-1}}.$$

Thus by the general Hölder inequality (in the case $n = 3$ simply by Cauchy-Schwarz)

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \le \left(\int_{-\infty}^{\infty} |\partial_1 u| dx_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_i u| dx_1 dx_i \right)^{\frac{1}{n-1}}.$$

Using the same argument for repeated integrations over $x_2, \dots, x_n$ gives

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \le \left(\prod_{i=1}^{n} \int_{\mathbb{R}^n} |\partial_i u| dx \right)^{\frac{1}{n-1}}.$$

Thus

$$\|u\|_{\frac{n}{n-1}} \le \|\nabla u\|_1.$$

Now set $u = \phi^{\frac{2(n-1)}{n-2}}$. (The reader may at this point worry about the fact that u is not necessarily C^1. One can easily convince oneself that the above argument works for this u too. Alternatively , in the case $n = 3$ which is the one of interest here $\frac{2(n-1)}{n-2}$ is an integer and thus u is actually C^1.) We then get

$$\|\phi\|_{\frac{2n}{n-2}}^{\frac{2(n-1)}{n-2}} \le \frac{2(n-1)}{n-2} \|\phi\|_{\frac{2n}{n-2}}^{\frac{n}{n-2}} \|\nabla \phi\|_2.$$

□

Especially for $n = 3$ we get

$$\|\phi\|_6 \le 4 \|\nabla \phi\|_2.$$

The sharp constant in the Sobolev inequality was found by Talenti, G. *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372. See the book *Analysis*, AMS Graduate Studies in Mathematics Vol. **114** by Lieb and Loss for a simple proof. In the case $n = 3$ the sharp version of the Sobolev inequality is

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$$\|\phi\|_6 \le \frac{\sqrt{3}}{2} (4\pi)^{1/3} \|\nabla \phi\|_2 \approx 2.01 \|\nabla \phi\|_2.$$

4.8 THEOREM (Sobolev lower bound on Schrödinger). *Assume that* $V \in L^1_{\rm loc}(\mathbb{R}^n)$, $n \geq 3$ *and that the positive part* $V_+ = \max V, 0$ *of the potential satisfies* $V_+ \in L^{\frac{2+n}{2}}(\mathbb{R}^n)$. *Then for all* $\phi \in C^1_0(\mathbb{R}^n)$ *we get*

$$Q(\phi) = \int_{\mathbb{R}^n} |\nabla\phi|^2 - \int_{\mathbb{R}^n} V|\phi|^2 \geq -\frac{2}{n+2}\left(\frac{n}{n+2}\right)^{n/2}\left(\frac{2(n-1)}{n-2}\right)^n \left(\int V_+^{\frac{2+n}{2}}\right)\|\phi\|_2^2$$

Proof. In order to prove a lower bound we may of course replace V by V_+. We use the Sobolev Inequality and Hölder's inequality

$$Q(\phi) \geq \left(\frac{2(n-1)}{n-2}\right)^{-2}\|\phi\|^2_{\frac{2n}{n-2}} - \|V_+\|_{\frac{2+n}{2}}\|\phi\|^{\frac{2n}{n+2}}_{\frac{2n}{n-2}}\|\phi\|_2^{\frac{4}{n+2}}.$$

We get a lower bound by minimizing over $t = \|\phi\|^2_{\frac{2n}{n-2}}$, i.e.,

$$Q(\phi) \geq \min_{t\geq 0}\left\{\left(\frac{2(n-1)}{n-2}\right)^{-2} t - \|V_+\|_{\frac{2+n}{2}}\|\phi\|_2^{\frac{4}{n+2}} t^{\frac{n}{n+2}}\right\},$$

which gives the answer above. □ Something is worng here

For $n = 3$ we find

$$Q(\phi) \geq -(2/5)(3/20)^{3/2}\left(\int V_+^{5/2}\right)\|\phi\|_2^2.$$

or using the sharp constant in the Sobolev inequality

$$Q(\phi) \geq -\frac{2}{5}(4\pi)^{-1/2}\left(\frac{6}{5\sqrt{3}}\right)^{3/2}\left(\int V_+^{5/2}\right)\|\phi\|_2^2. \qquad (6)$$

4.9 PROBLEM (Positivity of Schrödinger quadratic form). *Show that if* $V_+ \in L^{3/2}(\mathbb{R}^3)$ *and if the norm* $\|V_+\|_{3/2}$ *is small enough then*

$$Q(\phi) = \int_{\mathbb{R}^3} |\nabla\phi|^2 - \int_{\mathbb{R}^3} V|\phi|^2 \geq 0$$

for all $\phi \in C^1_0(\mathbb{R}^3)$.

4.10 EXAMPLE (Sobolev lower bound on hydrogen). We now use the Sobolev inequality to give a lower bound on the hydrogen quadratic form

$$Q(\phi) = \int_{\mathbb{R}^3} |\nabla\phi|^2 - \int_{\mathbb{R}^3} Z|x|^{-1}|\phi|^2.$$

In Corollary 4.6 we of course already found the sharp lower bound for the hydrogen energy. This example serves more as a test of the applicability of the Sobolev inequality.

For all $R > 0$

$$Q(\phi) \geq \int_{\mathbb{R}^3} |\nabla\phi|^2 - \int_{|x|\leq R} Z|x|^{-1}|\phi|^2 - ZR^{-1}\int_{\mathbb{R}^3}|\phi|^2.$$

Using (6) we find

$$\begin{aligned} Q(\phi) &\geq \left[-\frac{2}{5}(4\pi)^{-1/2}\left(\frac{6}{5\sqrt{3}}\right)^{3/2}\left(\int_{|x|<R}(Z|x|^{-5/2}\right) - ZR^{-1}\right]\int_{\mathbb{R}^3}|\phi|^2 \\ &= -2.22Z^{5/2}R^{1/2} - ZR^{-1} \geq -3.22Z^2, \end{aligned}$$

where we have minimized over R. This result should be compared with the sharp value $-0.25Z^2$.

4.11 PROBLEM (Hardy's Inequality)**.** *Show that for all $\phi \in C_0^1(\mathbb{R}^3)$ we have*

$$\int_{\mathbb{R}^3} |\nabla\phi(x)|^2 dx \geq \frac{1}{4}\int_{\mathbb{R}^3}|x|^{-2}|\phi(x)|^2 dx.$$

Show also that $1/4$ is the sharp constant in this inequality.

4.12 EXAMPLE (Dirichlet and Neumann Boundary conditions). Consider the quadratic form

$$Q(\phi) = \int_0^1 |\phi'|^2$$

on the Hilbert space $L^2([0,1])$ with the domain

$$D_0(Q) = \left\{\phi \in C^1([0,1]) : \phi(0) = \phi(1) = 0\right\}$$

or

$$D_1(Q) = C^1([0,1]).$$

The operator A_0 corresponding to Q according to Problem 3.10 with domain $D_0(Q)$ satisfies that

$$\left\{\phi \in C^2([0,1]) : \phi(0) = \phi(1) = 0\right\} \subset D(A_0)$$

and if $\phi \in C^2([0,1])$ with $\phi(0) = \phi(1) = 0$ then $A_0\phi = -\phi''$. This follows by integration by parts since if $\psi \in D_0(Q)$ then

$$Q(\psi,\phi) = \int_0^1 \overline{\psi'}\phi' = \phi'(1)\overline{\psi(1)} - \phi'(0)\overline{\psi(0)} - \int_0^1 \overline{\psi}\phi'' = -\int \overline{\psi}\phi''.$$

The condition $\phi(0) = \phi(1) = 0$ is called Dirichlet boundary condition.

The operator A_1 corresponding to Q with domain $D_1(Q)$ satisfies that

$$\left\{\phi \in C^2([0,1]) : \phi'(0) = \phi'(1) = 0\right\} \subset D(A_1)$$

and if $\phi \in C^2([0,1])$ with $\phi'(0) = \phi'(1) = 0$ then $A_1\phi = -\phi''$. Note that this time there was no boundary condition in the domain $D_1(Q)$, but it appeared in the domain of A_1. The boundary condition $\phi'(0) = \phi'(1) = 0$ is called Neumann boundary condition. The statement again follows by integration by parts as above. This time the boundary terms do not vanish automatically. However since the map $(\psi,\phi) \mapsto \phi'(1)\overline{\psi(1)}$ is not bounded on L^2 we have to ensure the vanishing of the boundary terms in the definition of the domain of A_1.

4.13 PROBLEM. *Show that the eigenvalues of the Dirichlet operator A_0 in Example 4.12 are $n^2\pi^2$, $n = 1, 2, \ldots$. Show that the eigenvalues of the Neumann operator A_0 in Example 4.12 are $n^2\pi^2$, $n = 0, 1, 2, \ldots$.*

5 The min-max principle form bounded operators

The *min-max principle* is a method for determining the eigenvalues of an operator A in terms of the corresponding quadratic form $Q_A(\psi,\phi) = (\psi, A\phi)$. We illustrate it first for compact operators.

5.1 DEFINITION (Compact operators)**.** A bounded operator $K : \mathcal{H} \to \mathcal{H}$ is said to be a normal compact operator if there is an orthonormal basis $\phi_1, \phi_2, \ldots$ for $\mathcal{H}$ consisting of eigenvectors of K, i.e., such that Changed Oct6/97

$$K\phi_i = \lambda_i\phi_i$$

and such that $\lim_j \lambda_j = 0$.

5.2 THEOREM (Max-min principle for positive compact operators)**.** *Let $K : \mathcal{H} \to \mathcal{H}$ be a positive definite compact operator. Then*

$$\mu_k = \sup_{M,\ \dim M = k} \inf_{\psi \in M \setminus \{0\}} \frac{(\psi, K\psi)}{\|\psi\|^2},$$

(where the sup is over subspaces M of $\mathcal{H}$), are the eigenvalues for K in decreasing order.

The sup and the inf are really a max and min. Indeed, if, $\phi_1, \phi_2, \ldots$ are the eigenvectors of K ordered such that the corresponding eigenvalues are decreasing then

$$\mu_k = \min_{\psi \in M \setminus \{0\}} \frac{(\psi, K\psi)}{\|\psi\|^2}$$

where $M = \mathrm{span}\{\phi_1, \ldots, \phi_k\}$ and the min is attained for $\psi = \phi_k$.

REMARK: Although the principle is usually called the min-max principle (for reasons that will become clear in Theorem 7.19 below) we here call it the max-min principle since the sup comes before the inf.

Proof. Note that the numbers μ_k *are* decreasing.

Let $\phi_1, \phi_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ be the eigenvectors and corresponding eigenvalues (in decreasing order).

If we set $M = \mathrm{span}\{\phi_1, \ldots, \phi_k\}$ we easily see that

$$\inf_{\psi \in M \setminus \{0\}} \frac{(\psi, K\psi)}{\|\psi\|^2} = \lambda_k.$$

Thus, $\mu_k \geq \lambda_k$.

On the other hand if M is *any* subspace with $\dim M = k$, then $M \cap \mathrm{span}\{\phi_1, \ldots, \phi_{k-1}\}^\perp$ contains a non-zero unit vector ψ (proof: exercise). Thus $\psi = \alpha_k \phi_k + \alpha_{k+1}\phi_{k+1} + \ldots$, with $|\alpha_k|^2 + |\alpha_{k+1}|^2 + \ldots = 1$ and we have

$$(\psi, K\psi) = |\alpha_k|^2 \lambda_k + |\alpha_{k+1}|^2 \lambda_{k+1} + \ldots \leq \lambda_k.$$

Thus to any k-dimensional subspace M there is a non-zero $\psi \in M$ with $Q(\psi)/\|\psi\|^2 \leq \lambda_k$. Therefore $\mu_k \leq \lambda_k$ and hence $\mu_k = \lambda_k$. $\square$

5.3 PROBLEM. *What would a correct formulation of this theorem be if K was only positive semi-definite?*

5.4 PROBLEM. *Let K_1 and K_2 be two positive definite compact operators. Use the max-min principle to show that the k-th eigenvalue of $K_2 + \alpha K_1$ is a continuous function of α for $\alpha > 0$. Here k-th refers to the decreasing order.*

Changed Oct6/97

5.5 PROBLEM. *Show that if K is a normal compact operator, but not necessarily positive or positive semi-definite, then*

$$\max_{\psi \in \mathcal{H}\setminus\{0\}} \frac{|(\psi, K\psi)|}{(\psi,\psi)}$$

is the modulus of the eigenvalue of largest modulus. Show also that the max is attained for the corresponding eigenvector. Use this to prove the Perron-Frobenius Theorem: Assume A is a normal $n \times n$-matrix with strictly positive elements $a_{ij} > 0$. Then the eigenvalue λ of A with greatest modulus $|\lambda|$ is positive (i.e, $\lambda = |\lambda|$), non-degenerate and has an eigenvector with strictly positive components.

The same conclusions hold with the weaker assumptions that A is any $n \times n$-matrix (not necessarily normal) such that there is a positive power A^k with strictly positive elements.

We now consider converse statements. Thus we assume that we have a quadratic form Q, but we do not assume that it is given by a compact operator. We want to use a max-min principle to determine the eigenvalues of the operator A defined in Problem 3.10 by $(\psi, A\psi) = Q(\psi)$. The operator A may not have a single eigenvalue. The following theorem gives a sufficient (but certainly not necessary) condition for the existence of an eigenvalue.

5.6 THEOREM (The largest eigenvalue of a positive quadratic form)**.** *Let Q be a positive semi-definite quadratic form defined everywhere, i.e., with $D(Q) = \mathcal{H}$. Let A be the operator given by $(\psi, A\psi) = Q(\psi)$.*

If there is a unit vector ϕ_1 such that

$$Q(\phi_1) = \frac{Q(\phi_1)}{\|\phi_1\|^2} = \sup_{\psi \in \mathcal{H}\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2}$$

then ϕ_1 is an eigenvector of A with eigenvalue

$$\mu_1 = Q(\phi_1).$$

Proof. Since μ_1 is the maximal possible value for $Q(\psi)/\|\psi\|^2$ we have for all $\psi \in \mathcal{H}$ that

$$0 = \frac{d}{dt}\left[\frac{Q(\phi_1 + t\psi)}{\|\phi_1 + t\psi\|^2}\right]\bigg|_{t=0} = \|\phi_1\|^{-2}\left[Q(\phi_1, \psi) + Q(\psi, \phi_1) - \mu_1(\phi_1, \psi) - \mu_1(\psi, \phi_1)\right].$$

Thus

$$Q(\phi_1, \psi) + Q(\psi, \phi_1) - \mu_1(\phi_1, \psi) - \mu_1(\psi, \phi_1) = 0.$$

Since this is true for all ψ we can insert $i\psi$. If we use the identity for ψ as well as for $i\psi$ and use that Q is sesquilinear we easily get

$$(\psi, A\phi_1) = Q(\psi, \phi_1) = \mu_1(\psi, \phi_1).$$

Since this is true for all ψ we conclude that $A\phi_1 = \mu_1\phi_1$. □

5.7 COROLLARY. *Let $Q(\psi)$ be a positive semi-definite quadratic form defined on all of $\mathcal{H}$ and let again A be the corresponding operator. Consider the numbers*

$$\mu_k = \sup_{M,\ \dim M = k}\ \inf_{\psi \in M\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2}. \tag{7}$$

If $\mu_1 < \infty$ and is not an eigenvalue for A then

$$\mu_1 = \mu_2 = \ldots = \mu_k = \ldots \tag{8}$$

Proof. Assume that all the μ's are not the same i.e., $\mu_1 = \ldots = \mu_{k_0} > \mu_{k_0+1}$. From the definition of the μ's it follows that there is a sequence of k_0-dimensional spaces M_n such that

$$\inf_{\psi \in M_n\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2} \to \mu_{k_0} = \mu_1. \tag{9}$$

We shall prove that we may find unit vectors $\psi_n \in M_n$ such that they form a Cauchy sequence and hence converge to some vector $\phi_1 \in \mathcal{H}$. But let us first show that ϕ_1 is an eigenvector of A.

Note that by (7) for $k = 1$ (here the inf is irrelevant since M is one-dimensional)we have that $Q(\psi) \le \mu_1\|\psi\|^2$ for all $\psi \in \mathcal{H}$. It follows from (9) that $Q(\psi_n) \to \mu_1$ as $n \to \infty$. By the Cauchy-Schwarz inequality $Q(\psi, \phi) \le Q(\psi, \psi)^{1/2}Q(\phi, \phi)^{1/2}$ (which holds since Q is positive) we have that

$$|Q(\psi_n)^{1/2} - Q(\phi_1)^{1/2}|^2 \le Q(\psi_n - \phi_1) \le \mu_1\|\psi_n - \phi_1\|^2 \to 0$$

as $n \to \infty$. Therefore $Q(\phi_1)/\|\phi_1\|^2 = \mu_1$ and Theorem 5.6 implies that ϕ_1 is an eigenvector with eigenvalue μ_1.

It still remains to prove the existence of the Cauchy sequence ψ_n. In order to prove this it is enough to show that for any two k_0-dimensional subspaces N_1 and N_2 satisfying

$$\inf_{\psi \in N_i \setminus \{0\}} \frac{Q(\psi)}{\|\psi\|^2} > \mu_1 - \varepsilon, \quad i = 1, 2 \tag{10}$$

for some $\varepsilon > 0$ (small enough) and any unit vector $\psi_1 \in N_1$ we can find a unit vector $\psi_2 \in N_2$, such that

$$\|\psi_1 - \psi_2\|^2 \leq \frac{16\varepsilon}{\mu_1 - \mu_{k_0+1}}. \tag{11}$$

If $\psi_1 \in N_2$ this is trivial. Otherwise $\dim \operatorname{span}\{\psi_1\} \cup N_2 \geq k_0 + 1$ and hence we can find a vector $\widetilde{\psi} \in \operatorname{span}\{\psi_1\} \cup N_2$ such that $Q(\widetilde{\psi}) \leq \mu_{k_0+1}\|\widetilde{\psi}\|^2$. We may write $\widetilde{\psi} = u_1 - u_2$, where $u_1/\|u_1\| = \psi_1$ and $u_2 \in N_2$. We therefore have

$$\begin{aligned}
\mu_{k_0+1}\|u_1 - u_2\|^2 &\geq Q(u_1 - u_2) = 2Q(u_1) + 2Q(u_2) - Q(u_1 + u_2) \\
&\geq 2(\mu_1 - \varepsilon)\|u_1\|^2 + 2(\mu_1 - \varepsilon)\|u_2\|^2 - Q(u_1 + u_2) \quad \text{by (10)} \\
&\geq 2(\mu_1 - \varepsilon)\|u_1\|^2 + 2(\mu_1 - \varepsilon)\|u_2\|^2 - \mu_1\|u_1 + u_2\|^2,
\end{aligned}$$

where the last inequality follows since $Q(\psi) \leq \mu_1\|\psi\|^2$ for all $\psi \in \mathcal{H}$. We can rewrite this as

$$\mu_{k_0+1}\|u_1 - u_2\|^2 \geq \mu_1\|u_1 - u_2\|^2 - 2\varepsilon(\|u_1\|^2 + \|u_2\|^2)$$

or

$$\|u_1 - u_2\|^2 \leq \frac{2\varepsilon}{\mu_1 - \mu_{k_0+1}}(\|u_1\|^2 + \|u_2\|^2). \tag{12}$$

Assume without loss of generality that $\|u_2\| \leq \|u_1\|$. Then $u_1 \neq 0$ and it follows from (12) that if ε is small enough then $\|u_2\| \neq 0$. Using

$$\left\|\frac{u_1}{\|u_1\|} - \frac{u_2}{\|u_2\|}\right\| \leq 2\frac{\|u_1 - u_2\|}{\|u_1\|},$$

and (12) we have that

$$\left\|\frac{u_1}{\|u_1\|} - \frac{u_2}{\|u_2\|}\right\|^2 \leq \frac{8\varepsilon}{\mu_1 - \mu_{k_0+1}}\left(1 + \frac{\|u_2\|^2}{\|u_1\|^2}\right) \leq \frac{16\varepsilon}{\mu_1 - \mu_{k_0+1}}$$

and (11) follows with $\psi_2 = u_2/\|u_2\|$. $\square$

We shall now explain how the method in Theorem 5.6 can be continued inductively.

5.8 LEMMA. *Let again $Q(\psi) = (\psi, A\psi)$ be a positive semi-definite quadratic form defined on all of $\mathcal{H}$ and let μ_k be defined as in (7). If μ_1 is an eigenvalue for A and ϕ_1 is the corresponding eigenvector then A maps $\mathcal{H}_1 = \mathrm{span}\{\phi_1\}^\perp$ to itself, i.e., $A(\mathcal{H}_1) \subset \mathcal{H}_1$ and*

$$\mu_{k+1} = \sup_{M\subset\mathcal{H}_1,\ \dim M=k} \inf_{\psi\in M\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2} \tag{13}$$

Proof. That $A(\mathcal{H}_1) \subset \mathcal{H}_1$ follows because A is symmetric hence $\psi \in \mathcal{H}_1$ implies $(A\psi, \phi_1) = (\psi, A\phi_1) = \mu_1(\psi, \phi_1) = 0$, i.e., $A\psi \perp \phi_1$ and thus $A\psi \in \mathcal{H}_1$.

To prove (13) notice that if $\psi \perp \phi_1$ then as we just saw $Q(\psi, \phi_1) = (\psi, A\phi_1) = 0$ hence if $\psi \neq 0$

$$\inf_{t\in\mathbb{C}} \frac{Q(\psi + t\phi_1)}{\|\psi + t\phi_1\|^2} = \inf_{t\in\mathbb{C}} \frac{Q(\psi) + |t|^2\mu_1}{\|\psi\|^2 + |t|^2} = \frac{Q(\psi)}{\|\psi\|^2}, \tag{14}$$

where we also used that $Q(\psi) \leq \mu_1\|\psi\|^2$.

For all $M \subset \mathcal{H}_1$ we therefore have that

$$\inf_{\psi\in M\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2} = \inf_{\psi\in\mathrm{span}(M\cup\{\phi_1\})\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2}.$$

This shows that

$$\sup_{M\subset\mathcal{H}_1,\ \dim M=k} \inf_{\psi\in M\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2} \leq \mu_{k+1}.$$

On the other hand if M' is a $k+1$ dimensional subspace of $\mathcal{H}$ with $1 \leq k$ then $M' \cap \mathcal{H}_1$ is a non-trivial subspace of $\mathcal{H}_1$ (proof: same exercise as in the proof of Thereom 5.2) and it follows from (14) that

$$\begin{aligned}
\inf_{\psi\in M'\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2} &= \inf_{\psi\in(M'\cap\mathcal{H}_1)\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2} \\
&\leq \sup_{M\subset M'\cap\mathcal{H}_1,\ \dim M=k} \inf_{\psi\in M\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2} \\
&\leq \sup_{M\subset\mathcal{H}_1,\ \dim M=k} \inf_{\psi\in M\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2}.
\end{aligned}$$

Since this is true for all $k+1$ dimensional subspaces M' we have

$$\mu_{k+1} = \sup_{M',\dim M'=k+1} \inf_{\psi\in M'\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2} \leq \sup_{M\subset\mathcal{H}_1,\ \dim M=k} \inf_{\psi\in M\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2}$$

and we have shown (13). □

We can now give the complete converse to Theorem 5.2.

5.9 THEOREM (Inverse max-min)**.** *If Q is a positive quadratic form defined on $D(Q) = \mathcal{H}$ and if the numbers μ_k defined in (7) satisfy* $\lim_k \mu_k = 0$*. Then the operator K defined by $Q(\psi) = (\psi, K\psi)$ is compact.*

Proof. If all the μ_k are identical they must all be zero and there is nothing to prove because then $Q(\psi) = 0$. Otherwise we have from Corollary 5.7 that μ_1 is an eigenvalue. Using Lemma 5.8 we inductively prove that all the non-zero μ_k are eigenvalues of K. Since K is symmetric it follows that we may choose the corresponding eigenvectors $\phi_1, \phi_2, \ldots$ so that they form an orthonormal family.

Consider the orthogonal complement to the eigenvectors with non-zero eigenvalues $M_\infty = \mathrm{span}\{\phi_1, \phi_2 \ldots\}^\perp$. It follows inductively from (13) that

$$\sup_{\psi \in \mathrm{span}\{\phi_1, \ldots, \phi_{k-1}\}^\perp} \frac{Q(\psi)}{\|\psi\|^2} = \mu_k.$$

Since $M_\infty \subset \mathrm{span}\{\phi_1, \ldots, \phi_{k-1}\}^\perp$ we have for $\psi \in M_\infty \setminus \{0\}$ that

$$\frac{Q(\psi)}{\|\psi\|^2} \leq \inf_k \mu_k = 0.$$

Thus $Q(\psi) = 0$ for all $\psi \in M_\infty$ and since $K(M_\infty) \subset M_\infty$ it follows that M_∞ is simply the kernel of K. we can therefore supplement $\phi_1, \phi_2, \ldots$ with an orthonormal basis for the kernel and we have found a basis of eigenvectors for K. □

5.10 THEOREM (Max-min for powers)**.** *For a positive semi-definite and bounded operator A let $\mu_k(A)$, $k = 1, 2, \ldots$ be the numbers defined as in (7) with $Q(\psi) = (\psi, A\psi)$. Then $\mu_k(A^m) = (\mu_k(A))^m$ for all positive integers m and k.* New Oct6/97

This is a simple consequence of the following lemma.

5.11 LEMMA. *Let A be a bounded positive semi-definite operator on a Hilbert space $\mathcal{H}$ and $\mu = \sup_{\|\phi\|=1}(\phi, A\phi)$. Then $(\phi, A^m\phi) \leq \mu(\phi, A^{m-1}\phi)$ for all $\phi \in \mathcal{H}$. And for all sequences of unit vectors ϕ_n with $(\phi_n, A\phi_n) \to \mu$ as $n \to \infty$ we have $(\phi_n, A^m\phi_n) \to \mu^m$ as $n \to \infty$.*

Proof. If $m = 2k+1$ we have $(\phi, A^{2k+1}\phi) = (A^k\phi, AA^k\phi) \leq \mu(\phi, A^{2k}\phi)$. If $m = 2k$ we find from the Cauchy-Schwarz inequality for the quadratic form $(\phi, A\phi)$ that

$$\begin{aligned}(\phi, A^{2k}\phi) &= (A^k\phi, AA^{k-1}\phi) \leq (A^k\phi, AA^k\phi)^{1/2}(A^{k-1}\phi, AA^{k-1}\phi)^{1/2} \\ &\leq \mu^{1/2}(\phi, A^{2k}\phi)^{1/2}(\phi, A^{2k-1}\phi)^{1/2}.\end{aligned}$$

This proves the first statement.

Since $\mu - A$ is a positive semi-definite operator we find using the Cauchy-Schwarz inequality for the quadratic form $(\phi, (\mu - A)\phi)$ that

$$0 \leq (\phi_n, (\mu - A)^m \phi_n) \leq (\phi_n, (\mu - A)\phi_n)^{1/2} (\phi_n, (\mu - A)^{2m-1} \phi_n)^{1/2} \to 0.$$

as $n \to \infty$. This clearly implies the second statement. $\square$

We shall often have to estimate the trace of positive operators.

5.12 DEFINITION (Trace of positive operator)**.** The trace of a positive (semi-definite) operator A can be defined as

$$\mathrm{Tr} A = \sum_{k=1}^{\infty} (u_k, A u_k),$$

where $u_1, u_2, \ldots \in D(A)$ is an orthonormal basis for $\mathcal{H}$. If $\mathrm{Tr} A < \infty$ we say that A is trace class.

Question: Why is it important to say that A is positive?

5.13 PROBLEM. *Prove that since $D(A)$ is dense in $\mathcal{H}$ we can always find a basis for $\mathcal{H}$ which belongs to $D(A)$.*

5.14 PROBLEM. *Show that the definition of trace is independent of the choice of basis (see also Theorem 5.17) below.*

5.15 THEOREM (Variational principle for eigenvalue sum)**.** *Let K be a positive semi-definite compact operator and let $\lambda_1 \geq \lambda_2, \ldots$ be the eigenvalues in decreasing order. Then*

$$\sum_{k=1}^{N} \lambda_k = \sup \left\{ \sum_{k=1}^{N} (\psi_k, K \psi_k) : \psi_1, \ldots, \psi_N \textit{ are orthonormal} \right\}.$$

Moreover, if $\psi_1, \ldots, \psi_k$ are orthonormal and

$$\sum_{k=1}^{N} \lambda_k = \sum_{k=1}^{N} (\psi_k, K \psi_k)$$

then $\mathrm{span}\{\psi_1, \ldots, \psi_k\} = \mathrm{span}\{\phi_1, \ldots, \phi_k\}$*, where $\phi_1, \ldots, \phi_k$ are eigenvectors for K corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$.*

5.16 PROBLEM. *Prove Theorem 5.15.*

5.17 THEOREM (Variational principle for trace)**.** *If A is a positive semi-definite operator then*

$$\mathrm{Tr} A = \sup_N \sup \left\{ \sum_{k=1}^{N} (\psi_k, A\psi_k) : \psi_1, \ldots, \psi_N \in D(A) \text{ are orthonormal} \right\}.$$

Proof. Easy consequence of Problem 5.14. □

6 The Banach-Alaoglu Theorem

We shall prove the Banach-Alaoglu Theorem. It will beone of the most useful tools from abstract functional analysis.

Usually this is proved using Tychonov's Theorem and thus relies on the axiom of choice. In the separable case this is however not necessary and we give a straightforward proof here.

6.1 THEOREM (Banach-Alaoglu)**.** . *Let X be a Banach space and X^* the dual Banach space of continuous linear functionals. Assume that the space X is separable, i.e., has a countable dense subset. Then to any sequence $\{x_n^*\}$ in X^* which is bounded, i.e., with $\|x_n^*\| \le M$ for some $M > 0$ there exists a weak-$*$ convergent subsequence $\{x_{n_k}^*\}$. Weak-$*$ convergent means that there exists $x^* \in X^*$ such that $x_{n_k}^*(x) \to x^*(x)$ as $k \to \infty$ for all $x \in X$.*

Proof. Let $x_1, x_2, \ldots$ be a countable dense subset of X. Since $\{x_n^*\}$ is a bounded sequence we know that all the sequences

$$x_1^*(x_1), x_2^*(x_1), x_3^*(x_1), \ldots$$

$$x_1^*(x_2), x_2^*(x_2), x_3^*(x_2), \ldots$$

$$\vdots$$

are bounded. We can therefore find convergent subsequences

$$x_{n_{11}}^*(x_1), x_{n_{12}}^*(x_1), x_{n_{13}}^*(x_1) \ldots$$

$$x_{n_{21}}^*(x_2), x_{n_{22}}^*(x_2), x_{n_{23}}^*(x_2) \ldots$$

$$\vdots$$

with the property that the sequence $n_{k+11}, n_{k+12}, \ldots$, is a a subsequence of $n_{k1}, n_{k2}, \ldots$. It is then clear that the tail $n_{kk}, n_{k+1k+1}, \ldots$ of the diagonal sequence $n_{11}, n_{22}, \ldots$ is a subsequence of $n_{k1}, n_{k2}, \ldots$ and hence that for all $k \geq 1$ the sequence

$$x^*_{n_{11}}(x_k), x^*_{n_{22}}(x_k), x^*_{n_{33}}(x_k) \ldots$$

is convergent. Now let $x \in X$ be any element of the Banach space then

$$\begin{aligned}
|x^*_{n_{pp}}(x) - x^*_{n_{qq}}(x)| &\leq |x^*_{n_{pp}}(x) - x^*_{n_{pp}}(x_k)| + |x^*_{n_{qq}}(x) - x^*_{n_{qq}}(x_k)| \\
&\quad + |x^*_{n_{pp}}(x_k) - x^*_{n_{qq}}(x_k)| \\
&\leq 2M\|x - x_k\| + |x^*_{n_{pp}}(x_k) - x^*_{n_{qq}}(x_k)|.
\end{aligned}$$

Since $\{x_k\}$ is dense we conclude that $x^*_{n_{pp}}(x)$ is a Cauchy sequence for all $x \in X$. Hence $x^*_{n_{pp}}(x)$ is a convergent sequence for all $x \in X$. Let $\Lambda(x) = \lim_{p\to\infty} x^*_{n_{pp}}(x)$. Then Λ is clearly a linear map and $|\Lambda(x)| \leq M\|x\|$. Hence $\Lambda \in X^*$. □

6.2 COROLLARY (Banach-Alouglu on Hilbert spaces). *If $\{x_n\}$ is a bounded sequence in a Hilbert space $\mathcal{H}$ (separable or not) then there exists a subsequence $\{x_{n_k}\}$ that converges weakly in $\mathcal{H}$.*

Proof. Consider the space X which is the closure of the space spanned by x_n, $n+1, \ldots$. This space X is a separable Hilbert space and hence is its own dual. Thus we may find a subsequence $\{x_{n_k}\}$ and an $x \in X$ such that $x_{n_k} \to x$ weakly in X. If $y \in \mathcal{H}$ let y' be its orthogonal projection onto X. We then have

$$\lim_k (x_{n_k}, y) = \lim_n (x_n, y') = (x, y') = (x, y).$$

Thus $x_{n_k} \to x$ weakly in $\mathcal{H}$. □

7 The Friedrichs' extension and the min-max principle for semi bounded operators

We now turn to the min-max principle for unbounded operators. We shall always consider operators that are bounded below. For simplicity we begin by considering operators A with $A \geq I$. As usual we consider also the corresponding quadratic form $Q(\psi) = (\psi, A\psi)$ defined on $D(Q) = D(A)$. Then our assumption is that

$$Q(\psi) = (\psi, A\psi) \geq \|\psi\|^2. \tag{15}$$

Before we can state the min-max principle it may first be necessary to enlarge the definiton set $D(A)$. This is a very important construction because it will allow us later to define *weak derivatives.* These are derivatives of functions that are not differentiable in the classical sense. A more general treatment can be done in the context of distribution theory, but we shall not discuss that here. When A is a differential operator, as it will be in many of our applications, then A is a-priori defined only on differentiable functions. The procedure that we now discuss is one way of enlarging the set.

The first observation has to do with Cauchy sequences.

7.1 DEFINITION (Cauchy for quadratic forms)**.** Let Q be a (not necessarily positive semi-definite) quadratic form with domain $D(Q)$. We shall say that $\psi_n \in D(Q)$ is a Cauchy sequence for Q if for all $\varepsilon > 0$

$$|Q(\psi_n - \psi_m)| < \varepsilon$$

for all n, m large enough.

If ψ_n is a Cauchy-sequence for Q and if Q satisfies (15) we see that $\|\psi_n - \psi_m\|^2 < \varepsilon$ for n, m large enough and ψ_n is also a Cauchy sequence for $\|\psi\|^2$. It follows (since a Hilbert space is complete) that there is a $\psi_\infty \in \mathcal{H}$ such that ψ_n converges to ψ_∞ as $n \to \infty$.

Since $Q(\psi_n)$ forms a Cauchy sequence of numbers we may define $Q(\psi) = \lim_n Q(\psi_n)$ even if $\psi \notin D(A)$. We must however argue that this does not depend on the sequence ψ_n. For this it is enough to consider the case $\psi = 0$ (if we have two Cauchy sequences ψ_n and ϕ_n for Q then the difference is also a Cauchy sequence). We therefore introduce the notion of being closable.

7.2 DEFINITION (Closable forms)**.** A quadratic form Q is said to be closable if

$$\psi_n \text{ Cauchy for } Q, \quad \text{and} \quad \lim_{n\to\infty} \psi_n = 0 \text{ in } \mathcal{H} \tag{16}$$

imply that

$$\lim_{n\to\infty} Q(\psi_n) \to 0.$$

7.3 PROPOSITION (Automatic closability). *Let $A \geq I$. Then the quadratic form defined by $Q(\psi, \phi) = (\psi, A\phi)$ for $\phi, \psi \in D(A)$ is closable.*

Proof. Consider a sequence $\psi_n \in \mathcal{H}$ satisfying (16). Then by Cauchy-Schwarz

$$Q(\psi_n) = Q(\psi_m, \psi_n) + Q(\psi_n - \psi_m, \psi_n) \leq (\psi_m, A\psi_n) + Q(\psi_n - \psi_m)^{1/2} Q(\psi_n)^{1/2}.$$

Given $\varepsilon > 0$ we can choose n_0 and m_0 such that for $n > n_0$ and $m > m_0$ we have $Q(\psi_n) < \varepsilon$. □

7.4 DEFINITION (Closure of quadratic form). Any closable quadratic form Q satisfying that $Q(\psi) \geq \|\psi\|^2$ can be extended as described above to a quadratic form on the set

$$\overline{D}(Q) = \{\phi \in \mathcal{H} : \text{ There exists a Cauchy sequence for } Q \text{ converging to } \psi \ \}.$$

The extended quadratic form is called the closure of Q.

WARNING: Even though $D(A)$ is dense it is not true that $\overline{D}(Q)$ is all of $\mathcal{H}$. The reason is that it is more restrictive to be a Cauchy sequence for Q than to be a Cauchy sequence for $\|\psi\|^2$. There may therefore be sequences in $D(A)$ which converge in $\mathcal{H}$ but which are not Cauchy sequences for Q and therefore the limiting vector may be outside of $\overline{D}(Q)$.

We shall next see that closability is closely related to norm lower semi-continuity. This will allow us to talk about quadratic forms defined on all of $\mathcal{H}$ possibly with the value $+\infty$.

Changed Oct6/97

7.5 PROPOSITION (norm lower continuity implies lower bound). *Let $F : D \to \mathbb{R} \cup \{\infty\}$ be a norm lower semi-continuous function defined on a subspace D of a Hilbert space. If F satisfies $F(\alpha\phi) = |\alpha|^2 F(\phi)$ for all $\alpha \in \mathbb{R}$ and all $\phi \in D$ then F is bounded below in the sense that there exists $M < \infty$ such that $F(\phi) \geq -M\|\phi\|^2$ for all $\phi \in D$.*

Proof. We first note that the set $\{\phi \in D : F(\phi) \leq -1\}$ is closed relative to D. This is true since if $\{\phi_n\}$ is a sequence from this set and if $\phi_n \to \phi$ in D then by the norm lower semi-continuity $F(\phi) \leq \liminf_n F(\phi_n) \leq -1$ and thus ϕ is also in the set. Hence $\{\phi \in D : F(\phi) > -1\}$ is open relative to D. We can therefore find $c > 0$ such that $\|\phi\| < c \Rightarrow F(\phi) > -1$ for all $\phi \in D$ and hence $\|\phi\| = c/2 \Rightarrow F(\phi) > -4c^{-2}\|\phi\|^2$. Thus for all $\phi \in D \setminus \{0\}$ we have

$$F(\phi) = F\left(\frac{2\|\phi\|}{c}\frac{c\phi}{2\|\phi\|}\right) = \left|\frac{2\|\phi\|}{c}\right|^2 F\left(\frac{c\phi}{2\|\phi\|}\right) \geq -4c^{-2}\|\phi\|^2.$$

□

Changed Oct6/97

7.6 THEOREM (Lower semi-continuity and closability)**.** *A quadratic form Q defined on a subspace $D(Q)$ of a Hilbert space $\mathcal{H}$ is closable and bounded below if and only if $\phi \mapsto Q(\phi)$ is norm lower semi-continuous on $D(Q)$.*

Proof. Using Proposition 7.5 we may by, if necessary, replacing Q by the quadratic form $Q(\phi) + (M+1)\|\phi\|^2$ assume that $Q(\phi) \geq \|\phi\|^2$.

We first assume that Q is closable. We may then close Q to a quadratic form defined on the set $\overline{D}(Q)$. Then $\overline{D}(Q)$ is a Hilbert space with the norm $Q(\phi)^{1/2}$. Assume now that $\phi_n \to \phi_\infty$ within $D(Q)$, where the convergence is in the norm of $\mathcal{H}$. We shall prove that $Q(\phi_\infty) \leq \liminf_n Q(\phi_n)$. Assume on the contrary that $Q(\phi_\infty) > \lim_n Q(\phi_n)$ (we may of course assume that $Q(\phi_n)$ converges). Then $Q(\phi_n)$ is bounded and thus by the Banach-Alaoglu Theorem (or rather its corollary to HIlbert spaces Corollary 6.2) we may assume (by if necessary passing to a subsequence) that ϕ_n converges weakly in the Q sense, i.e., in the space $\overline{D}(Q)$ to some $\psi_\infty \in \overline{D}(Q)$. It follows from Mazur's Theorem 3.14 that $Q(\psi_\infty) \leq \liminf_n Q(\phi_n)$. We shall now prove that $\psi_\infty = \phi_\infty$ and thus that $Q(\phi_\infty) \leq \liminf_n Q(\phi_n)$, which is a contradiction.

The important observation is that $\overline{D}(Q)$ is a *Hilbert space* with the inner product $Q(\phi, \psi)$. What we did above when discussing Cauchy sequences was to ensure that $\overline{D}(Q)$ is complete.

On this *new* Hilbert space we may consider the *old* inner product (ϕ, ψ) as a quadratic form defined on all of $\overline{D}(Q)$. It satisfies

$$|(\phi, \psi)| \leq \|\phi\|\|\psi\| \leq Q(\phi)^{1/2}Q(\psi)^{1/2}.$$

It folows by Riesz' representation Theorem that to each $\phi \in \mathcal{H}$ there is a $\phi' \in \overline{D}(Q)$ such that $Q(\phi', \psi) = (\phi, \psi)$ for all $\psi \in \overline{D}(Q)$. Thus for all $\phi \in \mathcal{H}$ we get

$$(\phi, \phi_\infty) = \lim_n (\phi, \phi_n) = \lim_n Q(\phi', \phi_n) = Q(\phi', \psi_\infty) = (\phi, \psi_\infty).$$

Hence $\psi_\infty = \phi_\infty$.

Assume now that Q is norm lower semi-continuous on $D(Q)$. Given a sequence $\{\phi_n\}$ from $D(Q)$ which is Cauchy for Q and satisfies $\phi_n \to 0$. We must conclude that $Q(\phi_n) \to 0$ as $n \to \infty$. If this is not the case we can assume, by possibly passing to a subsequence, that $\liminf_n Q(\phi_n) \neq 0$. On the other hand using the norm-lower semi-continuity we have

$$0 \leq \liminf_n Q(\phi_n) \leq \liminf_n \liminf_m Q(\phi_n - \phi_m) = 0,$$

where we used that we are dealing with a Cauchy sequence. □

7.7 DEFINITION (Norm lower continuous quadratic forms)**.** A norm lower semi-continuous function $Q : \mathcal{H} \to \mathbb{R}$ on a Hilbert space is said to be a quadratic form if

(i) The set $\{\phi \in \mathcal{H} : Q(\phi) < \infty\}$ is a subspace

(ii) The restriction of Q to this set agrees with the diagonal values of a finite quadratic form.

7.8 REMARK. It is obvious that if Q is a norm lower semi-continuous quadratic form then $Q(\alpha\phi) = |\alpha|^2 Q(\phi)$ for all $\phi \in \mathcal{H}$ and $\alpha \in \mathbb{C}$ and that $Q(\phi + \psi)^{1/2} \leq Q(\phi)^{1/2} + Q(\psi)^{1/2}$ for all $\phi, \psi \in \mathcal{H}$.

Changed Oct6/97

7.9 THEOREM (Extensions of semi-continuous quadratic forms)**.** *A quadratic form Q defined on a dense subspace $D(Q)$ of a Hilbert space $\mathcal{H}$ can be extended to a norm lower semi-continuous quadratic form on $\mathcal{H}$ if and only if Q is norm lower semi-continuous on $D(Q)$.*

There is a (unique) maximal norm lower semi-continuous extension $\hat{Q}$. (Maximal means that if Q' is any other norm lower semi continuous extension then $Q'(\phi) \leq \hat{Q}(\phi)$ for all $\phi \in \mathcal{H}$.)

Moreover, if $Q(\phi) \geq \|\phi\|^2$ then

$$\overline{D}(Q) = \left\{ \phi \in \mathcal{H} : \hat{Q}(\phi) < \infty \right\}$$

and the closure of Q to $\overline{D}(Q)$ agrees with the norm lower semi-continuous extension $\hat{Q}$.

Proof. That Q must be norm lower semi-continuous on $D(Q)$ in order to have a chance to extend to a norm lower semi-continuous function on all of $\mathcal{H}$ is obvious.

Assume now that Q is norm lower semi-continuous on $D(Q)$. We may then define an extension $\hat{Q}$ of Q to all of $\mathcal{H}$ by the definition

$$\hat{Q}(\phi) = \lim_{\varepsilon \searrow 0} \inf \{Q(\phi') : \phi' \in D(Q), \|\phi - \phi'\| < \varepsilon\}.$$

From Proposition 7.5 we know that the inf is not $-\infty$. Moreover as $\varepsilon \searrow 0$ then the inf is an increasing function hence the limit exists. It is left as an exercise to the reader to show that $\hat{Q}$ is a norm lower semi-continuous quadratic form. Clearly all other norm lower semi-continuous extensions Q' must satisfy that $Q' \leq \hat{Q}$.

That

$$\overline{D}(Q) \subset \left\{\phi \in \mathcal{H} : \hat{Q}(\phi) < \infty\right\}$$

is obvious since if $\phi \in \overline{D}(Q)$ then we can find a sequence $\phi_n \in D(Q)$ such that $\|\phi_n - \phi\| \to 0$ and $Q(\phi_n)$ converges. Thus $\hat{Q}(\phi) \leq \lim_n Q(\phi_n)$.

On the other hand if $\phi \in \left\{\phi \in \mathcal{H} : \hat{Q}(\phi) < \infty\right\}$ then there is a sequence of $\phi_n \in D(Q)$ such that $\|\phi_n - \phi\| \to 0$ and $\lim_n Q(\phi_n) = \hat{Q}(\phi)$. It follows from the Banach-Alaoglu Theorem that there is a $\psi \in \overline{D}(Q)$ such that (by maybe passing to a subsequence) ϕ_n converges weakly in $\overline{D}(Q)$ to ψ. As in the proof of Theorem 7.6 we conclude that $\phi = \psi$ and by Mazur's Theorem that $Q(\psi) \leq \lim_n Q(\phi_n) = \hat{Q}(\phi)$. Here $Q(\phi)$ refers to the value of the closure of Q on ψ. □

7.10 DEFINITION (Friedrichs' extension for $A \geq I$)**.** Let A be a densely defined operator satisfying $(\psi, A\psi) \geq \|\psi\|^2$. As in Problem 3.10 the relation $Q(\psi) = (\psi, A\psi)$ applied to the above extension of Q defines an extension of A to the set

$$\overline{D}(A) = \left\{\phi \in \overline{D}(Q) : \sup_{\psi \in \overline{D}(Q) \setminus \{0\}} \frac{|Q(\psi, \phi)|}{\|\psi\|} < \infty\right\}. \tag{17}$$

This extension of A which we still denote by A is called the Friedrichs extension.

7.11 PROBLEM. *Show that the Friedrichs extension still satisfies $A \geq I$. In Problem 7.13 we shall see that the Friedrichs extension is a maximal extension with this property.*

The Friedrichs extension of an operator A with $A \geq I$ can conveniently be realized as the inverse of a bounded operator.

7.12 LEMMA (Bounded inverse of Friedrichs extension)**.** *Consider a closable form Q satisfying $Q(\psi) \geq \|\psi\|^2$. Let A be the operator defined by Q on the set $\overline{D}(A)$ given in (17). There exists a bounded operator $B : \mathcal{H} \to \mathcal{H}$ (defined on all of $\mathcal{H}$) such that* Changed Oct6/97

$$\overline{D}(A) = R(B) = \{\phi \in \mathcal{H} : \phi = B\psi \textit{ for some } \psi \in \mathcal{H}\}$$

and

$$\begin{aligned} BA\phi &= \phi \textit{ for all } \phi \in \overline{D}(A) \\ AB\psi &= \psi \textit{ for all } \psi \in \mathcal{H}. \end{aligned}$$

We simply denote $B = A^{-1}$ and $A = B^{-1}$. The kernel $N(B) = \{0\}$ and hence A is densely defined. Moreover, B is bounded as a map $B : \mathcal{H} \to \overline{D}(Q)$.

The point is that if $\overline{D}(A) \neq \mathcal{H}$ then the operator $B : \mathcal{H} \to \mathcal{H}$ that we construct in the proof below is one-to-one but not onto (something which can not happen if $\dim \mathcal{H} < \infty$). The inverse therefore makes sense only as a densely defined (unbounded) operator.

Proof. We use again that $\overline{D}(Q)$ is a *Hilbert space* with the inner product $Q(\phi, \psi)$.

We may therefore apply Problem 3.10 with the roles of Q and $(\cdot, \cdot)$ interchanged to conclude that there exists an operator $B : \overline{D}(Q) \to \overline{D}(Q)$ such that

$$(\phi, \psi) = Q(\phi, B\psi) \tag{18}$$

for all $\phi, \psi \in \overline{D}(Q)$.

Note that $Q(B\psi) = Q(B\psi, B\psi) = (B\psi, \psi) \leq \|B\psi\|\|\psi\| \leq Q(B\psi)^{1/2}\|\psi\|$. Thus $Q(B\psi) \leq \|\psi\|^2$ and B may in fact be extended to a bounded operator (which we still denote by B) $B : \mathcal{H} \to \overline{D}(Q)$. With this extension (18) holds for all $\phi \in \overline{D}(Q)$ and all $\psi \in \mathcal{H}$. It is now clear that B has the stated properties. $\square$

7.13 PROBLEM. *Use the above lemma to show that if A' is any operator on $\mathcal{H}$ satisfying* *(i)* $A' > 0$, *i.e.,* $(\phi, A'\phi) > 0$ *for all* $\phi \neq 0$

(ii) $\overline{D}(A) \subset D(A')$

(iii) $A\phi = A'\phi$ *for all* $\phi \in D(A)$

then $D(A') = \overline{D}(A)$.

In particular, we have that Friedrichs extending twice gives the same result as Friedrichs extending once and we may therefore talk about an operator being Friedrichs extended.

As a corollary of Lemma 7.12 we get that A and A^{-1} share their eigenvectors and moreover that the max-min principle for A^{-1} turns into a min-max principle for A.

7.14 COROLLARY (Eigenvectors and eigenvalues of A and A^{-1})**.** *If* $A \geq I$ *is Friedrichs extended then* A *and* A^{-1} *have the same eigenvectors and reciprocal eigenvalues. Moreover,*

$$\inf_{M\subset D(A),\ \dim M=k}\ \sup_{\phi\in M\setminus\{0\}} \frac{(\phi, A\phi)}{\|\phi\|^2} = \left[\sup_{N\subset\mathcal{H},\ \dim N=k}\ \inf_{\psi\in N\setminus\{0\}} \frac{(\psi, A^{-1}\psi)}{\|\psi\|^2}\right]^{-1}. \tag{19}$$

Changed Oct6/97

Proof. Note that if $\phi = A^{-1}\psi$ then

$$\frac{(\phi, A\phi)}{\|\phi\|^2} = \frac{(\psi, A^{-1}\psi)}{(A^{-1}\psi, A^{-1}\psi)}.$$

It is an easy consequence of Lemma 5.11 that

$$\sup_{N\subset\mathcal{H},\ \dim N=k}\ \inf_{\psi\in N\setminus\{0\}} \frac{(\psi, A^{-1}\psi)}{\|\psi\|^2} = \sup_{N\subset\mathcal{H},\ \dim N=k}\ \inf_{\psi\in N\setminus\{0\}} \frac{(A^{-1}\psi, A^{-1}\psi)}{(\psi, A^{-1}\psi)}.$$

□

It is easy to see from the construction of the extension that

$$\begin{aligned}\inf_{\substack{M\subset D(A),\\ \dim M=k}}\ \sup_{\phi\in M\setminus\{0\}} \frac{(\phi, A\phi)}{\|\phi\|^2} &= \inf_{\substack{M\subset \overline{D}(A),\\ \dim M=k}}\ \sup_{\phi\in M\setminus\{0\}} \frac{(\phi, A\phi)}{\|\phi\|^2} = \inf_{\substack{M\subset \overline{D}(Q),\\ \dim M=k}}\ \sup_{\phi\in M\setminus\{0\}} \frac{Q(\phi)}{\|\phi\|^2}\\ &= \inf_{\substack{M\subset \mathcal{H},\\ \dim M=k}}\ \sup_{\phi\in M\setminus\{0\}} \frac{Q(\phi)}{\|\phi\|^2},\end{aligned}$$

where Q in the last step is the norm lower semi-continuous quadratic form defined on all of $\mathcal{H}$.

Next we turn to operators that are bounded below and we explain how to generalize the above constructions. Assume that $A \geq -cI$. Then $A+(c+1)I \geq I$ and we can use the above arguments on this operator.

7.15 DEFINITION (Friedrichs extension for $A \geq -cI$)**.** The Friedrichs extension of an operator A with $A \geq -cI$ is simply the extension of A to $\overline{D}(A) = \overline{D}(A+(c+1)I)$ given by

$$A\phi = (A+(c+1)I)\phi - (c+1)\phi,$$

where $(A+(c+1)I)\phi$ is given by the Friedrichs extended action of $A+(c+1)I$.

Using Corollary 7.14 we can now immediately translate all our max-min results (applied to $(A+(c+1)I)^{-1}$) to min-max results for A.

7.16 THEOREM (The lowest eigenvalue of a semi-bounded operator)**.** *Let A be an operator which is bounded below $A \geq -cI$. If there is a unit vector $\phi_1 \in \overline{D}(A)$ such that*

$$\frac{(\phi_1, A\phi_1)}{\|\phi_1\|^2} = \inf_{\psi \in \overline{D}(A)\setminus\{0\}} \frac{(\psi, A\psi)}{\|\psi\|^2}$$

then ϕ_1 is an eigenvector of A.

7.17 COROLLARY. *Let A be an operator which is bounded below $A \geq -cI$ and let Q be the corresponding quadratic form $Q(\psi) = (\psi, A\psi)$. Consider the numbers*

$$\mu_k = \inf_{M \subset D(A),\ \dim M = k} \sup_{\psi \in M\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2}. \tag{20}$$

If μ_1 is not an eigenvalue for the Friedrichs extension of A then

$$\mu_1 = \mu_2 = \ldots = \mu_k = \ldots \tag{21}$$

7.18 LEMMA. *Let $A \geq -cI$ be a Friedrichs extended operator and Q the corresponding quadratic form. Let μ_k be defined as in (20). Assume μ_1 is an eigenvalue for A and ϕ_1 is the corresponding eigenvector. Let $\mathcal{H}_1 = \operatorname{span}\{\phi_1\}^\perp$. Then A restricted to $\overline{D}(A) \cap \mathcal{H}_1$ is a Friedrichs extended operator on $\mathcal{H}_1$. In particular, $A(\overline{D}(A) \cap \mathcal{H}_1) \subset \mathcal{H}_1$. Moreover,*

$$\mu_{k+1} = \inf_{\substack{M \subset \mathcal{H}_1 \cap \overline{D}(A) \\ \dim M = k}} \sup_{\psi \in M\setminus\{0\}} \frac{Q(\psi)}{\|\psi\|^2}. \tag{22}$$

7.19 THEOREM (Compact resolvent). *Let $A \geq -cI$ be a Friedrichs extended operator and Q the corresponding quadratic form. If the numbers μ_k defined in (20) satisfy $\lim_k \mu_k = \infty$. Then the operator $K = (A + (c+1)I)^{-1}$ defined in Lemma 7.12 is compact.*

8 The Abstract Birman-Schwinger Principle

The Birman-Schwinger principle which we now discuss is a method for estimating the number of eigenvalues in an interval at the bottom of the spectrum of a perturbed operator.

To be more precise let A and B be operators with $D(B) = D(A)$ and assume that A is positive semi-definite. Given $\alpha > 0$, we shall be interested in estimating the number of eigenvalues less than or equal to $-\alpha$ for the Friedrichs extension of the operator $A - B$.

It is of course in general not true that $A-B$ is even bounded below. It is even less certain that the negative part of the spectrum of $A - B$ consists of discrete eigenvalues.

We shall introduce the notion that B is (form)-bounded relative to A, which can ensure that $A-B$ is bounded below and the stronger notion that B is (form)-compact relative to A which will ensure that the negative part of the spectrum of $A - B$ consists of discrete eigenvalues.

Instead of dealing directly with operators we shall be slightly more general and consider quadratic forms. Let Q^A be a positive semi-definite *closable* quadratic form and let Q^B be any symmetric quadratic form (not necessarily bounded below, above or closable) with domains $D(Q^B) \subset D(Q^A)$. That Q^B is symmetric, i.e., satisfies (5) is equivalent to the fact that $Q^B(\psi)$ is real. Considering quadratic forms is more general than operators since as a special case we could take $Q^A(\psi) = (\psi, A\psi)$ with $D(Q^A) = D(A)$ and $Q^B(\psi) = (\psi, B\psi)$ with $D(Q^B) = D(B)$, where A and B are as above.

Consider the quadratic form $Q^A_\alpha(\psi) = Q^A(\psi) + \alpha\|\psi\|^2$ defined on $D(Q^A)$. Consider the Hilbert space $\overline{D}(Q^A_1)$ with norm $Q^A_1(\psi)^{1/2}$, which we introduced when we discussed the Friedrichs extension. Note that if $\alpha_1 < \alpha_2$ then

$$(\alpha_1/\alpha_2)Q^A_{\alpha_2}(\psi) \le Q^A_{\alpha_1}(\psi) \le Q^A_{\alpha_2}(\psi). \tag{23}$$

Therefore Q^A_α for all $\alpha > 0$ define inner procucts with equivalent norms on $\overline{D}(Q^A_1)$.

8.1 DEFINITION (Relative form boundedness)**.** We say that a symmetric quadratic form Q^B is relative form-bounded with respect to a closable positive semi-definite form Q^A if $D(Q^B)$ is a dense subset (in the norm $(Q^A_1)^{1/2}$) of $D(Q^A)$ and there exist $a \ge 0$ and $\alpha > 0$ such that

$$|Q^B(\psi)| \le aQ^A_\alpha(\psi) \tag{24}$$

for all $\psi \in D(Q^A)$.

8.2 PROPOSITION. *If Q^B is relative form-bounded with respect to Q^A then Q^B extends to a quadratic form defined on all of $\overline{D}(Q^A_1)$ and defines a bounded operator on this space.*

Proof. This is clear by (23) since $D(Q^A)$ is dense in the Hilbert space $\overline{D}(Q^A_1)$. □

In the special case when $a \le 1$ then the quadratic form $Q^A - Q^B$ defined on $D(Q^B)$ is bounded below but it may not be closable. It is closable if $a < 1$. This result is part of what is sometimes refered to as the KLMN Theorem (K=Kato, L=Lax, Lions, M=Milgram, N=Nelson). Note also that since both Q^A and Q^B are symmetric $Q^A - Q^B$ will also be symmetric.

8.3 THEOREM (KLMN Theorem)**.** *If Q^B is relative form-bounded with respect to Q^A and if $a < 1$ in (24) then there exists $\beta > 0$ such that*

$$Q^{A-B}_\beta(\psi) = Q^A(\psi) - Q^B(\psi) + \beta\|\psi\|^2$$

defined on $D(Q^B)$ is a closable quadratic form with $Q^{A-B}_\beta(\psi) \ge \|\psi\|^2$. Moreover, $\overline{D}(Q^{A-B}_\beta) = \overline{D}(Q^A_1)$.

Proof. This is a simple consequence of the inequalities

$$(1-a)\left[Q^A_{\frac{\beta - a\alpha}{1-a}}\right] \le (1-a)Q^{A-B}_\beta \le (1+a)\left[Q^A_{\frac{\beta+a\alpha}{1+a}}\right],$$

which follow from (24). □

Note that the KLMN theorem implies that Q^{A-B} defines a Friedrichs extended operator. In the case described above when Q^A and Q^B were defined by operators then this is the Friedrichs extension of the operator $A - B + \beta I$ defined on $D(A) = D(B)$.

We now come to a stronger condition than relative form-boundedness. Namely relative form-compactness.

8.4 DEFINITION (Relative form compactness)**.** We say that a symmetric quadratic form Q^B is relative form-compact with respect to a closable positive semi-definite form Q^A if $D(Q^B) = D(Q^A)$ and there exists $\alpha_0 > 0$ such that Q^B defines a compact operator on the Hilbert space $\overline{D}(Q_1^A)$ with respect to the norm $(Q_{\alpha_0}^A)^{1/2}$.

8.5 THEOREM (Compactness impliess infinitesimal boundedness)**.** *Define*

$$\lambda_1(\alpha) = \sup_{\substack{\phi \in D(Q^B) \\ \phi \neq 0}} \frac{|Q^B(\phi)|}{Q_\alpha^A(\phi)}. \tag{25}$$

Then $\lambda_1(\alpha) \to 0$ as $\alpha \to \infty$. In particular, (24) is satisfied with arbitrarily small (positive) a if α is large enough.

Proof. This is somewhat complicated. Let $u_1, u_2, \ldots$ be the orthonormal basis for $\overline{D}(Q_1^A)$, with respect to the inner product $Q_{\alpha_0}^A$, consisting of eigenvectors of the compact operator defined by Q^B. Let $\mu_1, \mu_2, \ldots$ be the corresponding eigenvalues. Assume that $|\mu_1|, |\mu_2|, \ldots$ is in decreasing order. If $\phi \in D(Q^B)$ then $\phi = \sum_k \nu_k u_k$, where $\sum_k |\nu_k|^2 = Q_{\alpha_0}^A(\phi) < \infty$. Moreover,

$$|Q^B(\phi)| = |\sum_{k=1}^{\infty} \mu_k |\nu_k|^2| \leq \lambda_1(\alpha_0) \sum_{k=1}^{N} |\nu_k|^2 + |\mu_{N+1}| Q_{\alpha_0}^A(\phi).$$

Given $\varepsilon > 0$. Since $\mu_{N+1} \to 0$ as $N \to \infty$ (by compactness) we can choose N so large that if $\alpha > \alpha_0$ we have

$$|Q^B(\phi)| \leq \lambda_1(\alpha_0) \sum_{k=1}^{N} |\nu_k|^2 + \varepsilon Q_\alpha^A(\phi).$$

Here we used that $Q_\alpha^A(\phi)$ is an increasing function of α.

The difficult part of the proof is to show that for each fixed k

$$\sup_{\substack{\phi\in D(Q^B)\\ \phi\neq 0}} \frac{|\nu_k|^2}{Q^A_\alpha(\phi)} = \sup_{\substack{\phi\in D(Q^B)\\ \phi\neq 0}} \frac{|Q^A_{\alpha_0}(u_k,\phi)|^2}{Q^A_\alpha(\phi)} \to 0 \tag{26}$$

as $\alpha\to\infty$. The theorem clearly follows from (26).

Let A be the Friedrichs extended operator defined by Q^A. Then $\overline{D}(A)$ is dense in $\overline{D}(Q^A_1)$ with respect the all the equivalent norms $Q^{1/2}_\alpha$. [This requires a proof: the operator $(A+I)^{-1}$ is a continuous map from $\overline{D}(Q^A_1)$ to itself with range $R((A+I)^{-1})\subset\overline{D}(A)$ (see Lemma 7.12 and its proof). We shall show that $R((A+I)^{-1})$ is dense in $\overline{D}(Q^A_1)$. The operator $(A+I)^{-1}$ is positive definite with respect to the inner product Q^A_1. Consider $\psi\in\overline{D}(Q^A_1)$ which satisfies that $Q^A_1(\psi,(A+I)^{-1}\psi')=0$ for all $\psi'\in\overline{D}(Q^A_1)$, i.e., ψ is orthogonal to $R((A+I)^{-1})$ with respect to the inner product Q^A_1. Then since $(A+I)^{-1}$ is symmetric we conclude that $(A+I)^{-1}\psi=0$ and since $(A+I)^{-1}$ is positive definite this implies that $\psi=0$.]

We can therefore choose $\widetilde{u}_k\in\overline{D}(A)$ such that $Q^A_{\alpha_0}(\widetilde{u}_k-u_k)\leq\varepsilon$. Thus by Cauchy-Schwarz

$$\begin{aligned}
|Q^A_{\alpha_0}(u_k,\phi)|^2 &\leq 2|Q^A_{\alpha_0}(\widetilde{u}_k,\phi)|^2+2|Q^A_{\alpha_0}(\widetilde{u}_k-u_k,\phi)|^2\\
&\leq 2|((A+\alpha_0 I)\widetilde{u}_k,\phi)|^2+2\varepsilon Q^A_{\alpha_0}(\phi)\\
&\leq 2\|(A+\alpha_0 I)\widetilde{u}_k\|^2\|\phi\|^2+2\varepsilon Q^A_\alpha(\phi)\\
&\leq \left(2\alpha^{-1}\|(A+\alpha_0 I)\widetilde{u}_k\|^2+2\varepsilon\right)Q^A_\alpha(\phi).
\end{aligned}$$

This proves (26) and hence the theorem. □

We shall now restrict to the case when Q^B is positive semi-definite. In this case we may use Theorem 5.9 to give an equivalent formulation of being relatively form-compact.

8.6 PROPOSITION (Positivity and relative form-compactness)**.** *Let Q^B be a positive semi-definite quadratic form and Q^A a closable positive semi-definite form. Then Q^B is relative form-compact with respect to Q^A if $D(Q^A)=D(Q^B)$ and there exists $\alpha_0>0$ such that $\lim_{k\to\infty}\lambda_k(\alpha_0)=0$, where*

$$\lambda_k(\alpha)=\sup_{\substack{M\subset D(Q^A)\\ \dim M=k}}\ \inf_{\substack{\phi\in M\\ \phi\neq 0}}\frac{Q^B(\phi)}{Q^A_{\alpha_0}(\phi)}. \tag{27}$$

Note that (27) for $k = 1$ agrees with (25).

8.7 PROPOSITION (Compactness for all α)**.** *Assume that Q^B is positive semi-definite. If $\lim_{k\to\infty}\lambda_k(\alpha_0) = 0$ for one $\alpha_0 > 0$ then $\lim_{k\to\infty}\lambda_k(\alpha) = 0$ for all $\alpha > 0$. Thus if Q^B defines a compact operator with respect to $Q^A_{\alpha_0}$ for one α_0 then it defines a compact operator with respect to Q^A_α for all α.*

Proof. This is an immediate consequence of the inequalities

$$\left(\frac{\alpha'}{\alpha''}\right)\lambda_k(\alpha') \leq \lambda_k(\alpha'') \leq \lambda_k(\alpha')$$

for $\alpha' \leq \alpha''$, which follow from

$$\left(\frac{\alpha'}{\alpha''}\right)\frac{Q^B(\phi)}{Q^A_{\alpha'}(\phi)} \leq \frac{Q^B(\phi)}{Q^A_{\alpha''}(\phi)} \leq \frac{Q^B(\phi)}{Q^A_{\alpha'}(\phi)}$$

$\square$

8.8 THEOREM (Properties of λ_k)**.** *Let Q^B be positive semi-definite and relative compact with respect to Q^A_α. The $\lambda_k(\alpha)$, $k = 1, 2, \ldots$ defined in (27) are non-negative, strictly decreasing, continuous functions of $\alpha > 0$ satisfying that* $\lim_{\alpha\to\infty}\lambda_k(\alpha) = 0$.

Proof. That they are non-negative is obvious.

To prove that they are strictly decreasing recall that since Q^B defines a compact operator there exists an $M_\alpha \subset \overline{D}(Q^A_1)$ with $\dim M_\alpha = k$ such that

$$\lambda_k(\alpha) = \min_{\substack{\phi\in M_\alpha\\ \phi\neq 0}} \frac{Q^B(\phi)}{Q^A_\alpha(\phi)}.$$

We here write a min rather than an inf since there clearly is a minimizing ϕ on the finite dimensional space M_α. Thus if $\alpha' < \alpha$ we have

$$\lambda_k(\alpha') \geq \min_{\substack{\phi\in M_\alpha\\ \phi\neq 0}} \frac{Q^B(\phi)}{Q^A_{\alpha'}(\phi)} > \min_{\substack{\phi\in M_\alpha\\ \phi\neq 0}} \frac{Q^B(\phi)}{Q^A_\alpha(\phi)} = \lambda_k(\alpha),$$

where the strict inequality follows since the minima are achieved.

To prove the continuity we write

$$\left|\frac{Q^B(\phi)}{Q^A_{\alpha'}(\phi)} - \frac{Q^B(\phi)}{Q^A_\alpha(\phi)}\right| = \frac{Q^B(\phi)}{Q^A_{\alpha'}(\phi)Q^A_\alpha(\phi)}|Q^A_\alpha(\phi) - Q^A_{\alpha'}(\phi)|.$$

Since, $Q^A_\alpha(\phi) - Q^A_{\alpha'}(\phi) = (\alpha - \alpha')\|\phi\|^2$ and $Q^A_\alpha(\phi) \geq \alpha\|\phi\|^2$ we have

$$\frac{Q^B(\phi)}{Q^A_{\alpha'}(\phi)Q^A_\alpha(\phi)}|Q^A_\alpha(\phi) - Q^A_{\alpha'}(\phi)| \leq \lambda_1(\alpha')\alpha^{-1}|\alpha - \alpha'|.$$

Therefore

$$\frac{Q^B(\phi)}{Q^A_\alpha(\phi)} - \lambda_1(\alpha')\alpha^{-1}|\alpha - \alpha'| \leq \frac{Q^B(\phi)}{Q^A_{\alpha'}(\phi)} \leq \frac{Q^B(\phi)}{Q^A_\alpha(\phi)} + \lambda_1(\alpha')\alpha^{-1}|\alpha - \alpha'|,$$

and hence $|\lambda_k(\alpha') - \lambda_k(\alpha)| \leq \lambda_1(\alpha')\alpha^{-1}|\alpha - \alpha'|$ which implies continuity.

That $\lim_{\alpha\to\infty} \lambda_k(\alpha) = 0$ follows from Theorem 8.5 and the fact that $\lambda_k(\alpha) \leq \lambda_1(\alpha)$. □

We now come to the Birman-Schwinger principle for estimating the eigenvalues of the operator defined from the quadratic form Q^{A-B}. Let

$$\mu_k = \inf_{\substack{M\subset\overline{D}(Q^A_1)\\ \dim M = k}} \sup_{\substack{\phi\in M\\ \phi\neq 0}} \frac{Q^A(\phi) - Q^B(\phi)}{\|\phi\|^2}. \tag{28}$$

Since $\overline{D}(Q^{A-B}_\beta) = \overline{D}(Q^A_1)$ we know from Corollary 7.17 and Lemma 7.18 that all the μ_k which are not repeated infintely often are eigenvalues for Q^{A-B}.

8.9 THEOREM (Birman-Schwinger relation)**.** *With the definitions given above we have for all $\alpha > 0$*

$$\lambda_k(\alpha) = 1 \Leftrightarrow \mu_k = -\alpha$$

Proof. ($\Rightarrow$) Asume that $\lambda_k(\alpha) = 1$. We can then find a subspace $M_\alpha \subset \overline{D}(Q^A_1)$ with $\dim M_\alpha = k$ such that

$$1 = \inf_{\substack{\phi\in M_\alpha\\ \phi\neq 0}} \frac{Q^B(\phi)}{Q^A_\alpha(\phi)}.$$

Thus for all $\phi \in M_\alpha$ we have $Q^A(\phi) - Q^B(\phi) \leq -\alpha\|\phi\|^2$. Therefore

$$\mu_k \leq \sup_{\substack{\phi\in M_\alpha\\ \phi\neq 0}} \frac{Q^A(\phi) - Q^B(\phi)}{\|\phi\|^2} \leq -\alpha.$$

We next show that $\mu_k \geq -\alpha$. Assume on the contrary that $\mu_k \leq -\alpha - 2\varepsilon$ for some positive ε. Then there exists $M \subset \overline{D}(Q^A_1)$ with $\dim M = k$ such that

$$\sup_{\substack{\phi\in M\\ \phi\neq 0}} \frac{Q^A(\phi) - Q^B(\phi)}{\|\phi\|^2} \leq -\alpha - \varepsilon.$$

Then for all $\phi \in M$ we have $Q^B(\phi) \geq Q^A_{\alpha+\varepsilon}\phi)$. Thus

$$\lambda_k(\alpha+\varepsilon) \geq \inf_{\substack{\phi\in M\\ \phi\neq 0}} \frac{Q^B(\phi)}{Q^A_{\alpha+\varepsilon}} \geq 1.$$

Thus since λ_k is strictly decreasing we have $1 = \lambda_k(\alpha) > \lambda_k(\alpha+\varepsilon) \geq 1$, which is a contradiction.

($\Leftarrow$). Assume that $\mu_k = -\alpha$. Then for all $\varepsilon > 0$ there exists a subspace $M \subset \overline{D}(Q^A_1)$ with $\dim M = k$ such that

$$\sup_{\substack{\phi\in M\\ \phi\neq 0}} \frac{Q^A(\phi) - Q^B(\phi)}{\|\phi\|^2} \leq -\alpha + \varepsilon.$$

Thus

$$\lambda_k(\alpha-\varepsilon) \geq \inf_{\substack{\phi\in M\\ \phi\neq 0}} \frac{Q^B(\phi)}{Q^A_{\alpha-\varepsilon}} \geq 1.$$

We know, however, that λ_k is decreasing and tends to zero at infinity. Therefore there must be $\alpha' \geq \alpha - \varepsilon$ such that $\lambda(\alpha') = 1$. It follows from the first part of the proof that $\mu_k = -\alpha'$. Thus $\alpha' = \alpha$ and hence $\lambda_k(\alpha) = 1$. □

8.10 COROLLARY (Birman-Schwinger Principle)**.** *Let Q^B be positive semi-definite and relative compact with respect to Q^A. Consider the Friedrichs extended operator $A - B$ defined by the quadratic form $Q^{A-B} = Q^A - Q^B$. For each $k \geq 1$ such that $\mu_k < 0$ there corresponds an eigenvector ϕ_k for $A - B$ with eigenvalue μ_k. Moreover, the operator $A - B$ restricted to the orthogonal complement $\{\phi_1, \phi_2, \ldots\}^\perp$ of all these eigenvalues is positive semi-definite. The negative eigenvalues are discrete but may accumulate at zero. More precisely, the number of eigenvalues of $A - B$ less than or equal to $-\alpha < 0$ is equal to the number of k for which $\lambda_k(\alpha)$ is greater than or equal to 1.*

Proof. This is a simple consequence of Corollary 7.17, Lemma 7.18, Theorem 8.8 and Theorem 8.9. The Birman-Schwinger principle is the fact that the number of eigenvalues of $A-B$ less than or equal to $-\alpha$ is equal to the number of eigenvalues greater than or equal to 1 for the compact operator defined by Q^B relative to the norm $(Q^A_\alpha)^{1/2}$. This follows from Theorem 8.8 and Theorem 8.9 because the eigenvalues are continuous, decreasing and tend to zero as α tends to infinity. All eigenvalues greater than or equal to 1 will therefore eventually pass through 1 as α increases. □

9 Hilbert Schmidt Operators

9.1 DEFINITION (L^2 Integral kernels)**.** We define a map $k \mapsto K$ from $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ to the bounded operators on $L^2(\mathbb{R}^n)$ by

$$(K\phi)(x) = \int_{\mathbb{R}^n} k(x,y)\phi(y)dy.$$

The function k is said to be the integral kernel of the operator K. We call an operator K with integral kernel $k \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ a Hilbert-Schmidt operator.

That the above map is well-defined easily follows from the estimate

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} k(x,y)\phi(y)dy \right| dx &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |k(x,y)|^2 dydx \int_{sr^n} |\phi(y)|^2 dy \\ &= \|k\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

This also proves that $\|K\| \leq \|k\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n}$ where $\|K\|$ refers to the operator norm.

We need the following important result about Hilbert-Schmidt operators. It concerns the Hilbert-Schmidt norm $\|K\|_{HS} = \mathrm{Tr}[K^*K]$.

9.2 THEOREM (Hilbert-Schmidt norm)**.** *If K is a Hilbert-Schmidt operator with integral kernel k then* $\mathrm{Tr}[K^*K] = \|k\|^2_{L(\mathbb{R}^n\times\mathbb{R}^n)}$.

Proof. We shall use the fact that if ϕ_j, $j = 1,2,\ldots$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ then $\phi_i \otimes \overline{\phi_j}$, $i,j = 1,2,\ldots$ is an orthonormal basis for $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Here $\phi_i \otimes \overline{\phi_j}(x,y) = \phi_i(x)\overline{\phi_j(y)}$.

Therefore,

$$\begin{aligned} \mathrm{Tr}[K^*K] &= \sum_j (\phi_j, K^*K\phi_j) = \sum_j\sum_i (\phi_j, K^*\phi_i)(\phi_i, K\phi_j) = \sum_j\sum_i |(\phi_i, K\phi_j)|^2 \\ &= \sum_j\sum_i \left| \iint \overline{\phi_i(x)} k(x,y)\phi_j(y)dydx \right| = \sum_j\sum_i \left(\phi_i \otimes \overline{\phi_j}, k\right)_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}, \end{aligned}$$

where we have emphasized that the last inner product is $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Note that since $\phi_i \otimes \overline{\phi_j}$, $i,j = 1,2,\ldots$ form an orthonormal basis we have that

$$\sum_j\sum_i \left(\phi_i \otimes \overline{\phi_j}, k\right)_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = \|k\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^n)}$$

□

Note that the integral kernel of K^* is the function $k^*(x,y) = \overline{k(y,x)}$ and that

$$\mathrm{Tr}[K^*K] = \mathrm{Tr}[KK^*].$$

10 The Lieb-Thirring inequality

We shall now estimate the number of negative eigenvalues of the operator $-\Delta - V$, where $V \in L^1_{\rm loc}(\mathbb{R}^3)$ and $-\Delta = -(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. The Laplacian $-\Delta$ corresponds to the *Dirichlet* quadratic form

$$Q_0^{-\Delta}(\phi) = \int |\nabla\phi|^2.$$

defined on $D(Q_0^{-\Delta}) = C_0^1(\mathbb{R}^3)$.

10.1 PROBLEM. *Closability of Dirichlet form Show that the quadratic form*

$$Q_0^{-\Delta}(\phi) = \int |\nabla\phi|^2$$

defined on $C_0^1(\mathbb{R}^n)$ is norm lower semi-continuous and hence that $Q_1{-}\Delta(\phi) = Q_0^{-\Delta}(\phi) + \|\phi\|^2$ is closable.

10.2 PROPOSITION (Closability of Schrödinger quadratic form)**.** *Let $V \in L^1_{\rm loc}(\mathbb{R}^n)$ with $V_+ = \max\{V, 0\} \in L^{\frac{n+2}{2}}(\mathbb{R}^n)$. Then the Schrödinger quadratic form*

$$Q(\phi) = \int |\nabla\phi|^2 - \int V|\phi|^2$$

defined on $C_0^1(\mathbb{R}^n)$ is norm lower semi-continuous.

Proof. We first prove that $\phi \mapsto \int V_-|\phi|^2$ is norm lower semi-continuous, where $V_- = \max\{-V, 0\}$. Assume on the contrary that $\phi_n \to \phi$ in $L^2(\mathbb{R}^n)$, where $\phi_n, \phi \in C_0^1(\mathbb{R}^n)$ and that $\int V_-|\phi|^2 > \lim_n \int V_-|\phi_n|^2$. Since ϕ_n converges to ϕ in L^2 there is a subsequence ϕ_{n_k} which converges pointwise almost everywhere (recall the proof that L^p is complete). Then by Fatou's Lemma

$$\int V_-|\phi|^2 \leq \lim_k \int V_-|\phi_{n_k}|^2 < \int V_-|\phi|^2,$$

which is a contradiction.

Note now that for all $\varepsilon > 0$ we have as in the Sobolev bound on Schrödinger Theorem 4.8 that

$$\int V_+|\phi|^2 \leq \varepsilon \int |\nabla\phi|^2 + C_\varepsilon\|\phi\|^2.$$

Writing

$$Q(\phi) = (1-\varepsilon)\int |\nabla\phi|^2 + \int V_-|\phi|^2 + \varepsilon\int |\nabla\phi|^2 - \int V_+|\phi|^2$$

we therefore immediately conclude the norm lower semi-continuity. from Problem 10.1. □

The space we get by considering the closure of the Dirichlet quadratic form is called the Sobolev space.

10.3 DEFINITION (Sobolev space of order 1)**.** The quadratic form $Q_1^{-\Delta}$ defines the first order Sobolev Space $H^1(\mathbb{R}^3) = \overline{D}(Q_1^{-\Delta})$.

10.4 THEOREM. *Sobolev space in terms of Fourier transform If $\hat{\phi}$ denotes the Fourier trabsform of ϕ. Then we have*

$$H^1(\mathbb{R}^3) = \{\phi \in L^2(\mathbb{R}^3) \mid (1+|p|^2)^{1/2}\hat{\phi}(p) \in L^2(\mathbb{R}^3)\}.$$

Moreover, $\phi \mapsto (1+|p|^2)^{1/2}\hat{\phi}(p)$ is an isometric isomorphism from $H^1(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$.

10.5 PROBLEM. *(a) Show that if the Fourier transform is defined by*

$$\hat{\phi}(p) = (2\pi)^{-3/2}\int e^{-ip\cdot x}\phi(x)dx$$

then we have

$$Q_1^{-\Delta}(\phi) = (2\pi)^{-3/2}\int (1+|p|^2)|\hat{\phi}(p)|^2 dp.$$

(b) Use (a) to prove Theorem 10.4.

We can also go one step further and define the operators

$$[(1-\Delta)^{1/2}\phi](x) = (2\pi)^{-3}\int (1+|p|^2)^{1/2}e^{ip(x-y)}\phi(y)dydp$$

and

$$[(1-\Delta)^{-1/2}\phi](x) = (2\pi)^{-3}\int (1+|p|^2)^{-1/2}e^{ip(x-y)}\phi(y)dydp.$$

Here $(1-\Delta)^{1/2}$ can be considered as an unbounded operator on $L^2(\mathbb{R}^3)$ with domain $H^1(\mathbb{R}^3)$ or as an isometric isomorphism $(1-\Delta)^{1/2} : H^1(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The operator $(1-\Delta)^{-1/2}$ is the inverse map. It is a bounded operator on $L^2(\mathbb{R}^3)$ with range $H^1(\mathbb{R}^3)$.

Note that if $\phi \in C_0^\infty(\mathbb{R}^3)$ then $(1-\Delta)^{1/2}(1-\Delta)^{1/2}\phi = (1-\Delta)\phi$ where the Laplacian $-\Delta$ on the right side is the usual second derivative operator.

We now give a representation of $(1-\Delta)^{-1}$ in terms of an integral kernel

10.6 THEOREM (Integral kernel for the resolvent of the Laplacian). *If $\phi \in C_0^\infty(\mathbb{R}^3)$ then for all $m \in \mathbb{R}$ we have*

$$(m^2 - \Delta)^{-1}\phi(x) = (4\pi)^{-1} \int \frac{e^{-m|x-y|}}{|x-y|}\phi(y)dy.$$

Proof. The function $Y_m(x) = \frac{e^{-m|x|}}{|x|}$ is called the *Yukawa potential*. In the case $m = 0$ it is the Coulomb potential. For $m \neq 0$ we have that $Y_m \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. We compute the Fourier transform

$$\hat{Y}_m(p) = (2\pi)^{-3/2} \int_{|x|=0}^{\infty} \int_0^{2\pi} \int_0^{\pi} e^{-ip\cdot x - m|x|}|x|^{-1}|x|^2 \sin\theta d\theta d\phi d|x|.$$

We may choose a coordinate system such that $x \cdot p = |x||p|\cos\theta$. Then (substituting $u = -\cos\theta$)

$$\begin{aligned}
\hat{Y}_m(p) &= (2\pi)^{-1/2} \int_{|x|=0}^{\infty} \int_{u=-1}^{1} e^{i|p||x|u - m|x|}|x|dud|x| \\
&= (2\pi)^{-1/2} \int_{|x|=0}^{\infty} [e^{i|p||x|} - e^{-i|p||x|}]e^{-m|x|}(i|p|)^{-1}d|x| \\
&= 2(2\pi)^{-1/2}|p|^{-1}\Im\left[\int_{|x|=0}^{\infty} e^{i|p||x| - m|x|}d|x|\right] \\
&= 2(2\pi)^{-1/2}|p|^{-1}\Im\left[\frac{1}{m - i|p|}\right] = 2(2\pi)^{-1/2}\frac{1}{|p|^2 + m^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
(m^2 - \Delta)^{-1}\phi(x) &= (2\pi)^{-3/2} \int e^{ip\cdot x}(m^2 + |p|^2)^{-1}\hat{\phi}(p)dp \\
&= (2\pi)^{-3} \int\int e^{ip\cdot(x-y)}(m^2 + |p|^2)^{-1}\phi(y)dydp \\
&= (4\pi)^{-1} \int Y_m(x-y)\phi(y)dy
\end{aligned}$$

In the last step above we used that the inverse Fourier transform of $(m^2 + |p|^2)^{-1}$ is $2^{-1}(2\pi)^{1/2}Y_m$. Strictly speaking since $(m^2 + |p|^2)^{-1}$ is not in $L^1(\mathbb{R}^3)$ we would have to introduce a convergence factor to perform the last step.

The case $m = 0$ can be proved by going to the limit. □

We consider now also the quadratic form $Q^V(\phi) = \int V|\phi|^2$. Assume first that $V \geq 0$. In order to understand the negative eigenvalues of $-\Delta - V$ we know from the Birman-Schwinger principle that it is important to consider the fraction

$$\frac{Q^V(\phi)}{Q_\alpha^{-\Delta}(\phi)} = \frac{\int V|\phi|^2}{\int |\nabla\phi|^2 + \alpha \int |\phi|^2}.$$

If we set $\psi = (\alpha - \Delta)^{1/2}\phi$ and thus $\phi = (\alpha - \Delta)^{-1/2}\psi$ we get

$$\frac{Q^V(\phi)}{Q_\alpha^{-\Delta}(\phi)} = \frac{\int \overline{\psi}(\alpha - \Delta)^{-1/2} V (\alpha - \Delta)^{-1/2}\psi}{\int |\psi|^2}.$$

It is therfore clear that the eigenvalues of the operator defined by Q^V with respect to the norm $\left(Q_\alpha^{-\Delta}\right)^{1/2}$ are identical to the eigenvalues of the operator

$$(\alpha - \Delta)^{-1/2} V (\alpha - \Delta)^{-1/2} \tag{29}$$

on $L^2(\mathbb{R}^3)$.

We claim now that the operator (29) has the same non-zero eigenvalues as the operator

$$K_{BS} = \sqrt{V}(\alpha - \Delta)^{-1}\sqrt{V}.$$

This operator is called the Birman-Schwinger operator.

10.7 LEMMA (Birman-Schwinger kernel)**.** *Assume* $0 \leq V$ *is such that the function*

$$k_{BS}(x, y) = (4\pi)^{-1}\sqrt{V(x)}\frac{e^{-m|x-y|}}{|x - y|}\sqrt{V(y)}.$$

is in $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$*. Then* K_{BS} *is a Hilbert-Schmidt operator with integral kernel* k_{BS}*.*

Moreover, the non-zero eigenvalues of this operator are identical to the non-zero eigenvalues of the operator in (29).

Proof. It is clear from Theorem 10.6 that K_{BS} has k_{BS} as its integral kernel. Hence K_{BS} is a Hilbert-Schmidt operator.

Assume that ϕ is an eigenfunction of K_{BS} with eigenvalue $\lambda \neq 0$. Let $\psi = (\alpha - \Delta)^{-1/2}\sqrt{V}\phi$. Then $\|\psi\|^2 = (\phi, K_{BS}\phi) = \lambda\|\phi\|^2 \neq 0$ and

$$\begin{aligned} (\alpha - \Delta)^{-1/2} V (\alpha - \Delta)^{-1/2}\psi &= (\alpha - \Delta)^{-1/2}\sqrt{V} K_{BS}\phi = \lambda(\alpha - \Delta)^{-1/2}\sqrt{V}\phi \\ &= \lambda\psi. \end{aligned}$$

Hence λ is an eigenvalue of the operator in (29).

We can now reverse this argument. Let ψ be an eigenfunction of (29) with eigenvalue $\lambda \neq 0$. Let $\phi = \sqrt{V}(\alpha - \Delta)^{-1/2}\psi$. Then $\|\phi\|^2 = \lambda\|\psi\|^2$ and

$$\begin{aligned} K_{BS}\phi &= \sqrt{V}(\alpha - \Delta)^{-1}V(\alpha - \Delta)^{-1/2}\psi = \sqrt{V}(\alpha - \Delta)^{-1/2}\lambda\psi \\ &= \lambda\phi. \end{aligned}$$

Thus λ is an eigenvalue of K_{BS}. □

We are now ready to prove our main result

10.8 THEOREM (The Lieb-Thirring inequality). *If $V \in L^{5/2}(\mathbb{R}^3)$ then $-\Delta - V$ has discrete eigenvalues below zero. If $e_1(V), e_2(V), \ldots$ denote the negative eigenvalues of $-\Delta - V$ in increasing order then*

$$\sum_j |e_j(V)| \leq L \int [V(x)]_+^{5/2} dx,$$

where L is a constant and $[V(x)]_+ = \max\{V(x), 0\}$.

Proof. The proof of this is still not at all trivial. First of all the Birman-Schwinger principle allows us to count the number of eigenvalues, but not directly to sum them. The way to get to the sum is to realize that a sum can be expressed as an integral over the number. More precisely, let

$$N_\alpha(V) = \max\{j \mid e_j(V) \leq -\alpha\},$$

i.e., this is the number of eigenvalues of $-\Delta - V$ below $-\alpha$. Then

$$\sum_j |e_j(V)| = \int_0^\infty N_\alpha(V) d\alpha.$$

The (simple) proof of this is left as an exercise!

In order to estimate $N_\alpha(V)$ we use the Birman-Schwinger principle. This must, however, be done in a very clever way in order to get the estimate above. We estimate

$$-\Delta - V = -\Delta - \alpha/2 - (V - \alpha/2) \geq -\Delta - \alpha/2 - [V - \alpha/2]_+.$$

This is an operator inequality. It implies that the number of eigenvalues of $-\Delta - V$ below $-\alpha$ is smaller than the number of eigenvalues of $-\Delta - [V - \alpha/2]_+$ below $-\alpha/2$. I.e.,

$$N_\alpha(V) \leq N_{\alpha/2}([V - \alpha/2]_+).$$

According to the Birman-Schwinger principle and Lemma 10.7 we have that $N_{\alpha/2}([V-\alpha/2]_+)$ is equal to the number of eigenvalues of the operator

$$K_\alpha = [V-\alpha/2]_+^{1/2}(\alpha/2-\Delta)^{-1}[V-\alpha/2]_+^{1/2}$$

greater than or equal to one. Thus

$$N_{\alpha/2}([V-\alpha/2]_+) \leq \mathrm{Tr}[K_\alpha^2] = \int\int |k_\alpha(x,y)|^2 dxdy,$$

where

$$k_\alpha(x,y) = [V(x)-\alpha/2]_+^{1/2}(4\pi)^{-1}\frac{e^{-\sqrt{\alpha/2}|x-y|}}{|x-y|}[V(y)-\alpha/2]_+^{1/2}.$$

Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\int\int |k_\alpha(x,y)|^2 dxdy &\leq (\text{const. }) \int [V(x)-\alpha/2]_+^2 dx \int \frac{e^{-2\sqrt{\alpha/2}|x|}}{|x|^2}dx \\ &\leq (\text{const. })\alpha^{-1/2}\int [V(x)-\alpha/2]_+^2 dx.\end{aligned}$$

Thus $N_\alpha(V) \leq (\text{const. })\alpha^{-1/2}\int [V(x)-\alpha/2]_+^2 dx$.

We therefore get

$$\begin{aligned}\sum_j |e_j(V)| &\leq (\text{const. })\int_0^\infty \alpha^{-1/2}\int [V(x)-\alpha/2]_+^2 dxd\alpha \\ &= (\text{const. })\int\int_0^{2[V(x)]_+} \alpha^{-1/2}[V(x)-\alpha/2]_+^2 d\alpha dx \\ &= (\text{const. })\int [V(x)]_+^{5/2}dx.\end{aligned}$$

□

11 Solutions to selected problems

11.1 SOLUTION (Problem 5.14). Let ϕ_k and u_ℓ be orthonormal bases in $D(A)$, where A is a positive semi-definite operator. Then

$$\begin{aligned}
\sum_{k=1}^{K}(\phi_k, A\phi_k) &= \sum_{k=1}^{K}\sum_{\ell=1}^{\infty}(\phi_k, u_\ell)(u_\ell, A\phi_k) \\
&= \sum_{k=1}^{K}\sum_{\ell=1}^{\infty}(\phi_k, u_\ell)(Au_\ell, \phi_k) \\
&= \sum_{\ell=1}^{\infty}\left(Au_\ell, \sum_{k=1}^{K}(\phi_k, u_\ell)\phi_k\right) \\
&\leq \sum_{\ell=1}^{\infty}(Au_\ell, u_\ell)^{1/2}\left(A\sum_{k=1}^{K}(\phi_k, u_\ell)\phi_k, \sum_{k=1}^{K}(\phi_k, u_\ell)\phi_k\right)^{1/2}
\end{aligned}$$

where we applied the Cauchy-Schwarz inequality for the quadratic form $(A\phi, \phi)$ in the last step. Thus

$$\sum_{k=1}^{K}(\phi_k, A\phi_k) \leq \left(\sum_{\ell=1}^{\infty}(Au_\ell, u_\ell)\right)^{1/2}\left(\sum_{\ell=1}^{\infty}\left(A\sum_{k=1}^{K}(\phi_k, u_\ell)\phi_k, \sum_{k=1}^{K}(\phi_k, u_\ell)\phi_k\right)\right)^{1/2}.$$

However,

$$\begin{aligned}
\sum_{\ell=1}^{\infty}\left(A\sum_{k=1}^{K}(\phi_k, u_\ell)\phi_k, \sum_{k=1}^{K}(\phi_k, u_\ell)\phi_k\right) &= \sum_{\ell=1}^{\infty}\sum_{k=1}^{K}\sum_{k'=1}^{K}(A\phi_k, \phi_{k'})(\phi_{k'}, u_\ell)(u_\ell, \phi_k) \\
&= \sum_{k=1}^{K}(A\phi_k, \phi_k).
\end{aligned}$$

Hence,

$$\sum_{k=1}^{K}(\phi_k, A\phi_k) \leq \sum_{\ell=1}^{\infty}(Au_\ell, u_\ell)$$

which solves the problem

CORRECTIONS OCT. 7

(1) The statement and the proof of Lemma 5.11 should be corrected to

5.11 LEMMA *Let A be a bounded positive semi-definite operator on a Hilbert space $\mathcal{H}$ and $\mu = \sup_{\|\phi\|=1}(\phi, A\phi)$. Then $(\phi, A^m\phi) \le \mu(\phi, A^{m-1}\phi)$ for all $\phi \in \mathcal{H}$. And for all sequences of unit vectors ϕ_n with $(\phi_n, A\phi_n) \to \mu$ as $n \to \infty$ we have $(\phi_n, A^m\phi_n) \to \mu^m$ as $n \to \infty$.*

Proof. If $m = 2k+1$ we have $(\phi, A^{2k+1}\phi) = (A^k\phi, AA^k\phi) \le \mu(\phi, A^{2k}\phi)$. If $m = 2k$ we find from the Cauchy-Schwarz inequality for the quadratic form $(\phi, A\phi)$ that

$$\begin{aligned}(\phi, A^{2k}\phi) &= (A^k\phi, AA^{k-1}\phi) \le (A^k\phi, AA^k\phi)^{1/2}(A^{k-1}\phi, AA^{k-1}\phi)^{1/2} \\ &\le \mu^{1/2}(\phi, A^{2k}\phi)^{1/2}(\phi, A^{2k-1}\phi)^{1/2}.\end{aligned}$$

This proves the first statement.

Since $\mu - A$ is a positive semi-definite operator we find using the Cauchy-Schwarz inequality for the quadratic form $(\phi, (\mu - A)\phi)$ that

$$0 \le (\phi_n, (\mu - A)^m\phi_n) \le (\phi_n, (\mu - A)\phi_n)^{1/2}(\phi_n, (\mu - A)^{2m-1}\phi_n)^{1/2} \to 0.$$

as $n \to \infty$. This clearly implies the second statement. □

(2) The statement of Theorem 7.6 should be changed to

7.6 THEOREM (Lower semi-continuity and closability) *A quadratic form Q defined on a subspace $D(Q)$ of a Hilbert space $\mathcal{H}$ is closable <u>and bounded below</u> if and only if $\phi \mapsto Q(\phi)$ is norm lower semi-continuous on $D(Q)$.*

(3) The proof of Corollary 7.14 should be changed to

Changed Oct6/97

Proof. Note that if $\phi = A^{-1}\psi$ then

$$\frac{(\phi, A\phi)}{\|\phi\|^2} = \frac{(\psi, A^{-1}\psi)}{(A^{-1}\psi, A^{-1}\psi)}.$$

It is an easy consequence of Lemma 5.11 that

$$\sup_{N\subset\mathcal{H},\ \dim N = k}\ \inf_{\psi\in N\setminus\{0\}} \frac{(\psi, A^{-1}\psi)}{\|\psi\|^2} = \sup_{N\subset\mathcal{H},\ \dim N = k}\ \inf_{\psi\in N\setminus\{0\}} \frac{(A^{-1}\psi, A^{-1}\psi)}{(\psi, A^{-1}\psi)}.$$

□