

Ground state energy of dilute Bose gases in 1D

Johannes Agerskov¹, Robin Reuvers², and Jan Philip Solovej¹

1. Department of Mathematics, University of Copenhagen, Universitetsparken 5,
DK-2100 Copenhagen Ø, Denmark
 2. Università degli Studi Roma Tre, Dipartimento di Matematica e Fisica, L.go S. L.
Murialdo 1, 00146 Roma, Italy
- March 31, 2023

Abstract

We study the ground state energy of a gas of 1D bosons with density ρ , interacting through a general, repulsive 2-body potential with scattering length a , in the dilute limit $\rho|a| \ll 1$. The first terms in the expansion of the thermodynamic energy density are $\pi^2 \rho^3 / 3(1 + 2\rho a)$, where the leading order is the 1D free Fermi gas. This result covers the Tonks–Girardeau limit of the Lieb–Liniger model as a special case, but given the possibility that $a > 0$, it also applies to potentials that differ significantly from a delta function. We include extensions to spinless fermions and 1D anyonic symmetries, and discuss an application to confined 3D gases.

1 Introduction

The ground state energy of interacting, dilute Bose gases in 2 and 3 dimensions has long been a topic of study. Usually, a Hamiltonian of the form

$$-\sum_{i=1}^N \Delta_{x_i} + \sum_{1 \leq i < j \leq N} v(x_i - x_j) \tag{1.1}$$

is considered ($\hbar = 2m = 1$), in a box $[0, L]^d$ of dimension $d = 2, 3$, and with a repulsive 2-body interaction $v \geq 0$ between the bosons. Diluteness is defined by saying the density $\rho = N/L^d$ of the gas is low compared to the scale set by the scattering length a of the potential (see Appendix C in [30] for a discussion, and also Section 1.2 for $d = 1$ below). That is, $\rho a^2 \ll 1$ in 2D, and $\rho a^3 \ll 1$ in 3D.

In the thermodynamic limit, the diluteness assumption allows for surprisingly general expressions for the ground state energy. Take, for example, the famous energy expansion to second

order in $\rho a^3 \ll 1$ by Lee–Huang–Yang [27], derived for 3D bosons with a hard core of diameter a ,

$$4\pi N \rho^{2/3} (\rho a^3)^{1/3} \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3} + o\left(\sqrt{\rho a^3}\right) \right). \quad (1.2)$$

After early rigorous work by Dyson [11], Lieb and Yngvason [31] proved that the leading term in this expansion holds for a very general class of potentials v , and a similar result was obtained for the second-order term [3, 14, 15, 50].

The situation is similar in 2D. The leading order in the energy expansion for $\rho a^2 \ll 1$ derived by Schick [43] was proved rigorously by Lieb and Yngvason [36]. A second-order term has also been derived and was equally predicted to be general [1, 13, 38], resulting in the expansion

$$\frac{4\pi N \rho}{|\ln(\rho a^2)|} \left(1 - \frac{\ln |\ln(\rho a^2)|}{|\ln(\rho a^2)|} + \frac{C}{|\ln(\rho a^2)|} + o\left(|\ln(\rho a^2)|^{-1}\right) \right), \quad (1.3)$$

for some constant C . This was recently shown rigorously [12].

Remarkably, it seems the existence of a similar, general expansion in 1D was never studied in similar depth. It was, however, suggested in [2] by considering two exactly-known special cases, as we will do now as well.

The first is the famous Lieb–Liniger model [33]. Many of its features can be calculated explicitly with Bethe ansatz wave functions, but for our purpose we return to something basic: the ground state energy. Consider Lieb and Liniger’s Hamiltonian for a gas of N one-dimensional bosons on an interval of length L (periodic b.c.), with a repulsive point interaction of strength $2c > 0$,

$$-\sum_{i=1}^N \partial_{x_i}^2 + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j). \quad (1.4)$$

The ground state can be found explicitly [33], and in the thermodynamic limit $L \rightarrow \infty$ with density $\rho = N/L$ fixed, its energy is

$$E_{\text{LL}} = N \rho^2 e(c/\rho), \quad (1.5)$$

where $e(c/\rho)$ is described by integral equations. Since c/ρ is the only relevant parameter, diluteness, or low density ρ , should imply $c/\rho \gg 1$. In this case, the ground state energy can be expanded as ([33]; see, for example, [21, 25]),

$$E_{\text{LL}} = N \rho^2 e(c/\rho) = N \frac{\pi^2}{3} \rho^2 \left(\left(1 + 2 \frac{\rho}{c} \right)^{-2} + \mathcal{O}\left(\left(\frac{\rho}{c}\right)^3\right) \right). \quad (1.6)$$

Recall that the dilute limit is $\rho a^2 \ll 1$ in 2D and $\rho a^3 \ll 1$ in 3D. This seems easy to generalize

to 1D, but it turns out the Lieb–Liniger potential $2c\delta$ has scattering length $a = -2/c$. That is, in 1D the scattering length can be negative even if the potential is positive, and we should be careful to define the dilute limit as $\rho|a| \ll 1$. This then matches the limit $c/\rho \gg 1$ mentioned above, and we can write (1.6) as

$$\begin{aligned} E_{\text{LL}} &= N \frac{\pi^2}{3} \rho^2 \left((1 - \rho a)^{-2} + \mathcal{O}(\rho a)^3 \right) \\ &= N \frac{\pi^2}{3} \rho^2 (1 + 2\rho a + 3(\rho a)^2 + \mathcal{O}(\rho a)^3). \end{aligned} \quad (1.7)$$

This expansion should now be a good candidate for the 1D equivalent of (1.2) and (1.3). This is supported by the fact that 1D bosons with a hard core of diameter a have an exact thermodynamic ground state energy of [2, 17]

$$N \frac{\pi^2}{3} \left(\frac{N}{L - Na} \right)^2 = N \frac{\pi^2}{3} \rho^2 (1 - \rho a)^{-2}. \quad (1.8)$$

This is the 1D free Fermi energy on an interval shortened by the space taken up by the hard cores (the ground state is of Girardeau type; see Remark 2 and the discussion of the Girardeau wave function in Section 1.2).

With two explicit examples satisfying (1.7) to second order, it seems likely we can expect this expansion to be general [2], just like (1.2) and (1.3) in three and two dimensions. Indeed, our main result confirms the validity of (1.7) to first order, for a wide class of interaction potentials.

1.1 Main theorem

Throughout the paper, we will assume that the 2-body potential v is a symmetric measure with a finite range, $\text{supp}(v) \subset [-R_0, R_0]$. Furthermore, we assume $v = v_{\text{reg}} + v_{\text{h.c.}}$, where v_{reg} is a finite measure, and $v_{\text{h.c.}}$ is a positive linear combination of ‘hard-core’ potentials of the form

$$v_{[x_1, x_2]}(x) := \begin{cases} \infty & |x| \in [x_1, x_2] \\ 0 & \text{otherwise} \end{cases}, \quad (1.9)$$

for $0 \leq x_1 \leq x_2 \leq R_0$.¹ We will consider the N -body Hamiltonian

$$H_N = - \sum_{i=1}^N \partial_{x_i}^2 + \sum_{1 \leq i < j \leq N} v(x_i - x_j) \quad (1.10)$$

¹Note we allow $0 \leq x_1 = x_2 \leq R_0$, by which we mean that impenetrable delta potentials of the form $h(\delta_{-x_1} + \delta_{x_1})$ with $h \rightarrow \infty$ can freely be included. This amounts to a zero boundary condition at $|x| = x_1$.

on the interval $[0, L]$ with any choice of (local, self-adjoint) boundary conditions. Let $\mathcal{D}(H_N)$ be the appropriate bosonic domain of symmetric wave functions with these boundary conditions. The ground state energy is then

$$E(N, L) := \inf_{\substack{\Psi \in \mathcal{D}(H_N) \\ \|\Psi\|=1}} \langle \Psi | H_N | \Psi \rangle = \inf_{\substack{\Psi \in \mathcal{D}(H_N) \\ \|\Psi\|=1}} \mathcal{E}(\Psi), \quad (1.11)$$

with energy functional

$$\mathcal{E}(\Psi) = \int_{[0, L]^N} \sum_{i=1}^N |\partial_i \Psi|^2 + \sum_{1 \leq i < j \leq N} v_{ij} |\Psi|^2. \quad (1.12)$$

Theorem 1 (bosons). *Consider a Bose gas with repulsive interaction $v = v_{\text{reg}} + v_{\text{h.c.}}$ as defined above. Write $\rho = N/L$. For $\rho|a|$ and ρR_0 sufficiently small, the ground state energy can be expanded as*

$$E(N, L) = N \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a + \mathcal{O} \left((\rho|a|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3} \right) \right), \quad (1.13)$$

where a is the scattering length of v (see Lemma 4 below). A precise expression for the error is given in the upper and lower bounds (2.1) and (3.1).

To obtain this result, we prove an upper bound in the form of Proposition 8 in Section 2, and a matching lower bound in the form of Proposition 15 in Section 3. We use Dirichlet boundary conditions for the upper bound and Neumann boundary conditions for the lower bound, as these produce the highest and lowest ground state energy respectively. This way, Theorem 1 holds for a wide range of boundary conditions.

Remark 2. *As a special case, Theorem 1 covers the ground state energy expansion (1.6) of the Lieb–Liniger model (1.4) in the limit $c/\rho \gg 1$, as discussed in the introduction. This is known as the Tonks–Girardeau limit. Crucially, in this limit, the leading order term is the energy of the 1D free Fermi gas $N\pi^2\rho^2/3$, as first understood by Girardeau [17] (see also the discussion around (1.15) and (1.16) below).² Theorem 1 shows this holds for general potentials as well. That means that the dilute limit in 1D is very different from the one in two and three dimensions, where the zeroth-order term in the energy is that of a perfect condensate at zero momentum and the first-order term can be extracted using Bogoliubov theory [6]. In particular, the free Bose gas ($v = 0$) in 1D cannot be considered dilute, because it has infinite $|a|$.*

²Note that Girardeau studied the $c/\rho \rightarrow \infty$ case before Lieb and Liniger, who then generalized his work to obtain and solve the complete Lieb–Liniger model (1.4).

Remark 3. *An interesting feature of Theorem 1 is that the scattering length, a , can be both positive and negative. In this sense, our result covers cases that do not necessarily resemble the Lieb–Liniger model, which always has a negative scattering length. We discuss this further in Section 1.4.*

Note that zero scattering length can be achieved, which means the error in (1.13) cannot just be written in terms of $(\rho|a|)^s$ for some $s > 1$, but that $(\rho R_0)^s$ also appears.

1.2 Proof strategy

The most important ingredient in our proof is the following lemma, which follows from straightforward variational calculus. It is based on work by Dyson on the 3D Bose gas [11] and is present in Appendix C in [30].

Lemma 4 (The 2-body scattering solution and scattering length). *Suppose v is a repulsive interaction $v = v_{\text{h.c.}} + v_{\text{reg}}$ as defined in the previous section. In particular, v is symmetric and $\text{supp}(v) \subset [-R_0, R_0]$. Let $R > R_0$. For all $f \in H^1[-R, R]$ subject to $f(R) = f(-R) = 1$,*

$$\int_{-R}^R 2|\partial_x f|^2 + v(x)|f(x)|^2 dx \geq \frac{4}{R-a}. \quad (1.14)$$

There is a unique f_0 attaining the minimum energy: the scattering solution. It satisfies the scattering equation $\partial_x^2 f_0 = \frac{1}{2}v f_0$ in the sense of distributions, and $f_0(x) = (x-a)/(R-a)$ for $x \in [R_0, R]$. The parameter a is called the scattering length, which need not be positive in 1D.

Similar lemmas play an important role in the understanding of the ground state energy expansions (1.2) and (1.3) in higher dimensions [11, 31, 36], but there are a number of things we need to do differently. These relate to the fermionic behaviour of the bosons in the limit $\rho|a| \ll 1$ (see Remark 2 above).

What does this mean in practice? For the upper bound in Section 2, it suffices to find a suitable trial state by the variational principle (1.11). Good trial states for dilute bosons in 2D and 3D are close to a pure condensate, but in 1D the state will have to be close to the free Fermi ground state obtained in the limit $\rho|a| \rightarrow 0$. To achieve this, we can rely on Girardeau’s solution [17] of the $c/\rho \rightarrow \infty$ limit of the Lieb–Liniger model. In that case, the delta function in (1.4) enforces a zero boundary condition whenever two bosons meet, so the bosons are impenetrable. The wave function is then found by minimizing the kinetic energy subject to this boundary condition. If we only consider the sector $0 \leq x_1 \leq \dots \leq x_N \leq L$ (which suffices by symmetry), this is exactly the free Fermi problem. For periodic boundary conditions on the interval $[0, L]$,

the (unnormalized) free Fermi ground state is³

$$\Psi_F^{\text{per}}(x_1, \dots, x_N) = \prod_{1 \leq i < j \leq N} \sin\left(\pi \frac{x_i - x_j}{L}\right). \quad (1.15)$$

Of course, the ground state for impenetrable bosons should be symmetric rather than antisymmetric, and to correctly extend it beyond $0 \leq x_1 \leq \dots \leq x_N \leq L$ we need to remove the signs,

$$|\Psi_F^{\text{per}}|(x_1, \dots, x_N) = \prod_{1 \leq i < j \leq N} \left| \sin\left(\pi \frac{x_i - x_j}{L}\right) \right|. \quad (1.16)$$

This is Girardeau's ground state for impenetrable bosons, and it still produces the free Fermi kinetic energy $N\pi^2/3\rho^2$ in the thermodynamic limit.⁴

Returning to the problem of finding a suitable trial state, (1.16) should be a good departure point. To account for the effect of the interaction potential, we should modify the $\sin(\pi(x_i - x_j)/L)$ terms in (1.16) on the scale set by a . Lemma 4, and the scattering solution f_0 , are designed to provide the right 2-body wave function in the presence of the potential, so it seems natural to replace the sine by

$$\begin{cases} f_0(x) \sin(\pi b/L) & |x| \leq b \\ \sin(\pi(x_i - x_j)/L) & |x| > b \end{cases} \quad (1.17)$$

on some suitable scale $|a| \ll b \ll L$. This is the idea we rely upon for the upper bound proved in Section 2.

For the lower bound in Section 3, we equally need to find a way to obtain the free Fermi energy to leading order. We use Lemma 4 in combination with the known expansion (1.6) for the Lieb–Liniger model. Choosing a suitable $R > R_0$, the idea is that (1.14) can be written as

$$\int_{-R}^R 2|\partial_x f|^2 + v(x)|f(x)|^2 dx \geq \frac{2}{R-a} \int (\delta_R(x) + \delta_{-R}(x))|f(x)|^2 dx, \quad (1.18)$$

thus lower bounding the kinetic and potential energy on $[-R, R]$ by a symmetric delta potential at radius R . Heuristically, we proceed by repeatedly applying (1.18) to an N -body wave function Ψ to obtain the symmetric delta potential for any neighbouring pairs of bosons. Then—crucially—we throw away the regions where $|x_{i+1} - x_i| \leq R$, which is inspired by a similar step in [35]. This produces a lower bound since v is repulsive. With these regions removed, the two delta functions at radius $|x_{i+1} - x_i| = R$ collapse into a single delta at $|x_{i+1} - x_i| = 0$, with value

³This expression can be found by creating a Slater determinant of momentum eigenstates, and noting this is a Vandermonde determinant. See Section 2.1 for the calculation for Dirichlet boundary conditions.

⁴The wave functions Ψ_F^{per} and $|\Psi_F^{\text{per}}|$ have the same energy and that is all we will need in this paper. However, their momentum distributions are very different, which is discussed further in Section 1.5.

$4/(R - a)$. This gives the Lieb–Liniger model on a reduced interval, evaluated on some wave function, which can then be lower bounded using the Lieb–Liniger ground state energy (1.6) (appropriately corrected for finite N , and the loss of norm of Ψ from the thrown-out regions).

All this may seem rather radical, but the heuristics work out: starting with an interval of length L , we cut it back to length $L - (N - 1)R$, so that the Lieb–Liniger expansion (1.6) with $c = 2/(R - a)$ and new density $N/(L - (N - 1)R) = \rho(1 + \rho R + \dots)$ produce

$$N \frac{\pi^2}{3} \rho^2 (1 + 2\rho R + \dots)(1 - 2\rho(R - a) + \dots) = N \frac{\pi^2}{3} \rho^2 (1 + 2\rho a + \dots). \quad (1.19)$$

We show that, a priori, the ground state wave function has little weight in the regions that get thrown out, so that (1.19) is accurate. The rigorous procedure used to obtain the Lieb–Liniger model and the expansion (1.19) are outlined in Section 3.

1.3 Spinless fermions and anyons

The expansion in Theorem 1 generalizes to spinless fermions in 1D. Given the antisymmetry of the fermionic wave function, the result involves the odd-wave scattering length a_{odd} of v , obtained from Lemma 4 by replacing the symmetric boundary condition $f(R) = f(-R) = 1$ by an antisymmetric one, $f(R) = -f(-R) = 1$.

Theorem 5 (spinless fermions). *Consider a Fermi gas with repulsive interaction $v = v_{\text{reg}} + v_{\text{h.c.}}$ as defined before Theorem 1. Let a_{odd} be the odd-wave scattering length of v . Define $\mathcal{D}_F(H_N)$ to be the appropriate domain of antisymmetric wave functions, and let $E_F(N, L)$ be its associated ground state energy. Write $\rho = N/L$. For ρa_{odd} and ρR_0 sufficiently small, the ground state energy can be expanded as*

$$E_F(N, L) = N \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a_{\text{odd}} + \mathcal{O}\left((\rho R_0)^{6/5} + N^{-2/3}\right) \right). \quad (1.20)$$

This theorem follows from Theorem 1 by using Girardeau’s insight [17] that fermions and impenetrable bosons in 1D are unitarily equivalent, and hence have the same energy. It suffices to know the wave function on a single sector $0 \leq x_1 \leq \dots \leq x_N \leq L$, after which we can extend to any other sector by adding the correct sign for either bosons or fermions (note, any acceptable wave function is zero whenever $x_i = x_j$). Flipping these signs is exactly the nature of the unitary operator; see for example the equivalence between (1.15) and (1.16) discussed above. Given that Theorem 1 holds for impenetrable bosons, we can apply it as long as we use a zero boundary condition at $x = 0$ in Lemma 4. By similar reasoning, this produces the same scattering length as using the fermionic boundary condition $f(R) = -f(-R) = 1$ in Lemma 4. Theorem 5 is therefore a corollary of Theorem 1.

Remark 6 (spin-1/2 fermions). *Consider the case of spin-1/2 fermions. If we study the usual, spin-independent Lieb–Liniger Hamiltonian (1.4), the ground state will have a fixed total spin S . In fact, it is possible to study the ground state energy in each spin sector, and it will be monotone increasing in S according to work by Lieb and Mattis [32]. For each of these sectors, an explicit solution in terms of the Bethe ansatz exists [16, 49]. In certain cases, these can be expanded in the limit $c/\rho \gg 1$ [22], and the analogue to (1.6) and (1.7) can be obtained. The ground state energy for spin-1/2 fermions ($S = 0$ by Lieb–Mattis) gives [18, 22]*

$$N \frac{\pi^2}{3} \rho^2 \left(1 - 4 \frac{\rho}{c} \ln(2) + \mathcal{O}(\rho/c)^2 \right) = N \frac{\pi^2}{3} \rho^2 (1 + 2 \ln(2) \rho a + \mathcal{O}(\rho a)^2). \quad (1.21)$$

Both the Lieb–Liniger exact solution and the expansions can be generalized to higher spins (or Young diagrams) [23, 47]. Note the leading order will be the free Fermi $N\pi^2\rho^2/3$ in all cases, since the delta potential does not influence the energy for impenetrable particles.

For general potentials, the zeroth-order Fermi term is still expected to be correct, but the first-order term in (1.21) has to be more complicated. Given that two spin-1/2 fermions can form symmetric and antisymmetric combinations, both the even-wave scattering length $a_{\text{even}} = a$ and the odd-wave scattering length a_{odd} of the potential will play a role. In the Lieb–Liniger example (1.21), $a_{\text{odd}} = 0$, since the delta interaction does not affect antisymmetric wave functions. However, for hard-core fermions of diameter a , $a_{\text{odd}} = a_{\text{even}} = a$, and the energy should be (1.8) since the spin symmetry plays no role. These two examples suggest that the correct formula is

$$N \frac{\pi^2}{3} \rho^2 (1 + 2 \ln(2) \rho a_{\text{even}} + 2(1 - \ln(2)) \rho a_{\text{odd}} + \mathcal{O}(\rho \max(|a_{\text{even}}|, a_{\text{odd}}))^2). \quad (1.22)$$

We will discuss this expansion in a future publication.

The approach followed to obtain Theorem 5 can actually be taken further. What if, starting from some wave function on a sector $0 \leq x_1 < \dots < x_N \leq L$, we want to add anyonic phases $e^{i\kappa}$ with $0 \leq \kappa \leq \pi$, whenever two particles are interchanged? It turns out this can be made to work, going back to, amongst others, [26, 28] (see [7, 41] for a historical overview of this approach, comparisons with other versions of 1D anyonic statistics, and a discussion of experimental relevance). Just like fermions are unitarily equivalent to impenetrable bosons, these 1D anyons are equivalent to bosons with a certain choice of boundary conditions whenever two bosons meet. This can be related to the Lieb–Liniger model with certain c [41], since the delta function potential in (1.4) also imposes boundary conditions whenever two bosons meet. Hence, the (bosonic) Lieb–Liniger model can be viewed as a description of a non-interacting gas of anyons, with the $c/\rho \rightarrow \infty$ case being equivalent to fermions ($\kappa = \pi$) as understood by Girardeau.

Somewhat confusingly, this does not complete the picture, because many authors study gases of 1D anyons themselves interacting through a Lieb–Liniger potential, see for example [4, 24]. In this case, there are two parameters: the statistical parameter κ describing the phase $e^{i\kappa}$ upon particle exchange, and the Lieb–Liniger parameter c . Not surprisingly, this set-up is again unitarily equivalent to the bosonic Lieb–Liniger model, with an interaction potential of $2c\delta_0/\cos(\kappa/2)$.⁵ This means Theorem 1 can be applied. We provide more details about the set-up, and prove the following theorem as a corollary of Theorem 1, in Section 4.

Theorem 7 (anyons). *Let $c \geq 0$ and consider 1D anyons with statistical parameter $\kappa \in [0, \pi]$ and repulsive interaction $v = v_{\text{reg}} + v_{\text{h.c.}} + 2c\delta_0$, where $v_{\text{h.c.}}$ is defined before Theorem 1, and v_{reg} is a finite measure with $v_{\text{reg}}(\{0\}) = 0$. Define a_κ to be the scattering length associated with potential $v_\kappa = v_{\text{h.c.}} + v_{\text{reg}} + \frac{2c}{\cos(\kappa/2)}\delta_0$. Write $\rho = N/L$. For $\rho|a_\kappa|$ and ρR_0 sufficiently small, the ground state energy $E_{(\kappa,c)}(N, L)$ of the anyon gas can be expanded as*

$$E_{(\kappa,c)}(N, L) = N \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a_\kappa + \mathcal{O}\left((\rho|a_\kappa|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3}\right) \right). \quad (1.23)$$

1.4 Physical applications and confinement from 3D to 1D

Given the general expansions (1.2) and (1.3) for the energy of dilute Bose gases in three and two dimensions, it is perhaps surprising that the possibility of a 1D equivalent was only hinted at in [2], and never studied in depth. On the other hand, given the existence of the Lieb–Liniger model, this is perhaps not surprising at all. Not only can we calculate everything explicitly in that case, Lieb–Liniger physics also naturally shows up in experimental settings in which 3D particles are confined to a 1D environment [34, 35, 40, 45]. Nevertheless, we would like to argue that our result adds something that goes beyond the Lieb–Liniger model: it allows for positive scattering lengths a .

Mathematically, this seems clear. The scattering length of the Lieb–Liniger model with $c > 0$ is $a = -2/c < 0$, but Theorem 1 is also valid for potentials with a positive scattering length. There are plenty of interesting potentials with this property, and the energy shift has the opposite sign compared to the Lieb–Liniger case. (Note the Lieb–Liniger model with $c < 0$ can be solved explicitly [8], but that it has a clustered ground state of energy $-\mathcal{O}(N^2)$ [33, 37], so that scattering is irrelevant.)

Physically, the issue can seem more subtle. In the lab, 1D physics can be obtained by confining 3D particles with 3D potentials to a one-dimensional setting [19, 20, 39, 44]. As mentioned, the Lieb–Liniger model is very relevant to such set-ups [34, 35, 40, 45], but only in certain parameter regimes. In these references, the confinement length l_\perp in the trapping

⁵From the viewpoint of the energy, the combination $2c/\cos(\kappa/2)$ is the only relevant parameter. This is different for the momentum distribution, see Section 1.5.

direction (a length that is necessarily small on some scale to create 1D physics) is much bigger than the range of atomic forces (or 3D scattering length). This allows excited states in the trapping direction to play a role in the problem, making the mathematical analysis complicated. The assumption that $l_{\perp} \gg a_{3D}$ is sometimes referred to as weak confinement [5].

There should also be a ‘strong confinement’ regime $l_{\perp} \ll a_{3D}$, in which the excited states in the trapping direction play no role at all (presumably simplifying the mathematical steps needed to go from 3D to 1D). The problem would then essentially be 1D, and take on the form considered in Theorem 1, thus allowing for positive 1D scattering lengths. We do not know whether the strong confinement regime is currently experimentally accessible.

1.5 Open problems

1. **The second-order term.** The second-order expansions (1.2) and (1.3) of the ground state energy of the dilute Bose gas in 3D and 2D hold for a wide class of potentials. As motivated in the introduction, the same might be true in the 1D expansion (1.7), but the techniques used in higher dimensions are not expected to be applicable to 1D.
2. **Momentum distribution.** As mentioned in Footnote 4, even though the 1D free Fermi ground state (1.15) and Girardeau’s bosonic equivalent (1.16) have the same energy, their momentum distributions differ. In the thermodynamic limit, the free Fermi ground state has a uniform momentum distribution, up to the Fermi momentum $|k| \leq k_F = \pi\rho$. Girardeau’s state has the same quasi-momentum distribution, but the momentum distribution itself diverges like $1/\sqrt{k}$ for small k [29, 48]. At finite N , the $k = 0$ occupation is $O(1)$ for fermions, while it is $O(\sqrt{N})$ for bosons.
It is also possible to study the Lieb–Liniger ground state in this way [9]. The bosonic zero-momentum occupation λ_0 in the limit $c/\rho \gg 1$ is predicted to be

$$\lambda_0 \sim N^{\frac{1}{2} + \frac{2\rho}{c} + \mathcal{O}(\rho/c)^2} = N^{\frac{1}{2} - \rho a + \mathcal{O}(\rho a)^2}, \quad (1.24)$$

and one can ask if this holds for general potentials as well. The same question can be posed in the context of anyons [9], as the full prediction seems to be [4, 9]

$$\lambda_0 \sim N^{\left(\frac{1}{2} + \frac{2\rho}{c} \cos\left(\frac{\kappa}{2}\right)\right)\left(1 - \left(\frac{\kappa}{\pi}\right)^2\right) + \mathcal{O}(\rho \cos(\kappa/2)/c)^2} = N^{\left(\frac{1}{2} - \rho a_{\kappa}\right)\left(1 - \left(\frac{\kappa}{\pi}\right)^2\right) + \mathcal{O}(\rho a_{\kappa})^2}. \quad (1.25)$$

3. **Positive temperature.** For $T > 0$, one can again ask if quantities like the chemical potential and free energy only depend on ρa to lowest orders. Starting from the ideal Fermi gas and excluding volume as in the case of hard-core bosons (the equivalent of (1.8)), it is possible to generate appropriate expressions that might be universal [10].

Proving these for a wide class of potentials is an open problem.

2 Upper bound in Theorem 1

Proposition 8 (Upper bound in Theorem 1). *Consider a Bose gas with repulsive interaction $v = v_{\text{reg}} + v_{h.c.}$ as defined above Theorem 1, with Dirichlet boundary conditions. Write $\rho = N/L$. There exists a constant $C_U > 0$ such that for $\rho|a|$, $\rho R_0 \leq C_U^{-1}$, the ground state energy $E^D(N, L)$ satisfies*

$$E^D(N, L) \leq N \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a + C_U \left(\left((\rho|a|)^{6/5} + (\rho R_0)^{3/2} \right) \left(1 + \rho R_0^2 \int v_{\text{reg}} \right)^{1/2} + N^{-1} \right) \right). \quad (2.1)$$

As explained in Section 1.2, the proof relies on a trial state constructed from the free Fermi ground state. With Dirichlet boundary conditions, we cannot use $|\Psi_F^{\text{per}}|$ from (1.16), and shall instead have to construct its Dirichlet equivalent, denoted by $|\Psi_F|$ in this section. This will be done in Section 2.1. Given a suitable scale $b > R_0$ to be fixed later on, the trial state will be

$$\Psi_\omega(x) = \begin{cases} \omega(\mathcal{R}(x)) \frac{|\Psi_F(x)|}{\mathcal{R}(x)} & \text{if } \mathcal{R}(x) < b \\ |\Psi_F(x)| & \text{if } \mathcal{R}(x) \geq b, \end{cases} \quad (2.2)$$

where $\omega(x) = f_0(x)b$ is constructed from the scattering solution f_0 from Lemma 4 ($R = b$), and $\mathcal{R}(x) := \min_{i < j} (|x_i - x_j|)$ is the distance between the closest pair of particles (uniquely defined almost everywhere). In other words, we only modify $|\Psi_F|$ with the scattering solution for the closest pair. This is convenient for technical reasons, and will turn out to suffice if the number of particles N is not too big.

For this and other reasons, we will need another technical step: an argument that produces a trial state for arbitrary N (and L) using the Ψ_ω defined in (2.2). This is done in Section 2.4 by dividing $[0, L]$ into small intervals, and patching copies of Ψ_ω .

First, we focus on the small- N trial state Ψ_ω . Our goal will be the following lemma. In the following we will take b to satisfy $b > \max(2a, R_0)$.

Lemma 9. *Let $E_0 = N \frac{\pi^2}{3} \rho^2 (1 + \mathcal{O}(1/N))$ the ground state energy of the (Dirichlet) free Fermi gas. The energy of the trial state Ψ_ω defined in (2.2) can be estimated as*

$$\begin{aligned} \mathcal{E}(\Psi_\omega) &:= \int_{[0, L]^N} \sum_{i=1}^N |\partial_i \Psi_\omega|^2 + \sum_{1 \leq i < j \leq N} v_{ij} |\Psi_\omega|^2 \\ &\leq E_0 \left(1 + 2\rho a \frac{b}{b-a} + \text{const.} \left(N(\rho b)^3 \left[1 + \rho b^2 \int v_{\text{reg}} \right] + \rho a \frac{\ln(N)}{N} \right) \right). \end{aligned} \quad (2.3)$$

To prove this lemma, it is useful to divide the configuration space into various sets. For $i < j$, define

$$\begin{aligned} B &:= \{x \in \mathbb{R}^N \mid \mathcal{R}(x) < b\} \\ A_{ij} &:= \{x \in \mathbb{R}^N \mid |x_i - x_j| < b\} \\ B_{ij} &:= \{x \in \mathbb{R}^N \mid \mathcal{R}(x) < b, \mathcal{R}(x) = |x_i - x_j|\} \subset A_{ij}. \end{aligned} \quad (2.4)$$

Note that Ψ_ω equals $|\Psi_F|$ on the complement of B , and that B_{ij} equals B intersected with the set $\{\text{“particles } i \text{ and } j \text{ are closer than any other pair”}\}$. On the set A_{12} , we will use the shorthand $\Psi_{12} := \omega(x_1 - x_2) \frac{\Psi_F(x)}{(x_1 - x_2)}$, and define the energies

$$\begin{aligned} E_1 &:= \binom{N}{2} \int_{A_{12}} \sum_{i=1}^N |\partial_i \Psi_{12}|^2 + \sum_{1 \leq i < j \leq N} (v_{\text{reg}})_{ij} |\Psi_{12}|^2 - \sum_{i=1}^N |\partial_i \Psi_F|^2, \\ E_2^{(1)} &:= \binom{N}{2} 2N \int_{A_{12} \cap A_{13}} \sum_{i=1}^N |\partial_i \Psi_F|^2, \\ E_2^{(2)} &:= \binom{N}{2} \binom{N-2}{2} \int_{A_{12} \cap A_{34}} \sum_{i=1}^N |\partial_i \Psi_F|^2. \end{aligned} \quad (2.5)$$

Recall $E_0 = N \frac{\pi^2}{3} \rho^2 (1 + \mathcal{O}(1/N))$ is the ground state energy of the (Dirichlet) free Fermi gas. The following estimate then holds.

Lemma 10.

$$\mathcal{E}(\Psi_\omega) \leq E_0 + E_1 + E_2^{(1)} + E_2^{(2)}. \quad (2.6)$$

The plan to prove the upper bound for Theorem 1 (Proposition 8) is as follows. We first prove Lemma 10 below. We then study the Dirichlet free Fermi ground state Ψ_F in Section 2.1, laying the ground work for the estimates of E_1 , $E_2^{(1)}$ and $E_2^{(2)}$. We estimate E_1 in Section 2.2 and $E_2^{(1)}$ and $E_2^{(2)}$ in Section 2.3. Altogether, these prove Lemma 9, which will then be used to construct a successful trial state for large N in Section 2.4.

Proof of Lemma 10. Since v has supported in the interval $[-b, b]$, and $\Psi_\omega = |\Psi_F|$ except in the region $B = \{x \in \mathbb{R}^N \mid \mathcal{R}(x) < b\}$, we can write, using the diamagnetic inequality⁶,

$$\mathcal{E}(\Psi_\omega) \leq E_0 + \int_B \sum_{i=1}^N |\partial_i \Psi_\omega|^2 + \sum_{1 \leq i < j \leq N} v_{ij} |\Psi_\omega|^2 - \sum_{i=1}^N |\partial_i \Psi_F|^2, \quad (2.7)$$

with $E_0 = N \frac{\pi^2}{3} \rho^2 (1 + \mathcal{O}(1/N))$ the ground state energy of the free Fermi gas. Using symmetry under the exchange of particles and the fact that $B_{ij} \cap B_{kl} = \emptyset$ for $(i, j) \neq (k, l)$ and $(i, j) \neq (l, k)$,

⁶Strictly speaking, the diamagnetic inequality is not needed, as the estimate can be shown to be an equality in this case.

and using diamagnetic inequality in the first sum in the second line, we find

$$\begin{aligned}\mathcal{E}(\Psi_\omega) &\leq E_0 + \binom{N}{2} \int_{B_{12}} \sum_{i=1}^N |\partial_i \Psi_\omega|^2 + \sum_{1 \leq i < j \leq N} v_{ij} |\Psi_\omega|^2 - \sum_{i=1}^N |\partial_i \Psi_F|^2 \\ &\leq E_0 + \binom{N}{2} \int_{B_{12}} \sum_{i=1}^N |\partial_i \Psi_{12}|^2 + \sum_{1 \leq i < j \leq N} (v_{\text{reg}})_{ij} |\Psi_{12}|^2 - \sum_{i=1}^N |\partial_i \Psi_F|^2.\end{aligned}\tag{2.8}$$

where we have used that $\Psi_\omega = 0$ on the support of $(v_{\text{h.c.}})_{ij}$ for all i, j . Since we have $v_{\text{reg}} \geq 0$, it follows that

$$\begin{aligned}\mathcal{E}(\Psi_\omega) &\leq E_0 + \binom{N}{2} \int_{A_{12}} \sum_{i=1}^N |\partial_i \Psi_{12}|^2 + \sum_{1 \leq i < j \leq N} (v_{\text{reg}})_{ij} |\Psi_{12}|^2 - \sum_{i=1}^N |\partial_i \Psi_F|^2 \\ &\quad - \binom{N}{2} \int_{A_{12} \setminus B_{12}} \sum_{i=1}^N |\partial_i \Psi_{12}|^2 + \sum_{1 \leq i < j \leq N} (v_{\text{reg}})_{ij} |\Psi_{12}|^2 - \sum_{i=1}^N |\partial_i \Psi_F|^2 \\ &\leq E_0 + E_1 + \binom{N}{2} \int_{A_{12} \setminus B_{12}} \sum_{i=1}^N |\partial_i \Psi_F|^2.\end{aligned}\tag{2.9}$$

Noting that $x \in A_{12} \setminus B_{12}$ implies $x \in A_{ij}$ for some $(i, j) \neq (1, 2)$, we may, by antisymmetry of Ψ_F , estimate

$$\begin{aligned}\int_{A_{12} \setminus B_{12}} \sum_{i=1}^N |\partial_i \Psi_F|^2 &\leq 2N \int_{(A_{12} \setminus B_{12}) \cap A_{13}} \sum_{i=1}^N |\partial_i \Psi_F|^2 + \binom{N-2}{2} \int_{(A_{12} \setminus B_{12}) \cap A_{34}} \sum_{i=1}^N |\partial_i \Psi_F|^2 \\ &\leq 2N \int_{A_{12} \cap A_{13}} \sum_{i=1}^N |\partial_i \Psi_F|^2 + \binom{N-2}{2} \int_{A_{12} \cap A_{34}} \sum_{i=1}^N |\partial_i \Psi_F|^2.\end{aligned}\tag{2.10}$$

Thus we find $\mathcal{E}(\Psi_\omega) \leq E_0 + E_1 + E_2^{(1)} + E_2^{(2)}$ as desired. \square

2.1 The free Fermi ground state with Dirichlet b.c.

The Dirichlet eigenstates of the Laplacian are $\phi_j(x) = \sqrt{2/L} \sin(\pi j x / L)$. Thus, the Dirichlet free Fermi ground state is

$$\Psi_F(x) = \det(\phi_j(x_i))_{i,j=1}^N = \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \begin{vmatrix} e^{iy_1} - e^{-iy_1} & e^{i2y_1} - e^{-i2y_1} & \dots & e^{iNy_1} - e^{-iNy_1} \\ e^{iy_2} - e^{-iy_2} & e^{i2y_2} - e^{-i2y_2} & \dots & e^{iNy_2} - e^{-iNy_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{iy_N} - e^{-iy_N} & e^{i2y_N} - e^{-i2y_N} & \dots & e^{iNy_N} - e^{-iNy_N} \end{vmatrix},\tag{2.11}$$

where we defined $y_i = \frac{\pi}{L}x_i$. Defining $z = e^{iy}$ and using the relation $(x^n - y^n)/(x - y) = \sum_{k=0}^{n-1} x^k y^{n-1-k}$, we find

$$\Psi_F(x) = \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \prod_{i=1}^N (z_i - z_i^{-1}) \begin{vmatrix} 1 & z_1 + z_1^{-1} & \dots & \sum_{k=0}^{N-1} z_1^{2k-N+1} \\ 1 & z_2 + z_2^{-1} & \dots & \sum_{k=0}^{N-1} z_2^{2k-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_N + z_N^{-1} & \dots & \sum_{k=0}^{N-1} z_N^{2k-N+1} \end{vmatrix}. \quad (2.12)$$

Notice that $(z + z^{-1})^n = \sum_{k=0}^n \binom{n}{k} z^{2k-n}$. For $1 \leq i \leq N-1$, we add $\left(\binom{N-1}{i} - \binom{N-1}{i-1}\right)$ times column $N-i$ to column N . This does not change the determinant, so

$$\Psi_F(x) = \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \prod_{i=1}^N (z_i - z_i^{-1}) \begin{vmatrix} 1 & z_1 + z_1^{-1} & \dots & \sum_{k=0}^{N-2} z_1^{2k-N+1} & (z_1 + z_1^{-1})^{N-1} \\ 1 & z_2 + z_2^{-1} & \dots & \sum_{k=0}^{N-2} z_2^{2k-N+1} & (z_2 + z_2^{-1})^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & z_N + z_N^{-1} & \dots & \sum_{k=0}^{N-2} z_N^{2k-N+1} & (z_N + z_N^{-1})^{N-1} \end{vmatrix}. \quad (2.13)$$

For $1 \leq i \leq N-2$, we add $\left(\binom{N-2}{i} - \binom{N-2}{i-1}\right)$ times column $N-1-i$ to column $N-1$, and continue this process. That is, for $3 \leq j \leq N$ and $1 \leq i \leq N-j$, we add $\left(\binom{N-j}{i} - \binom{N-j}{i-1}\right)$ times column $N-1-i$ to column $N-j+1$. This gives

$$\Psi_F(x) = \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \prod_{i=1}^N (z_i - z_i^{-1}) \begin{vmatrix} 1 & z_1 + z_1^{-1} & (z_1 + z_1^{-1})^2 & \dots & (z_1 + z_1^{-1})^{N-1} \\ 1 & z_2 + z_2^{-1} & (z_2 + z_2^{-1})^2 & \dots & (z_2 + z_2^{-1})^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_N + z_N^{-1} & (z_N + z_N^{-1})^2 & \dots & (z_N + z_N^{-1})^{N-1} \end{vmatrix}. \quad (2.14)$$

This is a Vandermonde determinant and we conclude

$$\begin{aligned} \Psi_F(x) &= \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \prod_{k=1}^N (z_k - z_k^{-1}) \prod_{i < j}^N \left((z_i + z_i^{-1}) - (z_j + z_j^{-1})\right) \\ &= 2^{\binom{N}{2}} \sqrt{\frac{2}{L}}^N \prod_{k=1}^N \sin\left(\frac{\pi}{L}x_k\right) \prod_{i < j}^N \left[\cos\left(\frac{\pi}{L}x_i\right) - \cos\left(\frac{\pi}{L}x_j\right)\right] \\ &= -2^{\binom{N}{2}+1} \sqrt{\frac{2}{L}}^N \prod_{k=1}^N \sin\left(\frac{\pi}{L}x_k\right) \prod_{i < j}^N \sin\left(\frac{\pi(x_i - x_j)}{2L}\right) \sin\left(\frac{\pi(x_i + x_j)}{2L}\right). \end{aligned} \quad (2.15)$$

2.1.1 1-body reduced density matrix

The 1-particle reduced density matrix of the Dirichlet free Fermi ground state is

$$\gamma^{(1)}(x; y) = \frac{2}{L} \sum_{j=1}^N \sin\left(\frac{\pi}{L} jx\right) \sin\left(\frac{\pi}{L} jy\right) = \frac{\sin\left(\pi\left(\rho + \frac{1}{2L}\right)(x-y)\right)}{2L \sin\left(\frac{\pi}{2L}(x-y)\right)} - \frac{\sin\left(\pi\left(\rho + \frac{1}{2L}\right)(x+y)\right)}{2L \sin\left(\frac{\pi}{2L}(x+y)\right)}. \quad (2.16)$$

We can write $\gamma^{(1)}(x; y)$, as well as its translation invariant part $\tilde{\gamma}^{(1)}(x; y)$, in terms of the Dirichlet kernel $D_n(x) = \frac{1}{2\pi} \sum_{j=-n}^n e^{ijx} = \frac{\sin((n+1/2)x)}{2\pi \sin(x/2)}$,

$$\begin{aligned} \gamma^{(1)}(x; y) &= \frac{\pi}{L} \left(D_N\left(\pi \frac{x-y}{L}\right) - D_N\left(\pi \frac{x+y}{L}\right) \right), \\ \tilde{\gamma}^{(1)}(x; y) &:= \frac{\pi}{L} D_N\left(\pi \frac{x-y}{L}\right). \end{aligned} \quad (2.17)$$

A consequence is that

$$\left| \partial_x^{k_1} \partial_y^{k_2} \gamma^{(1)}(x; y) \right| \leq \frac{1}{\pi} (2N)^{k_1+k_2+1} \left(\frac{\pi}{L} \right)^{k_1+k_2+1} = \pi^{k_1+k_2} (2\rho)^{k_1+k_2+1}. \quad (2.18)$$

Combined with Wick's theorem, which we discuss in the next subsection, (2.18) implies bounds on (derivatives of) higher-order reduced density matrices of the free Fermi ground state, that are uniform in all coordinates. Note the relevant power of ρ can be obtained directly from dimensional analysis. This will be used later on to do Taylor expansions.

Other useful bounds, which will be used in the proof of Lemma 11 are

$$\begin{aligned} \int_{[0,L]} \left| \rho^{(1)'} \right| &\leq \text{const. } \rho \ln(N), \\ \int_{[0,L]} \left| \rho^{(1)''} \right| &\leq \text{const. } \rho^2 \ln(N), \end{aligned} \quad (2.19)$$

which follow from the textbook bound on the L^1 -norm of the m th derivative of Dirichlet's kernel

$$\|\partial^m D_N\|_{L^1([0,2\pi])} \leq \text{const. } N^m \ln(N).$$

2.1.2 k -body reduced density matrices and Wick's theorem

Given a wave function $\Psi \in L^2([0, L]^N)$, its k -particle reduced density matrix is given by

$$\gamma_{\Psi}^{(k)}(x_1, \dots, x_k; y_1, \dots, y_k) = \frac{N!}{(N-k)!} \int_{[0,L]^{N-k}} \overline{\Psi(x_1, \dots, x_N)} \Psi(y_1, \dots, y_k, x_{k+1}, x_N) dx_{k+1} \dots dx_N. \quad (2.20)$$

Similarly, we define the the k -particle reduced density by

$$\rho_{\Psi}^{(k)}(x_1, \dots, x_k) = \gamma_{\Psi}^{(k)}(x_1, \dots, x_k; x_1, \dots, x_k). \quad (2.21)$$

We will frequently abbreviate $\gamma_{\Psi_F}^{(k)}$ as $\gamma^{(k)}$ and $\rho_{\Psi_F}^{(k)}$ as $\rho^{(k)}$. For a quasi-free state, Wick's theorem states that the k -point function may be expressed solely in terms of sums of products of two-point functions, with appropriate signs (see e.g. [46], Theorem 10.2). For the free Fermi ground state (which has a fixed particle number), it implies

$$\gamma^{(k)}(x_1, \dots, x_k; y_1, \dots, y_k) = \begin{vmatrix} \gamma^{(1)}(x_1; y_1) & \gamma^{(1)}(x_1; y_2) & \cdots & \gamma^{(1)}(x_1; y_k) \\ \gamma^{(1)}(x_2; y_1) & \gamma^{(1)}(x_2; y_2) & \cdots & \gamma^{(1)}(x_2; y_k) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^{(1)}(x_k; y_1) & \gamma^{(1)}(x_k; y_2) & \cdots & \gamma^{(1)}(x_k; y_k) \end{vmatrix}. \quad (2.22)$$

We use this to compute $\rho^{(2)}$ below. Using Taylor expansion and (2.18), it will also be used to bound various reduced densities and density matrices.

2.1.3 Useful bounds on various reduced density matrices of Ψ_F

Lemma 11. *For the 2-body reduced density $\rho^{(2)}$ of the free Fermi ground state, it holds that*

$$\rho^{(2)}(x_1, x_2) = \left(\frac{\pi^2}{3} \rho^4 + f(x_2) \right) (x_1 - x_2)^2 + \mathcal{O}(\rho^6 (x_1 - x_2)^4), \quad (2.23)$$

with $\int_{[0, L]} |f(x_2)| dx_2 \leq \text{const. } \rho^3 \ln(N)$.

Proof. Note that by translation invariance, we may Taylor expand $\tilde{\gamma}^{(1)}(x; y)$, defined in (2.17), in $x - y$ around 0. Only even terms can appear as D_N is even. Using (2.18), we find

$$\tilde{\gamma}^{(1)}(x; y) - (\rho + 1/(2L)) = \frac{\pi^2}{6} (\rho^3 + \rho^2 \mathcal{O}(1/L)) (x_1 - x_2)^2 + \mathcal{O}(\rho^5 (x_1 - x_2)^4). \quad (2.24)$$

Furthermore, it is easy to check that $\gamma^{(1)}(x_1; x_2) - \rho^{(1)}((x_1 + x_2)/2) = \tilde{\gamma}^{(1)}(x_1; x_2) - (\rho + 1/(2L))$. Now, by Wick's theorem (2.22),

$$\rho^{(2)}(x_1, x_2) = \rho^{(1)}(x_1) \rho^{(1)}(x_2) - \gamma^{(1)}(x_1; x_2) \gamma^{(1)}(x_2; x_1). \quad (2.25)$$

Note that by Taylor's theorem and (2.18),

$$\begin{aligned}\rho^{(1)}(x_1) &= \rho^{(1)}((x_1 + x_2)/2) + \rho^{(1)'}((x_1 + x_2)/2) \frac{x_1 - x_2}{2} \\ &\quad + \frac{1}{2} \rho^{(1)''}((x_1 + x_2)/2) \left(\frac{x_1 - x_2}{2} \right)^2 + \mathcal{O}(\rho^4(x_1 - x_2)^3),\end{aligned}\tag{2.26}$$

$$\begin{aligned}\rho^{(1)}(x_2) &= \rho^{(1)}((x_1 + x_2)/2) + \rho^{(1)'}((x_1 + x_2)/2) \frac{x_2 - x_1}{2} \\ &\quad + \frac{1}{2} \rho^{(1)''}((x_1 + x_2)/2) \left(\frac{x_1 - x_2}{2} \right)^2 + \mathcal{O}(\rho^4(x_1 - x_2)^3),\end{aligned}\tag{2.27}$$

where both expressions can be expanded further if needed. Using that $\gamma^{(1)}$ is symmetric in its coordinates, we conclude from the previous three equations that

$$\begin{aligned}\rho^{(2)}(x_1, x_2) &= \rho^{(1)}((x_1 + x_2)/2)^2 - \gamma^{(1)}(x_1; x_2)^2 - \left[\rho^{(1)'}((x_1 + x_2)/2) \right]^2 \left(\frac{x_1 - x_2}{2} \right)^2 \\ &\quad + \rho^{(1)}((x_1 + x_2)/2) \rho^{(1)''}((x_1 + x_2)/2) \left(\frac{x_1 - x_2}{2} \right)^2 + \mathcal{O}(\rho^6(x_1 - x_2)^4).\end{aligned}\tag{2.28}$$

Terms of order $\mathcal{O}(\rho^5(x_1 - x_2)^3)$ must cancel due to symmetry.

Now, notice that $0 \leq \rho^{(1)} \leq 2\rho$ and $|\rho^{(1)'}| \leq 8\pi\rho^2$ by (2.18). Together with (2.19), this implies,

$$\rho^{(2)}(x_1, x_2) = \rho^{(1)}((x_1 + x_2)/2)^2 - \gamma^{(1)}(x_1; x_2)^2 + g_1(x_1 + x_2)(x_1 - x_2)^2 + \mathcal{O}(\rho^6(x_1 - x_2)^4),\tag{2.29}$$

for some function g_1 satisfying $\int_{[0,L]} |g_1| \leq \text{const. } \rho^3 \ln(N)$. Furthermore, notice that by (2.24) and the remark below it, we have

$$\begin{aligned}\rho^{(1)}((x_1 + x_2)/2)^2 - \gamma^{(1)}(x_1; x_2)^2 &= (\rho^{(1)}((x_1 + x_2)/2) - \gamma^{(1)}(x_1; x_2))(\rho^{(1)}((x_1 + x_2)/2) + \gamma^{(1)}(x_1; x_2)) \\ &= \left[\rho + 1/(2L) - \tilde{\gamma}^{(1)}(x_1; x_2) \right] \left[-\rho - 1/(2L) + \tilde{\gamma}^{(1)}(x_1; x_2) + 2\rho^{(1)}((x_1 + x_2)/2) \right] \\ &= - \left[\rho + 1/(2L) - \tilde{\gamma}^{(1)}(x_1; x_2) \right]^2 + 2 \left[\rho + 1/(2L) - \tilde{\gamma}^{(1)}(x_1; x_2) \right] \rho^{(1)}((x_1 + x_2)/2) \\ &= 2 \left(\frac{\pi^2}{6} (\rho + 1/(2L))^3 (x_1 - x_2)^2 + \mathcal{O}(\rho^5(x_1 - x_2)^4) \right) \left(\rho + \frac{1}{2L} - \frac{\pi}{L} D_N((x_1 + x_2)/(2L)) \right) \\ &= \frac{\pi^2}{3} \rho^4 (x_1 - x_2)^2 + g_2(x_1 - x_2)(x_1 - x_2)^2 + \mathcal{O}(\rho^6(x_1 - x_2)^4),\end{aligned}\tag{2.30}$$

where we have chosen $g_2(x) = \frac{\pi^2}{3} \rho^3 \left(\frac{\text{const.}}{2L} + \left| \frac{\pi}{L} D_N(x/(2L)) \right| \right)$ which clearly satisfies $\int_{[0,L]} g_2 \leq \text{const. } \rho^3 \ln(N)$. Combining (2.29) and (2.30) now proves the lemma. \square

Lemma 12.

$$\begin{aligned}
\rho^{(3)}(x_1, x_2, x_3) &\leq \text{const. } \rho^7(x_1 - x_2)^2(x_3 - x_2)^2, \\
\rho^{(4)}(x_1, x_2, x_3, x_4) &\leq \text{const. } \rho^8(x_1 - x_2)^2(x_3 - x_4)^2, \\
\left| \sum_{i=1}^2 \partial_{y_i}^2 \gamma^{(2)}(x_1, x_2; y_1, y_2) \Big|_{y=x} \right| &\leq \text{const. } \rho^6(x_1 - x_2)^2, \\
\left| \partial_{y_1}^2 \left(\frac{\gamma^{(2)}(x_1, x_2; y_1, y_2)}{y_1 - y_2} \right) \Big|_{y=x} \right| &\leq \text{const. } \rho^6 |x_1 - x_2|, \\
\left| \sum_{i=1}^2 (-1)^{i-1} \partial_{y_i} \left(\frac{\gamma^{(2)}(x_1, x_2; y_1, y_2)}{y_1 - y_2} \right) \Big|_{y=x} \right| &\leq \text{const. } \rho^6(x_1 - x_2)^2, \\
\sum_{i=1}^3 \left(\partial_{x_i} \partial_{y_i} \gamma^{(3)}(x_1, x_2, x_3; y_1, y_2, y_3) \right) \Big|_{y=x} &\leq \text{const. } \rho^9(x_1 - x_2)^2(x_3 - x_2)^2, \\
\left| \sum_{i=1}^3 \left(\partial_{y_i}^2 \gamma^{(3)}(x_1, x_2, x_3; y_1, y_2, y_3) \right) \Big|_{y=x} \right| &\leq \text{const. } \rho^9(x_1 - x_2)^2(x_3 - x_2)^2, \\
\left| \left[\partial_y \gamma^{(4)}(x_1, x_2, x_3, x_4; y, x_2, x_3, x_4) \Big|_{y=x_1} \right]_{x_1=x_2-b}^{x_1=x_2+b} \right| &\leq \text{const. } \rho^8 b(x_3 - x_4)^2.
\end{aligned} \tag{2.31}$$

Proof. The bounds follow straightforwardly from Taylor's theorem and the symmetries of the left-hand sides. We give, in the following, two examples which we find to be representative for the general strategy.

Example 1: Consider $\sum_{i=1}^2 \partial_{y_i}^2 \gamma^{(2)}(x_1, x_2; y_1, y_2) \Big|_{y=x}$. Notice first that $\sum_{i=1}^2 \partial_{y_i}^2 \gamma^{(2)}(x_1, x_2; y_1, y_2)$ is antisymmetric in (x_1, x_2) and in (y_1, y_2) . As we discussed after (2.18), all derivatives of $\gamma^{(k)}$ are bounded uniformly in its coordinates by a constant times ρ^k for some $k \in \mathbb{N}$, we can Taylor expand $\partial^2 \gamma^{(2)}$. By expanding x_1 around x_2 and y_1 around y_2 , we see that antisymmetry implies $\sum_{i=1}^2 \partial_{y_i}^2 \gamma^{(2)}(x_1, x_2; y_1, y_2) \leq \text{const. } \rho^6(x_1 - x_2)(y_1 - y_2)$, where the power of ρ can be found by dimensional analysis.

Example 2: Consider $\left| \sum_{i=1}^2 (-1)^{i-1} \partial_{y_i} \left(\frac{\gamma^{(2)}(x_1, x_2; y_1, y_2)}{y_1 - y_2} \right) \Big|_{y=x} \right|$. We start by defining the coordinates $z_y := (y_1 - y_2)/2$, $z'_y := (y_1 + y_2)/2$, $z_x := (x_1 - x_2)/2$, and $z'_x := (x_1 + x_2)/2$. Furthermore, define $\hat{\gamma}^{(2)}(z_x, z'_x; z_y, z'_y) := \gamma^{(2)}(z_x + z'_x, z'_x - z_x; z_y + z'_y, z'_y - z_y)$. By the antisymmetry of $\gamma^{(2)}$ in x_1, x_2 and y_1, y_2 , we see that $\hat{\gamma}^{(2)}$ is odd in z_x and z_y .

In this case, we notice that $\sum_{i=1}^2 (-1)^{i-1} \partial_{y_i} = \partial_{z_y}$ and thus we find

$$\partial_{z_y} \left(\frac{\hat{\gamma}^{(2)}(z_x, z'_x; z_y, z'_y)}{z_y} \right) = \frac{z_y \partial_{z_y} \hat{\gamma}^{(2)}(z_x, z'_x; z_y, z'_y) - \hat{\gamma}^{(2)}(z_x, z'_x; z_y, z'_y)}{z_y^2}.$$

Taylor expanding both terms the numerator in z_y and z_x around 0 to order $z_x z_y^3$ gives

$$\begin{aligned}
& \left| z_x z_y \partial_{z_x} \left[\partial_{z_y} \hat{\gamma}^{(2)}(z_x, z'_x; z_y, z'_y) \right] \Big|_{z_x=z_y=0} + z_x z_y^3 \partial_{z_x} \partial_{z_y}^2 \left[\partial_{z_y} \hat{\gamma}^{(2)}(z_x, z'_x; z_y, z'_y) \right] \Big|_{z_x=z_y=0} \right. \\
& \left. - z_x z_y \partial_{z_x} \partial_{z_y} \left[\hat{\gamma}^{(2)}(z_x, z'_x; z_y, z'_y) \right] \Big|_{z_x=z_y=0} - \frac{1}{2} z_x z_y^3 \partial_{z_x} \partial_{z_y}^3 \left[\hat{\gamma}^{(2)}(z_x, z'_x; z_y, z'_y) \right] \Big|_{z_x=z_y=0} \right. \\
& \left. + \mathcal{O}(\rho^8(z_x z_y^5 + z_x^3 z_y^3)) \right| \\
& \leq \text{const. } \rho^6 |z_x z_y^3|,
\end{aligned} \tag{2.32}$$

where we used that $\hat{\gamma}^{(2)}(z_x, z'_x; z_y, z'_y)$ is odd in z_x and z_y , to conclude that all even order terms vanish when Taylor expanding in these variables around 0. The desired result follows. \square

2.2 Estimating E_1

Recall $A_{12} = \{x \in \mathbb{R}^N \mid |x_1 - x_2| < b\}$ and $\Psi_{12}(x) = \frac{\omega(x_1 - x_2)}{(x_1 - x_2)} \Psi_F(x)$, as well as

$$E_1 := \binom{N}{2} \int_{A_{12}} \sum_{i=1}^N |\partial_i \Psi_{12}|^2 + \sum_{1 \leq i < j \leq N} (v_{\text{reg}})_{ij} |\Psi_{12}|^2 - \sum_{i=1}^N |\partial_i \Psi_F|^2. \tag{2.33}$$

We prove the following bound.

Lemma 13. *For $b > \max(2a, R_0)$ we have*

$$E_1 \leq E_0 \left(2\rho a \frac{b}{b-a} + \text{const.} \left(N(\rho b)^3 \left[1 + \rho b^2 \int v_{\text{reg}} \right] + \rho a \frac{\ln(N)}{N} \right) \right). \tag{2.34}$$

Proof. We estimate E_1 by splitting it into four terms $E_1 = E_1^{(1)} + E_1^{(2)} + E_1^{(3)} + E_1^{(4)}$, with

$$\begin{aligned}
E_1^{(1)} &:= \binom{N}{2} \int_{A_{12}} 2 |\partial_1 \Psi_{12}|^2, \\
E_1^{(2)} &:= - \binom{N}{2} \int_{A_{12}} \left(2 |\partial_1 \Psi_F|^2 + \sum_{i=3}^N |\partial_i \Psi_F|^2 \right), \\
E_1^{(3)} &:= \binom{N}{2} \int_{A_{12}} \sum_{1 \leq i < j \leq N} (v_{\text{reg}})_{ij} |\Psi_{12}|^2, \\
E_1^{(4)} &:= \binom{N}{2} \int_{A_{12}} \sum_{i=3}^N |\partial_i \Psi_{12}|^2.
\end{aligned} \tag{2.35}$$

By partial integration of x_1 in $E_1^{(1)}$, we find

$$E_1^{(1)} = 2 \binom{N}{2} \int_{A_{12}} \overline{\Psi}_{12} (-\partial_1^2 \Psi_{12}) + 2 \binom{N}{2} \int [\overline{\Psi}_{12} \partial_1 \Psi_{12}]_{x_1=x_2-b}^{x_1=x_2+b} dx_2 \dots dx_N. \quad (2.36)$$

The boundary term can be calculated explicitly, and we find

$$\begin{aligned} 2 \binom{N}{2} \int [\overline{\Psi}_{12} \partial_1 \Psi_{12}]_{x_1=x_2-b}^{x_1=x_2+b} dx_2 \dots dx_N &= \int \left[\frac{\omega(x_1 - x_2)}{|x_1 - x_2|} \partial_1 \left(\frac{\omega(x_1 - x_2)}{|x_1 - x_2|} \right) \rho^{(2)}(x_1, x_2) \right]_{x_2-b}^{x_2+b} dx_2 \\ &\quad + \int \left[\left(\frac{\omega(x_1 - x_2)}{|x_1 - x_2|} \right)^2 \partial_1 \left(\gamma^{(2)}(x_1, x_2; y, x_2) \right) \right]_{y=x_1} \Big|_{x_2-b}^{x_2+b} dx_2. \end{aligned} \quad (2.37)$$

Since the function $\frac{\omega(x_1 - x_2)}{|x_1 - x_2|}$ is continuously differentiable and satisfies $\frac{\omega(x_1 - x_2)}{|x_1 - x_2|} = \frac{|x_1 - x_2| - a}{b - a} \frac{b}{|x_1 - x_2|}$ for $|x_1 - x_2| > b$, we see that

$$\partial_1 \left(\frac{\omega(x_1 - x_2)}{|x_1 - x_2|} \right) \Big|_{x_1=x_2 \pm b} = \pm \frac{a}{b(b - a)}. \quad (2.38)$$

Using Lemma 11, we find

$$\int \left[\frac{\omega(x_1 - x_2)}{|x_1 - x_2|} \partial_1 \left(\frac{\omega(x_1 - x_2)}{|x_1 - x_2|} \right) \rho^{(2)}(x_1, x_2) \right]_{x_2-b}^{x_2+b} dx_2 \leq 2a \frac{b}{b - a} N \frac{\pi^2}{3} \rho^3 \left(1 + \text{const.} \frac{\ln(N)}{N} \right). \quad (2.39)$$

Furthermore, we denote

$$\begin{aligned} &\int \left[\left(\frac{\omega(x_1 - x_2)}{|x_1 - x_2|} \right)^2 \partial_1 \left(\gamma^{(2)}(x_1, x_2; y, x_2) \right) \right]_{y=x_1} \Big|_{x_2-b}^{x_2+b} dx_2 \\ &= \int \left[\partial_1 \left(\gamma^{(2)}(x_1, x_2; y, x_2) \right) \right]_{y=x_1} \Big|_{x_2-b}^{x_2+b} dx_2 =: \kappa_1. \end{aligned} \quad (2.40)$$

Thus, we have

$$E_1^{(1)} \leq \frac{\pi^2}{3} N \rho^3 (2a) \frac{b}{b - a} \left(1 + \text{const.} \frac{\ln(N)}{N} \right) + \kappa_1 + 2 \binom{N}{2} \int_{A_{12}} \overline{\Psi}_{12} (-\partial_1^2 \Psi_{12}). \quad (2.41)$$

For $E_1^{(2)}$, we find

$$\begin{aligned}
E_1^{(2)} &= -\binom{N}{2} \int_{A_{12}} \left(2 |\partial_1 \Psi_F|^2 + \sum_{i=3}^N |\partial_i \Psi_F|^2 \right) \\
&= -\binom{N}{2} \int_{A_{12}} \sum_{i=1}^N \overline{\Psi_F} (-\partial_i^2 \Psi_F) - 2 \binom{N}{2} \int [\overline{\Psi_F} \partial_1 \Psi_F]_{x_1=x_2-b}^{x_1=x_2+b} \\
&= -E_0 \binom{N}{2} \int_{A_{12}} |\Psi_F|^2 - \underbrace{\int [\partial_y \gamma^{(2)}(x_1, x_2; y, x_2)|_{y=x_1}]_{x_2-b}^{x_2+b} dx_2}_{\kappa_1} \\
&\leq -\kappa_1,
\end{aligned} \tag{2.42}$$

Part of $E_1^{(3)}$ can be estimated as follows. First, notice that using $|\omega| \leq b$, we find

$$\begin{aligned}
&\binom{N}{2} \int_{A_{12}} \left(\sum_{2 \leq i < j}^N (v_{\text{reg}})_{ij} |\Psi_{12}|^2 + \sum_{k=3}^N (v_{\text{reg}})_{1k} |\Psi_{12}|^2 \right) \\
&\leq \text{const. } b^2 \left(\int_{\{|x_1-x_2|<b\} \cap \text{supp}((v_{\text{reg}})_{34})} v_{\text{reg}}(|x_3-x_4|) \frac{1}{(x_1-x_2)^2} \rho^{(4)}(x_1, x_2, x_3, x_4) \right. \\
&\quad \left. + \int_{\{|x_1-x_2|<b\} \cap \text{supp}((v_{\text{reg}})_{23})} v_{\text{reg}}(|x_2-x_3|) \frac{1}{(x_1-x_2)^2} \rho^{(3)}(x_1, x_2, x_3) \right).
\end{aligned} \tag{2.43}$$

Hence, by Lemma 12,

$$\begin{aligned}
&\binom{N}{2} \int_{A_{12}} \left(\sum_{2 \leq i < j}^N (v_{\text{reg}})_{ij} |\Psi_{12}|^2 + \sum_{k=3}^N (v_{\text{reg}})_{1k} |\Psi_{12}|^2 \right) \\
&\leq \text{const. } \left(N^2 (\rho b)^3 \rho^3 \int x^2 v_{\text{reg}}(x) dx + N (\rho b)^3 \rho^3 \int x^2 v_{\text{reg}}(x) dx \right) \\
&\leq \text{const. } N^2 (\rho b)^5 \rho \int v_{\text{reg}} = \text{const. } E_0 N (\rho b)^3 \left(\rho b^2 \int v_{\text{reg}} \right),
\end{aligned} \tag{2.44}$$

and so

$$\begin{aligned}
E_1 &= E_1^{(1)} + E_1^{(2)} + E_1^{(3)} + E_1^{(4)} \\
&\leq \frac{2\pi^2}{3} N \rho^3 a \frac{b}{b-a} + 2 \binom{N}{2} \int_{A_{12}} \left(\overline{\Psi_{12}} (-\partial_1^2) \Psi_{12} + \frac{1}{2} \sum_{i=3}^N |\partial_i \Psi_{12}|^2 + \frac{1}{2} v_{12} |\Psi_{12}|^2 \right) \\
&\quad + \text{const. } E_0 \left(N (\rho b)^3 \left(\rho b^2 \int v_{\text{reg}} \right) + \rho a \frac{\ln(N)}{N} \right).
\end{aligned} \tag{2.45}$$

Using the two-body scattering equation $\partial^2 \omega = \frac{1}{2} v \omega$ from Lemma 4, this implies

$$\begin{aligned}
E_1 \leq & \frac{2\pi^2}{3} N \rho^3 a \frac{b}{b-a} + 2 \binom{N}{2} \int_{A_{12}} \frac{\overline{\Psi}_F}{(x_1 - x_2)} \omega^2 (-\partial_1^2) \frac{\Psi_F}{(x_1 - x_2)} \\
& + 4 \binom{N}{2} \int_{A_{12}} \frac{\overline{\Psi}_F}{(x_1 - x_2)} \omega (\partial_1 \omega) \partial_1 \frac{\Psi_F}{(x_1 - x_2)} \\
& + \binom{N}{2} \int_{A_{12}} \sum_{i=3}^N \overline{\Psi}_F \frac{\omega^2}{(x_1 - x_2)^2} (-\partial_i^2) \Psi_F \\
& + \text{const. } E_0 \left(N(\rho b)^3 \left(\rho b^2 \int v_{\text{reg}} \right) + \rho a \frac{\ln(N)}{N} \right).
\end{aligned} \tag{2.46}$$

Furthermore, we have

$$\begin{aligned}
& \binom{N}{2} \int_{A_{12}} \sum_{i=3}^N \overline{\Psi}_F \frac{\omega^2}{(x_1 - x_2)^2} (-\partial_i^2) \Psi_F \\
& = E_0 \binom{N}{2} \int_{A_{12}} \left| \frac{\omega}{(x_1 - x_2)} \Psi_F \right|^2 - 2 \binom{N}{2} \int_{A_{12}} \overline{\Psi}_F \frac{\omega^2}{(x_1 - x_2)^2} (-\partial_1^2) \Psi_F.
\end{aligned} \tag{2.47}$$

By Lemma 11 and $|\omega| \leq b$, it follows that, it follows that

$$\binom{N}{2} \int_{A_{12}} \left| \frac{\omega}{(x_1 - x_2)} \Psi_F \right|^2 \leq b^2 \int_{\{|x_1 - x_2| < b\}} \frac{\rho^{(2)}(x_1, x_2)}{|x_1 - x_2|^2} dx_1 dx_2 \leq \text{const. } N(\rho b)^3, \tag{2.48}$$

and by Lemma 12

$$\begin{aligned}
2 \binom{N}{2} \left| \int_{A_{12}} \overline{\Psi}_F \frac{\omega^2}{(x_1 - x_2)^2} (-\partial_1^2) \Psi_F \right| & \leq \frac{1}{2} \left| \sum_{i=1}^2 \int_{A_{12}} \frac{\omega^2}{(x_1 - x_2)^2} \partial_{y_i}^2 \gamma^{(2)}(x_1, x_2; y_1, y_2) \Big|_{y=x} \right| \\
& \leq \text{const. } N \rho^2 (\rho b)^3,
\end{aligned} \tag{2.49}$$

so that we find that the third line of (2.46) is bounded by $\text{const. } E_0 N(\rho b)^3$.

For the first line, again by Lemma 12, we find that

$$\begin{aligned}
2 \binom{N}{2} \int_{A_{12}} \frac{\overline{\Psi}_F}{(x_1 - x_2)} \omega^2 (-\partial_1^2) \frac{\Psi_F}{(x_1 - x_2)} & = \int_{A_{12}} \left| \frac{\omega^2}{x_1 - x_2} \right| \left[\partial_{y_1}^2 \left(\frac{\gamma^{(2)}(x_1, x_2; y_1, y_2)}{(y_1 - y_2)} \right) \right] \Big|_{y=x} \\
& \leq \text{const. } N \rho^2 (\rho b)^3.
\end{aligned} \tag{2.50}$$

For the second line of (2.46), by using the scattering equation $\partial^2 \omega = \frac{1}{2} v \omega \geq 0$ which implies

$0 \leq \omega'(x) \leq \omega'(b) = \frac{b}{b-a}$ for $|x| < b$, we find that

$$\begin{aligned}
& \left| 4 \binom{N}{2} \int_{A_{12}} \frac{\overline{\Psi_F}}{(x_1 - x_2)} \omega(\partial_1 \omega) \partial_1 \left(\frac{\Psi_F}{(x_1 - x_2)} \right) \right| \\
&= 2 \left| \frac{1}{2} \sum_{i=1}^2 \int_{A_{12}} \frac{\omega}{x_1 - x_2} (-1)^{i-1} \omega'(x_1 - x_2) \partial_{y_i} \left(\frac{\gamma^{(2)}(x_1, x_2; y_1, y_2)}{y_1 - y_2} \right) \right|_{y=x} \\
&\leq \text{const.} \cdot \frac{b}{b-a} N \rho^2 (\rho b)^3.
\end{aligned} \tag{2.51}$$

Combining everything, we get the desired result. \square

2.3 Estimating $E_2^{(1)} + E_2^{(2)}$

Recall that

$$\begin{aligned}
E_2^{(1)} &= \binom{N}{2} 2N \int_{A_{12} \cap A_{13}} \sum_{i=1}^N |\partial_i \Psi_F|^2, \\
E_2^{(2)} &= \binom{N}{2} \binom{N-2}{2} \int_{A_{12} \cap A_{34}} \sum_{i=1}^N |\partial_i \Psi_F|^2,
\end{aligned} \tag{2.52}$$

with $A_{ij} := \{x \in \mathbb{R}^N \mid |x_i - x_j| < b\}$. We prove the following bound.

Lemma 14.

$$E_2^{(1)} + E_2^{(2)} \leq E_0 (N(\rho b)^4 + N^2(\rho b)^6). \tag{2.53}$$

Proof. We start by splitting $E_2^{(1)}$ and $E_2^{(2)}$ in two terms each and using partial integration. Consider first $E_2^{(1)}$,

$$\begin{aligned}
E_2^{(1)} &= \binom{N}{2} 2N \int_{A_{12} \cap A_{23}} \sum_{i=1}^N |\partial_i \Psi_F|^2 \\
&= \binom{N}{2} 2N \int_{A_{12} \cap A_{23}} \sum_{i=1}^3 |\partial_i \Psi_F|^2 + \binom{N}{2} 2N \int_{A_{12} \cap A_{23}} \sum_{i=4}^N |\partial_i \Psi_F|^2.
\end{aligned} \tag{2.54}$$

For the second term, we perform partial integration to find

$$\begin{aligned}
\binom{N}{2} 2N \int_{A_{12} \cap A_{23}} \sum_{i=4}^N |\partial_i \Psi_F|^2 &= \binom{N}{2} 2N \int_{A_{12} \cap A_{23}} \sum_{i=4}^N \overline{\Psi_F} (-\partial_i^2 \Psi_F) \\
&\leq E_0 \binom{N}{2} 2N \int_{A_{12} \cap A_{23}} |\Psi_F|^2 - \binom{N}{2} 2N \int_{A_{12} \cap A_{23}} \sum_{i=1}^3 \overline{\Psi_F} (-\partial_i^2 \Psi_F) \\
&\leq 3E_0 \int_{[0,L]} \int_{[x_2-b, x_2+b]} \int_{[x_2-b, x_2+b]} \rho^{(3)}(x_1, x_2, x_3) dx_3 dx_1 dx_2 \\
&\quad - \binom{N}{2} 2N \int_{A_{12} \cap A_{23}} \sum_{i=1}^3 \overline{\Psi_F} (-\partial_i^2 \Psi_F).
\end{aligned} \tag{2.55}$$

Lemma 12 implies

$$3E_0 \int_{[0,L]} \int_{[x_2-b, x_2+b]} \int_{[x_2-b, x_2+b]} \rho^{(3)}(x_1, x_2, x_3) dx_3 dx_1 dx_2 \leq \text{const. } NE_0(\rho b)^6. \tag{2.56}$$

Furthermore, Lemma 12 and antisymmetry imply

$$\binom{N}{2} 2N \int_{A_{12} \cap A_{23}} \sum_{i=1}^3 \left(|\partial_i \Psi_F|^2 - \overline{\Psi_F} (-\partial_i^2 \Psi_F) \right) \leq \text{const. } \rho^9 L b^6 = \text{const. } E_0(\rho b)^6. \tag{2.57}$$

Collecting everything, we find

$$E_2^{(1)} \leq \text{const. } NE_0(\rho b)^6. \tag{2.58}$$

To estimate $E_2^{(2)}$, we use an identical strategy. Integration by parts and antisymmetry give

$$\begin{aligned}
E_2^{(2)} &= \binom{N}{2} \binom{N-2}{2} \int_{A_{12} \cap A_{34}} \left(\sum_{i=1}^4 |\partial_i \Psi_F|^2 + \sum_{i=5}^N |\partial_i \Psi_F|^2 \right) \\
&= \binom{N}{2} \binom{N-2}{2} \left(4 \int_{|x_3-x_4|<b} [\overline{\Psi_F} \partial_1 \Psi_F]_{x_1=x_2-b}^{x_1=x_2+b} + \int_{A_{12} \cap A_{34}} \sum_{i=1}^N \overline{\Psi_F} (-\partial_i^2 \Psi_F) \right) \\
&= 4 \int_{x_2 \in [0,L]} \int_{|x_3-x_4|<b} \left[\partial_{y_1} \gamma^{(4)}(x_1, x_2, x_3, x_4; y_1, x_2, x_3, x_4) \Big|_{y_1=x_1} \right]_{x_1=x_2-b}^{x_1=x_2+b} \\
&\quad + E_0 \int_{A_{12} \cap A_{34}} \rho^{(4)}(x_1, \dots, x_4).
\end{aligned} \tag{2.59}$$

Lemma 12 implies

$$4 \int_{x_2 \in [0,L]} \int_{|x_3-x_4|<b} \left[\partial_{y_1} \gamma^{(4)}(x_1, x_2, x_3, x_4; y_1, x_2, x_3, x_4) \Big|_{y_1=x_1} \right]_{x_1=x_2-b}^{x_1=x_2+b} \leq \text{const. } E_0 N(\rho b)^4, \tag{2.60}$$

and

$$E_0 \int_{A_{12} \cap A_{34}} \rho^{(4)}(x_1, \dots, x_4) \leq \text{const. } E_0 N^2 (\rho b)^6, \quad (2.61)$$

which finishes the estimate of $E_2^{(2)}$. \square

2.4 Constructing the trial state for arbitrary N

Together, Lemmas 10, 13 and 14 provide a proof of Lemma 9, which is the upper bound for small N obtained from the trial state Ψ_ω (2.2). To construct a trial state for arbitrary N , we glue together copies of Ψ_ω on small intervals. This is straightforward with Dirichlet boundary conditions since the wave functions vanish at the boundaries. We therefore consider the state $\Psi_{\text{full}} = \prod_{i=1}^M \Psi_{\omega, \ell}(x_1^i, \dots, x_{\tilde{N}}^i)$, where $(x_1^i, \dots, x_{\tilde{N}}^i)$ are the positions of the particles in box i and ℓ is the length of each box. Of course, $\cup_{i=1}^M \{x_1^i, \dots, x_{\tilde{N}}^i\} = \{x_1, \dots, x_N\}$ and $\{x_1^i, \dots, x_{\tilde{N}}^i\} \cap \{x_1^j, \dots, x_{\tilde{N}}^j\} = \emptyset$ for $i \neq j$, such that⁷ $M\tilde{N} = N$. The boxes are of length⁸ $\ell = L/M - b$, and are equally spaced throughout $[0, L]$, leaving a distance of b between each box. This is to prevent particles in different boxes from interacting.

We can now prove the upper bound needed for Theorem 1.

Proof of Proposition 8. From Lemma 9, the energy of the full trial state described above is bounded by

$$E \leq M e_0 \left(1 + 2\tilde{\rho} a \frac{b}{b-a} + \text{const.} \left(\tilde{N}(\tilde{\rho} b)^3 \left[1 + \tilde{\rho} b^2 \int v_{\text{reg}} \right] + \tilde{\rho} a \frac{\ln(\tilde{N})}{\tilde{N}} \right) \right) / \|\Psi_\omega\|^2, \quad (2.62)$$

with $e_0 = \frac{\pi^2}{3} \tilde{N} \tilde{\rho}^2 (1 + \text{const.} \frac{1}{\tilde{N}})$ and $\tilde{\rho} = \tilde{N}/\ell = \rho/(1 - \frac{bM}{L}) \leq \rho(1 + 2bM/L)$ for $bM/L \leq 1/2$. Notice that $\tilde{\rho} a \frac{\ln(\tilde{N})}{\tilde{N}} \leq C_\epsilon \max(\tilde{N}^{-1}, (\tilde{\rho} a)^{2-\epsilon})$ with some ϵ dependent constant C_ϵ . This is easily seen by considering the cases $N \leq (\tilde{\rho} a)^{-1}$ and $N > (\tilde{\rho} a)^{-1}$ separately. Thus this term is sub-leading, and we will absorb it into other error terms. Clearly, we have $\|\Psi_\omega\|^2 \geq 1 - \int_B |\Psi_F|^2 \geq 1 - \int_{|x_1 - x_2| < b} \rho^{(2)}(x_1, x_2) \geq 1 - \text{const.} \tilde{N}(\rho b)^3$, where the last inequality follows from Lemma 11. Thus, choosing M such that $bM/L \ll 1$, we have

$$E \leq N \frac{\pi^2}{3} \rho^2 \frac{\left(1 + \frac{2\rho ab}{b-a} + \text{const.} \frac{M}{\tilde{N}} + \text{const.} \frac{2\rho abM/L + \text{const.} \tilde{N}(\rho b)^3 (1 + \rho b^2 \int v_{\text{reg}})}{1 - \tilde{N}(\tilde{\rho} b)^3} \right)}{1 - \tilde{N}(\tilde{\rho} b)^3}. \quad (2.63)$$

⁷Of course there might not, for a given N , exist desirable integers \tilde{N} and M such that this relation is satisfied. However, below when choosing \tilde{N} , we think of M as being $\lceil N/\tilde{N} \rceil$. In this case the number of particles in each box will be $\lceil N/M \rceil$ or $\lceil N/M - 1 \rceil$. The energy, in the two cases, will differ only at sub-leading order, and the difference may be absorbed in the error terms.

⁸In fact, given that a boxes can have $\lceil N/M \rceil$ or $\lceil N/M - 1 \rceil$ particles, we may choose the respective length of these boxes as $\ell_{\lceil N/M \rceil} = \rho^{-1} \lceil N/M \rceil - b$ and $\ell_{\lceil N/M - 1 \rceil} = \rho^{-1} \lceil N/M - 1 \rceil - b$.

First assume that $N \geq (\rho b)^{-3/2} (1 + \rho b^2 \int v_{\text{reg}})^{1/2}$. Now, we would choose $\tilde{N} = N/M = \rho L/M \gg 1$, or equivalently $M/L \ll \rho$. Setting $x = M/N$, we see that the error is

$$\text{const.} \left[(1 + 2\rho^2 ab^2/(b-a))x + x^{-1}(b\rho)^3 \left(1 + \rho b^2 \int v_{\text{reg}} \right) \right], \quad (2.64)$$

Here, we used the fact that $\tilde{N}(\rho b)^3 \leq 1/2$, so that we have

$1/(1 - \tilde{N}(\rho b)^3) \leq 1 + 2\tilde{N}(\rho b)^3$. Optimizing in x , we find $x = M/N = \frac{(b\rho)^{3/2}(1+\rho b^2 \int v_{\text{reg}})^{1/2}}{1+2\rho^2 ab} \simeq (b\rho)^{3/2} (1 + \rho b^2 \int v_{\text{reg}})^{1/2}$, which gives the error

$$\text{const.} (b\rho)^{3/2} \left(1 + \rho b^2 \int v_{\text{reg}} \right)^{1/2}. \quad (2.65)$$

Now, choose $b = \max(\rho^{-1/5} |a|^{4/5}, R_0)$. Then, for $(\rho |a|)^{1/5} \leq 1/2$,

$$\frac{b}{b-a} \leq 1 + 2a/b \leq 1 + 2(\rho |a|)^{1/5}. \quad (2.66)$$

Notice that

$$(\rho b)^{3/2} = \max \left((\rho |a|)^{6/5}, (\rho R_0)^{3/2} \right) \leq (\rho |a|)^{6/5} + (\rho R_0)^{3/2}. \quad (2.67)$$

Now, for $N < (\rho b)^{-3/2} (1 + \rho b^2 \int v_{\text{reg}})^{1/2}$, the result follows from (2.62) with $M = 1$, as well as $\rho a \frac{\ln(N)}{N} \leq C_\epsilon \max(N^{-1}, (\rho a)^{2-\epsilon})$. \square

3 Lower bound in Theorem 1

Proposition 15 (Lower bound in Theorem 1). *Consider a Bose gas with repulsive interaction $v = v_{\text{reg}} + v_{h.c.}$ as defined above Theorem 1, with Neumann boundary conditions. Write $\rho = N/L$. There exists a constant $C_L > 0$ such that the ground state energy $E^N(N, L)$ satisfies*

$$E^N(N, L) \geq N \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a - C_L \left((\rho |a|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3} \right) \right). \quad (3.1)$$

As mentioned in Section 1.2, the proof is based on a reduction to the Lieb-Liniger model combined with Lemma 4. Similar to the upper bound, this idea only provides a useful lower bound for small N , which we obtain in Proposition 24 and Corollary 25 at the end Section 3.2, after preparatory estimates on the Lieb-Liniger model in Section 3.1. Then, in Section 3.3, this lower bound will be generalized to arbitrary N , proving Proposition 15.

3.1 Lieb-Liniger model: preparatory facts

The thermodynamic ground state energy of the Lieb-Liniger model is determined by the system of equations [33]

$$e(\gamma) = \frac{\gamma^3}{\lambda^3} \int_{-1}^1 g(x) x^2 dx, \quad (3.2)$$

$$2\pi g(y) = 1 + 2\lambda \int_{-1}^1 \frac{g(x)}{\lambda^2 + (x-y)^2} dx, \quad (3.3)$$

$$\lambda = \gamma \int_{-1}^1 g(x) dx, \quad (3.4)$$

with $g \geq 0$. This allows for a rigorous lower bound.

Lemma 16 (Lieb-Liniger lower bound). *For $\gamma > 0$,*

$$e(\gamma) \geq \frac{\pi^2}{3} \left(\frac{\gamma}{\gamma+2} \right)^2 \geq \frac{\pi^2}{3} \left(1 - \frac{4}{\gamma} \right). \quad (3.5)$$

Proof. Neglecting $(x-y)^2$ in the denominator of (3.3), we see that $g \leq \frac{1}{2\pi} + \frac{1}{\pi\lambda} \int_{-1}^1 g(x) dx$. On the other hand, (3.2) and (3.4) imply $e(\gamma) = \frac{\int_{-1}^1 g(x) x^2 dx}{(\int_{-1}^1 g(x) dx)^3}$. Denote $\int_{-1}^1 g(x) dx = M$, so that $g \leq \frac{1}{2\pi} \left(1 + \frac{2M}{\lambda} \right) = \frac{1}{2\pi} \left(1 + \frac{2}{\gamma} \right)$. Now, we minimize the expression for $e(\gamma)$ in g subject to this bound. This gives $g = K \mathbb{1}_{[-\frac{M}{2K}, \frac{M}{2K}]}$ with $K = \frac{1}{2\pi} \left(1 + \frac{2}{\gamma} \right)$, resulting in $\int_{-1}^1 g(x) x^2 dx = \frac{1}{3} \frac{M^3}{4K^2}$. Now, $e(\gamma) \geq \frac{1}{3} \frac{1}{4K^2}$ for $\gamma > 0$, and (3.5) follows. \square

The thermodynamic Lieb-Liniger energy behaves like $n\rho^2 e(c/\rho)$, and the next result corrects the lower bound from (3.5) to obtain an estimate for finite particle numbers n .

Lemma 17 (Lieb-Liniger lower bound for finite n). *The Lieb-Liniger ground state energy with Neumann boundary conditions can be estimated by*

$$E_{LL}^N(n, \ell, c) \geq \frac{\pi^2}{3} n \rho^2 \left(1 - 4\rho/c - \text{const.} \frac{1}{n^{2/3}} \right). \quad (3.6)$$

This will be proved after the following lemma due to Robinson. Note we use the superscripts N and D to denote Neumann and Dirichlet boundary conditions, respectively.

Lemma 18 (Robinson [42]). *For simplicity, we will consider the Lieb-Liniger model on $[-L/2, L/2]$ in this subsection, and use the notation $\Lambda_s := [-s/2, s/2]$. Let v be symmetric and decreasing (that is, $v \circ \mathbf{c} \geq v$ for any contraction \mathbf{c}). For any $b > 0$,*

$$E_{\Lambda_{L+2b}}^D \leq E_{\Lambda_L}^N + \frac{2n}{b^2}. \quad (3.7)$$

Proof. The idea of the proof is given on page 66 of [42], but we shall give a more explicit proof here. In order to compare energies with different boundary conditions, consider a cut-off function h with the property that

1. h is real, symmetric, and continuously differentiable on Λ_{3L} ,
2. $h(x) = 0$ for $|x| > L/2 + b$,
3. $h(x) = 1$ for $|x| < L/2 - b$,
4. $h(L/2 - x)^2 + h(L/2 + x)^2 = 1$ for $0 < x < b$,
5. $\left|\frac{dh}{dx}\right|^2 \leq \frac{1}{b^2}$, and $h^2 \leq 1$.

Let $f \in \mathcal{D}(\mathcal{E}_{\Lambda_L}^N)$. Define \tilde{f} by extending f to Λ_{3L} by reflecting f across each face of its domain in Λ_{3L} . Define then $V : L^2(\Lambda_L) \rightarrow L^2(\Lambda_{L+2b})$ by $Vf(x) := \tilde{f}(x) \prod_{i=1}^n h(x_i)$. It is not hard to show that V is an isometry, this is shown in Lemma 2.1.12 of [42]. Also, we clearly have $Vf \in \mathcal{D}(\mathcal{E}_{\Lambda_{L+2b}}^D)$. Let ψ be the ground state for $\mathcal{E}_{\Lambda_L}^N$, and define the trial state $\psi_{\text{trial}} = V\psi$. Without the potential, the bound (3.7) is obtained in Lemma 2.1.13 of [42]. Hence, we need only prove that no energy is gained by the potential in the trial state. To see this, define $\tilde{\psi}$ to be ψ extended by reflection as above and notice that for $|x_2| < L/2 - b$, we have

$$\begin{aligned}
& \int_{-L/2-b}^{L/2+b} v(|x_1 - x_2|) \left| \tilde{\psi}(x) \right|^2 h(x_1)^2 h(x_2)^2 dx_1 \leq \\
& \int_{-L/2+b}^{L/2-b} v(|x_1 - x_2|) \left| \tilde{\psi}(x) \right|^2 dx_1 + \sum_{s \in \{-1, 1\}} s \int_{s(L/2-b)}^{s(L/2)} v(|x_1 - x_2|) \left| \tilde{\psi}(x) \right|^2 (h(x_1)^2 + h(L - x_1)^2) dx_1 \\
& = \int_{-L/2}^{L/2} v(|x_1 - x_2|) \left| \tilde{\psi}(x) \right|^2 dx_1,
\end{aligned} \tag{3.8}$$

where we used that v is symmetric decreasing in the first inequality, as well as the fact that $h(x)^2 + h(L - x)^2 = 1$ for $L/2 - b \leq x \leq L/2$, which is just property 4 of h . Furthermore, when

$|x_2| \geq L/2 - b$ we find

$$\begin{aligned}
& \int_{L/2-b}^{L/2+b} \int_{L/2-b}^{L/2+b} v(|x_1 - x_2|) \left| \tilde{\psi}(x) \right|^2 h(x_1)^2 h(x_2)^2 dx_2 dx_1 \\
&= \sum_{(s_1, s_2) \in \{-1, 1\}^2} s_1 s_2 \int_{L/2-s_1 b}^{L/2} \int_{L/2-s_2 b}^{L/2} v(|x_1 - x_2|) \left| \tilde{\psi}(x) \right|^2 h(x_1)^2 h(x_2)^2 dx_2 dx_1 \\
&= \sum_{(s_1, s_2) \in \{-1, 1\}^2} \int_0^b \int_0^b v(|s_1 y_1 - s_2 y_2|) \left| \tilde{\psi}(L/2 - s_1 y_1, L/2 - s_2 y_2, \bar{x}^{1,2}) \right|^2 \\
&\quad \times h(L/2 - s_1 y_1)^2 h(L/2 - s_2 y_2)^2 dy_2 dy_1 \\
&\leq \int_0^b \int_0^b v(|y_1 - y_2|) \left| \tilde{\psi}(L/2 - y_1, L/2 - y_2, \bar{x}^{1,2}) \right|^2 \\
&\quad \times \sum_{(s_1, s_2) \in \{-1, 1\}^2} h(L/2 - s_1 y_1)^2 h(L/2 - s_2 y_2)^2 dy_2 dy_1 \\
&= \int_0^b \int_0^b v(|y_1 - y_2|) \left| \tilde{\psi}(L/2 - y_1, L/2 - y_2, \bar{x}^{1,2}) \right|^2 dy_2 dy_1,
\end{aligned} \tag{3.9}$$

where we write $\bar{x}^{1,2}$ as shorthand for (x_3, \dots, x_N) . In the third line, we use the definition of $\tilde{\psi}$, as well as the fact that $|s_1 y_1 - s_2 y_2| \geq |y_1 - y_2|$ for $y_1, y_2 \geq 0$. In the last, line we used property 4 of h . By combining the two bounds above, we clearly have

$$\begin{aligned}
& \int_{-L/2-b}^{L/2+b} \int_{-L/2-b}^{L/2+b} v(|x_1 - x_2|) \left| \tilde{\psi}(x) \right|^2 h(x_1)^2 h(x_2)^2 dx_1 dx_2 \\
&\leq \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} v(|x_1 - x_2|) \left| \tilde{\psi}(x) \right|^2 dx_1 dx_2.
\end{aligned} \tag{3.10}$$

The result now follows from the fact that V is an isometry. \square

Proof of Lemma 17. Lemma 18 implies that for any $b > 0$

$$E_{LL}^N(n, \ell, c) \geq E_{LL}^D(n, \ell + b, c) - \text{const.} \frac{n}{b^2}. \tag{3.11}$$

Since the range of the interaction in the Lieb-Liniger model is zero, we see that $e_{LL}^D(2^m n, 2^m \ell, c) := \frac{1}{2^m \ell} E_{LL}^D(2^m n, 2^m \ell, c)$ is a decreasing sequence. To see this, simply split the box of size $2^m \ell$ in two boxes of size $2^{m-1} \ell$. Now, there are no interactions between the boxes so by using the product state of the two $2^{m-1} n$ -particle ground states in each box as a trial state, we see that $E_{LL}^D(2^m n, 2^m \ell) \leq 2 E_{LL}^D(2^{m-1} n, 2^{m-1} \ell)$. Since we also have $e_{LL}^D(2^m n, 2^m \ell, c) \geq$

$e_{LL}(2^m n, 2^m \ell, c) \rightarrow e_{LL}(n/\ell, c)$ as $m \rightarrow \infty$ [33], we see that

$$\begin{aligned} E_{LL}^N(n, \ell, c) &\geq e_{LL}(n/(\ell + b), c)(\ell + b) - \text{const.} \frac{n}{b^2} \\ &\geq \frac{\pi^2}{3} n \rho^2 \left(1 - 4\rho/c - \text{const.} \left(3b/\ell - \frac{1}{\rho^2 b^2} \right) \right). \end{aligned} \quad (3.12)$$

Here, $\rho = n/\ell$, and the second inequality follows from Lemma 16. Optimizing in b , we find

$$E_{LL}^N(n, \ell, c) \geq \frac{\pi^2}{3} n \rho^2 \left(1 - 4\rho/c - \text{const.} \frac{1}{n^{2/3}} \right). \quad (3.13)$$

□

3.2 Lower bound for small particle numbers n

In this subsection, we work our way towards Proposition 24 and Corollary 25, which provide lower bounds on the Neumann ground state energy. The proof strategy followed is that in Section 1.2.

We start by removing the relevant regions of the wave function. Throughout this section, let Ψ be the Neumann ground state of \mathcal{E} and let $R > \max(R_0, 2|a|)$ be a length, to be fixed later. Define the continuous function $\psi \in L^2([0, \ell - (n-1)R]^n)$ by

$$\psi(x_1, x_2, \dots, x_n) := \Psi(x_1, R + x_2, \dots, (n-1)R + x_n) \quad \text{for} \quad 0 \leq x_1 \leq \dots \leq x_n \leq \ell - (n-1)R, \quad (3.14)$$

extended symmetrically to other orderings of the particles. Our first goal is to prove that almost no weight is lost in going from Ψ to ψ , so that the heuristic calculation (1.19) has a chance of success. The following lemma will be useful.

Lemma 19. *For any function $\phi \in H^1(\mathbb{R})$ such that $\phi(0) = 0$,*

$$\int_{[0, R]} |\partial \phi|^2 \geq \max_{[0, R]} |\phi|^2 / R. \quad (3.15)$$

Proof. Write $\phi(x) = \int_0^x \phi'(t) dt$, and find that

$$|\phi(x)| \leq \int_0^x |\phi'(t)| dt. \quad (3.16)$$

Hence $\max_{x \in [0, R]} |\phi(x)| \leq \int_0^R |\phi'(t)| dt \leq \sqrt{R} \left(\int_0^R |\phi'(t)|^2 dt \right)^{1/2}$. □

We can estimate the norm loss in the following way

$$\langle \psi | \psi \rangle = 1 - \int_B |\Psi|^2 \geq 1 - \sum_{i < j} \int_{D_{ij}} |\Psi|^2, \quad (3.17)$$

where $B := \{x \in \mathbb{R}^n | \min_{i,j} |x_i - x_j| < R\}$ and $D_{ij} := \{x \in \mathbb{R}^n | \mathbf{r}_i(x) = |x_i - x_j| < R\}$ with $\mathbf{r}_i(x) := \min_{j \neq i} (|x_i - x_j|)$. Note D_{ij} is not symmetric in i and j , and that for $j \neq j'$, $D_{ij} \cap D_{ij'} = \emptyset$ up to sets of measure zero. Also note $B = \cup_{i < j} D_{ij}$. To give a good bound on the right-hand side of (3.17), we need the following lemma, upper bounding the norm loss to an energy.

Lemma 20. *For ψ defined in (3.14), we have*

$$1 - \langle \psi | \psi \rangle \leq 8 \left(R^2 \sum_{i < j} \int_{D_{ij}} |\partial_i \Psi|^2 + R(R-a) \sum_{i < j} \int v_{ij} |\Psi|^2 \right). \quad (3.18)$$

Proof. Note that (3.15) implies that for any $\phi \in H^1$,

$$||\phi(x)| - |\phi(x')||^2 \leq |\phi(x) - \phi(x')|^2 \leq R \left(\int_{[0,R]} |\partial \phi|^2 \right), \quad (3.19)$$

for $x, x' \in [0, R]$. Furthermore,

$$|\phi(x)|^2 - |\phi(x')|^2 = (|\phi(x)| - |\phi(x')|)^2 + 2(|\phi(x)| - |\phi(x')|)|\phi(x')| \leq 2(|\phi(x)| - |\phi(x')|)^2 + |\phi(x')|^2. \quad (3.20)$$

It follows that

$$\max_{x \in [0,R]} |\phi(x)|^2 \leq 2R \int_{[0,R]} |\partial \phi|^2 + 2 \min_{x' \in [0,R]} |\phi(x')|^2. \quad (3.21)$$

Viewing Ψ as a function of x_i , we have

$$2 \min_{\mathbf{r}_i(x) = |x_i - x_j| < R} |\Psi|^2 \geq \max_{\mathbf{r}_i(x) = |x_i - x_j| < R} |\Psi|^2 - 4R \left(\int_{\mathbf{r}_i(x) = |x_i - x_j| < R} |\partial_i \Psi|^2 \right). \quad (3.22)$$

Hence,

$$\begin{aligned} 2 \sum_{i < j} \int v_{ij} |\Psi|^2 &\geq 2 \sum_{i < j} \int_{D_{ij}} v_{ij} |\Psi|^2 \\ &\geq \left(\int v \right) \sum_{i < j} \int \left(\max_{D'_{ij}} |\Psi|^2 - 4R \left(\int_{D'_{ij}} |\partial_i \Psi|^2 dx_i \right) \right) d\bar{x}^i \\ &\geq \frac{4}{R-a} \sum_{i < j} \left(\frac{1}{2R} \int_{D_{ij}} |\Psi|^2 - 4R \int_{D_{ij}} |\partial_i \Psi|^2 \right), \end{aligned} \quad (3.23)$$

where $D'_{ij} := \{x_i \in \mathbb{R} | \mathbf{r}_i(x) = |x_i - x_j| < R\}$ and $d\bar{x}^i$ is shorthand for integration with respect to all variables except x_i . In the last line we used $\int v \geq 4/(R - a)$. Now, rewriting and (3.17) give the result. \square

To make (1.19) in the proof outlined in Section 1.2 precise, we relate the Neumann ground state energy to the Lieb–Liniger energy in Lemma 22. First, we state a direct adaptation of Lemma 4, more suited to our purpose here.

Lemma 21 (Dyson’s lemma). *Let $R > R_0 = \text{range}(v)$ and $\varphi \in H^1(\mathbb{R})$, then for any interval $\mathcal{I} \ni 0$*

$$\int_{\mathcal{I}} |\partial\varphi|^2 + \frac{1}{2}v|\varphi|^2 \geq \int_{\mathcal{I}} \frac{1}{R-a} (\delta_R + \delta_{-R}) |\varphi|^2, \quad (3.24)$$

where a is the scattering length.

Lemma 22. *Let $R > \max(R_0, 2|a|)$ and $\epsilon \in [0, 1]$. For ψ defined in (3.14),*

$$\int \sum_i |\partial_i \Psi|^2 + \sum_{i \neq j} \frac{1}{2} v_{ij} |\Psi|^2 \geq E_{LL}^N \left(n, \tilde{\ell}, \frac{2\epsilon}{R-a} \right) \langle \psi | \psi \rangle + \frac{(1-\epsilon)}{R^2} \text{const.} (1 - \langle \psi | \psi \rangle). \quad (3.25)$$

where $\tilde{\ell} := \ell - (n-1)R$.

Proof. Splitting the energy functional into two parts, and using Lemma 20 on one term and Lemma 21 on the other (see also (1.18)), we find

$$\begin{aligned} & \int \sum_i |\partial_i \Psi|^2 + \sum_{i \neq j} \frac{1}{2} v_{ij} |\Psi|^2 \geq \\ & \int \sum_i |\partial_i \Psi|^2 \mathbb{1}_{\mathbf{r}_i(x) > R} + \epsilon \sum_i \frac{1}{R-a} \delta(\mathbf{r}_i(x) - R) |\Psi|^2 \\ & + (1-\epsilon) \left(\sum_{i < j} \int_{D_{ij}} |\partial_i \Psi|^2 + \int \sum_{i < j} v_{ij} |\Psi|^2 \right), \end{aligned} \quad (3.26)$$

where $\mathbf{r}_i(x) = \min_{j \neq i} (|x_i - x_j|)$ and the nearest neighbor delta interaction can be written $\delta(\mathbf{r}_i(x) - R) = \left(\sum_{j \neq i} [\delta(x_i - x_j - R) + \delta(x_i - x_j + R)] \right) \mathbb{1}_{\mathbf{r}_i(x) \geq R}$. The nearest-neighbor interaction is obtained by, for each i in the sum above, dividing the integration domain of x_i into Voronoi cells around x_k with $k \neq i$. Then, for each k , restricting to the cell around particle k and using Lemma 21 gives the desired nearest neighbor interaction. This technique is also used in [30]. With the use of Lemma 20 with $R > 2|a|$ in the last term, and by realizing that the first two terms can be obtained by using ψ as a trial state in the Lieb–Liniger model (since the two delta functions collapse to a single delta of twice the strength when volume R is removed

between particles), we obtain

$$\int \sum_i |\partial_i \Psi|^2 + \sum_{i \neq j} \frac{1}{2} v_{ij} |\Psi|^2 \geq E_{LL}^N \left(n, \tilde{\ell}, \frac{2\epsilon}{R-a} \right) \langle \psi | \psi \rangle + \frac{(1-\epsilon)}{R^2} \text{const.} (1 - \langle \psi | \psi \rangle). \quad (3.27)$$

□

The next lemma will continue the process of bounding the norm loss in going from Ψ of norm 1 to ψ in (3.14).

Lemma 23. *For $n(\rho R)^2 \leq \frac{3}{16\pi^2} \frac{1}{8}$, $\rho R \leq \frac{1}{2}$ and $R > 2|a|$ we have*

$$\langle \psi | \psi \rangle \geq 1 - \text{const.} \left(n(\rho R)^3 + n^{1/3}(\rho R)^2 \right). \quad (3.28)$$

Proof. From the known upper bound, *i.e.* Proposition 8, and by Lemma 22 with $\epsilon = 1/2$, it follows that

$$n \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a + \text{const.} (\rho R)^{6/5} \right) \geq E_{LL}^N \left(n, \tilde{\ell}, \frac{1}{R-a} \right) \langle \psi | \psi \rangle + \frac{1}{16R^2} (1 - \langle \psi | \psi \rangle). \quad (3.29)$$

Subtracting $E_{LL}^N \left(n, \tilde{\ell}, \frac{1}{R-a} \right)$ on both sides, and using Lemma 17 on the left-hand side, we find

$$\begin{aligned} & n \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a + \text{const.} (\rho R)^{6/5} \right) - n \frac{\pi^2}{3} \tilde{\rho}^2 \left(1 - 4\tilde{\rho}(R-a) - \text{const.} n^{-2/3} \right) \\ & \geq \left(\frac{1}{16R^2} - E_{LL}^N \left(n, \tilde{\ell}, \frac{1}{R-a} \right) \right) (1 - \langle \psi | \psi \rangle), \end{aligned} \quad (3.30)$$

with $\tilde{\rho} = n/\tilde{\ell} = \rho/(1 - (\rho - 1/\ell)R)$. Using the upper bound $E_{LL}^N \left(n, \tilde{\ell}, \frac{1}{R-a} \right) \leq n \frac{\pi^2}{3} \tilde{\rho}^2$ on the left-hand side, as well as $2\rho \geq \tilde{\rho} \geq \rho(1 + \rho R)$, we find

$$\text{const.} n \rho^2 R^2 \left(\rho R + (\rho R)^{6/5} + n^{-2/3} \right) \geq \left(\frac{1}{16} - R^2 n \frac{4\pi^2}{3} \rho^2 \right) (1 - \langle \psi | \psi \rangle). \quad (3.31)$$

It follows that we have

$$\langle \psi | \psi \rangle \geq 1 - \text{const.} \left(n(\rho R)^3 + n^{1/3}(\rho R)^2 \right). \quad (3.32)$$

□

For $n \leq \kappa(\rho R)^{-9/5}$ with $\kappa = \frac{3}{16\pi^2} \frac{1}{8}$ and $\rho R \leq \frac{1}{2}$, we find

$$\langle \psi | \psi \rangle \geq 1 - \text{const.} n(\rho R)^3 = 1 - \text{const.} (\rho R)^{6/5}. \quad (3.33)$$

It is now straightforward to show the following two results, finishing the bounds for small n .

Proposition 24. *For $n(\rho R)^2 \leq \frac{3}{16\pi^2} \frac{1}{8}$, $\rho R \leq \frac{1}{2}$ and $R > 2|a|$ we have*

$$E^N(n, \ell) \geq n \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a + \text{const.} \left(\frac{1}{n^{2/3}} + n(\rho R)^3 + n^{1/3}(\rho R)^2 \right) \right). \quad (3.34)$$

Proof. By Lemma 22 with $\epsilon = 1$, we reduce to a Lieb-Liniger model with volume $\tilde{\ell}$, density $\tilde{\rho}$, and coupling c , and we have $\tilde{\ell} = \ell - (n-1)R$, $\tilde{\rho} = \frac{n}{\tilde{\ell}}$ and $c = \frac{2}{R-a}$. Notice that $\rho(1 + \rho R) \leq \tilde{\rho} \leq \rho(1 + 2\rho R)$. Hence, by Lemmas 17 and 23,

$$\begin{aligned} E^N(n, \ell) &\geq E_{LL}^N(n, \tilde{\ell}, c) \langle \psi | \psi \rangle \\ &\geq n \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a - \text{const.} \frac{1}{n^{2/3}} \right) \left(1 - \text{const.} \left(n(\rho R)^3 + n^{1/3}(\rho R)^2 \right) \right). \end{aligned} \quad (3.35)$$

□

Corollary 25. *For $\frac{\tau}{2}(\rho R)^{-9/5} \leq n \leq \tau(\rho R)^{-9/5}$ with $\tau = \frac{3}{16\pi^2} \frac{1}{8}$ and $\rho R \leq \frac{1}{2}$,*

$$E^N(n, \ell) \geq n \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a - \text{const.} \left((\rho R)^{6/5} + (\rho R)^{7/5} \right) \right). \quad (3.36)$$

3.3 Lower bound for arbitrary N

The lower bound in Corollary 25 only applies to particle numbers of order $(\rho R)^{-9/5}$. In this subsection, we generalize to any number of particles by performing a Legendre transformation in the particle number and going to the grand canonical ensemble. First, we justify that only particle numbers of order less than or equal to $(\rho R)^{-9/5}$ are relevant for a certain choice of μ .

Lemma 26. *Let $\Xi \geq 4$ be fixed. Also let $n = m\Xi\rho\ell + n_0$ with $n_0 \in [0, \Xi\rho\ell)$ for some $m \in \mathbb{N}$, with $\frac{\tau}{2\Xi}(\rho R)^{-9/5} \leq \rho\ell =: n^* \leq \frac{\tau}{\Xi}(\rho R)^{-9/5}$ and $\tau = \frac{3}{16\pi^2} \frac{1}{8}$. Furthermore, assume that $\rho R \leq 1$ and $\Xi\rho a \geq -1/4$ and let $\mu = \pi^2 \rho^2 \left(1 + \frac{8}{3}\rho a \right)$. Then,*

$$E^N(n, \ell) - \mu n \geq E^N(n_0, \ell) - \mu n_0. \quad (3.37)$$

Proof. By Corollary 25, we have

$$E^N(\Xi\rho\ell, \ell) \geq \frac{\pi^2}{3} \Xi^3 \ell \rho^3 \left(1 + 2\Xi\rho a - \text{const.} (\rho R)^{6/5} \right). \quad (3.38)$$

Superadditivity caused by the positive potential implies

$$E^N(n, \ell) - \mu n \geq m \left(E^N(\Xi\rho\ell, \ell) - \mu\Xi\rho\ell \right) + E^N(n_0, \ell) - \mu n_0. \quad (3.39)$$

The result, therefore, follows from the fact that

$$\frac{\pi^2}{3} \Xi^3 \ell \rho^3 \left(1 + 2\Xi \rho a - \text{const. } (\rho R)^{6/5} \right) \geq \pi^2 \rho^2 \left(1 + \frac{8}{3} \rho a \right) \Xi \rho \ell. \quad (3.40)$$

□

We are ready to prove the lower bound for general particle numbers.

Proof of Proposition 15. For the case $N < \tau(\rho R)^{-9/5}$, the result follows from Proposition 24. For $N \geq \tau(\rho R)^{-9/5}$, notice that

$$E^N(N, L) \geq F^N(\mu, L) + \mu N, \quad (3.41)$$

where $F^N(\mu, L) = \inf_{N'} (E^N(N', L) - \mu N')$. Clearly, since v is repulsive, we have

$$F^N(\mu, L) \geq M F^N(\mu, \ell), \quad (3.42)$$

with $\ell = L/M$ and $M \in \mathbb{N}_+$. Now, let $\Xi = 4$ and choose M such that $\frac{\tau}{2\Xi} (\rho R)^{-9/5} \leq n^* := \rho \ell \leq \frac{\tau}{\Xi} (\rho R)^{-9/5}$ and $\mu = \pi^2 \rho^2 (1 + \frac{8}{3} \rho a)$ (notice that $\mu = \frac{d}{d\rho} (\frac{\pi^2}{3} \rho^3 (1 + 2\rho a))$). Furthermore, assume that $\Xi \rho a \geq -1/4$ (the case of $\Xi \rho a < -1/4$ is trivial, by choosing a sufficiently large constant in the error term). By Lemma 26,

$$F^N(\mu, \ell) := \inf_n (E^N(n, \ell) - \mu n) = \inf_{n < \Xi n^*} (E^N(n, \ell) - \mu n). \quad (3.43)$$

It is known from Proposition 24 that for $n < \Xi n^*$,

$$\begin{aligned} E^N(n, \ell) &\geq n \frac{\pi^2}{3} \bar{\rho}^2 \left(1 + 2\bar{\rho} a - \text{const. } \left(\frac{1}{n^{2/3}} + n(\bar{\rho} R)^3 + n^{1/3}(\bar{\rho} R)^2 \right) \right) \\ &\geq \frac{\pi^2}{3} n \bar{\rho}^2 (1 + 2\bar{\rho} a) - n^* \rho^2 \mathcal{O} \left((\rho R)^{6/5} \right), \end{aligned} \quad (3.44)$$

where $\bar{\rho} = n/\ell$ (notice that now $\rho = N/L = n^*/\ell \neq n/\ell$) and where we used $\bar{\rho} < \Xi \rho$. Thus, we have

$$F^N(\mu, \ell) \geq \inf_{\bar{\rho} < \Xi \rho} (g(\bar{\rho}) - \mu \bar{\rho}) \ell - n^* \rho^2 \mathcal{O} \left((\rho R)^{6/5} \right), \quad (3.45)$$

where $g(\bar{\rho}) = \frac{\pi^2}{3} \bar{\rho}^3 (1 + 2\bar{\rho} a)$ for $\bar{\rho} < \Xi \rho$. Note that g is a convex C^1 -function with invertible derivative for $\Xi \rho a \geq -1/4$. Hence,

$$\begin{aligned} E^N(N, L) &\geq M(F^N(\mu, \ell) + \mu n^*) \geq M n^* \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a - \mathcal{O} \left((\rho R)^{6/5} \right) \right) \\ &= \frac{\pi^2}{3} N \rho^2 \left(1 + 2\rho a - \mathcal{O} \left((\rho R)^{6/5} \right) \right), \end{aligned} \quad (3.46)$$

where the equality follows from the specific choice of $\mu = g'(\rho)$. \square

4 Anyons and proof of Theorem 7

In Theorem 5 and below, we discussed the fact that the fermionic ground state energy can be found from Theorem 1 by means of a unitary transformation. It was also mentioned that this concept can be generalized to a version of 1D anyonic symmetry [7, 28, 41]. We will now define our interpretation of such anyons, depending on a statistical parameter $\kappa \in [0, \pi]$ that defines the phase $e^{i\kappa}$ accumulated upon particle exchange. We also include a Lieb–Liniger interaction of strength $2c > 0$, such as in [4, 24, 26].

To start, divide the configuration space into sectors $\Sigma_\sigma := \{x_{\sigma_1} < x_{\sigma_2} < \dots < x_{\sigma_N}\} \subset \mathbb{R}^N$ indexed by permutations $\sigma = (\sigma_1, \dots, \sigma_N)$, and the diagonal $\Delta_N := \bigcup_{1 \leq i < j \leq N} \{x_i = x_j\}$. Consider the kinetic energy operator on $\mathbb{R}^N \setminus \Delta_N$,

$$H_N = - \sum_{i=1}^N \partial_{x_i}^2, \quad (4.1)$$

with domain

$$\mathcal{D}(H_N) = \left\{ \varphi = e^{-i\frac{\kappa}{2}\Lambda(x)} f(x) \mid \begin{array}{l} f \text{ is continuous, symmetric in } x_1, \dots, x_N, \text{ smooth on each } \Sigma_\sigma, \\ \text{and } (\partial_i - \partial_j)\varphi|_+^{ij} - (\partial_i - \partial_j)\varphi|_-^{ij} = 2c e^{-i\frac{\kappa}{2}\Lambda(x)} f|_0^{ij} \text{ for all } i \neq j \end{array} \right\}. \quad (4.2)$$

Here, $|_\pm^{ij}$ and $|_0^{ij}$ mean the function should be evaluated for $x_i \rightarrow x_j^\pm$ and $x_i = x_j$ respectively, and

$$\Lambda(x) := \sum_{i < j} \epsilon(x_i - x_j) \quad \text{with} \quad \epsilon(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases}. \quad (4.3)$$

The idea is that the (perhaps rather artificial) boundary condition in (4.2) encodes the presence of a delta potential of strength $2c$, just like it would for bosons.

Proposition 27. *Let $0 < \kappa < \pi$. H_N is symmetric with corresponding quadratic form*

$$\mathcal{E}_{\kappa,c}(\varphi) = \sum_{i=1}^N \int_{\mathbb{R}^N \setminus \Delta_N} |\partial_{x_i} \varphi(x)|^2 + \frac{2c}{\cos(\kappa/2)} \sum_{i < j} \delta(x_i - x_j) |\varphi(x)|^2 dx. \quad (4.4)$$

Proof. Let $\varphi, \vartheta \in \mathcal{D}(H_N)$. Then, by partial integration,

$$\begin{aligned}
\langle \vartheta | H_N \varphi \rangle &= - \sum_{i=1}^N \int_{\mathbb{R}^N \setminus \Delta_N} \overline{\vartheta} \partial_{x_i}^2 \varphi \\
&= \sum_{i=1}^N \int_{\mathbb{R}^N \setminus \Delta_N} \overline{\partial_{x_i} \vartheta} \partial_{x_i} \varphi - \int_{\mathbb{R}^{N-1} \setminus \Delta_{N-1}} \sum_{i \neq j} \left(\overline{\vartheta} \partial_{x_i} \varphi|_-^{ij} - \overline{\vartheta} \partial_{x_i} \varphi|_+^{ij} \right) \\
&= \sum_{i=1}^N \int_{\mathbb{R}^N \setminus \Delta_N} \overline{\partial_{x_i} \vartheta} \partial_{x_i} \varphi + \int_{\mathbb{R}^{N-1} \setminus \Delta_{N-1}} \sum_{i < j} \left(\overline{\vartheta} (\partial_{x_i} - \partial_{x_j}) \varphi|_+^{ij} - \overline{\vartheta} (\partial_{x_i} - \partial_{x_j}) \varphi|_-^{ij} \right).
\end{aligned} \tag{4.5}$$

Let f, g be the functions such that $\varphi = e^{-i\frac{\kappa}{2}\Lambda} f$ and $\vartheta = e^{-i\frac{\kappa}{2}\Lambda} g$. Then,

$$\begin{aligned}
\langle \vartheta | H_N \varphi \rangle &= \sum_{i=1}^N \int_{\mathbb{R}^N \setminus \Delta_N} \overline{\partial_{x_i} \vartheta} \partial_{x_i} \varphi + \int_{\mathbb{R}^{N-1} \setminus \Delta_{N-1}} \sum_{i < j} \left(\overline{g} (\partial_{x_i} - \partial_{x_j}) f|_+^{ij} - \overline{g} (\partial_{x_i} - \partial_{x_j}) f|_-^{ij} \right) \\
&= \sum_{i=1}^N \int_{\mathbb{R}^N \setminus \Delta_N} \overline{\partial_{x_i} \vartheta} \partial_{x_i} \varphi + \int_{\mathbb{R}^{N-1} \setminus \Delta_{N-1}} 2 \sum_{i < j} \left(\overline{g} (\partial_{x_i} - \partial_{x_j}) f|_+^{ij} \right),
\end{aligned} \tag{4.6}$$

where the last equality follows from the symmetry of f . Note that the boundary condition on $\mathcal{D}(H_N)$ imply

$$(\partial_i - \partial_j) \varphi|_+^{ij} - (\partial_i - \partial_j) \varphi|_-^{ij} = e^{-i\frac{\kappa}{2}(-1+S)} (\partial_i - \partial_j) f|_+^{ij} - e^{-i\frac{\kappa}{2}(1+S)} (\partial_i - \partial_j) f|_-^{ij} = 2c \varphi|_0^{ij} = e^{-i\frac{\kappa}{2}S} 2c f|_0^{ij}, \tag{4.7}$$

where $S := \Lambda - \epsilon(x_i - x_j)$. By symmetry of f , it follows that

$$\begin{aligned}
e^{-i\frac{\kappa}{2}(-1+S)} (\partial_i - \partial_j) f|_+^{ij} - e^{-i\frac{\kappa}{2}(1+S)} (\partial_i - \partial_j) f|_-^{ij} &= e^{-i\frac{\kappa}{2}(-1+S)} (\partial_i - \partial_j) f|_+^{ij} + e^{-i\frac{\kappa}{2}(1+S)} (\partial_i - \partial_j) f|_+^{ij} \\
&= e^{-i\frac{\kappa}{2}S} 2 \cos(\kappa/2) (\partial_i - \partial_j) f|_+^{ij} \\
&= e^{-i\frac{\kappa}{2}S} 2c f|_0^{ij},
\end{aligned} \tag{4.8}$$

so that

$$2(\partial_i - \partial_j) f|_+^{ij} = \frac{2c}{\cos(\kappa/2)} f|_0^{ij}. \tag{4.9}$$

Hence, it follows that

$$\langle \vartheta | H_N \varphi \rangle = \sum_{i=1}^N \int_{\mathbb{R}^N \setminus \Delta_N} \overline{\partial_{x_i} \vartheta} \partial_{x_i} \varphi(x) + \frac{2c}{\cos(\kappa/2)} \sum_{i < j} \delta(x_i - x_j) \overline{\vartheta(x)} \varphi(x) \, d^N x. \tag{4.10}$$

Starting from $\langle H_N \vartheta | \phi \rangle$, we can arrive at (4.10) by the same steps, proving that H_N is symmetric. \square

Remark 28. Since $\mathcal{E}_{\kappa,c}$ is non-negative and closable, it follows that H_N has a self-adjoint Friedrichs extension, \tilde{H}_N . This is what we regard as the Hamiltonian of the 1D anyon gas with statistical parameter κ and Lieb–Liniger interaction of strength $2c\delta_0$ that is relevant for Theorem 7.

We are now ready to provide a proof of Theorem 7 along the lines outlined in Section 1.3.

Proof of Theorem 7. Let \mathcal{E}_c denote the bosonic quadratic form with potential $v_c = v + 2c\delta_0$. By Proposition 27 and the observation that the quadratic form is independent of the phase factors, we see that the unitary operator $U_\kappa : f \mapsto e^{-i\frac{\kappa}{2}\Lambda} f$ provides a unitary equivalence of the bosonic and anyonic set-ups. That is, $U_\kappa \mathcal{D}(\mathcal{E}_{0,c/\cos(\kappa/2)}) = \mathcal{D}(\mathcal{E}_{\kappa,c})$ with $\mathcal{E}_{\kappa,c}(U_\kappa f) = \mathcal{E}_{0,c/\cos(\kappa/2)}(f)$. Hence, the result follows from Theorem 1. \square

5 Acknowledgements

JA and JPS were partially supported by the Villum Centre of Excellence for the Mathematics of Quantum Theory (QMATH). RR was supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (ERC CoG UniCoSM, Grant Agreement No. 724939). JA is grateful to IST Austria for its hospitality during a visit and to Robert Seiringer for interesting discussions. RR thanks the University of Copenhagen for the hospitality during a visit.

References

- [1] J O Andersen, *Ground state pressure and energy density of an interacting homogeneous Bose gas in two dimensions*, The European Physical Journal B-Condensed Matter and Complex Systems **28** (2002), no. 4, 389–396.
- [2] GE Astrakharchik, J Boronat, IL Kurbakov, Yu E Lozovik, and F Mazzanti, *Low-dimensional weakly interacting bose gases: Nonuniversal equations of state*, Physical Review A **81** (2010), no. 1, 013612.
- [3] Giulia Basti, Serena Cenatiempo, and Benjamin Schlein, *A new second-order upper bound for the ground state energy of dilute Bose gases*, Forum of Mathematics, Sigma, vol. 9, Cambridge University Press, 2021.
- [4] Murray T Batchelor, X-W Guan, and Norman Oelkers, *One-dimensional interacting anyon gas: low-energy properties and Haldane exclusion statistics*, Physical Review Letters **96** (2006), no. 21, 210402.

- [5] Immanuel Bloch, Jean Dalibard, and Wilhelm Zwerger, *Many-body physics with ultracold gases*, Reviews of modern physics **80** (2008), no. 3, 885.
- [6] N N Bogoliubov, *On the theory of superfluidity*, Proc. Inst. Math. Kiev **9** (1947), 89–103, Rus. Trans Izv. Akad. Nauk Ser. Fiz. 11, 77 (1947), Eng. Trans. J. Phys. (USSR), 11, 23 (1947).
- [7] Martin Bonkhoff, Kevin Jägering, Sebastian Eggert, Axel Pelster, Michael Thorwart, and Thore Posske, *Bosonic continuum theory of one-dimensional lattice anyons*, Physical Review Letters **126** (2021), no. 16, 163201.
- [8] Pasquale Calabrese and Jean-Sébastien Caux, *Correlation functions of the one-dimensional attractive Bose gas*, Physical Review Letters **98** (2007), no. 15, 150403.
- [9] Andrea Colcelli, Giuseppe Mussardo, and Andrea Trombettoni, *Deviations from off-diagonal long-range order in one-dimensional quantum systems*, EPL (Europhysics Letters) **122** (2018), no. 5, 50006.
- [10] Giulia De Rosi, Pietro Massignan, Maciej Lewenstein, and Grigori E Astrakharchik, *Beyond-luttinger-liquid thermodynamics of a one-dimensional bose gas with repulsive contact interactions*, Physical Review Research **1** (2019), no. 3, 033083.
- [11] Freeman John Dyson, *Ground-state energy of a hard-sphere gas*, Physical Review **106** (1957), no. 1, 20.
- [12] S Fournais, T Girardot, L Junge, L Morin, and M Olivieri, *The ground state energy of a two-dimensional bose gas*, arXiv preprint arXiv:2206.11100 (2022).
- [13] Søren Fournais, Marcin Napiórkowski, Robin Reuvers, and Jan Philip Solovej, *Ground state energy of a dilute two-dimensional Bose gas from the bogoliubov free energy functional*, Journal of Mathematical Physics **60** (2019), no. 7, 071903.
- [14] Søren Fournais and Jan Philip Solovej, *The energy of dilute Bose gases*, Annals of Mathematics **192** (2020), no. 3, 893–976.
- [15] ———, *The energy of dilute Bose gases II: The general case*, arXiv preprint arXiv:2108.12022 (2021).
- [16] M Gaudin, *Un système à une dimension de fermions en interaction*, Physics Letters A **24** (1967), no. 1, 55–56.
- [17] Marvin Girardeau, *Relationship between systems of impenetrable bosons and fermions in one dimension*, Journal of Mathematical Physics **1** (1960), no. 6, 516–523.

- [18] MD Girardeau, *Ground and excited states of spinor Fermi gases in tight waveguides and the Lieb-Liniger-Heisenberg model*, Physical Review Letters **97** (2006), no. 21, 210401.
- [19] A Görlitz, JM Vogels, AE Leanhardt, C Raman, TL Gustavson, JR Abo-Shaeer, AP Chikkatur, S Gupta, S Inouye, T Rosenband, et al., *Realization of Bose-Einstein condensates in lower dimensions*, Physical Review Letters **87** (2001), no. 13, 130402.
- [20] Markus Greiner, Immanuel Bloch, Olaf Mandel, Theodor W Hänsch, and Tilman Esslinger, *Exploring phase coherence in a 2D lattice of Bose-Einstein condensates*, Physical Review Letters **87** (2001), no. 16, 160405.
- [21] Xi-Wen Guan and Murray T Batchelor, *Polylogs, thermodynamics and scaling functions of one-dimensional quantum many-body systems*, Journal of Physics A: Mathematical and Theoretical **44** (2011), no. 10, 102001.
- [22] Xi-Wen Guan and Zhong-Qi Ma, *Analytical study of the fredholm equations for 1D two-component Fermions with delta-function interaction*, arxiv: 1110.2821 (2011).
- [23] Xi-Wen Guan, Zhong-Qi Ma, and Brendan Wilson, *One-dimensional multicomponent fermions with δ -function interaction in strong-and weak-coupling limits: κ -component fermi gas*, Physical Review A **85** (2012), no. 3, 033633.
- [24] Yajiang Hao, Yunbo Zhang, and Shu Chen, *Ground-state properties of one-dimensional anyon gases*, Physical Review A **78** (2008), no. 2, 023631.
- [25] Yu-Zhu Jiang, Yang-Yang Chen, and Xi-Wen Guan, *Understanding many-body physics in one dimension from the Lieb-Liniger model*, Chinese Physics B **24** (2015), no. 5, 050311.
- [26] Anjan Kundu, *Exact solution of double δ function bose gas through an interacting anyon gas*, Physical Review Letters **83** (1999), no. 7, 1275.
- [27] Tsin D Lee, Kerson Huang, and Chen N Yang, *Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties*, Physical Review **106** (1957), no. 6, 1135.
- [28] Jon M Leinaas and Jan Myrheim, *On the theory of identical particles*, Il Nuovo Cimento B (1971-1996) **37** (1977), no. 1, 1–23.
- [29] Andrew Lenard, *Momentum distribution in the ground state of the one-dimensional system of impenetrable bosons*, Journal of Mathematical Physics **5** (1964), no. 7, 930–943.
- [30] E.H. Lieb, R. Seiringer, J.P. Solovej, and J. Yngvason, *The Mathematics of the Bose Gas and its Condensation*, Oberwolfach Seminars, Birkhäuser Basel, 2006.

- [31] Elliott H Lieb and Jakob Yngvason, *Ground state energy of the low density Bose gas*, The Stability of Matter: From Atoms to Stars, Springer, 1998, pp. 755–758.
- [32] Elliott Lieb and Daniel Mattis, *Theory of ferromagnetism and the ordering of electronic energy levels*, Physical Review **125** (1962), no. 1, 164–172.
- [33] Elliott H Lieb and Werner Liniger, *Exact analysis of an interacting bose gas. i. the general solution and the ground state*, Physical Review **130** (1963), no. 4, 1605.
- [34] Elliott H Lieb, Robert Seiringer, and Jakob Yngvason, *One-dimensional bosons in three-dimensional traps*, Physical Review Letters **91** (2003), no. 15, 150401.
- [35] ———, *One-dimensional behavior of dilute, trapped Bose gases*, Communications in mathematical physics **244** (2004), no. 2, 347–393.
- [36] Elliott H Lieb and Jakob Yngvason, *The ground state energy of a dilute two-dimensional Bose gas*, Journal of Statistical Physics **103** (2001), no. 3, 509–526.
- [37] James B McGuire, *Study of exactly soluble one-dimensional N -body problems*, Journal of Mathematical Physics **5** (1964), no. 5, 622–636.
- [38] Christophe Mora and Yvan Castin, *Ground state energy of the two-dimensional weakly interacting Bose gas: first correction beyond Bogoliubov theory*, Physical Review Letters **102** (2009), no. 18, 180404.
- [39] Henning Moritz, Thilo Stöferle, Michael Köhl, and Tilman Esslinger, *Exciting collective oscillations in a trapped 1D gas*, Physical Review Letters **91** (2003), no. 25, 250402.
- [40] Maxim Olshanii, *Atomic scattering in the presence of an external confinement and a gas of impenetrable bosons*, Physical Review Letters **81** (1998), no. 5, 938.
- [41] Thore Posske, Björn Trauzettel, and Michael Thorwart, *Second quantization of Leinaas-Myrheim anyons in one dimension and their relation to the lieb-liniger model*, Physical Review B **96** (2017), no. 19, 195422.
- [42] D.W. Robinson, *The Thermodynamic Pressure in Quantum Statistical Mechanics*, Lecture Notes in Physics, Springer Berlin Heidelberg, 1971.
- [43] M Schick, *Two-dimensional system of hard-core bosons*, Physical Review A **3** (1971), no. 3, 1067.
- [44] F Schreck, Lev Khaykovich, KL Corwin, G Ferrari, Thomas Bourdel, Julien Cubizolles, and Christophe Salomon, *Quasipure bose-einstein condensate immersed in a Fermi sea*, Physical Review Letters **87** (2001), no. 8, 080403.

- [45] Robert Seiringer and Jun Yin, *The Lieb-Liniger model as a limit of dilute bosons in three dimensions*, Communications in mathematical physics **284** (2008), no. 2, 459–479.
- [46] Jan Philip Solovej, *Many body quantum mechanics*, Lecture Notes, <https://web.math.ku.dk/~solovej/MANYBODY/mbnotes-ptn-5-3-14.pdf> (2014).
- [47] Bill Sutherland, *Further results for the many-body problem in one dimension*, Physical Review Letters **20** (1968), no. 3, 98.
- [48] Hemant G Vaidya and CA Tracy, *One particle reduced density matrix of impenetrable bosons in one dimension at zero temperature*, Journal of Mathematical Physics **20** (1979), no. 11, 2291–2312.
- [49] Chen-Ning Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Physical Review Letters **19** (1967), no. 23, 1312.
- [50] Horng-Tzer Yau and Jun Yin, *The second order upper bound for the ground energy of a Bose gas*, Journal of Statistical Physics **136** (2009), no. 3, 453–503.