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Problem 1

(a) $T: X \to Y$ a linear map.

WTS $\|x\|_0 = \|x\|_X + \|Tx\|_Y$. is a norm on X.

(1) $\|x+y\|_0 = \|x+y\|_X + \|T(x+y)\|_Y$

$\leq \|x\|_X + \|y\|_X + \|Tx + Ty\|_Y$

$\leq \|x\|_X + \|Tx\|_X + \|y\|_X + \|Ty\|_Y = \|x\|_0 + \|y\|_0$.

(2) $\|\alpha(x+y)\|_0 = \|\alpha(x+y)\|_X + \|T(\alpha(x+y))\|_Y$

$= |\alpha|\|x+y\|_X + |\alpha|\|T(x+y)\|_Y$

$= |\alpha|\|x+y\|_0$.

(3) $\|x\|_0 = 0 \iff \|x\|_X + \|Tx\|_Y = 0 \iff \|x\|_X = 0$, $\|Tx\|_Y = 0$. since $\|x\|_X \geq 0$, $\|Tx\|_Y \geq 0$

$\iff x = 0$.

To show $\|\cdot\|_X$. $\|\cdot\|_0$ equivalent $\iff$ T is bounded

"$\Rightarrow$" If $\|\cdot\|_X$. $\|\cdot\|_0$ are equivalent $\exists C > 0$.

$C\|\cdot\|_0 \leq \|\cdot\|_X \leq \|\cdot\|_0$ Then for all $x \in X$.

$C\|x\|_0 \leq \|x\|_X$.

$C\|x\|_X + C\|Tx\|_Y \leq \|x\|_X$.

$\|Tx\|_Y \leq \frac{1-C}{C}\|x\|_X$.

$\|T\| = \sup\{\|Tx\|_Y . \|x\|_X = 1\} \leq \frac{1-C}{C}$

So T is bounded

"$\Leftarrow$" If T is bounded then $\exists C$. for all $x \in X$.

$\|Tx\|_Y \leq C\|x\|_X$.

$\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq (1+C)\|x\|_X$.

$\frac{\|x\|_0}{1+C} \leq \|x\|_X \leq \|x\|_0$.

then $\|\cdot\|_X$. $\|\cdot\|_0$ are equivalent.

(b). Show any linear map $T: X \to Y$ is bounded. if X is finite dimentional.

By theorem 1.6. any two norm are equivalent. Take $\|\cdot\|_\infty$ on X. And choose a basis of X. $\{e_1 \dots e_n\}$.

$$Tx = \sum a_i Te_i$$

$$\|Tx\|_Y \le \|\sum_i a_i Te_i\| \le \sum_i |a_i| \, \|Te_i\|_Y.$$

Let $C = \sup\{\|Te_i\|_Y\}$.

$$\|Tx\|_Y \le \sum_i |a_i| \, \|Te_i\|_Y \le \sum_i |a_i| C \le n\max\{a_i \; i=1 \dots n.\} C$$

$$= nC\|x\|_\infty$$

$\square$

(C). Suppose that X is infinite dimentional.

WTS $\exists \, T: X \to Y$ which is not bounded.

Then we can take a Hamel basis for X. $\{e_i\}_{i\in I}$ where I is infinite. then we can take a countable subset $_{\mathbb{N}}$ of I

Choose $y \in Y$ define map $T: X \to Y$. $T(e_i) = iy$ $i \in \mathbb{N}$.

So $\nexists r$ $T(\overline{B(0.1)}_X) \subseteq \overline{B(0.r)}$ then T is not bounded.

(d) Suppose that X is infinite dimensional. By (a) and (c) we define $\|\cdot\|_0$ by $\|x\|_0 = \|x\|_X + \|Tx\|_Y$. where $T(e_i) = iy$ $y \in Y$. $i \in \mathbb{N}$.

$\|x\|_X \le \|x\|_0 + \|Tx\|_Y$ Since T is not bounded so $\nexists \, C$

$\|Tx\|_Y \le C\|x\|_X$. then $\nexists \, M$ $M\|x\|_0 \le \|x\|_X$.

Moreover if $(X. \|\cdot\|_X)$ is a banach space. WTS $(X. \|\cdot\|_0)$ is not a banach space.

if $(x_i)_{i\in\mathbb{N}}$ is a cauchy sequence in $(X. \|\cdot\|_0)$ then for $\forall \varepsilon > 0$ $\exists \, N \in \mathbb{N}$ which $\forall m.n > N$ $\|x_m - x_n\|_X \le \|x_m - x_n\|_0 < \varepsilon$.

So $(x_i)_{i\in\mathbb{N}}$ is a cauchy sequence in $(X. \|\cdot\|_0)$

If x is a limit point of the cauchy sequence $(x_i)_{i\in\mathbb{N}}$ in $(X, \|\cdot\|_0)$

then $\lim_{n\to\infty} \|x_n - x\|_X \leq \lim_{n\to\infty} \|x_n - x\|_0 = 0$.

So x is also the limit point of $(x_i)_{i\in\mathbb{N}}$ in $(X, \|\cdot\|_X)$.

Construct a cauchy sequence by $x_n = \frac{e_n}{n}$ where $\{e_i\}_{i\in\mathbb{N}}$ is the subset of Hamel basis of X with norm 1.

$$\begin{aligned}
\|x_m - x_n\|_0 &= \|x_m - x_n\|_X + \|T(x_m - x_n)\|_Y \\
&= \|x_m - x_n\|_X + \left\|\frac{my}{m} - \frac{ny}{n}\right\|_Y \\
&= \|x_m - x_n\|_X \\
&\leq \|x_m\|_X + \|x_n\|_X = \frac{1}{n} + \frac{1}{m}
\end{aligned}$$

$\forall \varepsilon > 0$. ~~take~~ $N > \frac{2}{\varepsilon}$ $N \in \mathbb{N}$ $\|x_m - x_n\|_0 \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \varepsilon$ $\forall m, n > N$.

So it is a cauchy sequence in both $(X, \|\cdot\|_0)$. $(X, \|\cdot\|_X)$.

In $(X, \|\cdot\|_X)$ we have $x_n \to 0$ $n \to \infty$

$\lim_{n\to\infty} \|x_n - 0\| = \lim_{n\to\infty} \left\|\frac{e_n}{n}\right\| = 0$.

So If $(X, \|\cdot\|_0)$ is complete then 0 is also the limit point of $(x_i)_{i\in\mathbb{N}}$ in $(X, \|\cdot\|_0)$

$\lim_{n\to\infty} \|x_n\|_0 = \lim_{n\to\infty} \left\|\frac{e_n}{n}\right\| + \lim_{n\to\infty} \left\|T\left(\frac{e_n}{n}\right)\right\|_Y = \|y\|_Y$.

$\|0\|_0 = 0 \neq \|y\|_Y$

So 0 is not the limit point of $(x_i)_{i\in\mathbb{N}}$. So $(X, \|\cdot\|_0)$ is not complete

(e). Take $\|\cdot\|'$ in $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|_1)$.

$\|\cdot\|' = \sum_n \frac{|x_n|}{n}$. $e_i = (0 \cdots \underset{i\text{-th}}{1} \cdots 0 \cdots)$.

Indeed. $\|\cdot\|' \leq \|\cdot\|_1$ $\forall c > 0$ $\exists i > \frac{1}{c}$ $\|e_i\|' \leq c\|e_i\|_1$

So $\|\cdot\|'$ is not equivalent to $\|\cdot\|_1$

If x is the limit point of a cauchy sequence in $(\ell_1(N). \|\cdot\|_1)$

$$\|x_m - x_n\|' \leq \|x_m - x_n\|_1 \qquad \|x_n - x\|' \leq \|x_n - x\|_1$$

x should also be the limit point of the same cauchy sequence in $(\ell_1(\mathbb{N}). \|\cdot\|')$

2.

(a) $M = \{(a.b.0 \cdots)\ a.b \in \mathbb{C}\} \subseteq (\ell_p(\mathbb{N}) . \|\cdot\|_p)$

$f(a.b.0 \cdots) = a+b.$ let $b \neq 0$ $a > b$

$$\|f\| = \sup\left\{\frac{\|f(x)\|}{\|x\|}\right\} = \sup \frac{|a+b|}{(|a|^p + |b|^p)^{\frac{1}{p}}}$$

By Hölders inequality we have

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}$$

$y_1 = 1 \quad y_2 = 1 \quad x_1 = a \quad x_2 = b$

$$|a| + |b| \leq (|a|^p + |b|^p)^{\frac{1}{p}} (1+1)^{\frac{1}{q}}$$

So $\|f\| = \sup \frac{|a+b|}{(|a|^p + |b|^p)^{\frac{1}{p}}} \leq \sup \frac{|a| + |b|}{2^{-\frac{1}{q}}(|a| + |b|)} \leq (1+1)^{\frac{1}{q}} = 2^{1-\frac{1}{p}}$

$\|f\|$ is bounded

we can take $a = b$

$$\|f\| \geq \frac{|2a|}{(|a|^p + |a|^p)^{\frac{1}{p}}} = \frac{|2a|}{2^{\frac{1}{p}} |a|} = 2^{1-\frac{1}{p}}$$

so $\|f\| = 2^{1-\frac{1}{p}}$

(2) $1 < p < \infty$ $f \in L(M.K)$. by cor 2.6 $\exists$ F. on $\ell_p(N)$ st $F|_M = f$ $\|F\| = \|f\|$. Then we have to show the uniqueness we naturly have one $F(a.b.c \cdots) = a+b$.

Claim $F' = F + \alpha_3 \delta_3 + \alpha_4 \delta_4 + \cdots$ δ_i is the dual basis defined by $\delta_i(e_m) = \delta_{im} = \begin{cases} 1 & m = i \\ 0 & m \neq i \end{cases}$

$\|F'\| > \|f\|$ if $\exists\, \alpha_i \neq 0$. we just prove $\alpha_3 \neq 0$.

$$\|F'\| = \|a\delta_1 + b\delta_2 + \alpha_3 \cdot \delta_3 \qquad \|$$

$$= \sup \frac{|a+b+\alpha_3|}{(|a|^p+|b|^p+| \quad c|^p \quad)^{\frac{1}{p}}}$$

$$= \sup \frac{|a+b+\alpha_3|}{(a^p+b^p+ \quad c^p)^{\frac{1}{p}}(\alpha_3^q+1^q+1^q)^{\frac{1}{q}}}$$

$$\leq \sup \frac{|a|+|b|+|\alpha_3|}{(\frac{1}{2^q+\alpha_3^q})(|a|+|b|+|\alpha_3|)} = 2^q+\alpha_3^q \Leftarrow \text{by holder inequality.}$$

$$q = \frac{p-1}{p}$$

take $a=b=\alpha_3$ $\quad a+b+\alpha_3=1$

we get $\quad F'(a.b.\alpha_3 \ .0\ldots) = 2^q+\alpha_3^q$

$$\|F'\| = 2^q+\alpha_3^q > \|f\|.$$

In general $\|F'\|_p > \|f\|_p$ when $\exists\, \alpha_i \neq 0$.

(3)

we know that when $p=1$ $\quad \|F\|=1$

$$\|F'\|_1 = \|a\delta_1+b\delta_2+\alpha_3\delta_3+\cdots+\alpha_n\delta_n+\cdots\|_1 = \|\delta_1+\delta_2\|_1 = 1 \qquad \forall\, \alpha_i \in \mathbb{C}$$

$$\|F'\| = \sup \frac{|a+b+\alpha_3+\cdots|}{|a|+|b|+|\alpha_3|+\cdots} \leq \sup \frac{|a|+\cdots+}{|a|+\cdots+} = 1$$

take $a=b=\alpha_3=\cdots$

we have $\|F'\| \geq 1$

so $\|F'\| = 1 = \|f\|$.

$\square$

3.

(a) X is an infinite dimensional normed vector space over $\mathbb{K}$.

We can take $(n+1)$ linearly independent elements $x_1 \cdots x_{n+1}$. from X. Let $S = \{x_1, \cdots x_{n+1}\}$ $M = \operatorname{Span} S$.

$T|_M : X \to \mathbb{K}^n$. Suppose T is injective. then $T|_M$ is also injective. M is finite dimensional. then we have.

$$\dim M = \dim \operatorname{Im} T|_M + \dim \ker T|_M$$

$$T|_M \text{ injective} \Rightarrow \dim \ker T|_M = 0.$$

$$\dim M = \dim \operatorname{Im} T|_M = n+1 > \dim k^n = n, \quad (k = \mathbb{R})$$

for $k = \mathbb{C}$ we just choose $2n+1$ elements for S.

(b) $f_1 \cdots f_n \in X^*$. WTS. $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$

Consider $F(x) = (f_1(x), f_2(x), \cdots, f_n(x))$. $x \in X$

X infinite dimension $\Rightarrow F: X \to \mathbb{K}^n$ is not injective

So $\ker F \neq \{0\}$. $x \in \ker F$ $F(x) = (f_1(x) \cdots f_n(x)) = 0$.

$x \in \bigcap_{i=1}^{n} \ker f_i$

$\Rightarrow \bigcap_{i=1}^{n} \ker f_i \neq \{0\}$.

(C) Let $x_1 \cdots x_n \in X$.

For each x_i. By thm 2.7 (b).

we can find a map $f_i \in X^*$.

$\|f_i\| = 1$ $f_i(x) = \|x_i\|$ we have If $y \in \ker f_i$

$\|y - x_i\| = \|f_i\| \|y - x_i\| \geq \|f_i(y - x_i)\| = \|-f_i(x_i)\| = \|x_i\|$.

then we choose y from $\ker F$ (F is defined in (b) for f_i).

$y \in \ker F \Rightarrow y \in \bigcap_{i=1}^{n} \ker f_i \Rightarrow \|y - x_i\| \geq \|x_i\|$

$\ker F$ is a subspace of X.

So we can take $\|y\| = 1$.

(d). WTS One can not cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X s.t. none of the balls contain 0.

Suppose $\exists$ finite closed ball $\{B(x_i, r_i), r_i < \|x_i\|\}_{i \in I}$ By (c)

we can find $y \in S$. $\|y - x_i\| \geq \|x_i\| > r_i$ $\quad y \notin B(x_i, r) \cap S$.

$y \notin \bigcup_i (B(x_i, r) \cap S)$ so $\{B(x_i, r), r_i < \|x_i\|\}_{i \in I}$ can not cover

S

(e). WTS S is non-compact. We choose point step by step.

Take arbitrary $x_1 \in S$. use (c) find x_2. $x_2 \in S$. $\|x_2 - x_1\| \geq \|x_1\| = 1$.

And then choose $x_3, x_4, \dots \in S$. for any finite sequence $x_1 \dots x_n$ we can always choose x_{n+1}.

Therefore we get an infinite sequence $(x_i)_{i \in \mathbb{N}}$ which satisfies $\forall m, n \in \mathbb{N}$

$\|x_m - x_n\| \geq \|x_n\| = 1$

for all $x \in S$ take a neighborhood of x on S

$N_x(x, \frac{1}{2}) = B(x, \frac{1}{2}) \cap S$ $\quad$ if $\exists x_m \in (x_i)_{i \in \mathbb{N}}$ $\quad x_m \in N_x(x, \frac{1}{2})$

for all $x_n \in (x_i)_{i \in \mathbb{N}}$ $n \neq m$.

$\|x_n - x\| \geq \|x_n - x_m\| - \|x_m - x\| \geq 1 - \frac{1}{2} = \frac{1}{2}$ $\quad$ so $x_n \notin N_x(x, \frac{1}{2})$

So $\forall x \in S$ is not the limit point of any subsequence of $(x_i)_{i \in \mathbb{N}}$.

$(x_i)_{i \in \mathbb{N}}$ is an infinite sequence that has not convergent subsequence.

so S is non-compact.

For closed unit ball use the same construction of the sequence and choose neighborhood for $\forall x \in \overline{B(0,1)}$ $\quad N_x(x, \frac{1}{2}) = B(x, \frac{1}{2}) \cap \overline{B(0,1)}$

we still have $\|x_n - x\| \geq \|x_n - x_m\| - \|x_m - x\| \geq \frac{1}{2}$

So $\overline{B(0,1)}$ is non-compact.

(4) $E_n = \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 \, dm \leq n \right\}$.

(a) $n \geq 1$ Claim E_n is convex but not absorbing.

$f_1, f_2 \in E_n$ $\quad \|\alpha f_1 + (1-\alpha) f_2\|_p^p \leq \left(\|\alpha f_1\|_p + \|(1-\alpha) f_2\|_p\right)^p$

$$\leq \left(\alpha^p \sqrt{n} + (1-\alpha)^p \sqrt{n}\right)^p$$

$$= \left(\alpha^p + (1-\alpha)^p\right) n$$

$$= \left(\alpha^3 + (1-\alpha)^3\right) n \leq n.$$

Suppose E_n is absorbing

$\exists t > 0$. $\forall x \in L_1([0,1], m)$ $tx \in E_n$.

for $f \in L_1([0,1], m) \setminus E_n$ $\quad \int_{[0,1]} |f| \, dm < \infty$. $\quad \int_{[0,1]} |f|^3 \, dm > n$.

We have $\int_{[0,1]} |tf|^3 \, dm \leq n$. Let $f' = \frac{f}{t}$

We find that $\int_{[0,1]} |f'| \, dm = \frac{1}{t} \int_{[0,1]} |f| \, dm < \infty$. $f' \in L_1$

But $\int_{[0,1]} |tf'|^3 \, dm = \int_{[0,1]} |f|^3 \, dm \geq n$ $\quad tf' \notin E_n$.

$\Rightarrow E_n$ is not absorbing.

(b) Show that E_n has empty interior in $L_1([0,1], m)$ for all $n \geq 1$.

It is equivalent to show that $\forall \varepsilon > 0$ $\forall f \in E_n$ $\exists f' \in L_1 \setminus E_n$.

$\|f - f'\| < \varepsilon$.

To construct f' let $f' - f = \begin{cases} \frac{6\sqrt{n}}{\sqrt{\varepsilon}} & [0, \frac{\sqrt[2]{\varepsilon^3}}{8\sqrt{n}}] \\ 0 & (\frac{\sqrt[2]{\varepsilon^3}}{8n}, 1] \end{cases}$

$\int |f' - f| \, dm = \frac{6\sqrt{n}}{\sqrt{\varepsilon}} \cdot \frac{\sqrt[2]{\varepsilon^3}}{8\sqrt{n}} = \frac{3}{4}\varepsilon < \varepsilon$.

$\int |f' - f|^3 \, dm = \left(\frac{6\sqrt{n}}{\sqrt{\varepsilon}}\right)^3 \cdot \frac{\sqrt[2]{\varepsilon^3}}{8\sqrt{n}} = 27n$.

$\|f' - f\|_3 = 3\sqrt[3]{n}$

$\|f\|_3 \leq \sqrt[3]{n}$

$\|f'\|_3 \geq \|f' - f\|_3 - \|f\|_3 \geq 2\sqrt[3]{n}$

$\|f'\|_3^3 \geq 8n$.

So $f' \notin E_n$.

(C) Show that E_n is closed in $L_1([0,1].m)$ all $n \geq 1$

$f_n \in E_n$ $(x_n)_{n\in\mathbb{N}}$ is a convergent cauchy sequence in $L_1([0,1].m)$ converge to f

take a subsequence $\{f_{nk}\}$ converge pointwise to f a.e.

By fatou's lemma.

$$\int_{[0,1]} \liminf |f_{nk}|^3 \, dm = \int_{[0,1]} |f|^3 \, dm < \liminf \int_{[0,1]} |f_{nk}|^3 \, dm \leq n.$$

So f is in E_n.

(d) $L_3([0,1].m)$ is of the first category in $L_1([0,1].m)$

$E_n \subset L_3([0,1].m)$. $\bar{E}_n = E_n \Leftarrow E_n$ closed in L_1.

By (b) E_n has empty interior $Int(\bar{E}_n) = Int(E_n) = \phi$

$L_3([0,1].m) = \bigcup_{i=1}^{\infty} E_n$ $\forall f \in L_3$ $\exists M \in \mathbb{N} \, \|f\|_3 < M$.

$f \in E_{M^3}$ $L_3([0,1].m) \subseteq \bigcup_{i=1}^{\infty} E_n$ so $L_3([0,1].m)$ is of the first category in $L_1([0,1].m)$.

Problem 5.

H is an infinite dimentional separable Hilbert space.

(a) $x_n \to X$ in norm. it follows that $\|X_n\| \to \|X\|$.

as $n \to \infty$

Pf: $\lim_{n\to\infty} \|x_n - X\| = 0$ $\|x\| \leq \|x_n - X\| + \|x_n\|$.

$\|x_n\| \leq \|X\| + \|x_n - x\|$.

which implies $\lim_{n\to\infty} \|x_n\| \to \|x\|$.

(b) $x_n \to x$ weakly. it does not follow that $\|x_n\| \to \|x\|$ as $n \to \infty$.

By Homework 4. Pr2 and Pr3.

$x_n \to x$ weakly $\Leftrightarrow$.

$\langle x_n . y \rangle \to \langle x . y \rangle$ for all $y \in H$. as $n \to \infty$.

$\Leftrightarrow$

$\langle x_n . e_i \rangle \to \langle x . e_i \rangle$ for all $e_i \in \{e_i\}_{i \in \mathbb{N}}$

Choose $x_i = e_i$. $x = 0$.

$\langle x_n . e_i \rangle \to 0 = \langle x . e_i \rangle$.

$\|x_n\| = \|e_i\| = 1$. which does not converge to $\|x\| = 0$.

(c). It is true

By proposition 2.7 (b)

$0 \neq x \in X \; \exists \; f \in X^*$ $\quad f(x) = \|x\|$ $\quad \|f\| = 1$.

Then the weak convergence of $x_n \to x$.

$$\|x\| = \underset{\substack{\uparrow \\ \text{weak conv.}}}{f(x) = \lim_{n\to\infty} f(x_i)} = \underset{\substack{\uparrow \\ \text{Conv.} \Rightarrow \text{abs. conv.}}}{\lim_{n\to\infty} |f(x_i)|} \overset{\sup}{\leq} \|f\| \, \|x_i\| \overset{\sup}{=} \|x_i\| \leq 1.$$