Funk.An. Mandatory assignment 1.

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14 12 2020

Problem 1 [24 points]

Let (X, $|| \cdot ||_X$) and (Y, $|| \cdot ||_Y$) be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

a) [5 p]. Let T: X $\to$ Y be a linear map. Set $|| x ||_0 = || x ||_X + || Tx ||_Y$, for all $x \in X$. Show that $|| \cdot ||_0$ is a norm on X. Show next that the two norms $|| \cdot ||_X$ and $|| \cdot ||_0$ are equivalent if and only if T is bounded.

b) [4 p]. Show that any linear map T: X $\to$ Y is bounded, if X is finite dimensional.

c) [5 p]. Suppose that X is infinite dimensional. Show that there exists a linear map T: X $\to$ Y , which is not bounded (= not continuous). [Hint: Take a Hamel basis for X (see below).]

d) [5 p]. Suppose again that X is infinite dimensional. Argue that there exists a norm $|| \cdot ||_0$ on X, which is not equivalent to the given norm $|| \cdot ||_X$, and which satisfies $|| x ||_X \leq || x ||_0$, for all $x \in X$. Conclude that (X,$|| \cdot ||_0$) is not complete if (X,$|| \cdot ||_X$) is a Banach space.

e) [5 p]. Give an example of a vector space X equipped with two inequivalent norms $|| \cdot ||$ and $|| \cdot ||'$ satisfying $|| x ||' \leq || x ||$, for all $x \in X$, such that (X, $|| \cdot ||$) is complete, while (X, $|| \cdot ||'$) is not. [Hint: Take (X, $|| \cdot ||$) = $(\ell_1(\mathbb{N}), || \cdot ||_1)$ with a suitable choice of $|| \cdot ||'$; or take (X, $|| \cdot || = (L_2([0, 1], m), || \cdot ||_2)$ with a suitable choice of $|| \cdot ||'$, where m is the Lebesgue measure.]

Answers

a)

We have that, since $|| \cdot ||_X$ and $|| \cdot ||_Y$ are norms that $|| \cdot ||_0$: X $\to (0, \infty)$ by definition, and then we check the first condition from definition 1.1 of the lecture notes, $|| x + x' ||_0 = || x + x' ||_X + || Tx + Tx' ||_Y \leq || x ||_X + || x' ||_X + || Tx ||_Y + || Tx' ||_Y = || x ||_0 + || x' ||_0 \; \forall x, x' \in X$, since $|| \cdot ||_X$ and $|| \cdot ||_Y$ are norms. Then we chech the second condition, $|| \alpha x ||_0 = || \alpha x ||_X + || T\alpha x ||_Y = || \alpha x ||_X + || \alpha Tx ||_Y = | \alpha | \; || x ||_X + | \alpha | \; || Tx ||_Y = | \alpha | \; (|| x ||_X + || Tx ||_Y) = | \alpha | \; || x ||_0$, since T is a linear map and $|| \cdot ||_X$ and $|| \cdot ||_Y$ are norms.
Now we check the third and last condition of the definition, $|| x ||_0 = 0 \Leftrightarrow || x ||_X + || Tx ||_Y = 0$, and $|| x ||_X = 0 \Leftrightarrow x = 0$, and $|| Tx ||_Y = 0 \Leftrightarrow Tx = 0 \Leftrightarrow x = 0$, since T is linear and since $|| \cdot ||_X$ and $|| \cdot ||_Y$ are norms, so by definition of $|| x ||_0$ we have that $|| x ||_0 = 0 \Leftrightarrow x = 0$.
So $|| \cdot ||_0$ is a norm.

Now we need to show that $|| x ||_0$ and $|| x ||_X$ are equivalent $\Leftrightarrow$ T is bounded. First lets assume that $|| x ||_0$ and $|| x ||_X$ are equivalent. This means that $C_0 \; || x ||_0 \leq || x ||_X \leq C_X \; || x ||_0$, for $0 < C_0 \leq C_X < \infty$ by definition 1.4 in the lecture notes. So we have that $C_0 \; || x ||_X + C_0 \; || Tx ||_Y \leq || x ||_X \leq C_X \; || x ||_0 \implies || x ||_X + || Tx ||_Y \leq \frac{1}{C_0} \; || x ||_x \leq \frac{C_X}{C_0} \; || x ||_0 \implies || Tx ||_Y \leq \frac{1}{C_0} \leq \frac{C_X}{C_0} \; || x ||_0 - || x ||_X \leq \frac{C_X}{C_0} \; || x ||_0 \; \forall x \in X$, since $0 < C_0 \leq C_X < \infty$ and since the norms are non-zero by assumption. So T is bounded by Proposition 1.10 (3) from the lecture notes. Then let's assume that T is bounded, this means that there exists $C > 0$ such that $|| Tx ||_Y \leq C \; || x ||_0$, for all $x \in X$. So since $|| x ||_0 = || x ||_X + || Tx ||_Y$ we have that $|| x ||_0 \leq || x ||_X + C \; || x ||_0 \implies || x ||_0 - C \; || x ||_0 \leq || x ||_X \implies C_0 \; || x ||_0 \leq || x ||_X$ for $C_0 = 1 - C$ for $0 < C \leq 1$. And since $|| x ||_X \leq || x ||_0$ by definition of $|| x ||_0$, then we can pick a $0 < C_0 \leq C_X < \infty$ such that $C_0 \; || x ||_0 \leq || x ||_X \leq C_X \; || x ||_0$. So $|| \cdot ||_X$ and $|| \cdot ||_0$ are equivalent.

b)

By theorem 1.6 we have that if X is a finite dimensional vector space, then any two norms are equivalent. Which by (a) means that T is bounded. Or more generally, we have that if X is finite dimensional we can find a minimal distance min($\| x-y \|$) = D, $\forall x, y \in X$ Then if we take some point $x_0 \in X$ and a $\epsilon > 0$, we can let $\delta = \frac{D}{2}$. Then if $\| x - x_0 \| < \delta \implies \| Tx - Tx_0 \| < \epsilon$. So T is continuous when X is finite dimensional, which by proporsition 1.10 in the lecture notes means that T is bounded for all linear maps T: X $\to$ Y, when X is finite dimensional.

c)

Lets suppose that X is infinite dimensional. Then by Zorn's lemma X admits a Hamel basis, which means that $(e_i)_{i\in I}$ of elements in X for with the property that for each vector space Y over $\mathbb{K}$, and each family $(y_i)_{i\in I}$ in Y, there exists precisely one linear map $T : X \to Y$ satisfying $T(e_i) = y_i$ for all $i \in I$, or equivalently that for each $x \in X$, there is a unique family $(\lambda_i)_{i\in I}$ in $\mathbb{K}$ for which the set $\{i \in I : \lambda_i \neq 0\}$ is finite and $x = \sum_{i\in I} \lambda_i e_i$.
The existence of a linear map is clear from the definition of an algebraic basis, so we only need to show that it has to be not bounded (not continuous). Since X is infinite dimensional we must have that some of the $\lambda_i' s$ for all $i \in I$ has to be zero since we have finitely many $\lambda_i' s$ which are non-zero for $i \in I$ and since the family of $(\lambda_i)_{i\in I}$ are unique, So we can for example look at the function $T : X \to Y$ defined by $T(x) = \frac{1}{||0-x||}$, where we define $T(x) = 0$ for $x = 0$. This map is obviously discontinuous in 0, so T is therefore not bounded.

d)

Since we by problem (a) showed that $\| \cdot \|_0$ was a norm on X so it exists, and that the norms $\| \cdot \|_X$ and $\| \cdot \|_0$ only are equivalent if and only if T was bounded, and by problem (b) we had that any linear map T was bounded if X was finite dimensional and problem (c) tells us that there exists a linear map which is not bounded when X is infinite dimensional. This means that since we can find a linear map T which is not bounded, so not every linear map is bounded when X is infinite dimensional. So since we can find such a linear map T which isn't bounded we have that the two norms can not be eqivalent by problem (a). And by definition of $\| x \|_0$ and the fact that (X, $\| \cdot \|_X$) and (Y, $\| \cdot \|_Y$) are (non-zero) normed vector space over $\mathbb{K}$, we have that $0 \leq \| Tx \|_Y$ for all $x \in X$, so obviously $\| x \|_X \leq \| x \|_0$, for all $x \in X$.
If (X,$\| \cdot \|_X$) is a Banach space, we can find a Cauchy sequence $(x_n)_{n\geq 1}$ with respect to the metric d i.e., $\forall \epsilon > 0 \exists n_\epsilon \geq 1$ such that $\forall m, n \geq n_\epsilon$, $d(x_n, x_m) = \| x_n - x_m \|_X < \epsilon$, then there exists $x \in X$ such that $lim_{n\to\infty} \| x_n - x \|_X = 0$. And since T is unbounded and the two norms are not equivalent, then we wouldn't be able to find such a limit for a cauchy sequence with respect to the norm $\| \cdot \|_0$ since the limit wouldn't exist. We can for example look at the map I mentioned in problem (c) which was discontinuous at 0.

e)

Let us look at the vector space in the hint, i.e. the vector space $(X, \| \cdot \|) = (\ell_1(\mathbb{N}), \| \cdot \|_1)$. So I need to find a norm such that $\| x \|_1 \geq \| x \|_n$, where $\| x \|_n$ and $\| x \|_1$ are inequivalent and where $\| x \|_n$ makes the normed vector space $(\ell_1(\mathbb{N}), \| \cdot \|_n)$ not complete. We know that $\| x \|_2 \leq \| x \|_1$, so by taking the two norm $\| \cdot \|_2$ we would get that the normed vector space $(\ell_1(\mathbb{N}), \| \cdot \|_2)$ would not be complete since we could find a cauchy sequence in $\ell_1(\mathbb{N})$ with no limit inside $\ell_1(\mathbb{N})$ with respect to the two normn since the completion of $(\ell_1(\mathbb{N}), \| \cdot \|_2)$ with respect to $\| \cdot \|_2)$ is $\ell_2(\mathbb{N})$, where $\ell_1(\mathbb{N}) \subset \ell_2(\mathbb{N})$. And these two norms are inequivalent with respect to $\ell_1(\mathbb{N})$, since any two p norms are not equivalent on $\ell_1(\mathbb{N})$ for different p. So we have what we wanted

Problem 2 [20 points]

Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), ||\cdot||_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, \dots) : a, b \in \mathbb{C}\}.$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, 0, \dots) = a + b$, for all $a, b \in \mathbb{C}$.

a) [8 p]. Show that f is bounded on $(M, ||\cdot||_p)$ and compute $||\ f\ ||$. (answer depends on p.)

b) [7 p]. Show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $||\ F\ || = ||\ f\ ||$.

c) [5 p]. Show that if $p = 1$, then there are infinitely many linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $||\ F\ || = ||\ f\ ||$.

Answers

a)

f is obviously linear, since we can find $|\ x - x_0\ | < \delta$ for $\delta > 0$ such that $|\ f(x) - f(x_0)\ | < \epsilon$ for all $\epsilon > 0$, since every x in $\ell_p(\mathbb{N})$ is bounded and so is the sum by definition. So in particular $|\ (a, b, 0, 0, \dots) - (a_0, b_0, 0, 0, \dots)\ | < \delta$ for $\delta > 0$ such that $|\ f(a, b, 0, 0, \dots) - f(a_0, b_0, 0, 0, \dots)\ | = |\ a + b - (a_0 + b_0)\ | = |\ a - a_0 + b - b_0\ | < \epsilon$ for all $\epsilon > 0$ by definition of f. So f is bounded on $(M, ||\cdot||_p)$. Then we compute $||\ f\ || = \sup\{||\ fx\ ||\ :\ ||\ x\ ||\ \leq 1\}$ $= \inf\{C > 0 :\ ||\ fx\ ||\ \leq C\ ||\ x\ ||, x \in \ell_p(\mathbb{N})\}$. So we have that $||\ (a, b, 0, 0, \dots)\ ||_p = (|\ a\ |^p + |\ b\ |^p + |\ 0\ |^p + \dots)^{\frac{1}{p}} = (|\ a\ |^p + |\ b\ |^p)^{\frac{1}{p}}$, and $||\ f(a, b, 0, 0, \dots)\ ||_p = ||\ a + b\ ||_p = (|\ a + b\ |^p)^{\frac{1}{p}} = |\ a + b\ |$.
So for p = 1 we have that $||\ f\ || = \inf\{C > 0 :\ |\ a + b\ |\ \leq C(|\ a\ | + |\ b\ |), a, b \in \mathbb{C}\}$,
and for 1 < p < ∞ we have that $||\ f\ || = \inf\{C > 0 :\ |\ a + b\ |\ \leq C(|\ a\ |^p + |\ b\ |^p)^{\frac{1}{p}}, a, b \in \mathbb{C}\}$.

b)

Since f is bounded and hence continuous we have that $f \in M^* = \mathcal{L}(M, \mathbb{C})$ by definition of f, then by corollary 2.6 in the lecture notes we have that there exists $F \in (\ell_p(\mathbb{N}), ||\cdot||_p)^* = \mathcal{L}((\ell_p(\mathbb{N}), ||\cdot||_p)), \mathbb{C})$ such that $F\ |_M = f$ and $||\ F\ || = ||\ f\ ||$. So we only need to show the uniqeness of F on $\ell_p(\mathbb{N})$ for 1 < p < ∞. By example 2.11 in the lecture notes we have that $L_p(X, \mu)$ is reflexive for $1 < p < \infty$, so the same is the case for $(\ell_p(\mathbb{N}), ||\cdot||_p)$.

We know that there is an isometric isomorphism between $\ell_p(\mathbb{N})$ and $\ell_q(\mathbb{N})$ for every $1 < p < \infty$ by HW.1 problem 5. And we know that F exists by corollary 2.6 in the lecture notes, so by isometry there exists a $y \in \ell_q(\mathbb{N})$ such that $F(x) = \sum_{n=1}^{\infty} x_n y_n$, for all $x \in \ell_p(\mathbb{N})$. Where y is such that $||\ Fx\ || = ||\ x\ ||$ and $||\ f\ || = ||\ F\ ||$ and $F\ |_M = f$.

c)

Since f is bounded and hence continuous we have that $f \in M^* = \mathcal{L}(M, \mathbb{C})$ by definition of f, then by corollary 2.6 in the lecture notes we have that there exists $F \in (\ell_p(\mathbb{N}), ||\cdot||_p)^* = \mathcal{L}((\ell_p(\mathbb{N}), ||\cdot||_p), \mathbb{C})$ such that $F\ |_M = f$ and $||\ F\ || = ||\ f\ ||$. So we only need to show that there are infinitely many F on $\ell_1(\mathbb{N})$ such that this is the case for p = 1. By example 2.11 in the lecture notes we have that $L_p(X, \mu)$ is not reflexive for $p = 1$, so the same is the case for $(\ell_p(\mathbb{N}), ||\cdot||_p)$ for p = 1.
So for F being the continuous extension on $\ell_1(\mathbb{N})$, i.e. $F \in \ell_1(\mathbb{N}) \cong \ell_\infty(\mathbb{N})$ we have that the duality gives us some $u \in \ell_\infty(\mathbb{N})$ such that $\forall\ x = (x_n) \in \ell_1(\mathbb{N})$ being a sequence, we have that $F_k(x) = \sum_{i=1}^{k} x_i$. These $F_k(x)$ are obviously linear by construction, since $F_k(\alpha x + \beta y) = \sum_{i=1}^{k} \alpha(x_i) + \beta(y_i) = \sum_{i=1}^{k} \alpha(x_i) + \sum_{i=1}^{k} \beta(y_i) = \alpha F_k(x) + \beta F_k(y)$. And since the $x = (x_n) \in \ell_1(\mathbb{N})$ the F_k are extentsion of f with the same norm as f. So we have infinitely many extensions F of f in this case.

Problem 3 [25 points]

Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. a) [5 p]. Let n ≥ 1 be an integer. Show that no linear map $F : X \to \mathbb{K}^n$ is injective.

b) [5 p]. Let n ≥ 1 be an integer and let $f_1, f_2, \ldots, f_n \in X^*$. Show that $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$. [Hint: Consider the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), x \in X$.]

c) [5 p]. Let $x_1, x_2, \ldots, x_n \in X$. Show that there exists $y \in X$ such that $\| y \| = 1$ and $\| y - x_j \| \geq \| x_j \|$ for all $j = 1, 2, \ldots, n$. [Hint: Use Theorem 2.7 (b) from lectures to get started.]

d) [5 p]. Show that one cannot cover the unit sphere $S = \{x \in X : \| x \| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

e) [5 p]. Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

Answers

a)

We proof this by contradiction.
Let's suppose that $F : X \to \mathbb{K}^n$ is injective. Then we can take $x_1, \ldots, x_{n+1} \in X$, where $x_1, \ldots, x_{n+1}$ are linear independent in X, since X is infinite dimensional and we have that $F(x_1), \ldots, F(x_{n+1})$ is linear dependent in $\mathbb{K}^{n+1}$, since we have that $n+1$ vectors in a $n+1$ dimensional vector space are linear dependent. Then $\exists\ \alpha_1, \ldots, \alpha_{n+1} \in \mathbb{K}^{n+1}$ not all being 0 such that $\sum_{i=1}^{n+1} \alpha_i F(x_i) = \alpha_1 F(x_1) + \cdots + \alpha_{n+1} F(x_{n+1}) = F(\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}) = 0$ by linear dependence and since F is a linear map. Then since F was assumed injective we deduce that $\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0$, but $\alpha_i = 0$ for some $i \in \mathbb{N}$, since $x_1, \ldots, x_{n+1}$ is linear independent, which is a contradiction, so there is no linear map $F : X \to \mathbb{K}^n$ which is injective.

b)

Let's look at the opposite of what we want. For $\bigcap_{j=1}^{n} \ker(f_j) = \{0\}$, means that the only $x \in X$ making $f_j(x) = 0$ for $1 \leq j \leq n$ where $j, n \in \mathbb{N}$ would be $x = \{0\}$, by definition of the kernel and intersection. If we look at the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), x \in X$ as in the hint, we get that F is a linear map since it consists of linear maps by definition of the dual space which says that $X^* = \mathcal{L}(X, \mathbb{K})$. So by these facts we actually have that F isn't injective. This means that f_j aren't injective either $\forall\ j$.

So lets assume that $f_j(x) = 0\ \forall\ j$ for $x = 0$ since f_j are linear maps $\forall\ j$, so in particular we have that $F(\{0\}) = (f_1(0), f_2(0), \ldots, f_n(0)) = \{0\}$ then by the non-injectivety we have that $\exists x_i \in X$ such that $f_j(x_i) = 0$ $\forall\ j$ and for some i, so in particular we have that $\exists x_i \in X$ such that $F(x_i) = (f_1(x_i), f_2(x_i), \ldots, f_n(x_i)) = \{0\}$. This means that $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$, since there is another point in X where $F(x) = 0$ by the injectivity of F.

c)

We have by Theorem 2.7 in the lecture notes, that if $0 \neq x \in X$, then there exists $f \in X^*$ such that $\| f \| = 1$ and $f(x) = \| x \|$, so since X is infinite dimensional we can find a $0 \neq y \in X$ so we get that $\exists$ $f \in X^* = \mathcal{L}(X, \mathbb{K})$ such that $\| f \| = 1$ and $f(y) = \| y \|$. And by remark 1.11 from the lecture notes we also have that $\| f \| = \sup\{\| f(y) \| : \| y \| \leq 1\}$, which should be equal to 1 when we combine these two. So this means that $\sup\{\| \| y \| \| : \| y \| = 1\} = \{\| y \| : \| y \| \leq 1\} = 1$, which means that $\| y \| = 1$. And by the previous results we have that there is finitely many $0 \neq x_j \in X$ for $1 \leq j \leq n$ since we can find a Hamel basis. This means that $\| x_j \| \leq 1$ by theorem 2.7 (b) in the lecture notes. Then we use remark 1.2 from the notes, which gives that, $\| y - x_j \| \geq \| y \|$ - $\| x_j \| \geq 1 - \| x_j \| \geq 1 \geq \| x_j \|$.

d)

By the note below remark 5.3 in the lecture notes we have that S is weakly dense in the closed unit ball $\overline{B_X(0,1)} = \{x \in X : \ || \, x \, || \ \leq 1\}$ of X.

S is dense in $\overline{B_X(0,1)}$ in the weak topology means that the closure of S in this particular topology is equal to $\overline{B_X(0,1)}$. This, by basics of point-set topology, means that every point in $\overline{B_X(0,1)}$ is a limit (in the weak topology) of a net of points in S.

If we let B_i for $i = 1, \ldots, n$ be closed balls not containing 0, which are closed convex sets, since any closed ball in a normed vector space is convex. In particular $|| \, tx + (1-t)y - x_0 \, || \leq t \, || \, x - x_0 \, || + (1-t) \, || \, y - x_0 \, || \leq r$ for $x, y \in B(x_0, r), 0 \leq t \leq 1$. Hence we can find continuous functionals λ_i, such that Re $\lambda_i(x) \geq 1$ for $x \in B_i$. The vector space $V = \bigcap_{i=1}^n \ker(\lambda_i)$ does not intersect any of the B_i, since if $x \in V$, then $\lambda_i(x) = 0$, for all i. But $x \in B_i$ implies that Re $\lambda_i(x) \geq 1$. But $V \neq 0$, because X is infinite-dimensional. So we find an $x \in V \cap S$.
And in particular we have by subproblem (c) that there exists $y \in B_i$ such that $|| \, y \, || = 1$ and $|| \, y - x_j \, || \geq || \, x_j \, ||$ for all $j = 1, 2, \ldots, n$, where $|| \, y - x_j \, || = 0$ means that $|| \, x_j \, || = 0$ which can only be the case if $x_j = 0$. Therefore, no finite number of closed balls can cover S without one of them containing 0.

e)

We have that S is a subset of the closed unit ball $S \subset \overline{B_X(0,1)} = \{x \in X : \ || \, x \, || \ \leq 1\}$ of X.

For S being compact means that every infinite subset of S has a complete accumulation point, but since S is dense in $\overline{B_X(0,1)}$ in the weak topology, this cann't be true, so S is non-compact.

By Riesz's lemma which says that for X being a normed space and S being a closed proper subspace of X and a be a real number with $0 < a < 1$, then there exists an $x \in X$ with $|| \, x \, || = 1$ such that $|| \, x - y \, || \geq a$ for all $y \in S$. So we have that since X is an infinite dimensional normed vectorspace, the closed unit ball $\overline{B_X(0,1)}$ of X is non-compact, since we can take an element $x_1 \in S$, and pick an element $x_n \in S$ such that $\mathrm{d}(x_n, S_{n-1}) > a$ for a constant $0 < a < 1$ where S_{n-1} is the linear span of $\{x_1, \ldots, x_{n-1}\}$ and $\mathrm{d}(x_n, S) = \inf_{y \in S} | \, x_n - y \, |$. We easily see that $\{x_n\}$ contains no convergent subsequence, since S is non-compact, which means that the closed unit ball in X is non-compact.

Problem 4 [20 points]

Let $L_1([0,1],m)$ and $L_3([0,1],m)$ be the Lebesgue spaces on $[0,1]$. Recall from HW2 that $L_3([0,1],m) \subsetneq L_1([0,1],m)$. For $n \geq 1$, define

$$E_n := \{f \in L_1([0,1],m) : \int_{[0,1]} \mid f \mid^3 \, dm \leq n\}.$$

a) [5 p]. Given $n \geq 1$, is the set $E_n \subset L_1([0,1],m)$ absorbing? Justify.

b) [5 p]. Show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.

c) [7 p]. Show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$.

d) [3 p]. Conclude from (b) and (c) that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

Answers

a)

First we check that E_n is convex. We see that $\forall\ f_1, f_2 \in E_n$ and $\forall\ 0 \leq \alpha \leq 1$, $\alpha f_1 + (1-\alpha)f_2 \in E_n$, since $\int_{[0,1]} \mid \alpha f_1 + (1-\alpha)f_2 \mid^3 dm \leq \int_{[0,1]} \mid \alpha f_1 \mid^3 + \mid (1-\alpha)f_2 \mid^3 dm \leq \int_{[0,1]} \mid \alpha f_1 \mid^3 dm + \int_{[0,1]} \mid (1-\alpha)f_2 \mid^3 dm$ $\leq \int_{[0,1]} \mid \alpha \mid^3 \mid f_1 \mid^3 dm + \int_{[0,1]} \mid (1-\alpha) \mid^3 \mid f_2 \mid^3 dm \leq \mid \alpha \mid^3 \int_{[0,1]} \mid f_1 \mid^3 dm + \mid (1-\alpha) \mid^3 \int_{[0,1]} \mid f_2 \mid^3 dm \leq$ $\alpha^3 n + (1-\alpha)^3 n \leq \alpha n + (1-\alpha)n = n$, since $0 \leq \alpha \leq 1$ for all α. So E_n is convex.

E_n is absorbing if and only if $\forall\ 0 \neq f \in L_1([0,1],m)$, $\exists t > 0$ such that $f \in tE_n$, equivalently $t^{-1}f \in E_n$. To show this we can take $f \in L_1([0,1],m)$, then $\int_{[0,1]} \mid f \mid dm < \infty$ and then $\int_{[o,1]} \mid \frac{1}{t} f \mid^3 dm =$ $\int_{[o,1]} \mid \frac{1}{t^3} \mid \mid f \mid^3 dm = \mid \frac{1}{t^3} \mid \int_{[0,1]} \mid f \mid^3 dm \leq$ n, for t large enough where $0 < 1 \leq t$, since that $\frac{1}{t} \int_{[0,1]} \mid f \mid dm < \infty$ for $t \geq 1$ by assumption.

b)

Firstly we notice that $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n$, and we can find an open subset of E_n for every $n \geq 1$. The subset $U_1 \subset E_1$, where $U_1 = \{f \in L_1([0,1],m) : \int_{[0,1]} \mid f \mid^3 dm < n\}$. By definition of an interior point we have that if f is an interior point of E_n, then E_n is a neighbourhood of f, i.e. $f \in U_n \subset E_n$. So we easily see that $U_1 \subset E_1$ where U_1 also is an absorbing set since E_1 is absorbing in $L_1([0,1],m)$ by (a).
Then lemma 3.5 in the lecture notes gives us that $f \in U_1 \Leftrightarrow p_{U_1}(f) < 1$, where $p_{U_1}(f) = \inf\{t > 0 : f \in tU_1\}$ $= \inf\{t > 0 : t^{-1}f \in U_1\}$. Then by the same calculations as in problem (a) we can get that $t^{-1}f \in U_1 \Rightarrow$ $\mid \frac{1}{t^3} \mid \int_{[0,1]} \mid f \mid^3 dm < 1$, but this is only true for $t \geq 1$ and t large enough, so this means that $p_{U_1}(f) \geq 1 \Leftrightarrow$ $f \notin U_1$. So E_n has empty interior in $L_1([0,1],m)$ for all $n \geq 1$.

c)

For E_n to be closed in $L_1([0,1],m)$ for all $n \geq 1$, we need to have that any cauchy sequence in E_n has limit in E_n. We take (f_n) to be any cauchy sequence of funtions where each $f_n \in E_n$. Then there exists f such that $\lim(f_n) = f$ and there exists $n \geq 1$ such that (f_n) converges uniformly to f since $f_n \in E_n$ and by definition of E_n and f is continuous by definition, since $f \in L_1([0,1],m)$.
Then we can let $\mid f(x) - f_n(x) \mid < \frac{\epsilon}{2}$ and $\mid f_n(x) - n \mid < \frac{\epsilon}{2}$, for $\epsilon > 0$. So we have that $\mid f(x) - n \mid =$ $\mid f(x) - f_n(x) + f_n(x) - n \mid \leq \mid f(x) - f_n(x) \mid + \mid f_n(x) - n \mid < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, for $\epsilon > 0$. So $f \in E_n$ which means that E_n is closed in $L_1([0,1],m)$ for all $n \geq 1$.

d)

By definition 3.12 in the lecture notes, we need to show that there exists a sequence $(E_n)_{n\geq 1}$ of nowhere dense sets such that $L_3([0,1],m) = \cup_{n\geq 1}^{\infty} E_n$.
If we combine the result from (b) and (c) we get that $E_n = \bar{E_n}$, since E_n is closed in $L_1([0,1],m)$ for all $n \geq 1$ and that $\text{Int}(E_n) = \text{Int}(\bar{E_n}) = \emptyset$ for all $n \geq 1$, which means that $E_n \subset L_1([0,1],m)$ is nowhere dense for all $n \geq 1$ by definition 3.12 (i) in the lecture notes.
And we have that $\cup_{n\geq 1}^{\infty} E_n = \cup_{n\geq 1}^{\infty} \{f \in L_1([0,1],m) : \int_{[0,1]} \mid f \mid^3 dm \leq n\} = \{f : [0,1] \to \mathbb{K} \text{ measureable } : \mid\mid f \mid\mid_1 := (\int_{[0,1]} \mid f(x) \mid^3 dm) < \infty\} = \{f : [0,1] \to \mathbb{K} \text{ measureable } : \mid\mid f \mid\mid_3 := (\int_{[0,1]} \mid f(x) \mid^3 dm)^{\frac{1}{3}} < \infty\} = L_3([0,1],m)$. So $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$ by definition 3.12 (ii) in the lecture notes.

Problem 5 [11 points]

Let H be an infinite dimensional separable Hilbert space with associated norm $|| \cdot ||$, let $(x_n)_{n \geq 1}$ be a sequence in H, and let $x \in H$.

a) [2 p]. Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $|| x_n || \to || x ||$, as $n \to \infty$? Give a proof or a counterexample.

b) [5 p]. Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $|| x_n || \to || x ||$, as $n \to \infty$? Give a proof or a counterexample. [Hint: Consider an orthonormal basis $(e_n)_{n \geq 1}$ in H, and use HW4.]

c) [4 p]. Suppose that $|| x_n || \leq$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Is it true that $|| x || \leq 1$? Give a proof or a counterexample.

Answers

a)

Since $x_n \to x$ in norm, as $n \to \infty$, then $\lim_{n \to \infty} || x_n - x || = 0$. And we have that $|| x_n ||$ - $|| x || \leq || x_n - x ||$, so by the squeeze lemma, we get that $|| x_n || \to || x ||$ as $n \to \infty$.

b)

By proposition 5.28 and 5.29 in Folland we have that any Hilbert space has an orthonormal basis where any orthonomal basis countable when H is separable. So we can find an countable basis $(e_n)_{n \geq 1}$ in H.
And we have by definition of weak convergense that $x_n \to x$ weakly, as $n \to \infty$ means that $< x_n, y > \to < x, y > \forall\ y \in H$. Then if we consider an orthonormal basis $(e_n)_{n \geq 1}$ in H such that $< e_n, e_m >= 1$ if $n = m$ and 0 otherwise. Then for $x \in H$ we have that $\sum_{n \geq 1} |< e_n, x >|^2 \ \leq \ || x ||^2$, with equality when e_n is a basis for a Hilbert space as it is in our case. So we have that $|< e_n, x >|^2 \to 0$, i.e. $< e_n, x > \to 0$. Which means that since H is an infinite dimensional separable Hilbert space we have that $x_n \to 0$ as $n \to \infty$.
Then by HW4 problem 4 we have that the Hilbert space $\ell_2(\mathbb{N})$ is separable. And by HW4 problem 3 (a) we have that the sequence $(x_n)_{n \geq 1}$ is bounded in $|| \cdot ||_2$, which means that there is a constant $K > 0$ such that $|| x_n ||_2 \leq K$, for all $n \geq 1$. So we have that $|| x_n || \to || 0 ||$ as $n \to \infty$, since $|| 0 ||_2 = 0 \lneq K$ for $K > 0$.
So the statement that $|| x_n || \to || x ||$, as $n \to \infty$ as $x_n \to x$ weakly, as $n \to \infty$ is true.

c)

This is also true by calculations and arguments in problem (b), since we can choose $K > 0$ where $K = 1$ such that $|| x_n || \leq 1$ for all $n \geq 1$, since we are in the same situation as in problem (b) since we again assume that $x_n \to x$ weakly, as $n \to \infty$.