# Mandatory Assignment 2 FunkAn

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Before commencing on the problems, we wish to prove two preliminary lemmas:

**Lemma** (Norm Convergence Principle)**.** *Let $Z$ be a normed space, let $(z_n)_{n\in\mathbb{N}} \subseteq Z$ and $z \in Z$. If every subsequence of $(z_n)_{n\in\mathbb{N}}$ has a subsequence convergent to $z$, then $z_n \to z$.*

*Proof.* We prove this by contraposition. Assume $(z_n)_{n\in\mathbb{N}}$ does not converge to $z$. Then there exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$, there exists $m > N$ with $||z - z_m|| \geq \varepsilon$. Take any such $z_m =: z_{n_0}$, and iteratively construct $z_{n_{k+1}}$ by choosing $N = n_k$ and finding $n_{k+1} := m > n_k$ such that $||z - z_m|| \geq \varepsilon$. This yields a subsequence, $(z_{n_k})_{k\in\mathbb{N}}$, where for each element, $||z - z_{n_k}|| \geq \varepsilon$ for some fixed $\varepsilon$. This shows in particular that $z$ is not an accumulation point for this sequence, and so no subsequence of $(z_{n_k})_{k\in\mathbb{N}}$ can converge to $z$. This completes our proof. $\square$

The above proof can be used to show that the same principle holds for weak convergence in a Banach space. Indeed, instead of using the norm on $Z$, one can use Homework 4 Problem 2(a) to pick some $g \in Z^*$ such that $g(z_n)$ does not converge to $g(z)$, and so for every $N$ you may find $m > N$ with $|g(z) - g(z_m)| \geq \varepsilon$. The completely analogous argument for the rest of the proof then works. This shows the following statement:

**Lemma** (Weak Convergence Principle)**.** *Let $Z$ be a Banach space, let $(z_n)_{n\in\mathbb{N}} \subseteq Z$ and $z \in Z$. If every subsequence of $(z_n)_{n\in\mathbb{N}}$ has a subsequence weakly convergent to $z$, then $z_n \to z$ weakly.*

Now, let us begin the problems.

## Problem 1

Let H be an infinite dimensional separable Hilbert space with ONB $(e_n)_{n\in\mathbb{N}}$. Define $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$ for all $N \in \mathbb{N}$.

(a) Let us show that $f_N \to 0$ weakly for $N \to \infty$, but $||f_N|| = 1$ for all $N \in \mathbb{N}$.

We start out by calculating the norm:

$$||f_N||^2 = \langle f_N, f_N\rangle = \langle N^{-1}\sum_{n=1}^{N^2} e_n, N^{-1}\sum_{n=1}^{N^2} e_n\rangle = N^{-2}\sum_{n=1}^{N^2}\langle e_n, e_n\rangle = 1\,.$$

✓

Here we used that $(e_n)_{n\in\mathbb{N}}$ is orthonormal. Careful, should be something along to $\sum_{n,k=1}^{N^2}\langle e_n, e_k\rangle$.

For weak convergence, we wish to use the Weak Convergence Principle (see page 1). Firstly, as $H$ is reflexive, the closed unit ball is weakly compact by Theorem 6.3, and by the above norm calculation, $(f_N)_{n\in\mathbb{N}}$ is a sequence herein. Now take any subsequence of $(f_{N_k})_{n\in\mathbb{N}}$. Since it lies in a weakly compact set, it has a weakly convergent subsequence, $(f_{N_{k_s}})_{n\in\mathbb{N}}$. Let $x$ be the weak limit. By HW 4 Problem 2(a), we see that any $g \in H^*$ must satisfy $g(f_{N_{k_s}}) \to g(x)$. In particular, we may pick $g_n := \langle -, e_n\rangle$ for $n \in \mathbb{N}$. Note that $g_n(f_N) = 0$, if $n > N^2$, and $g_n(f_N) = N^{-1}$ else. Collecting, we see that

How do you know it is a subseq and not a subnet?

You need this to ensure $\langle x, e_n\rangle \geq 0$.

$$0 \leq g_n(f_{N_{k_s}}) \leq N_{k_s}^{-1} \to 0\,.$$

This shows that $\langle x, e_n\rangle = 0$ for all $n \in \mathbb{N}$, proving that $x = 0$. Now the Weak Convergence Principle tells us that $f_N \to 0$ weakly, and we are done. (✓)

(b) Let us define $K := \overline{\mathrm{co}\{f_N : N \in \mathbb{N}\}}$, and show that $K$ is weakly compact, with $0 \in K$.

First, note that any convex combination of $f_N$ has norm at most 1: What is $\alpha$?

$$\|\sum_{k=1}^{n} \underset{\alpha_k?}{\alpha} f_{N_k}\| \leq \sum_{k=1}^{n}\alpha\|f_{N_k}\| = \sum_{k=1}^{n}\alpha = 1\,,$$

where the second to last equality follows by (a). Thus, $\mathrm{co}\{f_N : N \in \mathbb{N}\} \subseteq \overline{B(0,1)}$, and since the latter set is closed, we see that $K \subseteq \overline{B(0,1)}$. As Hilbert spaces are reflexive, Theorem 6.3 yields that $\overline{B(0,1)}$ is weakly compact. As norm closure coincides with weak closure for convex sets (Theorem 5.7), we see that $K$ is a weakly closed subset of a weakly compact set, thus weakly compact itself.

To see that $0 \in K$, simply note that $K$ is weakly closed as argued above, and $(f_N)_{N\in\mathbb{N}} \subseteq K$ with $f_N \to 0$ weakly. As $K$ contains all its weak limit points, $0 \in K$. ✓

(c) Let us show that $f_N$ and 0 are extreme points of $K$.

Let us first attend to 0. Note that any convex combination of $\{f_i : i \in \mathbb{N}\}$ has positive inner product with each $e_n$, as each $f_N$ has. This was calculated in (a). Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of such convex combinations, converging to $x$ in norm. Note that any point in $K$ is such a limit. Then we see that, as each $g_n$ is continuous, that $\langle s_k, e_n \rangle \to \langle x, e_n \rangle$ for every $n \in \mathbb{N}$, and as $\langle s_k, e_n \rangle \geq 0$, so must $\langle x, e_n \rangle \geq 0$. Now, let us give any convex combination of 0 in $K$, $0 = \alpha x + (1-\alpha)y$. This means that

$$\alpha \langle x, e_n \rangle + (1-\alpha)\langle y, e_n \rangle = 0$$

for each $n \in \mathbb{N}$. But as both these inner products are positive, this can only be achieved if $\langle x, e_n \rangle = \langle y, e_n \rangle = 0$. As $n$ was arbitrary, we see that $x = y = 0$, proving 0 is an extreme point of $K$. ✓

Before turning our attention to $f_N$, we wish to further describe the points in $K$. Any $x \in K$ is a limit of some sequence $(s_n)_{n\in\mathbb{N}}$ of convex combinations of $\{f_N\}$. In fact, any such $s_n$ can be described as an infinite sum

$$s_n = \sum_{i=1}^{\infty} \alpha_{n,i} f_i \ ,$$

where only finitely many $\alpha_{n,i}$ are non-zero, and $\sum_{i=1}^{\infty} \alpha_{n,i} = 1$. In this way, for such a sequence $(s_n)_{n\in\mathbb{N}}$, we get a sequence for each $N \in \mathbb{N}$ of coefficients $(\alpha_{n,N})_{n\in\mathbb{N}}$ related to $f_N$. In fact, it is not hard to see that $(s_n)_{n\in\mathbb{N}}$ converges in norm to $f_N$ if $\alpha_{n,N} \to 1$, as then every other sequence of coefficients $(\alpha_{n,i})_{n\in\mathbb{N}}$ must converge to 0.

Then show it!

Now, let $\alpha x + (1-\alpha)y = f_N$ be an arbitrary convex combination of $f_N$ in $K$. Let $s_n \to x$ and $t_n \to y$ in norm for $n \to \infty$, with each $s_n$ and $t_n$ a convex combination of $\{f_i \ : \ i \in \mathbb{N}\}$. Then $\alpha s_n + (1-\alpha)t_n \to \alpha x + (1-\alpha)y = f_N$ in norm. In particular, for $g_k = \langle -, e_k \rangle$, we see that

$$g_{N^2}(\alpha s_n + (1-\alpha)t_n) = \alpha g_{N^2}(s_n) + (1-\alpha)g_{N^2}(t_n) \to g_{N^2}(f_N) = \frac{1}{N} \ , \tag{1}$$

as $g_k$ is continuous for all $k \in \mathbb{N}$. We claim that $g_{N^2}(s_n) \leq \frac{1}{N}$. Indeed, note that $g_{N^2}(f_M) = 0$ if $M^2 < N^2$, and $g_{N^2}(f_M) = \frac{1}{M} \leq \frac{1}{N}$ for $M^2 \geq N^2$. If $s_n = \sum_{i=1}^{r_n} \alpha_{n,N_i} f_{N_i}$, where we have chosen only the non-zero coefficients, we see that

$$g_{N^2}(s_n) = \sum_{i=1}^{r_n} \alpha_{n,N_i} g_{N^2}(f_{N_i}) \leq \sum_{i=1}^{r_n} \alpha_{n,N_i} \frac{1}{N} = \frac{1}{N} \ .$$

As the same could be argued for $g_{N^2}(t_n)$, this clearly shows that (1) can only hold if $g_{N^2}(s_n) \to \frac{1}{N}$. We claim that this implies that the related sequence $\alpha_{n,N}$ converges to 1, and therefore $s_n \to f_N$ in norm.

Be more explicit.

Assume for contradiction that $\alpha_{n,N}$ does not converge to 1. Then there exists a constant $\varepsilon > 0$ such that for every $K$, we have some $n > K$ with $|1 - \alpha_{n,N}| > \varepsilon$. Let $d_n := 1 - \alpha_{n,N}$. As $\alpha_{n,N} \leq 1$, the above can be reformulated $d_n > \varepsilon$. By our previous arguments, $g_{N^2}(\alpha s_n + (1-\alpha)t_n) \leq \frac{1}{N}$, so we may calculate

$$
\begin{aligned}
&\left|\frac{1}{N} - g_{N^2}(\alpha s_n + (1-\alpha)t_n)\right| = \frac{1}{N} - (\alpha g_{N^2}(s_n) + (1-\alpha)g_{N^2}(t_n)) \\
\geq &\frac{1}{N} - \left(\alpha g_{N^2}(s_n) + (1-\alpha)\frac{1}{N}\right) = \alpha\frac{1}{N} - \alpha\sum_{i=1}^{r_n}\alpha_{n,i}g_{N^2}(f_i) \\
= &\alpha\left(\frac{1}{N}(1-\alpha_{n,N}) - \sum_{i=1, i\neq N}^{r_n}\alpha_{n,i}g_{N^2}(f_i)\right) \overset{(i)}{\geq} \alpha\left(\frac{1}{N}d_n - d_n\frac{1}{N+1}\right) \\
\geq &\alpha\varepsilon\left(\frac{1}{N} - \frac{1}{N+1}\right).
\end{aligned}
$$

(i): We use that $g_{N^2}(f_i) \leq \frac{1}{N+1}$ for $i \neq N$, and that $\sum_{i=1, i\neq N}^{r_n}\alpha_{n,i} = 1 - \alpha_{n,N} = d_n$. This is a contradiction to the convergence in (1); Indeed, we have found an $\varepsilon_1$, such that for every $K \in \mathbb{N}$ there exists $n \geq K$ such that

$$\left|\frac{1}{N} - g_{N^2}(\alpha s_n + (1-\alpha)t_n)\right| \geq \varepsilon_1 .$$

The idea seems correct, but it is difficult to follow,

Thus, we get $\alpha_{n,N} \to 1$, i.e. $s_n \to f_N$, and so $x = f_N$. As the same argument can be repeated with $t_n$, we get that $y = f_N$, and so the convex combination $\alpha x + (1-\alpha)y = f_N$ is trivial. As this convex combination was arbitrary, this shows $f_N$ is an extreme point of $K$, completing our proof. ✓

(d) Let us show that $K$ has no other extreme points than those shown in $(c)$.

Since $(H, \tau_w)$ is a LCTVS (as argued in Lecture 5 p. 27), Theorem 7.9 immediately yields that $\text{Ext}(K) \subseteq \overline{\{f_N \ : \ N \in \mathbb{N}\}}^{\tau_w}$. As $f_N \to 0$ weakly, clearly we have

$$\{f_N \ : N \in \mathbb{N}\} \cup \{0\} \subseteq \overline{\{f_N \ : \ N \in \mathbb{N}\}}^{\tau_w} ,$$

and we argue this is an equality. Indeed, it is sufficient to show that no sequence in $\{f_N \ : \ N \in \mathbb{N}\}$ has any other weak limit. If $x$ is a weak limit for such a sequence, then $g(x)$ is a limit for some sequence in $(g(\{f_N \ : \ N \in \mathbb{N}\})$ for any $g \in H^*$, in particular $g_{e_1} := \langle -, e_1\rangle$. We see that

$$g_{e_1}(\{f_N \ : \ N \in \mathbb{N}\}) = \{N^{-1} \ : \ N \in \mathbb{N}\} . \tag{2}$$

This is a familiar set, and it clearly has no accumulation points other than 0 and $N^{-1}$ for all $N \in \mathbb{N}$. But any sequence $(f_{N_k})_{k\in\mathbb{N}}$, where $g_{e_1}(f_{N_k}) = N_k^{-1} \to N^{-1}$, must eventually be $f_N$, and thus weakly converge to $f_N$. This follows from the fact that the set $\{N_k^{-1}\}_{k\in\mathbb{N}}$ is discrete.

Now, if $g_{e_1}(f_{N_k}) \to 0$, then $N_k \to \infty$ for $k \to \infty$. Thus $(f_{N_k})_{k\in\mathbb{N}}$ has a subsequence $(f_{N_{k_t}})_{t\in\mathbb{N}}$, where $N_{k_{t_1}} < N_{k_{t_2}}$ for $t_1 < t_2$. This is then a subsequence of $(f_N)_{N\in\mathbb{N}}$, and so it must weakly converge to 0. But it is also a subsequence of the sequence $(f_{N_k})_{k\in\mathbb{N}}$. Then, by uniqueness of weak convergence, 0 is the only point to which $(f_{N_k})_{k\in\mathbb{N}}$ can weakly converge. This completes our proof. ✓

## Problem 2

Let $X$ and $Y$ be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X,Y)$, and assume $x_n \to x$ weakly for $n \to \infty$, for $x_n, x \in X$ for all $n \in \mathbb{N}$. Let us show that $Tx_n \to Tx$ weakly for $n \to \infty$.

By Homework 4 Problem 2(a), it is sufficient to show that $f(Tx_n) \to f(Tx)$ for any $f \in Y^*$. But this is exactly $(f \circ T)(x_n) \to (f \circ T)(x)$, and as $f \circ T \in X^*$, Homework 4 Problem 2(a) assures us of this convergence. As $f$ was arbitrary, we are done.

(b) Assume now $T$ is compact. In the same setup as above, let us show that $Tx_n \to Tx$ in norm for $n \to \infty$.

By Homework 4 Problem 2(b), we see that

$$r := \sup\{||x_n|| \;:\; n \in \mathbb{N}\} < \infty\,,$$

thus $(x_n)_{n\in\mathbb{N}} \subseteq \overline{B_X(0,r)}$. As $T$ is compact, $\overline{T(\overline{B_X(0,1)})}$ is compact, and so it is an easy consequence that $C := \overline{T(\overline{B_X(0,r)})}$ is also compact. Now, $(Tx_n)_{n\in\mathbb{N}}$ converges weakly to $Tx$ by (a), and so in particular does every subsequence. Let $(Tx_{n_k})_{k\in\mathbb{N}}$ be any subsequence. As this sequence lies in the (norm-)compact set $C$, it has a norm-convergent subsequence. But this subsequence, too, must weakly converge to $Tx$, and so the point to which it converges in norm must also be $Tx$. Then the Norm Convergence Principle (see page 1) tells us that $Tx_n \to Tx$ in norm, and we are done.

How so?

(c) Let $H$ be a separable infinite dimensional Hilbert space, and let $T \in \mathcal{L}(H,Y)$. We prove here the converse of (b), i.e. if $Tx_n \to Tx$ in norm for every weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$, then $T \in \mathcal{K}(X,Y)$.

$y_n \in \overline{T(B_H(0,1))}$ and hence not necessarily in the image of $T$.

Assume the setup, and assume for contradiction $T$ not compact. Then $\overline{T(B_H(0,1))}$ is not compact, thus has a sequence $(y_n)_{n\in\mathbb{N}}$ with no convergent subsequences. For each $y_n$, pick some $x_n$ with $||x_n|| \leq 1$ and $Tx_n = y_n$, and consider the sequence $(x_n)_{n\in\mathbb{N}} \subseteq \overline{B_H(0,1)} \subseteq H$. By Theorem 6.3, as any Hilbert space is reflexive, the closed ball is weakly compact. Thus, it has a weakly convergent subsequence $(x_{n_k})_{n\in\mathbb{N}}$. Taking the image yields a sequence $(Tx_{n_k})_{n\in\mathbb{N}}$, which is a subsequence of $(y_n)_{n\in\mathbb{N}}$. By assumption, this subsequence converges in norm, as it is the image of a weakly convergent sequence. But the contradicts our choice of $(y_n)_{n\in\mathbb{N}}$, namely that is has no norm-convergent subsequence. Thus our claim is proven.

Why is this a subseq and not a subnet?

(d) Let us show that every $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

Recall that $\ell_2(\mathbb{N})$ is a separable infinite dimensional Hilbert space, thus (c) applies. Then it is sufficient to show that for every $x_n \to x$ weakly in $\ell_2(\mathbb{N})$, $Tx_n \to Tx$ in norm in $\ell_1(\mathbb{N})$. Let such an $(x_n)_{n\in\mathbb{N}}$ be given. By (a), we have $Tx_n \to Tx$ weakly, and by Remark 5.3, a sequence converges weakly in $\ell_1(\mathbb{N})$ if and only if it converges in norm, thus $Tx_n \to Tx$ in norm.

(e) Let us show that no $T \in \mathcal{K}(X,Y)$ is surjective.

Assume $T$ is compact- Let us show that $T$ is not open - then it follows by the Open Mapping Theorem that T is not surjective.

By assumption, $\overline{T(B_X(0,1))}$ is compact. If T was open, then $T(B_X(0,1))$ would be open, and in particular contain an open ball in $Y$ around 0, say $B_Y(0,r)$. But by from Mandatory Assigment 1 Problem 3(e), $\overline{B_Y(0,1)}$ is not compact, and thus neither is $\overline{B_Y(0,r)}$ for any $r > 0$. Indeed, you can simply take a sequence with no convergent subsequence in $\overline{B_Y(0,1)}$ and scale it by $r$, showing $\overline{B_Y(0,r)}$ has a sequence with no convergent subsequence. Then, if $B_Y(0,r) \subseteq T(B_X(0,1))$, we would get $\overline{B_Y(0,r)} \subseteq \overline{T(B_X(0,1))}$, and as the latter is compact, we would get $\overline{B_Y(0,r)}$ compact, a contradiction. Thus, $T$ is not open, completing our proof. ✓

(f) Let $H = L_2([01], m)$, and let us show that $M \in \mathcal{L}(H,H)$, given by $Mf(t) = tf(t)$ for $f \in H$, is self-adjoint but not compact.

Let us show $M$ is self-adjoint. We calculate, for $f, g \in H$:

$$\begin{aligned}\langle Mf, g\rangle &= \int_{[0,1]} Mf\overline{g}dm = \int_{[0,1]} tf(t)\overline{g(t)}dm(t) \\ &= \int_{[0,1]} f(t)\overline{tg(t)}dm(t) = \int_{[0,1]} f(t)\overline{Mg(t)}dm(t) = \langle f, Mg\rangle .\end{aligned}$$

He were used that $t$ is real, thus $t = \overline{t}$. Now, if $M$ was compact, the Spectral Theorem would give us that there exists an ONB for $H$ of eigenvectors of $M$. But by Homework 6 Problem 3(a), $M$ has no eigenvectors, which shows it cannot be compact. Thus, we are done. ✓

## Problem 3

Let us consider the Hilbert space $H = L_2([0,1], m)$, and define $K : [0,1]^2 \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t & \text{if } 0 \leq t \leq s \leq 1 \\ (1-t)s & \text{if } 0 \leq s < t \leq 1 \end{cases} .$$

Define $T\mathcal{L}(H,H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t)\,, \quad s \in [0,1],\ f \in H\,.$$

(a) Let us show that $T$ is compact.

Firstly, $K \in L_2([0,1]^2, m \otimes m)$, as it is positive, bounded by 1 and piecewise continuous. In fact, it is continuous, as the piecewise definitions agree on the limit at $t = s$: Indeed, in this case, $(1-s)t = (1-t)s$.

No, $T = T_{\tilde{K}}$ for $\tilde{k}(s,t) = k(t,s)$

Now we see that $T$ is the associated kernel operator $T_K$ to $K$, as defined in Lecture Notes 9. As $[0,1]$ is compact and Hausdorff, and $m$ is finite on $[0,1]$, Theorem 9.6 yields that $T = T_K$ is compact.

(b) Let us show that $T = T^*$. Let $f, g \in H$. We calculate, using Fubini as the integrals are a.e. finite, shown at the beginning of Lecture Notes 9.

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) dm(t) \overline{g(s)} dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \overline{g(s)} dm(t) dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \overline{g(s)} dm(s) dm(t) \\
&= \int_{[0,1]} \int_{[0,1]} \overline{K(s,t) g(s)} dm(s) f(t) dm(t) \\
&= \int_{[0,1]} \overline{\int_{[0,1]} K(s,t) g(s) dm(s)} f(t) dm(t) = \langle f, Tg\rangle \,.
\end{aligned}$$

Here you use $k(s,t) = k(t,s)$

We used that $K$ is real, thus $K = \overline{K}$. Furthermore that the integration variable is real, thus we may take conjugation "outside"the integral. This shows $T = T^*$.

(c) Let us show that $Tf$ is continuous for $f \in H$, and that $(Tf)(0) = (Tf)(1) = 0$.

In the following, we will use that all one-point sets are $m$-null sets. First we prove an identity for $Tf$:

$$\begin{aligned}
(Tf)(s) = &\int_{[0,1]} K(s,t) f(t) dm(t) = \int_{[0,s]} K(s,t) f(t) dm(t) + \int_{[s,1]} K(s,t) f(t) dm(t) \\
=&(1-s) \int_{[0,s]} t f(t) dm(t) + s \int_{[s,1]} (1-t) f(t) dm(t) \,.
\end{aligned}$$

Here we used the piecewise definition of $K$. Now it is easy to see that

$$\begin{aligned}
(Tf)(0) &= \int_{[0,0]} t f(t) dm(t) + 0 \cdot \int_{[0,1]} (1-t) f(t) dm(t) = 0 \,, \\
(Tf)(1) &= 0 \cdot \int_{[0,1]} t f(t) dm(t) + \int_{[1,1]} (1-t) f(t) dm(t) = 0 \,,
\end{aligned}$$

justify this claim

To prove continuity, it seems most natural to use the Continuity lemma (e.g. 11.4 Schilling, First Edition). Note that the proof holds for closed intervals as well. $t \mapsto K(s,t) f(t)$ is integrable over $[0,1]$ for constant $s$ by the calculations in Lecture 9 p. 46. $s \mapsto K(s,t) f(t)$ is also clearly continuous for constant $t$. Now, we simply have to show that $|K(s,t) f(t)|$ is bounded by an integrable function independent of $s$. Take here $|K(s,t) f(t)| \leq |f(t)|$, which is integrable by assumption. This proves that $Tf$ is continuous by the Continuity lemma.

for a.e. fixed $s$.



$f \in L_2([0,1])$ by assumption so $|f|^2$ is integrable. Then you should show $|f|$ is integrable

## Problem 4

We consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and the Fourier transform $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

(a) For $k \in \mathbb{N}$ we define $g_k := x^k e^{-\frac{x^2}{2}}$ for $x \in \mathbb{R}$. Let us show $g_k \in \mathscr{S}(\mathbb{R})$ for all $k \in \mathbb{N}$.

From Homework 7 Problem 1, we see that $g_0 \in \mathscr{S}(\mathbb{R})$. By Homework 7 Problem 1(a), $g_0 \in \mathscr{S}(\mathbb{R})$ implies that $x^k g_0 \in \mathscr{S}(\mathbb{R})$. But this is exactly what we wanted to show, as $g_k(x) = x^k g_0(x)$.

$g_0 \in \mathcal{S}(\mathbb{R})$ is not shown explicitly in HW7P62.

Let us compute $\mathcal{F}(g_k)$ for $k = 0, 1, 2, 3$. By Proposition 11.13(c), we have the formula

$$\mathcal{F}(xf)(\xi) = i\frac{d}{d\xi}\mathcal{F}(f)(\xi)$$

We will use this recursively to determine the transformations. First, note that $\mathcal{F}(g_0)(\xi) = g_0(\xi)$ by Proposition 11.4. Noting that $g_{k+1}(x) = xg_k(x)$, we calculate:

$$\begin{aligned}
\mathcal{F}(g_1) = \mathcal{F}(xg_0)(\xi) =& i\frac{d}{d\xi}g_0(\xi) = -i\xi g_0(\xi) = i^3 g_1(\xi)\,, \\
\mathcal{F}(g_2) = \mathcal{F}(xg_1)(\xi) =& i\frac{d}{d\xi}(-ig_1(\xi)) = i^2\frac{d}{d\xi}(-\xi g_0(\xi)) \\
=& i^2(\xi^2 g_0(\xi) - g_0(\xi)) = i^2(g_2(\xi) - g_0(\xi))\,, \\
\mathcal{F}(g_3) = \mathcal{F}(xg_2)(\xi) =& i\frac{d}{d\xi}(i^2(g_2(\xi) - g_0(\xi))) = i\frac{d}{d\xi}(-(\xi^2 g_0(\xi) - g_0(\xi))) \\
=& i(-2\xi g_0(\xi) + \xi^3 g_0(\xi) - \xi g_0(\xi)) = i(g_3(\xi) - 3g_1(\xi))\,.
\end{aligned}$$

(b) Let us find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$ for $k = 0, 1, 2, 3$.

We will make heavy use of the results of the calculations in (a). We see that $\mathcal{F}(g_0) = g_0$ and $\mathcal{F}(g_1) = i^3 g_1$, and so we may choose $h_0 = g_0$ and $h_3 = g_1$. We also claim that we may choose the following: $h_1 = g_3 - \frac{3}{2}g_1$ and $h_2 = g_2 - \frac{1}{2}g_0$. The fact that these choices work will be shown in a quick calculation, using that $\mathcal{F}$ is linear:

$$\begin{aligned}
\mathcal{F}(g_3 - \frac{3}{2}g_1) =& i(g_3(\xi) - 3g_1(\xi)) - \frac{3}{2}i^3 g_1(\xi) \\
=& i(g_3(\xi) - 3g_1(\xi)) + \frac{3}{2}ig_1(\xi) = i(g_3(\xi) - \frac{3}{2}g_1(\xi))\,, \\
\mathcal{F}(g_2 - \frac{1}{2}g_0) =& i^2(g_2(\xi) - g_0(\xi)) - \frac{1}{2}g_0(\xi) \\
=& i^2(g_2(\xi) - g_0(\xi)) + i^2\frac{1}{2}g_0(\xi) = i^2(g_2(\xi) - \frac{1}{2}g_0(\xi))\,.
\end{aligned}$$

Thus, we are done.

(c) Let us show that $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$.

Define $f_-(x) = f(-x)$. First, we show that $\mathcal{F}^*(f_-) = \mathcal{F}(f)$ for all $f \in \mathscr{S}(\mathbb{R})$ using the substitution $y = -x$ and the fact that $m$ is rotation invariant.

$$\mathcal{F}^*(f_-) = \int_{\mathbb{R}} f(-x) e^{ix\xi} dm(x) = \int_{\mathbb{R}} f(y) e^{-iy\xi} dm(y) = \mathcal{F}(f)\,.$$

Now we see that $\mathcal{F}^2(f) = \mathcal{F}(\mathcal{F}(f)) = \mathcal{F}(\mathcal{F}^*(f_-)) = f_-$, using Corollary 12.12(iii), and the fact that $f_- \in \mathscr{S}(\mathbb{R})$ clearly if $f \in \mathscr{S}(\mathbb{R})$. Then it is easy to see that

$$\mathcal{F}^4(f) = \mathcal{F}^2(\mathcal{F}^2(f)) = \mathcal{F}^2(f_-) = (f_-)_- = f\,.$$

This completes the proof.

(d) Let us show that $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ has exactly the eigenvalues $\{\pm 1, \pm i\}$.

By (b), all these values are eigenvalues. By (c), any eigenvalue $\lambda$ must satisfy $\lambda^4 f = \mathcal{F}^4(f) = f$, so $\lambda^4 = 1$. Then $\lambda \in \{\pm 1, \pm i\}$, and we are done.

## Problem 5

Let $(x_n)_{n\in\mathbb{N}}$ be a dense subset of $[0,1]$, and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0,1]$. Let us show that $\operatorname{supp}(\mu) = [0,1]$.

Recall $\operatorname{supp}(\mu)$ is the complement of $N$, where $N$ is the union of all open null-sets. Take any open null-set $U$. By definition of denseness, if $U \neq \emptyset$, $\{x_n : n \in \mathbb{N}\} \cap U \neq \emptyset$, so take some $x_k$ in the intersection. Then

$$\mu(U) = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}(U) \geq 2^{-k}\delta_{x_k}(U) = 2^{-k} > 0\,.$$

This contradicts the fact that $U$ is a null-set, so we must have $U = \emptyset$. Then $N$ is empty as well, and we get that $\operatorname{supp}(\mu) = [0,1]$ as wanted.