

# Functional Analysis

## Mandatory Assignment 1

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### Problem 1.

*Solution.* (a) Let  $T : X \rightarrow Y$  be a linear map and define  $\|-\|_0$  as  $\|x\|_0 := \|x\|_X + \|Tx\|_Y$ .

To see that  $\|x\|_0$  is a norm let  $x, y \in X$  and  $\alpha \in \mathbb{K}$ , then

$$\begin{aligned}\|x + y\|_0 &= \|x + y\|_X + \|T(x + y)\|_Y \\ &= \|x + y\|_X + \|Tx + Ty\|_Y, \text{ linearity of } T \\ &\leq \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y, \text{ since } \|-\|_X \text{ and } \|-\|_Y \text{ are norms} \\ &= \|x\|_0 + \|y\|_0\end{aligned}$$

and

$$\begin{aligned}\|\alpha x\|_0 &= \|\alpha x\|_X + \|T(\alpha x)\|_Y \\ &= |\alpha| \|x\|_X + |\alpha| \|Tx\|_Y, \text{ linearity of } T \text{ and } \|-\|_X \text{ and } \|-\|_Y \text{ are norms} \\ &= |\alpha| (\|x\|_X + \|Tx\|_Y) \\ &= |\alpha| \|x\|_0.\end{aligned}$$

Finally, if  $x = 0$ , then clearly

$$\|0\|_0 = \|0\|_X + \|T0\|_Y = 0 + 0 = 0$$

and if  $\|x\|_0 = 0$ , then  $\|x\|_X + \|Tx\|_Y = 0$ , but these are both norms so positive definite and so  $\|x\|_X, \|Tx\|_Y = 0$ . Thus,  $\|x\|_X = 0$  which implies  $x = 0$  since  $\|-\|_X$  is a norm. Therefore,  $\|-\|_0$  is a norm.

Now we claim that  $\|-\|_X$  and  $\|-\|_0$  are equivalent if and only if  $T$  is bounded. First, suppose  $\|-\|_X$  and  $\|-\|_0$  are equivalent, then by definition there exists  $0 < c_1 \leq c_2 < \infty$  such that for all  $x \in X$

$$c_1 \|x\|_X \leq \|x\|_0 = \|x\|_X + \|Tx\|_Y \leq c_2 \|x\|_X.$$

Thus, we see immediately that for all  $x \in X$

$$0 \leq \|Tx\|_Y \leq (c_2 - 1) \|x\|_X$$

from which it follows that  $c_2 - 1 \geq 0$ . If  $c_2 - 1 = 0$ , then  $0 \leq \|Tx\|_Y \leq 0$  so  $\|Tx\|_Y = 0$  for all  $x \in X$  in which case  $T$  is the zero map and so bounded. In the case that  $c := c_2 - 1 > 0$ , then  $T$  is bounded by definition since  $c > 0$  is such that

$$\|Tx\|_Y \leq c\|x\|_X$$

for all  $x \in X$ .

Conversely suppose  $T$  is bounded, then by definition there exists  $c > 0$  such that  $\|Tx\|_Y \leq c\|x\|_X$  for all  $x \in X$ . Thus,

$$\|x\|_X \leq \|x\|_0 = \|x\|_X + \|Tx\|_Y \leq (c + 1)\|x\|_X$$

so that  $\|\cdot\|_X$  and  $\|\cdot\|_0$  are equivalent.

*Solution.* (b) Let  $X$  be a finite dimensional normed vector space of dimension  $d$ . Now we recall that any two norms on  $X$  are equivalent, that is, generate the same topology (theorem 1.6, lecture notes) so it is sufficient to show that if  $T : X \rightarrow Y$  is linear, then  $T$  is bounded (equivalently, continuous) for the 1-norm on  $X$ . That is, let  $e_1, \dots, e_n$  be a basis for  $X$  and  $x = \sum_{i=1}^d \alpha_i e_i, \alpha_i \in \mathbb{K}$ , then by theorem 1.6 we may assume

$$\|x\|_X = \sum_{i=1}^d |\alpha_i|.$$

Hence, let  $c = \max_{1 \leq i \leq d} \|Te_i\|_Y$ , then we have

$$\begin{aligned} \|Tx\|_Y &= \left\| \sum_{i=1}^d \alpha_i Te_i \right\|_Y \leq \sum_{i=1}^d \|\alpha_i Te_i\|_Y \\ &= \sum_{i=1}^d |\alpha_i| \|Te_i\|_Y \\ &\leq \sum_{i=1}^d |\alpha_i| c \\ &= c\|x\|_X \end{aligned}$$

Therefore, since this holds for all  $x \in X$ , then  $T$  is bounded with respect to the topology induced by the one norm and therefore continuous with respect to any norm induced topology on  $X$ .

*Solution.* (c) Let  $X$  be infinite dimensional and  $(e_i)_{i \in I}$  be a Hamel basis for  $X$  where we note that  $I$  is at least countable since  $X$  is infinite dimensional. Furthermore, we may assume  $\|e_i\| = 1$  for all  $i \in I$  by normalization<sup>1</sup>. Thus, since  $Y \neq 0$  let  $0 \neq y \in Y$  be fixed and consider the unbounded sequence  $(ny)_{n \geq 1}$ . Let  $(y_i)_{i \in I}$  be any family of elements in  $Y$  which contains the sequence  $(ny)_{n \geq 1}$ . Note that such a family exists by letting  $\varphi : \mathbb{N} \hookrightarrow I$  be any injection, then let  $(y_i)_{i \in I}$  be the sequence

$$y_i = \begin{cases} ny, & \text{if } i = \varphi(n) \\ 0, & \text{otherwise} \end{cases}$$

which is well-defined since  $\varphi$  is an injection. Hence, since  $(e_i)_{i \in I}$  is a Hamel basis, then there exists a unique linear map  $T : X \rightarrow Y$  such that  $Te_i = y_i$ . Thus,  $T$  is clearly unbounded since for each  $c > 0$  there exists a sufficiently large  $n$  such that

$$c = c\|e_i\| = c\|e_{\varphi(n)}\| < n\|y\| = \|ny\| = \|Te_{\varphi(n)}\| = \|Te_i\|,$$

that is,  $T$  is unbounded since the sequence  $(ny)_{n \geq 1}$  is unbounded and therefore,  $T$  is not continuous.

*Solution.* (d) Let  $X$  be infinite dimensional and let  $T : X \rightarrow Y$  be a discontinuous linear map which exists by part (c). Then by part (a),  $\|x\|_0 = \|x\|_X + \|Tx\|_Y$  is a norm on  $X$  and  $\|\cdot\|_0$  and  $\|\cdot\|_X$  are not equivalent since  $T$  is not bounded. Additionally, it is clear that

$$\|x\|_X \leq \|x\|_0 = \|x\|_X + \|Tx\|_Y$$

for all  $x \in X$  as desired. Now suppose  $(X, \|\cdot\|_X)$  is complete, that is, a Banach space, and for the purpose of contradiction suppose that  $(X, \|\cdot\|_0)$  is also a Banach space, then by homework 3 problem 1 the norms  $\|\cdot\|_X$  and  $\|\cdot\|_0$  must be equivalent since  $\|\cdot\|_X \leq \|\cdot\|_0$ , a contradiction by part (a) since  $T$  is not bounded.

*Solution.* (e) Consider  $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|_1)$  and let  $\|\cdot\|' = \|\cdot\|_2$  the standard 2-norm. Then recall that  $\|\cdot\|_2 \leq \|\cdot\|_1$  and we have an inclusion  $\ell_1(\mathbb{N}) \subsetneq \ell_2(\mathbb{N})$ . Thus, to see that  $(\ell_1(\mathbb{N}), \|\cdot\|_2)$  is not complete it is sufficient to show that  $\ell_1(\mathbb{N})$  is not closed in  $\ell_2(\mathbb{N})$ . Hence, let  $(x_n)_{n \in \mathbb{N}}$  be the sequence in  $\ell_1(\mathbb{N})$  where

$$x_n(k) = \begin{cases} \frac{1}{k}, & k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

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<sup>1</sup>This follows since every element may be written as a unique linear combination of  $e_i$  over  $I$  with finite support which will clearly still hold after normalizing the basis vectors.

Then clearly  $x_n \in \ell_1(\mathbb{N})$  since  $x_n$  has compact support for all  $n$  and  $c_c(\mathbb{N}) \subsetneq \ell_1(\mathbb{N})$ . Now we claim that  $(x_n)_{n \in \mathbb{N}}$  converges to the sequence  $x = (1/k)_{k \in \mathbb{N}}$  under  $\|-\|_2$ . To see this observe that

$$\|x_n - x\|_2 = \left( \sum_{k=1}^{\infty} |x_n(k) - x(k)|^2 \right)^{1/2} = \left( \sum_{k=n+1}^{\infty} 1/k^2 \right)^{1/2}$$

so  $\|x_n - x\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ . Finally,  $x \notin \ell_1(\mathbb{N})$  since  $\sum_{k=1}^{\infty} |1/k|$  diverges. Therefore,  $\ell_1(\mathbb{N})$  is not closed in  $\ell_2(\mathbb{N})$  and so  $(\ell_1(\mathbb{N}), \|-\|_2)$  is not complete while  $(\ell_1(\mathbb{N}), \|-\|_1)$  is complete and  $\|-\|_2 \leq \|-\|_1$ .

## Problem 2.

*Solution.* (a) First, recall in part (b) of problem 1 we have that any linear map  $f : X \rightarrow Y$  is bounded if  $X$  is finite dimensional. Now it is clear that  $M$  is a finite dimensional subspace of  $\ell_p(\mathbb{N})$  and that  $f : M \rightarrow \mathbb{C}$  is linear so  $f$  is bounded.

Now we claim that  $\|f\| = n^{1-1/p}$  where  $n = 2$  is the dimension of  $M$ . To see this recall that for  $t > 1$  Hölder's inequality states that for  $x = (x_k)_{k=1}^n, y = (y_k)_{k=1}^n$ , then

$$\sum_{k=1}^n |x_k y_k| \leq \left( \sum_{k=1}^n |x_k|^t \right)^{1/t} \cdot \left( \sum_{k=1}^n |y_k|^{\frac{t}{t-1}} \right)^{1-\frac{1}{t}} \quad (1)$$

since  $\frac{1}{t} + \frac{1}{\frac{t}{t-1}} = 1$ . Thus, let  $0 < r < p$  and applying (1) with  $x_k = |x_k|^r, y_k = 1$ , and  $t = p/r > 1$ , then

$$\sum_{k=1}^n |x_k|^r \leq \left( \sum_{k=1}^n (|x_k|^r)^{\frac{p}{r}} \right)^{\frac{r}{p}} \cdot \left( \sum_{k=1}^n 1^{\frac{p}{p-r}} \right)^{1-\frac{r}{p}} = \left( \sum_{k=1}^n |x_k|^p \right)^{\frac{r}{p}} \cdot n^{1-\frac{r}{p}}.$$

Hence, taking the  $r^{th}$  root we get

$$\|x\|_r = \left( \sum_{k=1}^n |x_k|^r \right)^{\frac{1}{r}} \leq \left( \left( \sum_{k=1}^n |x_k|^p \right)^{\frac{r}{p}} \cdot n^{1-\frac{r}{p}} \right)^{\frac{1}{r}} = \left( \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} \cdot n^{\frac{1}{r}-\frac{1}{p}} = \|x\|_p \cdot n^{\frac{1}{r}-\frac{1}{p}}.$$

Now let  $r = 1$  and  $1 < p < \infty$ , then

$$\|x\|_p \leq \|x\|_1 \leq \|x\|_p \cdot n^{1-\frac{1}{p}}$$

and since the norm of  $f$  may be calculated as,  $\|f\| = \sup_{\|x\|_p=1} |fx|$ , then for the  $x = (a, b) \in M$  with  $\|x\|_p = 1$  we get

$$1 \leq |x| = |a| + |b| \leq 2^{1-1/p}.$$

Now by the triangle inequality for all  $x = (a, b) \in M$  with  $\|x\|_p = 1$  we have

$$|fx| = |a + b| \leq |a| + |b| \leq 2^{1-1/p}.$$

Hence,  $\|f\| \leq 2^{1-1/p}$  and to see that  $2^{1-1/p} \leq \|f\|$ , let  $a, b = \frac{2^{1-1/p}}{2}$  and let  $x = (a, b)$ , then

$$\|x\|_p = \left( \left( \frac{2^{1-1/p}}{2} \right)^p + \left( \frac{2^{1-1/p}}{2} \right)^p \right)^{1/p} = \left( \frac{2^{p-1}}{2^p} + \frac{2^{p-1}}{2^p} \right)^{1/p} = \left( \frac{1}{2} + \frac{1}{2} \right)^{1/p} = 1$$

and

$$|fx| = |a + b| = \frac{2^{1-1/p}}{2} + \frac{2^{1-1/p}}{2} = 2^{1-1/p}.$$

Therefore,  $2^{1-1/p} \leq \|f\|$  and so  $\|f\| = 2^{1-1/p}$ .

*Solution.* (b) First, observe that  $F : \ell_p(\mathbb{N}) \rightarrow \mathbb{C}$  defined by

$$F(a, b, x_1, x_2, \dots) = f(a, b)$$

is an extension of  $f$  with  $\|F\| = \|f\|$ . This follows since clearly  $F|_M = f$  and for  $x = (a_1, b_1, x_1, \dots), y = (a_2, b_2, y_1, \dots) \in \ell_p(\mathbb{N})$  and  $\alpha, \beta \in \mathbb{C}$  we have

$$F(\alpha x + \beta y) = f(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2) = \alpha f(a_1, b_1) + \beta f(a_2, b_2) = \alpha F(x) + \beta F(y)$$

since  $f$  is linear and

$$\|F\| = \sup_{\|x\|_p=1} |F(x)| = \sup_{\|x\|_p=1} |f(a_1, b_1)| = \sup_{\|(a_1, b_1)\|_p=1} |f(a_1, b_1)| = \|f\|.$$

Now recall from homework 1 problem 5 for  $1 < p < \infty$  there is an isometric isomorphism

$$(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$$

where  $q$  is such that  $1/p + 1/q = 1$  and the map  $\psi : (\ell_p(\mathbb{N}))^* \xrightarrow{\cong} \ell_q(\mathbb{N})$  is defined in the following way. Let  $(e_k)_{k \in \mathbb{N}}$  be the collection of elements in  $\ell_p(\mathbb{N})$  such that  $e_k(i) = \delta_{ik}$  where  $\delta_{ik}$  is the Kronecker delta, then for  $G \in (\ell_p(\mathbb{N}))^*$

$$\psi(G) = (G(e_k))_{k \in \mathbb{N}}.$$

Additionally,  $\psi$  is an isometry so  $\|\psi(G)\|_q = \|G\|$  where we also observe that

$$\psi(F) = (f(1, 0), f(0, 1), 0, 0, \dots) = (1, 1, 0, 0, \dots).$$

Now let  $G$  be any other extension of  $f$  such that  $\|G\| = \|f\|$ , then  $\psi(G) = (G(e_k))_{k \in \mathbb{N}}$  and  $\|\psi(G)\|_q = \|G\| = 2^{1-1/p} = 2^{1/q}$ . Thus, since  $G$  is an extension of  $f$ , then  $G(e_1) = f(e_1) = 1$  and  $G(e_2) = f(e_2) = 1$ . Hence,  $\psi(G) = (1, 1, G(e_3), \dots)$ , but then

$$\|\psi(G)\| = \left( \sum_{k=1}^{\infty} |G(e_k)|^q \right)^{1/q} = \left( 1 + 1 + \sum_{k=3}^{\infty} |G(e_k)|^q \right)^{1/q} = 2^{1/q} = 2^{1-1/p}.$$

However,  $(1+1)^{1/q} = 2^{1/q}$  so

$$\sum_{k=3}^{\infty} |G(e_k)|^q = 0$$

which since  $|G(e_k)| \geq 0$  implies that  $G(e_k) = 0$  for  $k \geq 3$ . Thus,  $\psi(F) = \psi(G)$  which is an isomorphism so in particular injective, therefore,  $F = G$ .

*Solution.* (c) Similarly to  $M$  let  $M_k$  denote the subspace of  $(\ell_1(\mathbb{N}), \|\cdot\|_1)$  defined as

$$M_k = \{(x_1, x_2, x_3, \dots, x_k, 0, \dots) : x_i \in \mathbb{C}\}$$

and let  $g_k : M_k \rightarrow \mathbb{C}$  denote the linear functional defined by  $g_k(x_1, \dots, x_k, 0, \dots) = \sum_{i=1}^k x_k$ . Now for the same reasons as in part (b) that  $F$  was an extension of  $f$ , then for each  $g_k$  we obtain an extension  $G_k : \ell_1(\mathbb{N}) \rightarrow \mathbb{C}$  defined by

$$G_k(x_1, \dots, x_k, x_{k+1}, \dots) = g_k(x_1, \dots, x_k).$$

Now observe that each  $G_k$  for  $k \geq 2$  is an extension of  $f$  since for  $x = (a, b, 0, 0, \dots)$  we have

$$G_k(a, b, 0, 0, \dots) = g_k(a, b, 0, 0, \dots) = a + b = f(a, b)$$

and note that for the elements  $(e_j)_{j \in \mathbb{N}}$  in  $\ell_1(\mathbb{N})$  where  $e_j(i) = \delta_{ji}$  we have

$$G_k(e_j) = g_k(e_j) = \begin{cases} 1, & j \leq k \\ 0, & \text{otherwise.} \end{cases}$$

Again from homework 1 problem 5 we have an isometric isomorphism

$$\begin{aligned} \psi : \ell_1(\mathbb{N})^* &\rightarrow \ell_\infty(\mathbb{N}) \\ G &\mapsto (G(e_j))_{j \in \mathbb{N}}. \end{aligned}$$

Thus, it follows that for  $k \geq 1$

$$\|G_k\| = \|\psi(G_k)\|_\infty = \sup_{j \in \mathbb{N}} |G_k(e_j)| = 1.$$

Therefore, the family  $(G_k)_{k \geq 2}$  is an infinite collection of linear functionals on  $\ell_1(\mathbb{N})$  which extend  $f$  and  $\|G_k\| = 1 = 2^{1-1/1} = \|f\|$ .

**Problem 3.**

*Solution.* (a) Let  $1 \leq n \in \mathbb{Z}$  and let  $F : X \rightarrow \mathbb{K}^n$  be a linear map where  $X$  is infinite dimensional. Let  $(e_i)_{i \in I}$  be a Hamel basis for  $X$  and choose  $e_1, \dots, e_d$  from  $(e_i)_{i \in I}$  with  $d > n$ . Then  $e_1, \dots, e_d$  are linearly independent<sup>2</sup> and span a subspace  $M \subset X$  of dimension  $d > n$ . Now the restriction,  $F|_M$  to  $M$  is necessarily linear and cannot be injective since  $\dim M = d > n$  and therefore,  $F$  is not injective.

*Solution.* (b) Let  $1 \leq n \in \mathbb{Z}$  and  $f_1, \dots, f_n \in X^*$ , then define

$$\begin{aligned} F : X &\rightarrow \mathbb{K}^n \\ x &\mapsto (f_1(x), \dots, f_n(x)). \end{aligned}$$

This is clearly well-defined and linear since  $f_1, \dots, f_n$  are. Thus,  $F : X \rightarrow \mathbb{K}^n$  is a linear map where  $\dim X = \infty$  so by part (a)  $F$  is not injective. Hence, there exists  $0 \neq x \in X$  such that

$$F(x) = (f_1(x), \dots, f_n(x)) = (0, \dots, 0) = 0.$$

Thus,  $x \in \ker f_i$  for  $1 \leq i \leq n$  and so

$$0 \neq x \in \bigcap_{i=1}^n \ker(f_i)$$

and we are done.

*Solution.* (c) Let  $x_1, \dots, x_n \in X$ , then by theorem 2.7(b) from the lectures there exists  $f_1, \dots, f_n \in X^*$  with  $f_i(x_i) = \|x_i\|$  and  $\|f_i\| = 1$ . Thus, by part (b) there exists  $0 \neq y \in K = \bigcap_{i=1}^n \ker(f_i)$  where we may assume that  $\|y\| = 1$  since if  $y \in \ker(f_i)$  for all  $1 \leq i \leq n$ , then so is  $y/\|y\|$  by linearity. Hence, we have  $f_i(y - x_i) = f_i(y) - f_i(x_i) = -\|x_i\|$  since  $y \in \ker(f_i)$  for  $1 \leq i \leq n$ . Therefore, it follows that

$$\|x_i\| = |f_i(y - x_i)| \leq \|f_i\| \cdot \|y - x_i\| = \|y - x_i\|$$

where the inequality follows from the fact that  $f_i$  is bounded and so  $\|f_i(x)\| \leq \|f_i\| \cdot \|x\|$  for all  $x \in X$  by lecture 1, equation 1.8. Thus, there exists  $y \in X$  such that  $\|y\| = 1$  and  $\|y - x_i\| \geq \|x_i\|$  for  $1 \leq i \leq n$ , as desired.

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<sup>2</sup>Note this clearly follows from the facts about Hamel bases introduced in problem 1.

*Solution.* (d) Let  $x_1, \dots, x_n \in X$  and let  $\overline{B}_i = \overline{B(x_i, r_i)}$  denote closed balls centered at  $x_i$  of radius  $r_i > 0$ , that is,

$$\overline{B}_i = \{y \in X : \|y - x_i\| \leq r_i\},$$

and suppose that  $\overline{B}_1, \dots, \overline{B}_n$  cover,  $S$ , the unit sphere in  $X$ . Then by part (c) there exists  $y \in X$  such that  $\|y\| = 1$  and  $\|y - x_i\| \geq \|x_i\|$  for  $1 \leq i \leq n$ . Now since  $\|y\| = 1$ , then  $y \in S$  so  $y \in \overline{B}_k$  for some  $1 \leq k \leq n$ . Thus,

$$\|x_k\| \leq \|y - x_k\| \leq r_k$$

which implies that  $0 \in \overline{B}_k$  since  $\|x_k\| = \|x_k - 0\| \leq r_k$ , as desired.

*Solution.* (e) Consider the cover of  $S$  given by  $\mathcal{B} = \{B(x, 1)\}_{x \in S}$  where  $B(x, 1)$  is the open unit ball in  $X$  centered at  $x \in S$  and we claim that  $\mathcal{B}$  has no finite subcover. Suppose, for the purpose of contradiction, that  $B_i = B(x_i, 1)$  for some  $x_1, \dots, x_n \in S$  covers  $S$ . Then by part (c) there exists  $y \in X$  with  $\|y\| = 1$  such that  $1 = \|x_i\| \leq \|y - x_i\|$  for all  $1 \leq i \leq n$ , but  $y \in S$  and since  $B_i$  cover  $S$ , then  $y \in B_k$  for some  $k$ . Thus,  $1 = \|x_i\| \leq \|y - x_i\| < 1$ , a contradiction. Therefore,  $\mathcal{B}$  is an open cover of  $S$  with no finite subcover and so  $S$  is non-compact.

Now to see that  $\overline{B(0, 1)}$ , the closed unit ball in  $X$  is not compact observe that  $S = \overline{B(0, 1)} \setminus B(0, 1)$ . Hence,  $S$  is closed in  $\overline{B(0, 1)}$  since  $B(0, 1)$  is open in  $\overline{B(0, 1)}$ . Thus, if  $\overline{B(0, 1)}$  were compact, then  $S$  must be compact since closed subsets of compact sets are compact, but  $S$  is a closed subset of  $\overline{B(0, 1)}$  which is not compact and therefore,  $\overline{B(0, 1)}$  is not compact.

Alternatively, the non-compactness of  $\overline{B(0, 1)}$  can be seen by considering the cover  $\{B(x, 1)\}_{x \in S} \cup \{B(0, 1)\}$  which has no finite subcover since  $\{B(x, 1)\}_{x \in S}$  has no finite subcover of  $S$  and  $S \cap B(0, 1) = \emptyset$ .

#### Problem 4.

*Solution.* (a) Let  $n \geq 1$ , then  $E_n$  is not absorbing. To see this recall that there is a strict inclusion  $L_3([0, 1], m) \subsetneq L_1([0, 1], m)$ . Hence, let  $f \in L_1([0, 1], m) \setminus L_3([0, 1], m)$ , then

$$\|f\|_1 = \int_{[0,1]} |f| dm < \infty \text{ and } \|f\|_3 = \left( \int_{[0,1]} |f|^3 dm \right)^{1/3} = \infty.$$

Thus, for any  $t > 0$

$$\|t^{-1}f\|_3^3 = \int_{[0,1]} |t^{-1}f|^3 dm = t^{-3} \int_{[0,1]} |f|^3 dm = \infty$$

so that  $t^{-1}f \notin E_n$ . Thus,  $E_n$  is not absorbing by definition.



*Solution.* (b) First, by definition  $E_n \subset L_3([0, 1], m)$  and we know that  $L_3([0, 1], m) \subsetneq L_1([0, 1], m)$ . Hence, there exists  $g \in L_1([0, 1], m) \setminus L_3([0, 1], m)$  so that in particular  $g \notin E_n$ . Now let  $f \in E_n$ , then we claim that for every open ball  $B_{L_1}(f, \epsilon)$  there exists  $\tilde{g} \notin E_n$  such that  $\tilde{g} \in B_{L_1}(f, \epsilon)$  which by definition implies that  $E_n$  has no interior points. Hence, let  $\epsilon > 0$  be arbitrary, then there exists  $g \in L_1([0, 1], m) \setminus L_3([0, 1], m)$  so  $g \notin E_n$ . Define

$$\tilde{g} = f + \frac{\epsilon}{2\|g\|_1}g,$$

then since  $\frac{\epsilon}{2\|g\|_1} > 0$  and  $E_n$  is not absorbing  $\tilde{g} \notin E_n$ . However,

$$\|f - \tilde{g}\|_1 = \left\| \frac{\epsilon g}{2\|g\|_1} \right\|_1 = \frac{\epsilon}{2} \left\| \frac{g}{\|g\|_1} \right\|_1 = \frac{\epsilon}{2} < \epsilon$$

so  $\tilde{g} \in B_{L_1}(f, \epsilon)$ . Therefore, for every  $f \in E_n$  and open ball  $B_{L_1}(f, \epsilon)$  at  $f$  there is a  $\tilde{g}$  such that  $\tilde{g} \in B(f, \epsilon)$  and  $\tilde{g} \notin E_n$  which implies that  $\text{int}(E_n) = \emptyset$  by definition.

*Solution.* (c) Let  $(f_k)_{k \in \mathbb{N}} \subset E_n$  be a sequence converging to  $f \in L_1([0, 1], m)$  under  $\|\cdot\|_1$ . Now we claim that  $f \in E_n$  so that  $E_n$  is closed. Recall that since  $f_k \rightarrow f$ , then there exists a subsequence  $(f_{k_i})_{i \in \mathbb{N}}$  such that  $f_{k_i}(x) \rightarrow f(x)$  almost everywhere (see for example Schilling, corollary 13.8). Hence, without loss of generality we may assume that  $f_k(x) \rightarrow f(x)$  almost everywhere since such a subsequence will converge to  $f$ . Now  $|f(x)|^3 = \lim_{k \rightarrow \infty} |f_k(x)|^3$  almost everywhere since  $f_k(x) \rightarrow f(x)$  almost everywhere. Hence, since  $|f_k|^3, |f|^3 \geq 0$  are positive measurable functions for all  $k$  we may apply Fatou's lemma to get

$$\int_{[0,1]} |f|^3 \leq \liminf_{k \rightarrow \infty} \int_{[0,1]} |f_k|^3 dm \leq n$$

for all  $k$  since  $f_k \in E_n$ . Therefore,  $f \in E_n$  as desired.

*Solution.* (d) First, by part (c)  $E = \overline{E}$  since  $E$  is closed and by part (b)  $\text{int}(\overline{E}) = \text{int}(E) = \emptyset$ . Thus,  $E_n \subset L_1([0, 1], m)$  is nowhere dense in  $L_1([0, 1], m)$ . Additionally, we clearly have  $L_3([0, 1], m) = \cup_{i=1}^{\infty} E_n$ . This follows since if  $f \in E_n$ , then by definition of  $E_n$ ,  $f \in L_3([0, 1], m)$  and if  $f \in L_3([0, 1], m)$ , then by definition

$$\|f\|_3^3 = \int_{[0,1]} |f|^3 dm < \infty$$

so there exists  $n \in \mathbb{N}$  such that

$$\|f\|_3^3 = \int_{[0,1]} |f|^3 dm \leq n$$

so that  $f \in E_n$ . Therefore,  $\{E_n\}_{n \in \mathbb{N}}$  is a collection of nowhere dense sets subsets of  $L_1$  such that  $L_3([0, 1], m) = \cup_{n \in \mathbb{N}} E_n \subset L_1([0, 1], m)$  so  $L_3([0, 1], m)$  is of first category in  $L_1([0, 1], m)$ .

**Problem 5.**

*Solution.* (a) We recall the reverse triangle inequality, that is, that for any  $x, y$  in a normed vector space  $X$  we have

$$|||x|| - ||y||| \leq \|x - y\|.$$

Now suppose  $x_n \rightarrow x$  as  $n \rightarrow \infty$  in norm, then for all  $\epsilon > 0$  there exists integer  $N > 0$  such that for all  $n > N$

$$\|x_n - x\| < \epsilon.$$

Thus, applying the reverse triangle inequality we see immediately that

$$|||x_n|| - ||x||| \leq \|x_n - x\| < \epsilon$$

so that  $\|x_n\| \rightarrow \|x\|$  as  $n \rightarrow \infty$ .

*Solution.* (b) Let  $(e_n)_{n \in \mathbb{N}}$  be a (countable) orthonormal basis in  $H$ , that is,  $\langle e_i, e_j \rangle = \delta_{ij}$  where  $\delta_{ij}$  is the Kronecker delta. Note that  $H$  has a countable orthonormal basis since  $H$  is an infinite dimensional separable Hilbert space (see Folland proposition 5.29, pg. 176). Now we claim that  $e_n \rightarrow 0$  weakly as  $n \rightarrow \infty$ . Observe that if this holds, then we have a counterexample since  $\|e_n\| = 1$  is constant so  $\|e_n\| \rightarrow 1$  as  $n \rightarrow \infty$ , but  $1 \neq 0 = \|0\|$ .

To show that  $e_n \rightarrow 0$  weakly as  $n \rightarrow \infty$  recall that 0 has a neighborhood basis given by sets of the form

$$B_X(0, f_1, \dots, f_n, r) = \bigcap_{i=1}^n \{x \in H : |f_i(x)| < r\}.$$

Hence, to see convergence of  $(e_k)_{k \in \mathbb{N}}$  to 0 in the weak topology it is sufficient to check that  $(e_k)_{k \in \mathbb{N}}$  is eventually in  $B_X(0, f_1, \dots, f_n, r)$ . Thus, since  $H$  is a Hilbert space, then by the Riesz representation theorem for each  $f_i \in X^*$  there exists a unique  $y_i \in H$  such that  $f_i = \langle -, y_i \rangle$ . Now  $(e_k)_{k \in \mathbb{N}}$  is an orthonormal basis so we may write  $y_i$  uniquely as

$$y_i = \sum_{k \geq 1} \alpha_k(i) e_k, \quad \alpha_k(i) \in \mathbb{K}$$

and where  $\alpha_k(i) \neq 0$  for finitely many  $k$ . Thus, for  $K$  sufficiently large  $\alpha_k(i) = 0$  for all  $k > K$ . Additionally, we have that

$$\alpha_k(i) = \langle y_i, e_k \rangle = \overline{\langle e_k, y_i \rangle} = \overline{f_i(e_k)}.$$

Thus, for all  $k > K$ ,  $f_i(e_k) = 0 < r$  for  $1 \leq i \leq n$  and so  $(e_k)_{k \in \mathbb{N}}$  is eventually in  $B_X(0, f_1, \dots, f_n, r)$  which was arbitrary so we are done.

*Solution.* (c) Suppose  $\|x_n\| \leq 1$  for all  $n \geq 1$  and that  $x_n \rightarrow x$  weakly as  $n \rightarrow \infty$ , then we claim that  $\|x\| \leq 1$ .

Let  $f \in H^*$  be the unique linear functional represented by  $x$ , that is,  $f(y) = \langle y, x \rangle$  for all  $y \in H$  and let  $\epsilon > 0$ , then

$$B(x, f, \epsilon) = \{y \in H : |f(y - x)| < \epsilon\}$$

is a neighborhood of  $x$  in the weak topology by definition of neighborhood basis at 0. Thus, since  $x_n \rightarrow x$  weakly, then eventually  $(x_n)_{n \geq 1}$  is in  $B(x, f, \epsilon)$ . Hence, there exists  $N \geq 1$  such that for all  $n \geq N$

$$|f(x_n - x)| = |\langle x_n - x, x \rangle| = |\langle x_n, x \rangle - \|x\|^2| < \epsilon$$

and so by the reverse triangle inequality we have

$$||\langle x_n, x \rangle| - \|x\|^2| \leq |\langle x_n, x \rangle - \|x\|^2| < \epsilon.$$

Thus, the sequence  $(|\langle x_n, x \rangle|)_{n \geq 1}$  converges to  $\|x\|^2$ . Additionally, by the Cauchy-Schwarz inequality and since  $\|x_n\| \leq 1$  for all  $n \geq 1$ , then

$$|\langle x_n, x \rangle| \leq \|x_n\| \cdot \|x\| \leq \|x\|.$$

Hence, we must have  $\|x\|^2 \leq \|x\|$  since  $|\langle x_n, x \rangle|$  converges to  $\|x\|^2$  and  $|\langle x_n, x \rangle|$  is bounded by  $\|x\|$ . Therefore,  $\|x\|^2 \leq \|x\|$  which implies  $\|x\| \leq 1$ , as desired.