Stability of the $N+1$ Fermi gas with point interactions

Advanced Mathematical Physics

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May 15, 2020.

Overview

1. Motivation

2. Moser and Seiringer: Stability of $N+1$ system

3. Model

Motivation

Model of fermions interacting via point interactions are of great interest as the appear as

- Models of cold atomic gases.
- Models of nuclear interaction.
- Approximations of models with short-range interactions.

However, they are mathematically not very well understood.

Thomas collapse

Thomas collaps: It it known that a bosonic system of three or more bosons with zero-range interactions is unstable (of the first kind) i.e. there is no ground state energy. This can be seen from the variational principle.

No Thomas collapse for (spin-1/2) fermions: The Thomas collapse is a collective phenomenon where three (or more) bosons interacts in a single point. This can never happen for spin-$1/2$ fermions, due to the Pauli principle.

Stability of the first kind: Is still an unsolved problem for general $N+M$ systems (N spin up and M spin down).

Results of Moser and Seiringer

Results

- Prove stability of the $N+1$ system, within a certain mass ratio interval.
- Prove existence of self-adjoint bounded from below Hamiltonian.
- Prove Tan relations.

We focus on the first two.

Formal Hamiltonian

Formal Hamiltonian

The Hamiltonian of a system of N fermions of one species of mass 1 interacting via 1 fermion of another species of mass m can be described by the formal Hamiltonian

$$H = -\frac{1}{2m}\Delta_{x_0} - \frac{1}{2}\sum_{i=1}^{N}\Delta_{x_i} + \gamma\sum_{i=1}^{N}\delta(x_i - x_0) \tag{1}$$

Formal Hamiltonian

Centre of mass separation

We can split the Hamiltonian in two

$$H = H_{\mathsf{CM}} + \frac{m+1}{2m} H_{\mathsf{rel}}, \tag{2}$$

with $x_{\mathsf{cm}} = (m x_0 + \sum_{i=1}^{N} x_i)/(m+N)$, $y_i = x_i - x_0$, and

$$\begin{aligned} H_{\mathsf{CM}} &= \frac{1}{2(N+m)} \Delta_{x_{\mathsf{cm}}}, \\ H_{\mathsf{rel}} &= -\sum_{i=1}^{N} \Delta_{y_i} - \frac{2}{m+1} \sum_{1 \le i < j \le N} \nabla_{y_i} \cdot \nabla_{y_j} + \tilde{\gamma} \sum_{i=1}^{N} \delta(y_i) \end{aligned} \tag{3}$$

Quadratic form

The formal Hamiltonian can be given precise meaning through a quadratic form, which can be obtained by considering more regularized models such as rank-one perturbations of a free Hamiltonian. One obtains

$$
\begin{aligned}
F_\alpha(u) = \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k)^{-1} |\hat{w}|^2 - \mu \|u\|_{L^2(\mathbb{R}^{3N})} \\
+ N \left(T_{\mathsf{diag}}(\xi) + T_{\mathsf{off}}(\xi) + \alpha \|\xi\|^2_{L^2(\mathbb{R}^{3(N-1)})} \right)
\end{aligned}
\tag{4}
$$

with $\hat{u}(k) = \hat{w}(k) + \sum_{i=1}^N (-1)^{i-1} \hat{G}(k) \xi(\bar{k}^i)$, $\mu > 0$, $\hat{G}(k) = \left(\sum_{i=1}^N k_i^2 + \frac{2}{m+1} \sum_{1 \le i < j \le N} k_i \cdot k_j + \mu \right)^{-1}$,

$$T_{\mathsf{diag}}(\xi) = \int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N L(\bar{k}^N) \left|\xi(\bar{k}^N)\right|^2,$$

$$T_{\mathsf{off}}(\xi) = (N-1) \int_{\mathbb{R}^{3(N-2)}} \mathrm{d}\bar{q} \int_{\mathbb{R}^3} \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}t \overline{\xi(s,\bar{q})} \hat{G}(s,t,\bar{q}) \xi(t,\bar{q}). \tag{5}$$

with

$$L(\bar{k}^N) = 2\pi^2 \left(\frac{m(m+2)}{(m+1)^2} \sum_{i=1}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{1 \le i < j \le N-1} k_i \cdot k_j + \mu \right)^{1/2}. \tag{6}$$

The domain is

$$\mathscr{D}(F_\alpha) = \Big\{ u \in L^2_{\mathsf{as}}(\mathbb{R}^{3N}) \;\Big|\; \hat{u} = \hat{w} + \widehat{\rho G},\ w \in H^1_{\mathsf{as}}(\mathbb{R}^{3N}),\ \xi \in H^{1/2}_{\mathsf{as}}(\mathbb{R}^{3(N-1)}) \Big\}. \tag{7}$$

Thank you for your attention.