Functional Analysis
Mandatory Assignment 2

Thais James Minet (vtp239)

Problem 1. Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{i=1}^{N^2} e_n$, for all $N \geq 1$.

(a) Show that $f_N \to 0$ weakly, as $N \to \infty$, while $\|f_N\| = 1$, for all $N \geq 1$.

(b) Let K be the norm closure of $\mathrm{co}\{f_N : N \geq 1\}$. Argue that K is weakly compact, and that $0 \in K$.

(c) Show that 0, as well as each f_N, $N \geq 1$, are extreme points in K.

(d) Are there any other extreme points in K? Justify your answer.

Solution. (a) Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$ and let $f_N = N^{-1}\sum_{i=1}^{N^2} e_n$. By homework 4 problem 2(a) to show that $f_N \to 0$ weakly as $N \to \infty$ it is sufficient to show that for all $g \in H^*$, $g(f_N) \to g(0) = 0$ as $N \to \infty$. Hence, let $g \in H^*$, then by the Riesz representation theorem there exists a unique $y \in H$ such that $g = \langle -, y\rangle$. Additionally, since $(e_n)_{n\geq 1}$ is an orthonormal basis for H, then y may be written as $y = \sum_{j=1}^{k} \lambda_j e_{t_j}$ for some finite k. Thus,

$$\begin{aligned} g(f_N) &= \langle f_N, y\rangle \\ &= \langle \frac{1}{N}\sum_{i=1}^{N^2} e_i, \sum_{j=1}^{k} \lambda_j e_{t_j}\rangle \\ &= \frac{1}{N}\sum_{i=1}^{N^2}\sum_{j=1}^{k} \overline{\lambda_j}\langle e_i, e_{t_j}\rangle \\ &= \frac{1}{N}\sum_{i=1}^{N^2}\sum_{j=1}^{k} \overline{\lambda_j}\delta_{it_j} \end{aligned}$$

and clearly

this is not immediate

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N^2}\sum_{j=1}^{k} \overline{\lambda_j}\delta_{it_j} = 0.$$

Therefore, $f_N \to 0$ weakly as $N \to \infty$. (✓)

Finally, we have $\|f_N\| = 1$ since

$$\begin{aligned}
\|f_N\|^2 &= \langle f_N, f_N \rangle \\
&= \langle \frac{1}{N}\sum_{i=1}^{N^2} e_i, \frac{1}{N}\sum_{j=1}^{N^2} e_j \rangle \\
&= \frac{1}{N^2}\sum_{i=1}^{N^2}\sum_{j=1}^{N^2} \langle e_i, e_j \rangle \\
&= \frac{1}{N^2}\sum_{i=1}^{N^2}\sum_{j=1}^{N^2} \delta_{ij} \\
&= \frac{1}{N^2} N^2 \\
&= 1.
\end{aligned}$$

✓

Solution. (b) Let $F = \{f_N : N \geq 1\}$ so that $K = \overline{\text{co}(F)}^{\|-\|}$. Now since $\|f_N\| = 1$, then $f_N \in \overline{B(0,1)}^{\|-\|}$ the closed unit ball in the norm and since $\overline{B(0,1)}^{\|-\|}$ is convex it follows that $\text{co}(F) \subset \overline{B(0,1)}^{\|-\|}$ by definition of $\text{co}(F)$. Thus, $K \subset \overline{B(0,1)}^{\|-\|}$ and since H is reflexive, then the w^*-topology and w-topologies agree so $\overline{B(0,1)}^{\|-\|}$ is compact in the weak topology. Now since $\text{co}(F)$ is convex, then so is $K = \overline{\text{co}(F)}$ since the closure of a convex set is convex. Thus, by theorem 5.7

Be explicit about H being a dual space when talking about w*-topology.

why?

$$K = \overline{\text{co}(F)}^{\|-\|} = \overline{\text{co}(F)}^{\tau_w}$$

so that K is weakly closed. It follows that since (H, τ_w) is a Hausdorff space, $\overline{B(0,1)}^{\|-\|}$ is compact, and $K \subset \overline{B(0,1)}^{\|-\|}$ is closed, then K is compact as desired. Finally, since K is closed in τ_w and $f_N \to 0$ weakly as $N \to \infty$ it follows that $0 \in K$ and we are done. ✓

Solution. (c) Let $x \in K = \overline{\text{co}(F)}$, then by definition we may write x as

$$x = \lim_k \sum_{i=1}^{n(k)} \alpha_i^{(k)} f_{N_i}^{(k)}$$

where $f_{N_i} \in F$, $\alpha_i^{(k)} \overset{\geq}{>} 0$, and $\sum_{i=1}^{n(k)} \alpha_i^{(k)} = 1$. Thus, since $\alpha_i^{(k)} \overset{\geq}{>} 0$ and $f_{N_i}^{(k)} = N_i^{-1} \sum_{j=1}^{N_i^2} e_j$ it follows that $x_i := \langle x, e_i \rangle \geq 0$, that is, since x is a limit of elements with only non-negative coefficients in the $(e_n)_{n\geq 1}$ basis. Now let $0 = \alpha x + (1-\alpha)y$ for some $x, y \in K$ and $0 < \alpha < 1$, then for all $i \in \mathbb{N}$ we have

$$\alpha x_i + (1-\alpha) y_i = 0$$

where $x_i := \langle x, e_i \rangle \geq 0$ and $y_i := \langle y, e_i \rangle \geq 0$ as before. Hence, it follows that $\alpha x_i, (1-\alpha)y_i \geq 0$ so necessarily $\alpha x_i = (1-\alpha)y_i = 0$ and therefore $x_i, y_i = 0$ since $0 < \alpha < 1$. Thus, since $x_i, y_i = 0$ for all i, then $x = y = 0$ and so $0 \in \operatorname{Ext}(K)$. ✓

Now to see that $f_N \in \operatorname{Ext}(K)$ suppose $f_N = \alpha x + (1-\alpha)y$ for some $x, y \in K$ and $0 < \alpha < 1$. Additionally, we claim that $\|x\| = \|y\| \leq 1$. Indeed since

$$x = \lim_k \sum_{i=1}^{n(k)} \alpha_i^{(k)} f_{N_i}^{(k)}$$

with $f_{N_i} \in F$, $\alpha_i^{(k)} \overset{\geq}{>} 0$, and $\sum_{i=1}^{n(k)} \alpha_i^{(k)} = 1$, then

$$\left\| \sum_{i=1}^{n(k)} \alpha_i^{(k)} f_{N_i}^{(k)} \right\| \leq \sum_{i=1}^{n(k)} \alpha_i^{(k)} \|f_{N_i}\| = \sum_{i=1}^{n(k)} = 1$$

since $\|f_N\| = 1$. Hence, $\|x\| \leq 1$ and similarly for y.

Thus, it follows that

$$1 = \|f_N\| = \|\alpha x + (1-\alpha)y\| \leq \alpha\|x\| + (1-\alpha)\|y\| \leq 1$$

since $0 < \alpha < 1$. Thus, since $\alpha\|x\| + (1-\alpha)\|y\| = 1$ with $\|x\|, \|y\| \leq 1$ and $0 < \alpha < 1$, then $\|x\|, \|y\| = 1$. Additionally, we recall from standard linear algebra that the equality

$$\|\alpha x + (1-\alpha)y\| = \alpha\|x\| + (1-\alpha)\|y\| = 1$$

implies αx is a non-negative scalar multiple of $(1-\alpha)y$, that is, for some $\lambda \geq 0$

$$\alpha x = \lambda(1-\alpha)y.$$

$\lambda \in \mathbb{C}$ generally.

Hence, we have $\alpha = \lambda(1-\alpha)$ since $\|x\| = \|y\| = 1$ so $\lambda = \alpha/(1-\alpha)$. Thus, $\alpha x = \alpha y$ and since $0 < \alpha < 1$, then $x = y = f_N$. Therefore, $f_N \in \operatorname{Ext}(K)$ by definition, as desired. ✓

Solution. (d) Recall, that (H, τ_w) is a LCTVS and additionally we have that

$$K = \overline{\operatorname{co}(F)}^{\|-\|} = \overline{\operatorname{co}(F)}^{\tau_w}$$

is non-empty compact and convex. Thus, it follows by Milman's theorem (theorem 7.9) that $\operatorname{Ext}(K) \subset \overline{F}^{\tau_w}$ and since $f_N \to 0$ weakly as $N \to \infty$, then $\overline{F}^{\tau_w} = F \cup \{0\}$. Therefore, by part (c) we have that $\operatorname{Ext}(K) = \overline{F}^{\tau_w} = F \cup \{0\}$.

Elaborate! (✓)

Problem 2. Let X and Y be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

(b) Let $T \in \mathcal{K}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$.

(c) Let H be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H,Y)$ satisfies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H,Y)$.

(d) Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

(e) Show that no $T \in \mathcal{K}(X,Y)$ is onto.

(f) Let $H = L_2([0,1], m)$, and consider the operator $M \in \mathcal{L}(H,H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. Justify that M is self-adjoint, but not compact.

Solution. (a) Let $T \in \mathcal{L}(X,Y)$ and $(x_n)_{\geq 1} \subset X$ a sequence in X such that $x_n \to x$ weakly as $n \to \infty$. We claim that $Tx_n \to Tx$ weakly as $n \to \infty$. First, by homework 4 problem 2(a) we have that $Tx_n \to Tx$ weakly if and only if for every $f \in Y^*$, $fTx_n \to fTx$ as $n \to \infty$. Now $f \circ T \in X^*$ so by proposition 5.4 $f \circ T$ is weakly continuous from which is follows that $fTx_n \to fTx$ since $x_n \to x$ weakly as $n \to \infty$. Therefore, since $f \in Y^*$ was arbitrary, then $Tx_n \to Tx$ weakly as desired. ✓

Solution. (b) Let $(x_n)_{n\geq 1}$ be a sequence in X such that $x_n \to x$ weakly as $n \to \infty$ and let $T \in \mathcal{K}(X,Y)$. Then we claim that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$, that is, $Tx_n \to Tx$ in norm as $n \to \infty$. To see this recall that if every subsequence $(Tx_{n_k})_{k\geq 1}$ of $(Tx_n)_{n\geq 1}$ has a subsequence which converges to Tx, then $(Tx_n)_{n\geq 1}$ converges to Tx. Now by homework 4 problem 2b $(x_n)_{n\geq 1}$ is a bounded sequence since it converges weakly and in particular every subsequence of $(x_n)_{n\geq 1}$ is then necessarily bounded. Hence, let $(x_{n_k})_{k\geq 1}$ be a subsequence of $(x_n)_{n\geq 1}$, then there is a subsequence $(x_{n_{k_j}})_{j\geq 1}$ such that $(Tx_{n_{k_j}})_{j\geq 1}$ converges by proposition 8.2-(4) since T is compact. In particular, by part (a) we necessarily have $Tx_{n_{k_j}} \to Tx$ as $n \to \infty$ since $Tx_n \to Tx$ weakly. Therefore, every subsequence of $(Tx_n)_{n\geq 1}$ has a subsequence converging to Tx so $Tx_n \to Tx$ in norm as $n \to \infty$, that is, $\|Tx_n - Tx\| \to 0$ as $n \to \infty$ as desired. ✓

Solution. (c) Let H be a separable infinite dimensional Hilbert space and Y an infinite dimensional Banach space. Let $T \in \mathcal{L}(H,Y)$ be a continuous linear map such that for any $(x_n)_{n\geq 1} \subset H$ which converges weakly to $x \in H$, then $\|Tx_n - Tx\| \to 0$ as $n \to \infty$, that is, $Tx_n \to Tx$ in norm, then the claim is that T is compact. To see this we will apply proposition 8.4-(4)

Hence, let $(x_n)_{n\geq 1} \subset H$ be bounded, then without loss of generalization by scaling we may assume $(x_n)_{n\geq 1} \subset \overline{B_H(0,1)}$. Now $\overline{B_H(0,1)}$ is compact in the weak topology and our claim is that $\overline{B_H(0,1)}$ is in fact sequentially compact in the weak topology. If $\overline{B_H(0,1)}$ is sequentially compact in the weak topology, then by definition of sequentially compact $(x_n)_{n\geq 1}$ has a subsequence $(x_{n_k})_{k\geq 1}$ such that $x_{n_k} \to x$ weakly as $k \to \infty$ for some $x \in \overline{B_H(0,1)}$. Hence, by the condition on T we have that $\|Tx_{n_k} - Tx\| \to 0$ as $k \to \infty$ so $T \in \mathcal{K}(H, Y)$ by proposition 8.2-(4).

Now to see that $\overline{B_H(0,1)}$ is sequentially compact in the weak topology recall that any compact metric space is sequentially compact so if $\overline{B_H(0,1)}$ is metrizable in the weak topology, then we are done. Thus, by theorem 5.13 the closed unit ball in H^*, $\overline{B_{H^*}(0,1)}$, is metrizable in the weak* topology if and only if H is separable. Hence, since H is a separable Hilbert space, then the weak and weak* topologies agree and $H \cong (H^*)^*$ so the result follows if H^* is separable. However, this follows by Folland proposition 5.29 since $(\langle -, e_n\rangle)_{n\geq 1}$ is clearly an orthonormal basis for H^* where $(e_n)_{n\geq 1}$ is an orthonormal basis for H.

Solution. (d) Let $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$. Let $(x_n)_{n\geq 1}$ be a sequence in $\ell_2(\mathbb{N})$ which converges weakly to x, then by part (a) $Tx_n \to Tx$ weakly as $n \to \infty$. Now by remark 5.3 we have that a sequence in $\ell_1(\mathbb{N})$ converges weakly if and only if it converges in norm. Hence, it follows that $Tx_n \to Tx$ in norm as $n \to \infty$, in particular, $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. Finally, since $\ell_2(\mathbb{N})$ is a separable Hilbert space, then by part (c) we get that $T \in \mathcal{K}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ as desired.

Solution. (e) Suppose $T \in \mathcal{K}(X, Y)$ is surjective, then clearly there exists $r > 0$ such that $B_Y(0, r) \subset T(B_X(0, 1))$. Thus,

This is not immediate, but a deep result.

$$\overline{B_Y(0,r)} \subset \overline{T(B_X(0,1))}$$

where since T is compact, then $\overline{T(B_X(0,1))}$ is compact in Y. However, then $\overline{B_Y(0,r)}$ is compact since it is a closed subset of a compact set, a contradiction, since $\overline{B_Y(0,r)}$ is compact if and only if Y is finite dimensional. Therefore, T is not surjective.

Why?

Solution. (f) First, to see the M is self-adjoint we have for $f, g \in L_2([0,1], M)$ that

$$\begin{aligned}\langle Mf, g\rangle &= \int_{[0,1]} Mf(t)\overline{g(t)}dm(t) \\ &= \int_{[0,1]} tf(t)\overline{g(t)}dm(t) \\ &= \int_{[0,1]} f(t)\overline{tg(t)}dm(t) \text{ since } t \in [0,1] \\ &= \int_{[0,1]} f(t)\overline{Mg(t)}dm(t) \\ &= \langle f, Mg\rangle \qquad \checkmark\end{aligned}$$

so $M = M^*$ by uniqueness of adjoints.

Now to see that M is not compact we claim that M is surjective. Observe that M is injective since if $Mf = Mg$, then $tf(t) = tg(t)$ so $f(t) = g(t)$ almost everywhere for any $f, g \in L_2([0,1], m)$ so $f = g$ in $L_2([0,1], m)$. Thus, $\ker M = \{0\}$ and by homework 6 problem 1 and since M is self-adjoint we have $(\operatorname{im} M)^\perp = \ker M = \{0\}$ so $\operatorname{im} T = L_2([0,1], m)$. Therefore, $\checkmark$ by part (e) M is not compact.

Problem 3. Consider the Hilbert space $H = L_2([0,1], m)$, where m is the Lebesgue measure. Define $K : [0,1] \times [0,1] \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1 \\ (1-t)s, & \text{if } 0 \le s < t \le 1 \end{cases}$$

and consider $T \in \mathcal{L}(H, H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t), \;\; s \in [0,1], f \in H.$$

(a) Justify that T is compact.

(b) Show that $T = T^*$.

(c) Show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t), \;\; s \in [0,1], f \in H.$$

Use this to show that Tf is continuous on $[0,1]$, and that $(Tf)(0) = (Tf)(1) = 0$.

Solution. (a) First, it is clear that K is continuous by the gluing theorem for topological spaces. Hence, since $[0,1]$ is a compact Hausdorff space and the Lebesgue measure on $[0,1]$ is finite, then T is compact by theorem 9.6.

well you should recognize $T = T_{\tilde{k}}$ for this to apply.
$\tilde{k}(s,t) = k(t,s)$

Solution. (b) Let $f, g \in H$, then by uniqueness of adjoints to see that $T = T^*$ it is sufficient to show that $\langle Tf, g\rangle = \langle f, Tg\rangle$. Hence, we have

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} (Tf)(s)\overline{g(s)}dm(s) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(s,t)f(t)dm(t) \right) \overline{g(s)}dm(s) \\
&= \int_{[0,1]\times[0,1]} f(t)\overline{K(s,t)g(s)}dm(s,t) \text{ by Fubini and } K(s,t) \in \mathbb{R} \\
&= \int_{[0,1]} \left(\overline{K(s,t)g(s)dm(s)} \right) f(t)dm(t) \\
&= \langle f, Tg\rangle,
\end{aligned}$$

$\leftarrow$ only if $k(s,t) = k(t,s)$

as desired. Note that we may apply Fubini's theorem by Tonelli's theorem, that is, since K is bounded we have for some $M > 0$

$$\int_{[0,1]\times[0,1]} \left| f(t)\overline{K(s,t)g(s)} \right| dm(s,t) \leq M \int_{[0,1]\times[0,1]} |f(t)||g(s)|dm(s,t) < \infty.$$

Solution. (c) First, by definition of $K(s,t)$ and T we have

$$\begin{aligned}
(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} s(1-t)f(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)
\end{aligned}$$

as desired. Additionally, we have that

$$(Tf)(0) = \int_{[0,0]} tf(t)dm(t) = 0 = \int_{[1,1]} (1-t)f(t)dm(t) = (Tf)(1).$$

Now to see that Tf is continuous recall that for $f \in L_1([0,1], m)$, then the function

$$s \mapsto \int_{[0,s]} f(t)dm(t)$$

is continuous[1]. Thus, since $L_2([0,1], m) \subset L_1([0,1], m)$, then the functions

should argue $tf \in L_2 \subset L_1$. $(1-t)f \in L_2 \subset L_1$ when $f \in L_2$

$$g(s) := \int_{[0,s]} tf(t)dm(t) \text{ and } h(s) := \int_{[s,1]} (1-t)f(t)dm(t)$$

are continuous. Therefore, since products and sums of continuous functions are continuous, then it follows that Tf is continuous since

$$(Tf)(s) = (1-s)g(s) + sh(s).$$

Problem 4. Consider the Schwartz space $\mathcal{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

(a) For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$. Justify that $g_k \in \mathcal{S}(\mathbb{R})$, for all integers $k \geq 0$. Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

(b) Find non-zero functions $h_k \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

(c) Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathcal{S}(\mathbb{R})$.

(d) Use (c) to show that if $f \in \mathcal{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.

Solution. (a) Let $g_k(x) = x^k e^{-x^2/2}$ for all $k \geq 0$. Now observe that for $x \in \mathbb{R}$ we have $\|x\|^2 = |x|^2 = x^2$ and so by homework 7 problem 1 we have that $e^{-x^2} \in \mathcal{S}(\mathbb{R})$. Additionally, by homework 7 problem 1(d) we get that

$$f := S_{\sqrt{2}}(e^{-x^2}) = e^{-x^2/2} \in \mathcal{S}(\mathbb{R}).$$

Finally, by homework 7 problem 1(a) since $g_k(x) = x^k f$ we get that $g_k \in \mathcal{S}(\mathbb{R})$ since $f \in \mathcal{S}(\mathbb{R})$.

First, by proposition 11.4 we have that

$$\hat{f} = \mathcal{F}(f) = f$$

[1]this follows from the dominated convergence theorem

and by proposition 11.13(d) it follows that

$$\mathcal{F}(g_k)(\xi) = (x^k f)^{\wedge}(\xi) = i^k(\partial^k \hat{f})(\xi) = i^k(\partial^k f)(\xi).$$

Thus, for g_0, g_1, g_2, g_3 we get the following

$$\begin{aligned}
\mathcal{F}(g_0) &= \mathcal{F}(f) = f = g_0 \\
\mathcal{F}(g_1) &= i\partial f = -ixe^{-x^2/2} = -ig_1 \\
\mathcal{F}(g_2) &= i^2\partial^2 f = i^2(x^2-1)e^{-x^2/2} = f - x^2 f = g_0 - g_2 \\
\mathcal{F}(g_3) &= i^3\partial^3 f = -i^3 x(x^2-3)e^{-x^2/2} = ix(x^2-3)e^{-x^2/2} = i(g_3 - 3g_1)
\end{aligned}$$

Solution. (b) First, we note that it is clear from the definition that $\mathcal{S}(\mathbb{R}) \subset C^\infty(\mathbb{R})$ is a linear subspace due to linearity and additivity of limits and derivatives. Hence, any linear combination of g_k is in $\mathcal{S}(\mathbb{R})$ since by part (a) $g_k \in \mathcal{S}(\mathbb{R})$ for all $k \geq 0$.

Thus, after solving a system of linear equations we let

$$\begin{aligned}
h_0 &= g_0 \\
h_1 &= -\frac{3}{2}g_1 + g_3 \\
h_2 &= -\frac{1}{2}g_0 + g_2 \\
h_3 &= g_1,
\end{aligned}$$

then by part (a) we have

$$\begin{aligned}
\mathcal{F}(h_0) &= \mathcal{F}(g_0) = g_0 \\
\mathcal{F}(h_1) &= -\frac{3}{2}\mathcal{F}(g_1) + \mathcal{F}(g_3) = -\frac{3}{2}(-ig_1) + ig_3 - 3ig_1 = i(-\frac{3}{2}g_1 + g_3) = ih_1 \\
\mathcal{F}(h_2) &= -\frac{1}{2}\mathcal{F}(g_0) + \mathcal{F}(g_2) = -\frac{1}{2}g_0 + g_0 - g_2 = i^2(-\frac{1}{2}g_0 + g_2) = i^2 h_2 \\
\mathcal{F}(h_3) &= \mathcal{F}(g_1) = -ig_1 = i^3 g_1 = i^3 h_3,
\end{aligned}$$

as desired.

Solution. (c) Let $f \in \mathcal{S}(\mathbb{R})$, then by definition

$$\begin{aligned}\mathcal{F}^2(f)(t) &= \int_{\mathbb{R}} \hat{f}(\xi)e^{-i\langle \xi,t\rangle}dm(\xi)\\ &= \int_{\mathbb{R}} \hat{f}(\xi)e^{i\langle \xi,-t\rangle}dm(\xi)\\ &= (\mathcal{F}^* \circ \mathcal{F})(f)(-t)\\ &= f(-t).\end{aligned}$$

Thus, $\mathcal{F}^2(f)(t) = f(-t)$ so $\mathcal{F}^4(f)(t) = f(-(-t)) = f(t)$ and therefore, $\mathcal{F}^4(t) = f$ as desired. ✓

Solution. (d) Let $0 \neq f \in \mathcal{S}(\mathbb{R})$ and suppose $\mathcal{F}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$, then by linearity of $\mathcal{F}$ we have $\mathcal{F}^4(f) = \lambda^4 f$. Now by part (d) $\mathcal{F}^4(f) = f$ so $f = \lambda^4 f$. Thus, $\lambda^4 = 1$ so λ is a 4^{th}-root of unity, that is, $\lambda \in \{1, i, -1, -i\}$. Therefore, it follows by definition that the eigenvalues of $\mathcal{F}$ are precisely $\{1, i, -1, -i\}$ as desired.

It follow that any eigenvalue is an element in $\{1, i, -1, -i\}$
But that they are all eigenvalues follows from b)

Problem 5. Let $(x_n)_{n\geq 1}$ be a dense subset of $[0,1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} = 2^{-n}\delta_{x_n}$ on $[0,1]$. Show that $\mathrm{supp}(\mu) = [0,1]$.

Solution. Let $N = \cup_{i\in I} U_i$ be the union of all $U \subset [0,1]$ open such that $\mu(U) = 0$ as in homework 8 problem 3(a). Then $\mathrm{supp}(\mu) = [0,1] \setminus N$ by definition and we claim that $N = \emptyset$ so that $\mathrm{supp}(\mu) = [0,1]$.

To see this let $U \subset [0,1]$ be open, non-empty then we claim that $\mu(U) \neq 0$. This follows since $(x_n)_{n\geq 1}$ is dense in $[0,1]$ so $x_k \in U$ for some $k \in \mathbb{N}$ which implies

"only if $U \neq \emptyset$

$$\mu(U) = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}(U) = 2^{-k}\delta_{x_k}(U) = 2^{-k} \neq 0.$$

(✓)

Therefore, there are no open sets such that $\mu(U) = 0$ and so $N = \emptyset$, as desired.

Yes, $\mu(\emptyset) = 0$