Mandatory Assignment 1 for FunAn

Problem 1

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|)$ be (non-zero) normed vector spaces over $\mathbb{K}$

a) Let $T : X \to Y$ be a linear map, and set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ for all $x \in X$. We want to show that $\|\cdot\|_0$ is a norm on X:

$$\begin{aligned} \|x+y\|_0 &= \|x+y\|_X + \|T(x+y)\| \\ &\leq \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y \\ &= \|x\|_0 + \|y\|_0, \quad x, y \in X \end{aligned}$$

since $\|\cdot\|_X, \|\cdot\|_Y$ are norms and T is linear, so the triangular inequality holds.

$$\begin{aligned} \|\alpha x\|_0 &= \|\alpha x\|_X + \|T(\alpha x)\|_Y \\ &= |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y \\ &= |\alpha|\|x\|_0, \quad \alpha \in \mathbb{K}, x \in X \end{aligned}$$

again since $\|\cdot\|_X, \|\cdot\|_Y$ are norms and T is linear.
Let $x = 0$ then $\|0\|_0 = 0 + \|T0\|_Y = 0$. And if $\|x\|_0 = 0$ then $\|x\|_X = -\|Tx\|_Y$ so $x = 0$. Thus $\|\cdot\|_0$ is a norm on X.

We want to show that $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent iff T is bounded.
Assume there exists c_1, c_2 where $0 < c_1 \leq c_2$ s.t $c_\|x\|_X \leq \|x\|_0 \leq c_2\|x\|_X$ for $x \in X$. So $\|x\|_X + \|Tx\|_Y \leq c_2\|x\|_X$.
But then $\|Tx\|_Y \leq c_2\|x\|_X - \|x\|_X \leq c_2\|x\|_X + \|x\|_X = (c_2+1)\|x\|_X$, where $c_2 + 1 > 0$. So T is bounded.
Assume T is bounded. So there exists a $C > 0$ s.t. $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$.
We then have that $\|x\|_X + \|Tx\|_Y \leq \|x\|_X + C\|x\|_X = (C+1)\|x\|_X$, where $C + 1 > 0$.
And $\|x\|_X \leq \|x\|_X + \|Tx\|_Y$, since $\|Tx\|_Y \leq 0$. So we have that $\|x\|_X \leq \|x\|_0 \leq (C+1)\|x\|_X$ where $0 < 1 \leq C + 1$. Thus $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent.

b) We want to show that if X is finite dimensional then any linear map $T : X \to Y$ is bounded.
Assume X is finite dimensional ($\dim X = n$). Let $\{e_1, \ldots, e_n\}$ be a basis for X. Given $x \in X$ there exists unique scalars $x_1, \ldots, x_n \in \mathbb{K}$ s.t. $x = \sum_{i=1}^n x_i e_i$.

$$\begin{aligned} \|Tx\|_Y &= \left\| T\left(\sum_{i=1}^n x_i e_i\right)\right\|_Y = \left\|\sum_{i=1}^n x_i Te_i\right\|_Y \\ &\leq \sum_{i=1}^n |x_i|\|Te_i\|_Y \leq \|x\|_\infty \sum_{i=1}^n \|Te_i\|_Y \\ &= C\|x\|_\infty \end{aligned}$$

where $C = \sum_{i=1}^n \|Te_i\|_Y$. But since X is finite dimensional we have by theorem 1.6 that any two norms on X are equivalent.
So there exists c_1, c_2, $0 < c_1 \leq c_2$ s.t

$$c_1\|x\|_x \leq \|x\|_\infty \leq c_2\|x\|_X$$

So $\|Tx\|_Y \leq C\|x\|_\infty \leq C \cdot c_2\|x\|_X$, where $C \cdot c_2 = K > 0$. Thus T is bounded.

c) Suppose X is infinite dimensional. We want to show that there exists a linear map $T : X \to Y$, which is not bounded.
Take a Hamel basis $(e_i)_{i\in I}$ for X. So $(\lambda_i)_{i\in I}$ is unique family with $x = \sum_{i\in I} \lambda_i e_i$ and $\{i \in I \,.\, \lambda_i \neq 0\}$ is finite.
Let $\left(\frac{e_i}{\|e_i\|_X}\right)_{i\in I}$ be a family of elements in X. Then we have that $(\lambda_i\|e_i\|_X)_{i\in I}$ is a unique family in $\mathbb{K}$ where $\sum_{i\in I}(\lambda_i\|e_i\|)\frac{e_i}{\|e_i\|_X} = \sum_{i\in I} \lambda_i e_i = x$ and $\{i \in I \,:\, \lambda_i\|e_i\|_X \neq 0\}$. So $\left(\frac{e_i}{\|e_i\|_X}\right)_{i\in I}$ also a Hamel basis for X with $\left\|\frac{e_i}{\|e_i\|_X}\right\|_X = 1$.
So we can chose a Hamel basis $(e_i)_{i\in I}$ s.t $\|e_i\|_X = 1$ for all $i \in I$.
Let $\left(\frac{iy_i}{\|y_i\|_Y}\right)_{i\in I}$ be a family in Y where $\left\|\frac{iy_i}{\|y_i\|_Y}\right\|_Y = i$ for all $i \in I$, but since X is infinite dimensional we have that I contains infinite elements. So $\left\|\frac{iy_i}{\|y_i\|_Y}\right\|_Y \to \infty$ as $i \to \infty$. Then we can choose a family $(y_i)_{i\in I}$ in Y where $\|y_i\|_Y \to \infty$ as $i \to \infty$.
Because $(e_i)_{i\in I}$ is Hamel basis we have there exists a linear map $T : X \to Y$ s.t. $T(e_i) = y_i$ for all $i \in I$.
Let $N \in \mathbb{N}$, then there exists a $n \in I$ s.t. $\|y_i\|_Y > N$ for all $i \geq n$.
But then $\|T(e_i)\|_Y = \|y_i\|_Y > N = N\|e_i\|_X$ since $\|e_i\|_X = 1$ for all $i \geq n$. So T is not bounded.

d) Suppose X is infinite dimensional. Then there exists a norm $\|\cdot\|_0$ on X, which is not equivalent to $\|\cdot\|_X$ and which satisfies $\|x\|_X \leq \|x\|_0$.
Let $\|\cdot\|_0$ be as in (a) with the linear map T from (c) then $\|x\|_X \leq \|x\|_0$ for all $x \in X$. Since X is infinite dimensional we have that by (c) that T is not bounded, and then by (a) we have that $\|\cdot\|_X$ and $\|\cdot\|_0$ are not equivalent.
We now want to show that if $(X, \|\cdot\|_X)$ is a Banach space then $(X, \|\cdot\|_0)$ is not complete. Let $f : (X, \|\cdot\|_0) \to (X, \|\cdot\|_X)$ given by $f(x) = x$ in the other norm.
Since $\|f(x)\|_X = \|x\|_X \leq \|x\|_0$ we have that f is continuous by Proposition 1.10.
We have that f is not homeomorphism since the norms $\|\cdot\|_0$ and $\|\cdot\|_X$ are not equivalent. But then f is not open.
Assume that $(X_{\|} \cdot \|_X)$ is a Banach space, and assume for contradiction that $(X, \|\cdot\|_0)$ is also a Banach space.
Since f is continuous we have $f \in \mathcal{L}((X, \|\cdot\|_0), (X, \|\cdot\|_X))$ and f is surjective so by Theorem 3.15 (the Open mapping theorem) we have that f is open, but this a contradiction. So $(X, \|\cdot\|_0)$ is not complete.

f) Take $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|_1)$ and $\|\cdot\|' = \|\cdot\|_\infty$. Let $(x_n)_{n\geq 1} \in \ell_1(\mathbb{N})$. Then

$$\|(x_n)_{n\geq 1}\|_\infty = \max\{|x_n| \,:\, n \in \mathbb{N}\} \leq \sum_{n=1}^{\infty} |x_n| = \|(x_n)_{n\geq 1}\|_1$$

We have that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent.
Assume for contradiction that there exists $C > 0$ s.t $\|(x_n)_{n\geq 1}\|_1 \leq C\|(x_n)_{n\geq 1}\|_\infty$.
Take the sequence

$$x_n = \begin{cases} 1 & n \leq \lceil C + 1 \rceil \\ 0 & otherwise \end{cases}$$

Then we have that $\|(x_n)_{n\geq}\| = \lceil C+1\rceil > C = C\|(x_n)_{n\geq}\|_\infty$,
since $\|(x_n)_{n\geq 1}\|_\infty = 1$. So $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent.

We have that $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is a Banach space. Assume for contradiction that $(\ell_1, \|\cdot\|_\infty)$ is a Banach space. We repeat the argument from (d) with the function $f : (\ell_1(\mathbb{N}), \|\cdot\|_\infty) \to (\ell_1(\mathbb{N}), \|\cdot\|_1)$ given by $f(x) = x$. And get a contradiction. And so $(\ell_1, \|\cdot\|_\infty)$ is not complete.

Problem 2

Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), \|\cdot\|_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, \dots) \ : \ a, b \in \mathbb{C}\}$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0 \dots) = a + b$ for all $a, b \in \mathbb{C}$

a) We want to show that f is bounded on $(M, \|\cdot\|_p)$. We have that

$$|f(a, b, 0, \dots)| = |a + b| \leq |a| + |b| = (|a|^p)^{\frac{1}{p}} + (|b|^p)^{\frac{1}{p}}$$

But $(|a|^p)^{\frac{1}{p}} \leq (|a|^p + |b|^p)^{\frac{1}{p}}$, since $|b|^p \leq 0$. The same inequality holds for $(|b|^p)^{\frac{1}{p}}$. So

$$\begin{aligned} (|a|^p)^{\frac{1}{p}} + (|b|^p)^{\frac{1}{p}} &\leq (|a|^p + |b|^p)^{\frac{1}{p}} + (|a|^p + |b|^p)^{\frac{1}{p}} \\ &= 2(|a|^p + |b|^p)^{\frac{1}{p}} = 2\|(a, b, 0, \dots)\|_p \end{aligned}$$

Hence $|f(a, b, 0, \dots)| \leq 2\|(a, b, 0, \dots)\|_p$, so f is bounded.
Now we want to compute $\|f\|$. We spilt it up in two cases.
For $p = 1$: Since f is bounded, we have by Remark 1.11 that:

$$\begin{aligned} \|f\| &= \sup\{|f(a, b, 0, \dots)| \ : \ \|(a, b, 0, \dots)\|_1 = 1\} \\ &= \sup\{|a + b| \ : \ |a| + |b| = 1\} \end{aligned}$$

Then for $|a| + |b| = 1$ we have that $|a + b| \leq |a| + |b| = 1$, so $\|f\| \leq 1$. And we see that $|f(1, 0, \dots)| = 1$ where $\|(1, 0, \dots)\|_1 = 1$ so $\|f\| \geq 1$. Then for $p = 1$ we have that $\|f\| = 1$.
For $1 < p < \infty$: Again by Remark 1.11 we have that

$$\begin{aligned} \|f\| &= \sup\{|f(a, b, 0, \dots)| \ : \ \|(a, b, 0, \dots)\|_p = 1\} \\ &= \sup\{|a + b| \ : \ (|a|^p + |b|^p)^{\frac{1}{p}} = 1\} \\ &= \sup\{|a + b| \ : \ |a|^p + |b|^p = 1\} \end{aligned}$$

We see for $a = b = \frac{1}{2^{\frac{1}{p}}}$ that $\left\| \left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, \dots \right) \right\|_p = \left| \frac{1}{2^{\frac{1}{p}}} \right|^p + \left| \frac{1}{2^{\frac{1}{p}}} \right|^p = 1.$
We also have that $\left| f\left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}} \right) \right| = \frac{2}{2^{\frac{1}{p}}}$. Thus $\|f\| \geq \frac{2}{2^{\frac{1}{p}}} = 2^{\frac{p-1}{p}}$.
Now we want to show that $\|f\| \leq 2^{\frac{p-1}{p}} = 2^{\frac{1}{q}}$ for where we have assumed that

$\frac{1}{p} + \frac{1}{q} = 1$.
Recall Hölder's inequality from HW1. For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|(x_n)_{n\geq 1}\|_p \|(y_n)_{n\geq 1}\|_q$$

where $(x_n)_{n\geq 1} \in \ell_p(\mathbb{N})$ and $(y_n)_{n\geq 1} \in \ell_q(\mathbb{N})$. We use Hölder's inequality for $(a, b, 0, \dots) \in \ell_p(\mathbb{N})$ and $(1, 1, 0 \dots) \in \ell_q(\mathbb{N})$. We then get

$$|a+b| \leq |a| + |b| \leq (|a|^p + |b|^p)^{\frac{1}{p}} \cdot (1+1)^{\frac{1}{q}}$$

And $(|a|^p + |b|^p)^{\frac{1}{p}} \cdot (1+1)^{\frac{1}{q}} = 1 \cdot 2^{\frac{1}{q}}$ for $|a|^p + |b|^p = 1$. This means that $\|f\| \leq 2^{\frac{1}{q}} = 2^{\frac{p-1}{p}}$. Thus $\|f\| = 2^{\frac{p-1}{p}}$ for $1 < p < \infty$.

b) We want to show that for $1 < p < \infty$ there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.
We know such a functional exists by Corollary 2.6. Since by (a) we have that f is bounded so $f \in \mathcal{L}(M, \mathbb{K})$, and the conditions follows from Corollary 2.6.
Show we want to show uniqueness. By HW1 Pb 5 we have that $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$ isometrically isometric when $\frac{1}{p} + \frac{1}{q} = 1$.
Let $F : \ell_p(\mathbb{N}) \to \mathbb{C}$ be extension. Then by HW1 Pb 5

$$F((y_n)_{n\geq 1}) = \sum_{n=1}^{\infty} x_n y_n, \qquad \text{for some } (x_n)_{n\geq 1} \in \ell_q(\mathbb{N})$$

We see that for $(1, 0, \dots), (0, 1, 0, \dots) \in M$

$$\begin{aligned} x_1 &= F(1, 0, \dots) = f(1, 0, \dots) = 1 \\ x_2 &= F(0, 1, 0, \dots) = f(0, 1, 0, \dots) = 1 \end{aligned}$$

Since $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$ isometrically isometric we have that $\|(x_n)_{n\geq 1}\| = \|F\|$, and $\|F\| = \|f\| = 2^{\frac{1}{q}}$, thus $\|(x_n)_{n\geq 1}\| = 2^{\frac{1}{q}}$. We also have that

$$\begin{aligned} \|(x_n)_{n\geq 1}\| &= \left(\sum_{n=1}^{\infty} |x_n|^q \right)^{\frac{1}{q}} = \left(x_1 + x_2 \sum_{n=3}^{\infty} |x_n|^q \right)^{\frac{1}{q}} \\ &= \left(2 + \sum_{n=3}^{\infty} |x_n|^q \right)^{\frac{1}{q}} \end{aligned}$$

Since $(2 + \sum_{n=3}^{\infty} |x_n|^q)^{\frac{1}{q}} = \|(x_n)_{n\geq 1}\| = 2^{\frac{1}{q}}$ then $\sum_{n=3}^{\infty} |x_n|^q = 0$ so $x_n = 0$ for $n \geq 2$.
Thus $F((y_n)_{n\geq 1}) = y_1 + y_2$

c) We want to show that if $p = 1$ then there are infinitely many linear functional F in $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.
So assume $p = 1$. Let $F_t : \ell_1(\mathbb{N}) \to \mathbb{N}$. We want to show that
$F_t((y_n)_{n\geq 1}) = y_1 + y_2 + \frac{1}{2} y_t$ for $t \geq 3$ is an infinite family of such extensions.

We see that $F_t\big|_M = f$.
We have that

$$\begin{aligned}\|F_t\| &= \sup\{|F_t((y_n)_{n\geq 1}) \ : \ \|(y_n)_{n\geq 1}\|_1 = 1\}\\ &= \sup\left\{|y_1 + y_2 + \frac{1}{2}y_t| \ : \ \sum_{n=1}^{\infty}|y_n| = 1\right\}\\ &\leq \sup\left\{|y_1| + |y_2| + \frac{1}{2}|y_t| \ : \ |y_1| + |y_2| + |y_t| \leq 1\right\}\\ &\leq 1\end{aligned}$$

since if $\sum_{n=1}^{\infty}|y_n| = 1$ then we must have that $|y_1| + |y_2| + |y_t| \leq 1$, and since $|y_1| + |y_2| + \frac{1}{2}|y_t| \leq |y_1| + |y_2| + |y_t| \leq 1$.
We also see that $\|(1,0,\dots)\|_1 = 1$ and $|F_t(1,0,\dots)| = 1$ so $\|F_t\| \geq 1$. This means that $\|F_t\| = 1 = \|f\|$.

Problem 3

Let X be an infinite dimensional normed vector space over $\mathbb{K}$.

a) Let $n \geq 1$ be an integer. We want to show that no linear map $F : X \to \mathbb{K}^n$ is injective.
Let $x_1, x_2, \dots, x_{n+1} \in X$ be linear independent. The we have that $span\{x_1, x_2, \dots, x_{n+1}\} \subset X$. But this is $n+1$ dimensional so there is no injective linear map $span\{x_1, x_2, \dots, x_{n+1}\} \to \mathbb{K}^n$. And any injective map $F : X \to \mathbb{K}^n$ would restrict to injective linear map on the subspace. So there is no such injective map $F : X \to \mathbb{K}^n$.

b) Let $n \geq 1$ be an integer, and let $f_1, \dots, f_n \in X^*$. We want to show that

$$\bigcap_{j=1}^{n} \ker f_j \neq \{0\}$$

Consider the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x, \dots, f_n(x)))$ for $x \in X$. We have that

$$\begin{aligned}\ker F &= \{x \in X \ : \ F(x) = 0\}\\ &= \{x \in X \ : \ (f_1(x), \dots, f_n(x) = 0)\}\\ &= \{x \in X \ : \ f_j(x) = 0 \text{ for all } j = 1, \dots, n\}\\ &= \left\{x \in X \ : \ \bigcap_{j=1}^{n} f_j(x) = 0\right\}\\ &= \bigcap_{j=1}^{n} \ker f_j \neq \{0\}\end{aligned}$$

since $\ker F \neq \{0\}$ since by (a) F is not injective.

c) Let $x_1, \dots, x_n \in X$. We want to show there exists a $y \in X$ s.t. $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, 2, \dots, n$.

Assume $x_1, x_2, \dots, x_n \neq 0$. Since if $x_j = 0$ for $j = 1, \dots, n$ then let y be an unit vector, and then the two conditions are met.
So since $x_j \neq 0$ for all $j = 1, \dots, n$ we have by Theorem 2.7 (b) that for each x_j there exists $f_j \in X^*$ where $\|f_j\| = 1$ and $f_j(x_j) = \|x_j\|$ for $j = 1, \dots, n$.
So there exists $x' \neq 0$ since by (b) we can choose $x' \in \bigcap \ker f_j \neq \{0\}$. Then let $y = \frac{x'}{\|x'\|}$, so $\|y\| = \left\| \frac{x'}{\|x'\|} \right\| = 1$.
Note $y - x_j \in X$. Then since f_j are bounded for all $j = 1, \dots, n$, we have that $|f(y - x_j)| \leq \|f_j\| \|y - x_j\| = \|y - x_j\|$, since by Remark 1.11 $\|f_j\| = \inf\{C > 0 \ : \ |f_j(z)| \leq C\|z\|, \ z \in X\}$ and $\|f_j\| = 1$.
We have that

$$|f(y - x_j)| = |f(y) - f(x_j)| = |-f(x_j)| = f(x_j)$$

since f_j are linear, and $f(y) = 0$ since $f(x') = 0$. And so we have that

$$\|y - x_j\| \geq f(x_j) = \|x_j\|$$

for all $j = 1, \dots, n$.

d) We want to show that we cannot cover the unit sphere $\mathbb{S} = \{x \in X \ : \ \|x\| = 1\}$ with a finite family of closed balls in X s.t. none of the balls contains 0.
Assume we can cover $\mathbb{S}$ with a finite family of closed balls $\{\overline{B_j}(x_j, r_j)\}_{j=1}^n$ where we let $x_1, \dots, x_n$ be the center of these balls. Then by (c) we know there exists a $y \in X$ s.t. $\|y\| = 1$. This means that $y \in \mathbb{S}$, and since the balls cover $\mathbb{S}$ we have that y much lie in one of these balls. So assume $y \in \overline{B_j}(x_j, r_j)$. Then $\|y - x_j\| \leq r_j$, but then $\|x_j - 0\| \leq r_j$, since by (c) we have that $\|y - x_j\| \geq \|x_j - 0\|$ for all $j = 1, \dots, n$. So $0 \in \overline{B_j}(x_j, x_j)$.
Thus we have that if we have that a finite family of closed ball cover the unit sphere, then one of the balls much contain 0. So we cannot cover the unit sphere with a family of closed balls, where none contain zero.

e) We want to show that $\mathbb{S}$ is non-compact. Let $\{B(x, \frac{1}{2})\}_{x \in \mathbb{S}}$ be a open cover of $\mathbb{S}$.
Assume for contradiction that the sets $B(x_1, \frac{1}{2}), \dots, B(x_n, \frac{1}{2})$ are a finite subcover. Then we have that the sets $\overline{B}(x_1, \frac{1}{2}), \dots, \overline{B}(x_n, \frac{1}{2})$ also are a finite subcover of $\mathbb{S}$ but none of these balls contain 0, which contradicts (d). So $\mathbb{S}$ is non-compact.

Problem 4

Let $L_1([0,1], m)$ and $L_3([0,1], m)$ be Lebesgue spaces on $[0,1]$. For $n \geq 1$, define

$$E_n := \left\{ f \in L_1([0,1], m) \ : \ \int_{[0,1]} |f|^3 dm \leq n \right\}$$

a) Given $n \geq 1$. We have that the $E_n \subset L_1([0,1], m)$ is not absorbing.
If $E_n \subset L_1([0,1], m)$ was absorbing then for all $0 \neq f \in L_1([0,1], m)$ there exists $t > 0$ s.t. $t^{-1} f \in E_n$.
Take $f \in L_1([0,1], m)$ but where $f \notin L_3([0,1], m)$. We can choose such a f since

$L_3([0,1],m) \subsetneq L_1([0,1],m)$ by HW2 Pb 2.
This means that $\left(\int_{[0,1]} |f|^3 dm\right)^{\frac{1}{3}} = \infty$, but then $\int_{[0,1]} |f|^3 dm = \infty$.
And so we cannot find a $t > 0$ s.t. $t^{-1}f \in E_n$. So E_n is not absorbing.

b) We want to show that E_n has empty interior in $L_1([0,1],m)$ for all $n \geq 1$.
Let $f \in E_n$. Then we take an open ball with f as its center.
So $B(f,r) = \{g \in L_1([0,1],m) \ : \ \|f-g\| < r\}$.
Let $g \in L_1([0,1],m)$ be given by $g(x) := f(x) + \frac{r}{2x^{\frac{1}{3}}}$.
We then have that

$$\begin{aligned}\|f-g\|_1 &= \left\|\frac{r}{2x^{\frac{1}{3}}}\right\|_1 = \int_{[0,1]} \left|\frac{r}{2x^{\frac{1}{3}}}\right| dm \\ &= \frac{r}{r}\int_{[0,1]} x^{-\frac{1}{3}} dm = \frac{r}{2}\left[\frac{3}{2}x^{\frac{2}{3}}\right]_0^1 \\ &= \frac{3r}{4} < r\end{aligned}$$

So $g \in B(f,r)$. Now we want to show that $g \notin E_n$.
It is enough to so that $\left(\int_{[0,1]} \left|f(x) + \frac{r}{2x^{\frac{1}{3}}}\right|^3 dm\right)^{\frac{1}{3}} = \infty$.
We have that

$$\left(\int_{[0,1]} \left|f(x) + \frac{r}{2x^{\frac{1}{3}}}\right|^3 dm\right)^{\frac{1}{3}} = \left\|f + \frac{r}{2x^{\frac{1}{3}}}\right\|_3 \leq \left|\|f\|_3 - \left\|\frac{r}{2x^{\frac{1}{3}}}\right\|_3\right| = \infty$$

where we uses the reverse triangular inequality and that $\|f\|_3 < \infty$ since $f \in E_n$.
But we have that $\int_{[0,1]} \left|\frac{r}{2x^{\frac{1}{3}}}\right|^3 dm = \left(\frac{r}{2}\right)^3 \int_{[0,1]} \frac{1}{x} dm = \infty$, so $\left\|\frac{r}{2x^{\frac{1}{3}}}\right\|_3 = \infty$.
Therefore $g \notin E_n$. So E_n has empty interior.

c) Show that E_n is closed in $L_1([0,1],m)$ for all $n \geq 1$. Since we are in a metric space it is enough to show that for $f_n \to f$ as $n \to \infty$, where $(f_n)_{n\geq 1} \subset E_n$ and $f \in L_1([0,1],m)$ we have that $f \in E_n$. Let $(f_k)_{k\geq 1} \subset E_n$ be convergent in $L_1([0,1],m)$ so $\|f_k - f\|_1 \to 0$ as $k \to \infty$. Then $|f_n - f| \to 0$ as $n \to \infty$.
Now we want to show that $f \in E_n$.
We have there exists a subsequence $(f_t)_{t\geq 1} \subset (f_k)_{k\geq 1}$ s.t $|f_t(x) - f(x)| \to 0$ as $t \to \infty$ a.e for all $x \in [0,1]$.
Let $g : [0,1] \to [0,1]$ be given by $g(x) = |x|^3$, which is continuous. Since g is continuous we have that

$$\lim_{t\to\infty} |f_t(x)|^3 = \lim_{t\to\infty} g(f_t(x)) = g(\lim_{t\to\infty} f_t(x)) = g(f(x)) = |f(x)|^3 \ a.e$$

This means that

$$\int_{[0,1]} |f|^3 dm = \int_{[0,1]} \lim_{t\to\infty} |f_t|^3 dm \leq \lim_{t\to\infty} \int_{[0,1]} |f_t|^3 dm$$

where we use Fatou's lemma. We can do this since $|f_t|^3$ is positive measureble function. But then since $f_t \in E_n$ we have that $\lim_{t\to\infty} \int_{[0,1]} |f_t|^3 dm \leq$

$\lim_{t\to\infty} n = n$. So $\int_{[0,1]} |f|^3 dm \le n$. Hence $f \in E_n$. So E_n is closed.

d) We want to conclude that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$.
We have that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$ if it is a countable union of nowhere dense sets.
By (c) we have E_n is closed so its closure is itself E_n which has empty interior by (b) so E_n is nowhere dense. So now we just need to show that $L_3([0,1], m) = \bigcup_{n\ge 1}^{\infty} E_n$.

Assume $f \in L_3([0,1], m)$. Then $\left(\int_{[0,1]} |f|^3 dm\right)^{\frac{1}{3}} < \infty$.

So we set $k := \left(\int_{[0,1]} |f|^3 dm\right)^{\frac{1}{3}}$. Then $\int_{[0,1]} |f|^3 dm = k^3$, but that means that for some $c \ge k^3$ we have $\int_{[0,1]} |f|^3 dm \le c$. Hence $f \in E_c$, so $f \in \bigcup_{n\ge 1}^{\infty} E_n$.

Assume $f \in \bigcup_{n\ge 1}^{\infty} E_n$. Then $f \in E_k$, so $\int_{[0,1]} |f|^3 dm \le k$. But then $\left(\int_{[0,1]} |f|^3 dm\right)^{\frac{1}{3}} \le k^{\frac{1}{3}} < \infty$. And so $f \in L_3([0,1], m)$.

Problem 5

Let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n\ge 1}$ be a sequence in H, and let $x \in H$.

a) Suppose that $x_n \to x$ in norm, as $n \to \infty$. This means that $\|x_n - x\| \to 0$ as $n \to \infty$.
By the reverse triangular inequality we have that $\|x_n - x\| \ge \Big|\|x_n\| - \|x\|\Big|$.
So $0 \le \Big|\|x_n\| - \|x\|\Big| \le \|x_n - x\| \to 0$ as $n \to \infty$. So $\Big|\|x_n\| - \|x\|\Big| \to 0$ as $n \to \infty$.
This means that $\|x_n\| \to \|x\|$ as $n \to \infty$.

b) Suppose that $x_n \to x$ weakly, as $n \to \infty$.
Let $(e_n)_{n\ge 1}$ be a countable orthonormal basis in H. H is separable so there is such a basis. Assume $e_n \to x$ weakly, as $n \to \infty$ for $x \in H$. Then by HW4 Pb 2 we have that $f(e_n) \to f(x)$ as $n \to \infty$ for all $f \in H^*$.
By Riesz representation (HW2 Pb 1) we have that there exists a $y \in H$ s.t $f(e_n) = \langle e_n, y\rangle$ for $e_n \in H$. So by Bessel's inequality we have that

$$\sum_{n=1}^{\infty} |f(e_n)|^2 = \sum_{n=1}^{\infty} |\langle e_n, y\rangle|^2 \le \|e_n\|^2 = 1$$

since $(e_n)_{n\le 1}$ is a orthonormal basis. Because the series is bounded, we have that the tail goes of $|\langle e_n, y\rangle$ to zero. So $\lim_{n\to\infty} |f(e_n)| = \lim_{n\to\infty} |\langle e_n, y\rangle| = 0$ for all $f \in H^*$. So $e_n \to 0$ weakly, as $n \to \infty$.
But then

$$1 = \lim_{n\to\infty} \|e_n\| \ne \left\|\lim_{n\to\infty} e_n\right\| = \|0\| = 0$$

c) Suppose $\|x_n\| \le 1$, for all $n \ge 1$, and that $x_n \to x$ weakly, as $n \to \infty$.
Assume $0 \ne 0 \in H$. Then by Theorem 2.7 (b) we have there exists a $f \in H^*$ s.t. $\|f\| = 1$ and $f(x) = \|x\|$. Since $x_n \to x$ weakly, as $n \to \infty$ we have by HW4 Pb 2 that $f(x_n) \to f(x)$ for all $f \in H^*$. So $|f(x_n) - f(x)| \to 0$ as $n \to \infty$.

Since $\|x_n\| \leq 1$ we have that $|f(x_n)| \leq \|f\| = 1$.
Then $|f(x)| - |f(x_n)| \leq |f(x_n) - f(x)|$ so $|f(x)| \leq |f(x_n) - f(x)| + |f(x_n)| \leq 1$, as $n \to \infty$. But then we have that $\|x\| = f(x) \leq |f(x)| \leq 1$.