Functional Analysis - Mandatory Assignment 1

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Problem 1

Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a)

Let $T : X \to Y$ be a linear map. Set $||x||_0 = ||x||_X + ||Tx||_Y$, for all $x \in X$. Show that $||\cdot||_0$ is a norm on X. Show next that the two norms $||\cdot||_X$ and $||\cdot||_0$ are equivalent if and only if T is bounded.

To show that $||\cdot||_0$ is a norm we use definition 1.1. As $||\cdot||_X$ and $||\cdot||_Y$ are norms we have that $||\cdot||_0$ maps into $[0, \infty)$.
Furthermore we have

$$\begin{aligned}||x + x'||_0 &= ||x + x'||_X + ||T(x + x')||_Y \\ &\leq ||x||_X + ||x'||_X + ||Tx||_Y + ||Tx'||_Y \\ &= ||x||_0 + ||x'||_0.\end{aligned}$$

$$\begin{aligned}||\alpha x||_0 &= ||\alpha x||_X + ||T(\alpha x)||_Y \\ &= |\alpha|(||x||_X + ||Tx||_Y) \\ &= |\alpha|||x||_0.\end{aligned}$$

$$||x||_0 = 0 \Leftrightarrow ||x||_X + ||Tx||_Y = 0 \Leftrightarrow ||x||_X = ||Tx||_Y = 0 \Leftrightarrow x = 0.$$

Here we have used that $||\cdot||_X$ and $||\cdot||_Y$ are norms and that T is linear several times. Hence we have shown that $||\cdot||_0$ is a norm as desired. We now show that $||\cdot||_0$ and $||\cdot||_X$ are equivalent iff T is bounded.

"$\Rightarrow$": By assumption there exist $0 \leq c_1 \leq c_2 \leq \infty$ s.t. $c_1||x||_X \leq ||x||_0 \leq c_2||x||_X$. Note that $c_2 \geq 1$ by the construction of $||\cdot||_0$.
We now have

$$||Tx||_Y = ||x||_0 - ||x||_X \leq c_2||x||_X - ||x||_X = (c_2 - 1)||x||_X.$$

Hence by proposition 1.10 T is bounded.

"$\Leftarrow$": As T is bounded there exists c s.t. $||Tx||_Y \leq c||x||_X$, hence we have

$$||x||_0 = ||x||_X + ||Tx||_Y \leq (1+c)||x||_X.$$

Moreover by construction we have $||x||_X \leq ||x||_0$, hence the two norms are equivalent by definition 1.4.

(b)

Show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.

Given a linear map $T : X \to Y$, we can define $|| \cdot ||_0$ as in (a), which is a norm. As X is finite dimensional all norms on X are equivalent by Theorem 1.6. In particular $|| \cdot ||_X$ and $|| \cdot ||_0$ are equivalent hence T is bounded by (a).

(c)

Suppose that X is infinite dimensional. Show that there exists a linear map $T : X \to Y$, which is not bounded.

Let $(e_i)_{i\geq 1}$ be a Hamel basis for X. By normalizing the basis we can assume that $||e_i|| = 1$ for all i.
Let $y \in Y$ be an element with $||y||_Y = 1$. Such an element exists as $(Y, || \cdot ||_Y)$ is a non-zero normed space. We define a family of elements $(y_i)_{i\geq 1}$ in Y by $y_i = iy$. Then as $(e_i)_{i\geq 1}$ is a Hamel basis there exists a unique linear map $T : X \to Y$ s.t. $T(e_i) = y_i = iy$. Suppose T is bounded, i.e. there exists a C s.t. $||Tx||_Y \leq C||x||_X$ for all $x \in X$. Then there exists $i > C$ and hence $||T(e_i)||_Y = ||y_i||_Y = ||iy||_Y = |i|||y||_Y = i > C = C||e_i||_X$ which is a contradiction. Thus T is not bounded and we have shown that there exists a linear map $T : X \to Y$ which is not bounded.

(d)

Suppose again that X is infinite dimensional. Argue that there exists a norm $|| \cdot ||_0$ on X, which is not equivalent to the given norm $|| \cdot ||_X$, and which satisfies $||x||_X \leq ||x||_0$, for all $x \in X$. Conclude that $(X, ||x||_0)$ is not complete if $(X, || \cdot ||_X)$ is a Banach space.

Let $T : X \to Y$ be a linear map which is not bounded. Such a map exists as we have just shown. Then we can define the norm $|| \cdot ||_0$ as we did in (a). As T is not bounded (a) gives that $|| \cdot ||_X$ and $|| \cdot ||_0$ are not equivalent and furthermore by construction we have $||x||_X \leq ||x||_0$.

As the two norms are not equivalent, problem 1 in HW3 gives that $(X, ||\cdot||_0)$ cannot be complete if $(X, ||\cdot||_X)$ is a Banach space and in particular complete.

(e)

Give an example of a vector space X equipped with two inequivalent norms $||\cdot||$ and $||\cdot||'$ satisfying $||x||' \leq ||x||$, for all $x \in X$, such that $(X, ||\cdot||)$ is complete, while $(X, ||\cdot||')$ is not.

Let $(\ell_1(\mathbb{N}), ||\cdot||_1) = (X, ||\cdot||_X)$ which is a Banach space. Consider also $||\cdot||_\infty$ and recall $||(x_i)_{i\geq 1}||_\infty = \sup\{|x_i| : i \geq 1\}$. Note that $||\cdot||_\infty \leq ||\cdot||_1$ by definition. We have to find a Cauchy sequence with respect to $||\cdot||_\infty$ which does not converge. Consider the sequence $(y_j)_{j\geq 1}$ where each y_j is a sequence $(y_{j,i})_{i\geq 1}$ defined by

$$y_{j,i} = \begin{cases} \frac{1}{i} & i \leq j \\ 0 & \text{otherwise.} \end{cases}$$

To see that this sequence is Cauchy, let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$. Then we have for $n > m \geq N$ that

$$||y_n - y_m||_\infty = \sup\{|y_{n,i} - y_{m,i}| : i \geq 1\} = \frac{1}{m+1} < \varepsilon.$$

So it is a Cauchy sequence and furthermore it is clear by a virtually identical argument that the limit of this sequence is $(x_i)_{i\geq 1}$ where $x_i = \frac{1}{i}$, but it is well known that $(x_i)_{i\geq 1} \notin \ell_1(\mathbb{N})$, hence $\ell_1(\mathbb{N})$ is not complete with respect to $||\cdot||_\infty$. Thus by problem 1 in HW3, the 1-norm and the infinity norm are not equivalent.

Problem 2

Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), ||\cdot||_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, 0, \ldots) : a, b \in \mathbb{C}\}.$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, \ldots) = a + b$, for all $a, b \in \mathbb{C}$.

(a)

Show that f is bounded on $(M, ||\cdot||_p)$ and compute $||f||$.

First we consider $p = 1$. Let $x = (x_i)_{i\geq 1} \in M$, then $|f(x)| = |x_1 + x_2| \leq |x_1| + |x_2| = ||x||_1$, hence $||f|| \leq 1$. If $x_1 = 1$ and $x_2 = 0$, then $|f(x)| = 1$, hence we also have $||f|| \geq 1$ and thus $||f|| = 1$.

Now suppose $p > 1$. To show that f is bounded and to determine $||f||$ we note that for a sequence $x = (x_i)_{i\geq 1} \in M$ we have that $|f(x)| = |x_1 + x_2| \leq |x_1| + |x_2|$. If we now let $(y_i)_{i\geq 1} \in M$ be a sequence with $y_1 = y_2 = 1$ we can use Hölders inequality to get

$$|f(x)| \leq |x_1| + |x_2| = \sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |y_i|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} = ||x||_p \cdot 2^{1-\frac{1}{p}}.$$

Hence f is bounded and furthermore we have $||f|| \leq 2^{1-\frac{1}{p}}$. Now if we let $x_1 = x_2 = 2^{-\frac{1}{p}}$ we have $|f(x)| = |2^{-\frac{1}{p}} + 2^{-\frac{1}{p}}| = 2^{1-\frac{1}{p}}$ and

$$||x||_p = ((2^{-\frac{1}{p}})^p + (2^{-\frac{1}{p}})^p)^{\frac{1}{p}} = 1$$

hence $||f|| \geq 2^{1-\frac{1}{p}}$ and thus $||f|| = 2^{1-\frac{1}{p}}$.

(b)

Show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $||F|| = ||f||$.

Corollary 2.6 gives the existence of $F \in (\ell_p(\mathbb{N}))^*$, such that F extends f and $||F|| = ||f||$.
Now we want to use HW1 problems 4 and 5. In problem 5 we use the course of action from problem 4 to show that each $F \in (\ell_p(\mathbb{N}))^*$ corresponds bijectively to F_x for an $x \in \ell_q(\mathbb{N})$ where $F_x : \ell_p(\mathbb{N}) \to \mathbb{K}$ is defined by

$$(y_i)_{i\geq 1} \mapsto \sum_{i=1}^{\infty} x_i y_i$$

and q is the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Thus let $F = F_x$ be an extension of f. We want to show that it is unique, so it suffices to show that the sequence x is unique.
Let $y \in M$, then $y_1 + y_2 = f(y) = F_x(y) = \sum_{i=1}^{\infty} x_i y_i$ implying that $x_1 = x_2 = 1$. To determine the rest of the coordinates we use that $||F_x|| = ||f||$. We also use that the isomorphism $x \mapsto F_x$ from HW1 is an isometry.

$$2^{1-\frac{1}{p}} = ||F_x|| = ||x||_q = \left(\sum_{i=1}^{\infty} |x_i|^q\right)^{\frac{1}{q}} = \left(2 + \sum_{i=3}^{\infty} |x_i|^q\right)^{\frac{1}{q}}.$$

Hence as $\frac{1}{q} = 1 - \frac{1}{p}$ we must have $\sum_{i=3}^{\infty} |x_i|^q = 0$ and thus as the absolute value is a norm on $\mathbb{K}$ we get $x_i = 0$ for all $i \geq 3$. Hence the sequence x is uniquely determined and thus by HW1 the extension $F = F_x$ is unique as desired.

(c)

Show that if $p = 1$, then there are infinitely many linear functionals F on $\ell_1(\mathbb{N})$ extending f and satisfying $||F|| = ||f||$.

From the same HW1 problems we also have $(\ell_1(\mathbb{N}))^* \cong \ell_\infty(\mathbb{N})$. And similarly as in (b), corollary 2.6 gives an extension F, which is equal to F_x for some $x \in \ell_\infty$. Furthermore we get $x_1 = x_2 = 1$ as before. Now we have

$$1 = ||f|| = ||F_x|| = ||x||_\infty = \sup\{|x_i| : i \geq 1\}$$

which implies that we do not need $x_i = 0$ for all $i \geq 3$, instead we just need $|x_i| \leq 1$. Clearly there are infinitely many sequences satisfying this, and furthermore each of them defines an extension of f. Hence there are infinitely many extensions F of f with $||F|| = ||f||$.

Problem 3

Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a)

Let $n \geq 1$ be an integer. Show that no linear map $F : X \to \mathbb{K}^n$ is injective.

Suppose for contradiction that $F : X \to \mathbb{K}^n$ is linear and injective. Then F is a bijection onto its image, hence we have an isomorphism of vector spaces $X \cong \mathrm{Im}(F)$. But the image of F is a subspace of $\mathbb{K}^n$, hence it is finite dimensional whereas X is infinite dimensional. This is a contradiction and thus no injective linear map $X \to \mathbb{K}^n$ exists.

(b)

Let $n \geq 1$ be an integer and let $f_1, f_2, \ldots, f_n \in X^*$. Show that $\bigcap_{j=1}^n \ker(f_j) \neq \{0\}$.

Define $F : X \to \mathbb{K}^n$ by $F(x) = (f_1(x), \ldots, f_n(x))$. F is linear as each f_i is linear. Now (a) gives that F is not injective, hence as F is linear, we have

$$\{0\} \neq \ker(F) = \bigcap_{j=1}^n \ker(f_j)$$

as desired.

(c)

Let $x_1, x_2, \ldots, x_n \in X$. Show that there exists $y \in X$ such that $||y|| = 1$ and $||y - x_j|| \geq ||x_j||$ for all $j = 1, 2, \ldots, n$

If $x_i = 0$, then any y with $||y|| = 1$ would satisfy the desired. Hence we can assume all x_i are non-zero. Note that we can always find an element with norm 1 as X is infinite dimensional in particular non-zero.
Then for each x_i we can use theorem 2.7 (b) to obtain a linear functional f_i for each x_i, such that $||f_i|| = 1$ and $f_i(x_i) = ||x_i||$. Now consider $F : X \to \mathbb{K}^n$ defined by $F(x) = (f_1(x), \ldots, f_n(x))$ which is not injective by (a) and (b). Hence there exists an $\alpha \neq 0$ s.t. $F(\alpha) = 0$. Now we set $y := \frac{\alpha}{||\alpha||}$. Then $||y|| = 1$. We want to show that y has the desired property. Note that $f_i(y) = 0$ for all i as $F(y) = 0$. Now

$$||x_j|| = f_j(x_j) = f_j(x_j - y)$$

and as $||f|| = 1$ we have $|f_j(x_j - y)| \leq ||x_j - y||$ and thus $||x_j|| \leq ||x_j - y||$. This holds for each j and hence we have found a y satisfying the desired.

(d)

Show that one cannot cover the unit sphere $S = \{x \in X : ||x|| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

Let $B_1, \ldots, B_n$ be a finite family of closed balls covering S. Let z_i and r_i be the center and radius of B_i respectively. Then by (c) there exists a y with norm 1 such that $||y - z_i|| \geq ||z_i||$ for all i. Furthermore as $||y|| = 1$ we have $y \in S$ and thus $y \in B_j$ for some j. Thus we have $r_j \geq ||y - z_j|| \geq ||z_j|| = ||z_j - 0||$, hence $0 \in B_j$. Thus we cannot find a finite family of closed balls in X such that none of them contain 0.

(e)

Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

Suppose S is compact. Then consider $B_{\frac{1}{2}}(x)$ for all $x \in S$, which are the open balls of radius $\frac{1}{2}$ around each x.

Note that these open balls constitutes an open covering of S, hence by assumption there exists a finite subcover of S. If we take the closed version of the balls in this finite subcover, we then have a finite family of closed balls, which still covers S, however 0 is not in any of them which contradicts (d). Thus S cannot be compact.

Now the closed unit ball has S as a closed subset, hence as S is not compact, neither is the closed unit ball, as a closed subset of compact space is compact. To see that S is closed in the closed unit ball, note that its complement is the open unit ball, which is open since it is an open ball.

Problem 4

Let $L_1([0,1],m)$ and $L_3([0,1],m)$ be the Lebesgue spaces on $[0,1]$.
Recall from HW2 that $L_3([0,1],m) \subsetneq (L_1([0,1],m)$. For $n \geq 1$, define

$$E_n := \left\{ f \in L_1([0,1],m) : \int_{[0,1]} |f|^3 dm \leq n \right\}. \tag{1}$$

(a)

Given $n \geq 1$, is the set $E_n \subset L_1([0,1],m)$ absorbing?

Let $g \in L_1([0,1],m) \backslash L_3([0,1],m)$. Then if E_n was absorbing there would exist $t > 0$ s.t. $t^{-1}g \in E_n$, hence

$$\int_{[0,1]} |t^{-1}g(x)|^3 dm(x) = t^{-3} \int_{[0,1]} |g(x)|^3 dm(x) \leq n$$

but this is clearly a contradiction as $g \notin L_3([0,1],m)$. Thus E_n is not absorbing.

(b)

Show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.

We want to show that E_n^c is dense in $L_1([0,1],m)$, i.e. we want to show that for any $f \in L_1([0,1],m)$ there is a sequence in E_n^c which converges to f. If we have this, then clearly for any open ball $B_\varepsilon(f) \subset \text{Int}(E_n)$ it would contain an element of E_n^c as there exists a sequence in E_n^c converging to f. Thus there are no open balls and $\text{Int}(E_n) = \varnothing$.

Let $f \in L_1([0,1],m)$ we want to construct a sequence in E_n^c converging to f.

If $f \in E_n^c$ the constant sequence works, so suppose $f \in E_n$.
Let $g \in L_1([0,1],m) \backslash L_3([0,1],m)$ and define a sequence $(h_i)_{i \geq 1}$ by $h_i = f + \frac{1}{i}g$.
We want to show $h_i \in E_n^c$, and that the sequence converges to f.

As $g \notin L_3([0,1],m)$ it is clear that $\frac{1}{i}g \notin L_3([0,1],m)$, but also that $h_i = f + \frac{1}{i}g$, which just adds $f \in E_n$, is also not in $L_3([0,1],m)$. In particular $h_i \in E_n^c$. Let us now show that $(h_i)_{i\geq 1}$ converges to f in $L_1([0,1],m)$.
Given $\varepsilon > 0$, we have that

$$\int_{[0,1]} |g| dm = s < \infty.$$

There exists $i \in \mathbb{N}$ such that $\frac{s}{i} < \varepsilon$. Thus we have

$$||h_i - f||_1 = \int_{[0,1]} |h_i - f| dm = \int_{[0,1]} \left| \frac{1}{i} g \right| dm = \frac{1}{i} \int_{[0,1]} |g| dm = \frac{s}{i} < \varepsilon.$$

Thus the sequence does converge to f showing that E_n^c is dense and hence $\text{Int}(E_n) = \varnothing$.

(c)

Show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$.

Let $(f_i)_{i\geq 1}$ be a sequence in E_n with limit f. We want to show that $f \in E_n$.
We use corollary 12.8 from Schilling (An2), which gives us that there exists a subsequence $(f_{i_j})_{j\geq 1}$ converging pointwise to f almost everywhere. Hence also the $(|f_{i_j}|^3)_{j\geq 1}$ converges pointwise almost everywhere to $|f|^3$, as taking the absolute value and cubing a number are both continuous functions. We now want to use Fatou's lemma to get

$$\int_{[0,1]} |f|^3 dm = \int_{[0,1]} \liminf_{j\to\infty} |f_{i_j}|^3 dm \leq \liminf_{j\to\infty} \int_{[0,1]} |f_{i_j}|^3 dm \leq n$$

Hence $f \in E_n$ as desired. Thus E_n is closed and hence with (b) we have shown that E_n is nowhere dense.

(d)

Conclude from (b) and (c) that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

We want to show

$$L_3([0,1],m) = \bigcup_{n=1}^{\infty} E_n.$$

"$\subseteq$": Let $f \in L_3([0,1],m)$, then

$$\left(\int_{[0,1} |f|^3 dm \right)^{\frac{1}{3}} < \infty$$

hence it is equal to some $k \in \mathbb{R}$. Then there exists $n \in \mathbb{N}$ such that $n > k^3$, hence $f \in E_n$.

"$\supseteq$": Clear as each E_n is contained in $L_3([0,1], m)$.

Thus $L_3([0,1], m)$ is the union of a sequence of nowhere dense sets in $L_1([0,1], m)$, and hence it is of first category in $L_1([0,1], m)$.

Problem 5

Let H be an infinite dimensional separable Hilbert space with associated norm $||\cdot||$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

(a)

Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $||x_n|| \to ||x||$, as $n \to \infty$?

Given $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ s.t. $\varepsilon > ||x_n - x|| \geq |||x_n|| - ||x|||$ where we have used the reverse triangle inequality. Hence $||x_n||$ converges to $||x||$ as desired.

(b)

Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $||x_n|| \to ||x||$, as $n \to \infty$?

We want to give a counterexample.
Let $(e_n)_{n\geq 1}$ be a countable orthonormal basis, and regard it as a sequence in H. Such a countable orthonormal basis in H exists as H is separable (top of p. 44 in lecture notes). Recall that the Riesz representation theorem gives that any $F \in H^*$ is given by F_y where $F_y(x) = \langle x, y\rangle$ for some $y \in H$. We want to show that $(e_n)_{n\geq 1} \xrightarrow{w} 0$.
By problem 2 in HW4 this is the case iff. $F(e_n) \to F(0) = 0$ for any $F \in H^*$. By Riesz this means that $\langle e_n, y\rangle \to 0$ for all $y \in H$.

Bessel's inequality gives that $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq ||x||^2$. This series converges as $||x||^2$ is finite, which implies that $|\langle x, e_n\rangle| \to 0$, hence as $\langle a, b\rangle = \overline{\langle b, a\rangle}$ and $|z| = |\overline{z}|$, we also have $\langle e_n, x\rangle \to 0$ and as mentioned this implies that $(e_n)_{n\geq 1}$ converges weakly to 0.

We have $||e_n|| = 1$ for all n, hence $||e_n|| \to 1 \neq 0 = ||0||$ and this is a counterexample to the statement.

(c)

Suppose that $||x_n|| \leq 1$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Is it true that $||x|| \leq 1$?

If $x = 0$, then the statement holds, so suppose $x \neq 0$.
Then Theorem 2.7 (b) gives $f \in H^*$ s.t. $||f|| = 1$ and $f(x) = ||x||$. Thus we have

$$||x|| = f(x) = \lim_{n\to\infty} f(x_n).$$

Here we used problem 2 HW4 again. As $||f|| = 1$ we have

$$\frac{|f(x_n)|}{||x_n||} \leq 1 \Leftrightarrow |f(x_n)| \leq ||x_n|| \Rightarrow \lim_{n\to\infty} |f(x_n)| \leq \sup_{n\in\mathbb{N}} ||x_n|| \leq 1.$$

Combining this with the above we get $||x|| \leq 1$ as desired.