Mandatory Assignment 1
Functional Analysis

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Problem 1

In this problem I let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be non-zero vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a)

First I will show that $\|\cdot\|_0$ is a norm on X.
Let $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ for all $x \in X$. For $\|\cdot\|_0$ to be a norm on X it has to satisfy 3 conditions.

(1) I have to show that $\|x+y\|_0 \leq \|x\|_0 + \|y\|_0 \ \forall x, y \in X$.
So

$$\|x+y\|_0 = \|x+y\|_X + \|T(x+y)\|_Y = \|x+y\|_X + \|Tx+Ty\|_Y \leq (\|x\|_X + \|y\|_X) + (\|Tx\|_Y + \|Ty\|_Y =$$

$$= \|x\|_X + \|Tx\|_Y + \|y\|_X + \|Ty\|_Y = \|x\|_0 + \|y\|_0 \ \forall x, y \in X$$

where I use the triangle inequality and the fact that T is linear. Hence I have shown that $\|x+y\|_0 \leq \|x\|_0 + \|y\|_0 \ \forall x, y \in X$.

(2) Furtermore I will show that $\|\alpha x\|_0 = |\alpha|\|x\|_0$.
So

$$\|\alpha x\|_0 = \|\alpha x\|_X + \|T(\alpha x)\|_Y = |\alpha|\|x\|_X + \|\alpha Tx\|_Y = |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y$$

$$= |\alpha|(\|x\|_X + \|Tx\|_Y) = |\alpha| \cdot \|x\|_0$$

Hence $\|\alpha x\|_0 = |\alpha|\|x\|_0$.

(3) I will now show that $\|x\|_0 = 0 \Leftrightarrow x = 0$.
$\Rightarrow$: Assume that $\|x\|_0 = 0$ which implies that $\|x\|_X + \|Tx\|_Y = 0$. This will only apply if $\|x\|_X = 0$ and $\|Tx\|_Y$ because $\|\cdot\|_X \geq 0$ and $\|\cdot\|_Y \geq 0$. So this gives that $\|x\|_X = 0 \Rightarrow x = 0$.

$\Leftarrow$: I know assume that $x = 0$. This gives that

$$\|0\|_0 = \|0\|_X + \|T \cdot 0\|_Y = \|0\|_X + \|0\|_Y$$

So $\|0\|_0 = \|0\|_X + \|0\|_Y$.

Hence I have shown that $\|x\|_0 = 0 \Leftrightarrow x = 0$. I have now shown that $\|x\|_0$ is a norm on X.

I will now show that the two norms $\|x \cdot\|_X$ and $\|x \cdot\|_0$ are equivalent $\Leftrightarrow$ T is bounded.
$\Rightarrow$:
I assume that the two norms $\|x \cdot\|_X$ and $\|x \cdot\|_0$ are equivalent and I want to show that T is bounded.

Since the two norms $\|x \cdot\|_X$ and $\|x \cdot\|_0$ are equivalent then by definition of equivalent (definition 1.4 in lecture notes) we then have that there exist $0 < C_1 < C_2 < \infty$ such that

$$C_1\|x\|_X \leq \|x\|_0 \leq C_2\|x\|_X \text{ for } x \in X$$

By using these inequalities I get

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y \Rightarrow \|Tx\|_Y = \|x\|_0 - \|x\|_X \leq C_2\|x\|_X - \|x\|_X \leq C_2\|x\|_X$$

I have now shown that there exists $C = C_2 > 0$ such that $\|Tx\|_Y \leq C_2\|x\|_X$ $\forall x \in X$. Hence I have now shown that T is bounded.

$\Leftarrow$:
I now assume that T is bounded and I want to show that the two norms $\|x \cdot\|_X$ and $\|x \cdot\|_0$ are equivalent. This means, by definition 1.4 in lecture notes, that I want to show that there exists $0 < C_1 \leq C_2 < \infty$ such that:

$$C_1\|x\|_X \leq \|x\|_0 \leq C_2\|x\|_X, \ x \in X$$

The fact that T is bounded gives that there exists $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$. By using this I observe:

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq \|x\|_X + C\|x\|_X = (C+1)\|x\|_X$$

And

$$\|x\|_X = \|x\|_0 - \|Tx\|_Y \leq \|x\|_0$$

The last inequality applies since $\|Tx\|_Y \geq 0$. Then there exists $0 < C_1 \leq C_2 < \infty$ such that

$$C_1\|x\| \leq \|x\|_0 \leq C_2\|x\|_X$$

Where $C > 0$ and it applies that $C_1 = 1$ and $C_2 = C + 1$. I can now conclude by using definition 1.4 from lecture notes that $\|\cdot\|_X$ and $\|\cdot\|$. are equivalent norms on X.

(b)

I assume that X is finite dimensional and I want to show that any linear map $T : X \to Y$ is bounded.
Notice that from theorem 1.6 we know that if X is a finite dimensional vector space then any two norms on X are equivalent. So since we have assumed that X is finite dimensional then theorem 1.6 gives that any two norms on X are equivalent. In problem 1a I have shown that if the two norms $\|x \cdot \|_X$ and $\|x \cdot \|_0$ are equivalent then $T : X \to Y$ is bounded. Since T is an arbitrary linear map it is now possible to conclude that if X is finite dimensional then any linear map is bounded.

(c)

I assume that X is infinite dimensional and I want to show that there exists a linear map $T : X \to Y$ which is not bounded (=continuous).
By the fact that X is infinite dimensional, I notice that X admits a Hamel basis, which is a consequence of Zorn's lemma. This Hamel basis is defined as $B_x = (e_i)_{i \in I}$, for some index set I and e_i where $i \in I$, where e_i are elements in X.
I now define a linear map $T : X \to Y$ and I want to show that it is not bounded.
I let every e_i in X be normalized sucht that:

$$T\bigg(\frac{e_i}{\|e_i\|}\bigg) = i \cdot y$$

where y is fixed and $0 \neq y \in Y$ and $i \in \mathbb{N}$. Furthermore notice that if $i \notin \mathbb{N}$ I set $T\bigg(\frac{e_i}{\|e_i\|}\bigg) = 0$. We have that $T\bigg(\frac{e_i}{\|e_i\|}\bigg) = 0$ is well-defined since $\{\frac{e_i}{\|e_i\|}\}$ is a linearly independent subset of X. This applies because $\{\frac{e_i}{\|e_i\|}\}$ is in B_x. Furthermore

$$\bigg\{\frac{e_i}{\|e_i\|}\bigg\}_{i \in I} \subseteq \{x \in X : \|x\| \leq 1\} := B$$

Hence

$$T\bigg\{\frac{e_i}{\|e_i\|}\bigg\}_{i \in I} \subseteq TB$$

Finally notice

$$0 < i\|y\| \leq \sup_{x \in B} \|Tx\|$$

I have now shown that for each $i \in I$ there exists a linear map $T : X \to Y$ which is not bounded.

(d)

Assume that X is infinite dimensional. By using this, we know from problem 1c that there exist a linear map $T : X \to Y$ which is not bounded. By using this, we have from problem 1a that the two norms $\| \cdot \|_X$ and $\| \cdot \|_0$ are not equivalent. Hence for X infinite dimensional there exist a norm $\| \cdot \|_0$ on X which is not equivalent to the given norm $\| \cdot \|_X$.
The norm $\|x\|_X$ fulfill that

$$|x\|_X \leq |x\|_X + |Tx\|_Y = |x\|_0 \ \forall x \in X$$

The inequality applies since $\|Tx\|_Y \geq 0$.
Finally I will argue that $(X, \| \cdot \|_0)$ is not complete if $(X, \| \cdot \|_X)$ is a Banach space.
By using problem 1 in HW3 we can say that $(X, \| \cdot \|_0)$ is not complete, since we have that the norm $\| \cdot \|_0$ on X is not equivalent to the given norm $\| \cdot \|_X$, which gives that $(X, \| \cdot \|_0)$ is not complete.
So if we have $(X, \| \cdot \|_X)$ is a Banach space and hence complete then $(X, \| \cdot \|_0)$ cannot be complete, because of the fact that the norms are not equivalent.

(e)

Let $X = \ell(\mathbb{N})$ which is equipped with the two norms $\| \cdot \|_1$ and $\| \cdot \|_\infty$.
I will now show that the two norms $\| \cdot \|_1$ and $\| \cdot \|_\infty$ are inequivalent. This will be done by taking a finite sequence $(y_n)_{n \in \mathbb{N}} \subset \ell(\mathbb{N})$. Hence

$$\|y\|_1 = \sum_{i=1}^{n} |y_i| \geq \max_{i=1,\ldots,n} \{|y_i|\} = \|y\|_\infty$$

Hence $\|y\|_1 \geq \|y\|_\infty$.
To show that these two norms are inequivalent I observe a sequence $(b_n)_{n \in N}$. For this sequence it holds that $\nexists C > 0$ such that

$$\|b_n\|_1 \leq C \|b_n\|_\infty$$

We have

$$(b_n)_{n \in N} = (b_1, ..., b_k, 0, 0, ..., 0) = (1, 1, ..., 1, 0, 0, ..., 0)$$

So

$$\|b_n\|_1 = \sum_{i=1}^{k} |1| = \sum_{i=1}^{k} 1 = k$$

So we have

$$\|b_n\|_\infty = \max_{i\in\mathbb{N}}\{|b_i|\} = 1$$

I can now conclude that the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are inequivalent. This applies since it is possible for every $C > 0$ to find a $k > C$. Hence $\nexists C > 0$ such that $\|b_n\|_1 \leq C\|a_n\|_\infty$.

I will now argue that $(\ell_1(\mathbb{N}, \|\cdot\|_1)$ is complete while $(\ell_p(\mathbb{N}, \|\cdot\|_\infty)$ is not complete.
By Riesx-Fischer theorem it applies that $(\ell_p(\mathbb{N}, \|\cdot\|_p)$ is a Banach space for $1 \leq p < \infty$. This gives that $(\ell_1(\mathbb{N}, \|\cdot\|_1)$ is a Banach space, and since a Banach space is complete, I can now conclude that $(\ell_1(\mathbb{N}, \|\cdot\|_1)$ is complete.

I will now show that $(\ell_p(\mathbb{N}, \|\cdot\|_\infty)$ is not complete.To show this I observe the sequence of sequences $(y_n(k))_{n\in\mathbb{N}}$ where it applies that

$$y_n(k) = \begin{cases} \frac{1}{k} & when\ 1 \leq k \leq n \\ 0 & when\ k > n \end{cases}$$

Notice that $(y_n(k))_{n\in\mathbb{N}} \subseteq \ell_1(\mathbb{N})$ for all n and k. This applies since $y_n(k)$ is finite with respect to the norm $\|\cdot\|_1$ for all n and for each k.
I now claim that $y(k) = \frac{1}{k}\ \forall k \in \mathbb{N}$ and notice

$$\|y_n(k) - y(k)\|_\infty = \max_{n\in\mathbb{N}}\{|y_n(k) - y(k)|\} = \left|\frac{1}{n+1}\right| \to 0$$

This gives that $(y_n(k))_{n\in\mathbb{N}}$ is a cauchy sequence with respect to the norm $\|\cdot\|_\infty$. From the fact that $\left|\frac{1}{n+1}\right| \to 0$, I can conclude that $y(k) \notin \ell_1(\mathbb{N})$. Hence $(\ell_1(\mathbb{N}, \|\cdot\|_\infty$ is not complete.

Problem 2

In this problem I let $1 \leq p < \infty$ be fixed and I consider the subspace M of the Banach space $(\ell_p(\mathbb{N}, \|\cdot\|_p)$ which is considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, ...) : a, b \in \mathbb{C}\}$$

Furthermore I let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, 0, ...) = a + b$, for all $a, b \in \mathbb{C}$

(a)

I will show that f is bounded on $(M, \|\cdot\|_p)$ and furthermore compute $\|f\|$.

I will show that f is bounded on $(M, \|\cdot\|_p)$. First I will show that f is linear and next I will show that there exist $C > 0$ such that $\|f(x)\| \leq C\|x\| \; \forall x \in M$.
I the following I will show that f is linear:
I let $\alpha, \beta \in \mathbb{C}$. Furthermore I define

$$\gamma = (a_1, b_1, 0, 0, ...) \in M$$

$$\delta = (a_2, b_2, 0, 0, ...) \in M$$

Now I look at $f(\alpha \cdot \gamma + \beta \cdot \delta)$

$$f(\alpha \cdot \gamma + \beta \cdot \delta) = f(\alpha \cdot a_1 + \beta \cdot a_2, \alpha \cdot b_1 + \beta \cdot b_2, 0, 0, ...) = \alpha \cdot a_1 + \beta \cdot a_2 + \alpha \cdot b_1 + \beta \cdot b_2$$

$$= \alpha(a_1 + b_1) + \beta(a_2 + b_2) = \alpha \cdot f(\gamma) + \beta \cdot f(\delta)$$

Hence I have shown that f is linear.

I will now show that f is bounded. This means that I have show that there exist $C > 0$ such that

$$\|a + b\|_1 \leq C \cdot \|(a, b, 0, 0, ...)\|_p = C \cdot \|(a, b)\|_p$$

Notice that $\|(a, b)\|_p$ is a norm on $\mathbb{C}^2$.

We now observe that

$$\|a + b\|_1 = |a + b| \leq |a| + |b| = \|(a, b)\|_1 \leq C \cdot \|(a, b)\|_p = C\|(a, b, 0, 0, ...)\|_p$$

Notice that $\|(a, b)\|_1$ is a norm on $\mathbb{C}^2$.
So now we see that the first inequality comes from the triangular inequality. Furtermore the second inequality comes from the fact that $\mathbb{C}^2$ is a finite dimensional vector space. This gives, by using theorem 1.6, that every norm on $\mathbb{C}^2$ is equivalent. Hence $\exists C > 0$ such that the inequality I have shown applies for all $(a, b) \in \mathbb{C}^2$

I will now compute $\|f\|$
The claim is that $\|f\| = 2^{1-\frac{1}{p}}$. I will show this by showing that $\|f\| \leq 2^{1-\frac{1}{p}}$ and $\|f\| \geq 2^{1-\frac{1}{p}}$.
Now let $t = \left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, 0, ...\right)$. Then $\|t\|_p = 1$ since

$$\|t\|_p = \left(\left|\frac{1}{2^{\frac{1}{p}}}\right|^p + \left|\frac{1}{2^{\frac{1}{p}}}\right|^p\right)^{\frac{1}{p}} = \left(\frac{1}{2} + \frac{1}{2}\right)^{\frac{1}{p}} = 1$$

And furthermore

$$\|f\| = \sup\{|a+b| : \|(a,b,0,0,...)\|_p = 1\} \geq \left|\frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}}\right| = \frac{2}{2^{\frac{1}{p}}} = 2^{1-\frac{1}{p}}$$

The first inequality comes from the fact that

$$\left|\frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}}\right| \in \{|a+b| : \|(a,b,0,0,...)\|_p = 1\}$$

So now I have shown that $\|f\| \geq 2^{1-\frac{1}{p}}$.

I will now show that $\|f\| \leq 2^{1-\frac{1}{p}}$.
This claim applies because

$$|a+b| \leq |a|+|b| = \|(a,b,0,0,...)\|_1 = \|(a\cdot 1, b\cdot 1, 0, 0, ...)\|_1 \leq \|(a,b,0,0,...)\|_p \cdot \|(1,1,0,0,...)\|_q$$

The last inequality comes from Hölders inequality for $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow q = \frac{p}{p-1}$ where p is fixed

Now let $\|(a,b,0,0,...)\|_p = 1$ and

$$|a+b| \leq \|(1,1,0,0,...)\|_q = \sum_{i=1}^{2}(|1|^q)^{\frac{1}{q}} = 2^{\frac{1}{q}} = 2^{1-\frac{1}{p}}$$

So this inequality is true for all $(a,b,0,0,...)$ with norm 1 so

$$\sup\{|a+b| : \|(a,b,0,0,...)\|_p = 1\} \leq 2^{1-\frac{1}{p}}$$

which shows that for all $|a+b| \in S$ we have that $\|a+b\| \leq 2^{1-\frac{1}{p}}$.
Hence $\|f\| \geq 2^{1-\frac{1}{p}}$ and hence $\|f\| = 2^{1-\frac{1}{p}}$

(b)

I will show that if $1 < p < \infty$ then there is a unique linear functional F on $\ell_p(\mathbb{N}$ extending f and satisfying $\|F\| = \|f\|$.

I start by showing the existence.
I notice that $(\ell_p(\mathbb{N}), \|\cdot\|_p)$ is a normed vectorspace and that $M \subseteq (\ell_p(\mathbb{N}), \|\cdot\|_p)$. Furtermore $f \in M^*$ because in problem 1a I have shown that f is linear and bounded. From all these, I can now use corollary 2.6 to the Hahn-Banach extension theorem and then conclude that there exist $F \in \ell_p(\mathbb{N})^*$ such that $F_{|_M} = f$ and $\|F\| = \|f\|$.
I will now show the uniqueness of such a F.

Note $1 < p < \infty$. From problem 5 in HW1 I have that $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$ if $\frac{1}{p} + \frac{1}{q} = 1$. Let

$$F(x) = \sum_{n=1}^{\infty} x_n y_n$$

for $y = (y_n)_{n\geq 1} \in \ell_q(\mathbb{N})$ and $x = (x_n)_{n\geq 1} \in \ell_p(\mathbb{N})$

We have that $2^{\frac{1}{q}} = 2^{1-\frac{1}{p}} = \|f\| = \|F\|$.

Since F is represented by $y \in \ell_q(\mathbb{N})$ then we may have that $\|y\|_q = 2^{\frac{1}{q}}$ and this is what I will show now.

Observer that $F_{|M}(x) = f(x) = x_1 + x_2$ which gives that $y = (1, 1, y_3, y_4, ...)$. Furthermore we have

$$\|y\|_q = \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{\frac{1}{q}} = (|1|^q + |1|^q + |y_3|^q + |y_4|^q + ...+)^{\frac{1}{q}} = \|F\| = 2^{\frac{1}{q}}$$

So for $\|y\| = \|F\|$ to be valid, based on the isometry criteria, $y_3, y_4, ... = 0$ and I can now conclude that $y = (1, 1, 0, 0, ...)$.

It is now time to argue for the uniqueness.

We choose a $F' \in (\ell_p(\mathbb{N})^*)$, which is another linear functional such that $F'_{|M} = f$ and $\|F'\| = \|f\|$. Because I have argue that $y = (1, 1, y_3, y_4$ where y_3 and y_4 is arbitrary, I can do the same step with F' at get that $F'_{|M} = x_1 + x_2$.Hence the uniqueness is shown and $F'(x) = F(x)$.

(c)

I let $p = 1$ and I define $F_i : \ell_1(\mathbb{N}) \to \mathbb{C}$ given by $(x_1, x_2, x_3, ...) \mapsto x_1 + x_2 + x_i$ for $i > 2$. This functional is clearly linear on $\ell_1(\mathbb{N})$. Furthermore it is an extension on $\ell_1(\mathbb{N})$. This applies because if we look at $F_{i\ |M}(x) = x_1 + x_2 = f(x)$ for $x \in M$. It must apply that

$$\|F_i\| \geq \|f\| = 2^{1-\frac{1}{1}} = 1$$

since F_i extends f. Notice that $p = 1$.

I now look at

$$\|F_i\|_1 = \sup\{|F_i x| : \|x\|_1 = 1\} = \sup\{|x_1 + x_2 + x_i| : \|x\|_1 = 1\}$$

$$\leq \sup\{|x_1| + |x_2| + |x_i| : \|x\|_1 = 1\} \leq 1$$

So I have now shown that $\|F_i\|_1 \leq 1$ and I have earlier argue that $\|F_i\|_1 \geq 1$. This gives that $\|F_i\|_1 = 1 = \|f\|$

Hence F_i is linear functional extending f. Furthermore there are infinitely many linear functional F on $\ell_1(\mathbb{N})$ extending f and which satisfy $\|F\| = \|f\|$ since $i > 2$.

Problem 3

In this problem X will be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a)

Let $n \geq 1$ be an integer. In this section I will show that no linear map $F : X \to \mathbb{K}^n$ is injective.
This will be shown by contradiction, so I assume that $F : X \to \mathbb{K}^n$ is injective. I set $x_1, ..., x_{n+1}$ to be linearly indenpendent in X and $F(x_1), ..., F(x_{n+1})$ to be linearly dependent. Then there exists $a_1, ..., a_{n+1}$, where at least one of these is different from zero, and since $F(x_1), ..., F(x_{n+1})$ is linearly dependent it all gives that

$$a_1F(x_1) + ... + a_{n+1}F(x_{n+1}) = 0$$

The linearity of F furtermore gives that

$$a_1F(x_1) + ... + a_{n+1}F(x_{n+1}) = F(a_1x_1 + ... + a_{n+1}x_{n+1})$$

And then

$$F(a_1x_1 + ... + a_{n+1}x_{n+1}) = 0$$

Furthermore the injectivity of F gives that $a_1x_1 + ... + a_{n+1}x_{n+1} = 0$. From the fact that $x_1, ..., x_{n+1}$ are linearly indenpendent in X it gives that all $a_i = 0$. This gives the contradiction since I have notice that there exists $a_1, ..., a_{n+1}$, where at least one of these is different from zero. The contradiction gives that no linear map $F : X \to \mathbb{K}^n$ is injective.

(b)

I let $n \geq 1$ be an integer and I let $f_1, f_2, ..., f_n \in X^*$. In this part I will show that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$$

I start by looking at the map $F : X \to \mathbb{K}^n$ which is given by $F(x) = (f_1(x), f_2(x), ..., f_n(x))$ for $x \in X$.
The map is linear because notice

$$a \cdot F(x) = a(f_1(x), ..., f_n(x)) = (af_1(x), ..., af_n(x))$$

Hence $f_1(x), ..., f_n(x)$ is linear, and then this linear map $F : X \to \mathbb{K}^n$ is not injective because in problem 3a I have shown that no linear map is injective. By this I now notice that

$$\ker(F) \neq \{0\} \Rightarrow \ker(f_1(x), f_2(x), ..., f_n(x)) \neq \{0\}$$

From the fact that $\ker(F) \neq \{0\}$ notice that there exist $x \neq 0$ such that

$$F(x) = (f_1(x), f_2(x), ..., f_n(x)) = 0$$

which gives that $F(x)$ is equal to zero if and only $f_1(x) = 0, ..., f_n(x) = 0$.
From this I can now conclude that

$$0 \neq \ker(F) = \bigcap_{j=1}^{n} \ker(f_j)$$

(c)

I let $x_1, x_2, ..., x_n \in X$. I will show that there exist $y \in X$ such that $\|y\| = 1$ and that for all $j = 1, 2, ..., n$ $\|y - x_j\| \geq \|x_j\|$.

I start by picking a non-zero $z \in \bigcap_{j=1}^{n} \ker(f_j)$. This is possible becuse in problem 3b I have shown that $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$. Furthermore I define $y := \frac{z}{\|z\|}$. So I now have that

$$f_j(y) = f_j\left(\frac{z}{\|z\|}\right) = \frac{1}{\|z\|} \cdot f_j(z) = 0$$

where the second equality comes from the linearity of f_j, and the last equality comes from the fact that $f_j(z) = 0$ since $z \in \bigcap_{j=1}^{n} \ker(f_j)$.

Notice then that $y \in \bigcap_{j=1}^{n} \ker(f_j) \subseteq X$. Observere now

$$\|y\| = \left\| \frac{z}{\|z\|} \right\| = \frac{\|z\|}{\|z\|} = 1$$

And I have now shown that there exists $y \in \bigcap_{j=1}^{n} \ker(f_j) \subseteq X$ such that $\|y\| = 1$.

I will now show that $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, 2, ..., n$. I take $y \in \bigcap_{j=1}^{n} \ker(f_j)$ where it applies that $\|y\| = 1$.
From theorem 2.7(b) it applies that $\|f_j\| = 1$ since X is an infinite dimensional normed vector space over $\mathbb{K}$ and since $f_j \in X^*$. By using $\|f_j\| = 1$ notice

$$\|y - x_j\| = \|f_j\| \cdot \|y - x_j\| \geq \|f_j(y - x_j)\| = |f_j(y - x_j)| = |f_j(y) - f_j(x_j)|$$

where the inequality comes from the definition of the operator norm and where the last equality comes from the linearity of f_j.
So now, since $y \in \bigcap_{j=1}^{n} \ker(f_j)$ it gives $f_j(y) = 0$ and furthermore by theorem 2.7 (b) $f_j(x_j) = \|x_j\|$. By using all these gives us that

$$\|y-x_j\| = \|f_j\|\cdot\|y-x_j\| \geq \|f_j(y-x_j)\| = |f_j(y-x_j)| = |f_j(y)-f_j(x_j)| = |0-\|x_j\|| = \|x_j\|$$

Hence I have now shown that for all $j = 1, 2, ..., n$, it applies that $\|y - x_j\| \geq \|x_j\|$.

(d)

In this part I will show that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.
More specifically I will show that $S \not\subseteq \bigcup_{i=1}^{n} B_i$, where B_i is closed balls, where none of the balls contain 0.

I will solve this problem by taking $x \in S$ and show that $x \notin \bigcup_{i=1}^{n} B_i$.

Before I show this, I will show that B_i is convex. B_i is convex if it applies that for all $x, y \in B_i$ and for all $0 \leq \alpha \leq 1$ that $\alpha x + (1-\alpha)y \in B_i$.
Observe now that

$$\|\alpha x + (1-\alpha)y - p\| = \|\alpha x - \alpha p + (1-\alpha)y - p + \alpha p\| = \|\alpha(x-p) + (1-\alpha)y - p(1-\alpha)\|$$

$$= \|\alpha(x-p)+(1-\alpha)(y-p)\| \leq \|\alpha(x-p)\|+\|(1-\alpha)(y-p)\| = |\alpha|\|x-p\|+|(1-\alpha)|\cdot\|y-p\|$$

$$= \alpha\|x-p\| + (1-\alpha)\|y-p\| \leq \alpha r + (1-\alpha)r = \alpha r + r - \alpha r = r$$

So I have shown that $\|\alpha x + (1-\alpha)y - p\| \leq r$, hence $\alpha x + (1-\alpha)y \in B_i$. This gives that B_i is convex.

I will now take $x \in S$ and show that $x \notin \bigcup_{i=1}^{n} B_i$.
Specifically take $x \in \bigcap_{j=1}^{n} \ker(f_j) \cap S \subseteq S$.
For all $i \geq 1$ and for B_i which is convex, $x \in B_i$ by Hahn-Banach Theorem if $Re(f_j(x)) \geq 1$. I will use this to show that $x \notin \bigcup_{i=1}^{n} B_i$.

For $x \in \bigcap_{j=1}^{n} \ker(f_j)$ it applies that $f_j(x) = 0$, which will give that $Re(f_j(x)) = 0$. So for $x \in \bigcap_{j=1}^{n} \ker(f_j)$ it applies that $Re(f_j(x)) = 0$ which is not greater or equal to 1. Hence by using the earlier mentioned Hahn-Banach Theorem $x \notin B_i$ and hence $x \in \bigcap_{j=1}^{n} \ker(f_j)$ gives that $x \notin B_i$.
From this I can notice that

$$\bigcap_{j=1}^{n} \ker(f_j) \cap B_i = \emptyset$$

Hence furthermore

$$\bigcap_{j=1}^{n} \ker(f_j) \cap B_i \cap S = \emptyset$$

By using this result, I can now conclude that if we take

$$x \in \bigcap_{j=1}^{n} \ker(f_j) \cap S \subseteq S \Rightarrow x \notin B_i \text{ for all } i \geq 1 \Rightarrow x \notin \bigcup_{i=1}^{n} B_i$$

Hence I have shown that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

(e)

In this problem, I will show that S is non-compact and I will deduce further that the closed unit ball in X is non-compact.
This proof will be done by contradiction, so I assume that S is compact. I start by taking an arbitrarily $x \in S$ and I look at the open ball

$$B_x\left\{v \in X : \|x - v\| < \frac{1}{2}\right\}$$

Notice that $\{B_x\}_{x\in S}$ is an open covering of S. This is true because if we take a $x \in S$ then it applies that $\|x - x\| = 0 < \frac{1}{2}$, which gives that $x \in B_x$. Hence $S \subset \bigcup_{x\in S} B_x$. All these gives that $\{B_x\}_{x\in S}$ is an open covering of S.

The definition of compactness says that every open cover of S has a finite subcover. Because S is compact by assumption, then I notice that there exists a finite subcover of S, which is $\{B_{x_i}\}_{x_i\in S}$ for all $1 \leq i \leq n$.
I know that $B_{x_i} \subseteq \overline{B_{x_i}}$ for all $i = 1, .., n$. This gives that

$$\bigcup_{i=1}^{n} B_{x_i} \subseteq \bigcup_{i=1}^{n} \overline{B_{x_i}}$$

for all $i = 1, .., n$. From the fact that B_{x_i} is a finite subcover, I have

$$S \subseteq \bigcup_{i=1}^{n} B_{x_i}$$

which gives that

$$S \subseteq \bigcup_{i=1}^{n} \overline{B_{x_i}}$$

Hence I have know shown that there exists a closed ball $\{\overline{B_{x_i}}\}_{x_i \in S}$ which covers S. But none of these $\{\overline{B_{x_i}}\}_{x_i \in S}$ contains zero since we have that $\|x - 0\| = 1 \nleq \frac{1}{2}$. This contradicts with problem 3d, and hence S is non-conpact

I will now deduce further that the closed unit ball in X is non-compact. I am noticing that $S \subseteq B$, where B is defined as the closed unit ball. It applies that B is non-compact, because S is non-compact from earlier in this problem. This holds because we know that a closed subset of compact space is compact.

Problem 4

In this problem I let $L_1([0,1], m)$ and $L_3([0,1], m)$ be the Lebesgue spaces on $[0,1]$, and from HM2 I notice that $L_3([0,1], m) \subsetneq L_1([0,1], m)$. For $n \geq 1$ I define

$$E_n := \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 \, dm \leq n \right\}$$

(a)

Given $n \geq 1$, I want to justify whether the set
$E_n := \{f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 \, dm \leq n\}$ is absorbing.
Given $n \geq 1$, for E_n to be absorbing it has to be convex and satisfy this condition:

$$\forall f \in L_1([0,1], m) \ \exists t > 0 : t^{-1} f \in E_n$$

Before I show that E_n does not satifsy this condition I first justify that E_n is convex. E_n is convex if for all $f, g \in E_n$ and for all $0 \leq \alpha \leq 1$

$$\alpha f + (1 - \alpha) g \in E_n$$

This will be shown by showing that

$$\left(\int_{[0,1]} |\alpha f + (1 - \alpha) g|^3 \, dm \right) \leq n$$

Look at

$$\left(\int_{[0,1]} |\alpha f + (1 - \alpha) g|^3 \, dm \right)^{\frac{1}{3}}$$

Now the first inequality comes from Minkowskis inequality and we get

$$\left(\int_{[0,1]} |\alpha f + (1 - \alpha) g|^3 \, dm \right)^{\frac{1}{3}} \leq \left(\int_{[0,1]} |\alpha f|^3 \, dm \right)^{\frac{1}{3}} + \left(\int_{[0,1]} |(1 - \alpha) g|^3 \, dm \right)^{\frac{1}{3}}$$

$$= \Big(\int_{[0,1]} \alpha^3 |f|^3 \, dm \Big)^{\frac{1}{3}} + \Big(\int_{[0,1]} (1-\alpha)^3 |g|^3 \, dm \Big)^{\frac{1}{3}} = \alpha \Big(\int_{[0,1]} |f|^3 \, dm \Big)^{\frac{1}{3}} + (1-\alpha) \Big(\int_{[0,1]} |g|^3 \, dm \Big)^{\frac{1}{3}}$$

$$\leq \alpha n^{\frac{1}{3}} + (1-\alpha) n^{\frac{1}{3}} = n^{\frac{1}{3}}$$

where the last inequality applies since $f, g \in E_n$. So now it has been shown that

$$\Big(\int_{[0,1]} |\alpha f + (1-\alpha) g|^3 \, dm \Big) \leq n$$

and hence $\alpha f + (1-\alpha) g \in E_n$ and then E_n is convex.
I will now show that E_n does not satisfy this condition:

$$\forall f \in L_1([0,1], m) \; \exists t > 0 : t^{-1} f \in E_n$$

Let $f(t) = t^{-\frac{1}{3}}$. Then $f \in L_1([0,1], m)$ because

$$\|f\|_1 = \int_{[0,1]} |f| \, dm = \int_0^1 x^{-\frac{1}{3}} dx = \frac{3}{2} < \infty$$

So since $\|f\|_1 < \infty$ and since $f(t)$ is measurable,I have that $f \in L_1([0,1], m)$. Furthermore, for any $t > 0$ notice that

$$\int_{[0,1]} |f|^3 dm = \int_0^1 \frac{1}{x} dx \approx \infty$$

This shows that $f \notin L_3([0,1], m)$ so there does not exists a $t > 0$ such that $t^{-1} f \in E_n$ because $\int_{[0,1]} |f|^3 \, dm \approx \infty$ implies that $\int_{[0,1]} |t^{-1} f|^3 \, dm \approx \infty$. From this I can conclude that $\int_{[0,1]} |t^{-1} f|^3 \, dm \nleq n$ and hence I have point out hat E_n is not absorbing.

(b)

In this section I will show that for all $n \geq 1$, E_n has empty interior in $L_1([0,1], m)$. To show this, I have to show that $E_n^\circ = \emptyset$. This will be shown by contradiction, so I assume for some $n \geq 1$ that $E_n^\circ \neq \emptyset$.
For $E_n^\circ \neq \emptyset$ we have a $f \in E_n^\circ$ and for some $\varepsilon > 0$ we have the open ball

$$B(f, \varepsilon) := \{ g \in L_1([0,1], m) : \|f - g\|_1 < \varepsilon \} \subseteq E_n$$

So for $g \in L_1([0,1], m)$ where $g \neq 0$ we have

$$\Big\| f - (f + \frac{\varepsilon}{2\|g\|_1} g) \Big\|_1 = \Big\| f - f - \frac{\varepsilon}{2\|g\|_1} g \Big\|_1 = \Big\| \frac{-\varepsilon}{2\|g\|_1} g \Big\|_1 = \Big| \frac{\varepsilon}{2\|g\|_1} \Big| \|g\|_1 = \frac{\varepsilon}{2\|g\|_1} \|g\|_1 = \frac{\varepsilon}{2} < \varepsilon$$

The second last equality comes from the fact that $\frac{\varepsilon}{2\|g\|_1} > 0$ From this we now have $h := f + \frac{\varepsilon}{2\|g\|_1} g \in B(f, \varepsilon)$. So

$$h = f + \frac{\varepsilon}{2\|g\|_1} g \Rightarrow g = (h - f) \cdot \frac{2\|g\|_1}{\varepsilon}$$

Since $h \in B(f, \varepsilon) \subseteq E_n$ and since any function in E_n is in $L_3([0,1], m)$, we have that $h \in L_3([0,1], m)$. And since $f \in E_n$ then $f \in L_3([0,1], m)$. All these gives that $g \in L_3([0,1], m)$. So I have now shown that $L_1([0,1], m) \subseteq L_3([0,1], m)$, but from HW2 we know that $L_3([0,1], m) \subsetneq L_1([0,1], m)$. This gives the contradiction and I have now shown that $E_n^\circ = \emptyset$ and hence I have shown that for all $n \geq 1$ E_n has empty interior in $L_1([0,1], m)$.

(c)

In this section I will show that for all $n \geq 1$ E_n is closed in $L_1([0,1], m)$. For showing this, I have to show that if I take a sequence $(f_k)_{k \in \mathbb{N}} \subseteq E_n$ then the limit of this sequence is in E_n.
So I starting by taking a sequence $(f_k)_{k \in \mathbb{N}} \subseteq E_n$, where it applies that $\|f_k - f\|_1 \to 0$. From the Bolzano-Weistrass property I notice that there is a subsequence $(f_{n_k})_{n_k \in \mathbb{N}}$, where it converges pointwise.
So now we have

$$\|f\|_3^3 = \int_{[0,1]} |f|^3 \, dm \leq \liminf_{n_k \to \infty} \int_{[0,1]} |f_{n_k}|^3 \, dm \leq \liminf_{n_k \to \infty} n = n$$

The first inequality comes from Fatous lemma. Furthermore by using that $f_{n_k} \in E_n$ it gives the last inequality.
Notice $\int_{[0,1]} \liminf_{n_k \to \infty} |f_{n_k}|^3 \, dm = \int_{[0,1]} |f|^3 \, dm$.
From the earlier inequalities I have now shown that $\|f\|_3^3 \leq n$. This gives that $f \in E_n$. Hence E_n is closed in $L_1([0,1], m)$.

(d)

By using problem 4b and 4c I will now argue why $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$.
By using definition 3.12 (ii),$L_3([0,1], m)$ is of first category in $L_1([0,1], m)$ if there exist a sequence $(E_n)_{n \geq 1}$ of nowhere dense sets such that $L_3([0,1], m) = \cup_{n=1}^{\infty} E_n$.

If I can show that $Int(\overline{E_n}) = \emptyset$, then I have shown that E_n is nowhere dense set for all $n \geq 1$.
From problem 4(b) we know that for all $n \geq 1$ $Int(E_n) = \emptyset$, and furthermore from problem 4(c) we have that for all $n \geq 1$ E_n is closed, and from this we have that $E_n = \overline{E_n}$ for all $n \geq 1$. By using these result we can now conclude that

$$Int(\overline{E_n}) = Int(E_n) = \emptyset$$
so $Int(\overline{E_n}) = \emptyset$, which shows that for all $n \geq 1$ E_n is nowhere dense set.

To show that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$ I have to show that
$$L_3([0,1],m) = \bigcup_{n=1}^{\infty} E_n$$
This applies since
$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \Big\{ f \in L_1([0,1],m) : \int_{[0,1]} |f|^3 \, dm \leq n \Big\} =$$
$$\Big\{ f \in L_1([0,1],m) : \int_{[0,1]} |f|^3 \, dm \leq \infty \Big\} = \Big\{ f \in L_1([0,1],m) : f \in L_3([0,1],m) \Big\} = L_3([0,1],m)$$
Where the last equality comes from the fact that $L_3([0,1],m) \subsetneq L_1([0,1],m)$.
From this, it has been showed that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

Problem 5

In this problem I let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$, and I let $(x_n)_{n\geq 1}$ be a sequence in H and I let $x \in H$.

(a)

I assume that $x_n \to x$ in norm as $n \to \infty$ and I want to proof that $\|x_n\| \to \|x\|$ as $n \to \infty$.

To begin with, I observe that
$$\|x\| = \|x - x_n + x_n\| \leq \|x - x_n\| + \|x_n\|$$
Furthermore I observe that
$$\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\|$$
By combining these two expressions and by using the reverse triangle inequality I get
$$\Big| \|x\| - \|x_n\| \Big| \leq \|x - x_n\|$$
By assumption $x_n \to x$ in norm as $n \to \infty$ so for $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that
$$n_\varepsilon \leq n \Rightarrow \Big| \|x\| - \|x_n\| \Big| \leq \|x - x_n\| < \varepsilon$$

The last inequality comes from the fact that $x_n \to x$ in norm as $n \to \infty$. Then it follows that $\|x_n\| \to \|x\|$ as $n \to \infty$

(b)

In this problem I suppose that $x_n \to x$ weakly as $n \to \infty$. I will show by a counterexample that it does not follow that $\|x_n\| \to \|x\|$ as $n \to \infty$.
I define $H = \ell_2(\mathbb{N})$ and since H is separable it is possible to consider an orthonormal basis $(e_n)_{n\geq 1}$ in H. Because of that I let $x_n = e_n$

I claim that $e_n \to 0$ weakly and I want to show that this is true.

For $x \in H$ and for an orthonormal basis of H which is $(e_n)_{n\geq 1}$, I use Bessels inequality and get

$$\sum_{n\in\mathbb{N}} |\langle e_n, x\rangle|^2 \leq \|x\|^2$$

So the fact that

$$\sum_n |\langle e_n, x\rangle|^2 \leq \|x\|^2 \leq \infty$$

gives that $\sum_n |\langle e_n, x\rangle|^2$ converges. Hence the corresponding sequence $|\langle e_n, x\rangle|^2$ converges to $0 = \langle 0, x\rangle$. Hence $\langle e_n, x\rangle \to \langle o, x\rangle$.

Since we have that H is a Hilbert space with associated norm $\|\cdot\|$ then H is a Banach space, since we know that a Hilbert space is a Banach space too.
Furthermore we have that $(e_n)_{n\geq 1}$ is a sequence in H, hence $(e_n)_{n\geq 1}$ is a net. I can now use HW4 problem 2a. From this homework I can now conclude that because $\langle e_n, x\rangle \to \langle 0, x\rangle$ applies then $e_n \to$. Finally notice that since $(e_n)_{n\geq 1}$ is an orthonormal basis then $\|e_n\| = 1$. This gives that $\|e_n\| \to 1 \neq 0 = \|0\|$. Hence I have shown that for $x_n \to x$ weakly as $n \to \infty$ it does not follow that $\|x_n\| \to \|x\|$.

(c)

I assume that $\|x_n\| \leq 1$ for all $n \geq 1$ and that $x_n \to x$ weakly as $n \to \infty$. With a proof, I will show that $\|x\| \leq 1$.

A property of weak convergence is that if x_n converges weakly to x then

$$\|x\| \leq \liminf_{x\to\infty} \|x_n\|$$

By assumption I have that $x_n \to x$ weakly as $n \to \infty$ so then it applies that

$$\|x\| = \langle x, x\rangle = \lim_{n\to\infty} \langle x, x_n\rangle$$

where the last equality comes from definition of weak convergence.
Furthermore it applies that

$$\langle x, x_n\rangle \leq \|x_n\|$$

All these remarks now gives that

$$\|x\| = \lim_{n\to\infty} \langle x, x_n \rangle \leq \liminf_{n\to\infty} \|x_n\|$$

where the last inequality comes from the property of weak convergence. For $\|x_n\| \leq 1$ it now follows that $\|x\| \leq 1$.