Mandatory Assignment for Functional Analysis 2020
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Problem 1

(a)

Since $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms we see that $\|\cdot\|_0$ is a map from X to $[0,\infty)$. Let $x, y \in X$. Then, since $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms and hence satisfy the triangle inequality and T is linear, we see that

$$\|x+y\|_0 = \|x+y\|_X + \|T(x+y)\|_Y \leq \|x\|_X + \|y\|_Y + \|T(x)\|_Y + \|T(y)\|_Y = \|x\|_0 + \|y\|_0 . \tag{1}$$

So $\|\cdot\|_0$ satisfies the triangle inequality. Also, let $\alpha \in \mathbb{K}$ and $x \in X$ then we have

$$\|\alpha x\|_0 = \|\alpha x\|_X + \|\alpha T(x)\|_Y = |\alpha|(\|x\|_X + \|T(x)\|_Y) = |\alpha|\|x\|_0 , \tag{2}$$

where again we used that $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms and T is linear. And lastly, suppose $\|x\|_0 = 0$ for some $x \in X$. This is equivalent to having both $\|x\|_X = 0$ and $\|T(x)\|_Y = 0$ and from the first of these we see that it is equivalent to x being 0. This shows that $\|\cdot\|_0$ is a norm.

Notice that we, due to the definition of $\|\cdot\|_0$, have that $\|x\|_0 \geq \|x\|_X$ for all $x \in X$. Now, suppose T is bounded. Then there exists $C > 0$ such that $\|T(x)\|_Y \leq C\|x\|_X$ for all $x \in X$. This means that $\|x\|_0 \leq (1+C)\|x\|_X$ for all $x \in X$ such that $\|x\|_X \leq \|x\|_0 \leq (1+C)\|x\|_X$ for all $x \in X$ and hence $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent. On the other hand, if $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent then we know that there exists $C' > 0$ such that $\|x\|_0 = \|x\|_X + \|T(x)\|_Y \leq C'\|x\|_X$ for all $x \in X$ which implies that $\|T(x)\|_Y \leq (C'-1)\|x\|_X$ for all $x \in X$ and hence T is bounded.

(b)

Let X have dimension $n < \infty$. Then there exists a basis $\{e_1, ..., e_n\} \subset X$ for X and every element $x \in X$ can be written as a unique linear combination $x = x_1 e_1 + ... + x_n e_n$ where $x_1, ..., x_n \in \mathbb{K}$. Now, for any norm, $\|\cdot\|_Y$ on Y we have

$$\|T(x)\|_Y \leq |x_1|\|T(e_1)\|_Y + ... + |x_n|\|T(e_n)\|_Y \tag{3}$$

where we used the definition of a norm and linearity of T. Let $C = \max_{i\in\{1,...,n\}}\|T(e_i)\|$. Then we have

$$\|T(x)\|_Y \leq C(|x_1| + ... + |x_n|) = C\|x\|_1 . \tag{4}$$

where $\|x\|_1 = |x_1| + ... + |x_n|$ for all $x = x_1 e_1 + ... + x_n e_n \in X$ is the usual 1-norm. We know from Theorem 1.6 of the Lecture Notes (LN) that any two norms on a finite dimensional vector space are equivalent, i.e. for any norm, $\|\cdot\|_X$ on X there exists $C' > 0$ such that $\|x\|_1 \leq C'\|x\|_X$ for all $x \in X$. Let $K = CC'$, then from (4) we then see that

$$\|T(x)\|_Y \leq K\|x\|_X \tag{5}$$

for all $x \in X$, which was the desired.

(c)

Let $(e_i)_{i\in I}$ be a Hamel basis of X consisting of normalized vectors and consider an infinite countable subset Λ of I with elements $\lambda_1, \lambda_2, \dots$ Pick $0 \neq y \in Y$ and let the family, $(y_i)_{i\in I}$, of elements of Y be given by $y_i = ny$ if $i = \lambda_n$ and $y_i = 0$ if $i \in I\backslash\Lambda$. Then, according to the comment in the assignment, there exists a unique linear extension $T : X \to Y$ with $T(e_i) = y_i$. This has norm

$$\|T\| = \sup\{\|T(x)\|_Y \mid x \leq 1\} \geq n\|y\| \tag{6}$$

so it is unbounded.

(d)

By **(c)**, since X is infinite-dimensional, we can pick an unbounded operator, T, from X to Y. By **(a)**, we have that $\|x\|_0 = \|x\|_X + \|T(x)\|_Y$ (for all $x \in X$) is a norm on X that fulfills $\|x\|_0 \geq \|x\|_X$ (for all $x \in X$) that is not equivalent to $\|x\|_X$. Now, in HW3 problem 1 we showed that if $(X,\|\cdot\|_X)$ and $(X,\|\cdot\|_0)$ are both complete and $\|x\|_0 \geq \|x\|_X$ for all $x \in X$ then $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent. Hence, if we assume that $(X,\|\cdot\|_X)$ is complete, then, by counter-position, $(X,\|\cdot\|_0)$ is not complete.

(e)

Let $X = \ell_1(\mathbb{N})$ (space of sequences with a series that is absolutely convergent) with $\|x\| = \sum_{n\in\mathbb{N}} |x_n|$ for all $x \in X$ and $\|x\|' = \sum_{n\in\mathbb{N}} \frac{|x_n|}{n}$ for all $x \in X$. Then we have $\|x\|' \leq \|x\|$ for all $x \in X$ and $\|\cdot\|'$ is a norm since it is a map from X to $[0,\infty)$ fulfilling: (1) for any two $x, y \in X$ we have $\|x+y\|' = \sum_{n\in\mathbb{N}} \frac{|x_n+y_n|}{n} \leq \sum_{n\in\mathbb{N}} \frac{|x_n|}{n} + \frac{|y_n|}{n} = \|x\|' + \|y\|'$. (2) for any pair $\alpha \in \mathbb{K}$ and $x \in X$ we have $\|\alpha x\|' = \sum_{n\in\mathbb{N}} \frac{|\alpha x_n|}{n} = |\alpha| \sum_{n\in\mathbb{N}} \frac{|x_n|}{n} = |\alpha|\|x\|'$. (3) We have for all $x \in X$ that $\|x\|' = 0$ if and only if $x = 0$.

Now, consider a sequence consisting of truncated versions of the sequence, $x = (\frac{1}{n})_{n\in\mathbb{N}}$, defined by $(x_n)_{n\in\mathbb{N}}$ where $x_n = (\frac{1}{m}\mathbf{1}_{m\leq n})_{m\in\mathbb{N}} \in X$. With respect to $\|\cdot\|'$, this is a Cauchy sequence since we know from An1 or An2 that $\lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n^2} = \frac{\pi^2}{6}$ and hence, given $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for all $n > m \geq N_\epsilon$ we have $\|x_n - x_m\|' = \sum_{i=m}^{n} \frac{1}{i^2} \leq \frac{\pi^2}{6} - \sum_{i=1}^{N_\epsilon} \frac{1}{i^2} < \epsilon$. We see, though, that $(x_n)_{n\in\mathbb{N}}$ does not converge to an element in X. Hence X is not complete with respect to $\|\cdot\|'$.

Problem 2

(a)

We have

$$\|f\| = \sup\{|a+b| \mid (|a|^p + |b|^p)^{1/p} = 1\} \leq \sup\{|a| + |b| \mid (|a|^p + |b|^p)^{1/p} = 1\}. \tag{7}$$

So we certainly have $\|f\| \leq 2$ and f is hence bounded. Now, in case $p = 1$, we trivially have $||f|| \leq 1$ as can be seen from (7). Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $x = (a, b, 0, \ldots)$ and $y = (1, 1, 0, \ldots)$ where $(|a|^p + |b|^p)^{1/p} = 1$. Then Hölder's inequality gives

$$|a| + |b| \leq 2^{1/q}(|a|^p + |b|^p)^{1/p} = 2^{1/q} = 2(\tfrac{1}{2})^{1/p}. \tag{8}$$

This means that we have $||f|| \leq 2(\frac{1}{2})^{1/p}$. On the other hand, let $x' = ((\frac{1}{2})^{1/p}, (\frac{1}{2})^{1/p}, 0, \ldots)$. Then $||x'||_p = 1$ and $|f(x')| = 2(\frac{1}{2})^{1/p}$ which implies that $||f|| \geq 2(\frac{1}{2})^{1/p}$. Hence we see that $||f|| = 2(\frac{1}{2})^{1/p}$.

(b)

We have the existence of a bounded linear functional $F : \ell_p(\mathbb{N}) \to \mathbb{C}$ satisfying $\|f\| = \|F\|$ and $F|_M = f$ directly from Corollary 2.6 of the LN. As we have seen in the first exercise class, for $1 < p < \infty$ we have that $(\ell_p(\mathbb{N}))^*$ is isometrically isomorphic to $\ell_q(\mathbb{N})$ where $\frac{1}{p} + \frac{1}{q} = 1$ such that there exists $y \in \ell_q(\mathbb{N})$ for which F is given by $F(x) = \sum_{i\in\mathbb{N}} x_i y_i$ with $\|F\| = \|y\|_q$. The requirement that $F|_M = f$ implies that the first two entries of y be one. And $\|f\| = \|y\|_q$ implies that

$$2^{1/q} = (2 + \sum_{i\geq 3} |y_i|^q)^{1/q}, \tag{9}$$

such that $\sum_{i\geq 3} |y_i|^q = 0$ and hence $y_i = 0$ for all $i \geq 3$. Therefore y is uniquely determined by $(1, 1, 0, 0, ...)$ and hence F is unique.

(c)

When $p = 1$ we have, also from HW1, that $(\ell_1(\mathbb{N}))^*$ is isometrically isomorphic to $\ell_\infty(\mathbb{N})$. The situation is as in **(b)** except for the fact that now any $y = (1, 1, y_3, y_4, ...) \in \ell_\infty(\mathbb{N})$ with $y_i \leq 1$ for all $i \geq 3$ satisfies $\|f\| = \|y\|_\infty = 1$. Hence there are infinitely many linear extensions.

Problem 3

(a)

Suppose, for the sake of a contradiction, that F is injective. Let $x_1, ..., x_{n+1}$ be linearly independent vectors in X. We then have that $F(x_1), ..., F(x_{n+1}) \neq 0$ (since F is assumed to be injective) are linearly dependent in $\mathbb{K}^n$, hence there exist $c_1, ..., c_{n+1} \in \mathbb{K}$ (not all equal to zero) such that $c_1 F(x_1) + ... + c_{n+1} F(x_{x+1}) = F(c_1 x_1 + ... + c_{n+1} x_{n+1}) = 0$ which implies that $\ker(F) \neq \{0\}$ and therefore F is not injective and we have a contradiction.

(b)

Let $F : X \to \mathbb{K}^n$ be given by $F(x) = (f_1(x), ..., f_n(x))$. Since, from **(a)** we have that F is not injective, we know that there exists $0 \neq x' \in X$ such that $F(x') = (f_1(x'), ..., f_n(x')) = 0$. This implies that $f_j(x') = 0$ for all $j \in \{1, ..., n\}$ and hence $0 \neq x' \in X$ is in the kernel of f_j for all $j \in \{1, ..., n\}$ which shows the desired.

(c)

For all the $x_1, ..., x_n \in X$ there exist $f_1, ... f_n \in X^*$ such that $\|f_i\| = 1$ and $f_i(x_i) = \|x_i\|$ according to Theorem 2(b) of LN. As we saw in **(b)** there exists a non-zero element in $\cap_{i=1}^n \ker(f_i)$ - call it y'. Then also $y = \frac{y'}{\|y'\|}$ is in $\cap_{i=1}^n \ker(f_i)$ and has $\|y\| = 1$. Now we see that

$$\|y - x_j\| = \|f_j\| \|y - x_j\| \geq |f_j(y - x_j)| = |f_j(x_j)| = \|x_j\|, \tag{10}$$

which was the desired.

(d)

Let $\{B_i\}_{i=1}^n$ be closed balls not containing 0. Since $\{0\}$ is compact and B_i (for all $i \in \{1, ..., n\}$) is convex and closed and they are disjoint we see from Thm 3.6 in the LN (and Remark 3.8) that there exists $f_i \in X^*$ such that $0 = f_i(0) < f_i(x)$ for all $x \in B_i$. Then from **(b)** we have that $\cap_{i=1}^n \ker(f_i)$ is a non-trivial subspace of X. Hence there exists $x \in S \cap (\cap_{i=1}^n \ker(f_i))$ which also fulfills $x \notin \cup_{i=1}^n B_i$.

(e)

The unit sphere, S, is not compact. This follows from **(d)** since $S \subset \cup_{x \in S} B(x, r)$, where $B(x, r)$ is an open ball centered in $x \in S$ with radius $r < 1$. Suppose this has a finite subcover, i.e. there exists $F \subset S$ (finite) such that $S \subset \cup_{x \in F} B(x, r)$. Then we certainly have that $S \subset \cup_{x \in F} \overline{B}(x, r)$ which contradicts what we found in **(d)**.

The closed unit ball, $\overline{B}(0, 1)$, cannot be compact since that would imply that S is compact because S is a closed subset of $\overline{B}(0, 1)$.

Problem 4

(a)

Suppose $f \in L_1([0,1], m) \backslash L_3([0,1], m)$ and that E_n is absorbing. This means that there exists some $t > 0$ such that $t^{-1} f \in E_n$. Then we have

$$\int_{[0,1]} |t^{-1} f|^3 dm \leq n < \infty, \tag{11}$$

which contradicts the fact that f is not an element of $L_3([0,1], m)$. Hence E_n is not absorbing.

(b)

Suppose that $E_n^\circ \neq \varnothing$ and take $f \in E_n^\circ$. Then for some $\epsilon > 0$ we have that the open ball

$$B(f, \epsilon) = \{k \in L_1([0,1], m) \mid \|k - f\|_1 < \epsilon\}, \tag{12}$$

is a subset of E_n° by definition. Now, for any non-zero $\tilde{f} \in L_1([0,1], m)$ and any $0 < \epsilon' < \epsilon$ we have that $h = \epsilon' \frac{\tilde{f}}{\|\tilde{f}\|_1} + f \in B(f, \epsilon)$. Since both h and f lie in $B(f, \epsilon) \subset E_n \subset L_3([0,1], m)$ we have that $\tilde{f} = \frac{\|\tilde{f}\|_1}{\epsilon'}(h - f) \in L_3([0,1], m)$. But this implies that $L_1([0,1], m) \subset L_3([0,1], m)$ which is a contradiction and hence $E_n^\circ = \varnothing$.

(c)

Let $f_j \in E_n$ be a sequence converging to $f \in L_1([0,1], m)$. We want to show that $f \in E_n$.

(d)

We have from **(b)** that E_n is nowhere dense for all $n \geq 1$ so we just need that $\cup_{i=1}^\infty E_n = L_3([0,1], m)$ in order to show that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$. We have already $\cup_{i=1}^n E_n \subset L_3([0,1], m)$ trivially.