Stability of the $N+1$ Fermi gas with point interactions

Advanced Mathematical Physics

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Overview

Motivation

Model of fermions interacting via point interactions are of great interest as they appear as

- Models of cold atomic gases.
- Models of nuclear interaction.
- Approximations of models with short-range interactions.

However, they are mathematically not very well understood.

Thomas collapse

Thomas collaps: It it known that a bosonic system of three or more bosons with attractive zero-range interactions is unstable (of the first kind) i.e. there is no ground state energy. This can be seen from the variational principle.

No Thomas collapse for (spin-1/2) fermions: The Thomas collapse is a collective phenomenon where three (or more) bosons interacts in a single point. This can never happen for spin-1/2 fermions, due to the Pauli principle.

Stability of the first kind: Is still an unsolved problem for general $N + M$ systems (N spin up and M spin down).

Results of Moser and Seiringer

Results

Moser, T., Seiringer, R. Stability of a Fermionic N + 1 Particle System with Point Interactions. Commun. Math. Phys. 356, 329–355 (2017).

- Prove stability of the $N + 1$ system, within a certain mass ratio interval.
- Prove existence of self-adjoint bounded from below Hamiltonian.
- Prove Tan relations.

We focus on the first two.

Formal Hamiltonian

Formal Hamiltonian

The Hamiltonian of a system of N fermions of one species of mass 1 interacting with 1 fermion of another species of mass m can be described by the formal Hamiltonian

$$H = -\frac{1}{2m}\Delta_{x_0} - \frac{1}{2}\sum_{i=1}^{N}\Delta_{x_i} + \gamma\sum_{i=1}^{N}\delta(x_i - x_0) \tag{1}$$

Formal Hamiltonian

Centre of mass separation

We can split the Hamiltonian in two

$$H = H_{\mathsf{CM}} + \frac{m+1}{2m} H_{\mathsf{rel}}, \tag{2}$$

with $x_{\mathsf{cm}} = (mx_0 + \sum_{i=1}^{N} x_i)/(m+N)$, $y_i = x_i - x_0$, and

$$\begin{aligned} H_{\mathsf{CM}} &= \frac{1}{2(N+m)} \Delta_{x_{\mathsf{cm}}}, \\ H_{\mathsf{rel}} &= -\sum_{i=1}^{N} \Delta_{y_i} - \frac{2}{m+1} \sum_{1 \le i < j \le N} \nabla_{y_i} \cdot \nabla_{y_j} + \tilde{\gamma} \sum_{i=1}^{N} \delta(y_i) \end{aligned} \tag{3}$$

Quadratic form

The formal Hamiltonian can be given precise meaning through a quadratic form, which can be obtained by considering more regularized models such as rank-one perturbations of a free Hamiltonian. One obtains

$$
\begin{aligned}
F_\alpha(u) = \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k)^{-1} |\hat{w}|^2 - \mu \left\| u \right\|_{L^2(\mathbb{R}^{3N})} \\
+ N \left(T_{\mathsf{diag}}(\xi) + T_{\mathsf{off}}(\xi) + \alpha \left\| \xi \right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \right)
\end{aligned}
\tag{4}
$$

with $\hat{u}(k) = \hat{w}(k) + \sum_{i=1}^{N} (-1)^{i-1} \hat{G}(k) \xi(\bar{k}^i)$, $\mu > 0$,
$\hat{G}(k) = \left(\sum_{i=1}^{N} k_i^2 + \frac{2}{m+1} \sum_{1 \leq i < j \leq N} k_i \cdot k_j + \mu \right)^{-1}$,

$$T_{\mathsf{diag}}(\xi) = \int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N L(\bar{k}^N) \left|\xi(\bar{k}^N)\right|^2,$$

$$T_{\mathsf{off}}(\xi) = (N-1) \int_{\mathbb{R}^{3(N-2)}} \mathrm{d}\bar{q} \int_{\mathbb{R}^3} \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}t \overline{\xi(s,\bar{q})} \hat{G}(s,t,\bar{q}) \xi(t,\bar{q}). \tag{5}$$

with

$$L(\bar{k}^N) = 2\pi^2 \left(\frac{m(m+2)}{(m+1)^2} \sum_{i=1}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{1 \le i < j \le N-1} k_i \cdot k_j + \mu \right)^{1/2}. \tag{6}$$

The domain is

$$\mathscr{D}(F_\alpha) = \Big\{ u \in L^2_{\mathsf{as}}(\mathbb{R}^{3N}) \ \Big| \ \hat{u} = \hat{w} + \widehat{\rho G},\ w \in H^1_{\mathsf{as}}(\mathbb{R}^{3N}),\ \xi \in H^{1/2}_{\mathsf{as}}(\mathbb{R}^{3(N-1)}) \Big\}. \tag{7}$$

Theorem 1

Introduce the function for $m > 0$

$$\Lambda(m) = \sup_{\substack{s, K \in \mathbb{R}^3, \\ Q \geq 0}} \frac{s^2 + Q^2}{\pi^2(1+m)} \ell_m(s, K, Q)^{-1/2} \int_{\mathbb{R}^3} \mathrm{d}t \frac{1}{t^2} \ell_m(t, K, Q)^{-1/2}$$
$$\times \frac{|(s + AK) \cdot (t + AK)|}{\left[(s + AK)^2 + (t + AK)^2 + \frac{m}{1+m}(Q^2 + AK^2)\right]^2 - \left[\frac{2}{(1+m)}(s + AK) \cdot (t + AK)\right]^2} \tag{8}$$

where $A = (2 + m)^{-1}$ and

$$\ell_m(s, K, Q) = \left(\frac{m}{(1+m)^2}(s + K)^2 + \frac{m}{1+m}(s^2 + Q^2)\right). \tag{9}$$

It is then showed that

$$\Lambda(m) \leq \frac{4(1+m)^2(2 + 4m + m^2)^{3/2}}{\sqrt{2}\pi\left[m(m+2)\right]^3}. \tag{10}$$

Theorem 1

Theorem 1, Moser, T., Seiringer, R., 2017.

For any $\xi \in H^{1/2}_{\mathsf{as}}(\mathbb{R}^{3(N-1)})$, $\mu > 0$ *and* $N \geq 2$,

$$T_{\mathsf{off}}(\xi) \geq -\Lambda(m) T_{\mathsf{diag}}(\xi). \tag{11}$$

In particular, if $\Lambda(m) < 1$, *then* F_α *is closed and bounded from below by*

$$F_\alpha(u) \geq \begin{cases} 0 & \text{for } \alpha \geq 0, \\ -\left(\frac{\alpha}{2\pi^2(1-\Lambda(m))}\right)^2 \|u\|^2_{L^2(\mathbb{R}^{3N})} & \text{for } \alpha < 0. \end{cases} \tag{12}$$

Proof idea

- Define $\phi = L^{1/2}\xi$ such that $T_{\mathsf{diag}} = \|\phi\|^2_{L^2(\mathbb{R}^{3(N-1)})}$.
- Notice that $T_{\mathsf{off}}(\xi) =$ $\int_{\mathbb{R}^{3(N-2)}} \mathrm{d}q \int_{\mathbb{R}^3} \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}t \overline{\phi(s,q)}\phi(t,q)L(s,q)^{-1/2}L(t,q)^{-1/2}\hat{G}(s,t,q)$.
- Throw away positive part.
- Use Schur test $\|\sigma\| \leq \sup_t \left(h(t)\int \mathrm{d}s\sigma(s,t)\frac{1}{h(s)}\right)$ on $(L^{-1/2}GL^{-1/2})_-$.
- Choose h carefully and use (anti-)symmetry of ϕ to arrive at $\left\|(L^{-1/2}GL^{-1/2})_-\right\| \leq \Lambda(m)$.
- Closedness is now standard argument since all terms are indvidually bounded from below.

Physical consequences

$\Lambda(m) < 1$ for $m \geq 0.36$ so the critical mass ratio is less that 0.36

- Stability of first and second kind.
- Existence of self-adjoint bounded from below Hamiltonian.

For $m \to \infty$ we have $\Lambda(m) \to 0$. Thus, the system with $\alpha < 0$ has energy bounded from below by $-\big(\alpha/(2\pi^2)\big)^2$.

- For $m \to \infty$ the heavy fermion can bind at most one light fermion. This is the Pauli principle.

Theorem 2

Since $T_{\mathsf{diag}} + T_{\mathsf{off}}$ is symmetric, closed, and bounded from below, we may define the unique self-adjoint operator Γ by

$$T_{\mathsf{diag}}(\xi) + T_{\mathsf{off}}(\xi) = \langle \xi | \Gamma \xi \rangle \,. \tag{13}$$

It can be shown that $H^1_{\mathsf{as}}(\mathbb{R}^{3(N-1)}) \subset \mathscr{D}\,(\Gamma)$

Theorem 2

Theorem 2, Moser, T., Seiringer, R., 2017

For any $\xi \in H^1_{\mathsf{as}}(\mathbb{R}^{3(N-1)}), \mu > 0$, *and* $N \geq 2$

$$\|\Gamma\xi\|^2_{L^2(\mathbb{R}^{3(N-1)})} \geq (1 - \Lambda_1(m)) \|L\xi\|^2_{L^2(\mathbb{R}^{3(N-1)})} . \tag{14}$$

~~In particular~~, if $\Lambda_1(m) < 1$, *Then* $\mathscr{D}(\Gamma) = \mathscr{D}(L) = H^1_{\mathsf{as}}(\mathbb{R}^{3(N-1)})$.
More generally for $0 \leq \beta \leq 2$,

$$\left\|L^{(\beta-1)/2}\Gamma\xi\right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \geq (1 - \Lambda_\beta(m)) \left\|L^{(\beta+1)/2}\xi\right\|^2_{L^2(\mathbb{R}^{3(N-1)})} , \tag{15}$$

for all $\xi \in H^{(\beta+1)/2}_{\mathsf{as}}(\mathbb{R}^{3(N-1)})$.

Proof idea

- Write $\Gamma = L + J$, i.e. $\langle \xi | J\xi \rangle = T_{\mathsf{off}}(\xi)$.
- Notice that define $\phi = L^{(\beta+1)/2}\xi$ and notice that $\left\| L^{(\beta-1)/2}\Gamma\xi \right\|^2 = \langle \phi | L^{-(\beta+1)/2}(L+J)L^{\beta-1}(L+J)L^{-(\beta+1)/2}|\phi\rangle$
- Throw away positive term $\langle \phi | L^{-(\beta+1)/2}JL^{\beta-1}JL^{-(\beta+1)/2}|\phi\rangle$.
- Claim is now equivalent to $\langle \phi | L^{(\beta-1)/2}JL^{-(\beta+1)/2} + L^{-(\beta+1)/2}JL^{(\beta-1)/2}|\phi\rangle \geq -\Lambda_\beta(m)\left\|\phi\right\|_2^2$.
- Use Cauchy-Schwartz, and similar proof to that of theorem 1 to obtain the desired result.

Consequences

Knowing that the quadratic form F_α is symmetric, closed, and bounded from below, it is straightforward to obtain the Hamiltonian:

$$\mathscr{D}(H_\alpha) = \left\{ u \in \mathscr{D}(F_\alpha) \mid F_\alpha(\cdot, u) \text{ is } L^2 \text{ bounded on } \mathscr{D}(F_\alpha) \right\}. \tag{16}$$

Notice that for $u \in \mathscr{D}(H_\alpha)$, we have $F_\alpha(\cdot, u) = \langle \cdot, x \rangle$ and we set $H_\alpha u = x$. Moser and Seiringer obtains

$$\begin{aligned}\mathscr{D}(H_\alpha) = \Big\{ & u \in L^2_{\mathsf{as}}(\mathbb{R}^{3N}) \mid u = w + G\xi, w \in H^2_{\mathsf{as}}(\mathbb{R}^{3(N-1)}), \\ & \xi \in \mathscr{D}(\Gamma), w|_{y_N=0} = (2\pi)^{-3/2}(-1)^{N+1}(\alpha + \Gamma)\xi \Big\},\end{aligned} \tag{17}$$

$(H_\alpha + \mu)(w + G\xi) = (H_{\mathsf{free}} + \mu)w$, with $(G\xi)(x) := \left(\sum_{i=1}^N (-1)^{i-1} \hat{G}(k) \xi(\bar{k}^i) \right)^\vee (x)$

Conclusion

Thus stability of the fermionic $N + 1$ system is established for $m \geq 0.36$, and a rigorous version of the formal Hamiltonian is found.

Thank you for your attention.

$$\Lambda_\beta(m) = \sup_{\substack{s,K\in\mathbb{R}^3,\\ Q\geq 0}} \frac{s^2+Q^2}{\pi^2(1+m)} \int_{\mathbb{R}^3} \mathrm{d}t \, \frac{1}{t^2} \left(\frac{\ell_m(s,K,Q)^{(\beta-1)/2}}{\ell_m(t,K,Q)^{(\beta+1)/2}} + \frac{\ell_m(t,K,Q)^{(\beta-1)/2}}{\ell_m(s,K,Q)^{(\beta+1)/2}} \right)$$

$$\times \frac{|(s+AK)\cdot(t+AK)|}{\left[(s+AK)^2+(t+AK)^2+\frac{m}{1+m}(Q^2+AK^2)\right]^2 - \left[\frac{2}{(1+m)}(s+AK)\cdot(t+AK)\right]^2} \tag{18}$$