Functional Analysis
Mandatory Assignment 1

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Problem 1.

Solution. (a) Let $T : X \to Y$ be a linear map and define $\|-\|_0$ as $\|x\|_0 := \|x\|_X + \|Tx\|_Y$.

To see that $\|x\|_0$ is a norm let $x, y \in X$ and $\alpha \in \mathbb{K}$, then

$$\begin{aligned}
\|x+y\|_0 &= \|x+y\|_X + \|T(x+y)\|_Y \\
&= \|x+y\|_X + \|Tx+Ty\|_Y, \text{ linearity of } T \\
&\leq \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y, \text{ since } \|-\|_X \text{ and } \|-\|_Y \text{ are norms} \\
&= \|x\|_0 + \|y\|_0
\end{aligned}$$

and

$$\begin{aligned}
\|\alpha x\|_0 &= \|\alpha x\|_X + \|T(\alpha x)\| \\
&= |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y, \text{ linearity of } T \text{ and } \|-\|_X \text{ and } \|-\|_Y \text{ are norms} \\
&= |\alpha|(\|X\|_X + \|Tx\|_Y) \\
&= |\alpha|\|x\|_0.
\end{aligned}$$

Finally, if $x = 0$, then clearly

$$\|0\|_0 = \|0\|_X + \|T0\|_Y = 0 + 0 = 0$$

and if $\|x\|_0 = 0$, then $\|x\|_X + \|Tx\|_Y = 0$, but these are both norms so positive definite and so $\|x\|_X, \|Tx\|_Y = 0$. Thus, $\|x\|_X = 0$ which implies $x = 0$ since $\|-\|_X$ is a norm. Therefore, $\|-\|_0$ is a norm.

Now we claim that $\|-\|_X$ and $\|-\|_0$ are equivalent if and only if T is bounded. First, suppose $\|-\|_X$ and $\|-\|_0$ are equivalent, then by definition there exists $0 < c_1 \leq c_2 < \infty$ such that for all $x \in X$

$$c_1\|x\|_X \leq \|x\|_0 = \|x\|_X + \|Tx\|_Y \leq c_2\|x\|_X.$$

Thus, we see immediately that for all $x \in X$

$$0 \leq \|Tx\|_Y \leq (c_2 - 1)\|x\|_X$$

from which it follows that $c_2 - 1 \geq 0$. If $c_2 - 1 = 0$, then $0 \leq \|Tx\|_Y \leq 0$ so $\|Tx\|_Y = 0$ for all $x \in X$ in which case T is the zero map and so bounded. In the case that $c := c_2 - 1 > 0$, then T is bounded by definition since $c > 0$ is such that

$$\|Tx\|_Y \leq c\|x\|_X$$

for all $x \in X$.

Conversely suppose T is bounded, then by definition there exists $c > 0$ such that $\|Tx\|_Y \leq c\|x\|_X$ for all $x \in X$. Thus,

$$\|x\|_X \leq \|x\|_0 = \|x\|_X + \|Tx\|_Y \leq (c+1)\|x\|_X$$

so that $\|-\|_X$ and $\|-\|_0$ are equivalent.

Solution. (b) Let X be a finite dimensional normed vector space of dimension d. Now we recall that any two norms on X are equivalent, that is, generate the same topology (theorem 1.6, lecture notes) so it is sufficient to show that if $T : X \to Y$ is linear, then T is bounded (equivalently, continuous) for the 1-norm on X. That is, let $e_1, \ldots, e_n$ be a basis for X and $x = \sum_{i=1}^{d} \alpha_i e_i, \alpha_i \in \mathbb{K}$, then by theorem 1.6 we may assume

$$\|x\|_X = \sum_{i=1}^{d} |\alpha_i|.$$

Hence, let $c = \max_{1 \leq i \leq d} \|Te_i\|_Y$, then we have

$$\begin{aligned}
\|Tx\|_Y = \left\| \sum_{i=1}^{d} \alpha_i Te_i \right\|_Y &\leq \sum_{i=1}^{d} \|\alpha_i Te_i\|_Y \\
&= \sum_{i=1}^{d} |\alpha_i| \|Te_i\|_Y \\
&\leq \sum_{i=1}^{d} |\alpha_i| c \\
&= c\|x\|_X
\end{aligned}$$

Therefore, since this holds for all $x \in X$, then T is bounded with respect to the topology induced by the one norm and therefore continuous with respect to any norm induced topology on X.

Solution. (c) Let X be infinite dimensional and $(e_i)_{i\in I}$ be a Hamel basis for X where we note that I is at least countable since X is infinite dimensional. Furthermore, we may assume $\|e_i\| = 1$ for all $i \in I$ by normalization[1]. Thus, since $Y \neq 0$ let $0 \neq y \in Y$ be fixed and consider the unbounded sequence $(ny)_{n\geq 1}$. Let $(y_i)_{i\in I}$ be any family of elements in Y which contains the sequence $(ny)_{n\geq 1}$. Note that such a family exists by letting $\varphi : \mathbb{N} \hookrightarrow I$ be any injection, then let $(y_i)_{i\in I}$ be the sequence

$$y_i = \begin{cases} ny, & \text{if } i = \varphi(n) \\ 0, & \text{otherwise} \end{cases}$$

which is well-defined since φ is an injection. Hence, since $(e_i)_{i\in I}$ is a Hamel basis, then there exists a unique linear map $T : X \to Y$ such that $Te_i = y_i$. Thus, T is clearly unbounded since for each $c > 0$ there exists a sufficiently large n such that

$$c = c\|e_i\| = c\left\|e_{\varphi(n)}\right\| < n\|y\| = \|ny\| = \left\|Te_{\varphi(n)}\right\| = \|Te_i\|,$$

that is, T is unbounded since the sequence $(ny)_{n\geq 1}$ is unbounded and therefore, T is not continuous.

Solution. (d) Let X be infinite dimensional and let $T : X \to Y$ be a discontinuous linear map which exists by part (c). Then by part (a), $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ is a norm on X and $\|-\|_0$ and $\|-\|_X$ are not equivalent since T is not bounded. Additionally, it is clear that

$$\|x\|_X \leq \|x\|_0 = \|x\|_X + \|Tx\|_Y$$

for all $x \in X$ as desired. Now suppose $(X, \|-\|_X)$ is complete, that is, a Banach space, and for the purpose of contradiction suppose that $(X, \|-\|_0)$ is also a Banach space, then by homework 3 problem 1 the norms $\|-\|_X$ and $\|-\|_0$ must be equivalent since $\|-\|_X \leq \|-\|_0$, a contradiction by part (a) since T is not bounded.

Solution. (e) Consider $(X, \|-\|) = (\ell_1(\mathbb{N}), \|-\|_1)$ and let $\|-\|' = \|-\|_2$ the standard 2-norm. Then recall that $\|-\|_2 \leq \|-\|_1$ and we have an inclusion $\ell_1(\mathbb{N}) \subsetneq \ell_2(\mathbb{N})$. Thus, to see that $(\ell_1(\mathbb{N}), \|-\|_2)$ is not complete it is sufficient to show that $\ell_1(\mathbb{N})$ is not closed in $\ell_2(\mathbb{N})$. Hence, let $(x_n)_{n\in\mathbb{N}}$ be the sequence in $\ell_1(\mathbb{N})$ where

$$x_n(k) = \begin{cases} \frac{1}{k}, & k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

[1]This follows since every element may be written as a unique linear combination of e_i over I with finite support which will clearly still hold after normalizing the basis vectors.

Then clearly $x_n \in \ell_1(\mathbb{N})$ since x_n has compact support for all n and $c_c(\mathbb{N}) \subsetneq \ell_1(\mathbb{N})$. Now we claim that $(x_n)_{n\in\mathbb{N}}$ converges to the sequence $x = (1/k)_{k\in\mathbb{N}}$ under $\|{-}\|_2$. To see this observe that

$$\|x_n - x\|_2 = \left(\sum_{k=1}^{\infty}|x_n(k) - x(k)|^2\right)^{1/2} = \left(\sum_{k=n+1}^{\infty} 1/k^2\right)^{1/2}$$

so $\|x_n - x\|_2 \to 0$ as $n \to \infty$. Finally, $x \notin \ell_1(\mathbb{N})$ since $\sum_{k=1}^{\infty}|1/k|$ diverges. Therefore, $\ell_1(\mathbb{N})$ is not closed in $\ell_2(\mathbb{N})$ and so $(\ell_1(\mathbb{N}), \|{-}\|_2)$ is not complete while $(\ell_1(\mathbb{N}), \|{-}\|_1)$ is complete and $\|{-}\|_2 \leq \|{-}\|_1$.

Problem 2.

Solution. (a) First, recall in part (b) of problem 1 we have that any linear map $f : X \to Y$ is bounded if X is finite dimensional. Now it is clear that M is a finite dimensional subspace of $\ell_p(\mathbb{N})$ and that $f : M \to \mathbb{C}$ is linear so f is bounded.

Now we claim that $\|f\| = n^{1-1/p}$ where $n = 2$ is the dimension of M. To see this recall that for $t > 1$ Hölder's inequality states that for $x = (x_k)_{k=1}^n, y = (y_k)_{k=1}^n$, then

$$\sum_{k=1}^{n}|x_k y_k| \leq \left(\sum_{k=1}^{n}|x_k|^t\right)^{1/t} \cdot \left(\sum_{k=1}^{n}|y_k|^{\frac{t}{t-1}}\right)^{1-\frac{1}{t}} \tag{1}$$

since $\frac{1}{t} + \frac{1}{\frac{t}{t-1}} = 1$. Thus, let $0 < r < p$ and applying (1) with $x_k = |x_k|^r, y_k = 1$, and $t = p/r > 1$, then

$$\sum_{k=1}^{n}|x_k|^r \leq \left(\sum_{k=1}^{n}(|x_k|^r)^{\frac{p}{r}}\right)^{\frac{r}{p}} \cdot \left(\sum_{k=1}^{n} 1^{\frac{\frac{p}{r}}{\frac{p}{r}-1}}\right)^{1-\frac{r}{p}} = \left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{r}{p}} \cdot n^{1-\frac{r}{p}}.$$

Hence, taking the r^{th} root we get

$$\|x\|_r = \left(\sum_{k=1}^{n}|x|^r\right)^{\frac{1}{r}} \leq \left(\left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{r}{p}} \cdot n^{1-\frac{r}{p}}\right)^{\frac{1}{r}} = \left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}} \cdot n^{\frac{1}{r}-\frac{1}{p}} = \|x\|_p \cdot n^{\frac{1}{r}-\frac{1}{p}}.$$

Now let $r = 1$ and $1 < p < \infty$, then

$$\|x\|_p \leq \|x\|_1 \leq \|x\|_p \cdot n^{1-\frac{1}{p}}$$

and since the norm of f may be calculated as, $\|f\| = \sup_{\|x\|_p=1}|fx|$, then for the $x = (a, b) \in M$ with $\|x\|_p = 1$ we get

$$1 \leq |x| = |a| + |b| \leq 2^{1-1/p}.$$

Now by the triangle inequality for all $x = (a, b) \in M$ with $\|x\|_p = 1$ we have

$$|fx| = |a + b| \leq |a| + |b| \leq 2^{1-1/p}.$$

Hence, $\|f\| \leq 2^{1-1/p}$ and to see that $2^{1-1/p} \leq \|f\|$, let $a, b = \frac{2^{1-1/p}}{2}$ and let $x = (a, b)$, then

$$\|x\|_p = \left(\left(\frac{2^{1-1/p}}{2} \right)^p + \left(\frac{2^{1-1/p}}{2} \right)^p \right)^{1/p} = \left(\frac{2^{p-1}}{2^p} + \frac{2^{p-1}}{2^p} \right)^{1/p} = \left(\frac{1}{2} + \frac{1}{2} \right)^{1/p} = 1$$

and

$$|fx| = |a + b| = \frac{2^{1-1/p}}{2} + \frac{2^{1-1/p}}{2} = 2^{1-1/p}.$$

Therefore, $2^{1-1/p} \leq \|f\|$ and so $\|f\| = 2^{1-1/p}$. You only considered $p > 1$. $(\checkmark)$

Solution. (b) First, observe that $F : \ell_p(\mathbb{N}) \to \mathbb{C}$ defined by

$$F(a, b, x_1, x_2, \dots) = f(a, b)$$

is an extension of f with $\|F\| = \|f\|$. This follows since clearly $F_{|M} = f$ and for $x = (a_1, b_1, x_1, \dots), y = (a_2, b_2, y_1, \dots) \in \ell_p(\mathbb{N})$ and $\alpha, \beta \in \mathbb{C}$ we have

$$F(\alpha x + \beta y) = f(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2) = \alpha f(a_1, b_1) + \beta f(a_2, b_2) = \alpha F(x) + \beta F(y)$$

since f is linear and

$$\|F\| = \sup_{\|x\|_p = 1} |F(x)| = \sup_{\|x\|_p = 1} |f(a_1, b_1)| = \sup_{\|(a_1, b_1)\|_p = 1} |f(a_1, b_1)| = \|f\|.$$

Now recall from homework 1 problem 5 for $1 < p < \infty$ there is an isometric isomorphism

$$(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$$

where q is such that $1/p + 1/q = 1$ and the map $\psi : (\ell_p(\mathbb{N}))^* \xrightarrow{\cong} \ell_q(\mathbb{N})$ is defined in the following way. Let $(e_k)_{k \in \mathbb{N}}$ be the collection of elements in $\ell_p(\mathbb{N})$ such that $e_k(i) = \delta_{ik}$ where δ_{ik} is the Kronecker delta, then for $G \in (\ell_p(\mathbb{N}))^*$

$$\psi(G) = (G(e_k))_{k \in \mathbb{N}}.$$

Additionally, ψ is an isometry so $\|\psi(G)\|_q = \|G\|$ where we also observe that

$$\psi(F) = (f(1, 0), f(0, 1), 0, 0, \dots) = (1, 1, 0, 0, \dots).$$

Now let G be any other extension of f such that $\|G\| = \|f\|$, then $\psi(G) = (G(e_k))_{k\in\mathbb{N}}$ and $\|\psi(G)\|_q = \|G\| = 2^{1-1/p} = 2^{1/q}$. Thus, since G is an extension of f, then $G(e_1) = f(e_1) = 1$ and $G(e_2) = f(e_2) = 1$. Hence, $\psi(G) = (1, 1, G(e_3), \dots)$, but then

$$\|\psi(G)\| = \left(\sum_{k=1}^{\infty} |G(e_k)|^q\right)^{1/q} = \left(1 + 1 + \sum_{k=3}^{\infty} |G(e_k)|^q\right)^{1/q} = 2^{1/q} = 2^{1-1/p}.$$

However, $(1+1)^{1/q} = 2^{1/q}$ so

$$\sum_{k=3}^{\infty} |G(e_k)|^q = 0$$

which since $|G(e_k)| \geq 0$ implies that $G(e_k) = 0$ for $k \geq 3$. Thus, $\psi(F) = \psi(G)$ which is an isomorphism so in particular injective, therefore, $F = G$. ✓

Solution. (c) Similarly to M let M_k denote the subspace of $(\ell_1(\mathbb{N}), \|-\|_1)$ defined as

$$M_k = \{(x_1, x_2, x_3, \dots, x_k, 0, \dots) : x_i \in \mathbb{C}\}$$

and let $g_k : M_k \to \mathbb{C}$ denote the linear functional defined by $g_k(x_1, \dots, x_k, 0, \dots) = \sum_{i=1}^{k} x_k$. Now for the same reasons as in part (b) that F was an extension of f, then for each g_k we obtain an extension $G_k : \ell_1(\mathbb{N}) \to \mathbb{C}$ defined by

$$G_k(x_1, \dots, x_k, x_{k+1}, \dots) = g_k(x_1, \dots, x_k).$$

Now observe that each G_k for $k \geq 2$ is an extension of f since for $x = (a, b, 0, 0, \dots)$ we have

$$G_k(a, b, 0, 0, \dots) = g_k(a, b, 0, 0, \dots) = a + b = f(a, b)$$

and note that for the elements $(e_j)_{j\in\mathbb{N}}$ in $\ell_1(\mathbb{N})$ where $e_j(i) = \delta_{ji}$ we have

$$G_k(e_j) = g_k(e_j) = \begin{cases} 1, & j \leq k \\ 0, & \text{otherwise.} \end{cases}$$

Again from homework 1 problem 5 we have an isometric isomorphism

$$\begin{aligned} \psi : \ell_1(\mathbb{N})^* &\to \ell_\infty(\mathbb{N}) \\ G &\mapsto (G(e_j))_{j\in\mathbb{N}}. \end{aligned}$$

Thus, it follows that for $k \geq 1$

$$\|G_k\| = \|\psi(G_k)\|_\infty = \sup_{j\in\mathbb{N}} |G_k(e_j)| = 1.$$

Therefore, the family $(G_k)_{k\geq 2}$ is an infinite collection of linear functionals on $\ell_1(\mathbb{N})$ which extend f and $\|G_k\| = 1 = 2^{1-1/1} = \|f\|$. ✓

Problem 3.

Solution. (a) Let $1 \leq n \in \mathbb{Z}$ and let $F : X \to \mathbb{K}^n$ be a linear map where X is infinite dimensional. Let $(e_i)_{i\in I}$ be a Hamel basis for X and choose $e_1, \ldots, e_d$ from $(e_i)_{i\in I}$ with $d > n$. Then $e_1, \ldots, e_d$ are linearly independent[2] and span a subspace $M \subset X$ of dimension $d > n$. Now the restriction, $F_{|M}$ to M is necessarily linear and cannot be injective since $\dim M = d > n$ and therefore, F is not injective.

Solution. (b) Let $1 \leq n \in \mathbb{Z}$ and $f_1, \ldots, f_n \in X^*$, then define

$$\begin{aligned} F : X &\to \mathbb{K}^n \\ x &\mapsto (f_1(x), \ldots, f_n(x)). \end{aligned}$$

This is clearly well-defined and linear since $f_1, \ldots, f_n$ are. Thus, $F : X \to \mathbb{K}^n$ is a linear map where $\dim X = \infty$ so by part (a) F is not injective. Hence, there exists $0 \neq x \in X$ such that

$$F(x) = (f_1(x), \ldots, f_n(x)) = (0, \ldots, 0) = 0.$$

Thus, $x \in \ker f_i$ for $1 \leq i \leq n$ and so

$$0 \neq x \in \bigcap_{i=1}^{n} \ker(f_i)$$

and we are done.

Solution. (c) Let $x_1, \ldots, x_n \in X$, then by theorem 2.7(b) from the lectures there exists $f_1, \ldots, f_n \in X$ with $f_i(x_i) = \|x_i\|$ and $\|f_i\| = 1$. Thus, by part (b) there exists $0 \neq y \in K = \cap_{i=1}^{n} \ker(f_i)$ where we may assume that $\|y\| = 1$ since if $y \in \ker(f_i)$ for all $1 \leq i \leq n$, then so is $y/\|y\|$ by linearity. Hence, we have $f_i(y - x_i) = f_i(y) - f(x_i) = -\|x_i\|$ since $y \in \ker(f_i)$ for $1 \leq i \leq n$. Therefore, it follows that

$$\|x_i\| = |f(y - x_i)| \leq \|f_i\| \cdot \|y - x_i\| = \|y - x_i\|$$

where the inequality follows from the fact that f_i is bounded and so $\|f_i(x)\| \leq \|f_i\| \cdot \|x\|$ for all $x \in X$ by lecture 1, equation 1.8. Thus, there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_i\| \geq \|x_i\|$ for $1 \leq i \leq n$, as desired.

[2]Note this clearly follows from the facts about Hamel bases introduced in problem 1.

Solution. (d) Let $x_1, \ldots, x_n \in X$ and let $\overline{B}_i = \overline{B(x_i, r_i)}$ denote closed balls centered at x_i of radius $r_i > 0$, that is,

$$\overline{B}_i = \{y \in X : \|y - x_i\| \leq r_i\},$$

and suppose that $\overline{B}_1, \ldots, \overline{B}_n$ cover, S, the unit sphere in X. Then by part (c) there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_i\| \geq \|x_i\|$ for $1 \leq i \leq n$. Now since $\|y\| = 1$, then $y \in S$ so $y \in \overline{B}_k$ for some $1 \leq k \leq n$. Thus,

$$\|x_k\| \leq \|y - x_k\| \leq r_k$$

which implies that $0 \in \overline{B}_k$ since $\|x_k\| = \|x_k - 0\| \leq r_k$, as desired.

Solution. (e) Consider the cover of S given by $\mathcal{B} = \{B(x,1)\}_{x \in S}$ where $B(x,1)$ is the open unit ball in X centered at $x \in S$ and we claim that $\mathcal{B}$ has no finite subcover. Suppose, for the purpose of contradiction, that $B_i = B(x_i, 1)$ for some $x_1, \ldots, x_n \in S$ covers S. Then by part (c) there exists $y \in X$ with $\|y\| = 1$ such that $1 = \|x_i\| \leq \|y - x_i\|$ for all $1 \leq i \leq n$, but $y \in S$ and since B_i cover S, then $y \in B_k$ for some k. Thus, $1 = \|x_i\| \leq \|y - x_i\| < 1$, a contradiction. Therefore, $\mathcal{B}$ is an open cover of S with no finite subcover and so S is non-compact.

Now to see that $\overline{B(0,1)}$, the closed unit ball in X is not compact observe that $S = \overline{B(0,1)} \setminus B(0,1)$. Hence, S is closed in $\overline{B(0,1)}$ since $B(0,1)$ is open in $\overline{B(0,1)}$. Thus, if $\overline{B(0,1)}$ were compact, then S must be compact since closed subsets of compact sets are compact, but S is a closed subset of $\overline{B(0,1)}$ which is not compact and therefore, $\overline{B(0,1)}$ is not compact.

Alternatively, the non-compactness of $\overline{B(0,1)}$ can be seen by considering the cover $\{B(x,1)\}_{x \in S} \cup \{B(0,1)\}$ which has no finite subcover since $\{B(x,1)\}_{x \in S}$ has no finite subcover of S and $S \cap B(0,1) = \emptyset$.

Problem 4.

Solution. (a) Let $n \geq 1$, then E_n is not absorbing. To see this recall that there is a strict inclusion $L_3([0,1], m) \subsetneq L_1([0,1], m)$. Hence, let $f \in L_1([0,1], m) \setminus L_3([0,1], m)$, then

$$\|f\|_1 = \int_{[0,1]} |f| dm < \infty \text{ and } \|f\|_3 = \left(\int_{[0,1]} |f|^3 dm \right)^{1/3} = \infty.$$

Thus, for any $t > 0$

$$\left\|t^{-1} f\right\|_3^3 = \int_{[0,1]} \left|t^{-1} f\right|^3 dm = t^{-3} \int_{[0,1]} |f|^3 dm = \infty$$

so that $t^{-1} f \notin E_n$. Thus, E_n is not absorbing by definition.

Solution. (b) First, by definition $E_n \subset L_3([0,1],m)$ and we know that $L_3([0,1],m) \subsetneq L_1([0,1],m)$. Hence, there exists $g \in L_1([0,1],m) \setminus L_3([0,1],m)$ so that in particular $g \notin E_n$. Now let $f \in E_n$, then we claim that for every open ball $B_{L_1}(f,\epsilon)$ there exists $\tilde{g} \notin E_n$ such that $\tilde{g} \in B_{L_1}(f,\epsilon)$ which by definition implies that E_n has no interior points. Hence, let $\epsilon > 0$ be arbitrary, then there exists $g \in L_1([0,1],m) \setminus L_3([0,1],m)$ so $g \notin E_n$. Define

$$\tilde{g} = f + \frac{\epsilon}{2\|g\|_1} g,$$

then since $\frac{\epsilon}{2\|g\|_1} > 0$ and E_n is not absorbing $\tilde{g} \notin E_n$. However,

En could in principle absorb g even though it is not absorbing. Also $\tilde{g} = f + t^{-1} g$

so how do you use absorbing here?

$$\|f - \tilde{g}\|_1 = \left\| \frac{\epsilon g}{2\|g\|_1} \right\|_1 = \frac{\epsilon}{2} \left\| \frac{g}{\|g\|_1} \right\|_1 = \frac{\epsilon}{2} < \epsilon$$

so $\tilde{g} \in B_{L_1}(f,\epsilon)$. Therefore, for every $f \in E_n$ and open ball $B_{L_1}(f,\epsilon)$ at f there is a $\tilde{g}$ such that $\tilde{g} \in B(f,\epsilon)$ and $\tilde{g} \notin E_n$ which implies that $\text{int}(E_n) = \emptyset$ by definition.

Solution. (c) Let $(f_k)_{k\in\mathbb{N}} \subset E_n$ be a sequence converging to $f \in L_1([0,1],m)$ under $\|-\|_1$. Now we claim that $f \in E_n$ so that E_n is closed. Recall that since $f_k \to f$, then there exists a subsequence $(f_{k_i})_{i\in\mathbb{N}}$ such that $f_{k_i}(x) \to f(x)$ almost everywhere (see for example Schilling, corollary 13.8). Hence, without loss of generality we may assume that $f_k(x) \to f(x)$ almost everywhere since such a subsequence will converge to f. Now $|f(x)|^3 = \lim_{k\to\infty} |f_k(x)|^3$ almost everywhere since $f_k(x) \to f(x)$ almost everywhere. Hence, since $|f_k|^3, |f|^3 \geq 0$ are positive measurable functions for all k we may apply Fatou's lemma to get

$$\int_{[0,1]} |f|^3 \leq \liminf_{k\to\infty} \int_{[0,1]} |f_k|^3 dm \leq n$$

for all k since $f_k \in E_n$. Therefore, $f \in E_n$ as desired.

Solution. (d) First, by part (c) $E = \overline{E}$ since E is closed and by part (b) $\text{int}(\overline{E}) = \text{int}(E) = \emptyset$. Thus, $E_n \subset L_1([0,1],m)$ is nowhere dense in $L_1([0,1],m)$. Additionally, we clearly have $L_3([0,1],m) = \cup_{i=1}^{\infty} E_n$. This follows since if $f \in E_n$, then by definition of E_n, $f \in L_3([0,1],m)$ and if $f \in L_3([0,1],m)$, then by definition

$$\|f\|_3^3 = \int_{[0,1]} |f|^3 dm < \infty$$

so there exists $n \in \mathbb{N}$ such that

$$\|f\|_3^3 = \int_{[0,1]} |f|^3 dm \leq n$$

so that $f \in E_n$. Therefore, $\{E_n\}_{n\in\mathbb{N}}$ is a collection of nowhere dense sets subsets of L_1 such that $L_3([0,1],m) = \cup_{n\in\mathbb{N}} E_n \subset L_1([0,1],m)$ so $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

Problem 5.

Solution. (a) We recall the reverse triangle inequality, that is, that for any x, y in a normed vector space X we have

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

Now suppose $x_n \to x$ as $n \to \infty$ in norm, then for all $\epsilon > 0$ there exists integer $N > 0$ such that for all $n > N$

$$\|x_n - x\| < \epsilon.$$

Thus, applying the reverse triangle inequality we see immediately that

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| < \epsilon$$

✓

so that $\|x_n\| \to \|x\|$ as $n \to \infty$.

Solution. (b) Let $(e_n)_{n\in\mathbb{N}}$ be a (countable) orthonormal basis in H, that is, $\langle e_i, e_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta. Note that H has a countable orthonormal basis since H is an infinite dimensional separable Hilbert space (see Folland proposition 5.29, pg. 176). Now we claim that $e_n \to 0$ weakly as $n \to \infty$. Observe that if this holds, then we have a counterexample since $\|e_n\| = 1$ is constant so $\|e_n\| \to 1$ as $n \to \infty$, but $1 \neq 0 = \|0\|$.

To show that $e_n \to 0$ weakly as $n \to \infty$ recall that 0 has a neighborhood basis given by sets of the form

$$B_X(0, f_1, \ldots, f_n, r) = \bigcap_{i=1}^{n} \{x \in H : |f_i(x)| < r\}.$$

Hence, to see convergence of $(e_k)_{k\in\mathbb{N}}$ to 0 in the weak topology it is sufficient to check that $(e_k)_{k\in\mathbb{N}}$ is eventually in $B_X(0, f_1, \ldots, f_n, r)$. Thus, since H is a Hilbert space, then by the Riesz representation theorem for each $f_i \in X^*$ there exists a unique $y_i \in H$ such that $f_i = \langle -, y_i \rangle$. Now $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis so we may write y_i uniquely as

$$y_i = \sum_{k\geq 1} \alpha_k(i) e_k, \;\; \alpha_k(i) \in \mathbb{K}$$

This need not be the case.

and where $\alpha_k(i) \neq 0$ for finitely many k. Thus, for K sufficiently large $\alpha_k(i) = 0$ for all $k > K$. Additionally, we have that

$$\alpha_k(i) = \langle y_i, e_k \rangle = \overline{\langle e_k, y_i \rangle} = \overline{f_i(e_k)}.$$

Thus, for all $k > K$, $f_i(e_k) = 0 < r$ for $1 \leq i \leq n$ and so $(e_k)_{k\in\mathbb{N}}$ is eventually in $B_X(0, f_1, \ldots, f_n, r)$ which was arbitrary so we are done.

(✓)

Solution. (c) Suppose $\|x_n\| \leq 1$ for all $n \geq 1$ and that $x_n \to x$ weakly as $n \to \infty$, then we claim that $\|x\| \leq 1$.

Let $f \in H^*$ be the unique linear functional represented by x, that is, $f(y) = \langle y, x \rangle$ for all $y \in H$ and let $\epsilon > 0$, then

$$B(x, f, \epsilon) = \{y \in H : |f(y - x)| < \epsilon\}$$

is a neighborhood of x in the weak topology by definition of neighborhood basis at 0. Thus, since $x_n \to x$ weakly, then eventually $(x_n)_{n\geq 1}$ is in $B(x, f, \epsilon)$. Hence, there exists $N \geq 1$ such that for all $n \geq N$

$$|f(x_n - x)| = |\langle x_n - x, x \rangle| = \left|\langle x_n, x \rangle - \|x\|^2\right| < \epsilon$$

and so by the reverse triangle inequality we have

$$\left||\langle x_n, x \rangle| - \|x\|^2\right| \leq \left|\langle x_n, x \rangle - \|x\|^2\right| < \epsilon.$$

Thus, the sequence $(|\langle x_n, x \rangle|)_{n\geq 1}$ converges to $\|x\|^2$. Additionally, by the Cauchy-Schwarz inequality and since $\|x_n\| \leq 1$ for all $n \geq 1$, then

$$|\langle x_n, x \rangle| \leq \|x_n\| \cdot \|x\| \leq \|x\|.$$

Hence, we must have $\|x\|^2 \leq \|x\|$ since $|\langle x_n, x \rangle|$ converges to $\|x\|^2$ and $|\langle x_n, x \rangle|$ is bounded by $\|x\|$. Therefore, $\|x\|^2 \leq \|x\|$ which implies $\|x\| \leq 1$, as desired.