The ground state energy of dilute 1D many-body quantum systems

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Overview

Background

The scattering length

Theorem 1 (Lieb-Yngvason 2001)

For $B_R = \{0 \le |x| < R\} \subset \mathbb{R}^d$ with $R > R_0 := \text{range}(v)$, let $\phi \in H^1(B_R)$ satisfy

$$-\Delta\phi + \frac{1}{2}v\phi = 0, \qquad \text{on } B_R, \tag{1}$$

with boundary condition $\phi(x) = 1$ for $|x| = R$. Then $\phi(x) = f(|x|)$ for some $f : (0, R] \to [0, \infty)$, and for range$(v) < r < R$, we have

$$f(r) = \begin{cases} (r-a)/(R-a) & \text{for } d = 1 \\ \ln(r/a)/\ln(R/a) & \text{for } d = 2 \\ (1-ar^{2-d})/(1-aR^{2-d}) & \text{for } d \ge 3, \end{cases} \tag{2}$$

with some constant a called the (s-wave) **scattering length**.

Model

We consider a many-body system of bosons that interacts via a repulsive pair potential $v_{ij} = v(|x_i - x_j|)$, with $v = v_{\mathsf{reg}} + v_{\mathsf{h.c.}}$

$$\mathcal{E}(\psi) = \int_{\Lambda_L} \left(\sum_{i=1}^{N} |\nabla_i \psi|^2 + \sum_{i<j} v_{ij} |\psi|^2 \right) \quad \text{on } L^2(\Lambda_L)^{\otimes_{\mathsf{sym}} N}. \quad (3)$$

The ground state energy is defined by

$$E(N,L) := \inf_{\psi \in \mathcal{D}(\mathcal{E}),\ \|\psi\|^2 = 1} \mathcal{E}(\psi).$$

2D and 3D

For $\Lambda_L = [0, L]^d$, let $e(\rho) := \lim_{\substack{L\to\infty \\ N/L^d \to \rho}} E(N, L)/L^d$.

Theorem 2 ($d = 3$ result, Lee-Huang-Yang 1957[1])

$$e(\rho) = 4\pi\rho^2 a \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho a^3})\right). \tag{4}$$

Theorem 3 ($d = 2$ result[2])

$$e(\rho) = 4\pi\rho^2 Y \left(1 - Y|\log Y| + \left(2\Gamma + \frac{1}{2} + \ln(\pi)\right) Y\right) + o\left(\rho^2 Y^2\right), \tag{5}$$

$Y = \left|\ln(\rho a^2)\right|^{-1}$.

[1] Upper bound: Yau-Yin 2009, Basti-Cenatiempo-Schlein 2021. Lower bound: Fournais-Solovej 2021

[2] Fournais-Girardot-Junge-Morin-Olivieri 2022

Main result

Theorem 4 (A.-Reuvers-Solovej, 2022)

Consider a 1D Bose gas with repulsive interaction $v = v_{\mathit{reg}} + v_{\mathit{h.c.}}$ as defined above. Define the denisty $\rho = N/L$. For $\rho|a|$ and ρR_0 sufficiently small, the ground state energy can be expanded as

$$E(N,L) = N\frac{\pi^2}{3}\rho^2\left(1 + 2\rho a + \mathcal{O}\left((\rho|a|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3}\right)\right), \tag{6}$$

where a is the scattering length of v.

Examples in 1D

The hard core gas

Energy behaves like free Fermi energy in volume $L - NR$, *i.e.*

$$\begin{aligned} E_{\text{hard core}}(N, L) &= N\frac{\pi^2}{3}\rho^2(1 - NR/L)^{-2} \\ &= E_0\left(1 + 2\rho R + \mathcal{O}\left((\rho R)^2\right)\right). \end{aligned} \tag{7}$$

Scattering length is $a = R$.

Lieb-Liniger model $(v(\cdot) = 2c\delta(\cdot))$

Energy behaves asymptotically like

$$E_{\text{LL}}(N, L, c) = N\frac{\pi^2}{3}\rho^2\left(1 - 4\rho/c + \mathcal{O}\left((\rho/c)^2\right)\right), \tag{8}$$

with scattering length $a = -\frac{2}{c}$.

Spinless/spin-polarized fermions

Spinless Fermions are unitarily equivalent to Bosons with a zero b.c. at all planes of intersection, *i.e.* with an infinite delta potential. As a consequence we have the following corollary.

Theorem 5 (A.-Reuvers-Solovej 2022)

Consider a 1D Fermi gas with repulsive interaction $v = v_{\text{reg}} + v_{\text{h.c.}}$ as defined before. Let $E_F(N, L)$ be its associated ground state energy. Write $\rho = N/L$. For ρa_o and ρR_0 sufficiently small, the ground state energy can be expanded as

$$E_F(N, L) = N \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a_o + \mathcal{O}\left((\rho R_0)^{6/5} + N^{-2/3} \right) \right), \quad (9)$$

where $a_o \geq 0$ is the odd wave scattering length of v.

This is consistent with lower bound $E_F(N, L) \geq E_0$, since $a_o \geq 0$.

A conjecture for spin-1/2 fermions

Two solvable model for spin-1/2 fermion:

The hard core gas

Ground state energy is independent of spin so

$$E_{\text{hard core}}(N, L) = N\frac{\pi^2}{3}\rho^2(1 - NR/L)^{-2} \approx E_0(1 + 2\rho R). \quad (10)$$

Scattering length is $a_e = a_o = R$.

Yang-Gaudin model

Is the spin-1/2 version of the LL model, *i.e.* $H_{\text{YG}} = H_{\text{LL}}$. Behaves asymptotically like

$$E_{\text{YG}}(N, L, c) = N\frac{\pi^2}{3}\rho^2\left(1 - 4\rho\ln(2)/c + \mathcal{O}\left((\rho/c)^2\right)\right), \quad (11)$$

with scattering length $a_e = -\frac{2}{c}$, $a_o = 0$.

A conjecture for spin-1/2 fermions

Based on the two solvable cases:

Conjecture

$$E(N,L) = N\frac{\pi^2}{3}\rho^2\Big(1 + 2\ln(2)\rho a_e + 2(1-\ln(2))\rho a_o + \mathcal{O}\left((\rho\max(|a_e|, a_o))^2\right)\Big) \tag{12}$$

Thanks for your attention!