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1

a) Write $\|f_N\|^2 = \dots$

The norm squared of f_N is $\langle \sum_{n=1}^{N^2} N^{-1}e_n, \sum_{n=1}^{N^2} N^{-1}e_n\rangle = \sum_{n=1}^{N^2}\langle N^{-1}e_n, \sum_{n=1}^{N^2} N^{-1}e_n\rangle = \sum_{n=1}^{N^2} N^{-2} = 1$, since e_n is an ONB. The square root of this is also 1 and this is the norm of f_N. ✓

Note that the linear map φ sending e_n to $1_{\{n\}}$ is an isometric isomorphism into $\ell^2(\mathbb{N})$. Linearity is clear and if $x = \sum_{n=1}^{\infty} x_n e_n \in H$ we have $\|x\|^2 = \sum_{n=1}^{\infty} |x_n|^2 = \|\sum_{n=1}^{\infty} x_n 1_{\{1\}}\|^2 = \|\varphi(x)\|_2^2$, which also implies injectivity as only 0 (in either space) has norm 0, so the kernel is trivial. For surjectivity note that for an ℓ_2 sequence (x_n) we have $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ so we know $\sum_{n=1}^{\infty} x_n e_n$ defines an element in H that is mapped to (x_n).

Now we prove $\varphi(f_N) \to 0$ weakly and use this to prove $f_N \to 0$ weakly. Since f_n is bounded $\varphi(f_n)$ is bounded in ℓ_2 norm and clearly $\lim_{n\to\infty} \varphi(f_n)_i \leq \lim_{n\to\infty} 1/n = 0$ for every i, since the i'th element in the sequence $\varphi(f_n)$ is zero when $i > n^2$ and otherwise $1/n$. At the same time we see $0 \leq \varphi(f_n)_i$ so the limit for $n \to \infty$ is zero for every i. By HW4 problem 3 this implies weak convergence of $\varphi(f_n)$ to zero. Next note that φ has an equally nice inverse so for $g \in X^*$ we have $g \circ \varphi^{-1} \in \ell^2(\mathbb{N})^*$ so $g(f_n) = (g \circ \varphi^{-1})(\varphi(f_n))$ converges to $g \circ \varphi^{-1}(0) = 0$ by weak convergence of $\varphi(f_n)$ to 0, which shows $g(f_n)$ converges to zero so f_n converges weakly to zero. ✓

b)

The norm closure of a convex set is the same as the weak closure by thm 5.7, so since $f_n \to 0$ weakly and $f_n \in \mathrm{co}\{f_n : n \geq 1\}$ 0 must be in $\overline{\mathrm{co}\{f_n : n \geq 1\}}^{\tau_w} = \overline{\mathrm{co}\{f_n : n \geq 1\}} = K$. Please note this also implies K is closed in the weak topology, since it is the weak closure of a set. ✓

$x \in \mathrm{co}\{f_n : n \geq 1\}$ can be written $\sum_{n=1}^{N} \alpha_n f_n$, where $\sum_{n=1}^{N} \alpha_n = 1, \alpha_n \geq 0$. By the triangle inequality $\|x\| \leq \sum_{n=1}^{N} \alpha_n \|f_n\| = 1$. So $\mathrm{co}\{f_n : n \geq 1\} \subseteq \overline{\mathrm{B}(0,1)}$ and the same inclusion holds for its closure K. By theorem 6.3 in the lecture notes and the fact that all Hilbert spaces are reflexive (prop 2.10), the closed unit ball is weakly compact. So K is closed in a weakly compact set and therefore weakly compact set. ✓ This holds since we know the weak topology is Hausdorff.

c)

0 is an extreme point since all elements in K have positive coordinates, meaning $\langle x, e_i\rangle \geq 0$ for all i. So if $0 = \alpha x + (1-\alpha)y, \alpha \in (0,1)$ then unless both x, y are zero one of them, say x, has a strictly positive coordinate i'th coordinate for some i. Multiplying by $\alpha > 0$ and adding the weakly positive i'th coordinate of $(1-\alpha)y$ preserves strict positivity. So since 0 does not have any strictly positive coordinates, it is not a proper convex combination of some other points in K. So it is extreme.

Write this earlier...

We see all $x \in K$ have positive coordinates as follows. Clearly all $y \in \mathrm{co}\{f_n : n \geq 1\}$ have positive coordinates since each f_n scaled by a positive constant has positive coordinates and so does a sum of them. This implies if y has a stricly negative coordinate ($\langle y, e_i\rangle < -\varepsilon$ for some i, ε $\varepsilon > 0$?), it cannot be in K.

Write this before stating that $y \notin K$ or be more explicit that you are proving it...

The distance from y to $x \in \mathrm{co}(\{f_n : n \geq 1\})$ is less than the difference in the i'th coordinate which is bounded from below by ε, so a sequence from $\text{co}(\{f_n : n \geq 1\})$ cannot converge to y so it is not in K. Since any point with a strictly negative coordinate cannot be in K, all elements in K must have positive coordinates as I claimed in the preceding paragraph. Write out the math.

Write out the math.

f_i is extreme because it has maximal i^2 coordinate. All other elements in K have strictly smaller i^2 coordinate and their convex combination will also have strictly smaller i^2 coordinate. Since a proper convex combination of other elements from K cannot have equality in this coordinate such a combination cannot equal f_n and thus it is extreme.

Again, write this earlier...

The i^2 coordinate of $\sum_{n=1}^{N} \alpha_n f_n \in \text{co}(\{f_n : n \geq 1\})$ is less than $\alpha_i/i+(1-\alpha_i)/(i+1)$ since the f_n's where $n \neq i$ have i^2'th coordinate at most $1/(i+1)$. So we see for a converging sequence $x_n \in \text{co}(\{f_n : n \geq 1\})$ unless $\alpha_{i,n} \to 1$ the limit will have a strictly smaller i^2'th coordinate than f_i. But if $\alpha_{i,n} \to 1$ the $x_n \to f_i$ which is not another element in K. Here $\alpha_{i,n}$ is α_i in x_n, written as a sum of f_ns. We conclude f_i has strictly maximal i^2 coordinate in K, and is therefore extreme by the preceding paragraph.

Why?

Express it explicitly!

3 times you've stated something before proving it afterwards without making it clear you were going to prove it. That is sloppy writing.

d)

Let F be $\{0\} \cup \{f_n : n \geq 1\}$. We know K is nonempty, convex and compact in the weak topology and can therefore apply Milman theorem 7.9 in the lecture notes. We have $K = \overline{\text{co}\{f_n : n \geq 1\}}^{\tau_w} \subseteq \overline{\text{co}(F)}^{\tau_w}$ so by Milmal $\text{Ext}(K) \subseteq \overline{F}^{\tau_w} = F$ since it is a finite set and thus closed since the weak topology is Hausdorff, which proves the other inclusion. So no, there are no other extreme points.

No it is not.

0%

2

a)

Weak convergence in Y is equivalent with $f(Tx_n) \to f(Tx), f \in Y^*$. Since composition preserves both linearity and continuity $f \circ T \in X^*$ so we have $f(Tx_n) = (f \circ T)x_n \to (f \circ T)x = f(Tx)$ by weak convergence of x_n to x. So we have weak convergence of $Tx_n \to Tx$.

✓

b)

By HW4 weak convergence of a sequence implies boundedness and by prop 8.2 in the lecture notes compactness of T means that every bounded sequence (y_n) has a subsequence so (Ty_{n_k}) converges in norm. Suppose for contradiction that (Tx_n) does not converge in norm. Negating convergence we see $\exists \varepsilon > 0 : \forall N \exists n \geq N : \|Tx_n - Tx\| > \varepsilon$, and so for appropriate ε we have a subsequence (x_{n_k}) where $\|Tx_{n_k} - Tx\| > \varepsilon$. But this subsequence is bounded since (x_n) is bounded, so (Tx_{n_k}) has a norm convergent subsequence. Since norm convergence implies weak convergence to the same limit and that (Tx_n) (and thus all subsequence) converges weakly to Tx by a), we see the subsequence of (Tx_{n_k}) converges in norm to Tx. This is a contradiction as $\|Tx_{n_k} - Tx\| > \varepsilon$, so we conclude (Tx_n) must converge in norm.

✓

c)

Suppose for contradiction that T is not compact. Then by prop 8.2 there is a bounded sequence x_n that does not contain a subsequence x_{n_k} so Tx_{n_k} converges in norm. Since it is bounded we can scale the sequence so it is in $\overline{\mathrm{B}(0,1)}$. By theorem 6.3 and that all Hilbert spaces are reflexive (prop 2.10) the closed unit ball is weakly compact. Suppose for a moment the weak topology restricted to the closed unit ball is a metric topology. Then compactness implies sequential compactness, so x_n must have a weakly convergent subsequence x_{n_k}. But by the assumption on T, Tx_{n_k} converges in norm, but this is a contradiction as x_n did not contain a subsequence so its image under T converged in norm. We conclude T is compact. ✓

Proof the weak topology restricted to the closed unit ball is metrizable. Consider the function $d(x,y) = \sum_{n=1}^{\infty} \frac{|\langle x-y,e_n\rangle|}{|\langle x-y,e_n\rangle|+1} 2^{-n}$ where e_n is a fixed ONB for H which is a assumed to be separable. d is well defined since the first factor in each term is less than 1 and $\sum_{n=1}^{\infty} 2^{-n}$ converges. d is also clearly positive and symmetric and $d(x,y) = 0 \implies x = y$ since they would agree in every coordinate. The function $\frac{x}{1+|x|}$ is an increasing function (its derivative is positive). Therefore since $|\langle x-y,e_n\rangle| = |\langle x-z,e_n\rangle + \langle z-y,e_n\rangle| \leq |\langle x-z,e_n\rangle| + |\langle z-y,e_n\rangle|$ we get

$$\frac{|\langle x-y,e_n\rangle|}{1+|\langle x-y,e_n\rangle|} \leq \frac{|\langle x-z,e_n\rangle|}{|\langle x-z,e_n\rangle|+1} + \frac{|\langle z-y,e_n\rangle|}{|\langle z-y,e_n\rangle|+1},$$

and further

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|\langle x-y,e_n\rangle|}{|\langle x-y,e_n\rangle|+1} 2^{-n} \leq \sum_{n=1}^{\infty} \Big(\frac{|\langle x-z,e_n\rangle|}{|\langle x-z,e_n\rangle|+1} + \frac{|\langle z-y,e_n\rangle|}{|\langle z-y,e_n\rangle|+1} \Big) 2^{-n} = d(x,z) + d(z,y),$$

so d is a metric.

Now we prove that the topology generated by d τ_d is τ_w. In τ_d the neighborhood basis at 0 is balls of the form $\{x : \sum_{n=1}^{\infty} \frac{|\langle x,e_n\rangle|}{|\langle x,e_n\rangle|+1} 2^{-n} < r\}$. For N sufficiently large $\sum_{n=N}^{\infty} 2^{-n} < r/2$. Consider the finitely many functions of the form $\sum_{n=1}^{N} \lambda_n \langle x,e_n\rangle, \lambda \in \{\pm 1, \pm i\}$ and the element of the Neigborhood basis a zero consisting the x so that the absolute value of all these functions in x is less than $r/4$, and take x in this set. Clearly $\sum_{n=1}^{N} \frac{|\langle x-z,e_n\rangle|}{|\langle x-z,e_n\rangle|+1} \leq \sum_{n=1}^{N} |x_n|$ where $x_n = \langle x,e_n\rangle$. For every complex number $z = a+ib$ one choice of λ will make $\Re(\lambda z) = \max(|a|,|b|) \geq |z|/2$. This generalizes to for a finite number of complex number $z_1,...,z_N$ one choice of $\lambda_1,...,\lambda_N$ makes $\Re(\sum_{n=1}^{N} \lambda_n z_n) = \sum_{n=1}^{N} \max(|a_n|,|b_n|) \geq \sum_{n=1}^{N} |x_n|/2$. So $\sum_{n=1}^{N} |x_n| \leq 2\sum_{n=1}^{N} \max(|a_n|,|b_n|) \leq 2|f_\lambda(x)| < 2r/4 = r/2$ where f_λ is the function where $\lambda_1,...,\lambda_N$ are chosen so that $\Re(\sum_{n=1}^{N} \lambda_n x_n) = \sum_{n=1}^{N} \max(|a_n|,|b_n|)$. This is less than $|f_\lambda(x)|$ since there is also an imaginary part that contributes to the norm, and finally $|f_\lambda(x)| < r/4$ since it is one of the function used to define the open set we took x from. ? Not sure what is happening here?

So in summary, for x in the previously specified open set we have $d(0,x) = \sum_{n=1}^{N} \frac{|\langle x,e_n\rangle|}{|\langle x,e_n\rangle|+1} 2^{-n} + \sum_{n=N}^{\infty} \frac{|\langle x,e_n\rangle|}{|\langle x,e_n\rangle|+1} 2^{-n}$. For the first N terms we can estimate their contribution by $\sum_{n=1}^{N} |x_n|$ and our functions and radius was chosen in a clever way so this sum is less than $r/2$, and for the remaining terms their sum is less than $r/2$ by choice of N. This proves the weak topology is finer than the metric topology as it contains a basis for the metric topology in the form of these balls, translated appropriately.

Now for the other inclusion. Our neighborhood basis at 0 for the weak topology is sets of the form $\cap_{n=1}^{N} \{x : |f_n(x)| < r\}$. But if we can find a (metric) open ball in each of them, we can use the minimal

Write out the name.

radius as a ball contained in all of them. Therefore I will only consider the case where there is only one function denoted f. By RF $f(x) = \langle x, y\rangle$ for some y and we can find N so $\sum_{n=N}^{\infty} |y_n|^2 < (r/2)^2$ where $y_n = \langle y, e_n\rangle$ since this is the tail of the series that converges to the finite number $\|y\|^2$. Then we have

$$|f(x)| \leq |\langle x, \sum_{n=N}^{\infty} y_n e_n\rangle| + |\langle x, \sum_{n=1}^{N} y_n e_n\rangle|$$

and the first term can be estimated by $\|x\| \cdot \|\sum_{n=N}^{\infty} y_n e_n\| \leq 1 \cdot r/2$ since we are in the closed unit ball and choice of N. For the second term note that $d(o, x) \leq \varepsilon \implies \frac{|x_n|}{|x_n|+1} \leq 2^n\varepsilon$ for all n. Rearranging this we get $|x_n|(1 - 2^n\varepsilon) \leq 2^n\varepsilon$. We can choose ε so small $1 - 2^N\varepsilon > 0$. For $n \leq N$ we get $|x_n| \leq \frac{2^n\varepsilon}{1-2^n\varepsilon} \leq \frac{2^N\varepsilon}{1-\varepsilon}$. Since it has limit 0 as $\varepsilon \to 0$ we can choose ε so small it is smaller than $r/(2NC)$, where $C = \max(|y_1|, ..., |y_N|)$. Then we can estimate the second term by

$$|\langle x, \sum_{n=1}^{N} y_n e_n\rangle| \leq \sum_{n=1}^{N} |x_n||y_n| \leq \sum_{n=1}^{N} \frac{r}{2NC} C = r/2$$

So if $d(0, x) < \varepsilon$ we have $|f(x)| < r$ so we have found the required open ball. This proves the metric topology on the closed unit ball is finer than the weak topology so they are the same. Therefore the weak topology restricted to $\overline{\mathrm{B}(0,1)}$ is a metric topology and my argument using sequential compactness is justified. (✓)

Sorry if this is too messy. I misremembered topology and had previously assumed compactness always implied sequential compactness. If it did, this argument would be a lot shorter haha.

Think you'd enjoy Thm 5.13 in the notes.

d)

Let $x_n \to x$ weakly in ℓ_2 and $T \in \mathcal{L}(\ell_2, \ell_1)$. By a) we have weak convergence of Tx_n to Tx and by remark 5.3 this implies norm convergence in ℓ_1. So whenever we have weak convergence, Tx_n converges in norm. to Tx
Then by c) T is compact and we are done. (✓)

e)

Suppose T is compact and surjective. Let $Z = X/\ker T$ and note $\tilde{T}$ defined by $T = \tilde{T} \circ \pi$, where $\pi : X \to Z$ is the quotient map, is invertible. I will show it is compact and this leads to a contradiction. By the remark after proposition 8.2 (proven in HW4) if $\mathcal{K}(Z, Y)$ contains an invertible operator, then Z, Y must be finite dimensional, but Y is assumed to be infinite dimensional.

Compactness follows if the image of the unit ball is precompact. We have $\pi(B(0,1)) = \{[x] : \|x\| < 1\} \subseteq B_Z(0,1) = \{[x] : \inf_{y \in \ker T} \|x + y\| < 1\}$ since the infimum is less than $\|x + 0\|$which is already strictly smaller than 1. For the other inclusion, fix $x \in B_Z(0,1)$ and note for $\varepsilon > 0$ there is a y so $\|x + y\| < \inf_{y \in \ker T} \|x + y\| + \varepsilon$ so for $\varepsilon = 1 - \inf_{y \in \ker T} \|x + y\| > 0$ we conclude there is a y so $\|x + y\| < 1$, so $x + y \in B(0,1)$. This means $[x] = [x + y] \in \pi(B(0,1))$ proving $\pi(B(0,1)) = B_Z(0,1)$. So $\tilde{T}(B_Z(0,1)) = \tilde{T} \circ \pi(B(0,1)) = T(B(0,1))$, which we know is precompact by compactness of T. So $\tilde{T}$ is compact and we have the contradiction described before. ✓

These should have a different name.

f)

What are f_n, g?

We have $\langle Mf, g\rangle = \int_{[0,1]} tf(f)\overline{g(t)}dt = \int_{[0,1]} f(f)\overline{tg(t)}dt = \langle f, Mg\rangle$ since t is real. So M is self adjoint.
I will find a sequence f_n that converges weakly to 0, but Mf_n does not converge to 0 in norm. By b) this proves M is not compact. Define $f_n = \sqrt{n}\cdot 1_{[0,1/n]}$ and recall that L_2 is a Hilbert space so the functionals are given as inner products with another element from L_2 by RF representation theorem. So for $g \in L_2$ we show $\langle f_n, g\rangle \to 0$. We have

Riesz-Fisher

$$|\int_0^1 f_n\bar{g}dt| \le \int_0^{1/n} \sqrt{n}|g|dt \le \|\sqrt{n}\|_{[0,1/n]}\|g\|_{[0.1/n]} \le \|g\|/\sqrt{n} \to 0,$$

But this is the 1-norm?

Where I use Cauchy-Schwartz and that $\sqrt{n}, g \in L_2([0,1/n])$ by HW 2 problem 2. This proves weak convergence to 0. Note that $g_n(t) = f_n(1-t) = \sqrt{n}\cdot 1_{[1-1/n,1]}(t)$ also converges to 0 weakly because this change of variable does not change the integral. Alternatively $\langle g_n, g\rangle = \langle f_n, g(1-t)\rangle \to 0$ since $g(1-t) \in L_2([0,1])$ and $f_n \to 0$ weakly.

Notation.

Now I prove that Mg_n does not converge in norm to 0. Since we are integrating near the point one we can use 1/2 as a lower estimate for t. Then we have $\|Mg_n\| = \|tg_n(t)\| \ge \|g_n\|/2 = 1/2$. So we have a weakly converging sequence whose image does not converge in norm. Then M is not compact. g_n has norm one since $\|g_n\|^2 = \int_{[1-1/n,1]} |\sqrt{n}|^2 dt = n\cdot 1/n = 1$.

?

(✓)

Why?

Why not write that earlier...

3

a)

K is bounded by one as both $(1-s)t, (1-t)s$ are bounded by 1 as each factor is bounded by 1. So $K \in L_2([0,1]^2, m^2)$ since [0,1] has finite measure it is a fortiori a σ-finite measure space. Then by proposition 9.12 in the lecture notes T_K is a compact operator. $T_K(f)$ is defined as $\int_0^1 K(x,t)f(t)dt$ so this is just T, which is then compact

Not in lecture notes.

$T_K f(s) = \int_{[0,1]} k(t,s)f(t)dm(t)$

b)

Since K is real we have $\langle Tf, g\rangle = \int_0^1 K(s,t)f(t)\overline{g(t)}dt = \int_{[0,1]} f(t)\overline{K(s,t)g(t)}dt = \langle f, Tg\rangle$ so T is self adjoint.

s?

1/2

c)

We have $Tf(s) = \int_{[0,1]} K(s,t)f(t)dt = \int_{[0,s]} K(s,t)f(t)dt + \int_{[s,1]} K(s,t)f(t)$, and in the first integral K is equal to $(1-s)t$ and in the second it is almost everywhere equal to $(1-t)s$. So
$Tf(s) = \int_{[0,s]}(1-s)tf(t)dt + \int_{[s,1]}(1-t)sf(t) = (1-s)\int_{[0,s]} tf(t)dt + s\int_{[s,1]}(1-t)f(t)$. For the second part note that the $t, (1-t)$ is less than one so since $f \in L_1([0,1], m)$ by HW2 problem 2 so $|f|$ is an integrable majorant for $1_{[0,s]}tf(t), 1_{[s,1]}(1-t)f(t)$. So by dominated convergence if $s_n \to s$ we have $\lim_{n\to\infty}\int 1_{[0,s_n]}tf(t)dt = \int \lim_{n\to\infty} 1_{[0,s_n]}tf(t)dt = \int_{[0,s]} tf(t)dt$ which is continuity in s of the first integral. By the same argument the other integral is continuous in s and multiplying by the continuous functions $(1-s), s$ preserve continuity so $Tf(s)$ is continuous as it is a sum of continuous functions.

✓

$T(0) = T(1) = 0$?

4

a)

We know that $e^{-x^2/2} =: f(x) \in \mathscr{S}(\mathbb{R})$ by example 11.3 since $x^2 = |x|^2$. We will show that any monomial times any derrivative of g_k goes to zero as $|x| \to \infty$. By Leibniz formula $\partial_x^n g_k = \sum_{i=0}^n \binom{n}{i} (x^k)^{(i)} f^{(n-1)}(x)$. So we see any monomial times any derivative of g_k is a finite sum whose terms are a constant times a monomial times a derivative of f. Since $f \in \mathscr{S}(\mathbb{R})$ any monomial times a derivative of f has limit zero as $|x| \to \infty$ and multiplying by a constant does not change this. Since this holds for every term in the finite sum, the sum has limit zero as $|x| \to \infty$. So $g_k \in \mathscr{S}(\mathbb{R})$.

By prop 11.13 we have $\mathcal{F}(x^k f) = i^k \partial_x^k \hat{f}$, but by example 11.3 f is its own Fourier transform so we can drop the hat. By straightforward calculation

$$f'(x) = -xf(x), f''(x) = -f(x) + x^2 f(x), f'''(x) = 3xf(x) - x^3 f(x)$$

So

$$\begin{aligned}
\mathcal{F}(g_0) &= i^0 f(x) = g_0(x) \\
\mathcal{F}(g_1) &= i^1(-x)f(x) = -ig_1(x) \\
\mathcal{F}(g_2) &= i^2(-f(x) + x^2 f(x)) = f(x) - x^2 f(x) = g_0(x) - g_2(x) \\
\mathcal{F}(g_3) &= i^3(3xf(x) - x^3) = -3ig_1(x) + ig_3(x)
\end{aligned}$$

b)

We see $h_0 = g_0$ and $h_3 = g_1$ are valid choices. Less obvious is that

$$\mathcal{F}(g_2 - g_0/2) = g_0 - g_2 - g_0/2 = -g_2 + g_0/2$$

is a valid choice for h_2. Even less obvious is that

$$\mathcal{F}(-1.5g_1 + g_3) = 1.5ig_1 - 3ig_1 + ig_3 = -1.5ig_1 + ig_3$$

$\frac{3}{2}$

is a valid choice for h_1.

c)

The idea is to prove $\mathcal{F}^2 = (\mathcal{F}^*)^2$ as a linear map on $\mathscr{S}(\mathbb{R})$. Then we would have $\mathcal{F}^4(f) = (\mathcal{F}^*)^2 \mathcal{F}^2(f)$, and since $\mathcal{F}(f) \in \mathscr{S}(\mathbb{R})$ the Fourier inversion thm and continuity imply $\mathcal{F}^*\mathcal{F}^2(f) = (\mathcal{F}^*\mathcal{F})\mathcal{F}(f) = \mathcal{F}(f)$. So by applying the same argument again $\mathcal{F}^4(f) = (\mathcal{F}^*)^2\mathcal{F}^2(f) = f$. f ?

It follows from the definition that $\mathcal{F}^2(f)(\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) e^{-iyx} e^{-iy\xi} dm x dm y$, and from the theory of the Lebesgue integral we know changing to the variable $-y$ does not change the integral since the absolute value of its derivative is 1. But after this small change we have $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x) e^{iyx} e^{iy\xi} dm x dm y = (\mathcal{F}^*)^2(f)(\xi)$. Since $(\mathcal{F}^*)^2 = \mathcal{F}^2$, we are done by the comment above.

f

d)

By c) and linearity of $\mathcal{F}$, $\mathcal{F}^4(f) = f = \lambda^4 f$ and since $f \neq 0$ we must have $\lambda^4 = 1 \implies \lambda \in \{1, -1, i, -i\}$. So $\mathcal{F}$ has no other eigenvalues, but each of them is an eigenvalue by b).

5

Show this!

Let U be a non-empty open set in [0,1]. Since (x_n) is dense U must contain some x_n and thus have non-zero measure. This means that every non-empty open set is not contained in the union of open sets with measure zero, N, so $N = \emptyset$. So its complement, by definition the support of μ, is the entire space.