

# Stability of the $N + 1$ Fermi gas with point interactions

Advanced Mathematical Physics

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May 15, 2020.

# Overview

① Motivation

② Moser and Seiringer: Stability of  $N + 1$  system

③ Model

# Motivation

Model of fermions interacting via point interactions are of great interest as they appear as

- Models of cold atomic gases.
- Models of nuclear interaction.
- Approximations of models with short-range interactions.

However, they are mathematically not very well understood.

# Thomas collapse

**Thomas collapse:** It is known that a bosonic system of three or more bosons with zero-range interactions is unstable (of the first kind) i.e. there is no ground state energy. This can be seen from the variational principle.

**No Thomas collapse for (spin-1/2) fermions:** The Thomas collapse is a collective phenomenon where three (or more) bosons interact in a single point. This can never happen for spin-1/2 fermions, due to the Pauli principle.

**Stability of the first kind:** Is still an unsolved problem for general  $N + M$  systems ( $N$  spin up and  $M$  spin down).



# Results of Moser and Seiringer

## Results

Moser, T., Seiringer, R. Stability of a Fermionic  $N + 1$  Particle System with Point Interactions. Commun. Math. Phys. 356, 329–355 (2017).

- Prove stability of the  $N + 1$  system, within a certain mass ratio interval.
- Prove existence of self-adjoint bounded from below Hamiltonian.
- Prove Tan relations.

We focus on the first two.

# Formal Hamiltonian

## Formal Hamiltonian

The Hamiltonian of a system of  $N$  fermions of one species of mass 1 interacting via 1 fermion of another species of mass  $m$  can be described by the formal Hamiltonian

$$H = -\frac{1}{2m}\Delta_{x_0} - \frac{1}{2}\sum_{i=1}^N \Delta_{x_i} + \gamma \sum_{i=1}^N \delta(x_i - x_0) \quad (1)$$

# Formal Hamiltonian

## Centre of mass separation

We can split the Hamiltonian in two

$$H = H_{\text{CM}} + \frac{m+1}{2m} H_{\text{rel}}, \quad (2)$$

with  $x_{\text{cm}} = (mx_0 + \sum_{i=1}^N x_i)/(m+N)$ ,  $y_i = x_i - x_0$ , and

$$H_{\text{CM}} = \frac{1}{2(N+m)} \Delta_{x_{\text{cm}}}, \quad (3)$$
$$H_{\text{rel}} = - \sum_{i=1}^N \Delta_{y_i} - \frac{2}{m+1} \sum_{1 \leq i < j \leq N} \nabla_{y_i} \cdot \nabla_{y_j} + \tilde{\gamma} \sum_{i=1}^N \delta(y_i)$$



# Quadratic form

The formal Hamiltonian can be given precise meaning through a quadratic form, which can be obtained by considering more regularized models such as rank-one perturbations of a free Hamiltonian. One obtains

$$F_\alpha(u) = \int_{\mathbb{R}^{3N}} dk \hat{G}(k)^{-1} |\hat{w}|^2 - \mu \|u\|_{L^2(\mathbb{R}^{3N})}^2 + N \left( T_{\text{diag}}(\xi) + T_{\text{off}}(\xi) + \alpha \|\xi\|_{L^2(\mathbb{R}^{3(N-1)})}^2 \right) \quad (4)$$

with  $\hat{u}(k) = \hat{w}(k) + \sum_{i=1}^N (-1)^{i-1} \hat{G}(k) \xi(\bar{k}^i)$ ,  $\mu > 0$ ,  
 $\hat{G}(k) = \left( \sum_{i=1}^N k_i^2 + \frac{2}{m+1} \sum_{1 \leq i < j \leq N} k_i \cdot k_j + \mu \right)^{-1}$ ,





$$T_{\text{diag}}(\xi) = \int_{\mathbb{R}^{3(N-1)}} d\bar{k}^N L(\bar{k}^N) |\xi(\bar{k}^N)|^2,$$

$$T_{\text{off}}(\xi) = (N-1) \int_{\mathbb{R}^{3(N-2)}} d\bar{q} \int_{\mathbb{R}^3} ds \int_{\mathbb{R}^3} dt \overline{\xi(s, \bar{q})} \hat{G}(s, t, \bar{q}) \xi(t, \bar{q}). \quad (5)$$

with

$$L(\bar{k}^N) = 2\pi^2 \left( \frac{m(m+2)}{(m+1)^2} \sum_{i=1}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{1 \leq i < j \leq N-1} k_i \cdot k_j + \mu \right)^{1/2}. \quad (6)$$

The domain is

$$\mathcal{D}(F_\alpha) = \left\{ u \in L_{\text{as}}^2(\mathbb{R}^{3N}) \mid \right. \\ \left. \hat{u} = \hat{w} + \widehat{\rho G}, w \in H_{\text{as}}^1(\mathbb{R}^{3N}), \xi \in H_{\text{as}}^{1/2}(\mathbb{R}^{3(N-1)}) \right\}. \quad (7)$$



# Theorem 1

Introduce the function for  $m > 0$

$$\Lambda(m) = \sup_{\substack{s, K \in \mathbb{R}^3, \\ Q \geq 0}} \frac{s^2 + Q^2}{\pi^2(1+m)} \ell_m(s, K, Q)^{-1/2} \int_{\mathbb{R}^3} dt \frac{1}{t^2} \ell_m(t, K, Q)^{-1/2} \\ \times \frac{|(s + AK) \cdot (t + AK)|}{\left[ (s + AK)^2 + (t + AK)^2 + \frac{m}{1+m} (Q^2 + AK^2) \right]^2 - \left[ \frac{2}{(1+m)} (s + AK) \cdot (t + AK) \right]^2} \quad (8)$$

where  $A = (2 + m)^{-1}$  and

$$\ell_m(s, K, Q) = \left( \frac{m}{(1+m)^2} (s + K)^2 + \frac{m}{1+m} (s^2 + Q^2) \right). \quad (9)$$

It is then showed that

$$\Lambda(m) \leq \frac{4(1+m)^2(2+4m+m^2)^{3/2}}{\sqrt{2}\pi [m(m+2)]^3}. \quad (10)$$



# Theorem 1

Theorem 1, Moser, T., Seiringer, R., 2017.

For any  $\xi \in H_{\text{as}}^{1/2}(\mathbb{R}^{3(N-1)})$ ,  $\mu > 0$  and  $N \geq 2$ ,

$$T_{\text{off}}(\xi) \geq -\Lambda(m)T_{\text{diag}}(\xi). \quad (11)$$

In particular, if  $\Lambda(m) < 1$ , then  $F_\alpha$  is closed and bounded from below by

$$F_\alpha(u) \geq \begin{cases} 0 & \text{for } \alpha \geq 0, \\ -\left(\frac{\alpha}{2\pi^2(1-\Lambda(m))}\right)^2 \|u\|_{L^2(\mathbb{R}^{3N})}^2 & \text{for } \alpha < 0. \end{cases} \quad (12)$$

# Physical consequences

$\Lambda(m) < 1$  for  $m \geq 0.36$  so the critical mass ratio is less than 0.36

- Stability of first and second kind.
- Existence of self-adjoint bounded from below Hamiltonian.

For  $m \rightarrow \infty$  we have  $\Lambda(m) \rightarrow 0$ . Thus, the system with  $\alpha < 0$  has energy bounded from below by  $-(\alpha/(2\pi^2))^2$ .

- For  $m \rightarrow \infty$  the heavy fermion can bind at most one light fermion. This is the Pauli principle.



## Theorem 2

Since  $T_{\text{diag}} + T_{\text{off}}$  is symmetric, closed, and bounded from below, we may define the unique self-adjoint operator  $\Gamma$  by

$$T_{\text{diag}}(\xi) + T_{\text{off}}(\xi) = \langle \xi | \Gamma \xi \rangle. \quad (13)$$

It can be shown that  $H_{\text{as}}^1(\mathbb{R}^{3(N-1)}) \subset \mathcal{D}(\Gamma)$



## Theorem 2

Theorem 2, Moser, T., Seiringer, R., 2017

For any  $\xi \in H_{\text{as}}^1(\mathbb{R}^{3(N-1)})$ ,  $\mu > 0$ , and  $N \geq 2$

$$\|\Gamma\xi\|_{L^2(\mathbb{R}^{3(N-1)})}^2 \geq (1 - \Lambda_1(m)) \|L\xi\|_{L^2(\mathbb{R}^{3(N-1)})}^2. \quad (14)$$

~~In particular~~, if  $\Lambda_1(m) < 1$ , Then  $\mathcal{D}(\Gamma) = \mathcal{D}(L) = H_{\text{as}}^1(\mathbb{R}^{3(N-1)})$ .  
More generally for  $0 \leq \beta \leq 2$ ,

$$\left\| L^{(\beta-1)/2} \Gamma\xi \right\|_{L^2(\mathbb{R}^{3(N-1)})}^2 \geq (1 - \Lambda_1(m)) \left\| L^{(\beta+1)/2} \xi \right\|_{L^2(\mathbb{R}^{3(N-1)})}^2, \quad (15)$$

for all  $\xi \in H_{\text{as}}^{(\beta+1)/2}(\mathbb{R}^{3(N-1)})$ .

# Consequences

Knowing that the quadratic form  $F_\alpha$  is symmetric, closed, and bounded from below, it is straightforward to obtain the Hamiltonian:

$$\mathcal{D}(H_\alpha) = \{u \in \mathcal{D}(F_\alpha) \mid F_\alpha(\cdot, u) \text{ is } L^2 \text{ bounded on } \mathcal{D}(F_\alpha)\}. \quad (16)$$

Notice that for  $u \in \mathcal{D}(H_\alpha)$ , we have  $F_\alpha(\cdot, u) = \langle \cdot, x \rangle$  and we set  $H_\alpha u = x$ . Moser and Seiringer obtains

$$\begin{aligned} \mathcal{D}(H_\alpha) = \Big\{ u \in L^2_{\text{as}}(\mathbb{R}^{3N}) \mid u = w + G\xi, w \in H^2_{\text{as}}(\mathbb{R}^{3(N-1)}), \\ \xi \in \mathcal{D}(\Gamma), w|_{y_N=0} = (2\pi)^{-3/2}(-1)^{N+1}(\alpha + \Gamma)\xi \Big\}, \end{aligned} \quad (17)$$

$$\begin{aligned} (H_\alpha + \mu)(w + G\xi) &= (H_{\text{free}} + \mu)w, \text{ with} \\ (G\xi)(x) &:= \left( \sum_{i=1}^N (-1)^{i-1} \hat{G}(k) \xi(\bar{k}^i) \right)^\vee(x) \end{aligned}$$



Thank you for your attention.





$$\Lambda_{\beta}(m) = \sup_{\substack{s, K \in \mathbb{R}^3, \\ Q \geq 0}} \frac{s^2 + Q^2}{\pi^2(1+m)} \ell_m(s, K, Q)^{-1/2} \int_{\mathbb{R}^3} dt \frac{1}{t^2} \left( \frac{\ell_m(s, K, Q)^{(\beta-1)/2}}{\ell_m(t, K, Q)^{(\beta+1)/2}} + \frac{\ell_m(t, K, Q)^{(\beta-1)/2}}{\ell_m(s, K, Q)^{(\beta+1)/2}} \right) \\ \times \frac{|(s + AK) \cdot (t + AK)|}{\left[ (s + AK)^2 + (t + AK)^2 + \frac{m}{1+m}(Q^2 + AK^2) \right]^2 - \left[ \frac{2}{(1+m)}(s + AK) \cdot (t + AK) \right]^2} \quad (18)$$

