FunkAn Mandatory Assignment 1

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Problem 1

a)

First we want to check that $\|\cdot\|_0$ is a norm.
Let $x, x' \in X$, then

$$\begin{aligned}\|x+x'\|_0 =& \|x+x'\|_X + \|T(x+x')\|_Y = \|x+x'\|_X + \|Tx+Tx'\|_Y \\ \leq& \|x\|_X + \|Tx\|_Y + \|x'\|_X + \|Tx'\|_Y = \|x\|_0 + \|x'\|_0\end{aligned}$$

Since both $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms, and T is a linear map.
Now let $\alpha \in \mathbb{K}$, and $x \in X$, then

$$\begin{aligned}\|\alpha x\|_0 =& \|\alpha x\|_X + \|T\alpha x\|_Y = |\alpha|\|x\|_X + \|\alpha Tx\|_Y \\ =& |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y = |\alpha|(\|x\|_X + \|Tx\|_Y) = |\alpha|\|x\|_0\end{aligned}$$

Finally let $\|x\|_0 = 0$, then

$$\|x\|_X + \|Tx\|_Y = 0 \Rightarrow \|x\|_X = 0 \text{ and } \|Tx\| = 0$$

Since $\|\cdot\|_X$ is a norm, we get $x = 0$.
Next we want to show that T is bounded iff $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent norms.
"$\Rightarrow$"
Assume that T is bounded. By definition, this means that there exists some $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$.
We want to show that there exists constants C^* and C' such that

$$C^*\|x\|_X \leq \|x\|_0 \leq C'\|x\|_X$$

We start by noticing that

$$\|x\|_X \leq \|x\|_0 = \|x\|_X + \|Tx\|_Y$$

since $|Tx\|_Y \geq 0$ for all $x \in X$. So $C^* = 1$.
Next we note that since T is bounded we get

$$\|x\|_0 = \|x\| + \|Tx\|_Y \leq \|x\| + C\|x\|_X = (C+1)\|x\|_X$$

So $C' = C + 1$.
We conclude that $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent.
"$\Rightarrow$"
We now assume that $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent, and want to show that T is bounded.
This follows from the following:

$$\|Tx\|_Y = \|x\|_0 - \|x\|_X \leq C\|x\|_X - \|x\|_X = (C-1)\|x\|_X$$

Hence T is bounded.

b)

If X is finite dimensional, then theorem 1.6 in the notes, says that any two norms are equivalent. Hence $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent hence a) implies that T must be bounded.

c)

Let $(e_i)_{i\in I}$ be a Hamel basis for X which is infinite dimensional.
We know that for each family $(y_i)_{i\in I}$ in Y (Y is non-zero) there exists precisely one linear map T such that $T: X \to Y$ and $T(e_i) = y_i$ for all $i \in I$.
Since Y non-zero, there exists an element $y' \in Y$ with norm $\|y'\| \neq 0$.
We can then scale it. Thus there exists some element $y \in Y$ with norm $\|y\| = 1$.
Now choose the family of $y_i's$ such that $y_i = i \cdot y$ where $\|y\| = 1$.
Then $T(e_i) = y_i = i \cdot y$.
Further we have that $\|e_i\| = 1$ and $\|Te_i\| = \|i \cdot y\| = i$ for all e_i.
And T is not bounded by the following argument.
If T was bounded by some $C > 0$ we could just choose an index $i > C$. Then

$$\|Te_i\| = i \leq c = C\|e_i\|$$

We conclude that there exists a linear map T which is unbounded.

d)

Let $\|\cdot\|_0$ be as in a), and suppose that X is infinite dimensional.
If we take T as in c) such that it is not bounded then a) implies that $\|\cdot\|_X$ and $\|\cdot\|_0$ are not equivalent, and further $\|\cdot\|_X \leq \|\cdot\|_0$ for all $x \in X$ by definition of $\|\cdot\|_0$.
Now assume that $(X, \|\cdot\|)_X$ is complete. Then homework 3, problem 1 tells us that, X can not be complete with respect to both norms, since if it was, then $\|\cdot\|_X$ and $\|\cdot\|_0$ would be equal. We conclude that $(X, \|\cdot\|_0)$ can not be complete if $(X, \|\cdot\|)_X$ is.

e)

Consider the infinity norm $\|\cdot\|_\infty$ on $\ell_1(\mathbb{N})$.
Note that for any sequence $(x_n)_{n\geq 1}$

$$\|(x_n)_{n\geq 1}\|_\infty = \sup\{|x_1|, |x_2|, \dots\} \leq \sum_{i=1}^{\infty} |x_i| = \|(x_n)_{n\geq 1}\|_1$$

Hence $\|\cdot\|_\infty \leq \|\cdot\|_1$.
We want to show that $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is complete while $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete.
We know from the notes, that $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is a Banach space, hence complete.
Now let $(x_n)_{n\geq 1)}$ be the sequence in $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ given by $x_i = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{i}, 0, 0, \dots)$.
This sequence is Cauchy since, for $i > j$ we have that

$$\|x_i - x_j\|_\infty = \|(0, 0, \dots, 0, x_{j+1}, x_{j+2}, \dots, x_i, 0, 0, \dots)\|_\infty = |x_{j+1}|$$

Hence, given $\varepsilon > 0$ we can choose indices i, j such that $\|x_i - x_j\| < \varepsilon$
Now we have a cauchy sequence, but by noting that $(x_n)_{n\geq 1} \to (y_n)_{n\geq 1}$ where (y_n) is the sequence from the harmonic series i.e $\frac{1}{n}$. This however is not in $\ell_1(\mathbb{N})$, and hence can not be in $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$. Since we have shown that there exists a Cauchy sequence, which does not converge in the space, we conclude that $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete.

Problem 2

a)

Let $x = (x_n)_{n\geq 1} = (a, b, 0, 0, \dots)$ and $y = (y_n)_{n\geq 1} = (1, 1, 0, 0, \dots)$ which are elements of M.
Recall that $\|f\| = \sup\{|f(x)| : \|x\|_p \leq 1\} = \sup\{|f(x)| : \|x\|_p = 1\}$.
We consider $|f(x)|$ for $x = (x_n)_{n\geq 1} \in M$

$$\begin{aligned}|f(x)| =& |a+b| \leq |a| + |b| \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{\frac{1}{q}} = (|a|^p|b|^p)^{\frac{1}{p}} \left(|1|^q + |1|^q\right)^{\frac{1}{q}} \\ &= (|a|^p + |b|^p)^{\frac{1}{p}} 2^q = \|x_n\|_p \, 2^{(1-\frac{1}{p})}\end{aligned}$$

Were we used that

$$|a| + |b| = \sum_{n=1}^{\infty} |x_n y_n| = |x_1 y_1| + |x_2 y_2| \neq 0$$

which means, that we can use Hölders inequality. This states that if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{\frac{1}{q}}$$

Now consider the element $z = (z_n)_{n\geq 1} = \left(\frac{1}{2}^{\frac{1}{p}}, \frac{1}{2}^{\frac{1}{p}}, 0, 0, \dots\right) \in M$. This element has norm 1, since

$$\|z\|_p = \left\| \left(\frac{1}{2}^{\frac{1}{p}}, \frac{1}{2}^{\frac{1}{p}}, 0, 0, \dots\right) \right\|_p = \left(\left|\frac{1}{2}^{\frac{1}{p}}\right|^p + \left|\frac{1}{2}^{\frac{1}{p}}\right|^p \right)^{\frac{1}{p}} = 1^{\frac{1}{p}} = 1$$

Then

$$|f(z)| = \left| \left|\frac{1}{2}^{\frac{1}{p}}\right|^p + \left|\frac{1}{2}^{\frac{1}{p}}\right|^p \right| = \left| 2\left(\frac{1}{2}^{\frac{1}{p}}\right) \right| = 2\left(\frac{1}{2^{\frac{1}{p}}}\right) = 2^{(1-\frac{1}{p})} \leq \|f\|$$

which is less than $\|f\|$ since it is a part of the set, over which we take the supremum.
By looking at remark 1.11 from the notes, we see that an equivalent definition of $\|f\|$ is

$$\|f\| = \inf\{C > 0 : |f(z)| \leq C\|z\|_p, z \in M\} = \inf\{C > 0 : |f(z)| \leq C, \|z\|_p = 1, z \in M\}$$

Hence

$$\|f\| = \inf\{C > 0 : |f(z)| \leq C, \|z\|_p = 1, z \in M\} \leq 2^{(1-\frac{1}{p})} \leq \|f\|$$

Hence we conclude that $\|f\| = 2^{(1-\frac{1}{p})}$.

b)

By corollary 2.6, there exists a functional $F \in (\ell_p(\mathbb{N}))^*$ such that $F_{|M} = f$ and $\|F\| = \|f\|$.
Thus we need to show that this is unique when $1 < p < \infty$.
By homework 1 problem 5 we know that if $\frac{1}{p} + \frac{1}{q} = 1$ then $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$
This means, that we may write

$$F_y(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x = (x_n)_{n\geq 1)} \in \ell_p(\mathbb{N}), \quad y = (y_n)_{n\geq 1} \in \ell_q(\mathbb{N})$$

Since $F_{|M} = f$ we get that

$$F_{y|M} = \sum_{n=1}^{\infty} x_n y_n = f(x) = a + b$$

Hence $y = (1, 1, y_3, y_4, \dots)$.
By a) we know that $\|f\| = 2^{(1-\frac{1}{p})} = 2^{\frac{1}{q}} = \|F\|$.
But since F is represented by y we have $\|y\| = 2^{(1-\frac{1}{p})} = 2^{\frac{1}{q}}$.
Hence

$$\|y\|_q = \left(\sum_{n=1}^{\infty} |y_i|^q\right)^{\frac{1}{q}} = (|1|^q + |1|^q + |y_3|^q + \cdots)^{\frac{1}{q}} = 2^{\frac{1}{q}}$$

But for this to be true, then $|y_i| = 0$ for all $i > 2$, thus $y_i = 0$ for all $i > 2$ so $y = (1, 1, 0, 0, \dots)$.
Therefore $F(x) = a + b$.
Now let $F' \in (\ell_p(\mathbb{N}))^*$ be another function with the same properties as F.
Then $F'_{|M} = f$ and $\|F'\| = \|f\|$. But by the exact same argument as above, we would get $F'(x) = a+b$, hence $F'(x) = F(x)$ for all $x \in \ell_p(\mathbb{N})$. Thus F must be unique.

c)

Let $p = 1$, then by corollary 2.6, there exists a functional $F \in (\ell_p(\mathbb{N}))^*$ such that $F_{|M} = f$ and $\|F\| = \|f\|$.
Let $F_i : \ell_1(\mathbb{N}) \to \mathbb{K}$ given by

$$F_i((x_1, x_2, \dots)) = x_1 + x_2 + x_i$$

for $i > 2$.
This is clearly a linear function of $\ell_1(\mathbb{N})$ since if $x = (x_n)_{n\geq 1}$ and $x' = (x'_n)_{n\geq 1}$ in $\ell_1(\mathbb{N})$ then

$$F_i(x + x') = x_1 + x'_1 + x_2 + x'_2 + x_i + x'_i = F_i(x) + F_i(x')$$

Furtermore $F_{i|M} = x_1 + x_2 = f(x)$ for $x \in M$. Hence F is an extension.
Now since F extends f we must have

$$\|F_i\| \geq \|f\| = 2^{(1-\frac{1}{p})} = 2^0 = 1$$

But we also have that

$$\begin{aligned}\|F_i\| &= \sup\{|F_i(x)| : \|x\|_1 = 1\} = \sup\{|(x_1 + x_2 + x_i)| : \|x\|_1 = 1\}\\ &\leq \sup\{|x_1| + |x_2| + |x_i| : \|x\|_1\} \leq 1.\end{aligned}$$

since $x = (x_1, x_2, \dots)$ and $\|x\|_1 = \sum_{n=1}^{\infty} |x_i| = 1$
We are taking the supremum of only 3 terms, thus they must be less than 1, which means that $\|F_i\| = \|f\| = 1$.
We conclude that there exists infinitely many linear functionals on $\ell_1(\mathbb{N})$ which extend f and satisfy $\|F_i\| = \|f\|$.

Problem 3

a)

Assume that $F : X \to \mathbb{K}^n$ is injective and linear.
We know that all maps are surjective on their image. Hence, if F is injective it will be bijective on its image.
This is impossible, since X is infinite dimension and $\mathbb{K}^n$ is not.
We conclude that F can not be injective.

b)

Consider the map $F : X \in \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), x \in X$.
We note that the kernel of this map is exactly the set

$$\ker F = \{x \in X : F(x) = 0\} = \{x \in X : (f_1(x) = 0, f_2(x) = 0, \ldots, f_n(x) = 0)\} = \bigcap_{i=1}^{\infty} \ker(f_i)$$

But since $F : X \to \mathbb{K}^n$ a) implies that F is not injective, hence $\ker F \neq \{0\}$.

c)

Let $x_1, x_2, \ldots, x_n \in X$ which is infinite dimensional.
If $x_i = 0, i = 1, \ldots n$ then since X is infinite dimensional (and not $\{0\}$) we can find some element $x' \in X$ and scale it. Hence there exists some $y = \frac{x'}{\|x'\|}$ which has norm 1. Furthermore

$$\|y - x_j\| = \|y\| = 1 \geq \|x_j\| = 0$$

for all $j = 1, \ldots n$.
Next if $j < n$ of the n elements are 0, then the proof is the same but for $x_1, \ldots, x_k$ where instead there are $k = n - j$ non-zero elements.
We can therefore assume that all $x_1, x_2, \ldots, x_n$ are non-zero.
By the above argument we can find some element $y \in X$ with norm 1.
By 2.7 b) in the notes, there exists for each $0 \neq x \in X$ some functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.
Hence there exists $f_1, \ldots, f_n \in X^*$ such that $\|f_i\| = 1$ and $f_i(x_i) = \|x_i|$.
Recall from a) the map $F : X \in \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), x \in X$. We showed that this could never be injective. Hence there exists some element $0 \neq z' \in X$ such that $F(z') = 0$.
We can scale this, as with y so it has norm 1. Then $F(z) = 0$ since F is linear, i.e.

$$F(z) = \frac{F(z')}{\|z'\|} = 0.$$

We know that $\|f_i\| = \sup\{|f_i(x)|, \|x\| = 1\} = 1$ and that $f_i(x_i) = \|x_i\|$. Hence since

$$F(z) = 0 \Rightarrow f_1(z) = 0, f_2(z) = 0, \ldots, f_n(z) = 0$$

We get that

$$\|x_i\| = f_i(x_i) - f_i(z) = f_i(x_i - z)$$

Now recall that

$$\sup\{|f_i(x)| : \|x\| \leq 1\} = \sup\left\{\frac{|f_i(x)|}{\|x\|}, x \neq 0\right\}$$

But then we have

$$\frac{|f_i(x_i - z)|}{\|x_i - z\|} \leq \|f_i\| = 1 \Rightarrow |f_i(x_i - z)| = \|x_j\| \leq \|x_j - z\|$$

We conclude that our z fulfills the criteria of the problem.

d)

Assume that we can cover S with a finite family of closed balls in X i.e $S \subset \bigcup_{i\in I} \bar{B}_i$ such that none of the balls contain 0.
We denote by $\bar{B}(c,r)$ the closed ball at c with radius $r > 0$.
We notice that $0 \in \bar{B}(c,r) \Leftrightarrow \|c-0\| < r \Leftrightarrow \|c\| < r$
Now choose $x_1, \dots, x_n$ to be centers of each ball $\bar{B}_i$.
By c) there then exists some y with norm 1 and for which $\|y - x_j\| \geq \|x_j\|$
Since $\|y\| = 1$ it is in S hence it must be in one of the balls $\bar{B}_i(x_i, r)$ that cover S.
Hence $\|y - x_i\| \leq r$.
But since $\|x_i\| \leq \|y - x_i\| \leq r$ we conclude that $0 \in \bar{B}_i(x_i, r)$ since $\|x_i - 0\| = \|x_i\| \leq r$.
Thus we can not cover S with a family of closed balls, such that none of the balls contain 0.

e)

Assume for contradiction that S is compact. Hence each of its open covers has a finite subcover.
So for every collection C of open subsets of X such that $X \subset \bigcup_{x\in C} x$ there is a finite subset F of C such that $X \subset \bigcup_{x\in F} x$.
Note that this open cover always exists since any subset of X is contained in X which is open in itself.
Now consider the open balls of radius $r = \frac{1}{2}$, with center at some point in S. They provide an open covering of S, which does not contain 0.
Hence the closed balls of radius $r = \frac{1}{2}$ and center at some point in S is also a covering of S, which does not contain 0.
Since we assumed that S is compact, there must exist a finite subcover of this cover, which covers S. However this is not possible by d).
Thus we can conclude that S can not be compact.
For the last part, we note that any closed subset of a compact space is again compact. We note that S is closed in $\bar{B}(0,1)$ since it's complement is $B(0,1)$ which is open, since it is an open ball.
Hence $S \subset \bar{B}(0,1)$ is closed in $\bar{B}(0,1)$ and is not compact, so $\bar{B}(0,1)$ can never be compact.

Problem 4

a)

Assume that E_n is absorbing. That means, that for each $0 \neq f \in L_1([0,1], m)$ there exists some $t > 0$ such that $t^{-1}f \in E_n$.
By definition of E_n we have

$$\int_{[0,1]} |t^{-1}f|^3 dm \leq n \Leftrightarrow t^{-3}\int_{[0,1]} |f|^3 dm \leq n \Leftrightarrow \int_{[0,1]} |f|^3 dm \leq t^3 n$$

Consider $f(x) = x^{-\frac{1}{3}}$. Then

$$\int_{[0,1]} |x^{-\frac{1}{3}}| dm = \left[\frac{3}{2}x^{\frac{2}{3}}\right]_0^1 = \frac{3}{2} < \infty$$

So $f \in L_1([0,1], m)$.
But now

$$\int_{[0,1]} |x^{-\frac{1}{3}}|^3 dm = \int_{[0,1]} x^{-1} dm = [\ln(x)]_a^0 = \ln(1) - \ln(a) = -\ln(a)$$

So since $-\ln(a) \to \infty$ for $a \to 0$ there exists no $t > 0$ such that $t^{-1}f \in E_n$.
Hence it can not be absorbing.

b)

Assume for contradiction that $\text{Int}(E_n) \neq \emptyset$.
This means that there exists some element $x \in \text{Int}(E_n)$ and some $r > 0$ such that $B(x,r) \subset E_n$, i.e the ball at x with radius r is contained in E_n.
Now take $y \in L_1([0,1],m)$ and let $z = x - \frac{r}{2}\frac{y}{\|y\|}$. Then

$$\|z - x\| = \left\| -\frac{r}{2}\frac{y}{\|y\|} \right\| = \left|\frac{r}{2}\right| \frac{\|y\|}{\|y\|} = \frac{r}{2} < r$$

Hence we have the following inclusions

$$z \in B(x,r) \subset E_n \subset L_3([0,1],m) \subsetneq L_1([0,1],m)$$

Where we note that $E_n \subset L_3([0,1],m)$ since $L_3([0,1],m)$ is the set of measurable functions f from $[0,1] \to \mathbb{K}$ where $\|f\|_3 < \infty$, and E_n is the set of f in $L_1([0,1],m)$, hence measurable, for which $\|f\|_3 \leq \|f\|_3^3 \leq n < \infty$.
Since we are in a vector space, we can manipulate our expression of z and get that

$$y = \frac{2}{r}\|y\|(x - z)$$

Since both x and z lie in $E_n \subset L_3([0,1],m)$ we must have that y lies in $L_3([0,1],m)$, since y is just a scalar multiple of some element in $L_3([0,1],m)$, which is a vector space.
Now since we chose y arbitrarily we have that $L_1([0,1],m) \subset L_3([0,1],m)$ which is a contradiction.
Hence $\text{Int}(E_n)$ must be empty.

c)

To show that E_n is closed, we consider an arbitrary sequence $(f_k)_{k\geq 1}$ in E_n, and assume that $f_k \to f$ for $n \to \infty$ for some f in $L_1([0,1],m)$.
By Fatou's lemma, we have that

$$\int_{[0,1]} f dm \leq \liminf_{n\to\infty} f_k dm$$

And since $f_k \to f$ implies $|f_k| \to |f|$ which implies $|f_k|^3 \to |f|^3$ we have that

$$\int_{[0,1]} |f|^3 dm \leq \liminf_{n\to\infty} |f_k|^3 dm$$

Furthermore since $f_k \in E_n$ we have that

$$\int_{[0,1]} |f_k|^3 dm \leq n$$

For all $n \geq 1$. So now

$$\int_{[0,1]} |f|^3 dm \leq \liminf_{n\to\infty} |f_k|^3 dm \leq \liminf_{n\to\infty} n = n$$

Hence $\int_{[0,1]} |f|^3 dm \leq n$ and therefore $f \in E_n$. Thus since our sequence was chosen arbitrarily in E_n, and the limit was contained in E_n we conclude that E_n must be closed.

d)

In b) and c) we showed that $\text{Int}(E_n)$ was empty for each E_n, and since E_n was closed we have that

$$\text{Int}(\bar{E_n}) = \text{Int}(E_n) = \emptyset$$

Hence $(E_n)_{n\geq 1}$ is a sequence of nowhere dense sets.
If we can show that $\bigcup_{n=1}^{\infty} E_n = L_3([0,1],m)$ then by def. 3.12 in the notes $L_3([0,1],m)$ will be of first category in $L_1([0,1],m)$. We do this by showing both inclusions.
In c) we noted that E_n was contained in $L_3([0,1],m)$ for each $n \geq 1$.
Let $f \in \bigcup_{n=1}^{\infty} E_n$ then $f \in E_k$ for some $k \geq 1$. Hence

$$\int_{[0,1]} |f|^3 dm \leq k < \infty$$

and

$$\|f\|_3^3 = \int_{[0,1]} |f|^3 dm \leq k < \infty \Rightarrow \|f\|_3 \leq k^{\frac{1}{3}}$$

so $f \in L_3([0,1],m)$.
Now for the other inclusion. Let $f \in L_3([0,1],m)$. Then $\|f\|_3 = n < \infty$ for some $0 \leq n < \infty$.
Then

$$\|f\|_3^3 = n^3 \Rightarrow \int_{[0,1]} |f|^3 dm = n^3 < n^3 + 1$$

But since $f \in L_3([0,1],m) \subsetneq L_1([0,1],m)$ f must be in $L_1([0,1],m)$. So by definition of E_n f is in E_{n^3+1} which must be in the union.
We conclude that $L_3([0,1],m) = \bigcup_{n=1}^{\infty} E_n$ and therefore $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

Problem 5

a)

Suppose that $x_n \to x$ in norm. This means that $\|x_n - x\| \to 0$. But then we have

$$0 \leftarrow \|x_n - x\| \geq \|x_n\| - \|x\| \Rightarrow \|x_n\| - \|x\| \to 0 \Rightarrow \|x_n\| \to \|x\|$$

Hence if $x_n \to x$ in norm, then $\|x\| \to \|x\|$. Which was what we wanted.

b)

Since H is a separable Hilbert space, we can find an orthonormal basis $(e_n)_{n\geq 1}$ for H.
We assert that for this orthonormal basis, $(e_n)_{n\geq 1} \to 0$ for $n \to \infty$, and hence $1 = \|e_n\| \to \|0\| = 0$, which is not true, thus it would be counterexample.
By homework 4 problem 2 a), which applies since sequences are special cases of nets, we have that a sequence $(x_n)_{n\geq 1} \in X$ converges to x weakly iff the sequence $(f(x_n))_{n\geq 1}$ converges to $f(x)$ for every $f \in X^*$.
So if we can show that $(f(e_n))_{n\geq 1} \to f(0)$ for any $f \in X^*$, then HW 4.2 would imply that $(e_n)_{n\geq 1} \to 0$ and we would be done.
Let $f \in H^*$, then Riesz representation theorem says that there exists $y \in H$ such that

$$f_y(x) = \langle x, y \rangle, \quad \text{for all } \; x \in H$$

Then since we have an orthonormal sequence in H, we can use Bessels inequality: For any $x \in H$ one has

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2 < \infty$$

But this implies that $|\langle x, e_n \rangle|^2 \xrightarrow{n \to \infty} 0$ hence $\langle x, e_n \rangle \to 0 = f(0)$. Where $f(0) = 0$ since it is linear. This means that $f_{e_n}(x) = \langle x, e_n \rangle \to f(0)$. And since we chose f arbitrarily we are done.

c)

I was not able to solve this problem.