Problem 1

Let $(X, \|\cdot\|_X)$ and $(X, \|\cdot\|_Y)$ be non-zero normed vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $T : X \to Y$ be a linear map.

(a)

Set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ for all $x \in X$. We show $\|\cdot\|$ is a norm on X: Because $\|\cdot\|_X : X \to [0, \infty)$ and $\|T(\cdot)\|_Y : X \to [0, \infty)$ are well-defined maps also $\|\cdot\|_0 = \|\cdot\| + \|\cdot\|_Y : X \to [0, \infty)$ is a well-defined map. Let $x, y \in X$ and $\alpha \in \mathbb{K}$.

$$\|x + y\| = \|x + y\|_X + \|T(x + y)\|_Y \leq \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y = \|x\|_0 + \|y\|_0$$
$$\|\alpha x\|_0 = \|\alpha x\|_X + \|T(\alpha x)\|_Y = |\alpha|(\|x\|_X + \|Tx\|_Y) = |\alpha|\|x\|_0$$
$$\|0\|_0 = \|0\|_X + \|0\|_Y = 0$$

If $\|x\|_0 = \|x\|_X + \|Tx\|_Y = 0$ then $x = 0$ because $\|x\|_X > 0$ for $x \neq 0$ and $\|Tx\|_Y \geq 0$ for all $x \in X$. So $\|\cdot\|_X$ is a norm on X.

It is clear that $\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_0$. If T is bounded then there exists a $c > 0$ such that $\|Tx\|_Y \leq c\|x\|_X$ for all $x \in X$. So $\frac{1}{c+1}\|x\|_0 = \frac{1}{c+1}(\|x\|_X + \|Tx\|_Y) \leq \|x\|_X \leq \|x\|_0$

If $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent norms then $\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq c\|x\|_X$ for some $c > 0$ so $\|Tx\|_Y \leq (c - 1)\|x\|_X \leq c\|x\|_X$. So T is bounded if and only if $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent.

(b)

If X is finite dimensional then it has some finite basis $\{e_1, ..., e_n\}$. $\|x\|_\infty = \max\{|x_1|, ..., |x_n|\}$ where the x_i's are the scalers with respect to the basis is a norm on X and by Theorem 1.6 any norm on finite dimensional vector space is equivalent so there exist a $c > 0$ such that $\|x\|_\infty \leq c\|x\|_X$. Let $x = \sum_{i=1}^n x_i e_i$ then

$$\begin{aligned}\|Tx\|_Y &= \|T(\sum_{i=1}^n x_i e_i)\|_Y \leq \sum_{i=1}^n |x_i|\|T(e_i)\|_Y = \sum_{i=1}^n |x_i| \frac{\|T(e_i)\|_Y}{\|e_i\|_\infty}\|e_i\|_\infty \\ &= \sum_{i=1}^n \frac{\|T(e_i)\|_Y}{\|e_i\|_\infty}\|x_i e_i\|_\infty \leq \sum_{i=1}^n \frac{\|T(e_i)\|_Y}{\|e_i\|_\infty}\|x\|_\infty \leq (\sum_{i=1}^n \frac{\|T(e_i)\|_Y}{\|e_i\|_\infty})c\|x\|_X\end{aligned}$$

(c)

Because Y i non-zero there exists some $y \in Y$ such that $\|y\|_Y \neq 0$ assume $\|y\|_Y = 1$ by dividing y by the norm. Suppose X is infinite dimensional then there exists a infinite Hamel basis $(x_i)_{i \in I}$ for X. Let $(x_n)_{n \in \mathbb{N}} \subset (x_i)_{i \in I}$ be a subsequence. Define $T(x_n) := n\|x_n\|_X y$ for $x_n \in (x_n)_{n \in \mathbb{N}}$ and $T(x_i) := 0$ for $x_i \notin (x_n)_{n \in \mathbb{N}}$. T is a linear map. We cannot have that there exists a $c > 0$ such that $\|Tx\|_Y \leq c\|x\|_X$ for all $x \in X$ because $\|T(x_{\lceil c \rceil + 1})\|_Y = (\lceil c \rceil + 1)\|x_{\lceil c \rceil + 1}\|_X\|y\|_Y = (\lceil c \rceil + 1)\|x_{\lceil c \rceil + 1}\|_X > c\|x_{\lceil c \rceil + 1}\|_X$. So there exists a linear map $T : X \to Y$ which is not bounded.

(d)

Suppose X is infinite dimensional then by (c) Problem 1 there exists a linear map $T : X \to Y$ which is not bounded and therefore $\|\cdot\|_0$ is not equivalent to $\|\cdot\|_X$ by (a) Problem 1. We have $\|x\|_0 = \|x\|_X + \|Tx\|_Y \geq \|x\|_X$ for $x \in X$. Now by Problem 1 Homework for week 3 if $(X, \|\cdot\|_X)$ is a Banach space then $(X, \|\cdot\|_X)$ cannot be complete because otherwise $\|\cdot\|_0$ and $\|\cdot\|_X$ would be equivalent.

(e)

Consider the map $\|\cdot\|_s : \ell(\mathbb{N}) \to [0,\infty)$ defined by $\|(x_n)_{n\geq 1}\|_s = \sum_{n=1}^{\infty} (\frac{1}{2})^n |x_n|$ for $(x_n)_{n\geq 1} \in \ell(\mathbb{N})$. Clearly $\|\cdot\|_s$ is well-defined. We see that it is a norm: Let $(x_n)_{n\geq 1}, (y)_{n\geq 1} \in \ell(\mathbb{N})$ and $\alpha \in \mathbb{K}$.

$$\begin{aligned}
\|(x_n)_{n\geq 1} + (y_n)_{n\geq 1}\|_s &= \sum_{n=1}^{\infty} (\frac{1}{2})^n |x_n + y_n| = \sum_{n=1}^{\infty} (\frac{1}{2})^n |x_n| + \sum_{n=1}^{\infty} (\frac{1}{2})^n |y_n| \\
&= \|(x_n)_{n\geq 1}\|_s + \|(y_n)_{n\geq 1}\|_s \\
\|\alpha (x_n)_{n\geq 1}\|_s &= \sum_{n=1}^{\infty} (\frac{1}{2})^n |\alpha x_n| = |\alpha| \sum_{n=1}^{\infty} (\frac{1}{2})^n |x_n| = |\alpha| \|(x_n)_{n\geq 1}\|_s \\
\|0\|_s &= \sum_{n=1}^{\infty} (\frac{1}{2})^n |0| = 0
\end{aligned}$$

If $\|(x)_{n\geq 1}\|_s = 0$ then $\sum_{n=1}^{\infty} (\frac{1}{2})^n |x_n| = 0$ so $x_n = 0$ for $n \geq 1$ so $(x_n)_{n\geq 1} = 0$.

We have $\|(x_n)_{n\geq 1}\|_s = \sum_{n=1}^{\infty} (\frac{1}{2})^n |x_n| \leq \sum_{i=1}^{n} |x_n| = \|(x_n)_{n\geq 1}\|_1$ for all $(x)_{n\geq 1} \in \ell_1(\mathbb{N})$.

Assume for contradiction that $\|\cdot\|_s$ and $\|\cdot\|_1$ are equivalent. Then there must exist some $C > 0$ such that $\|(x)_{n\geq 1}\|_1 \leq C\|(x)_{n\geq 1}\|_s$ for all $(x)_{n\geq 1} \in \ell_1(\mathbb{N})$. Let $(y_n)_{n\geq 1} \in \ell_1(\mathbb{N})$ be the sequence given by

$$y_n := \begin{cases} 1 & n = \lceil \log_2(C+1) \rceil \\ 0 & \text{otherwise} \end{cases}$$

then $C\|(y)_{n\geq 1}\|_s = \sum_{n=1}^{\infty} (\frac{1}{2})^n |y_n| = C(\frac{1}{2})^{\lceil \log_2(C+1) \rceil} \leq \frac{C}{C+1} < 1 = \sum_{n=1}^{\infty} |y_n| = \|(y_n)_{n\geq 1}\|_1$. Which is a contradiction. So $\|\cdot\|_s$ and $\|\cdot\|_1$ are not equivalent. $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is complete by HW1 problem 5. If $(\ell_1(\mathbb{N}), \|\cdot\|_s)$ was also complete then $\|\cdot\|_s$ and $\|\cdot\|_1$ would be equivalent by HW3 Problem 1 (and because $\|(x_n)_{n\geq 1}\|_s \leq \|(x_n)_{n\geq 1}\|_1$).

Problem 2

Let $1 \leq p < \infty$ be fixed and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}, \|\cdot\|_p)$, considered as a vector space over $\mathbb{C}$, give by

$$M = \{(a, b, 0, 0, ...) : a, b \in \mathbb{C}\}.$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, ...) = a + b$, for all $a, b \in \mathbb{C}$.

(a)

f is bounded on $(M, \|\cdot\|_p)$: Let $m = (a, b, 0, ...) \in M$ then

$$|fm| = |a + b| \leq |a| + |b| \leq 2\text{max}\{|a|, |b|\} = 2(|b|^p)^{\frac{1}{p}} \leq 2(|a|^p + |b|^p)^{\frac{1}{p}} = 2\|m\|_p$$

We calculate $\|f\|$: By the triangle inequality we have $|f(a, b, 0, ...)| = |a+b| \leq |a| + |b| = ||a| + |b|| = |f(|a|, |b|)|$. We also have $\|(a, b, 0, ...)\|_p = (|a|^p + |b|^p)^{\frac{1}{p}} = (|\ |a|\ |^p + |\ |b|\ |^p)^{\frac{1}{p}} = \|(|a|, |b|, 0, ...)\|_p$. Define $M' = \{m \in M | a, b \in \mathbb{R}, a, b \geq 0\}$. So we can calculate $\|f\|$ as $\sup\{|fm| : \|m\|_p = 1, m \in M'\}$ i.e. by only considering the nonnegative real numbers.

Let $m = (a, b, 0, ...) \in M'$ and assume that $a \leq b$ (the case with $b \leq a$ is similar) then there exist $\epsilon \geq 0$ such that $b = a + \epsilon$. We see

$$\begin{aligned}
\|m\|_p - \|(a + \frac{\epsilon}{2}, a + \frac{\epsilon}{2}, 0, ...)\|_p &= |a|^p + |a + \epsilon|^p - |a + \frac{\epsilon}{2}|^p - |a + \frac{\epsilon}{2}|^p \\
&= a^p + \sum_{i=0}^{p} \binom{p}{i} a^i \epsilon^{p-i} - 2\sum_{i=0}^{n} \binom{p}{i} a^i \left(\frac{\epsilon}{2}\right)^{p-i} \\
&= \sum_{i=0}^{p-1} \binom{p}{i} a^i \epsilon^{p-i} - 2\sum_{i=0}^{p-1} \binom{p}{i} a^p \left(\frac{\epsilon}{2}\right)^{p-i} \\
&= \sum_{i=0}^{p-1} \binom{p}{i} a^i \left(\epsilon^{p-i} - 2\left(\frac{\epsilon}{2}\right)^{p-i}\right) \\
&= \sum_{i=0}^{p-1} \binom{p}{i} a^i \left(\frac{\epsilon}{2}\right)^{p-i} (2^{p-i} - 2) \geq 0
\end{aligned}$$

Where the inequality comes from the fact that $2^{p-i} - 2 \geq 0$ for $i \in \{0, 1, ..., p-1\}$. But we have $|f(m)| = |a + a + \epsilon| = |a + \frac{\epsilon}{2} + a + \frac{\epsilon}{2}| = |f(a + \frac{\epsilon}{2}, a + \frac{\epsilon}{2}, 0, ...)|$. So we need only consider sequences of the form $(a, a, 0, ...) \in M'$ to calculate $\|f\|$ (there exist $(a, a, 0, ...) \in M'$ such that $\|(a, a, 0, ...)\|_p = 1$ by scalar multiplication). Now assume $(a, a, 0, ...)$ is the sequence in M' such that $\|(a, a, 0, ...)\|_p = 1$ then $(|a|^p + |a|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}} a = 1$ so $a = \frac{1}{2^{\frac{1}{p}}}$. We now know that $\|f\| = |f(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, ...)| = ||\frac{1}{2^{\frac{1}{p}}}| + |\frac{1}{2^{\frac{1}{p}}}|| = 2^{1-\frac{1}{p}}$ because $(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, ...) \in M$ and because $|f(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, ...)| \geq |fm|$ for all $m \in M$ with $\|m\|_p = 1$.

(b)

Because f is linear and bounded, we have $f \in M^*$ and therefore by corrolary 2.6 we know that there exists a map $F \in (\ell_p(\mathbb{N}))^*$ such that $F|_M = f$ and $\|F\| = \|f\|$. By HW1 problem 5 there exist for $1 < p < \infty$ a conjugate number q (i.e. p, q such that $\frac{1}{p} + \frac{1}{q} = 1$) and an isometric isomorphism $T : \ell_q(\mathbb{N}) \to (\ell_p(\mathbb{N}))^*$ given by $T(x) = f_x$ where $f_x(y) = \sum_{n=1}^{\infty} x_n y_n$, $x \in \ell_q(\mathbb{N})$ and $y \in \ell_p(\mathbb{N})$. So there exist a $x = (x_n)_{n \geq 1} \in \ell_q(\mathbb{N})$ such that $T(x) = F$. Because $F|_M = f$ we must have $x_1 = 1$ and $x_2 = 1$ because $f_x(\delta_{n1}) = x_1$ and $f_x(\delta_{n2}) = x_2$ so $x_1 = 1 = x_2$ as $f(\delta_{n1}) = f(\delta_{n2}) = 1$ and $\delta_{n1}, \delta_{n2} \in M$ ($\delta_{nk} = 1$ if $k = n$ and $\delta_{nk} = 0$ if $n \neq k$). Assume $x_n \neq 0$ for some $n \geq 3$ then because the isomorphism is isometric, we have

$$\begin{aligned}\|F\| = \|T(x)\| = \|x\|_q &= \left(\sum_{n=1}^{\infty} |x_n|^q\right)^{\frac{1}{q}} = \left(\sum_{n=1}^{\infty} |x_n|^{(1-\frac{1}{p})^{-1}}\right)^{(1-\frac{1}{p})} \\ &= \left(1 + 1 + \sum_{n=3}^{\infty} |x_n|^{(1-\frac{1}{p})^{-1}}\right)^{(1-\frac{1}{p})} > 2^{(1-\frac{1}{p})} = \|f\|\end{aligned}$$

So $x_n = 0$ for $n \geq 3$ if $\|F\| = \|T(x)\|$. So $x = (x_n)_{n\geq 1} \in \ell_q(\mathbb{N})$ with $x_1 = x_2 = 1$ and $x_n = 0$ for $n \geq 3$ is the only $x \in \ell_q(\mathbb{N})$ with $T(x) = F$ for any $F \in (\ell_p(\mathbb{N}))^*$ that extends f and has $\|F\| = \|f\|$ and therefore F is unique because T is an isomorphism. So if $1 < p < \infty$ there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

(c)

Define $f_i : \ell_1(\mathbb{N}) \to \mathbb{K}$ by $f_i((x_n)_{n\geq 1}) := x_1 + x_2 + x_i$ for $i \geq 3$. We see f_i is linear: Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \in \ell_1(\mathbb{N})$ and $\alpha, \beta \in \mathbb{K}$ then:

$$\begin{aligned}f_i(\alpha(x_n)_{n\geq 1} + \beta(y_n)_{n\geq 1}) &= (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + (\alpha x_i + \beta y_i) \\ &= \alpha(x_1 + x_2 + x_i) + \beta(y_1 + y_2 + y_i) \\ &= \alpha f_i((x_n)_{n\geq 1}) + \beta f_i((y_n)_{n\geq 1})\end{aligned}$$

For $m \in M$ we have $f_i(m) = a + b = f(m)$ for $i \geq 3$. So f_i extends f. We can calculate $\|f_i\| = \sup\{|f_i((x_n)_{n\geq 1})| : \|x\|_1 = 1\}$ as $\sup\{|f_i((x_n)_{n\geq 1})| : \|(x_n)_{n\geq 1}\|_1 = 1, x_n \in \mathbb{R}, x_n \geq 0 \text{ for all } n \geq 1\}$ because $\|(|x_n|)_{n\geq 1}\|_1 = \sum_{n=1}^{\infty} ||x_n|| = \sum_{n=1}^{\infty} |x_n| = \|(x_n)_{n\geq 1}\|_1$ and $|f_i((x_n)_{n\geq 1})| = |x_1 + x_2 + x_i| \leq ||x_1| + |x_2| + |x_i|| = |f_i((|x_n|)_{n\geq 1})|$. For $(x_n)_n \in \ell_1(\mathbb{N})$ with $x_n \in \mathbb{R}$ and $x_n \geq 0$ for all $n \geq 1$ we have $|f_i((x_n)_{n\geq 1})| = |x_1 + x_2 + x_i| = x_1 + x_2 + x_i = |x_1| + |x_2| + |x_i| \leq \|(x_n)_{n\geq 1}\|_1$ but $|f_i(\delta_{n1})_{n\geq 1})| = 1$ and $\|(\delta_{n1})_{n\geq 1}\|_1 = 1$ and $\delta_{n1} \in \mathbb{R}$ and $\delta_{n1} \geq 0$ for all $n \geq 1$. So $\|f_i\| = 1 = \|f\|$ for $i \geq 3$ and $f_i|_M = f$. So there exists infinitely many functional F such that F extends f and $\|F\| = \|f\|$.

Problem 3

Let X be an infinite dimensional vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

(a)

Let $n \geq 1$ be an integer. Then $\mathbb{K}^n$ considered as a vector space over $\mathbb{K}$ has a basis consisting of n element, namely the standard basis. So $\mathbb{K}^n$ has dimension n. Because X is infinite dimensional it has an infinite basis B. Let B_{n+1} be a subset of B with $n + 1$ elements then the elements in B_{n+1} are linearly independent. Assume for contradiction that there exists a linear map $F : X \to \mathbb{K}^n$ which is injective. Then $F(B_{n+1})$ must be a set of $n + 1$ linear independent vectors but this would mean that the dimension of $\mathbb{K}^n$ is greater than or equal to $n + 1$, which is a contradiction. So there exists no injective linear maps $F : X \to \mathbb{K}^n$.

(b)

Let $n \geq 1$ be an integer and let $f_1, f_2, ..., f_n \in X^*$. Define $F : X \to \mathbb{K}^n$ by $F(x) = (f_1(x), f_2(x), ..., f_n(x))$, $x \in X$. F is a linear map:

$$\begin{aligned} F(\alpha x + \beta y) &= (f_1(\alpha x + \beta y), f_2(\alpha x + \beta y), ..., f_n(\alpha x + \beta y)) \\ &= (\alpha f_1(x) + \beta f_1(y), \alpha f_2(x) + \beta f_2(y), ..., \alpha f_1(x) + \beta f_1(y)) \\ &= \alpha F(x) + \beta F(y) \end{aligned}$$

By (a) Problem 3 we have that a linear map $F : X \to \mathbb{K}$ cannot be injective, and therefore we have that

$$\ker(F) = \bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}.$$

(c)

Let $x_1, x_2, ..., x_n \in X$ and let $K = \{k \in \{1, 2, ..., n\} | x_k \neq 0\}$ then by Theorem 2.7 (b) there exist $f_i \in X^*$ with $\|f_i\| = 1$ and $f_i(x_i) = \|x_i\|$ for $i \in K$. By (b) Problem 3 there exists a $z \neq 0$ with $z \in \bigcap_{i \in K} \ker(f_i)$. Define $y := \frac{z}{\|z\|}$. So because $\|f_i\| = 1$ for $i \in K$, we have $|f_i(\frac{y+x_i}{\|y+x_i\|})| = |\frac{f_i(y)+f_i(x_i)}{\|y+x_i\|}| = |\frac{\|x_i\|}{\|y+x_i\|} \leq 1$ so $\|x_i\| \leq \|y - x_i\|$ for $i \in K$ with $\|y\| = \|\frac{z}{\|z\|}\| = 1$. We also have $\|y - 0\| = \|y\| \geq \|0\| = 0$. So for $x_1, x_2, ..., x_n \in X$ there exists a $y \in X$ with $\|y\| = 1$ and $\|y - x_i\| \geq \|x_i\|$ for all $i \in \{1, 2, ..., n\}$.

(d)

Let $\overline{B_{r_i}(x_i)}$ with $i \in \{1, ..., n\}$ be a finite family of closed balls with $0 \notin \overline{B_{r_i}(x_i)}$ and $S = \{x \in X : \|x\| = 1\}$ be the unit sphere. By (d) Problem 3 there exists a y with $\|y\| = 1$ and $\|y - x_i\| \geq \|x_i\|$, but because $0 \notin \overline{B_{r_i}(x_i)}$ we have $\|y - x_i\| \geq \|x_i\| = \|x_i - 0\| > r_i$, hence $y \in S$ but $y \notin \overline{B_{r_i}(x_i)}$ with $i \in \{1, ..., n\}$. So one cannot cover the unit sphere S with a finite family of closed balls in X such that none of the balls contains 0.

(e)

Let $\{B_{\frac{1}{2}}(x)\}_{x \in S}$ be an open cover of S. We see $0 \notin \overline{B_{\frac{1}{2}}(x)}$ for $x \in S$ because $\|x - 0\| = \|x\| = 1 > \frac{1}{2}$. A finite subcover $\{B_{\frac{1}{2}}(x_i) | x_i \in \{x_1, ..., x_n\} \subset S\}$ of $\{B_{\frac{1}{2}}(x)\}_{x \in S}$ cannot cover S as $\overline{B_{\frac{1}{2}}(x)}$ would be a finite cover of S of closed balls that do not contain 0, but this was seen in (d) Problem 3 to be impossible, hence S is not compact.

S is closed: Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in S converging to x then by continuity of the norm $\|x\| = \|\lim_{n\to\infty} x_n\| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} 1 = 1$, hence $x \in S$ and S is closed.

Because $\overline{B_1(0)}$ is closed, $S \subset \overline{B_1(0)}$ is a closed subset. So if $\overline{B_1(0)}$ was compact then S would be compact as it is a closed subset of a compact set, but we have seen this to not be case so $\overline{B_1(0)}$ is not compact.

Problem 4

Let $L_1([0,1],m)$ and $L_3([0,1],m)$ be the Lebesgue spaces on $[0,1]$.

(a)

Assume for contradiction that $E_n \subset L_1([0,1],m)$ is absorbing. Then for every $f \in L_1([0,1],m)$ there exist a $t > 0$ such that $tf \in E_n$ but then

$$n \geq \int_{[0,1]} |tf|^3 dm = t^3 \int_{[0,1]} |f|^3 dm$$

So

$$\int_{[0,1]} |f|^3 dm \leq \frac{n}{t^3} < \infty$$

and because f is measurable because $f \in L_1([0,1],m)$, we have $f \in L_3([0,1],m)$. So $L_1([0,1],m) \subset L_3([0,1],m)$ but this is a contradiction with HW 2 problem 2 (b) which says $L_3([0,1],m) \subsetneq L_1([0,1],m)$. So $E_n \subset L_1([0,1],m)$ is not absorbing.

(b)

Let $f \in E_n$ and define $B(f,r) = \{f \in L_1([0,1],m) | \|f-g\|_1 = \int_{[0,1]} |f-g| dm < r\}$. Let V be open neighborhood of f because the open balls is a basis for the topology there exists a $B(f,r) \subset V$. Define $g \in L_1([0,1],m)$ by $g = f + \frac{r}{2x^{\frac{1}{3}}}$ then

$$\|f-g\|_1 = \int_{[0,1]} |f-g| dm = \int_{[0,1]} |-\frac{r}{2x^{\frac{1}{3}}}| dm = \int_{[0,1]} \frac{r}{2x^{\frac{1}{3}}} dm = \frac{3}{4} r < r$$

So $g \in B(f,r) \subset V$ but by the reverse triangle inequality

$$\int_{[0,1]} |g|^3 dm = (\|-g\|_3)^3 = (\|\frac{r}{2x^{\frac{1}{3}}} - f\|_3)^3 \geq (\|\frac{r}{2x^{\frac{1}{3}}}\|_3 - \|f\|_3)^3 \geq (\|\frac{r}{2x^{\frac{1}{3}}}\|_3 - n)^3 \geq \infty$$

So $g \notin E_n$. So the interior of E_n is empty.

(c)

Let $(f_n)_{n \geq 1}$ be a sequence in E_n with $f_n \to f$ in $L_1([0,1],m)$ as $n \to \infty$. Because it converges to something in $L_1([0,1],m)$ by assumption f is measurable. By Measures, Integrals and Martingales Corollary 13.8 there exists a subsequence $(f_{n(k)})_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} f_{n(k)}(x) = f(x)$ for almost every $x \in [0,1]$. Then also $\lim_{k \to \infty} |f_{n(k)}(x)|^3 = |\lim_{k \to \infty} f_{n(k)}(x)|^3 = |f(x)|^3$ (by continuity of $|\cdot|^3$) for almost every $x \in [0,1]$. So

$$\begin{aligned} \int_{[0,1]} |f|^3 dm &= \int_{[0,1]} \lim_{k \to \infty} |f_{n(k)}(x)|^3 dm = \int_{[0,1]} \liminf_{k \to \infty} |f_{n(k)}(x)|^3 dm \\ &\leq \liminf_{k \to \infty} \int_{[0,1]} |f_{n(k)}(x)|^3 \leq \liminf_{k \to \infty} n \leq n \end{aligned}$$

Where we have used Fatou's lemma Theorem 9.11 and Corollary 11.3 in Measures, Integrals and Martingales. Before we noted that f is measurable so $f \in E_n$, hence E_n is closed.

(d)

Because E_n is nowhere dense for every $n \geq 1$ and for every $f \in L_3([0,1],m)$ there exists an $n \geq 1$ such that $\int_{[0,1]} |f|^3 dm \leq n$ so $\bigcup_{n\geq 1} E_n = L_3([0,1],m)$ therefore $(E_n)_{n\geq 1}$ is a sequence of nowhere den sets such that $L_3([0,1],m) = \bigcup_{n\geq 1} E_n$, hence $L_3([0,1],m)$ is of first category.

Problem 5

Let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

(a)

If $x_n \to x$ in norm as $n \to \infty$ then $\|x_n\| \to \|x\|$ as $n \to \infty$ as the norm is continuous (in this topology) and therefore $\lim_{n\to\infty} \|x_n\| = \|\lim_{n\to\infty} x_n\| = \|x\|$.

(b)

Because H is a separable Hilbert space it has a countable orthonormal basis $(e_n)_{n\geq 1}$. By Reisz representation Theorem Problem 1 HW2 the functionals in H are of the form $\langle \cdot, x\rangle$ for some $x \in H$. By Problem 2 (a) HW4 x_n converges to x weakly if and only if $f(x_n)$ converges to $f(x)$ for every $f \in H^*$. By Bessel's inequality (Measures, integrals and Martingales Theorem 26.19):

$$\sum_{n=1}^{\infty} |\langle e_n, x\rangle|^2 \leq \|x\|^2$$

Because the sum is bounded and every term $|\langle e_n, x\rangle|^2$ is non-negative, we must have $|\langle e_n, x\rangle|$ converge to 0 for every $x \in H$. So $(f(e_n))_{n\geq 1}$ converges to $f(0)$ for every $f \in H^*$. So $(e_n)_{n\geq 1}$ converges weakly to 0, but $(\|e_n\|)_{n\geq 1} = (1)_{n\geq 1}$ converges to $1 \neq 0 = \|0\|$. So no $x_n \to x$ weakly as $n \to \infty$ does not mean $\|x_n\| \to \|x\|$ as $n \to \infty$.

(c)

Suppose that $\|x_n\| \leq 1$ for all $n \geq 1$ and that $x_n \to x$ weakly as $n \to \infty$ then $(x_n)_{n\geq 1}$ is inside the closed (in the topolgy induced by the norm) ball $\overline{B_{\|\cdot\|}(0,1)} = \{x \in H | \|x\| \leq 1\}$. Assume for contradiction that $\|x\| > 1$ then $x \neq 0$ and x is not in the closed unit ball $\overline{B_{\|\cdot\|}(0,1)}$ and therefore by theorem 2.7 (a) there exists a $f \in H^*$ such that $f(x) \neq 0$ and $f|_{\overline{B_{\|\cdot\|}(0,1)}} = 0$. Let $p_f(x) = |f(x)|$ be the associated semi norm. The weak topology τ_w is defined as the coarsest topology making $p_\alpha(x)$ continuous for all $\alpha \in H^*$. So since $(0,\infty)$ is open in $[0,\infty)$, we must have that $p_f^{-1}((0,\infty))$ is open in the weak topology τ_w. Since $f(x) \neq 0$, we have $x \in p_f^{-1}((0,\infty))$, hence $p_f^{-1}((0,\infty))$ is an open neighborhood of x. But since $f(x_n) = 0$ for all $n \geq 1$, $(x_n)_{n\geq 1}$ will never be in the neighborhood $p_f^{-1}((0,\infty))$ of x, which contradicts that $x_n \to x$ weakly as $n \to \infty$. So we must have $\|x\| \leq 1$. So it is true that $\|x\| \leq 1$ if $x_n \to x$ weakly as $n \to \infty$ and $\|x_n\| \leq 1$ for all $n \geq 1$.