Functional Analysis Assignment 1

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December 14, 2020

Problem 1

a)

For $x, y \in X, \alpha \in \mathbb{K}$, we show

1. $\|x\|_0 = 0 \Leftrightarrow x = 0$
2. $\|\alpha x\|_0 = |\alpha| \, \|x\|_0$
3. $\|x + y\|_0 \leq \|x\|_0 + \|y\|_0$,

in order to prove $\|\cdot\|_0$ is a norm on X.
1. Assuming $\|x\|_0 = 0$, we get

$$0 = \|x\|_0 \equiv \|x\|_X + \|Tx\|_Y \, .$$

As $\|\cdot\|_X$ and $\|\cdot\|_Y$ are themselves norms, and particularly satisfy $\|\cdot\|_X , \|\cdot\|_Y \geq 0$, we get $\|x\|_X , \|Tx\|_Y = 0$ from the above equation. Either of these prove $x = 0$, as we are dealing with two norms that therefore themselves satisfy the first norm property, (using T linear for $\|\cdot\|_Y$.)
Assuming $x = 0$ gives

$$\|x\|_0 \, \|x\|_X + \|Tx\|_Y = 0 + 0,$$

as T linear, and $\|\cdot\|_X , \|\cdot\|_Y$ are norms.

2. For $\alpha \in \mathbb{K}$

$$\begin{aligned}
\|\alpha x\|_0 &\equiv \|\alpha x\|_X + \|T(\alpha x)\|_Y \\
&= |\alpha| \, \|x\|_X + \|\alpha T x\|_Y \\
&= |\alpha| \, \|x\|_X + |\alpha| \, \|Tx\|_Y \\
&= |\alpha| \left(\|x\|_X + \|Tx\|_Y\right) \\
&= |\alpha| \, \|x\|_0 \, ,
\end{aligned}$$

using $\|\cdot\|_X , \|\cdot\|_Y$ themselves norms and T linear.

3. For $x, y \in X$

$$\begin{aligned}
\|x + y\|_0 &\equiv \|x + y\|_X + \|T(x + y)\|_Y \\
&= \|x + y\|_X + \|Tx + Ty\|_Y \\
&\leq \|x\|_X + \|Tx\|_Y + \|y\|_X + \|Ty\|_Y \\
&= \|x\|_0 + \|y\|_0 \, ,
\end{aligned}$$

using $\|\cdot\|_X , \|\cdot\|_Y$ themselves norms and T linear, thus altogether proving $\|\cdot\|_0$ is a norm on X.

Assuming T bounded we prove $\|\cdot\|_0, \|\cdot\|_X$ equivalent on X based on Definition 1.4 in the Lecture Notes (denoted "LN" from here), and having shown $\|\cdot\|_0$ to be a norm on X above.
For some $c_1 \in \mathbb{R}$

$$c_1 \|x\|_0 \equiv c_1 (\|x\|_X + \|Tx\|_Y).$$

By P1.10 in LN, as T is bounded $\exists C > 0 \forall x \in X \ \ \|Tx\|_Y \leq C \|x\|_X$, hence

$$c_1 (\|x\|_X + \|Tx\|_Y) \leq c_1 (\|x\|_X + C \|x\|_X) = c_1 \|x\|_X (1 + C).$$

Our goal is choosing c_1 such that $c_1 \|x\|_X (1 + C) \leq \|x\|_X$, (with the need for $c_1 \leq c_2$ for some $c_2 > 0$ postponed for now.)
For $x = 0$ we are free to choose as

$$c_1 \|x\|_X (1 + C)|_{x=0} \equiv c_1 \cdot 0 \cdot (1 + C) = 0 \leq \|x\|_X |_{x=0} = 0.$$

For $x \neq 0$ we have

$$c_1 \|x\|_X (1 + C) \leq \|x\|_X$$
$$\Leftrightarrow$$
$$c_1 \leq \frac{\|x\|_X}{\|x\|_X (1 + C)} = \frac{1}{1 + C},$$

so we choose $c_1 := \frac{1}{1+C}$, which satisfies our requirement of $c_1 > 0$ as $C > 0$, while making sure $c_1 \|x\|_0 \leq \|x\|_X, \forall x \in X$.

We now need to show $\exists c_2 \geq c_1 > 0, c_2 < \infty$ such that $\|x\|_X \leq c_2 \|x\|_0, \forall x \in X$.
We are once again assisted by P1.10 from LN

$$c_2 \|x\|_0 \equiv c_2 (\|x\|_X + \|Tx\|_Y) \leq c_2 (\|x\|_X + C \|x\|_X) = c_2 \|x\|_X (1 + C).$$

As before with $x = 0$, we are fairly free to choose our c_2 as

$$c_2 \|x\|_X (1 + C)|_{x=0} \equiv c_2 \cdot 0 \cdot (1 + C) = 0 \geq \|x\|_X |_{x=0} = 0.$$

For $x \neq 0$ we have

$$c_2 \|x\|_X (1 + C) \geq \|x\|_X$$
$$\Leftrightarrow$$
$$c_2 \geq \frac{\|x\|_X}{\|x\|_X (1 + C)} = \frac{1}{1 + C} =: c_1,$$

so we might choose $c_2 := \frac{1}{1+C} = c_1$, which satisfies the requirement of Definition 1.4 from LN as we have shown that $\exists c_1, c_2 \in \mathbb{R} \,|\, 0 < c_1 \leq c_2 < \infty$, such that for all $x \in X$ we have $c_1 \|x\|_0 \leq \|x\|_X \leq c_2 \|x\|_0$.

Now, assume $\|\cdot\|_X, \|\cdot\|_0$ equivalent ie. we assume the existence of $c_1, c_2 \in \mathbb{R} \,|\, 0 < c_1 \leq c_2 < \infty$, such that for all $x \in X$ we have $c_1 \|x\|_X \leq \|x\|_0 \leq c_2 \|x\|_X$. Note that for some fixed $x \in X$

$$\|Tx\|_Y \leq \|x\|_X + \|Tx\|_Y \equiv \|x\|_0 \leq c_2 \|x\|_X,$$

with the first inequality due to $\|\cdot\|_X \geq 0$, and the second due to our assumption. As $c_2 > 0$ choosing $C := c_2$ grants us the desired result.

b)

Assuming X to be finite dimensional leads to the following desired implications, with the first derived from T1.6 in LN (which was also proved in An1), and the third coming from 1a).

$$\dim X < \infty \Rightarrow \text{all norms on } X \text{ are equivalent} \Rightarrow \|\cdot\|_X, \|\cdot\|_0 \text{ are equivalent} \Rightarrow T \text{ bounded.}$$

c)

From the supplied hint, we might choose a Hamel Basis $(x_i)_{i\in I}$ for X, for some set I. By dividing each of these x_i's with their own norm we might repick a Hamel Basis $(q_i)_{i\in I}$ that is normalized, with $q_i := \frac{x_i}{\|x_i\|}$. We will now use the fact that a Hamel Basis by definition grants us the existence of a unique linear function $T : X \to Y$, that pairs $(q_i)_{i\in I} \subseteq X$ with $(y_i)_{i\in I} \subseteq Y$ through $q_i \overset{T}{\longmapsto} y_i$. We do this, as we want to prove $\|T\| = \infty$, and that we want to choose our y_i's to bring this about, as we want to continue work on the expression

$$\begin{aligned} \|T\| \equiv \sup_{x\in X} (\|Tx\| \mid \|x\| \leq 1) &\geq \sup_{i\in I} (\|Tq_i\|) \\ &\equiv \sup_{i\in I} \left(\left\| T \frac{x_i}{\|x_i\|} \right\| \right) \\ &= \sup_{i\in I} \left(\frac{1}{\|x_i\|} \|Tx_i\| \right) \\ &= \sup_{i\in I} \left(\frac{1}{\|x_i\|} \|y_i\| \right) \end{aligned} \tag{1}$$

While the normalization of the basis via the q_i's tames it somewhat, the definition of a Hamel basis still allows for the possibility of I be uncountable, and we may therefore select some countable subset thereof $K := \{k_1, k_2, \ldots\}$, so that for $i \in K$ the y_i's are monotonically growing.
From (1) we see that we might for $i = k_n \in K$ we could define $y_i \equiv y_{k_n} \in Y$ to be some element in y with norm $\|y_{k_n}\| = \|x_{k_n} \cdot n\|$ and then choose to kill off the y_i's for $i \in I \setminus K$ with $y_i := 0$, $i \in I \setminus K$. We thus get from (1) that

$$\begin{aligned} \sup_{i\in I} \left(\frac{1}{\|x_i\|} \|y_i\| \right) &= \sup_{k_n\in K} \left(\frac{1}{\|x_{k_n}\|} \|y_{k_n}\| \right) \\ &= \sup_{n\in\mathbb{N}} \left(\frac{1}{\|x_{k_n}\|} \|y_{k_n}\| \right) \\ &= \sup_{n\in\mathbb{N}} \left(\frac{1}{\|x_{k_n}\|} \|x_{k_n}\| \cdot n \right) \\ &= \sup_{n\in\mathbb{N}} (n) = \infty, \end{aligned}$$

so that $\|T\| = \infty$ ie. T is unbounded.

d)

Using subproblem 1c) we may pick an unbounded linear map $T : X \to Y$, and define the norm $\|\cdot\|_0$ on X as in subproblem 1a); $\|x\|_0 := \|x\|_X + \|Tx\|_Y$. Subproblem 1a) tells us that this definition of $\|\cdot\|_0$ will not be equivalent with the given norm $\|\cdot\|_X$ as T is unbounded.
Furthermore, by the definition of the "0-norm" we get $\|x\|_0 := \|x\|_X + \|Tx\|_Y \geq \|x\|_X$, as $\|\cdot\|_Y \geq 0$.
Using the contrapositive statement of the result reached in HW3P1 we may conclude that as $\|\cdot\|_X, \|\cdot\|_0$ are not equivalent, X cannot be complete with respect to both norms. So, if $(X, \|\cdot\|_X)$ were to be complete (ie. a Banach Space), $(X, \|\cdot\|_0)$ could not be.

e)

1e) is unsolved

Problem 2

a)

In this problem we will be making liberal use of the conclusions drawn in HW1P5 (that themselves are very much based upon HW1P4) as an alternative to a Hahn-Banach - style argument. As always when dealing with conjugate numbers of $1 < \infty$, we will be splitting the cases in two, defining, as by regular convention, $\frac{1}{\infty} := 0$, we see that 1 and ∞ are conjugate, for the case $p = 1$, and that for $p > 1$, p and $q := \frac{p}{p-1}$ will be conjugate. Though we will only argue boundedness for $p > 1$, as $HW1P5$ does most of the preliminary work for us (again using the convention that ∞ and 1 are conjugate numbers).

Note that in the spirit of HW1P4-5 we may think of $f : M \to \mathbb{C}$, $f\left((a, b, 0, 0, \ldots)\right) := a + b$, as summing over some productset $X \times Y = (x_n)_{n\in\mathbb{N}} \times (y_n)_{n\in\mathbb{N}}$, where we in our case have

$$x_n = \begin{cases} a, & n = 1 \\ b, & n = 2 \\ 0, & n \geq 3, \end{cases}$$

and

$$y_n = \begin{cases} 1, & n = 1 \\ 1, & n = 2 \\ 0, & n \geq 3, \end{cases}$$

such that the y_n's act coefficients for the x_n's and that we consequently can write f as $f_y(x) = \sum_{n=1}^{\infty} x_n y_n = a+b$, $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ as defined above. When refering to some specific y in $\tilde{f}_y$ below, we will be considering this to be defined as $(y_n)_{n\in\mathbb{N}}$ is above. We will call this "fact1"

So as not to confuse notation we will rename the complex-valued, well defined, bounded linear functional $f : \ell_p \to \mathbb{C}$ from HW1P5 as $\tilde{f}$ such that $\forall y \equiv (y_n)_{n\in\mathbb{N}} \in \ell_q(\mathbb{N})$ we have $\tilde{f}_y(x) := \sum_{n=1}^{\infty} x_n y_n$, for $x \equiv (x_n)_{n\in\mathbb{N}} \in \ell_p(\mathbb{N})$. Notice that by the considerations above, we may consider $\tilde{f}_y$ an extension of f from M to the whole of ℓ_p. One way of formalizing this further could be to view $M \subseteq \ell_p(\mathbb{N})$ as consisting of all the elements of ℓ_p that are killed off after the second sequence term, ie. we may generate any element $x_m \in M$, as a surjective projection $\pi : \ell_p \twoheadrightarrow M$ of any element in $x \in \ell_p$ through

$$x_m = \pi(x) := (x_1, x_2, 0, 0, \ldots),$$

and by fact1, as,

$$\begin{aligned} \tilde{f}_y(x) \equiv \sum_{n=1}^{\infty} x_n y_n &\doteq x_1 \cdot 1 + x_2 \cdot 1 = x_1 + x_2 \\ &\equiv \tilde{f}_y\left(\pi\left(x\right)\right) \\ &\overset{\text{fact1}}{\equiv} f(\pi(x)). \end{aligned} \tag{2}$$

We will be refering to the fact that $\tilde{f}_y$ extends f this way, as calling "fact2"

Also proved in HW1P5 is the existence of an isometric isomorphism $\ell_q(\mathbb{N}) \ni y \overset{T}{\mapsto} \tilde{f}_y \in \left(\ell_p(\mathbb{N})\right)^*$, so as to

make $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$, for q and p conjugate numbers. By definition of T we therefore have $\left\|\tilde{f}_y\right\| \equiv \|T(y)\|$, and as T is an isometry from ℓ_q we further get $\|T(y)\| = \|y\|_q$. Let us call this "fact3"
Using fact1-3 combined allows us to bound f through the derivation

$$\begin{aligned} \|f\| &\overset{\text{by def}}{\equiv} \sup\{|f(x)| \mid \|x\| \leq 1, x \in M\} \overset{\text{fact1,2}}{=} \sup\left\{\left|\tilde{f}_y(x)\right| \mid \|x\| \leq 1, x \in M\right\} \\ &\overset{\text{fact2}}{\leq} \sup\left\{\left|\tilde{f}_y(x)\right| \mid \|x\| \leq 1, x \in \ell_p\right\} \\ &\equiv \left\|\tilde{f}_y\right\| \overset{\text{fact3}}{=} \|y\|_q \overset{\text{by def.}}{<} \infty. \end{aligned} \tag{3}$$

As we are following an alternative to a Hahn-Banach approach, we might want to show that the norm of f on M is equal to the norm of $\tilde{f}_y$ on ℓ_p, of which we have shown one inclusion by showing that f is bounded above.
With x_m defined as above, notice that

$$\begin{aligned} \|\pi(x)\|_M &\equiv \|x_m\|_M \\ &\equiv \left(|x_1|^p + |x_2|^p\right)^{\frac{1}{p}} \\ &\leq \left(|x_1|^p + |x_2|^p + \ldots\right)^{\frac{1}{p}} \\ &\equiv \|x\|_{\ell_p}, \end{aligned}$$

such that we may by (2) conclude that

$$\left|\tilde{f}_y(\pi(x))\right| \overset{(2)}{=} \left|\tilde{f}_y(x)\right| \overset{(2)}{=} |f(\pi(x))| \leq \|f\| \cdot \|\pi(x)\| \overset{(3)}{\leq} \|f\| \cdot \|x\|,$$

such that as $\left\|\tilde{f}_y\right\| = \inf\left\{C > 0 : \left\|\tilde{f}_y(x)\right\| \leq C\|x\|\right\}$, we get $\left\|\tilde{f}_y\right\| \leq \|f\|$, consequently $\|f\| = \left\|\tilde{f}_y\right\| = \|y\|_q \equiv \|(1,1,0,0,\ldots)\|_q$, for $p > 1$.
As mentioned the buildup for the case $p = 1$, (also of HW1P4-5 origin) is rather similar, with the implementation of the buildup most oftenly following along aswell, so that we get $\|f\| = \|y\|_\infty \equiv \|(1,1,0,0,\ldots)\|_\infty = 1$.

b)

Having shown existence of the desired functional (that we have chosen to name $\tilde{f}$) in subproblem a) we will now uniqueness by contradiction as standard.
Assume therefore that there exists some different bounded linear functional $F : \ell_p \to \mathbb{C}$ that extends f, such that $\|F\| = \|f\|$.

Further in 2b) is unsolved

The intuition for why assuming existence of a different functional F with the same properties leads to a contradiction for $p > 1$ being, that as the two functionals are different on a different $y \in \ell_q$ than $(1,1,0,\ldots)$ must be "assigned" to F, but with $p > 1$ (and the conjugate q) the norms are sensitive to this change, as it forces the norm of F to be different from the norm of f creating a contradiction. The argument should have gone through the use of the HW1P5 constructed function, that we in a) dubbed T, and in particular its bijectivity and it being a isometric isomorphism.

c)

As the supremum of $y \equiv (1,1,0,0,\ldots) \in \ell_\infty$, is 1, we may, by choosing some new element in ℓ_q y_c that in addition to having its first two coordinates be 1 also contains some $c \in \mathbb{C}$ with $|c| \leq 1$ such that $y_c := (1,1,c,0,\ldots)$. Notice in particular that this construction also has $\|y_c\|_\infty = 1$, so that we might by the use of T as introduced in a), find some $\tilde{f}_{y_c} = T(y_c)$ extending f (T being bijective, in particular surjective). As T is also injective we know a priori that $\tilde{f}_{y_c} \neq \tilde{f}_y$. By a) we thus have $\|y_c\| = \|y\| = \left\|\tilde{f}_y\right\| = \|f\|$. The construction will by definition also have the property that

$$\tilde{f}_{y_c} \equiv \sum_{n=1}^{\infty} x_n y_{c_n} \doteq a \cdot 1 + b \cdot 1 + 0 \cdot c = a + b \equiv f(x),$$

so that we may tag the norms $\left\|\tilde{f}_{y_c}\right\| \overset{\cong}{\equiv} \|T(y_c)\| = \|f\|$ onto the fold. Ie. we have found a different linear functional that extends f and that has the same norm. As $\#\{z \in \mathbb{C} | \, |z| \leq 1\}$ is infinite, we may by the bijectivity of T choose infinitely many different linear functionals on $\ell_1(\mathbb{N})$ extending f and having the same norm aswell.

Problem 3

a)

Using the Linear Algebra result (As an example, see Lineær Algebra by Hesselholt&Wahl T.4.3.11(1)) that injective linear maps take linearly independent sets to linearly independent sets, we may choose a set of $n+1$ linearly independent vectors $(x_i)_{i \in \{1,\dots,n,n+1\} =: I}$ in X.

Assume for contradiction that the linear map $F : X \to \mathbb{K}^n$ is injective.
By assumption we would therefore have that $F\left((x_i)_{i \in I}\right) \subseteq \mathbb{K}^n$ would be a set of $n+1$ linearly independent vectors in $\mathbb{K}^n$. As you cannot have $n+1 > n$ linearly independent vectors in $\mathbb{K}^n$, we get our required contradiction with F being injective and linear from X to $\mathbb{K}^n$.

b)

For $F : X \to \mathbb{K}^n$, with $F(x) = (f_1(x), \dots, f_n(x))$, $f_i \in X^*$, we note that F is linear, on account of the f_i's being linear, as we for $\alpha \in \mathbb{K}$, $x, y \in X$ have

$$\begin{aligned} F(\alpha x + y) &\equiv (f_1(\alpha x + y), \dots, f_n(\alpha x + y)) \\ &= (f_1(\alpha x + y), \dots, f_n(\alpha x + y)) \\ &= (f_1(\alpha x) + f_1(y), \dots, f_n(\alpha x) + f_n(y)) \\ &= (\alpha f_1(x) + f_1(y), \dots, \alpha f_n(x) + f_n(y)) \\ &= (\alpha f_1(x), \dots, \alpha f_n(x)) + (f_1(y), \dots, f_n(y)) \\ &= \alpha\,(f_1(x), \dots, f_n(x)) + (f_1(y), \dots, f_n(y)) \\ &= \alpha F(x) + F(y). \end{aligned}$$

By problem 3a) F is therefore non-injective ie. $\ker F \equiv \{x \in X \,|\, F(x) = 0 \in \mathbb{K}^n\} \neq \{0\}$.
Note that for any fixed $x \in X$ we have $\mathbb{K}^n \ni 0 = F(x) \equiv (f_1(x), \dots, f_n(x)) \Leftrightarrow \mathbb{K} \ni 0 = f_1(x) = f_2(x) = \dots = f_n(x)$, so $x_0 \in \ker F \Leftrightarrow x_0 \in \ker f_i \; \forall i \in \{1, \dots, n\} \Leftrightarrow x_0 \in \cap_{i \in \{1,\dots,n\}} \ker f_i$.
So as $\ker F \equiv \{x \in X \,|\, F(x) = 0 \in \mathbb{K}^n\} \neq \{0\}$, we get the desired result.

c)

Using T2.7b) in LN, we may for $0 \neq x \in X$ choose n functionals from the dual space of X; $f_i \in X^*, i \in I := \{1, \dots, n\}$ such that $\|f_i\| = 1$, and $f(x) = \|x\|$.
From subproblem 3b) we know that $\cap_{i \in I} \ker f_i \neq \{0\}$. We note that for each $i \in I$ that $\ker f_i$ will be a subspace of X, such that we may use the Linear Algebra result that in particular finite intersections

of subspaces are again a subspace (Proved in Exercises in the 2018-2019 LinAlg-course), we get that the intersections of the kernels is again a subspace of X.

Choose some $0 \neq y_0 \in \cap_{i\in I} \ker f_i$. Using the fact that $\cap_{i\in I} \ker f_i$ is a subspace of X we may pick some $\alpha \in \mathbb{K}$ such that $\|\alpha y_0\| = 1$, and define $\cap_{i\in I} \ker f_i \ni y := \alpha y_0$.
As $y \in \cap_{i\in I} \ker f_i$, we get $f_i(y) = 0$, $\forall i \in I$, so that for $x_1, x_2, \ldots, x_n \in X$ we may conclude that for some $i \in I$ such that $x_i \neq 0$ we get

$$|f_i(y - x_i)| = |f_i(y) - f_i(x_i)| = |f_i(x_i)| = |\|x_i\|| = \|x_i\|,$$

such that as

$$|f_i(y - x_i)| \leq \|f_i\| \|y - x_i\| = \|y - x_i\|,$$

we get $\|x_i\| \leq \|y - x_i\|$. If $\exists i_0 \in I : x_{i_0} = 0$, we get the desired result aswell, as $\|y - x_{i_0}\| |_{(x_{i_0}=0)} \doteq \|y\| \doteq 1 \geq \|x_{i_0}\| |_{(x_{i_0}=0)} \doteq 0$.

d)

The fact that you may not cover the unit-sphere S with a finite cover of closed balls without having any of the balls contain 0 follows from subproblem3c). Choosing once again a set of points $x_1, \ldots, x_n \in X$, along with a set of radii $r_1, \ldots, r_n$, the covering of S will take the form

$$\bigcup_{i=1}^{n} \overline{B}(x_i, r_i) \equiv \bigcup_{i=1}^{n} \{x \in X | \ \|x - x_i\| \leq r_i\} \supseteq S \equiv \{x \in X | \ \|x\| = 1\}.$$

However, by choosing such a covering, there will, by subproblem3c) exist some $y \in X$ with $\|y\| = 1 \Rightarrow y \in S$ such that $\|y - x_i\| \geq \|x_i - 0\|$, $\forall i \in \{1, \ldots, n\}$. ie. as $y \in S$ it is necessary for the covering to contain y, but by containing y the balls containing y in the convering will inevitably also contain 0, thus proving the desired result.

e)

That the unit ball in X is non-compact follows immediately from haven proven that the unit-sphere S is non-compact by the way of contraposition of the fact that a closed subspace of a compact space is compact (see for example Prop 4.22 of Folland).

The fact that the unit-sphere S is non-compact, follows from the Open Covering Theorem, and subproblem 3d); Assume towards a contraposition that S is compact. The Open Covering Theorem tells us that we may therefore reduce any open covering of S to a finite covering. Choose some covering of S consisting of the family of open balls (of for example radius 1/3), with center on S, and reduce this to a finite cover. Taking the closure of each of the finitely many open balls, we get a finite covering of S consisting of closed balls. As each of these are centered somewhere on S and each have norm 1/3 none of them will contain 0 which contradicts with the statement proved in subproblem 3d).

Problem 4

a)

Let $n \in \mathbb{N}$. Note that $E_n \subseteq L_1([0,1], m)$ will be absorbing in $L_1([0,1], m)$ (by definition) if and only if $E_n := \left\{ f \in L_1([0,1], m) | \int_{[0,1]} |f(x)|^3 \, dm(x) \leq n \right\}$ is convex and satisfies $\forall (f \neq 0) \in L_1([0,1], m) \, \exists t > 0$ such that $tf \in E_n$ ie. such that $\int_{[0,1]} |tf(x)|^3 \, dm(x) \leq n$. Note that $\int_{[0,1]} |tf(x)|^3 \, dm(x) = t^3 \int_{[0,1]} |f(x)|^3 \, dm(x)$,

for $t > 0$. We now show that E_n is not absorbing in $L_1([0,1], m)$.
By HW2P2b) we know that $L_3([0,1], m) \subset L_1([0,1], m)$ such that we might find some $\tilde{f} \in L_1([0,1], m) \setminus L_3([0,1], m)$. For $\|\cdot\|_3$ being the norm on $L_3([0,1], m)$, we would for such a function have $\left\|\tilde{f}\right\|_3^3 \equiv \int_{[0,1]} \left|\tilde{f}(x)\right|^3 dm(x) = \infty$, so for $t > 0$ we would a have

$$\left\|t\tilde{f}\right\|_3^3 \equiv \int_{[0,1]} \left|t\tilde{f}(x)\right|^3 dm(x) = t^3 \int_{[0,1]} \left|\tilde{f}(x)\right|^3 dm(x) = \infty.$$

ie. there does not exist a $t > 0$ that would let $t\tilde{f}$ be absorbed in E_n, for any $n \in \mathbb{N}$.

b)

To show that E_n has empty interior for any $n \in \mathbb{N}$ in $L_1([0,1], m)$ we show that as $E_n \subseteq L_3([0,1], m) \subset L_1([0,1], m)$, any sequence $(f_i)_{i\in\mathbb{N}}$ in the complement of E_n that converges (in L_1) to some arbitrary $f \in E_n$, will not be an interior point of E_n.
Inspired by solutions to HW1 and HW2, reimplementing $\tilde{f} \in L_1([0,1], m) \setminus L_3([0,1], m)$ and using the result $L_3([0,1], m) \subset L_1([0,1], m)$ derived from HW2P2b) to "seperate" $L_1([0,1], m) \setminus L_3([0,1], m)$ from E_n, we will choose our sequence to be of the form $f_i := f + \frac{\tilde{f}}{i}, i \in \mathbb{N}$, which serves both our desired purposes, of

$$f_i \in E_n^c,$$

and of

$$f_i \overset{\|\cdot\|_1}{\to} f.$$

Noting from An2 that the L_p-spaces are vector spaces and thus in particular stable under addition and multiplication, we see that assuming $f_i \in E_n$ for some $i \in \mathbb{N}$ leads to a contradiction by

$$f_i = f + \frac{\tilde{f}}{i} \Leftrightarrow \tilde{f} = i \cdot (f_i - f),$$

as $\tilde{f} \in L_1([0,1], m) \setminus L_3([0,1], m)$, but we have assumed $f_i \in E_n \subseteq L_3([0,1], m) \subset L_1([0,1], m)$, such that as $f \in E_n$ $f_i - f \in E_n \subseteq L_3([0,1], m) \Rightarrow i \cdot (f_i - f) \in L_3([0,1], m)$ $\lightning$.
The convergence will be satisfied as $\tilde{f} \in L_1([0,1], m) \Rightarrow \left\|\tilde{f}\right\|_1 < \infty$ so that

$$\|f_i - f\|_1 = \left\|\frac{\tilde{f}}{i}\right\|_1 = \frac{1}{i}\left\|\tilde{f}\right\|_1 \to 0,$$

for $i \to \infty$.

c)

Pick some $n \in \mathbb{N}$. We will once again pick some sequence $(f_i)_{i\in\mathbb{N}} \subseteq E_n$, that exhibits convergence in L_1 to some f such that we might show that actually $f \in E_n$. Notice as in b) that $f \in E_n$ if and only if $\int_{[0,1]} |f(x)|^3 dm(x) \leq n$, ie. if and only if $\int_{[0,1]} |f(x)|^3 dm(x) \in [0, n]$. Our core machinery will be Fatou's Lemma (see for example An2, T9.11.)

In order to qualify for using Fatou, we will need to find a sequence that converges (atleast liminf) pointwise to f. To aid in this, we will be using the Riesz Fisher-derived corollary, that as $f_i \overset{L_1}{\to} f$ there exists a subsequence $(f_{i_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f_{i_k}(x) \overset{a.s.}{=} f(x)$. (See Corollary 13.8 An2, which is good enough for

L_1 as well as its intended $\mathcal{L}_1$.) Notice that as the f_i's are measurable the subsequence will also be. Using continuous (and therefore measureable) transformation of f_{i_k} with $|\cdot|^3$, we may thus conclude that

$$0 \leq \int_{[0,1]} |f(x)|^3 \, dm(x) = \int_{[0,1]} \lim_{k\to\infty} |f_{i_k}(x)|^3 dm(x) = \int_{[0,1]} \liminf_{k\to\infty} |f_{i_k}(x)|^3 dm(x) \overset{\text{Fatou}}{\leq} \liminf_{k\to\infty} \int_{[0,1]} |f_{i_k}(x)|^3 dm(x),$$

but as $(f_i)_{i\in\mathbb{N}} \subseteq E_n \Rightarrow (f_{i_k})_{k\in\mathbb{N}} \subseteq E_n \Leftrightarrow \int_{[0,1]} |f_{i_k}(x)|^3 \leq n$, we get $\liminf_{k\to\infty} \int_{[0,1]} |f_{i_k}(x)|^3 dm(x) \leq \liminf_{k\to\infty} n = n$, such that $\int_{[0,1]} |f(x)|^3 \, dm(x) \in [0, n]$. As n was arbitrary, we how now shown the requested result.

d)

By D3.12 ii) in LN $L_3([0,1], m) \subset L_1([0,1], m)$ will be of first category, if there exists some sequence $(E_n)_{n\in\mathbb{N}}$ of nowhere dense sets, such that $L_3([0,1], m) = \bigcup_{n\in\mathbb{N}} E_n$. Notice, as we happen to have a sequence of this very moniker, that

$$\begin{aligned} \bigcup_{n\in\mathbb{N}} E_n \equiv \bigcup_{n\in\mathbb{N}} &\left\{ f \in L_1([0,1], m) \mid \int_{[0,1]} |f(x)|^3 \, dm(x) \leq n \right\} \\ &= \left\{ f \in L_1([0,1], m) \mid \int_{[0,1]} |f(x)|^3 \, dm(x) < \infty \right\} \equiv L_3([0,1], m). \end{aligned}$$

So what remains to be shown is that the E_n's are nowhere dense. To this end note that for some $n \in \mathbb{N}$ E_n will by definition By D3.12 i) in LN be nowhere dense if(f) the closure of E_n is empty, ie. if(f) $Int\left(\overline{E_n}\right) = \emptyset$. By subproblem c) we have that as the E_n's are closed, $\overline{E_n} = E_n$. By subproblem b), the E_n's have empty interior. We therefore have the requirements to say that the E_n's are nowhere dense, and as they union to be $L_3([0,1], m)$, we can say that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$.

Problem 5

a)

Note that $x_n \to x$ in norm as $n \to \infty \Leftrightarrow \|x_n - x\| \to 0$, for $n \to \infty$.
Note also that the absolute value on $\mathbb{R}$ is continuous, such that by an application of the inverse triangle inequality on $\|\cdot\|$, can do the following computation

$$\begin{aligned} \|x_n - x\| &\geq |\|x_n\| - \|x\|| \geq 0 \\ &\Rightarrow \\ 0 = \lim_{n\to\infty} \|x_n - x\| &\geq \lim_{n\to\infty} |\|x_n\| - \|x\|| \\ &= \left| \lim_{n\to\infty} (\|x_n\| - \|x\|) \right| \\ &= \left| \lim_{n\to\infty} (\|x_n\|) - \|x\| \right| \geq 0, \end{aligned}$$

which implies that $|\lim_{n\to\infty} (\|x_n\|) - \|x\|| = 0$, such that $\lim_{n\to\infty} (\|x_n\|) - \|x\| = 0$,
and hence $\lim_{n\to\infty} \|x_n\| = \|x\|$.

b)

As a Hilbert Space is in particular a Banach Space, and as nets generalize sequences, we get by HW4 Problem 2a), that as $x_n \rightharpoonup x$ in H, for $n \to \infty$, we have that $\forall f \in H^* \ (f(x_n))_{n\in\mathbb{N}} \to f(x)$. The Riesz Representation

Theorem proves that any such $f \in H^*$ can, for some $y \in H$ be written on the form $f_y(x) = \langle x, y \rangle$. By HW4 we thus have $f_y(x_n) = \langle x_n, y \rangle \to \langle x, y \rangle, \forall y \in H$. By An2 T26.24 every seperable Hilbert Space has a (countable) Orthonormal Basis $(e_n)_{n\in\mathbb{N}}$, and by An2 T26.21 this is equivalent to Parsevals identity,

$$\sum_{n=1}^{\infty} |\langle e_n, h \rangle|^2 = \|h\|^2$$

being satisfied $\forall h \in H$. We therefore get that $|\langle e_n, h \rangle|^2 \overset{n\to\infty}{\to} 0$, so that $\langle e_n, h \rangle \overset{n\to\infty}{\to} 0$, and as $\langle \cdot_1, \cdot_2 \rangle = \overline{\langle \cdot_2, \cdot_1 \rangle}$ get also get $\langle h, e_n \rangle \overset{n\to\infty}{\to} 0$. So as $\forall f \in H^* \exists x_f \in H : f(e_n) = \langle e_n, x_f \rangle$ we get that $\|x_n\| \to \|x\|$ doesn't follow.

c)

We will reach the desired result by showing that the norm is (atleast sequentially) weakly lower-semicontinuous as $x_n \rightharpoonup x$ in H. To this end, we use T2.7b) in LN such that for $\|x\| \leq 1, x \neq 0 \, \exists f \in H^* : \|f\| = 1$ and $f(x) = \|x\|$, as well as the result of problem 2a) in HW4. We thus have

$$\begin{aligned} \|x\| = f(x) &= |f(x)| \\ &= \left| \lim_{n\to\infty} f(x_n) \right| \\ &= \lim_{n\to\infty} |f(x_n)| \\ &\leq \liminf_{n\to\infty} \|f\| \, \|x_n\| \\ &= \liminf_{n\to\infty} \|x_n\| \leq 1, \end{aligned}$$

which is the desired result as the case $x = 0$ is immediate.