Functional Analysis, assignment 2

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Problem 1a

The first part is to show $f_N \to 0$ weakly as $N \to \infty$

Is F_y arbitrary or a concrete, specific map?

We have that $(e_n)_{n\geq 1} \in H$ since $(e_n)_{n\geq 1}$ is a ONB and since $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, then $f_N \in H$ for $N \geq 1$.
I will now define a linear bounded function $F_y : H \to \mathbb{C}$ and let $y = y_n e_n \in H$ such that $F_y(x) = \langle x, y\rangle$ by Riesz´s representations theorem

$y = \sum_{n=1} y_n e_n$?

I will now look at

$$\begin{aligned}
F_y(f_N) &= \langle f_N, y\rangle \\
&= \langle N^{-1}\sum_{n=1}^{N^2} e_n, \sum_{n=1}^{\infty} y_n e_n\rangle \\
&= N^{-1}\sum_{n=1}^{N^2}\langle e_n, \sum_{n=1}^{\infty} y_n e_n\rangle \\
&= N^{-1}\sum_{n=1}^{N^2} y_n < \infty
\end{aligned}$$

Generally, $y_n \in \mathbb{C}$ so this is not defined.

since I had that the function F_y was bounded.

To show $N^{-1}\sum_{n=1}^{N^2} e_n \to 0$ I will show $\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n < \infty$ Why would that suffice?
So I start by having:

You either mean weakly or mean $\frac{1}{N}\sum_{n=1}^{N^2} y_n$.

$$\Big(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n\Big)^2$$

and use the triangle inequality and Cauchy-Schwarz´:

$$\begin{aligned}
\Big(\frac{1}{\sqrt{N}}\Big|\sum_{n=1}^{N} y_n\Big|\Big)^2 &\leq \Big(\frac{1}{\sqrt{N}}\sum_{n=1}^{N}|y_n|\Big)^2 \\
&= \Big(\sum_{n=1}^{N}\frac{1}{\sqrt{N}}|y_n|\Big)^2 \\
&\leq \sum_{n=1}^{N}\Big(\frac{1}{\sqrt{N}}\Big)^2\sum_{n=1}^{N}|y_n|^2 \\
&= \sum_{n=1}^{N}|y_n|^2
\end{aligned}$$

From this I can get

$$|\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n| \leq \Big(\sum_{n=1}^{N} |y_n|^2\Big)^{1/2} < \infty$$

for $N \geq 1$ since $(y_n)_{n\geq 1} \in \ell_2(\mathbb{N})$ and $< \infty$ applies by definition of $\ell_p(\mathbb{N})$ since we had an y by Riesz´s theorem. ?

Further we can say that since $\sum_{n=1}^{N} |y_n|^2 < \infty$ we have that there exists a $C \in \mathbb{C}$ such that $\sum_{n=1}^{N} |y_n|^2 \to C$ for $n \to \infty$. No, this requires $\sum_{n=1} |y_n|^2 < \infty$, which you stated above without proof.

For $\varepsilon > 0 \exists\, M$ for which $\sum_{n=M+1}^{\infty} |y_n|^2 < \varepsilon$, then $K \geq 1$ for any constant will give us $\sum_{n=M+1}^{K+M} |y_n|^2 < \varepsilon$.

Now by $N \geq \frac{C^2}{\varepsilon^2}$ we will get

Where does the square go?

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{M} |y_n| \leq \frac{\varepsilon}{C}\cdot C = \varepsilon$$

I will now use Cauchy-Schwartz´ on the following

$$\begin{aligned}
|\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n| &\leq \frac{1}{\sqrt{N}}\sum_{n=1}^{N} |y_n| \\
&= \frac{1}{\sqrt{N}}\sum_{n=1}^{M} |y_n| + \frac{1}{\sqrt{N}}\sum_{n=M+1}^{N} |y_n| \\
&\leq \varepsilon + \frac{1}{\sqrt{N}}\sum_{n=M+1}^{N+M} |y_n| \\
&\leq \varepsilon + \sqrt{\Big(\sum_{n=M+1}^{N+M} \frac{1}{N}\Big)\Big(\sum_{n=M+1}^{N+M} |y_n|^2\Big)} \\
&= \varepsilon + \sqrt{1\cdot \sum_{n=M+1}^{N+M} |y_n|^2} \\
&\leq \varepsilon + \sqrt{\varepsilon}
\end{aligned}$$

this gives us

$$\Big|\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n\Big| \to 0$$

for $N \to \infty$ which implies

$$\Big|\frac{1}{\sqrt{N}}\sum_{n=1}^{N^2} y_n\Big| \to 0$$

for $N \to \infty$
So to conclude that $f_N \to 0$ weakly, we look at the limit.

$$\lim_{N\to\infty} F(f_N) = \lim_{N\to\infty} \underbrace{N^{-1}\sum_{n=1}^{N^2} e_n}_{\text{This is not } F(f_N).} = 0$$

Where do you use this?

as I had F was bounded, hence continuous. So now we can conclude that $f_N \to 0$ weakly for $N \to \infty$. (✓)

The second part is to show that $||f_N|| = 1 \ \forall \ N \geq 1$.

$$
\begin{aligned}
||f_N||^2 &= ||N^{-1} \sum_{n=1}^{N^2} e_n||^2 \\
&= |N^{-1}|^2 || \sum_{n=1}^{N^2} e_n||^2 \\
&= N^{-2} || \sum_{n=1}^{N^2} e_n||^2 \\
&= N^{-2} \sum_{n=1}^{N^2} ||e_n||^2 \\
&= N^{-2} \sum_{n=1}^{N^2} 1^2 \\
&= N^{-2} \cdot N^2 \\
&= 1
\end{aligned}
$$

What happens here?

I take the square root and get that $||f_N|| = 1$ (✓)

Problem 1b

I have that $K = \overline{co\{f_N | N \geq 1\}}^{||\cdot||}$ then by definition 7.7 is $co\{f_N | N \geq 1\}$ convex, this mean that the norm and the weak closures of $co\{f_N | N \geq 1\}$ will coincide by theorem 5.6 And then theorem 5.6 says that $\overline{co\{f_N | N \geq 1\}}^{||\cdot||} = \overline{co\{f_N | N \geq 1\}}^{||\tau w||}$ and this gives that K is weakly closed.

I will now consider a unit ball $\overline{B}_{H^*}(0,1) \subset H^*$ then is $\overline{B}_{H^*}(0,1)$ weak* compact by theorem 6.1, since we have that H is normed vector space. By lecture notes we have that Hilbert spaces are reflexive, then by thm. 5.9 we get $\tau w = \tau w^*$ for H^*. This gives us $\overline{B}_{H^*}(0,1)$ is weakly compact.

Notation.

Note that this is an antilinear isomorphism.

We know from Riesz´representations theorem that every element in H^* is in the form $F_y = \langle \cdot, y \rangle$ where $y \in H$. This gives us an isomorphism $H^* \to H$ where $F_y \to y$.
From this we will get the isomorphism $\overline{B}_{H^*}(0,1) \to \overline{B}_H(0,1)$. This implies $K \subset \overline{B}_H(0,1)$ is a weakly closed subset of a weakly compact space. We can now say that K is weakly compact and hence conclude that K is weakly closed and $f_N \to 0$ weakly as $N \to \infty$, hence $0 \in K$. (✓)

Problem 1c

From definition 7.1 I have that for $x \in K$ it applies $x = \alpha x_1 + (1-\alpha)x_2$, this implies $x = x_1 = x_2$ for $x_1, x_2 \in K$ and $0 < \alpha < 1$.

This is unclear.

I now observe $K \subseteq H$ is non-empty convex compact subset. Now I say that $g_n = \langle \cdot, e_n \rangle \in H^*$ for any $n \in \mathbb{N}$ where it is a continuous linear functional. I note $h_n(K)$ is a subset of $\mathbb{R}$ and we let $C = \sup_n \{\langle x, -e_n \rangle | x \in K\} = \sup_n \{-\langle x, e_n \rangle | x \in K\}$ and I will now get that $x \in K, x \geq 0, 0 \in K$ hence $C \leq 0$. Why?

Since I have fulfilled all the requirement I get from lemma 7.5 that $F_n := \{x \in K | Re\langle x, -e_n\rangle = 0\} \neq \emptyset$ is compact face of K for all $n \in \mathbb{N}$ I have $0 \in F_n$, hence $0 \in \bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Since 0 is the only element which is orthogonal on all elements e_n then I get

$$\bigcap_{n=1}^{\infty} F_n = \{x \in K | Re\langle x, -e_n\rangle = 0 \forall n \in \mathbb{N}\} = \{0\}$$

Earlier I said that F_n is compact face of K hence $\bigcap_{n=1}^{\infty} F_n = \{0\}$ is also a face of K by Remark 7.4(3). Then we can conclude that 0 is a extreme point in K by Remark 7.4(1).

I will now show that f_N is extreme point in K.
I start by fixing $N \geq 1$ and will suppose that $f_N = \alpha x_1 + (1-\alpha)x_2$ for $0 < \alpha < 1$ and $x_1, x_2 \in K$. Since I know that $1 = ||f_N||^2 = \langle f_N, f_N\rangle$, I examine the following:

$$1 = \langle f_N, f_N\rangle = \langle \alpha x_1 + (1-\alpha)x_2, f_N\rangle = \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle$$

and this gives me that:

$$\begin{aligned} 0 &= \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle - 1 \\ &= \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle - (\alpha + (1-\alpha)) \\ &= \alpha(\langle x_1, f_N\rangle - 1) + (1-\alpha)(\langle x_2, f_N\rangle - 1) \end{aligned}$$

Which ones? Elaborate!

from earlier I have that $0 < \alpha < 1$, and I know that $\langle x_1, f_N\rangle, \langle x_2, f_N\rangle \geq 0$ from our assumptions, so this gives me that $0 \leq \langle x_i, f_N\rangle \leq 1$ for $i = 1, 2$ which implies $\langle x_1, f_N\rangle = 1$ and $\langle x_2, f_N\rangle = 1$.

Why is this true? Why?

I will now observe that

$$1 = ||\langle x_1, f_N\rangle|| \leq ||x_1|| ||f_N|| = ||x_1||$$

? to show $x_1 = x_2 = f_N$. From our definition of extreme points I get $x_1 \in K \subseteq \overline{B}_H(0,1)$ which implies $||x_1|| \leq 1$.

Why?

Hence $1 = ||\langle x_1, f_N\rangle|| = ||x_1|| ||f_N|| = ||x_1||$.
I now have that both x_1 and f_N are linear dependent which gives me that $x_1 = \lambda f_N$ with the scalar λ. This implies

$$1 = \langle x_1, f_N\rangle = \langle \lambda x_1, f_N\rangle = \lambda\langle x_1, f_N\rangle = \lambda ||f_N||^2 = \lambda$$

This gives me $x_1 = f_N$, and for x_2 it is the same, and then I can say $x_1 = x_2 = f_N$. I can now conclude that f_N is extreme points in K for all $N \geq 1$.

Problem 1d

From 1.b I have that $K = \overline{co\{f_N | N \geq 1\}}^{\tau w}$ is non-empty convex subset of H. I can now say that $Ext(K) \subseteq \overline{co\{f_N | N \geq 1\}}^{\tau w}$ by thm 7.9. From 1.c I get $\{f_N | N \geq 1\} \cup \{0\} \subseteq \overline{co\{f_N | N \geq 1\}}^{\tau w}$. Since I knew that H is normed vector spaces, then H is metrizable and hence $\{f_N | N \geq 1\}$ is metrizable.

I now have that $\{f_N | N \geq 1\}$ is countable, this mean that I can look at sequences in $\{f_N | N \geq 1\}$ rather than looking at nets. I will now assume that $(x_n)_{n\geq 1}$ is a sequence in $\{f_N | N \geq 1\}$ which converges weakly to $x \in \overline{co\{f_N | N \geq 1\}}^{\tau w}$. This implies $x_i = f_N$ for some $N \geq 1$, then I see that x is either f_N or zero.

Why should (x_n) be constant?

From this I get $Ext(K) \subseteq \overline{\{f_N | N \geq 1\}}^{\tau w} = \{f_N | N \geq 1\} \cup \{0\}$. From 1.c I get $\{f_N | N \geq 1\} \cup \{0\} \subseteq Ext(K)$, which implies $Ext(K) = \{f_N | N \geq 1\} \cup \{0\}$. I can now conclude that there do not exists other extreme points.

Problem 2a

I start by noting $T : X \to Y$ and $g : Y \to \mathbb{K}$.
Next I see that from HW4P2 I have $x_n \to x$ weakly for $x \in X$ and $n \to \infty$ if and only if we have $f(x_n) \to f(x)$ $\forall f \in X^*$. From the functions T and g I can take a $g \in Y^*$ and will end with having ? $g \circ T \in X^*$ where $(g \circ T)(x_n) \to (g \circ T)(x)$ for $n \to \infty$.
This will also apply for for $g(Tx_n) \to g(Tx)$. Since I had that it was if and only if I can now conclude that $Tx_n \to Tx$ weakly as $n \to \infty$. ✓

Problem 2b

Like before I have that $x_n \to x$ weakly for $n \to \infty$. To show $||Tx_n - Tx|| \to 0$ for $n \to \infty$, will I do it by contradiction. So I say $||Tx_n - Tx|| \not\to 0$ for $n \to \infty$ there will exists a sequence $(x_{n_k})_{k \in \mathbb{N}}$ where $||Tx_{n_k} - Tx|| > \epsilon$ for $k \in \mathbb{N}$, which mean? that $x_{n_k} \to x$ weakly for $k \to \infty$ because $x_n \to x$ weakly for $n \to \infty$, so we have that $(x_{n_k})_{k \in \mathbb{N}}$ is bounded. This implies that since it is bounded it will have a subsequence $(x_{n_{k_i}})_{i \in \mathbb{N}}$ for which $||Tx_{n_{k_i}} - Tx'|| \to 0$ for some $x' \in X$, because we are in Banach space, this mean that it is complete. How do you know $Tx_{n_{k_i}}$ converges in $T(X)$?

What is ε?

Why?

Earlier we showed that $x_{n_k} \to x$ weakly for $k \to \infty$, and together with problem 2a I get that $Tx_{n_k} \to Tx$ weakly, which implies $Tx_{n_{k_i}} \to Tx$ weakly for $i \in \mathbb{N}$. I can now say that since we had $Tx_{n_{k_i}} \to Tx$ weakly it will imply that $||Tx_{n_{k_i}} - Tx|| \to 0$ for $i \to \infty$ which mean $||Tx_{n_k} - Tx|| < \epsilon$ which is a contradiction to $||Tx_{n_k} - Tx|| > \epsilon$. This gives us that $||Tx_{n_k} - Tx|| \to 0$ for $n \to \infty$.

Why? (✓) for which k?

Elaborate!

Problem 2c

I have to show T is compact, so I will do it by contradiction. I start by having that T is not compact, i.e $T \notin \mathcal{K}(H, Y)$. By assuming this I get that $T(\overline{B}_H(0,1))$ is not totally bounded by proposition 8.2. This mean that there exists an $\varepsilon > 0$ for which there do not exist union of finitely many open balls for which $T(\overline{B}_H(0,1))$ is covered by radius ε.

I will now show that there exists a sequence $(x_n)_{n \geq 1}$ in the closed unit ball of H such that $||Tx_n - Tx_m|| \geq \varepsilon$ for all $n \neq m$. I will now take a $x_1 \in \overline{B}_H(0,1)$ where I also have $x_1 \in (x_n)_{n \geq 1} \subset \overline{B}_H(0,1)$ and will suppose that for $x_2, x_3, ..., x_n$ it applies that $||Tx_q - Tx_r|| \geq \varepsilon$ $\forall q, r \leq n$. Now I look at

This is not defined yet.

$$S := T(\overline{B_H(0,1)}) \cap (\cup_{i=1}^n B_Y(T_{x_i}, \varepsilon))^C$$

where we notice that

$$T(\overline{B_H(0,1)}) \nsubseteq (\cup_{i=1}^n B_Y(T_{x_i}, \varepsilon))$$

because $T(\overline{B_H(0,1)})$ is not totally bounded. From this I can say $S \neq \emptyset$. I will now take $x_{n+1} \in \overline{B_H(0,1)}$ for which it apply that $Tx_{n+1} \in S$, and I notice that $Tx_{n+1} \in (\cup_{i=1}^n B_Y(T_{x_i}, \varepsilon))^C$ which implies for any i that $Tx_{n+1} \notin B_Y(T_{x_i}, \varepsilon)$ We now get that $||Tx_{n+1} - Tx_i|| \geq \varepsilon$ $\forall i \leq n$. And if we continue the same process we will obtain the sequence $||Tx_n - Tx_m|| \geq \varepsilon$. ✓

Since it is given that H is separable, we get by thm 5.13 that H is metrizable and by proposition 2.10 we get that H is reflexive. We can now say that $\overline{B}(0,1)$ is weakly compact by 6.3. Hence I get that every sequence has a weakly convergent subsequence. So we let $(x_{n_k})_{k \geq 1}$ be a weakly convergent subsequence of $(x_n)_{n \geq 1}$ since $\overline{B}(0,1)$ is weakly sequentially compact.

What is x? Elaborate on this

Hence $||Tx_{n_k} - Tx|| \not\to 0$ for $k \to \infty$ since we had $||Tx_n - Tx_m|| \geq \varepsilon$ $\forall n \neq m$. I do now get a contradiction and can conclude that T must be compact. ✓

Problem 2d

To show that $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact I start by having $(x_n)_{n\geq 1} \in H$ and $x_n \to x$ weakly for $n \to \infty$. This implies that from 2a that $Tx_n \to Tx$ weakly in $\ell_1(\mathbb{N}))$ and specially say that $Tx_n \to Tx$ weakly iff $||Tx_n - Tx|| \to 0$ by Remark 5.3

I know that $\ell_2(\mathbb{N})$ is Hilbert space, infinite dimensional separable by HW4P4, then can I say that T is compact by 2c. ✓

Problem 2e

I want to show that no $T \in \mathcal{K}(X, Y)$ is onto.
So I start by assuming that T is onto, which is contradiction. I know that X, Y are infinite dimensional Banach space. I let $T \in \mathcal{L}(X, Y)$ be compact and open. Why does onto ⇒ open?
Since I have that X, Y are Banach spaces, are they also normed vector spaces, and since T is open, I will get that $\exists r > 0$ for which $B_Y(0,1) \subset T(B_X(0,1))$, the same with the closure $\overline{B_Y(0,1)} \subset \overline{T(B_X(0,1))}$. So I said that T is compact which gives us that $\overline{B_Y(0,1)}$ is also compact. I will now examine if $\overline{B_Y(0,1)}$ is compact for different values of r.

This is badly written.

I look at $r = 1$ and get that: $\overline{B_Y(0,1)} = \overline{B_Y(0,1)}$, i.e $\overline{B_Y(0,1)}$ is compact, but it cannot be compact since the unit ball of finite dimensional normed space Y is never compact by Riezs lemma.

For $r > 1$ I get $\overline{B_Y(0,1)}$ is a closed subset of the set $\overline{B_Y(0,1)}$ which is compact, so here I also get that $\overline{B_Y(0,1)}$ is compact which is a contradiction as before.

What is f?

For $r < 1$ I look at a continuous function $f : Y \to Y$. We know that the image $f\overline{B_Y(0,1)}$ under a continuous function of a compact set $\overline{B_Y(0,1)}$ is compact, i.e we have $f\overline{B_Y(0,1)} = \overline{B_Y(0,1)}$ is compact, but this is again a contradiction as before. ✓

I can now conclude that no $T \in \mathcal{K}(X, Y)$ can be onto.

Problem 2f

To show that M is self-adjoint, i.e $M = M^*$, I start by defining $t = \bar{t}$ because t can only have real values.

?

I will now look at the inner product on H, where $f, g \in H$

$$\begin{aligned}
\langle Mf, g\rangle &= \int_{[0,1]} Mf(t)(\overline{g(t)})dm(t) \\
&= \int_{[0,1]} tf(t)(\overline{g(t)})dm(t) \\
&= \int_{[0,1]} f(t)\bar{t}(\overline{g(t)})dm(t) \\
&= \int_{[0,1]} f(t)\overline{tg(t)}dm(t) \\
&= \int_{[0,1]} f(t)\overline{Mg(t)}dm(t) \\
&= \langle f, Mg\rangle
\end{aligned}$$

the adjoint M^*.

The definition of self-adjoint is that $\langle Mf, g\rangle = \langle f, M^*g\rangle$ and we have now shown $\langle Mf, g\rangle = \langle f, Mg\rangle$, ✓ hence we may have that $\langle f, Mg\rangle = \langle f, M^*g\rangle$ where $M = M^*$.

To show that M is not compact, I start by assuming that M is compact and show it by contradiction. I have just shown that M is self-adjoint, and I know that H is finite dimensional, and separable by HW4P4, so I now have that H has an ONB $(e_n)_{n\geq 1}$ consisting of eigenvectors for M with corresponding values $\lambda_n \in \mathbb{N}$ by thm 10.1. But in HW6P3 we have shown that M has no eigenvalues, so now there is a contradiction with our assumption. So I can now conclude that M is not compact.

infinite. ✓

Problem 3a

Since $[0, 1]$ is compact Hausdorff spaces, and since m is lebesgue-measure on Borel-sigma-algebra is it finite Borel-measure on $[0, 1]$ and since we know that K is continuous on $[0, 1] \times [0, 1]$ we will get that $K \in C([0, 1] \times [0, 1])$. I can now use theorem 9.6 to conclude that T is compact.

only if you show $T = T_K$ in fact $T = T_{\tilde{K}}$ $\tilde{K}(s,t) = K(t,s)$

check at least $s \to t$

Problem 3b

I will use Tonelli-Fubini to show that $T = T^*$

$$
\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} Tf(s)\overline{g(s)}dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s) \\
&= \int_{[0,1]} \Big(\int_{[0,1]} K(s,t)f(t)dm(t) \Big) \overline{g(s)}dm(s) \\
&= \int_{[0,1]\times[0,1]} K(s,t)f(t)\overline{g(s)}dm(s,t) \\
&= \int_{[0,1]\times[0,1]} K(t,s)\overline{g(s)}f(t)dm(t,s) \\
&= \int_{[0,1]} \Big(\int_{[0,1]} K(t,s)\overline{g(s)}dm(s) \Big) f(t)dm(t) \\
&= \int_{[0,1]} T\overline{g(t)}f(t)dm(t) \\
&= \langle f, Tg\rangle
\end{aligned}
$$

say $K(s,t) = K(t,s)$

you want $\overline{Tg(t)}$

I how have shown what I wanted.

Problem 3c

It is given how $Tf(s)$ and $K(s,t)$ is defined,so I use this to show

$$Tf(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)$$

$$
\begin{aligned}
Tf(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)
\end{aligned}
$$

The first part is now shown.

We know that $||f||_2 < \infty$ since we know that $f \in L_2([0,1], m)$ then we will see that

$$\Big(\int_{[0,1]} |f|^2 dm(t)\Big)^{\frac{1}{2}} < \infty$$

we will now get How?

$$\Big((1-s)\int_{[0,1]} |tf(t)|^2 dm(t)\Big)^{\frac{1}{2}} < \infty$$

and then we will have that

$$(1-s)\int_{[0,1]} tf(t)dm(t) < \infty$$

The same applies for the other integral

$$\Big(s\int_{[0,1]} |(1-t)f(t)|^2 dm(t)\Big)^{\frac{1}{2}} < \infty$$

then we will get

$$s\int_{[0,1]} (1-t)f(t)dm(t) < \infty$$

This does not make sense.

I can now conclude that Tf is continuous on $[0,1]$ by proposition 1.10 since Tf is bounded

Tf bounded does not imply cont.

only for linear operators not (non-linear) functions.

The next I have to show is $(Tf)(0) = (Tf)(1) = 0$.
For $s = 0$ I get:

$$(Tf)(0) = (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]} (1-t)f(t)dm(t) = \int_{[0,0]} tf(t)dm(t) = 0$$

For $s = 1$ I get:

$$(Tf)(1) = (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[0,1]} (1-t)f(t)dm(t) = \int_{[1,1]} (1-t)f(t)dm(t) = 0$$

Hence I get $(Tf)(0) = (Tf)(1) = 0$

✓

Problem 4a

First part:
I will justify that $g_k \in \mathscr{S}(\mathbb{R})$
I notice that $e^{-x^2} \in \mathscr{S}(\mathbb{R})$ from HW7P1 since $e^{-||x||^2} = e^{-x^2}$
Now I note that $(S_a f)(x) := f(\frac{x}{a})$ from lecture notes p.62 and then can I by HW7P1 say $S_{\sqrt{2}} e^{-x^2} \in \mathscr{S}(\mathbb{R})$, hence $e^{\frac{-x^2}{2}} \in \mathscr{S}(\mathbb{R})$ and can conclude that $x^\alpha e^{\frac{-x^2}{2}} \in \mathscr{S}(\mathbb{R})$ and hence we finish by getting $g_k \in \mathscr{S}(\mathbb{R})$.

Next part is to compute $\mathcal{F}(g_k)$ for $k = 0, ..., 3$
I start by letting $\phi(x) := e^{\frac{-x^2}{2}}$, and noting that both $e^{\frac{-x^2}{2}}$ and $x^k e^{\frac{-x^2}{2}}$ are integrable. Then ~~is~~ $\phi(x) = \hat{\phi}(x)$ by proposition 11.4. From this we get

$$\mathcal{F}(g_k)(\xi) = \hat{g_k}(x) = (x^k \phi)(\xi) = i^k(\partial^k \hat{\phi})(\xi) = i^k(\partial^k \phi)(\xi)$$

The first equality is from definition 11.1, because we have from HW7P1c that $\mathscr{S} \subset L_p$, so $f, x^\alpha f \in L_1(\mathbb{R}^n)$, the third equality is by proposition 11.3d where the argument is the same as before.

So for $k = 0$ we get:
$g_0 := \mathcal{F}(g_0)(\xi) = i^0(\partial^0 \phi)(\xi) = e^{\frac{-\xi^2}{2}}$

For $k = 1$ we get:
$g_1 := \mathcal{F}(g_1)(\xi) = i^1(\partial^1 \phi)(\xi) = -i\xi e^{\frac{-\xi^2}{2}}$

For $k = 2$ we get:
$g_2 := \mathcal{F}(g_2)(\xi) = i^2(\partial^2 \phi)(\xi) = i^2 e^{\frac{-\xi^2}{2}}(\xi^2 - 1)$

For $k = 3$ we get:
$g_3 := \mathcal{F}(g_3)(\xi) = i^3(\partial^3 \phi)(\xi) = i^3 e^{\frac{-\xi^2}{2}}(-\xi)(\xi^2 - 3) = i\xi^3 e^{\frac{-\xi^2}{2}} - 3i\xi e^{\frac{-\xi^2}{2}}$

Problem 4b

For $h_0 \in \mathscr{S}(\mathbb{R})$ I will show $\mathcal{F}(h_0) = i^0 h_0$

$$\mathcal{F}(g_0) = e^{\frac{-\xi^2}{2}} = i^0 h_0 = g_0$$

For $h_1 \in \mathscr{S}(\mathbb{R})$ I will show $\mathcal{F}(h_1) = i h_1$
I will start by looking at $\mathcal{F}(g_3)(\xi)$

$$\mathcal{F}(g_3)(\xi) = i(\xi^3 e^{\frac{-\xi^2}{2}} - 3\xi e^{\frac{-\xi^2}{2}}) = i(g_3(\xi) - 3g_1(\xi))$$

so then I have by linearity of Fourier transform that

$$\begin{aligned}\mathcal{F}(g_3 - \frac{3}{2}g_1)(\xi) &= \mathcal{F}(g_3)(\xi) - \frac{3}{2}\mathcal{F}(g_1)(\xi) \\ &= i(g_3(\xi) - 3g_1(\xi)) + \frac{3}{2} i\xi^{\frac{-\xi^2}{2}} \\ &= i(g_3(\xi) - \frac{3}{2}g_1(\xi))\end{aligned}$$

Hence I get that $h_1 = (g_3(\xi) - \frac{3}{2}g_1(\xi))$ and then $\mathcal{F}(h_1) = ih_1$

For $h_2 \in \mathscr{S}(\mathbb{R})$ I will show $\mathcal{F}(h_2) = i^2 h_2 = -h_2$

$$\mathcal{F}(g_2)(\xi) = -(g_2(\xi) - g_0(\xi))$$

so then I have by linearity of Fourier transform that

$$\begin{aligned}\mathcal{F}(g_2 - \frac{1}{2}g_0)(\xi) &= \mathcal{F}(g_2)(\xi) - \frac{1}{2}\mathcal{F}(g_0)(\xi) \\ &= -(g_2(\xi) - g_0(\xi)) - \frac{1}{2}\mathcal{F}(g_0)g_0(\xi) \\ &= -(g_2(\xi) - \frac{1}{2}g_0(\xi))\end{aligned}$$

Therefore is $h_2 = (g_2 - \frac{1}{2}g_0)$ and hence I have that $\mathcal{F}(h_2) = i^2 h_2 = -h_2$

For $h_3 \in \mathscr{S}(\mathbb{R})$ I will show $\mathcal{F}(h_3) = i^3 h_3 = -ih_3$

$$\mathcal{F}(g_1)(\xi) = -i\xi e^{\frac{-\xi^2}{2}} = -ig_1(\xi)$$

Therefore I get $h_3 = g_1$ and hence $\mathcal{F}(h_3) = i^3 h_3 = -ih_3$

Problem 4c

I want to show that $\mathcal{F}^4(f) = f$.
I know that $\mathscr{S}(\mathbb{R}) \subseteq L_1(\mathbb{R})$, and $f, \hat{f} \in L_1(\mathbb{R})$.
I will now look at $\mathcal{F}$:

$$\begin{aligned}\mathcal{F}(f)(\xi) &= \hat{f}(\xi) \\ &= \int_{\hat{\mathbb{R}}} f(x)e^{-ix\xi}dm(x) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}d(x)\end{aligned}$$

So then I will get

$$\mathcal{F}^*(f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}d(x) \quad \%$$

I will now look at $\mathcal{F}^2$

$$\begin{aligned}\mathcal{F}^2(f)(\xi) &= \mathcal{F}(\mathcal{F}(f)(\xi)) \\ &= \mathcal{F}(\hat{f}(\xi)) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(x)e^{-ix\xi}d(x)\end{aligned}$$

I will now define $\mathcal{T}(f) := S_{-1}f \in \mathscr{S}(\mathbb{R})$, and look at $\mathcal{T}^2$

$$(\mathcal{T}^2 f)(x) = \mathcal{T}(\mathcal{T}f)(x) = (\mathcal{T}f)(-x) = f(x)$$

we note that $f = \mathcal{F}^*\mathcal{F}(f)$ from corollary 12.12 so therefore I get

$$\begin{aligned}(\mathcal{T}f)(\xi) &= \mathcal{F}^*(\mathcal{F}(f)(-\xi)) \\ &= \mathcal{F}^*(\hat{f})(-\xi) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(x)e^{-ix\xi}d(x) \\ &= \mathcal{F}^2(f)(\xi)\end{aligned}$$

does not match above.

I will now look at $\mathcal{F}^4$ where I use $\tau(f)$
$\mathcal{F}^4 = (\mathcal{F}^2)^2 = f$
So the conclusion is that I have shown $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$ ✓

Problem 4d

I want to show that $\lambda \in \{1, i, -1, -i\}$, and it will be enough to show that $\lambda^4 = 1$. I start by supposing that $f \neq 0$. It is given that $\lambda f = \mathcal{F}(f)$, so I will get that $\lambda^4 f^4 = \mathcal{F}^4(f) = f$ and then I will have that $\lambda^4 = \frac{f}{f^4}$.

I notice that in 4c I showed that $\mathcal{F}^4(f) = f$ and this will now give me

$$f^2 = \mathcal{F}^8(f) = \mathcal{F}^4(\mathcal{F}^4(f)) = \mathcal{F}^4(f) = f$$

and hence $f^4 = (f^2)^2 = f^2 = f$. ? $\mathcal{F}^8(f) = \mathcal{F}^4(\mathcal{F}^4(f)) = \mathcal{F}^4(f) = f$.
I can now say that $\lambda^4 = \frac{f}{f^4} = \frac{f}{f} = 1$.
f need not be non-zero everywhere!

I can now see that $\lambda = 1$, $\lambda = -1$, $\lambda = i$ and also $\lambda = -i$, so now I have that these are the only values for which $\lambda f = \mathcal{F}(f)$. Therefore will the eigenvalues of $\mathcal{F}$ be $\{1, i, -1, -i\}$. no $\lambda \in \{1, -1, i, -i\}$

Problem 5

I want to show that $\text{supp}(\mu) = [0, 1]$. To do this I start by noting that it is given that μ is a Radon measure on $[0, 1]$ which is LCHT-space. I recall from HW8P3 where it will be enough for me to show that $\mu([0, 1]^c) = 0$. No
It is given that $\mu([0, 1]) = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}([0, 1])$ and know that This will only show that $[0,1]^c \subseteq \text{supp}(\mu)^c$; hence $\text{supp}(\mu) \subseteq [0,1]$, which is trivial.

$$\delta_{x_n}([0, 1]^c) = \begin{cases} 0 & x \in [0, 1]^c \\ 1 & x \notin [0, 1]^c \end{cases}$$

I will now get from HW8P3 that

$$\mu([0, 1]^c) = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}([0, 1]^c) = 0$$

From this can I conclude that $\text{supp}(\mu) = [0, 1]$ since $x_n \in [0, 1]$. 0/0