# From integers to real numbers

## eIUP lecture

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# Overview

# ILOs

After this lecture the student will be able to

- Define and construct the natural numbers, the integers and the rational numbers.
- Understand the concept of a *proof by contradiction*.
- Show that $\sqrt{2}$ is not a rational number.
- If time permits; be familiar with the construction of real numbers as the completion of the rational numbers.

# Naturals and integers

## Natural numbers

The *natural numbers* are defined as the infinite *totally ordered set*

$$\mathbb{N} = \{0, 1, 2, 3, ...\}. \tag{1}$$

These come defined with two compatible binary operations: addition and multiplication denoted by $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ respectively.

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We also define the set

$$\mathbb{N}_\times = \mathbb{N} \setminus \{0\} = \{1, 2, 3, ...\}. \tag{2}$$

# Naturals and integers

## Integers

Requiring all naturals to have additive inverses we obtain integers

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With addition and (extended to the negatives) multiplication.
We define the even and odd integers by

$$\mathbb{Z}^{\text{even}} = \{..., -4, -2, 0, 2, 4, ...\}, \quad \mathbb{Z}^{\text{odd}} = \{..., -3, -1, 1, 3, ...\}, \tag{4}$$

such that $\mathbb{Z} = \mathbb{Z}^{\text{even}} \cup \mathbb{Z}^{\text{odd}}$.

# The rational numbers

## Construction of rational numbers

By requiring all integers to have multiplicative inverses and extending addition to these inverses we obtain the rational numbers

$$\mathbb{Q} = \left\{ \frac{n}{m} \;\middle|\; n \in \mathbb{Z},\ m \in \mathbb{N}_\times \right\}. \tag{5}$$

## Reduced fractions

We notice that all rational numbers can be written in a reduced form $\frac{n}{m}$ where $n \in \mathbb{Z}$ and $m \in \mathbb{N}_\times$ have no common divisors.

# Square-roots and rational numbers

## The square-root

We say that $a = \sqrt{b}$ if and only if $a > 0$ and $a^2 = b$, *i.e.*

$$\sqrt{b}^2 = \sqrt{b} \cdot \sqrt{b} = b. \tag{6}$$

## Square-roots and rational numbers

**Question** (we will answer shortly together): *If $a$ is a rational number, is $\sqrt{a}$ then also always a rational numbers?*
*e.g.* $\sqrt{\frac{9}{16}} = \frac{3}{4}$, $\sqrt{\frac{1}{144}} = \frac{1}{12}$, *etc.* What about $\sqrt{2}$?

# Proof by contradiction

## Reductio ad absurdum

A proof by contradiction is a mathematical proof technique, that takes the following form: Let A be a logic statement,

- Assume A is true.
- Prove that A implies B.
- Prove that A implies -B (not B *or* opposite of B).
- Conclude that A is false (*i.e.* -A is true).

*e.g.* We want to show that there is no highest natural number: Thus A="There is a largest natural number (say $N$)". B="All natural numbers are smaller than $N$". However, if $N$ is a natural number then $N+1$ is a natural number and $N+1>N$ which proves -B . Thus we may conclude that -A is true.

# Is $\sqrt{2}$ a rational number?

**Theorem**

*$\sqrt{2}$ is not a rational number.*

**Proof**

- Assume that (A) $\sqrt{2}$ is a rational number.
- Then (B) $\sqrt{2} = \frac{a}{b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}_{\times}$ (can assume reduced).
- Notice that then $\frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2 = 2$.
- Continued on next slide...

# Is $\sqrt{2}$ a rational number?

## Proof continued

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- Knowing that $\frac{a^2}{2}$ is even, is $b^2$ even or odd? (**Even** since $b^2 = \frac{a^2}{2}$. Notice then $b$ is also **Even**.)
- We go to breakout rooms and finish the proof!

# Is $\sqrt{2}$ a rational number?

Go to Padlet

`https://padlet.com/johannesas/wrgamlv82s844qfs`

## Proof continued

Questions for Padlet:

1. Knowing that $a$ and $b$ are even, is $\frac{a}{b}$ a reduced fraction?
2. What may we then conclude about our initial assumption that $\sqrt{2}$ is a rational numbers?

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My answers:

1. No, $\frac{a}{b}$ is not reduced since $2$ is a common divisor of $a$ and $b$.
2. By reductio ad absurdum, we may conclude that $\sqrt{2}$ is **not** a rational number.

# The real numbers

## What number is $\sqrt{2}$?

- Representing $\sqrt{2}$ by decimal number $\sqrt{2} = 1.414...$ we find that $\sqrt{2}$ can always be approximated by a rational number by simply cutting off the decimals at some point, *e.g.* $1.41 = \frac{141}{100}$ and $\sqrt{2} - 1.41 < 0.01$.
- We call all numbers that can be approximated by rational numbers in such a way for real numbers, and we say that they are obtained by completing the rational numbers.

## Fun fact!

There are equally many natural numbers, integers, and rational numbers but the real numbers form a substantially larger set. If you were to say how big a percentage of the real numbers the rational numbers make up, it is $0\%$

Thank you for your attention.