Notes on 1D bosons

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We consider the dilute bose gas in one dimension, where we seek to prove the formula for the ground state energy

$$\frac{E}{L} = \frac{\pi^2}{3}\rho^3\left(1 + 2\rho a + \mathcal{O}\left((\rho a)^2\right)\right). \tag{0.1}$$

We assume that the interaction potential v has compact support, say in the ball of radius b, B_b.

1 Upper bound

We provide the upper bound for (0.1), by using the variational principle with a suitable trial state. Consider the trial state

$$\Psi(x) = \begin{cases} \omega(\mathcal{R}(x))\frac{\Psi_F(x)}{\sin\left(\frac{\pi}{L}\mathcal{R}(x)\right)} & \text{if } \mathcal{R}(x) < b, \\ \Psi_F(x) & \text{if } \mathcal{R}(x) \geq b, \end{cases} \tag{1.1}$$

where ω is the suitably normalized solution to the two-body scattering equation, *i.e.* $\omega(x) = f(x)\frac{\sin\left(\frac{\pi}{L}b\right)}{f(b)}$ where f is any solution of the two-body scattering equation. $\Psi_F(x) = \mathcal{N}^{1/2}\prod_{i<j}^N \sin\left(\frac{\pi}{L}(x_i - x_j)\right)$ is the free fermionic ground state, and $\mathcal{R}(x) = \min_{i<j}(|x_i - x_j|)$.
The energy of this trial state is then

$$\mathcal{E}(\Psi) = \int \sum_{i=1}^N |\nabla_i \Psi|^2 + \sum_{i<j}^N v_{ij}|\Psi|^2, \tag{1.2}$$

where $v_{ij}(x) = v(x_i - x_j)$. Since v is supported in B_b and $\Psi = \Psi_F$ except in the region $B = \{x \in \mathbb{R}^N | \mathcal{R}(x) < b\}$, we may rewrite this as

$$\mathcal{E}(\Psi) = E_0 + \int_B \sum_{i=1}^N |\nabla_i \Psi|^2 + \sum_{i<j}^N v_{ij}|\Psi|^2 - \sum_{i=1}^N |\nabla_i \Psi_F|^2, \tag{1.3}$$

where $E_0 = N\frac{\pi^2}{3}\rho^2$ is the ground state energy of the free Fermi gas.