Functional Analysis - Mandatory Assignment 2

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Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set

$$f_N = N^{-1}\sum_{n=1}^{N^2} e_n$$

for all $N \geq 1$.

(a)

Show that $f_N \to 0$ weakly, as $N \to \infty$, while $||f_N|| = 1$, for all $N \geq 1$. Let K be the norm closure of $co\{f_N : N \geq 1\}$.

We first show that $||f_N|| = 1$ for all $N \geq 1$.

$$\begin{aligned}
||f_N||^2 &= \langle f_N, f_N\rangle \\
&= \langle N^{-1}\sum_{n=1}^{N^2} e_n, N^{-1}\sum_{k=1}^{N^2} e_k\rangle \\
&= N^{-2}\sum_{n=1}^{N^2}\langle e_n, \sum_{k=1}^{N^2} e_k\rangle \\
&= N^{-2}\sum_{n=1}^{N^2} 1 \\
&= N^{-2}\cdot N^2 \\
&= 1.
\end{aligned}$$

Be more explicit here.

(✓)

Thus $||f_N|| = 1$ as desired.

To show that $f_N \to 0$ weakly as $N \to \infty$ we use problem 2 in HW4. So we have to show that $(g(f_N))_{N\geq 1}$ converges to $g(0)$ for every $g \in H^*$.

Note this holds for all $x \in H$ and not just for f_N.

Thus let $\varepsilon > 0$. By the Riesz representation theorem each $g \in H^*$ is of the form F_y, where $F_y(f_N) = \langle f_N, y\rangle$ for some $y \in H$. Thus we have to show that $\langle f_N, y\rangle \to 0$ as $N \to \infty$ for all $y \in H$.
Recall that we can write

$$y = \sum_{i=1}^{\infty} \langle y, e_i\rangle e_i$$

for any $y \in H$ (5.27 Folland). We also have Parceval's identity

$$||y||^2 = \sum_{i=1}^{\infty} |\langle y, e_i\rangle|^2.$$

As this sum converges, then for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$\sum_{i=m+1}^{\infty} |\langle y, e_i\rangle|^2 < \frac{\varepsilon^2}{4}.$$

Using this we can use the Pythagorean Theorem (5.23 Folland) to show that

$$\begin{aligned}
||\sum_{i=m+1}^{\infty} \langle y, e_i\rangle e_i|| &= \left(||\sum_{i=m+1}^{\infty} \langle y, e_i\rangle e_i||^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=m+1}^{\infty} ||\langle y, e_i\rangle e_i||^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=m+1}^{\infty} |\langle y, e_i\rangle|^2\right)^{\frac{1}{2}} \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Now we look at $\langle f_N, y\rangle$. We have

$$\begin{aligned}
\langle f_N, y\rangle &= \langle f_N, \sum_{i=1}^{\infty} \langle y, e_i\rangle e_i\rangle \\
&= \langle f_N, \sum_{i=1}^{m} \langle y, e_i\rangle e_i\rangle + \langle f_N, \sum_{i=m+1}^{\infty} \langle y, e_i\rangle e_i\rangle.
\end{aligned}$$

We first look at the second term, where we can use Cauchy-Schwarz to get

$$|\langle f_N, \sum_{i=m+1}^{\infty} \langle y, e_i\rangle e_i\rangle| \leq ||f_N||||\sum_{i=m+1}^{\infty} \langle y, e_i\rangle e_i|| < \frac{\varepsilon}{2}.$$

Now we consider the first term.

$$\begin{aligned}
|\langle f_N, \sum_{i=1}^{m} \langle y, e_i\rangle e_i\rangle| &= |\langle N^{-1} \sum_{n=1}^{N^2} e_n, \sum_{i=1}^{m} \langle e_i, y\rangle e_i\rangle| \\
&= |N^{-1} \sum_{i=1}^{m} \sum_{n=1}^{N^2} \langle e_n, \langle y, e_i\rangle e_i\rangle| \\
&= |N^{-1} \sum_{i=1}^{m} \langle y, e_i\rangle \sum_{n=1}^{N^2} \langle e_n, e_i\rangle| \\
&\leq N^{-1} |\sum_{i=1}^{m} \langle y, e_i\rangle| \\
&\leq N^{-1} \sum_{i=1}^{m} |\langle y, e_i\rangle|.
\end{aligned}$$

Should be $\overline{\langle y, e_i\rangle}$ if using antilinearity in second entry.

Where we used that

$$\sum_{n=1}^{N^2} \langle e_n, e_i\rangle \leq 1$$

and that the absolute value is a norm. Now we have that

$$\sum_{i=1}^{m} |\langle y, e_i\rangle| = M$$

for some $M \in \mathbb{R}_+$ as it is a finite sum. As $N^{-1} \to 0$ for $N \to \infty$ there exists some $Q \in \mathbb{N}$ such that $N^{-1} < \frac{\varepsilon}{2M}$ for all $N > Q$. Thus for $N > Q$ we have that

$$|\langle f_N, \sum_{i=1}^{m} \langle y, e_i\rangle e_i\rangle| \leq N^{-1} M < \frac{\varepsilon}{2}.$$

Thus we get for $N > Q$

$$|\langle f_N, y\rangle| \leq |\langle f_N, \sum_{i=1}^{m} \langle y, e_i\rangle e_i\rangle| + |\langle f_N, \sum_{i=m+1}^{\infty} \langle y, e_i\rangle e_i\rangle| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We have shown $|\langle f_N, y\rangle| \to 0$ as $N \to \infty$ for all $y \in H$ and hence we have shown $f_N \to 0$ weakly for $n \to \infty$. ✓

(b)

Argue that K is weakly compact, and that $0 \in K$.

Set $A := \{f_N | N \geq 1\}$. By proposition 2.10 we have that H is reflexive as it is a Hilbert space. Then Theorem 6.3 gives that $\overline{B}_H(0, 1)$ is weakly compact. We

want to show that K is a closed subset in the weak topology of $\overline{B}_H(0,1)$ as then also K is weakly compact.
Theorem 5.7 gives that K is the weak closure of $\mathrm{co}(A)$, hence we want to show that $\mathrm{co}(A) \subset \overline{B}_H(0,1)$ as then $K \subset \overline{B}_H(0,1)$ by 5(c) in mandatory assignment 1. Let $x = \sum_{i=1}^n \alpha_i f_{N_i} \in \mathrm{co}(A)$. Then we have that

$$||x|| \leq \sum_{i=1}^n \alpha_i ||f_{N_i}|| = \sum_{i=1}^n \alpha_i = 1.$$

Thus $x \in \overline{B}_H(0,1)$ as desired. Thus $K \subset \overline{B}_H(0,1)$ and K is a weakly closed subset of a weakly compact set, and hence K is weakly compact as desired. ✓

Note that in (a) we saw that $f_N \to 0$ weakly as $N \to \infty$, hence as K is the weak closure of $\mathrm{co}(A)$ we have $0 \in K$ as desired. ✓

(c)

Show that 0, as well as each f_N for $N \geq 1$, are extreme points in K.

Let us first show that 0 is an extreme point.
Suppose $0 = \alpha x + (1-\alpha)y$ for $0 < \alpha < 1$ and $x, y \in K$. Then as K is the (norm and weak) closure of $\mathrm{co}(A)$ we have sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n \geq 1}$ in $\mathrm{co}(A)$ converging to x and y respectively.
Then as taking the inner product is continuous we have for any $k \geq 1$

$$\begin{aligned}
\langle x, e_k \rangle &= \langle \lim_{n\to\infty} x_n, e_k \rangle \\
&= \lim_{n\to\infty} \langle \sum_{s=1}^p \beta_s f_{N_s}, e_k \rangle \\
&= \lim_{n\to\infty} \sum_{s=1}^p \beta_s \langle N_s^{-1} \sum_{m=1}^{N_s^2} e_m, e_k \rangle \\
&= \lim_{n\to\infty} \sum_{s=1}^p \beta_s N_s^{-1} \sum_{m=1}^{N_s^2} \langle e_m, e_k \rangle.
\end{aligned}$$

Careful with the notation, now it looks as though there is no dependence on n.

This is not what you obtain from above.

Thus $\langle x_n, e_k \rangle \geq 0$ for all n as we are taking sums consisting of non-negative terms. Hence we also have $\langle x, e_k \rangle \geq 0$.
Similarly we have that $\langle y, e_k \rangle \geq 0$ for all $k \geq 1$. Now we have

$$0 = \langle 0, e_k \rangle = \langle \alpha x + (1-\alpha)y, e_k \rangle = \alpha \langle x, e_k \rangle + (1-\alpha)\langle y, e_k \rangle$$

which implies that $\langle x, e_k \rangle = \langle y, e_k \rangle = 0$ as $\alpha > 0$. Hence by completeness (5.27 Folland) $x = y = 0$ as desired and 0 is an extreme point for K. ✓

We now show that f_N is an extreme point for any $N \geq 1$.
Let $f_N = \alpha x + (1-\alpha)y$ for $x, y \in K$ and $0 < \alpha < 1$. As $x, y \in K$ we have

from (b) that $||x||, ||y|| \leq 1$. Now we can use Cauchy-Schwarz to get $|\langle f_N, x\rangle| \leq ||f_N|| ||x|| \leq 1$ and similarly $|\langle f_N, y\rangle| \leq 1$. Then by using the triangle inequality we get

$$\begin{aligned} 1 &= |\langle f_N, f_N\rangle| \\ &= |\langle f_N, \alpha x + (1-\alpha)y\rangle| \\ &\leq \alpha|\langle f_N, x\rangle| + (1-\alpha)|\langle f_N, y\rangle| \end{aligned}$$

and as $\alpha > 0$ this implies that $|\langle f_N, x\rangle| = |\langle f_N, y\rangle| = 1$.
Now using Cauchy-Schwarz and then the triangle inequality yields

$$\begin{aligned} 1 &= |\langle f_N, f_N\rangle| \\ &= |\langle f_N, \alpha x + (1-\alpha)y\rangle| \\ &\leq ||f_N|| ||\alpha x + (1-\alpha)y|| \\ &\leq \alpha||x|| + (1-\alpha)||y|| \end{aligned}$$

and as above we get $||x|| = ||y|| = 1$.

Recall that Cauchy-Schwarz is an equality iff the two elements it is used on are linearly dependent.
Note that we have $1 = |\langle f_N, x\rangle| = ||f_N|| ||x|| = ||f_N|| ||y|| = |\langle f_N, y\rangle|$. Hence there exists s, t such that $f_N = sx = ty$. But then

$$|s| = |s| ||x|| = ||sx|| = ||ty|| = |t| ||y|| = |t|$$

and

$$1 = ||f_N|| = ||sx|| = |s| = |t|.$$

What if s,t are complex numbers?

Now if $s = t = -1$ we have $-f_N = x = y$ and thus $f_N = -\alpha f_N - (1-\alpha)f_N = -f_N$ implying $f_N = 0$ which is a contradiction. Hence $s = t = 1$ and thus $f_N = x = y$ and f_N is an extreme point as desired. (✓)

(d)

If K was not convex, then it could not have extreme points by definition...

Are there any other extreme points in K? Justify your answer.

We want to show that K is convex. Let $x, y \in K$. Is $\alpha x + (1-\alpha)y \in K$ for all $\alpha \in [0, 1]$?
As $x, y \in K$, we have sequences $(x_n)_{n\geq 1}$ and $(y_n)_n \geq 1$ in $\text{co}(A)$ converging to x and y respectively. Thus we have that the sequence $(w_n)_{n\geq 1}$ defined by

$$w_n := \alpha x_n + (1-\alpha)y_n$$

converges to $\alpha x + (1-\alpha)y$. However it is a sequence in $\text{co}(A)$ as that set is convex and $x_n, y_n \in \text{co}(A)$. Thus also $\alpha x + (1-\alpha)y \in K$ as desired.

Thus as H is a Hilbert space in particular it is a LCTVS and hence we can use Theorem 7.9 (Milman) to get that $\text{Ext}(K) \subset \overline{A}^{\tau_w}$.

Let $x \in \overline{A}^{\tau_w}$, such that $x \notin A \cup \{0\}$. Then there exists a sequence $(a_n)_{n\geq 1}$ in A converging to x. As $x \notin A$ none of the f_N occurs infinitely many times in the sequence, hence infinitely many different elements of A occurs in the sequence converging to x.

In (a) we showed that $f_N \to 0$ weakly for $N \to \infty$, that means for every neighbourhood U of 0, the sequence f_N is eventually in U. In particular this means that for every neighbourhood of 0, there are only finitely many of the elements in A that are not in the neighbourhood. Thus for any neighbourhood U of 0, we have that the sequence $(a_n)_{n\geq 1}$ has infinitely many elements that are in U as there are only finitely many elements of A that are not in U and $(a_n)_{n\geq 1}$ has infinitely many different elements. Thus 0 is a cluster point for $(a_n)_{n\geq 1}$, but as the weak topology is Hausdorff, this contradicts that the sequence converges to x. $\lightning$ ✓

Hence $\overline{A}^{\tau_w} = A \cup \{0\}$ and thus there are no more extreme points in K.

Problem 2

Let X and Y be infinite dimensional Banach spaces.

(a)

Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

By problem 2 in HW4 we have to show that $F(Tx_n) \to F(Tx)$ as $n \to \infty$ for all $F \in Y^*$.
Let $F \in Y^*$, then $F \circ T \in X^*$, hence as $x_n \to x$ weakly for $n \to \infty$, problem 2 in HW4 gives that $F \circ T(x_n) \to F \circ T(x)$ for $n \to \infty$, but that was exactly what we wanted to show. Hence $Tx_n \to Tx$ weakly as $n \to \infty$. ✓

(b)

Let $T \in \mathcal{K}(x, y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $||Tx_n - Tx|| \to 0$, as $n \to \infty$.

As $(x_n)_{n\geq 1}$ converges weakly to x problem 2 from HW4 gives that the sequence is bounded. In particular any subsequence $(x_{n_k})_{k\geq 1}$ is also bounded. Then Proposition 8.2 gives that there exists a subsequence $(x_{n_{k_i}})_{i\geq 1}$ of the subsequence $(x_{n_k})_{k\geq 1}$ such that $||Tx_{n_{k_i}} - y|| \to 0$ for $i \to \infty$. Hence it also converges weakly to y, but from (a) we have that it also converges weakly to Tx and as the weak topology is Hausdorff we have that $y = Tx$. Hence any subsequence of $(x_n)_{n\geq 1}$

has a subsequence, such that using T yields a sequence that converges in norm to Tx.

To show that $(Tx_n)_{n\geq 1}$ converges to Tx in norm, suppose for contradiction that it does not. Then there exists an $\varepsilon > 0$ such that for any positive integer N there exists an $s_N \geq N$ such that $||Tx_{s_N} - Tx|| \geq \varepsilon$.
Now let N run through the positive integers and for each number pick the smallest such s_N. That yields a sequence of natural numbers $(s_N)_{N\geq 1}$ such that the sequence $(x_{s_N})_{N\geq 1}$ is a subsequence of $(x_n)_{n\geq 1}$ satisfying $||Tx_{s_N} - Tx|| \geq \varepsilon$ for all $N \geq 1$. However as we have seen above it has a subsequence $(x_{s_{N_j}})_{j\geq 1}$ s.t. $(Tx_{s_{N_j}})_{j\geq 1}$ converges to Tx in norm which is clearly a contradiction. $\lightning$ ✓
Thus $||Tx_n - Tx|| \to 0$ for $n \to \infty$ as desired.

(c)

Let H be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H, Y)$ satisfies that $||Tx_n - Tx|| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

We prove this by contraposition. Thus we assume T is not compact. Then Proposition 8.2 gives that $T(\overline{B}_H(0,1)$ is not totally bounded i.e. there exists an $\varepsilon > 0$ such that $T(\overline{B}_H(0,1))$ can't be covered by a finite union of open ε-balls. We now construct a sequence $(x_n)_{n\geq 1}$ in $\overline{B}_H(0,1)$.
Let x_1 be any element in $\overline{B}_H(0,1)$. Then pick some $y \in T(\overline{B}_H(0,1)) \backslash B_Y(Tx_1, \varepsilon)$. This is possible as $T(\overline{B}_H(0,1))$ is not totally bounded. Then there exists some element $a \in \overline{B}_H(0,1)$ such that $T(a) = y$. Set $x_2 := a$. We continue in this fashion, i.e. if we have defined the first $n-1$ elements of the sequence, we can define x_n as follows; as $T(\overline{B}_H(0,1))$ is not totally bounded there exists an element $y' \in T(\overline{B}_H(0,1))$ such that

$$y' \notin \bigcup_{i=1}^{n-1} B_Y(x_i, \varepsilon).$$

Furthermore there exists $a' \in \overline{B}_H(0,1)$ such that $T(a') = y$. Then set $x_n := a'$. The constructed sequence clearly satisfies that $||Tx_i - Tx_j|| \geq \varepsilon$ for all $i \neq j$. ✓

Now we want to show that the constructed sequence has a convergent subsequence.
To show this we first note that H is reflexive by Proposition 2.10. Now Riesz representation Theorem gives each $F \in H^*$ is of the form F_y with $||F_y|| = ||y||$ and that each $y \in Y$ gives rise to an $F_y \in H^*$ i.e. $y \mapsto F_y$ is an isometric isomorphism. In particular it sends an ONB in H to an ONB in H^*, and hence as H is separable, H^* is also separable (5.29 Folland). Thus Theorem 5.13 gives that $(\overline{B}_H(0,1), \tau_w)$ is metrizable as H is relexive. Now Theorem 6.3 gives that $\overline{B}_H(0,1)$ is weakly compact. But then $\overline{B}_H(0,1)$ is a compact metric space, hence it is sequentially compact i.e. every sequence has a convergent subsequence.

This is an antilinear isomorphism!

in the weak topology

metrisable

In particular the constructed sequence has a convergent subsequence $(x_{n_k})_{k\geq 1}$. It is clear by construction that the corresponding sequence $(Tx_{n_k})_{k\geq 1}$ does not converge to Tx in norm for $k \to \infty$.
Thus we have shown the contrapositive statement.

(d)

Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

Note that $\ell_2(\mathbb{N})$ is a Hilbert space and furthermore that it is separable by problem 4 in HW4. Thus we can use (c).
First let $(x_n)_{n\geq 1}$ be a sequence in $\ell_2(\mathbb{N})$ converging weakly to x. Then by (a) $Tx_n \to Tx$ weakly for $n \to \infty$. Remark 5.3 then gives that $||Tx_n - Tx|| \to 0$ for $n \to \infty$, hence by (c) T is compact as desired.

(e)

Show that no $T \in \mathcal{K}(X, Y)$ is onto.

Suppose that T is onto. Then T is open by the Open Mapping Theorem. Then $T(B_X(0,1))$ is open in Y and thus as T is linear $0 \in T(B_X(0,1))$ and thus there exists $\varepsilon > 0$ such that $B_Y(0,\varepsilon) \subset T(B_X(0,1))$. Also we have that $T(\overline{B}_X(0,1))$ is compact in Y. $\overline{T(B_X(0,1))}$

Consider the map $f : Y \to Y$ defined by $y \mapsto \frac{1}{2\varepsilon}y$. It is clearly continuous and it maps $B_Y(0,\varepsilon)$ to $B_Y(0,2)$, hence $T(\overline{B}_X(0,1))$ is mapped to some compact set containing the closed unit ball of Y. Then the closed unit ball is a closed subset of a compact set, hence it is compact. However this contradicts problem 3(e) in the first mandatory assignment. ↯
Thus T is not onto.

(f)

Let $H = L_2([0,1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$ for $f \in H$ and $t \in [0,1]$. Justify that M is self-adjoint, but not compact.

What are f,g?

$$\langle Mf, g\rangle = \int_{[0,1]} tf(t)g(t)dm(t) = \int_{[0,1]} f(t)tg(t)dm(t) = \langle f, Mg\rangle.$$

Thus M is self-adjoint.
Note now that H is an infinite dimensional vector space, and that it is separable by problem 4 HW4. Thus if T is compact, then the Spectral Theorem gives that H has an ONB consisting of eigenvectors for M, however by problem 3 HW6 M has no eigenvalues. ↯
Thus M is not compact.

Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$ where m is the Lebesgue measure. Define $K : [0,1] \times [0,1] \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t, & 0 \le t \le s \le 1, \\ (1-t)s, & 0 \le s < t \le 1, \end{cases}$$

and consider $T \in \mathcal{L}(H,H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t) f(t) dm(t), \quad s \in [0,1],\ f \in H.$$

(a)

Justify that T is compact.

As $m([0,1]) = 1 < \infty$ m is a finite Borel measure. Note that $|K(s,t)| \le 1$ for $(s,t) \in [0,1] \times [0,1]$, hence by using Tonelli we have

$$\begin{aligned} \int_{[0,1]\times[0,1]} |K(s,t)| d(m \otimes m)(s,t) &= \int_{[0,1]} \int_{[0,1]} |K(s,t)| dm(t) dm(s) \\ &\le \int_{[0,1]} \int_{[0,1]} 1 dm(t) dm(s) \\ &= 1 \\ &< \infty. \end{aligned}$$

Thus $K \in L_2([0,1] \times [0,1], m \otimes m)$ and hence we see $T = T_K$, where T_K is the operator defined in ($\square$) on p. 46 in the notes. Furthermore it is clear that K is continuous by the pasting/gluing lemma as $(1-s)t$ and $(1-t)s$ are both continuous functions.

We also have that $[0,1]$ is compact as it is a closed and bounded subset of $\mathbb{R}$, furthermore it is Hausdorff. Thus Theorem 9.6 gives that T is compact.

$T = T_{\tilde{K}}$

$\tilde{K}(s,t) = K(t,s)$

$(= K(s,t))$

Show this

(b)

Show $T = T^*$.

Note first that $K(s,t) = K(t,s)$. This is immediate from the definition of K. ✓

Let $f, g \in H$, then we have

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} Tf \cdot g dm \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(s,t) f(t) dm(t) \right) g(s) dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) g(s) dm(t) dm(s)
\end{aligned}$$

We want to use Fubini's Theorem so we have to show, that

$$\int_{[0,1]\times[0,1]} |f(t)K(s,t)g(s)| d(m \otimes m)(t,s) < \infty.$$

We use Tonelli's Theorem to show this. Furthermore we use that $|K(s,t)| \leq 1$ for $s, t \in [0,1]$. We also use that $f, g \in L_1([0,1], m)$, which we know from problem 2 HW2.

$$\begin{aligned}
\int_{[0,1]\times[0,1]} |f(t)K(s,t)g(s)| d(m \otimes m)(t,s) &= \int_{[0,1]} \left(\int_{[0,1]} |f(t)K(s,t)g(s)| dm(t) \right) dm(s) \\
&= \int_{[0,1]} \left(\int_{[0,1]} |f(t)K(s,t)| dm(t) \right) |g(s)| dm(s) \\
&\leq \int_{[0,1]} \left(\int_{[0,1]} |f(t)| dm(t) \right) |g(s)| dm(s) \\
&\leq \int_{[0,1]} k |g(s)| dm(s) \\
&\leq kk' \\
&< \infty
\end{aligned}$$

where k and k' are finite as $f, g \in L_1([0,1], m)$. Thus we can use Fubini and we have

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) g(s) dm(t) dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) g(s) dm(s) dm(t) \\
&= \int_{[0,1]} f(t) \left(\int_{[0,1]} K(t,s) g(s) dm(s) \right) dm(t) \\
&= \langle f, Tg\rangle.
\end{aligned}$$

Here we used $K(s,t) = K(t,s)$. Thus T is self-adjoint as desired.

(✓)

(c)

Show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t),\ s\in[0,1],\ f\in H.$$

Use this to show that Tf is continuous on $[0,1]$ and $(Tf)(0) = (Tf)(1) = 0$.

To show this we note that K is a piecewise-defined function. Thus we get

$$\begin{aligned}(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t)\\ &= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t)\\ &= \int_{[0,s]}(1-s)tf(t)dm(t) + \int_{[s,1]} s(1-t)f(t)dm(t)\\ &= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t)\end{aligned}$$

as desired.
We now have

$$\begin{aligned}Tf(0) &= 1\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]}(1-t)f(t)dm(t) = 0+0 = 0.\\ Tf(1) &= 0\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]}(1-t)f(t)dm(t) = 0+0 = 0.\end{aligned}$$

Now we want to use this to show that Tf is continuous on $[0,1]$. We do this by showing that Tf is bounded.

$$\begin{aligned}|Tf(s)| &= |(1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t)|\\ &\le |\int_{[0,s]} tf(t)dm(t)| + |\int_{[s,1]}(1-t)f(t)dm(t)|\\ &\le \int_{[0,s]}|tf(t)|dm(t)| + \int_{[s,1]}|(1-t)f(t)|dm(t)\\ &\le \int_{[0,s]}|f(t)|dm(t)| + \int_{[s,1]}|f(t)|dm(t)\\ &= \int_{[0,1]}|f(t)|dm(t)\\ &= k\\ &< \infty.\end{aligned}$$

This is for linear operator. Tf is a (not necessarilly linear) function.

Thus $||Tf|| \le k$, hence by (1.8) we have $|Tf(s)| \le ||Tf|||s| \le k|s|$ and now Tf is continuous by Proposition 1.10 as desired.

Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

(a)

For each integer $k \geq 0$, set $g_k(x) = x^k e^{\frac{-x^2}{2}}$ for $x \in \mathbb{R}$.
Justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$.
Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

Define $f(x) = e^{\frac{-x^2}{2}}$. By HW7 problem 1 it suffices to show that $f \in \mathscr{S}(\mathbb{R})$. We want to show this by showing that $\partial^\beta f(x) = p(x)f(x)$ for some polynomial p as then it is clear that $p(x)f(x) \to 0$ for $|x| \to \infty$ as f goes to 0 faster than any polynomial goes to infinity.
We show this by induction on β. For $\beta = 0$ it is clear. Suppose then that it holds for β, then we have

$$
\begin{aligned}
\partial^{\beta+1} e^{-\frac{x^2}{2}} &= \partial \left(\partial^\beta e^{-\frac{x^2}{2}} \right) \\
&= \partial \left(p(x) e^{-\frac{x^2}{2}} \right) \\
&= p'(x) e^{-\frac{x^2}{2}} - x p(x) e^{-\frac{x^2}{2}} \\
&= (p'(x) - x p(x)) e^{-\frac{x^2}{2}} \\
&= q(x) f(x)
\end{aligned}
$$

as the derivative of a polynomial is a polynomial and the product and difference of polynomials is also a polynomial.
Thus we have shown the asserted and furthermore note that $x^\alpha \partial^\beta f(x)$ is also of the form $p(x)f(x)$, hence $f \in \mathscr{S}(\mathbb{R})$, and so is g_k for all $k \geq 1$ as desired.

First we note that by Proposition 11.4 we have $\mathcal{F}(f(x)) = f(x)$ and thus we can use (d) in Proposition 11.13 to compute $\mathcal{F}(g_k)$ for $k = 0, 1, 2, 3$. Furthermore we have $g_0 = f = \mathcal{F}(f)$ so $\mathcal{F}(g_0) = g_0$.

We have

$$\partial(f(x)) = -xe^{-\frac{x^2}{2}}.$$

$$\begin{aligned}\partial^2(f(x)) &= \partial(-xe^{-\frac{x^2}{2}}) \\ &= -e^{-\frac{x^2}{2}} - (-x)xe^{-\frac{x^2}{2}} \\ &= (x^2-1)e^{-\frac{x^2}{2}}.\end{aligned}$$

$$\begin{aligned}\partial^3(f(x)) &= \partial((x^2-1)e^{-\frac{x^2}{2}}) \\ &= 2xe^{-\frac{x^2}{2}} - x(x^2-1)e^{-\frac{x^2}{2}} \\ &= (3x-x^3)e^{-\frac{x^2}{2}}.\end{aligned}$$

Thus we can compute the Fourier transformations

$$\begin{aligned}\mathcal{F}(g_1(x)) &= i^1\partial(f(x)) \\ &= i\cdot(-xe^{-\frac{x^2}{2}}) \\ &= -ixe^{-\frac{x^2}{2}}.\end{aligned}$$

$$\begin{aligned}\mathcal{F}(g_2(x)) &= i^2\partial^2(f(x)) \\ &= -1\cdot(x^2-1)e^{-\frac{x^2}{2}} \\ &= (1-x^2)e^{-\frac{x^2}{2}}.\end{aligned}$$

$$\begin{aligned}\mathcal{F}(g_3(x)) &= i^3\partial^3(f(x)) \\ &= -i\cdot(3x-x^3)e^{-\frac{x^2}{2}} \\ &= i(x^3-3x)e^{-\frac{x^2}{2}}.\end{aligned}$$

Hence we have computed the desired Fourier transformations.

(b)

Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

Note first for $f, g \in \mathscr{S}(\mathbb{R})$, that $h := af + bg$ is in $\mathscr{S}(\mathbb{R})$ as

$$\begin{aligned} x^\alpha \partial^\beta h(x) &= x^\alpha \partial^\beta (af(x) + bg(x)) \\ &= x^\alpha (a\partial^\beta (f(x)) + b\partial^\beta (g(x))) \\ &= ax^\alpha \partial^\beta (f(x)) + bx^\alpha \partial^\beta (g(x)) \\ &\to a \cdot 0 + b \cdot 0 \\ &= 0 \end{aligned}$$

for $|x| \to \infty$. Note that we have used that ∂ is linear, hence also ∂^β is linear.
Now consider the functions

$$h_0 := g_0,\ h_1 := 2g_3 - 3g_1,\ h_2 := 2g_2 - g_0,\ h_3 := g_1.$$

By the above these functions are Schwarz functions. We want to show, that these functions satisfies the desired. We use that $\mathcal{F}$ is linear.
As we saw in (a) we have $\mathcal{F}(h_0) = \mathcal{F}(g_0) = g_0 = h_0 = i^0 h_0$.

$$\begin{aligned} \mathcal{F}(h_1) &= \mathcal{F}(2g_3 - 3g_1) \\ &= 2\mathcal{F}(g_3) - 3\mathcal{F}(g_1) \\ &= 2(i(x^3 - 3x)e^{-\frac{x^2}{2}}) - 3(-ixe^{-\frac{x^2}{2}}) \\ &= i(2x^3 - 3x)e^{-\frac{x^2}{2}} \\ &= i(2g_3 - 3g_1) \\ &= ih_1. \end{aligned}$$

$$\begin{aligned} \mathcal{F}(h_2) &= \mathcal{F}(2g_2 - g_0) \\ &= 2(1 - x^2)e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}} \\ &= (1 - 2x^2)e^{-\frac{x^2}{2}} \\ &= i^2(2x^2 - 1)e^{-\frac{x^2}{2}} \\ &= i^2(2g_2 - g_0) \\ &= i^2 h_2. \end{aligned}$$

$$\begin{aligned} \mathcal{F}(h_3) &= \mathcal{F}(g_1) \\ &= -ixe^{-\frac{x^2}{2}} \\ &= i^3 xe^{-\frac{x^2}{2}} \\ &= i^3 g_1 \\ &= i^3 h_3. \end{aligned}$$

As all of h_0, h_1, h_2, h_3 are non-zero we have found functions satisfying the desired. Furthermore as $g_k \in \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$, we have that $h_j \in \mathscr{S}(\mathbb{R})$ for $j = 0, 1, 2, 3$ as desired.

(c)

Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$.

Let $\hat{f} := \mathcal{F}(f)$ and let the inverse Fourier transformation be denoted $\check{f}$.
Then we have

$$\begin{aligned}
\hat{\hat{f}}(y) &= \mathcal{F}^2(f) \\
&= \mathcal{F}(\hat{f}) \\
&= \int_{\mathbb{R}} \hat{f}(x) e^{-iyx} dm(x) \\
&= \check{\hat{f}}(-y).
\end{aligned}$$

This is easily seen by the definition of the inverse Fourier transformation.
Now corollary 12.12 gives that $\check{\hat{f}}(-y) = f(-y)$ as $f \in \mathscr{S}(\mathbb{R})$. Thus we have

$$\mathcal{F}^4(f) = \mathcal{F}^2(\mathcal{F}^2(f)) = \mathcal{F}^2(f(-y)) = f(-(-y)) = f(y).$$

Hence we have shown that $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$ as desired.

(d)

Use (c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.

By (c) we have $f = \mathcal{F}^4(f) = \mathcal{F}^3(\lambda f) = \lambda \mathcal{F}^2(\lambda f) = \lambda^4 f$. Thus we have $\lambda^4 = 1$, but 1 has exactly four fourth roots and those are exactly $\{1, i, -1, -i\}$, hence $\lambda \in \{1, i, -1, -i\}$ as desired.

Note that $\{i^k | k = 0, 1, 2, 3\} = \{1, i, -1, -i\}$ and thus these are all eigenvalues for $\mathcal{F}$ by (b). Furthermore any eigenvalue λ of $\mathcal{F}$ satisfies $\lambda g = \mathcal{F}(g)$ for some non-zero g, hence there are no other eigenvalues of $\mathcal{F}$ than $\{1, i, -1, -i\}$.

Problem 5

Let $(x_n)_{n \geq 1}$ be a dense subset of $[0, 1]$ and consider the Radon measure

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}$$

on $[0, 1]$. *Show that supp*$(\mu) = [0, 1]$.

The support of the Radon measure is defined in problem 3 HW8. Note that $[0, 1]$ is compact in particular locally compact. It is also Hausdorff, so we are in the setting of problem 3 HW8. We have to determine all open null sets of $[0, 1]$ wrt. μ.
The empty set is open and has measure 0. Let $U \neq \varnothing$ be an open set. Then as $(x_n)_{n\geq 1}$ is a dense subset of $[0, 1]$ we have that $x_k \in U$ for some positive integer k. Then by definition of μ we have that

$$\mu(U) \geq \mu(\{x_k\}) = 2^{-k} > 0.$$

Hence the only open set with measure 0 is the empty set and thus by definition we have $\text{supp}(\mu) = \varnothing^c = [0, 1]$ as desired.