Mandatory Assignment 1, Functional Analysis

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Problem 1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $T : X \to Y$ be a linear map. Set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$, for all $x \in X$. Show that $\|\cdot\|_0$ is a norm on X. Show next that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded.

Proof. For $x, y \in X$, since $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed vector spaces, we have $\|x+y\|_X \leq \|x\|_X + \|y\|_X$ and $\|Tx + Ty\|_Y \leq \|Tx\|_Y + \|Ty\|_Y$. Hence,

$$\begin{aligned}
\|x+y\|_0 &= \|x+y\|_X + \|Tx+Ty\|_Y \\
&\leq \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y \\
&= (\|x\|_X + \|Tx\|_Y) + (\|y\|_X + \|Ty\|_Y) \\
&= \|x\|_0 + \|y\|_0.
\end{aligned}$$

✓

For $\alpha \in \mathbb{K}$ and $x \in X$,

Should be $|\alpha|$

$$\begin{aligned}
\|\alpha x\|_0 &= \|\alpha x\|_X + \|T(\alpha x)\|_Y \\
&= \alpha\|x\|_X + \alpha\|Tx\|_Y \\
&= \alpha(\|x\|_X + \|Tx\|_Y) \\
&= \alpha\|x\|_0.
\end{aligned}$$

(✓)

For every $x \in X$

$$\begin{aligned}
\|x\|_0 = 0 &\Leftrightarrow \|x\|_X + \|Tx\|_Y = 0 \\
&\Leftrightarrow \|x\|_X = 0 \text{ and } \|Tx\|_Y = 0, \text{ since } \|x\|_X \geq 0 \text{ and } \|Tx\|_Y \geq 0. \\
&\Leftrightarrow x = 0.
\end{aligned}$$

✓

Therefore, $\|\cdot\|_0$ is a norm on X.

Since $\|Tx\|_Y \leq \|T\|\|x\|_X$, $\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq \|x\|_X + \|T\|\|x\|_X = (1 + \|T\|)\|x\|_X$. Put $c = 1$ and $C = 1 + \|T\|$.

$$T \text{ is bounded} \Leftrightarrow \|T\| < \infty \Leftrightarrow C = 1 + \|T\| < \infty.$$

This only shows T bounded $\Rightarrow$ norms are equivalent.

Hence, T is bounded $\Leftrightarrow$ there exist $0 < c \leq C < \infty$ such that $c\|x\|_X \leq \|x\|_0 \leq C\|x\|_X \Leftrightarrow \|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent. $\square$

(b) Show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.

Proof. Suppose that X is finite dimensional, $\dim X = n$. Consider a basis of X, denoted as

$\{e_1, e_2, \ldots, e_n\}$. Then every $x \in X$ can be written as

$$x = \sum_{i=1}^{n} a_i e_i, \; a_i \in \mathbb{K}.$$

So

$$Tx = \sum_{i=1}^{n} a_i T e_i, \; a_i \in \mathbb{K}.$$

Then we have

$$\|Tx\|_Y = \|\sum_{i=1}^{n} a_i T e_i\|_Y \leq \sum_{i=1}^{n} |a_i| \|Te_i\|_Y.$$

Since X is a finite vector space, then any two norms on X are equivalent. Therefore, $\|\cdot\|_\infty$ and $\|\cdot\|_X$ are equivalent. It follows that there exist $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \|x\|_X \leq \|x\|_\infty \leq c_2 \|x\|_X.$$

Let $M = \max_i \|Te_i\|_Y$. Then

That is $\|x\|_1$, not $\|x\|_\infty$.

$$\|Tx\|_Y \leq M \underline{\sum_{i=1}^{n}} |a_i| = M\|x\|_\infty \leq Mc_2 \|x\|_X.$$

$$\|T\| = \sup_{x \in X, x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \leq Mc_2.$$

Therefore, T is bounded. □

(c) Suppose that X is infinite dimensional. Show that there exists a linear map $T : X \to Y$, which is not bounded (= not continuous).

This S is not well-defined before you have found your Hamel basis.

Proof. Take a linearly independent sequence $\{e_i\} \subseteq X$. Define a linear map $S : X \to \mathbb{K}$ by $S(e_i) = i\|e_i\|$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We can extend this sequence of linearly independent vectors to a Hamel basis of X, then every vector in X can be written as

How do you know that is possible?

$$x = \sum_i \lambda_i e_i, \; \lambda_i \in \mathbb{K}.$$

Define S at the other vectors in the basis to be 0. For $x \in X$,

$$S(x) = T\left(\sum_i \lambda_i e_i\right) = \sum_i \lambda_i T(e_i) = \sum_i \lambda_i i \|e_i\|.$$

$S(e_i)$ is bounded. $\{S(e_i)\}_{i \in \mathbb{N}}$ is not.

Assume that $\|e_i\| = 1$ for each i, then $S(e_i) = i\|e_i\| = i$. Therefore, $S(e_i)$ is not bounded, which means S is not bounded.

(✓)

Define a linear map $T : X \to Y$ by $T(x) = yS(x)$, where $0 \neq y \in Y$. Then T is a linear map from X to Y which is not bounded. □

(d) Suppose again that X is infinite dimensional. Argue that there exists a norm $\|x\|_0$ on X, which is *not* equivalent to the given norm $\|x\|_X$, and which satisfies $\|x\|_X \leq \|x\|_0$ for all $x \in X$. Conclude that $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space.

Proof. Suppose that X is a Banach space. Take $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ as is stated in (a), where $T : X \to Y$ defined the same as it in (c).

Let $(x_n)_{n\geq 1}$ be a sequence in $(X, \|\cdot\|_0)$, where $x_i = \frac{1}{i}e_i$. Then

$$\begin{aligned}
\|x_m - x_n\|_0 &= \|x_m - x_n\|_X + \|Tx_m - Tx_n\|_Y \\
&= \|x_m - x_n\|_X + \left\|yi\frac{1}{i} - yi\frac{1}{i}\right\|_Y \\
&= \|x_m - x_n\|_X \\
&= \left\|\frac{1}{m}e_m - \frac{1}{n}e_n\right\|_X \\
&\leq \left\|\frac{1}{m}e_m\right\|_X + \left\|\frac{1}{m}e_m\right\| \\
&= \frac{1}{m} + \frac{1}{n} \to 0, \text{ as } m,\ n \to \infty.
\end{aligned}$$

This implication is not true (but x_n does converge to 0 in $\|\cdot\|_X$).

Then $(x_n)_{n\geq 1}$ is Cauchy in $(X, \|\cdot\|_0)$ and $(X, \|\cdot\|_X)$. Since $(X, \|\cdot\|_X)$ is complete, the Cauchy sequence $(x_n)_{n\geq 1}$ has a limit, i.e. $\|x_n\|_X \to 0$ as $n \to \infty$.

In $(X, \|\cdot\|_0)$, $x_n = \frac{1}{n}e_n \to 0$ as $n \to 0$, while

The idea seems right, but it is a bit imprecisely written.

$$\|x_n - 0\|_0 = \left\|\frac{1}{n}e_n\right\|_X + \|y\|_Y \to \|y\|_Y, \text{ as } n \to 0.$$

Hence, the Cauchy sequence $(x_n)_{n\geq 1}$ is not convergent in $(X, \|\cdot\|_0)$. Therefore $(X, \|\cdot\|_0)$ is not complete. □

(✓)

(e) Give an example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ satisfying $\|x\|' \leq \|x\|$, for all $x \in X$, such that $(X, \|\cdot\|)$ is complete, while $(X, \|\cdot\|')$ is not.

Proof. Take $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|)$ and $\|\cdot\|_1$?

$$\|x\|' = \sum_n \frac{|x_n|}{n} \text{ for } x \in \ell_1(\mathbb{N}).$$

need a $\forall j$ quantifier here.

It is clear that $\|x\|' \leq \|x\|$, for all $x \in \ell_1(\mathbb{N})$. Consider a sequence $(\delta_j)_{j\geq 1}$, where j-th term is 1 and others are 0. $\|\delta_j\| = 1$ for all j and $\|\delta_j\|' = \frac{1}{j}$. We cannot find $0 < c_1 \leq c_2 < \infty$ such that $c_1\|\delta_j\|' \leq \|\delta_j\| \leq c_2\|\delta_j\|'$, hence $\|\cdot\|'$ is not equivalent to $\|\cdot\|$. Since $(\ell_1(\mathbb{N}), \|\cdot\|)$ is a Banach space, $(\ell_1(\mathbb{N}), \|\cdot\|')$ cannot be complete. How so? Therefore, we have found $\|\cdot\|'$ such that for every $x \in \ell_1(\mathbb{N})$, $\|x\|' \leq \|x\|$ but $(\ell_1(\mathbb{N}), \|\cdot\|')$ is not complete. □

(✓)

Problem 2 Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), \|\cdot\|_p)$, considered as a vector space over M, given by

$$M = \{(a, b, 0, 0, 0, \ldots) : a, b \in \mathbb{C}\}.$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, 0, \ldots) = a + b$, for all $a, b \in \mathbb{C}$.

(a) Show that f is bounded on $(M, \|\cdot\|_p)$ and compute $\|f\|$. (Answer depends on p.)

Proof.

$$\begin{aligned}
|f(a, b, 0, 0, 0, \ldots)| &= |a + b| \leq |a| + |b| \\
&\leq (|a|^p + |b|^p)^{\frac{1}{p}}(1 + 1)^{1-\frac{1}{p}} \\
&= 2^{\frac{p-1}{p}}\|(a, b, 0, 0, 0, \ldots)\|_p.
\end{aligned}$$

What are you using here?

It follows that

$$\|f\| = \sup_{0\neq a,b\in\mathbb{C}} \frac{|f(a,b,0,0,0,\dots)|}{\|(a,b,0,0,0,\dots)\|_p} \le 2^{\frac{p-1}{p}}. \tag{1}$$

Therefore, f is bounded on $(M, \|\cdot\|_p)$.

Set $a = b = \frac{1}{2^{\frac{1}{p}}}$. Then $\|(a,b,0,0,0,\dots)\|_p = 1$. We also have

$$\|f\| = \sup_{\|(a,b,0,0,0,\dots)\|_p=1} |f(a,b,0,0,0,\dots)| \ge \frac{2}{2^{\frac{1}{p}}} = 2^{\frac{p-1}{p}}. \tag{2}$$

You've already used a, b above.

According to (1) and (2), we conclude that $\|f\| = 2^{\frac{p-1}{p}}$.

(✓)

□

(*b*) Show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Proof. Suppose $F : \ell_p(\mathbb{N}) \to \mathbb{C}$ is a linear functional and $\|F\| = \|f\| = 2^{\frac{p-1}{p}}$ and $F|_M = f$. Let $(e_i)_{i\ge1}$ be an orthonormal basis of $\ell_p(\mathbb{N})$. Take $x = (a, b, x_3, x_4, \dots) \in \ell_p(\mathbb{N})$, then

$$x = e_1a + e_2b + e_3x_3 + e_4x_4 + \dots$$

Also not that what you have defined IS an extension.

Let $F(e_i) = \alpha_i$. For all $i \ge 3$,

What are you using here?

$$\begin{aligned}
|F(x)| = |F(e_1a + e_2b + e_ic_i)| &= |a + b + \sum_{i\ge3} \alpha_i c_i| = |1\cdot a + 1\cdot b + \sum_{i\ge3} \alpha_i c_i| \\
&\le (|a|^p + |b|^p + \sum_{i\ge3} |c_i|^p)^{\frac{1}{p}} (1 + 1 + \sum_{i\ge3} |\alpha_i|^{\frac{p}{p-1}})^{\frac{p-1}{p}} \\
&= (2 + \sum_{i\ge3} |\alpha_i|^{\frac{p}{p-1}})^{\frac{p-1}{p}} \|x\|_p.
\end{aligned}$$

Since equality can be obtained, How?

$$\|F\| = \sup_{x\in\ell_p(\mathbb{N})} \frac{|F(x)|}{\|x\|_p} = (2 + \sum_{i\ge3} |\alpha_i|^{\frac{p}{p-1}})^{\frac{p-1}{p}}.$$

Since $\|F\| = 2^{\frac{p-1}{p}}$, $\alpha_i = 0$, for every $i \ge 2$ ($i\ge3$). Therefore, there exists a unique linear functional $F : \ell_p(\mathbb{N}) \to \mathbb{C}$ defined by $F = a + b$. □ (✓)

(*c*) Show that if $p = 1$, then there are infinitely many linear functional F on $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Proof. When $p = 1$, $\|f\| = 1$. Let $(e_i)_{i\ge1}$ be an orthonormal basis of $\ell_1(\mathbb{N})$. Take $x = (a, b, x_3, x_4, \dots) \in \ell_p(\mathbb{N})$, then

$$x = e_1a + e_2b + e_3x_3 + e_4x_4 + \dots$$

Define $F : \ell_1(\mathbb{N}) \to \mathbb{C}$ satisfying

Is $\alpha_i, i\ge3$, fixed?

$$F(e_i) = \alpha_i, \ |\alpha_i| \le 1 \text{ for all } i \ge 3 \tag{3}$$

and

$$F(x) = a + b + \sum_{i=3}^{\infty} \alpha_i x_i. \tag{4}$$

Observe $F|_M = f$.

Shouldbe $|a+b+\sum \alpha_i x_i|$

Then

$$\|F\| = \sup_{x\in\ell_p(\mathbb{N})} \frac{|a|+|b|+\sum_{i=3}^{\infty}|\alpha_i x_i|}{|a|+|b|+\sum_{i=3}^{\infty}|x_i|} \le \sup_{x\in\ell_p(\mathbb{N})} \frac{|a|+|b|+\sum_{i=3}^{\infty}|\alpha_i||x_i|}{|a|+|b|+\sum_{i=3}^{\infty}|x_i|} \le 1.$$

Take $x = (1,0,0,0,\ldots)$, then $\|x\| = 1$.

$$\|F\| = \sup_{\|x\|=1} |F(x)| \ge 1 + 0 + \sum_{i=3}^{\infty} 0 = 1.$$

Hence, $\|F\| = 1$. Therefore, F defined above satisfying (3) and (4) is as required and there are infinitely many such F. $\square$

Problem 3 Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $n \ge 1$ be an integer. Show that no linear map $F : X \to \mathbb{K}^n$ is injective.

Proof. Assume that there is an injective linear map $F : X \to \mathbb{K}^n$. Since X is finite dimensional and $\mathbb{K}^n$ is n-dimensional, F is surjective. Take a linearly independent sequence $\{x_1, x_2, \ldots, x_{n+1}\} \subseteq X$. Then $F(x_1), F(x_2), \ldots, F(x_{n+1})$ are linearly dependent in Y since Y is n-dimensional. Hence there exist $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ (not all zero), such that

$$\alpha_1 F(x_1) + \alpha_2 F(x_2) + \ldots + \alpha_{n+1} F(x_{n+1}) = 0.$$

I.e.

$$F(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{n+1} x_{n+1}) = 0.$$

As assumed, F is injective. So $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{n+1} x_{n+1} = 0$. However, it is contradict to the fact that $x_1, x_2, \ldots, x_{n+1}$ are linearly independent. Therefore, no linear map $F : X \to \mathbb{K}^n$ is injective. $\square$

(b) Let $n \ge 1$ be an integer and let $f_1, f_2, \ldots, f_n \in X^*$. Show that

$$\bigcap_{j=1}^{n} \ker(f_i) \ne \{0\}.$$

Proof. Define a linear map $F : X \to \mathbb{K}^n$ by $F(x) = (f(x_1), f(x_2), \ldots, f(x_n))$. Then

$$\ker(F) = \bigcap_{j=1}^{n} \ker(f_i).$$

Show this maybe

Assume that $\bigcap_{j=1}^{n} \ker(f_i) = \{0\}$, i.e. $\ker(F) = 0$, which means that F is an injective linear map from X to $\mathbb{K}^n$. However, this is contradict to the conclusion of Problem 3 (a). Therefore, $\bigcap_{j=1}^{n} \ker(f_i) \ne \{0\}$. $\square$

(c) Let $x_1, x_2, \ldots, x_n \in X$. Show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \ge \|x_j\|$ for all $j = 1, 2, \ldots, n$.

Proof. From (b) we know that $\bigcap_{j=1}^{n} \ker(f_i) \ne \{0\}$. Then choose $z \in \bigcap_{j=1}^{n} \ker(f_i)$. Let $y = \frac{z}{\|z\|}$, so $\|y\| = 1$. For $0 \ne x_j \in X, j = 1, 2, \ldots, n$, there exists $f_j \in X^*$ such that $\|f_j\| = 1$ and

$f_j(x_j) = \|x_j\|$. Then we have

$$\begin{aligned}\|y - x_j\| &= \|f_j\|\|y - x_j\| \\ &\geq |f_j(y - x_j)| \\ &= |f_j y - f_j x_j| \\ &= |0 - \|x_j\|| = \|x_j\|.\end{aligned}$$

□

(d) Show that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

Proof. Suppose that there is a finite family of closed balls $\{B_j(x_j, \delta_j)\}$ ($j = 1, 2, \ldots, n$), none of which contains 0. Denote

$$M := \bigcap_{j=1}^{n} \ker(f_i).$$

Define

$$f_j(x) = \frac{\|x - x_j\|}{\|x_j\|} \quad (x_j \text{ are the centers of } B_j).$$

but f_i is already defined?

As is proved in (c), if $x \in M$, then

$$f_j(x) = \frac{\|x - x_j\|}{\|x_j\|} \geq 1, \text{ for all } j = 1, 2, \ldots, n.$$

little confusion with the f_i's.

Since $0 \notin B_j(x_j, \delta_j)$, for each $x \in B_j(x_j, \delta_j)$,

$$f_j(x) = \frac{\|x - x_j\|}{\|x_j\|} < 1.$$

Therefore, $M \cap B_j = \varnothing$, for every j. Since $M \neq \{0\}$, we can find $0 \neq v \in M$. Take $w = \frac{v}{\|v\|}$, then $\|w\| = 1$, so $w \in S \cap M$. However,

$$w \notin \bigcup_{j=1}^{n} B_j(x_j, \delta_j).$$

Therefore, S cannot be covered by a finite family of closed balls. □

(e) Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

Proof. Assume that S is compact. For any $x \in S$, we consider

$$B_x = \{y \in X \mid \|x - y\| < \frac{1}{2}\}.$$

Then $\{B_x\}_{x \in S}$ is an open cover of S. Since S is compact as assumed, there exists a finite subcover $\{B_{x_1}, B_{x_2}, \ldots, B_{x_n}\}$ of S. Take the closures of each B_{x_i}, then $\{\overline{B_{x_1}}, \overline{B_{x_2}}, \ldots, \overline{B_{x_n}}\}$ is a finite family of closed balls covering S. This is contradict to the conclusion of (d). Therefore, S is non-compact. □

Problem 4 Let $L_1([0,1], m)$ and $L_3([0,1], m)$ be the Lebesgue spaces on $[0,1]$. Recall from HW2 that $L_3([0,1], m) \subsetneq L_1([0,1], m)$. For $n \geq 1$, define

$$E_n := \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm \leq n \right\}.$$

(a) Given $n \geq 1$, is the set $E_n \subset L_1([0,1], m)$ absorbing? Justify.

Proof. E_n is not absorbing.

Firstly prove that E_n is a convex set. Let $0 \leq \alpha \leq 1$ and $f, g \in E_n$.

$$\begin{aligned}\alpha f + (1-\alpha) g &= \int_{[0,1]} |\alpha f|^3 dm + \int_{[0,1]} |(1-\alpha) g|^3 dm \\ &= \alpha^3 \int_{[0,1]} |f|^3 dm + (1-\alpha)^3 \int_{[0,1]} |g|^3 dm \\ &\leq \alpha^3 n + (1-\alpha)^3 n = (1 - 3\alpha + 3\alpha^2) n \leq n.\end{aligned}$$

Hence, E_n is a convex set.

$\forall t > 0$, $\exists h = t n^{\frac{1}{3}} + 1 \in L_1([0,1], m)$, such that

Need to find one h independent of t.

0/0

$$\int_{[0,1]} |t^{-1} h|^3 dm = \int_{[0,1]} |t^{-1} t n^{\frac{1}{3}} + 1|^3 dm = \int_{[0,1]} |n^{\frac{1}{3}} + 1|^3 dm \geq n.$$

$h = tn^{\frac{1}{3}} + 1$ is absorbed since $(tn^{\frac{1}{3}} + 1)^{-1} h \in E_1 \subset E_n$

Therefore, E_n is not absorbing in $L_1([0,1], m)$. □

(b) Show that E_n has empty interior in $L_1([0,1], m)$, for all $n \geq 1$.

4/0

Proof. □

(c) Show that E_n is closed in $L_1([0,1], m)$, for all $n \geq 1$.

2/0

Proof. □

(d) Conclude from (b) and (c) that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$.

Proof. According to (b) and (c), E_n is closed and E_n has empty interior in $L_1([0,1], m)$, so $\overline{E_n} = E_n$ is nowhere dense. And note that

$$L_3([0,1], m) = \bigcup_{n=1}^{\infty} E_n.$$

I.e. $L_3([0,1], m)$ can be expressed as the countable union of subsets which are nowhere dense in $L_1([0,1], m)$. Therefore, $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$. □

✓

Problem 5 Let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n \geq 1}$ be a sequence in H, and let $x \in H$.

(a) Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$? Give a proof or a counterexample.

Proof. Yes.

Since $x_n \to x$ in norm, $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for every $n > N$, $\|x_n - x\| < \varepsilon$. Notice that

$$\|x_n\| = \|(x_n - x) + x\| \leq \|x_n - x\| + \|x\|.$$

Thus,

$$\|x_n\| - \|x\| \leq \|x_n - x\| < \varepsilon.$$

It follows that $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for every $n > N$, $\|x_n\| - \|x\| < \varepsilon$. Therefore, $\|x_n\| \to \|x\|$, as $n \to \infty$. □

(✓)

Need $\overline{\big|\|x_n\| - \|x\|\big|} < \varepsilon$.

(*b*) Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$? Give a proof or a counterexample.

Proof. Counterexample: Take an orthonormal basis $(e_n)_{n\geq 1}$ in $(\ell_2(\mathbb{N}), \langle\cdot,\cdot\rangle)$. Note that $\ell_2(\mathbb{N}) \cong \ell_2(\mathbb{N})^*$. For every $y \in \ell_2(\mathbb{N})^*$,

This does not make sense without more justification.

$$y = (\eta_1, \eta_2, \ldots), \text{ where } \sum_{i=1}^{\infty} |\eta_i|^2 < \infty,$$

then we have

$$\langle e_n, y\rangle = \eta_n \to 0 \text{ as } n \to 0.$$

Therefore, $(e_n)_{n\geq 1}$ converges weakly to 0. However, $\|e_n\| = 1$, $\forall n = 1, 2, \ldots$, so $\|e_n\| \to 1$ as $n \to \infty$. Here you implicitly use that $H^* = \{\langle\cdot, y\rangle \mid y \in H\}$. □ (✓)

(*c*) Suppose that $\|x_n\| \leq 1$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Is it true that $\|x\| \leq 1$? Give a proof or a counterexample.

Proof. Suppose that $x_n \to x$ weakly and $\|x_n\| \leq 1$. Then we have

$$\left|\left\langle \frac{x}{\|x\|}, x_n \right\rangle\right| \leq \|x_n\|.$$

Since $x_n \to x$ weakly,

Once again uses H^*-identification without reference.

$$\left|\left\langle \frac{x}{\|x\|}, x_n \right\rangle\right| \to \left|\left\langle \frac{x}{\|x\|}, x \right\rangle\right| = \|x\|, \text{ as } n \to \infty.$$

Thus,

$$\|x\| \leq \varliminf_{n\to\infty} \|x_n\|.$$

Since $\|x_n\| \leq 1$, then $\|x\| \leq 1$. □ ✓