References: Throughout this mandatory assignment we will consider the lecture notes to be the "canonical" source material. Any source-less references to definitions, theorems and propositions (e.g. *"see Proposition 2.10"*) will thus implicitly refer to the lecture notes. When referencing *Measures, Integrals and Martingales by René L. Schilling* and *Real Analysis: Modern Techniques and Their Applications by Gerald B. Folland* we will simply append *(Schilling)* and *(Folland)*, respectively, to the relevant reference. Homework and mandatory problems will be abbreviated and should be fairly self-explanatory, e.g. *HW4-P2(a)* will refer to homework problem 4, problem 2, sub-problem (a).

Problem 1: Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.

(a) Show that $f_N \to 0$ weakly as $N \to \infty$, while $\|f_N\| = 1$ for all $N \geq 1$.

Proof. By combining the Riesz representation theorem (see *HW2-P1*) with *HW4-P2(a)*, it will suffice to show that

$$\langle f_N, y\rangle \to \langle 0, y\rangle = 0, \quad \text{for all } y \in H. \tag{1}$$

Since $(e_n)_{n\geq 1}$ is an orthonormal basis, it follows from Analysis 2 (see Theorem 26.21, Schilling) that any given $y \in H$ can be written as

This is generally false. $(e_n)_{n\in\mathbb{N}}$ is not a Hamel basis.

$$y = \sum_{m=1}^{\infty}\langle y, e_m\rangle e_m = \sum_{m=1}^{N_y}\langle y, e_m\rangle e_m, \tag{2}$$

for some $N_y \in \mathbb{N}$, where the latter equality used that only finitely many of the $\langle y, e_n\rangle$ terms are non-zero (see the *MA1-P1*). Plugging (2) and the expression for f_N into (1) yields

$$\begin{aligned}\langle f_N, y\rangle &= \langle N^{-1}\sum_{n=1}^{N^2} e_n, \sum_{m=1}^{N_y}\langle y, e_m\rangle e_m\rangle = N^{-1}\sum_{n=1}^{N^2}\sum_{m=1}^{N_y}\overline{\langle y, e_m\rangle}\langle e_n, e_m\rangle \\ &= N^{-1}\sum_{n=1}^{\min\{N^2, N_y\}}\overline{\langle y, e_n\rangle}. \end{aligned}\tag{3}$$

The sum in (3) is clearly finite since $\min\{N^2, N_y\} \to N_y$ for $N \to \infty$, and hence the entire expression will converge to zero due to the N^{-1} term. This holds for any arbitrarily chosen $y \in H$: we conclude that $f_N \to 0$ weakly as $N \to \infty$. (✓)

Further, we have

$$\begin{aligned}\|f_N\|^2 &= \langle N^{-1}\sum_{n=1}^{N^2} e_n, N^{-1}\sum_{m=1}^{N^2} e_m\rangle = N^{-2}\sum_{n=1}^{N^2}\sum_{m=1}^{N^2}\langle e_n, e_m\rangle \\ &= N^{-2}\sum_{n=1}^{N^2}\langle e_n, e_n\rangle = N^{-2}N^2 = 1,\end{aligned}$$

and hence $\|f_N\| = 1$ for all $N \geq 1$. □ ✓

Let K be the norm closure of $\text{co}\{f_N : N \geq 1\}$.

(b) Argue that K is weakly compact, and that $0 \in K$.

Recall the definition of the convex hull,

$$\text{co}\{f_N : N \geq 1\} = \left\{ \sum_{i=1}^{n} \alpha_i f_{N_i} \mid f_{N_i} \in \{f_N : N \geq 1\}, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N} \right\}.$$

Thus for any $f \in \text{co}\{f_N : N \geq 1\}$ we have

$$\|f\| = \|\sum_{i=1}^{n} \alpha_i f_{N_i}\| \leq \sum_{i=1}^{n} \|\alpha_i f_{N_i}\| = \sum_{i=1}^{n} |\alpha_i| \|f_{N_i}\| = \sum_{i=1}^{n} \alpha_i = 1,$$

where we used the result from (a) that $\|f_{N_i}\| = 1$ for all $f_{N_i} \in \{f_N : N \geq 1\}$. We conclude that any element in the convex hull is contained in the closed unit ball, i.e.

$$\text{co}\{f_N : N \geq 1\} \subset \overline{B_H(0,1)},$$

and thus the norm closure of the convex hull – i.e. K – will be contained within the closed unit ball:

$$K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} \subset \overline{B_H(0,1)}.$$

Convex hulls are (obviously) convex, and so it furthermore follows from Theorem 5.7 that

$$K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w}. \tag{4}$$

Since H is reflexive (see Proposition 2.10) we may apply Theorem 6.3 to conclude that $\overline{B_H(0,1)}$ is *weakly* compact. Finally we piece everything together: We have shown that K is closed in the weak topology (4), and that it is a subset of a weakly compact set $\overline{B_H(0,1)}$. Then it follows from Proposition 4.22 (Folland) that K is weakly compact. ✓

Then by *HW5-P1* combined with *MA2-P1(a)* there exists a sequence $(f_{N_i})_{i\geq 1} \subset \text{co}\{f_N : N \geq 1\}$ such that $f_{N_i} \to 0$ in norm, which implies that $0 \in \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = K$. (✓)

It is a sequence of convex combinations of the f_N's, not just the f_N's themselves.

(c) Show that 0, as well as f_N, $N \geq 1$, are extreme points in K.

Let us first show that $0 \in \text{Ext}(K)$. Notice that for any $f \in \text{co}\{f_N : N \geq 1\}$ and any basis vector $e_m \in (e_n)_{n\geq 1}$ we have

$$\langle e_m, f \rangle = \langle e_m, \sum_{i=1}^{k} f_{N_i} \alpha_i \rangle = \langle e_m, \sum_{i=1}^{k} \left(N_i^{-1} \sum_{n=1}^{N_i^2} e_n \right) \alpha_i \rangle = \sum_{i=1}^{k} N_i^{-1} \sum_{n=1}^{N_i^2} \alpha_i \langle e_m, e_n \rangle \geq 0. \tag{5}$$

Next assume that $0 = \alpha g + (1-\alpha)h$ for some $\alpha \in (0,1)$ and $g, h \in K$. Since $g, h \in K = \overline{\text{co}\{f_N : N \geq 1\}}$ we can approximate them arbitrarily well with elements

$g_i, h_i \in \text{co}\{f_N : N \geq 1\}$ for which $\langle e_m, g_i \rangle \geq 0$ and $\langle e_m, h_i \rangle \geq 0$ by (5), and so we must have $\langle e_m, g \rangle \geq 0, \langle e_m, h \rangle \geq 0$. But notice that

$$-\alpha g = (1-\alpha)h \;\Leftrightarrow\; g = \frac{\alpha - 1}{\alpha} h$$
$$\Rightarrow \langle e_m, g \rangle = \frac{\alpha - 1}{\alpha} \langle e_m, h \rangle,$$

where $(\alpha - 1)/\alpha < 0$. But because the inner products are positive the above equality is only possible when $\langle e_m, g \rangle = \langle e_m, h \rangle = 0$, and hence $g = h = 0$, i.e. 0 is an extreme point. Next let us show that $f_N, N \geq 1$ are extreme points. Define $F := \{f_N : N \geq 1\}$, and note that since K is weakly compact by *MA2-P1(b)* then Theorem 7.8 (Klein-Milman) implies

$$\overline{\text{co}(F)}^{\tau_w} = K = \overline{\text{co}(\text{Ext}(K))}^{\tau_w},$$

so if we can show that the *removal* of any element from F renders the above equality incorrect, then said element must be an extreme point. That is, for some $M \in \mathbb{N}$ define

$$F' = F \setminus \{f_M\},$$

and assume for contradiction that there exist $f \in \text{co}(F')$ such that $f = f_M$. Then

$$\langle f_M, f \rangle = \|f_M\|^2 = 1,$$

but

$$\begin{aligned}
\langle f_M, f \rangle &= \langle M^{-1} \sum_{i=1}^{M^2} e_i, \sum_{j=1}^{n} f_{N_j} \alpha_j \rangle = \langle M^{-1} \sum_{i=1}^{M^2} e_i, \sum_{j=1}^{n} (N_j^{-1} \sum_{k=1}^{N_j^2} e_k) \alpha_j \rangle \\
&= \sum_{j=1}^{n} \alpha_j M^{-1} N_j^{-1} \sum_{i=1}^{M^2} \sum_{k=1}^{N_j^2} \langle e_i, e_k \rangle = \sum_{j=1}^{n} \alpha_j M^{-1} N_j^{-1} \min\{M^2, N_j^2\} \\
&= \sum_{i=1}^{n} \alpha_j \frac{\min\{M^2, N_j^2\}}{M N_j} < 1,
\end{aligned}$$

where we used that $\frac{\min\{M^2, N_j^2\}}{M N_j} < 1$ since $f \in \text{co}(F')$, i.e. none of the f_{N_j}'s contributing to f contain the Mth term, and hence $M \neq N_j$ for any j. Clearly inner products cannot simultaneously equal 1 and be strictly less than 1, so we have reached a contradiction, i.e. f_M *cannot* be written as a convex sum of terms in F', and furthermore we have that the weak closure cannot coincide, i.e. * Why? What if f_M was a limit point of co(F')?

$$\overline{\text{co}(F')}^{\tau_w} \neq \overline{\text{co}(F)}^{\tau_w} = \overline{\text{co}(\text{Ext}(K))}^{\tau_w}.$$

(✓)

We conclude that $\{f_M\} \subset \text{Ext}(K)$, but since our $M \in \mathbb{N}$ was chosen arbitrarily we get that all $f_N, N \geq 1$ are extreme points of K.

(d) Are there any other extreme points in K?

Since K is a weakly compact, non-empty convex subset of H, and $F \subset K$ satisfies $K = \overline{\text{co}(F)}^{\tau_w}$, then by Theorem 7.9 (Milman) it follows that $\text{Ext}(K) \subset \overline{F}^{\tau_w}$. But since the weak closure of F only adds the element 0 (any other weakly convergent sequence in F that doesn't converge to 0 will eventually be constant in F), then in combination with *MA2-P1(c)* we conclude that $F \cup \{0\} = \text{Ext}(K)$, i.e. there are *no* other extreme points in K.

Problem 2: Let X and Y be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that if $x_n \to x$ weakly as $n \to \infty$, then $Tx_n \to Tx$ weakly as $n \to \infty$.

Proof. By reference to *HW4-P2(a)* it suffices to show that

$$y^*(Tx_n) \to y^*(Tx), \quad \text{for all } y^* \in Y^*.$$

By Theorem 7.13 there exists $T^\dagger \in \mathcal{L}(Y^*, X^*)$ such that

$$(T^\dagger y^*)(x) = y^*(Tx), \quad \text{for all } x \in X, y^* \in Y^*,$$

Thus for all $y^* \in Y^*$ we recognize that $T^\dagger y^* \in X^*$, and since $x_n \to x$ weakly we can apply *HW4-P2(a)* to conclude

$$y^*(Tx_n) = (T^\dagger y^*)(x_n) \to (T^\dagger y^*)(x) = y^*(Tx)$$

as desired. □

(b) Let $T \in \mathcal{K}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that if $x_n \to x$ weakly as $n \to \infty$, then $\|Tx_n - Tx\| \to 0$ as $n \to \infty$.

Proof. We wish to show that the further restriction to *compact* operators is enough to ensure convergence of $(Tx_n)_{n\geq 1}$ to Tx in norm. Note that $\mathcal{K}(X, Y) \subset \mathcal{L}(X, Y)$ (see Definition 8.1 and subsequent comments), so by the previous sub-problem *MA2-P2(a)* the sequence $(Tx_n)_{n\geq 1}$ will converge *weakly* to Tx. Let us establish the norm convergence via an application of the double thinning principle: Consider a *subsequence* $(Tx_{n_k})_{k\geq 1}$. Since the $(x_n)_{n\geq 1}$ sequence converges weakly it will be bounded according to *HW4-P2(b)*, and thus the subsequence $(x_{n_k})_{k\geq 1}$ is bounded as well. Then by Proposition 8.2(4) we can find a further subsequence $(x_{n_{k_p}})_{p\geq 1}$ such that $(Tx_{n_{k_p}})_{p\geq 1}$ converges in norm to *some* element $y \in Y$. But convergence in norm implies convergence in the weak topology (since $\tau_w \subset \tau_{\|\cdot\|}$, see Remark 5.3), so by the uniqueness of weak limits we conclude that y and Tx must coincide; i.e. $(Tx_{n_{k_p}})_{p\geq 1}$ converges to Tx in norm. But since our initial subsequence was chosen arbitrarily then the double thinning principle implies that $(Tx_n)_{n\geq 1}$ converges to Tx in norm, i.e. $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. □

(c) Let H be a separable infinite dimensional Hilbert space. Assume that $T \in \mathcal{L}(H, Y)$ satisfies $\|Tx_n - Tx\| \to 0$ as $n \to \infty$ whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$. Show that $T \in \mathcal{K}(H, Y)$.

Proof. Assume that $T \in \mathcal{L}(H,Y)$ satisfies $\|Tx_n - Tx\| \to 0$ as $n \to \infty$ whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, but that T is *not* compact. Then by Proposition 8.2 the set $T(\overline{B_H(0,1)})$ is *not* totally bounded, which means that there exists a $\delta > 0$ such that no matter how large of an $N \in \mathbb{N}$ we chose, it is *not* possible to cover $T(\overline{B_H(0,1)})$ with a union of N open balls with radius δ. Now take some arbitrary initial element $y_0 \in T(\overline{B_H(0,1)})$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $T(\overline{B_H(0,1)})$ such that

i-1.

$$y_i \in T(\overline{B_H(0,1)}) \setminus \left(\bigcup_{n=0}^{i} B_Y(y_i,\delta)\right), \quad \text{for } i \in \mathbb{N}.$$

y_n

Let us provide some intuition behind the construction: We start with the initial element $y_0 \in T(\overline{B_H(0,1)})$. Then we "surround it" with a ball $B_Y(y_0,\delta)$ and pick a $y_1 \in T(\overline{B_H(0,1)})$ that is *not* contained within the surrounding ball, i.e. y_1 has distance *at least* δ from y_0. Then we surround this y_1 with another ball $B_Y(y_1,\delta)$ and pick the next element in the sequence $y_2 \in T(\overline{B_H(0,1)}) \setminus \left(\bigcup_{n=0}^{1} B_Y(y_i,\delta)\right)$, and we repeat this process indefinitely. This ensures that every new element in the sequence has a distance of at least δ from any previous element. Note that the set $T(\overline{B_H(0,1)}) \setminus \left(\bigcup_{n=0}^{i} B_Y(y_i,\delta)\right)$ is never empty for any $i \in \mathbb{N}$, because if it were, then $T(\overline{B_H(0,1)})$ would be covered by a *finite* union of open balls with radius δ, contradicting our previous conclusion that it was *not* totally bounded.

Then given this sequence $(y_n)_{n\in\mathbb{N}} \subset T(\overline{B_H(0,1)})$ it follows that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \overline{B_H(0,1)}$ such that $(Tx_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}$ and with $\|Tx_n - Tx_m\| = \|y_n - y_m\| \geq \delta$ for all $n \neq m$. Next we aim to show that there exists a *subsequence* $(x_{n_k})_{k\in\mathbb{N}}$ that converges weakly to some element x. If successful, then it would follow from the initial assumptions that $\|Tx_{n_k} - Tx\| \to 0$, which implies that $(Tx_{n_k})_{k\in\mathbb{N}}$ is Cauchy. But this clearly contradicts what we showed earlier, because for any $k \neq l$ we have $\|Tx_{n_k} - Tx_{n_l}\| \geq \delta$, i.e. it is *not* Cauchy; and then we could conclude that T must be compact.

This identification is antilinear.

Identify the sequence $(x_n)_{n\in\mathbb{N}} \subset H$ with the sequence $f_n = (\langle \cdot, x_n\rangle)_{n\in\mathbb{N}} \subset H^*$ via Riesz representation. Since H is separable, then it follows from Theorem 5.13 that $(\overline{B_{H^*}(0,1)}, \tau_{w^*})$ is metrizable, and it furthermore follows from Theorem 6.1 (Alaoglu) that $\overline{B_{H^*}(0,1)}$ is compact in the w^*-topology: we conclude that $\overline{B_{H^*}(0,1)}$ is weakly sequentially compact. But we also know that the $(f_n)_{n\in\mathbb{N}}$ sequence is bounded since $\|f_n\| = \|x_n\| \leq 1$, where we used the comments on page 13 and the fact that $(x_n)_{n\in\mathbb{N}} \subset \overline{B_H(0,1)}$. Thus there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that w^*-converges to some $f = \langle \cdot, x\rangle \in H^*$. But then by *HW4-P2(c)* we know that $f_{n_k}(y) = \langle y, x_{n_k}\rangle$ converges to $\langle y, x\rangle = f(y)$ for all $y \in H$. But then we also have $\langle x_{n_k}, y\rangle \to \langle x, y\rangle$ for all $y \in H$, which allows us to apply *HW4-2(a)* to conclude that x_{n_k} converges weakly to x. This is what we wanted to show, and hence we're done with the proof. □ ✓

(d) Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

We wish to apply *MA2-P2(c)*: Consider a sequence $(x_n)_{n\geq 1}$ in $\ell_2(\mathbb{N})$ such that $x_n \to x$ weakly for $n \to \infty$. Then by *MA2-P2(a)* we know that $Tx_n \to Tx$ weakly for $n \to \infty$. But according to Remark 5.3, weak convergence in $\ell_1(\mathbb{N})$ is *equivalent* to convergence in norm, and hence we conclude that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. Note finally that $\ell_2(\mathbb{N})$ is a *separable* Hilbert space (see *HW4-P4(a)*), and so it follows that $T \in \mathcal{K}(H, Y)$. ✓

(e) Show that no $T \in \mathcal{K}(X, Y)$ is surjective.

Assume for contradiction that $T \in \mathcal{K}(X, Y)$ *is* surjective. Then by the Theorem 3.15 (The Open Mapping Theorem) the map T is open, which implies that $T(B_X(0, 1))$ is open. But then there must exist a radius $\delta > 0$ such that $B_Y(0, \delta) \subset T(B_X(0, 1))$. Furthermore, $\overline{T(B_X(0, 1))}$ is compact since T is compact (see Definition 8.1), and we have the relation

$$\overline{B_Y(0, \delta)} \subset \overline{T(B_X(0, 1))}.$$

Since $\overline{B_Y(0, \delta)}$ is a *closed* subset of a *compact* space $\overline{T(B_X(0, 1))}$, then by Theorem 4.22 (Folland) we conclude that $\overline{B_Y(0, \delta)}$ is compact, but this contradicts *MA1-P3(e)*, which stated that the closed (unit) ball in an infinite dimensional normed space was *not* compact (the particular radius is of no importance, any scaling of the closed unit ball will not be compact either). ✓

(f) Let $H = L_2([0, 1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$ for $f \in H$ and $t \in [0, 1]$. Justify that M is self-adjoint, but not compact.

Let g f,g $\in L_2([0, 1], m)$ and consider

$$\langle Mf, g\rangle = \langle tf, g\rangle = \int_{[0,1]} tf\overline{g}dm = \int_{[0,1]} f\overline{tg} = \langle f, tg\rangle = \langle f, Mg\rangle,$$

where we used the fact that t was real. We conclude that M is self-adjoint, i.e. $M = M^*$ (see comments in Theorem 10.1 and on page 41). ✓

If we assume for contradiction that M *is* compact, then since H is a separable (see *HW4-P4(a)*), infinite dimensional Hilbert space and M is self-adjoint, then it follows from the Spectral Theorem for self-adjoint compact operators (Theorem 10.1) that H has an orthonormal basis $(e_n)_{n\geq 1}$ consisting of eigenvectors for T with corresponding eigenvalues $\lambda_n \in \mathbb{R}$ for all $n \geq 1$. But this contradicts the results from *HW6-P3(a)*, which stated that M has no eigenvalues. We conclude that M cannot be compact. ✓

Problem 3: Consider the Hilbert space $H = L_2([0, 1], m)$, where m is the Lebesgue measure. Define $K : [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$K(s, t) = \begin{cases} (1 - s)t, & 0 \leq t \leq s \leq 1, \\ (1 - t)s, & 0 \leq s < t \leq 1, \end{cases}$$

and consider $T \in \mathcal{L}(H, H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s, t)f(t)dm(t), \quad s \in [0, 1], f \in H.$$

(a) Justify that T is compact.

First we recognize that T is defined analogously to a Kernel operator (see page 46): The Lebesgue measure satisfies the σ-finiteness condition, but we need to justify that $K \in L_2([0,1]^2, m_2)$. This follows easily since K is bounded by 1:

$$\int_{[0,1]^2} |K| dm_2 \leq \int_{[0,1]^2} 1 dm_2 = 1 < \infty.$$

Now if follows directly from Proposition 9.12 that T is Hilbert-Schmidt, and in particular (see comment in Proposition 9.12, or refer to Proposition 9.11) T is compact. ✓

(b) Show that $T = T^*$.

Proof. Start by recognizing by simple inspection that $K(s,t) = K(t,s)$. Indeed: if $t \leq s$, then

$$K(s,t) = (1-s)t = K(t,s),$$

whereas if $s < t$ then

$$K(s,t) = (1-t)s = K(t,s).$$

In combination with Fubini's theorem this yields why is Fubini justified?

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) dm(t) \overline{g(s)} dm(s) \\
&= \int_{[0,1]} f(t) \int_{[0,1]} K(s,t) \overline{g(s)} dm(s) dm(t) \\
&= \int_{[0,1]} f(t) \overline{\int_{[0,1]} K(t,s) g(s) dm(s)} dm(t) = \langle f, Tg\rangle,
\end{aligned}$$

where we also implicitly used that K was real-valued, and thus taking its conjugate wouldn't affect its value. We conclude that $T = T^*$. □

(c) Show that

$$(Tf)(s) = (1-s) \int_{[0,s]} t f(t) dm(t) + s \int_{[s,1]} (1-t) f(t) dm(t), \quad s \in [0,1], f \in H.$$

Use this to show that Tf is continuous on $[0,1]$, and that $(Tf)(0) = (Tf)(1) = 0$.

Proof. The first part can be established via straight-forward calculations - we start splitting the integrals into two parts:

$$(Tf)(s) = \int_{[0,1]} K(s,t) f(t) dm(t) = \int_{[0,s]} K(s,t) f(t) dm(t) + \int_{(s,1]} K(s,t) f(t) dm(t),$$

and then simply evaluate $K(s,t)$ in the two respective domains $t \in [0,s]$ and $t \in (s,1]$:

$$\begin{aligned}\int_{[0,s]} K(s,t)f(t)dm(t) &+ \int_{(s,1]} K(s,t)f(t)dm(t)\\ &= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{(s,1]} (1-t)sf(t)dm(t)\\ &= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{(s,1]} (1-t)f(t)dm(t), \end{aligned} \tag{6}$$

which is the expression we wanted (Note: whether or not the singleton $\{s\}$ is included in the latter integral domain is irrelevant since it is a Lebesgue null set).

The purpose of rewriting $(Tf)(s)$ in this manner is to remove the dependence on s from the integrands. Because now we notice that (6) is a composition of continuous functions: $(1-s)$ and s are clearly continuous, while we know from Analysis 2 that maps of the form $s \mapsto \int_{[0,s]} tf(t)dm(t)$ and $s \mapsto \int_{(s,1]}(1-t)f(t)dm(t)$ are continuous. We conclude that Tf is continuous on $[0,1]$. Alternatively one could have applied the continuity lemma (Theorem 12.4, Schilling).

Ref or Proof.

How ?

Finally we simply plug in the values 0 and 1 to conclude

$$\begin{aligned}(Tf)(0) &= (1-0)\int_{[0,0]} tf(t)dm(t) + 0\cdot\int_{(0,1]}(1-t)f(t)dm(t) = 0+0\\ &= 0\\ &= 0+0 = (1-1)\int_{[0,1]} tf(t)dm(t) + 1\cdot\int_{(1,1]}(1-t)f(t)dm(t) = (Tf)(1).\end{aligned}$$

✓

□

Problem 4: Consider the Schwarz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

(a) For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$. Justify that $g_k \in \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$, and compute $\mathcal{F}(g_k)$ for $k = 0,1,2,3$.

First we note that $x \mapsto \frac{|x|}{\sqrt{2}} := \|x\|_0$ is a norm; indeed it is simply a scaled version of the Euclidean norm. Then it follows from *HW7-P1* that the function

$$f(x) := e^{-x^2/2} = e^{-\left(\frac{|x|}{\sqrt{2}}\right)^2} = e^{-\|x\|_0^2}$$

is a Schwartz function. It furthermore follows from *HW7-P1(a)* that $g_k = x^k f \in \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$, and from *HW7-P1(c)* that the $g_k \in L_1(\mathbb{R})$ for all $k \geq 0$. Thus we are in a position to apply Proposition 11.13(d) to calculate the Fourier transforms. We know from Proposition 11.4 that $\hat{f}(\xi) = f(\xi) = e^{-\xi^2/2}$, so let us start by calculating $\partial^k \hat{f}$ for $k = 1, 2, 3$:

$$\frac{\partial}{\partial \xi}\hat{f}(\xi) = \frac{\partial}{\partial \xi}e^{-\xi^2/x} = -\xi e^{-\xi^2/2}$$
$$\frac{\partial^2}{\partial \xi^2}\hat{f}(\xi) = \frac{\partial}{\partial \xi}(-\xi e^{-\xi^2/2}) = -e^{-\xi^2/2} - \xi(-\xi e^{-\xi^2/2}) = (\xi^2 - 1)e^{-\xi^2/2}$$
$$\frac{\partial^3}{\partial \xi^3}\hat{f}(\xi) = \frac{\partial}{\partial \xi}\left((\xi^2 - 1)e^{-\xi^2/2}\right) = 2\xi e^{-\xi^2/2} + (\xi^2 - 1)(-\xi)e^{-\xi^2/2} = (3 - \xi^2)\xi e^{-\xi^2/2}.$$

With the above calculations in mind, we now apply Proposition 11.13(d) and find that

$$\mathcal{F}(g_0) = g_0 = e^{-\xi^2/2}$$
$$\mathcal{F}(g_1) = i^{|1|}\frac{\partial}{\partial \xi}\hat{f}(\xi) = -i\xi e^{-\xi^2/2}$$
$$\mathcal{F}(g_2) = i^{|2|}\frac{\partial^2}{\partial \xi^2}\hat{f}(\xi) = (1 - \xi^2)e^{-\xi^2/2}$$
$$\mathcal{F}(g_3) = i^{|3|}\frac{\partial^3}{\partial \xi^3}\hat{f}(\xi) = i(\xi^2 - 3)\xi e^{-\xi^2/2},$$

where the calculation of $\mathcal{F}(g_0)$ was just a reminder of Proposition 11.4.

(b) Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

After some tinkering one comes to the realisation that the functions

$$h_0 := g_0 = e^{-x^2/2}$$
$$h_1 := g_3 - \frac{3}{2}g_1 = x^3 e^{-x^2/2} - \frac{3}{2}xe^{-x^2/2} = (x^2 - \frac{3}{2})xe^{-x^2/2}$$
$$h_2 := g_0 - 2g_2 = e^{-x^2/2} - 2x^2 e^{-x^2/2} = (1 - 2x^2)e^{-x^2/2}$$
$$h_3 := g_1 = xe^{-x^2/2}$$

satisfy the desired conditions. Let us verify this using linearity of the Fourier transform and our previous calculations:

$$\mathcal{F}(h_0) = \mathcal{F}(g_0) = e^{-\xi^2/2} = i^0 e^{-\xi^2/2} = i^0 h_0(\xi)$$
$$\mathcal{F}(h_1) = \mathcal{F}(g_3) - \frac{3}{2}\mathcal{F}(g_1) = i(\xi^2 - 3)\xi e^{-\xi^2/2} - \frac{3}{2}(-i\xi e^{-\xi^2/2}) = i(\xi^2 - \frac{3}{2})\xi e^{-\xi^2/2} = ih_1(\xi)$$
$$\mathcal{F}(h_2) = \mathcal{F}(g_0) - 2\mathcal{F}(g_2) = e^{-\xi^2/2} - 2(1 - \xi^2)e^{-\xi^2/2} = -(1 - 2\xi^2)e^{-\xi^2/2} = i^2 h_2(\xi)$$
$$\mathcal{F}(h_3) = \mathcal{F}(g_1) = -i\xi e^{-x^2/2} = i^3 h_3(\xi).$$

(c) Show that $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$.

Consider first $\mathcal{F}(\mathcal{F}(f))$:

$$\mathcal{F}(\mathcal{F}(f)(y))(\xi) = \int\int f(x)e^{-ixy}dm(x)e^{-iy\xi}dm(y).$$

Now substitute $x = -z$ and use the fact that the Lebesgue measure is invariant under this rotation (or *reflection* if you prefer, see Analysis 2) and thus

$$\int\int f(x)e^{-ixy}dm(x)e^{-iy\xi}dm(y) = \int\int f(-z)e^{izy}dm(z)e^{-iy\xi}dm(y)$$
$$= \int\int S_{-1}f(z)e^{izy}dm(z)e^{-iy\xi}dm(y). \quad (7)$$

where S_{-1} is defined on page 62. The above is simply the Fourier transform of the *inverse* Fourier transform of the function $S_{-1}f$. But since f is assumed to be a Schwartz function, then $S_{-1}f \in \mathscr{S}(\mathbb{R})$ by *HW7-P1(d)*, and then it follows from Corollary 12.12(iii) that

$$\mathcal{F}(\mathcal{F}(f))(\xi) = \mathcal{F}(\mathcal{F}^*(S_{-1}f))(\xi) = S_{-1}f(\xi).$$

Then simply apply this transformation once again to yield

$$\mathcal{F}^4(f) = \mathcal{F}^2(\mathcal{F}^2(f)) = S_{-1}S_{-1}f = f.$$

(d) Use (c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ are $\{1, i, -1, -i\}$.

Given the above assumptions we have $\mathcal{F}^4(f) = \lambda^4 f = f$, which implies $(\lambda^4 - 1)f = 0$, which in turn implies $\lambda^4 = 1$ since f was assumed to be non-negative. The only solutions to this complex polynomial equation are $1, i, -1$ and $-i$, so *if* $\mathcal{F}$ has any eigenvalues $\lambda \in \mathbb{C}$, then we must have $\lambda \in \{1, i, -1, -i\}$. But we showed in *MA2-P4(b)* that these *are* indeed eigenvalues, since $1, i, -1 = i^2$ and $-i = i^3$ were precisely the values such that $\mathcal{F}(h_0) = 1 \cdot h_0, \mathcal{F}(h_1) = i \cdot h_1, \mathcal{F}(h_2) = -1 \cdot h_2, \mathcal{F}(h_3) = -i \cdot h_3$, and so we conclude that the eigenvalues of of $\mathcal{F}$ are precisely $\{1, i, -1, -i\}$.

non-zero

Problem 5: Let $(x_n)_{n\geq 1}$ be a dense subset of $[0, 1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0, 1]$. Show that $\text{supp}(\mu) = [0, 1]$.

non-empty!

Proof. Recall from *HW8-P3(a)* that the support of μ is defined as the *complement* to the set $N :=$ "the union of all open subsets $U \subset [0, 1]$ such that $\mu(U) = 0$". To conclude that $\text{supp}(\mu) = [0, 1]$ it suffices to show that *no* open subsets of $[0, 1]$ have measure zero, because then N would clearly be empty, in which case $\text{supp}(\mu) = N^c = \emptyset^c = [0, 1]$.

non-empty

Let $U \subset [0, 1]$ be an open set, and let us show that $\mu(U) > 0$. Because U is open, there exists a compact subset $K := [a, b] \subset U$ for some $0 < a < b < 1$. Consider some element in K^o – for instance $x := (a + b)/2$ – and note that x can be approximated arbitrarily well via elements in $(x_n)_{n\geq 1}$ due to the denseness assumption. In particular there must exist $N \in \mathbb{N}$ such that $|x - x_N| \leq (b - a)/2$, i.e. such that $x_N \in K \subset U$. Radon measures satisfy the *inner regularity* condition, so we can now bound the μ-measure of U from below via the calculation

$$\mu(U) = \sup\{\mu(C) : C \text{ compact}, C \subset U\} \geq \mu(K)$$
$$= \sum_{n\in\mathbb{N}} 2^{-n}\delta_{x_n}(K) \geq 2^{-N}\delta_{x_N}(K) = 2^{-N} > 0,$$

non-empty

where we used that K (by construction) was a compact subset of U. This concludes the proof, as we have shown that any given open set $U \subset [0, 1]$ will have a measure $\mu(U) > 0$, and hence not contribute to the union defining N.

$\square$