

## Assignment 2, Functional Analysis

Jan McGarry Furriol (sbh877)

January 2020

**Problem 1.** Let  $H$  be an infinite dimensional separable Hilbert space with orthonormal basis  $(e_n)_{n \geq 1}$ . Set  $f_N = N^{-1} \sum_{n=1}^{N^2} e_n$  for all  $N \geq 1$ .

(a) We show that the  $f_N$  converge weakly to 0 as  $N \rightarrow \infty$  using the characterization of weak convergence of HW4 Problem 2. That is, we prove that for any functional  $g: H \rightarrow \mathbb{K}$ , which by the Riesz representation theorem is of the form  $g = \langle \cdot, x \rangle$  for some  $x \in H$ , we have that  $\langle f_N, x \rangle$  converges weakly to 0 as  $N \rightarrow \infty$ . Look at

$$\lim_{N \rightarrow \infty} \langle f_N, x \rangle = \lim_{N \rightarrow \infty} N^{-1} \sum_{n=1}^{N^2} \langle e_n, x \rangle,$$

which is 0, because by Bessel's inequality  $\sum_{n=1}^{\infty} \langle e_n, x \rangle^2 \leq \|x\|^2$  bounded, hence convergent since the terms are positive, and the product of two convergent sequences converges to the product of the limits. On the other hand,  $\|f_N\| = 1$  because  $\|\cdot\| \geq 0$  and

$$\|f_N\|^2 = \langle N^{-1} \sum_{n=1}^{N^2} e_n, N^{-1} \sum_{n=1}^{N^2} e_n \rangle = \frac{1}{N^2} \sum_{n,m=1}^{N^2} \delta_n^m = \frac{1}{N^2} N^2 = 1.$$

(b) *Notation:*  $F = \{f_N : N \geq 1\}$  and  $B = \{x \in H : \|x\| \leq 1\}$ .

First of all, notice that since  $\text{co}(F)$  is convex, its norm closure equals its weak closure (Theorem 5.7), so  $K$  is closed in the weak topology. Also,  $F \subseteq B$  by (a), where the latter is convex, thus  $\text{co}(F) \subseteq B$ ; taking norm closure we get  $K \subseteq B$ . Thus, if we show that  $B$  is compact in the weak topology, we will obtain that  $K$  is also weakly compact, because closed subsets of compact ones are also compact. The norm closed unit ball  $B$  is weakly compact by Theorem 6.3, because  $H$  is reflexive, since it is a Hilbert space (Proposition 2.10).

The fact that  $0 \in K$  follows because  $(f_N)_{N \geq 1} \subseteq K$  converges weakly to 0, and we have just proven that  $K$  is weakly closed.

(c) Let  $(e_i)_{i \geq 1}$  be an orthonormal basis for  $H$ . First, we show that each  $f_N$  is an extreme point of  $K$ . Suppose that we have  $f_N = \alpha x_1 + (1 - \alpha)x_2$  for some  $0 < \alpha < 1$  and  $x_1, x_2 \in K$ . We must show that  $x_1 = x_2 = f_N$ . In part (a) we saw that  $\|f_N\| = 1$ , and it followed that  $K \subseteq \overline{B}_H(0, 1)$ . Thus, we have

$$1 = \|f_N\| = \|\alpha x_1 + (1 - \alpha)x_2\| \leq \alpha \|x_1\| + (1 - \alpha)\|x_2\| \leq 1,$$

so both inequalities must be equalities. From the first of them we deduce that  $\alpha x_1 = \lambda(1 - \alpha)x_2$  for some  $\lambda \in \mathbb{R}$ , since it is only then that the triangular

inequality is actually an equality; from the second we deduce that  $\|x_1\| = \|x_2\| = 1$ , because these norms are not greater than 1, as we have pointed out before. Putting both together, we get  $\lambda = \pm\alpha/(1-\alpha) \in \mathbb{R}$ , therefore  $x_1 = \pm x_2$ . Now we prove something that will allow us to conclude that  $x_1 = x_2$ .

*Claim:* If  $x \in K$ , then  $\langle x, e_i \rangle \geq 0$  for every  $i \geq 1$ .

*Proof.* If  $x \in K = \overline{\text{co}(F)}$ , then it is a limit of a convergent sequence in norm of points in the convex hull of  $F$ , i.e.,  $x = \lim_{k \rightarrow \infty} x^k$  with

$$x^k = \sum_{j=1}^n a_j f_{N_j}, \quad \text{with } a_j > 0, \text{ and } \sum_{j=1}^n a_j = 1,$$

where, in fact, all indices and coefficients depend on  $k$ . Fix  $e_i$  arbitrary in the orthonormal basis, and notice that the limit comes out in what follows because  $\langle \cdot, e_i \rangle: (H, \|\cdot\|) \rightarrow \mathbb{K}$  is continuous:

$$\langle x, e_i \rangle = \lim_{k \rightarrow \infty} \langle x^k, e_i \rangle.$$

This quantity is non-negative since  $\langle x^k, e_i \rangle$  is either  $a_i > 0$  as above, or zero.  $\square$

To finish, it remains to prove that 0 is also an extreme point of  $K$ ; this makes sense because  $0 \in K$  by part (b). Suppose that we have  $0 = \alpha x_1 + (1-\alpha)x_2$  with  $0 < \alpha < 1$ . By the claim, we have  $\langle x_1, e_i \rangle, \langle x_2, e_i \rangle \geq 0$  for every  $i \geq 1$ . Also,  $\alpha, (1-\alpha) > 0$ , so we deduce that  $\langle x_1, e_i \rangle = \langle x_2, e_i \rangle = 0$  for all  $i \geq 1$ . Thus  $x_1 = x_2 = 0$ .

(d) Write  $F = \{f_N\}_{N \geq 1}$ , and notice that, since the norm closure agrees with the weak closure on convex sets such as  $\text{co}(F)$ , we have that  $K = \overline{\text{co}(F)}^w$ . The other hypothesis of the Milman theorem are also satisfied:  $(H, \tau_w)$  is a LCTVS,  $K$  is non-empty, compact (in  $\tau_w$  by (b)) and convex subset of  $H$ . By the mentioned theorem, we conclude that  $\text{Ext}(K) \subseteq \overline{F}^w = F \cup \{0\}$ , so there are no other extreme points in  $K$ . The last equality holds because  $F \cup \{0\}$  is compact and  $(H, \tau_w)$  is Hausdorff (a convergent sequence with its limit point, which is unique in a Hausdorff space, is compact. This is a standard proof with an open cover argument), hence it is closed, while  $F$  is not because  $0 \notin F$ ; recall that the closure is the smallest closed set which contains  $F$ .

**Problem 2.** Let  $X$  and  $Y$  be infinite dimensional Banach spaces.

(a) Let  $T \in \mathcal{L}(X, Y)$ , and let  $(x_n)_{n \geq 1}$  be a sequence in  $X$  converging weakly to  $x \in X$ . We show that  $(Tx_n)_{n \geq 1}$  converges weakly to  $Tx$  in  $Y$ , by using the characterization of HW4 Problem 2 (a). Thus, let  $f: Y \rightarrow \mathbb{K}$  be any functional, and now consider the functional  $fT: X \rightarrow \mathbb{K}$ , obtained by precomposing with the bounded linear functional  $T$ . Then, using the characterization of weak convergence with  $(x_n)_{n \geq 1}$ , we obtain that  $f(Tx_n)$  converges to  $f(Tx)$  in  $\mathbb{K}$ .

(b) Let  $T \in \mathcal{L}(X, Y)$  compact, and let  $(x_n)_{n \geq 1}$  be a sequence in  $X$  converging weakly to  $x \in X$ . We show that  $\|Tx_n - Tx\| \rightarrow 0$ , by proving that every subsequence of  $(Tx_n)_{n \geq 1}$  admits a further subsequence which converges in norm to  $Tx$ .

Let  $(Tx_{n_k})_{k \geq 1}$  be a subsequence; note that  $(x_{n_k})_{k \geq 1}$  also converges weakly to  $x$ . By HW4 Problem 2(b), the sequence  $(x_{n_k})_{k \geq 1}$  is bounded, so there exists a further subsequence  $(x_{n_{k(i)}})_{i \geq 1}$  such that  $(Tx_{n_{k(i)}})_{i \geq 1}$  converges in norm (in

particular weakly) to some  $y \in Y$ , by the characterizations of compactness of  $T$  in Proposition 8.2. Finally, the limit point  $y$  of this sequence must be  $Tx$ , because  $(Tx_n)_{n \geq 1}$  converges weakly to  $Tx$  by part (a), so also does  $(Tx_{n_{k(i)}})_{i \geq 1}$ ; since  $(H, \tau_w)$  is a Hausdorff topological space, the limit is unique, and we get  $y = Tx$ .

(c) We prove that  $T$  is compact using the characterization of Proposition 8.2. Let  $(x_n)_{n \geq 1}$  be a bounded sequence in  $X$ , which by scaling if necessary, we may assume that it is contained in the unit ball  $B_H(0, 1)$ . Since  $H$  is a Hilbert space, it is reflexive (Proposition 2.10), hence  $\overline{B_H}(0, 1)$  is compact with respect to the weak topology, by Theorem 6.3.

*Claim:*  $\overline{B_H}(0, 1)$  with the weak topology is metrizable.

*Proof.* Since  $H$  is separable, it follows that the Hilbert space  $H^*$  is separable. Indeed, the map  $F: H \rightarrow H^*$  defined by  $y \mapsto \langle \cdot, y \rangle$  is onto, by the Riesz Representation Theorem, and continuous; indeed, it is actually an isometry, in particular continuous, because  $\|F(y)\| = \|\langle \cdot, y \rangle\| \leq \|y\|$ , by the Cauchy-Schwartz inequality after writing out the definition of the norm of the functional, and  $\|y\|^2 = \langle y, y \rangle \leq \|F(y)\| \|y\|$ , by boundedness of  $F(y)$ , implies  $\|y\| \leq \|F(y)\|$ . Thus, if  $A$  is a countable dense subset of  $H$ , then  $F(A)$  is a countable dense subset of  $H^*$ , by continuity and surjectivity; indeed,  $H^* = F(H) = F(\overline{A}) \subseteq \overline{F(A)} \subseteq H^*$ .

Now, by Theorem 5.13,  $\overline{B_{(H^*)^*}}(0, 1)$  with the weak\* topology is metrizable. Notice that the weak and weak\* topology agree on  $(H^*)^*$  because  $H^*$  is reflexive, by Theorem 5.9. The conclusion follows from reflexivity:  $\Lambda: (H^*)^* \cong H$ .  $\square$

Consequently, the compactness of  $\overline{B_H}(0, 1)$  is equivalent to its sequential compactness, so there exists a weakly convergent subsequence  $(x_{n_k})_{k \geq 1}$ . By the hypothesis, we get that  $(Tx_{n_k})_{k \geq 1}$  converges in norm in  $Y$ . This proves that  $T$  is compact.

(d) Let  $(x_n)_{n \geq 1}$  be a sequence converging to  $x$  weakly in  $X$ , and  $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$  be arbitrary. Then, by part (a),  $(Tx_n)_{n \geq 1}$  converges weakly to  $Tx$ , or equivalently, it converges in norm to  $Tx$ , by Remark 5.3. Notice that  $l_2(\mathbb{N})$  is a separable (HW4 P4) infinite Hilbert space, and that we have just shown that the conditions under which part (c) ensures that  $T$  is compact hold.

(e) Suppose that  $T \in \mathcal{L}(X, Y)$  is onto. Then  $T$  is open by the Open Mapping Theorem. In particular,  $TB_X(0, 1)$  is open, so there exists an open ball  $B_Y(0, r)$ , for small enough  $r > 0$ , centered at  $T(0) = 0$ , and which is contained in  $TB_X(0, 1)$ . Thus, we have

$$\overline{B_Y(0, r)} \subseteq \overline{TB_X(0, 1)}.$$

Now, notice that  $\overline{B_Y(0, r)}$  is not compact by Problem 3(e) in the first assignment, because  $Y$  is infinite dimensional and the mentioned set is homeomorphic to the closed unit ball by scaling.

If  $T$  were compact, i.e.,  $\overline{TB_X(0, 1)}$  compact, then  $\overline{B_Y(0, r)}$  would also be compact, because it is a closed subset. This is a contradiction, hence  $T$  is not compact.

(f)  $M$  is self-adjoint:

$$\begin{aligned}\langle Mf, g \rangle &= \int_{[0,1]} (tf(t)) \overline{g(t)} dm(t) \\ &= \int_{[0,1]} f(t) \overline{(tg(t))} dm(t) \\ &= \langle f, Mg \rangle.\end{aligned}$$

Suppose that  $M$  is compact. Since we have just shown that  $M$  is self-adjoint, and  $L_2([0, 1], m)$  is a separable (HW4 P4) infinite dimensional Hilbert space, the Spectral Theorem ensures the existence of infinitely many eigenvalues for  $M$ . However, we showed in HW6 Problem 3 that  $M$  has no eigenvalues. We conclude that  $M$  is not compact.

**Problem 3.** Consider the Hilbert space  $H = L_2([0, 1], m)$ , where  $m$  is the Lebesgue measure.

(a) The operator  $T$  is the kernel operator associated to  $K$ , and it is compact by Theorem 9.6. The hypothesis are satisfied because  $K$  is continuous on  $[0, 1] \times [0, 1]$  (it is clearly piece-wise continuous and these agree on the intersection), the topological space  $[0, 1]$  is compact and Hausdorff, and the Lebesgue measure is a finite Borel measure on  $[0, 1]$ .

(b) The operator  $T$  is self-adjoint:

$$\begin{aligned}\langle Tf, g \rangle &= \int_{[0,1]} \left( \int_{[0,1]} K(s, t) f(t) dm(t) \right) \overline{g(s)} dm(s) \\ &= \int_{[0,1]} \int_{[0,1]} f(t) \overline{K(s, t) g(s)} dm(s) dm(t) \\ &= \int_{[0,1]} f(t) \overline{\int_{[0,1]} K(s, t) g(s) dm(s)} dm(t) \\ &= \langle f, Tg \rangle,\end{aligned}$$

where in the second equality we first pulled  $\overline{g(s)}$  inside, then we applied Fubini's Theorem, and finally we have used that  $K(s, t) = \overline{K(s, t)}$  because it is real valued. The hypothesis of Fubini's Theorem are satisfied because the Lebesgue measure is  $\sigma$ -finite on  $[0, 1]$ , and the product  $K(s, t)f(t)\overline{g(s)}$  is integrable on  $[0, 1] \times [0, 1]$ ; this holds because  $K(s, t) \leq M$  is bounded on  $[0, 1] \times [0, 1]$ , because  $K$  is continuous and the domain is compact, so we have

$$\begin{aligned}\int_{[0,1] \times [0,1]} |K(s, t) f(t) \overline{g(s)}| dm(s, t) &\leq M \int_{[0,1] \times [0,1]} |f(t)| |g(s)| dm(s, t) \\ &= \|f\|_1 \|g\|_1 < \infty.\end{aligned}$$

In the last equality we have used that  $|f(t)|$  and  $|g(s)|$  are non-negative, and measurable functions because  $f, g \in L_2([0, 1], m)$ , so applying Tonelli and using that each is independent of the other variable, we get what we wrote. The  $\|\cdot\|_1$ -norms are finite because  $L_2([0, 1], m) \subseteq L_1([0, 1], m)$ , by HW2 Problem 2.

(c) The stated formula for  $(Tf)(s)$  is obtained directly from the definition of  $K$ : for a fixed  $s$ , the function  $K(s, t)$  is given by  $(1-s)t$  for  $t$  in the interval

$[0, s]$ , and  $s(1 - t)$  for  $t$  in the interval  $[s, 1]$ ; to get  $Tf$  we integrate over  $t$ , so  $(1 - s)$  and  $s$  come out of the integral.

With  $(Tf)(s)$  expressed in this way, we see that it is continuous because it is a product and sum of continuous functions. Indeed, the integrals are continuous in  $s$ . The argument for this, which works in more generality, is as follows. Let  $h \in L_2([0, 1], m)$ , and we want to see that  $H(s) := \int_{[0, s]} h dm$  is continuous in  $s \in [0, 1]$ . Consider any convergent sequence  $(s_n)_{n \geq 1}$  to a fixed  $s_0$ , and we show that the  $H(s_n)$  converge to  $H(s_0)$ . First, we consider every  $H(s_n) = \int_{[0, 1]} h \cdot \chi_{[0, s_n]} dm$  over the same measure space. Then, Lebesgue Dominated Convergence Theorem gives the result, since pointwise convergence is clear, and each function is dominated by  $|h|$ , which is integrable.

It is clear that  $(Tf)(0) = (Tf)(1) = 0$  because in both cases one of the integrals gets multiplied by zero and the other integrals is over a point, a set of measure zero.

**Problem 4.** Consider the Schwartz space  $\mathcal{S}(\mathbb{R})$ .

(a) In HW7 Problem 1 we first saw that  $e^{-x^2} \in \mathcal{S}(\mathbb{R})$ , and then, by part (d) in that problem,  $S_{\sqrt{2}}(e^{-x^2}) = e^{-x^2/2} \in \mathcal{S}(\mathbb{R})$ , and finally, by part (a) in the same, we get  $x^k e^{-x^2/2} \in \mathcal{S}(\mathbb{R})$  for all  $k \geq 0$ .

Write  $\varphi(x) := e^{-x^2/2}$  and  $g_k := x^k \varphi$ . Let us compute  $\mathcal{F}(g_k)$ . Notice that each  $g_k$  is in  $L_1(\mathbb{R})$ , by problem 1(c) in HW7. Hence, by Proposition 11.13, we have

$$\mathcal{F}(g_k) = i^k (\partial^k \hat{\varphi}).$$

This gives the following, using that  $\mathcal{F}(\varphi) = \varphi$  by Proposition 11.4:

$$\begin{aligned} \mathcal{F}(g_0) &= \varphi, \\ \mathcal{F}(g_1) &= -i\xi\varphi, \\ \mathcal{F}(g_2) &= i^2(-\varphi + \xi^2\varphi) = \varphi - \xi^2\varphi, \\ \mathcal{F}(g_3) &= i^3(\xi\varphi + 2\xi\varphi - \xi^3\varphi) = i(\xi^3\varphi - 3\xi\varphi). \end{aligned}$$

(b) We write the computations of part (a) as follows:

$$\begin{aligned} \mathcal{F}(g_0) &= g_0, \\ \mathcal{F}(g_1) &= -ig_1, \\ \mathcal{F}(g_2) &= g_0 - g_2, \\ \mathcal{F}(g_3) &= ig_3 - 3ig_1. \end{aligned}$$

We can already see that we can put  $h_0 = g_0$  and  $h_3 = g_1$ . For  $k = 1, 2$ , we set up a system of linear equations by imposing

$$\mathcal{F}(ag_1 + bg_2 + cg_3 + dg_4) = i^k(ag_1 + bg_2 + cg_3 + dg_4),$$

(use linearity of  $\mathcal{F}$  to develop the left hand side expression). All in all, we get

the following:

$$\begin{aligned} h_0 &= g_0, \\ h_1 &= -\frac{3}{2}g_1 + g_3, \\ h_2 &= -\frac{1}{2}g_0 + g_2, \\ h_3 &= g_1. \end{aligned}$$

(c) Let  $f \in \mathcal{S}(\mathbb{R})$ . We prove that  $\mathcal{F}^4(f) = f$ . First of all, notice that  $\mathcal{F}(f)$  is also a Schartz function by Proposition 11.13 (e), so it makes sense to iterate  $\mathcal{F}$ . Our claim follows from the fact that  $\mathcal{F}^2(f)(x) = f(-x)$  for all  $x \in \mathbb{R}$ , because then applying  $\mathcal{F}^2$  again gives back  $f$ . Proof of the fact:

$$\begin{aligned} \mathcal{F}^2(f)(x) &= \int_{\mathbb{R}} \hat{f}(\xi) e^{-i\langle \xi, x \rangle} dm(\xi) \\ &= \int_{\mathbb{R}} \hat{f}(\xi) e^{i\langle \xi, -x \rangle} dm(\xi) \\ &= \mathcal{F}^* \mathcal{F}(f)(-x) \\ &= f(-x), \end{aligned}$$

where in the last step we have used that  $\mathcal{F}$  and  $\mathcal{F}^*$  are mutual inverses on  $\mathcal{S}(\mathbb{R})$  (Corollary 12.12).

(d) Suppose that  $f \in \mathcal{S}(\mathbb{R})$  is non-zero and  $\mathcal{F}(f) = \lambda f$ , for  $\lambda \in \mathbb{C}$ . Then,  $\mathcal{F}^4(f) = \lambda^4 f$  by linearity, and by part (c) we get  $\lambda^4 = 1$ . The solutions to the later equation are  $\lambda = 1, i, -1, -i$ .

**Problem 5.** Let  $(x_n)$  be a dense subset of  $[0, 1]$  and consider the Radon measure  $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}$  on  $[0, 1]$ .

By definition (HW3 Problem 3),  $\text{supp}(\mu)$  is the complement of the union of all open sets of  $[0, 1]$  on which  $\mu$  vanishes. Since  $(x_n)_{n \geq 1}$  is dense in  $[0, 1]$ , any non-empty open set  $U$  in  $[0, 1]$  contains some  $x_i$ , thus  $2^{-i} = \delta_{x_i}(U) \leq \mu(U)$ , since all the summands are positive. So  $\mu$  only vanishes on the empty set, thus  $\text{supp}(\mu) = [0, 1]$ .