FunkAn, Mandatory 2

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Problem 1: Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$. Let K be the norm closure of $\text{co}\{f_N : N \geq 1\}$.

(a) Show that $f_N \longrightarrow 0$ weakly, as $N \longrightarrow \infty$, while $||f_N|| = 1$, for all $N \geq 1$.

First we show that $||f_N|| = 1$.

$$||f_N|| = |\langle f_N, f_N\rangle|^{1/2} = \left|\left\langle N^{-1}\sum_{l=1}^{N^2} e_l, N^{-1}\sum_{k=1}^{N^2} e_k\right\rangle\right|^{1/2} = \left|\left(N^{-1}\right)^2\left\langle \sum_{l=1}^{N^2} e_l, \sum_{k=1}^{N^2} e_k\right\rangle\right|^{1/2}$$

$$= \left|N^{-2}\sum_{l=1}^{N^2}\left\langle e_l, \sum_{k=1}^{N^2} e_k\right\rangle\right|^{1/2} = \left|N^{-2}\sum_{l=1}^{N^2}\sum_{k=1}^{N^2}\langle e_l, e_k\rangle\right|^{1/2}$$

Now since $(e_n)_{n\geq 1}$ is an orthonormal basis, $\langle e_l, e_k\rangle = 0$ when $l \neq k$ and $\langle e_l, e_l\rangle = 1$ so

$$\left|N^{-2}\sum_{l=1}^{N^2}\sum_{k=1}^{N^2}\langle e_l, e_k\rangle\right|^{1/2} = \left|N^{-2}\sum_{l=1}^{N^2}\langle e_l, e_l\rangle\right|^{1/2} = \left|N^{-2}\sum_{l=1}^{N^2} 1\right|^{1/2} = |N^{-2}N^2|^{1/2} = 1$$

Now if we can show that $h(f_N) \longrightarrow H(0) = 0$ as $N \longrightarrow \infty$ for all $h \in H^*$, then HW 4 problem 2 (a) gives that $f_N \longrightarrow 0$ weakly as $N \longrightarrow \infty$. But by the Riesz representation theorem for each $h \in H^*$ there exists $y \in H$ such that $h(x) = \langle x, y\rangle$ for all $x \in H$. So if we can show that $\langle f_N, y\rangle \longrightarrow 0$ as $N \longrightarrow \infty$ for all $y \in H$, we are done.

We see that if $y = \sum_{n=1}^{\infty}\alpha_i e_i$ then

$$\langle y, e_n\rangle = \langle\sum_{n=1}^{\infty}\alpha_i e_i, e_n\rangle = \sum_{i=1}^{\infty}\alpha_i\langle e_i, e_n\rangle = \alpha_i$$

so $y = \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i$. By Bessel's inequality we know that $\sum_{n=1}^{\infty}|\langle y, e_n\rangle|^2 \leq ||y||^2 < \infty$ hence given $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $\sum_{n=M}^{\infty}|\langle y, e_n\rangle|^2 < \frac{\varepsilon^2}{4}$.

Now for arbitrary $y \in H$ let $M \in \mathbb{N}$ be given as above. We consider

$$|\langle f_N, y\rangle| = \left|\left\langle f_N, \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i\right\rangle\right| \leq \left|\left\langle f_N, \sum_{i=1}^{M-1}\langle y, e_i\rangle e_i\right\rangle\right| + \left|\left\langle f_N, \sum_{i=M}^{\infty}\langle y, e_i\rangle e_i\right\rangle\right|$$

Now by using the Cauchy-Schwartz inequality we get

$$\left|\left\langle f_N, \sum_{i=M}^{\infty}\langle y, e_i\rangle e_i\right\rangle\right| \leq ||f_N||\left|\left|\sum_{i=M}^{\infty}\langle y, e_i\rangle e_i\right|\right| = \left|\left|\sum_{i=M}^{\infty}\langle y, e_i\rangle e_i\right|\right| = \left|\left\langle \sum_{i=M}^{\infty}\langle y, e_i\rangle e_i, \sum_{i=M}^{\infty}\langle y, e_i\rangle e_i\right\rangle\right|^{1/2}$$

$$= \left| \sum_{l=M}^{\infty} \sum_{k=M}^{\infty} \langle y, e_l \rangle \overline{\langle y, e_k \rangle} \langle e_l, e_k \rangle \right|^{1/2} = \left| \sum_{l=M}^{\infty} \langle y, e_l \rangle \overline{\langle y, e_l \rangle} \right|^{1/2} \leq \left(\sum_{l=M}^{\infty} |\langle y, e_l \rangle|^2 \right)^{1/2} < \left(\frac{\varepsilon^2}{4} \right)^{1/2} = \frac{\varepsilon}{2}$$

Now we consider

$$\left| \left\langle f_N, \sum_{i=1}^{M-1} \langle y, e_i \rangle e_i \right\rangle \right| = \left| \left\langle N^{-1} \sum_{j=1}^{N^2} e_j, \sum_{i=1}^{M-1} \langle y, e_i \rangle e_i \right\rangle \right|$$

$$= N^{-1} \left| \sum_{j=1}^{N^2} \sum_{i=1}^{M-1} \overline{\langle y, e_i \rangle} \langle e_j, e_i \rangle \right| \leq N^{-1} \left| \sum_{i=1}^{M-1} \overline{\langle y, e_i \rangle} \right|$$

but $\left| \sum_{i=1}^{M-1} \overline{\langle y, e_i \rangle} \right|$ is constant with respect to N, hence there exists $M' \in \mathbb{N}$ such that

$$\left| \left\langle f_N, \sum_{i=1}^{M-1} \langle y, e_i \rangle e_i \right\rangle \right| \leq N^{-1} \left| \sum_{i=1}^{M-1} \overline{\langle y, e_i \rangle} \right| < \frac{\varepsilon}{2}$$

whenever $N > M'$. Now putting this together, given $\varepsilon > 0$ we have for $y \in H$

$$|\langle f_N, y \rangle| \leq \left| \left\langle f_N, \sum_{i=1}^{M-1} \langle y, e_i \rangle e_i \right\rangle \right| + \left| \left\langle f_N, \sum_{i=M}^{\infty} \langle y, e_i \rangle e_i \right\rangle \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $N > \max\{M, M'\}$. But this means that $\langle f_N, y \rangle \longrightarrow 0$ as $N \longrightarrow \infty$ which is what we needed so $f_N \longrightarrow$ weakly as $N \longrightarrow \infty$.

(b) Argue that K is weakly compact, and that $0 \in K$.

By (a) $(f_N)_{N \geq 1} \subset \text{co}\{f_N | N \geq 1\}$ converges weakly to 0, so by HW5 problem 1 there exists a sequence $(x_n)_{n \geq 1} \subset \text{co}\{f_N | N \geq 1\} \subset K$ which converges to 0 in norm. Now since K is the norm closure of $\text{co}\{f_N | N \geq 1\}$ and $(x_n)_{n \geq 1} \subset K$ converges to 0 in norm, we must have $0 \in K$.

H is reflexive as it is a Hilbert space, so by theorem 6.3 in Magdalenas notes, $\overline{B}_H(0, 1)$ is weakly compact.

From (a) we know that $||f_N|| = 1$ for all $N \geq 1$, so by the definition of the convex hull we have for any $x \in \text{co}\{f_N | N \geq 1\}$

$$||x|| = \left|\left| \sum_{i=1}^{n} a_i f_i \right|\right| \leq \sum_{i=1}^{n} ||a_i f_i|| = \sum_{i=1}^{n} a_i ||f_i|| = \sum_{i=1}^{n} a_i = 1$$

Now suppose there exist $y \in K$ with $||y|| > 1$. Then there exists sequence $(x_n)_{n \geq 1} \subset \text{co}\{f_N | N \geq 1\}$ such that $||x_n - y|| \longrightarrow 0$ as $n \longrightarrow \infty$. But $||x_n - y|| \geq \Big| ||x_n|| - ||y|| \Big| = \Big| 1 - ||y|| \Big| > 0$ for all $n \in \mathbb{N}$, hence there is no $y \in K$ such that $||y|| > 1$ and thus $K \subset \overline{B}_H(0, 1)$. Since $\text{co}\{f_N | N \geq 1\}$ is convex we have by theorem 5.7 that the norm closure and the weak closure of $\text{co}\{f_N | N \geq 1\}$ coincide, hence since K is the norm closure it is also weakly closed. So K is a weakly closed subset of a weakly compact set, hence K is weakly compact.

(c) Show that 0, as well as each f_N, $N \geq 1$, are extreme points in K.

Suppose that $0 = \alpha x_1 + (1 - \alpha) x_2$ for some $\alpha \in (0, 1), x_1, x_2 \in K$. We claim that $\langle x, e_i \rangle \in [0, \infty)$ for all $x \in K, i \in \mathbb{N}$. Then for all $i \in \mathbb{N}$

$$0 = \langle 0, e_i \rangle = \langle \alpha x_1 + (1 - \alpha) x_2, e_i \rangle = \alpha \langle x_1, e_i \rangle + (1 - \alpha) \langle x_2, e_i \rangle$$

But by our claim $\alpha\langle x_1, e_i\rangle \geq 0$ and $(1-\alpha)\langle x_2, e_i\rangle \geq 0$ which means both terms must be 0, so since $\alpha \neq 0 \neq (1-\alpha)$ we must have $\langle x_1, e_i\rangle = \langle x_2, e_i\rangle = 0$ for all $i \in \mathbb{N}$. But that can only happen if $x_1 = x_2 = 0$, which means that 0 is an extreme point.

Here you need $\alpha_k \geq 0$, because else $\frac{1}{2}f_1 + \frac{1}{2}f_3$ is not possible

So we just need to prove our claim. We first show that for each $n \in \mathbb{N}$, we have $\langle F, e_n\rangle \in [0, \infty)$ for each $F \in \text{co}\{f_N | N \geq 1\}$. So let $F \in \text{co}\{f_N | N \geq 1\}$. Then $F = \sum_{k=1}^{m} \alpha_k f_k$ for some $m \in \mathbb{N}$, $\alpha_k > 0$ with $\sum_{k=1}^{m} \alpha_k = 1$. But then for any $n \in \mathbb{N}$

$$\langle F, e_n\rangle = \left\langle \sum_{k=1}^{m} \alpha_k f_k, e_n \right\rangle = \sum_{k=1}^{m} \alpha_k \left\langle k^{-1} \sum_{j=1}^{k^2} e_j, e_n \right\rangle = \sum_{k=1}^{m} \alpha_k k^{-1} \sum_{j=1}^{k^2} \langle e_j, e_n\rangle$$

Now we know that $\langle e_j, e_n\rangle$, k^{-1}, and α_k are all positive real numbers for all j, k. So this means that

$$\langle F, e_n\rangle = \sum_{k=1}^{m} \alpha_k k^{-1} \sum_{j=1}^{k^2} \langle e_j, e_n\rangle \in [0, \infty)$$

Now let $x \in K$. Then since K is the norm closure of $\text{co}\{f_N | N \geq 1\}$, there exists a sequence $(F_n)_{n\geq 1} \subset \text{co}\{f_N | N \geq 1\}$ such that $x = \lim_{n\to\infty} F_n$. But then

$$\langle x, e_i\rangle = \left\langle \lim_{n\to\infty} F_n, e_i \right\rangle = \lim_{n\to\infty} \langle F_n, e_i\rangle$$

Now we know that $[0, \infty)$ is sequentially closed, so since $\langle F_n, e_i\rangle \in [0, \infty)$ for all $n \geq 1$, we have that

$$\langle x, e_i\rangle = \lim_{n\to\infty} \langle F_n, e_i\rangle \in [0, \infty)$$

for all $i \geq 1$, which proves our claim. ✓

Now to show that f_N is an extreme point for each $N \in \mathbb{N}$, we see that given $n \in \mathbb{N}$,

$$\langle f_m, f_n\rangle = \left\langle m^{-1} \sum_{i=1}^{m^2} e_i, n^{-1} \sum_{j=1}^{n^2} e_j \right\rangle = m^{-1} n^{-1} \left(\sum_{i=1}^{m^2} \sum_{i=1}^{n^2} \langle e_i, e_j\rangle \right)$$

$$= m^{-1} n^{-1} \sum_{i=1}^{\min\{m^2, n^2\}} \langle e_i, e_i\rangle = \frac{\min\{m^2, n^2\}}{mn} = \frac{\min\{m, n\}}{\max\{m, n\}} \in [0, 1]$$

for every $m \in \mathbb{N}$. Now let $x \in \text{co}\{f_N | N \geq 1\}$. Then $x = \sum_{k=1}^{l} a_k f_{N_k}$ for some $a_k > 0, N_k \in \mathbb{N}, l \in \mathbb{N}$ with $\sum_{k=1}^{l} a_k = 1$. Then

Same as before.

$$\langle x, f_N\rangle = \left\langle \sum_{k=1}^{l} a_k f_{N_k}, f_N \right\rangle = \sum_{k=1}^{l} a_k \langle f_{N_k}, f_N\rangle$$

Now using that $a_k > 0$ for all $k = 1, ..., l$, $\sum_{k=1}^{l} a_k = 1$, and $\langle f_{N_k}, f_N\rangle \in [0, 1]$ for all $k = 1, ..., l$, we see that

$$0 \leq \langle x, f_N\rangle = \sum_{k=1}^{l} a_k \langle f_{N_k}, f_N\rangle \leq \sum_{k=1}^{l} a_k = 1$$

for all $x \in \text{co}\{f_N | N \geq 1\}$.

Now since $\langle -, f_N\rangle : H \longrightarrow \mathbb{C}$ is continuous for all $N \in \mathbb{N}$, and $K = \overline{\text{co}\{f_N | N \geq 1\}}$ we get that

$$\langle -, f_N\rangle(K) = \langle -, f_N\rangle(\overline{\text{co}\{f_N | N \geq 1\}}) \subset \overline{\langle -, f_N\rangle(\text{co}\{f_N | N \geq 1\})}$$

and by what we have just shown $\langle -, f_N\rangle(\text{co}\{f_N|N \geq 1\}) \subset [0,1]$ which is closed, hence

$$\langle -, f_N\rangle(K) \subset \overline{\langle -, f_N\rangle(\text{co}\{f_N|N \geq 1\})} \subset [0,1]$$

So for every $x \in K, N \in \mathbb{N}$ we have $\langle x, f_N\rangle \in [0,1]$.

Now suppose that $f_N = \alpha x_1 + (1-\alpha)x_2$ for some $x_1, x_2 \in K, \alpha \in (0,1)$. By (a), and the fact that $\langle f_N, f_N\rangle \in [0,1]$, we see that $||f_N||^2 = \langle f_N, f_N\rangle = 1$ so

Where do you use that here?

$$1 = \langle f_N, f_N\rangle = \langle \alpha x_1 + (1-\alpha)x_2, f_N\rangle = \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle$$

and since $\langle x_1, f_N\rangle, \langle x_2, f_N\rangle \in [0,1]$, this only holds if $\langle x_1, f_N\rangle = \langle x_2, f_N\rangle = 1$, since otherwise $\alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle < 1$.

Now by (b) $K \subset \overline{B}_H(0,1)$ hence $||x_1||, ||x_2|| \leq 1$. Then by the Cauchy-Schwartz inequality we get that

$$1 = |\langle x_1, f_N\rangle|^2 \leq ||x_1||^2||f_N||^2 = ||x_1||^2 \leq 1$$

so $||x_1|| = 1$, and we know that since $\langle x_1, f_N\rangle = ||x_1||\cdot||f_N||$ there exists $x \in \mathbb{C}$ such that $f_N = cx_1$. But then we have

$$1 = \langle f_N, f_N\rangle = \langle cx_1, f_N\rangle = c\langle x_1, f_N\rangle = c$$

hence $c = 1$ so $x_1 = f_N$, and then $f_N = \alpha f_N + (1-\alpha)x_2 \Rightarrow (1-\alpha)f_N = (1-\alpha)x_2 \Rightarrow f_N = x_2$, hence if $f_N = \alpha x_1 + (1-\alpha)x_2$ for some $x_1, x_2 \in K, \alpha \in [0,1]$ then $x_1 = x_2 = f_N$, so f_N is an extreme point for each $N \in \mathbb{N}$. ✓

(d) Are there any other extreme points in K? Justify your answer.

From (a) we know that $(f_N)_{N\geq 1}$ converges weakly to 0, hence the weak closure of $\{f_N|N \geq 1\}$ must be $\{0\} \cup \{f_N|N \geq 1\}$. These are exactly the extreme points in K we found in (c). Now by the arguments in (b) K is the weak closure of $\text{co}\{f_N|N \geq 1\}$. But then since K is non-empty weakly compact convex subset of H, theorem 7.9 says that the extreme points in K are contained in the weak closure of $\{f_N|N \geq 1\}$ which are exactly the points we found in (c), hence there are no other extreme points in K.

Problem 2: Let X and Y be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \longrightarrow x$ weakly, as $n \longrightarrow \infty$, implies that $Tx_n \longrightarrow Tx$ weakly, as $n \longrightarrow \infty$.

By HW4 problem 2 (a) $Tx_n \longrightarrow Tx$ weakly as $n \longrightarrow \infty$ if $\varphi(Tx_n) \longrightarrow \varphi(Tx)$ as $n \longrightarrow \infty$ for all $\varphi \in Y^*$. But since for each $\varphi \in Y^*$ we have $\varphi \circ T \in X^*$, HW4 problem 2 (a) says that since $x_n \longrightarrow x$ weakly as $n \longrightarrow \infty$ then $\varphi(Tx_n) = \varphi \circ T(x_n) \longrightarrow \varphi \circ T(x) = \varphi(Tx)$ as $n \longrightarrow \infty$ which is what we needed to show. ✓

(b) Let $T \in \mathcal{K}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \longrightarrow x$ weakly, as $n \longrightarrow \infty$, implies that $||Tx_n - Tx|| \longrightarrow 0$, as $n \longrightarrow \infty$.

Since $(x_n)_{n\geq 1}$ converges weakly we know from HW4 problem 2 (b) that it is bounded. Now any subsequence $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$ is again bounded and converges weakly to x. So since T is compact we have by proposition 8.2 that every subsequence $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$ has a further subsequence $(x_{n_{k_j}})_{j\geq 1} \subset (x_{n_k})_{k\geq 1}$ such that $(Tx_{n_{k_j}})$ converges in norm. Now by (a) $(Tx_n)_{n\geq 1}$ converges weakly to Tx so any subsequence converging in norm converges weakly to Tx, hence it must also converge to Tx in norm. So every subsequence $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$ has a further subsequence $(x_{n_{k_j}})_{j\geq 1} \subset (x_{n_k})_{k\geq 1}$ such that $||Tx_{n_{k_j}} - Tx|| \longrightarrow 0$ as $j \longrightarrow \infty$.

Now suppose that $(Tx_n)_{n\geq 1}$ does not converge to Tx in norm. Then there exist $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exist $n_k > n$ with $||Tx_{n_k} - Tx|| > \varepsilon$. But then the subsequence $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$ has no further subsequence $(x_{n_{k_j}})_{j\geq 1} \subset (x_{n_k})_{k\geq}$ such that $||Tx_{n_{k_j}} - Tx|| \longrightarrow 0$ as $j \longrightarrow \infty$ since $||Tx_{n_{k_j}} - Tx|| > \varepsilon$ for all $j \geq 1$. But this is a contradiction, as we have just shown every subsequence has a further subsequence with this property, so we must have $(Tx_n)_{n\geq 1}$ converges to Tx in norm, i.e. $||Tx_n - Tx|| \longrightarrow 0$ as $n \longrightarrow \infty$.

(c) Let H be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H, Y)$ satisfies that $||Tx_n - Tx|| \longrightarrow 0$, as $n \longrightarrow \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Suppose T is not compact. Then by proposition 8.2 there exists a bounded sequence $(x_n)_{n\geq 1} \subset H$ such that for every subsequence $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$, $(Tx_{n_k})_{k\geq 1}$ does not converge in $(Y, ||\cdot||)$. Now since $(x_n)_{n\geq 1}$ is bounded there exists $c > 0$ such that $||cx_n|| \leq 1$ for all $n \geq 1$, so we may assume without loss of generality that $(x_n)_{n\geq 1} \subset \overline{B}_H(0,1)$. Now since $(Tx_{n_k})_{k\geq 1}$ does not converge for any subsequence $(x_{n_k})_{k\geq 1} \subset (x_n)_{n\geq 1}$, in particular $(Tx_n)_{n\geq 1}$ does not converge in $(Y, ||\cdot||)$. But then there exists $\varepsilon > 0$ such that $||Tx_i - Tx_j|| > \varepsilon$ for all $i \neq j$. Now if we can show that $(x_n)_{n\geq 1}$ has a weakly convergent subsequence $(x_{n_k})_{k\geq 1}$ then by assumption $||Tx_{n_k} - Tx_{n_{k+1}}|| \longrightarrow 0$ as $k \longrightarrow \infty$, which contradicts that $||Tx_i - Tx_j|| > \varepsilon$ for all $i \neq j$, hence T must be compact.

Not necessarily all, but for infinitely many (as subsequence)

So we need to show that $(x_n)_{n\geq 1} \subset \overline{B}_H(0,1)$ has a weakly convergent subsequence. Now since H is a Hilbert space it is reflexive, hence theorem 6.3 states that $\overline{B}_H(0,1)$ is weakly compact. But then by the definition of compactness $(x_n)_{n\geq 1}$ has a weakly convergent subsequence, so we are done.

Generally, this will only be a weakly convergent subnet.

(d) Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

We note that $\ell_2(\mathbb{N})$ is a separable infinite dimensional Hilbert space, using the orthonormal basis $\{e_n\}_{n\geq 1}$ where e_n is the sequence with 1 in the n'th place and 0's everywhere else.

Now let $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ and let $(x_n)_{n\geq 1} \subset \ell_2(\mathbb{N})$ be a sequence such that $x_n \longrightarrow x$ weakly as $n \longrightarrow \infty$ for some $x \in \ell_2(\mathbb{N})$. Then by (a) $Tx_n \longrightarrow Tx$ weakly as $n \longrightarrow \infty$. Now remark 5.3 in Magdalenas notes tells us that any sequence $(y_n)_{n\geq 1} \subset \ell_1(\mathbb{N})$ converges weakly to $y \in \ell(\mathbb{N})$ if and only if it converges to y in norm. So in particular since $(Tx_n)_{n\geq 1}$ converges weakly to x it converges to x in norm, so $||Tx_n - Tx|| \longrightarrow 0$ as $n \longrightarrow \infty$. But then since $\ell_2(\mathbb{N})$ is a separable infinite dimensional Hilber space, (c) tells us that T is compact.

(e) Show that no $T \in \mathcal{K}(X, Y)$ is onto.

Suppose T is onto for some $T \in \mathcal{K}(X, Y)$. Then by the open mapping theorem T is an open map, so $T(B_X(0,1))$ is open in Y so there exists $r > 0$ such that $B_Y(0,r) \subset T(B_X(0,1))$, and therefore $\overline{B_Y(0,r)} \subset \overline{T(B_X(0,1))}$. But since T is a compact operator, $\overline{T(B_X(0,1))}$ is compact, hence $\overline{B_Y(0,r)}$ is a closed subset of a compact set and therefore compact. Now since the continuous map $\frac{1}{r} : Y \longrightarrow Y; y \mapsto \frac{1}{r}y$ maps $\overline{B_Y(0,r)}$ bijectively to $\overline{B_Y(0,1)}$, this is also compact. But $\overline{B_Y(0,1)}$ being compact implies that Y is finite dimensional which is a contradiction, hence no $T \in \mathcal{K}(X, Y)$ can be onto.

(f) Let $H = L_2([0,1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. Justify that M is self-adjoint, but not compact.

First we see that for $f, g \in L_2([0,1], m)$

$$\langle Mf, g\rangle = \int_{[0,1]} (Mf)(t)g(t)dm(t) = \int_{[0,1]} tf(t)g(t)dm(t)$$

$$= \int_{[0,1]} f(t)tg(t)dm(t) = \int_{[0,1]} f(t((Mg)(t)dm(t) = \langle f, Mg\rangle$$

✓

hence M is self-adjoint.

Note that $L^2([0,1],m)$ separable and infinite-dimensional!

Now suppose M is compact. Then the spectral theorem for self-adjoint compact operators tells us that there exists an orthonormal basis $(e_n)_{n\geq 1}$ for $L_2([0,1], m)$ such that $Me_n = \lambda_n e_n$ with $\lambda_n \longrightarrow 0$ as $n \longrightarrow \infty$, however this cannot be as for any orthonormal basis $(e_n)_{n\geq 1}$, $Me_n = te_n$ for all $n \geq 1$. So we have a contradiction hence M cannot be compact. ✓

Problem 3: Consider that Hilbert space $H = L_2([0,1], m)$, where m is the Lebesgue measure. Define $K : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \leq t \leq s \leq 1 \\ (1-t)s, & \text{if } 0 \leq s \leq t \leq 1 \end{cases}$$

and consider $T \in \mathcal{L}(H, H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t), \qquad s \in [0,1], f \in H.$$

(a) Justify that T is compact.

We see that since $K(s,t)$ is a non-negative measurable function on $[0,1] \times [0,1]$, so by the Tonelli theorem we get

Why?

$$\left(\int_{[0,1]\times[0,1]} |K(s,t)|^2 d(m \otimes m)(s,t)\right)^{1/2} = \left(\int_{[0,1]} \left(\int_{[0,1]} K(s,t)^2 dm(s)\right) dm(t)\right)^{1/2}$$

$$= \left(\int_0^1 \left(\int_0^1 K(s,t)^2 ds\right) dt\right)^{1/2} = \left(\int_0^1 \left(\int_0^t ((1-t)s)^2 ds + \int_t^1 ((1-s)t)^2 ds\right) dt\right)^{1/2}$$

using K Riemann integrable

$$\left(\int_0^1 \left(\left[\frac{(1-t)^2}{3}s^3\right]_0^t + \left[t^2\left(s - s^2 + \frac{s^3}{3}\right)\right]_t^1\right) dt\right)^{1/2} = \left(\int_0^1 \frac{t^4}{3} - \frac{2t^3}{3} + \frac{t^2}{3} dt\right)^{1/2}$$

$$= \left(\left[\frac{t^5}{15} - \frac{t^4}{6} + \frac{t^3}{9}\right]_0^1\right)^{1/2} = \left(\frac{1}{15} - \frac{1}{6} + \frac{1}{9}\right)^{1/2} = \sqrt{\frac{1}{90}} < \infty$$

so $K \in L_2([0,1] \times [0,1], m \otimes m)$, and we then recognize T as the associated kernel operator and then, by proposition 9.12 in Magdalenas notes, T is compact.

$T = T_{\tilde{K}}$

$\tilde{K}(s,t) = K(t,s)$

(b) Show that $T = T^*$. + σ-finiteness.

So we want to show that given $f, g \in L_2([0,1], m)$ we get that $\langle Tf, g\rangle = \langle f, Tg\rangle$. We notice that K is symmetric, and we consider

$$\langle Tf, g\rangle = \int_{[0,1]} (Tf)(s)\overline{g(s)}dm(s) = \int_{[0,1]} \int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s)$$

$$= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \overline{g(s)} dm(t) dm(s) \stackrel{*}{=} \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \overline{g(s)} dm(s) dm(t)$$

$$= \int_{[0,1]} f(t) \overline{\int_{[0,1]} K(t,s) g(s) dm(s)} dm(t) = \langle f, Tg \rangle$$

where the * equality follows from the Fubini theorem.

We now need to justify the use of Fubini. By HW2 problem 2 (b) we know that $L_2([0,1], m) \subset L_1([0,1], m)$, so for any $f \in L_2([0,1], m)$ we know that

$$\int_{[0,1]} |f(s)| dm(s) < \infty$$

We note that $|K(s,t)| < 1$ for all $(s,t) \in [0,1] \times [0,1]$. We then consider

$$\int_{[0,1]} \int_{[0,1]} |K(s,t) f(t) g(s)| dm(t) dm(s) < \int_{[0,1]} \int [0,1] |f(t)| dm(t) |g(s)| dm(s) = \int_{[0,1]} a|g(s)| dm(s) = a \int_{[0,1]} |g(s)| d$$

for some positive $a, b < \infty$ hence

$$\int_{[0,1]} \int_{[0,1]} |K(s,t) f(t) g(s)| dm(t) dm(s) < \infty$$

seems to be correct.

so we are allowed to use the Fubini theorem, so we conclude that

$$\langle Tf, g \rangle = \langle f, Tg \rangle$$

for $f, g \in L_2([0,1], m)$ hence $T = T^*$ as we wanted to show. ✓

(c) Show that

$$(Tf)(s) = (1-s) \int_{[0,s]} t f(t) dm(t) + s \int_{[s,1]} (1-t) f(t) dm(t), \qquad s \in [0,1], f \in H.$$

Use this to show that Tf is continuous on $[0,1]$, and that $(Tf)(0) = (Tf)(1) = 0$.

$$(Tf)(s) = \int_{[0,1]} K(s,t) f(t) dm(t) = \int_{[0,s]} K(s,t) f(t) dm(t) + \int_{[s,1]} K(s,t) f(t) dm(t)$$

$$= \int_{[0,s]} (1-s) t f(t) dm(t) - \int_{[s,1]} (1-t) s f(t) dm(t) = (1-s) \int_{[0,s]} t f(t) dm(t) + s \int_{[s,1]} (1-t) f(t) dm(t)$$

Now $(1-s) \int_{[0,s]} t f(t) dm(t)$ and $s \int_{[s,1]} (1-t) f(t) dm(t)$ are both continuous since both $tf(t) \in L_2([0,1], m)$ and $(1-t) f(t) \in L_2([0,1], m)$, hence Tf is a sum of continuous functions and is therfore continuous itself.

How does $tf \in L_2$, $(1-t)f \in L_2$ imply continuity?

Now since $m(\{0\}) = m(\{1\}) = 0$ we get

$$(Tf)(0) = (1-0) \int_{[0,0]} t f(t) dm(t) + 0 \int_{[0,1]} (1-t) f(t) dm(t) = \int_{\{0\}} t f(t) dm(t) =$$

and

$$(Tf)(1) = (1-1) \int_{[0,1]} t f(t) dm(t) + 1 \int_{[1,1]} (1-t) f(t) dm(t) = \int_{\{1\}} (1-t) f(t) dm(t) = 0$$

✓

Problem 4: Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathscr{F} : \mathscr{S}(\mathbb{R}) \longrightarrow \mathscr{S}(\mathbb{R})$.

(a) For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$. Justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$. Compute $\mathscr{F}(g_k)$, for $k = 0, 1, 2, 3$.

We notice that for each $k \geq 0$, g_k is smooth, as it is a product of smooth functions. We want to show that

$$\lim_{|x|\to\infty} x^n \frac{d^m}{d^m x} g_k(x) = \lim_{|x|\to\infty} x^n \frac{d^m}{d^m x} x^k e^{-x^2/2} = 0$$

We see that $x^n \frac{d^m}{d^m x} x^k e^{-x^2/2} = \frac{p(x)}{\sqrt{e}^{x^2}}$ for som polynomial $p(x)$. Now since $\sqrt{e} > 1$ we know that $\sqrt{e}^{x^2}$ grows faster than any polynomial, hence $\lim_{|x|\to\infty} x^n \frac{d^m}{d^m x} x^k e^{-x^2/2} = \lim_{|x|\to\infty} \frac{p(x)}{\sqrt{e}^{x^2}} = 0$, so $g_k \in \mathscr{S}(\mathbb{R})$.

By remark 11.12 in Magdalenas notes $\mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$, so for all $k \geq 0$, $g_k \in L_1(\mathbb{R})$ and $g_{k+1} = xg_k \in L_1(\mathbb{R})$. By proposition 11.4 $\mathscr{F}(g_0)(\xi) = g_0(\xi) = e^{-\xi^2/2}$. Now since $g_k, xg_k \in L_1(\mathbb{R})$ for all $k \geq 0$ we can use proposition 11.13 (d), to compute

$$\mathscr{F}(g_1)(\xi) = (\mathscr{F}(xg_0))(\xi) = (i(\frac{d}{dx}\mathscr{F}(g_0)(x)))(\xi) = (i(-xe^{-x^2/2}))(\xi) = -i\xi e^{-\xi^2/2}$$

Now by successive use of proposition 11.13 (d) we get

$$\mathscr{F}(g_2)(\xi) = (\mathscr{F}(xg_1))(\xi) = (i(\frac{d}{dx}\mathscr{F}(g_1)(x)))(\xi) = (i(-ie^{-x^2/2}+ix^2e^{-x^2/2}))(\xi) = e^{-\xi^2/2}-\xi^2e^{-\xi^2/2}$$

and

$$\mathscr{F}(g_3)(\xi) = (\mathscr{F}(xg_2))(\xi) = (i(\frac{d}{dx}\mathscr{F}(g_2)(x)))(\xi)$$
$$= (i(-xe^{-x^2/2} - 2xe^{-x^2/2} + x^3e^{-x^2/2}))(\xi) = i\xi^3e^{-\xi^2/2} - 3i\xi e^{-\xi^2/2}$$

(b) Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathscr{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

We notice from (a) that

$$\mathscr{F}(g_0) = g_0$$
$$\mathscr{F}(g_1) = -ig_1$$
$$\mathscr{F}(g_2) = g_0 - g_2$$
$$\mathscr{F}(g_3) = ig_3 - 3ig_1$$

We define

$$h_0 := g_0$$
$$h_1 := 2g_3 - 3g_1$$
$$h_2 := 2g_2 - g_0$$
$$h_3 := g_1$$

Then since the Schwartz space is closed under addition and scaling, $h_k \in \mathscr{S}(\mathbb{R})$ for $k = 0, 1, 2, 3$. Now using that the Fourier transform is linear, we get that

$$\mathscr{F}(h_0) = \mathscr{F}(g_0) = g_0 = h_0 = i^0 h_0$$
$$\mathscr{F}(h_1) = \mathscr{F}(2g_3 - 3g_1) = 2\mathscr{F}(g_3) - 3\mathscr{F}(g_1) = 2(ig_3 - 3ig_1) - 3(-ig_1) = 2ig_3 - 3ig_1 = ih_1 = i^1 h_1$$
$$\mathscr{F}(h_2) = \mathscr{F}(2g_2 - g_0) = 2\mathscr{F}(g_2) - \mathscr{F}(g_0) = 2(g_0 - g_2) - g_0 = g_0 - 2g_2 = -h_2 = i^2 h_2$$
$$\mathscr{F}(h_3) = \mathscr{F}(g_1) = -ig_1 = -ih_3 = i^3 h_3$$

hence $\mathscr{F}(h_k) = i^k h_k$ for $k = 0, 1, 2, 3$.

(c) Show that $\mathscr{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$.

We know by corollary 12.14 that $\mathscr{F} : \mathscr{S}(\mathbb{R}) \longrightarrow \mathscr{S}(\mathbb{R})$ is an isomorphism with inverse $\mathscr{F}^*$ given by $\mathscr{F}^*(f)(x) = \int_{\mathbb{R}} f(\xi)e^{ix\xi}dm(\xi)$. Now let $f \in \mathscr{S}(\mathbb{R})$. Then since $\mathscr{F}$ is an isomorphism, there exists a unique $g \in \mathscr{S}(\mathbb{R})$ such that $\mathscr{F}(f) = g$ (and also $f = \mathscr{F}^*(g)$). We then see that

$$\mathscr{F}^2(f)(\xi) = \mathscr{F}^2(\mathscr{F}^*(g))(\xi) = \mathscr{F}(g)(\xi) = \int_{\mathbb{R}} g(x)e^{-ix\xi}dm(x) = \mathscr{F}^*(g)(-\xi) = f(-\xi)$$

But then

$$\mathscr{F}^4(f)(\xi) = \mathscr{F}^2(\mathscr{F}^2(f))(\xi) = \mathscr{F}^2(f)(-\xi) = f(-(-\xi)) = f(\xi)$$

so $\mathscr{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$.

✓

(d) Use (c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathscr{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathscr{F}$ precisely are $\{1, i, -1, -i\}$.

$\neq 0$

Let $f \in \mathscr{S}(\mathbb{R})$ and suppose that $\mathscr{F}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$. Then by linearity of $\mathscr{F}$ and since by (c) $\mathscr{F}^4(f) = f$, we get that

$$f = \mathscr{F}^4(f) = \lambda^4 f$$

hence $\lambda^4 = 1$. But the only $\lambda \in \mathbb{C}$ such that $\lambda^4 = 1$ are $\{e^{\frac{2\pi i n}{4}} | n \in \mathbb{Z}\} = \{1, i, -1, -i\} = \{i^k | k = 0, 1, 2, 3\}$ so the eignevalues of $\mathscr{F}$ must be found here.

Now in (b) we have shown that for each $k = 0, 1, 2, 3$ there exists $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathscr{F}(h_k) = i^k h_k$ hence each of these must be eigenvalues, so the eigenvalues for $\mathscr{F}$ are exactly $\{1, i, -1, -i\}$ as we wanted.

✓

Problem 5: Let $\{x_n\}_{n\geq 1}$ be a dense subset of $[0, 1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0, 1]$. Show that $\text{supp}(\mu) = [0, 1]$.

If we can show that for any open set $\emptyset \neq A \subset [0, 1]$, $\mu(A) > 0$, then $\text{supp}(\mu) = (\bigcup\{A \subset [0, 1] | \mu(A) = 0\})^{\complement} = \emptyset^{\complement} = [0, 1]$ as wanted. So it is enough to show that for any non-empty open set $A \subset [0, 1]$ there is some $n \geq 1$ such that $x_n \in A$, since we then have that $\mu(A) > 2^{-n}\delta_{x_n}(A) = 2^{-n} > 0$.

Now let $A \subset [0, 1]$ be an open subset such that $\{x_n\}_{n\geq 1} \cap A = \emptyset$. Then since A is open, $A^{\complement} = [0, 1] \setminus A$ is closed and contains $\{x_n\}_{n\geq 1}$. Since $\{x_n\}_{n\geq 1}$ is dense in $[0, 1]$, the closure $\overline{\{x_n\}_{n\geq 1}} = [0, 1]$ hence $[0, 1]$ is the smallest closed set containing $\{x_n\}_{n\geq 1}$. But then since $[0, 1] \setminus A$ is a closed subset containing $\{x_n\}_{n\geq 1}$ we must have $[0, 1] \subset [0, 1] \setminus A$ which means $A = \emptyset$. So for any non-empty open subset $A \subset [0, 1]$ there is an $n \geq 1$ such that $x_n \in A$ which is what we needed to show. ✓