

# One-dimensional Dilute Quantum Gases and Their Ground State Energies

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# Overview

- 1 Motivation
- 2 Many-Body Quantum Mechanics
- 3 The Scattering Length
- 4 Bosons Main Result
- 5 Upper Bound
- 6 Lower Bound
- 7 Spin-1/2 Fermions
- 8 Conclusion and Outlook

# Motivation (Bosons)

- 2D and 3D dilute Bose gases are well-studied in the mathematical physics literature.
- 2D and 3D results are related to Bose-Einstein condensation (BEC).
- No BEC is expected in 1D.
- In 1D the hard core and Lieb-Liniger models are solvable.
- Our result is consistent with the absence of BEC in 1D.
- On the contrary it suggests that the 1D dilute bose gas shares features with the Fermi gas.

# Motivation (Spin-1/2 fermions)

- 2D and 3D dilute Fermi gases are well-studied in the mathematical physics literature.
- In 1D the hard core and Yang-Gaudin models are solvable.
- In 1962 E. H. Lieb and D. C. Mattis showed that one-dimensional Fermi gases are antiferromagnetic (contradicting standard perturbative tight-binding methods).
- Hence the standard justification of the Heisenberg model of magnetism is too simple.
- Our result will break ground in rigorously justifying the Heisenberg antiferromagnet as an effective model in 1D.

# Many-Body Quantum Mechanics

For a system of  $N$  bosons/fermions in region  $\Omega \in \mathbb{R}^d$ , we define for  $\sigma \in [0, \infty]$  **the energy quadratic forms**

$$\mathcal{E}_{(v,\sigma)}(\Psi) = \int_{\Omega^N} \sum_{i=1}^N |\nabla_i \Psi|^2 + \sum_{i < j} v(x_i - x_j) |\Psi|^2 + \sigma \int_{\partial(\Omega^N)} |\Psi|^2, \quad (1)$$

with domain  $\mathcal{D}(\mathcal{E}_{(v,\sigma)}) = \{\Psi \in (C^\infty(\Omega^N))_{\text{b/f}} | \mathcal{E}_{(v,\sigma)}(\Psi) < \infty\}$ .

## Definition 1

We say a potential  $v \geq 0$  is **allowed** in dimension  $d$ , if  $\mathcal{E}_{(v,\sigma)}$  is closable on  $\mathcal{H}_{(v,\sigma)} := \overline{\mathcal{D}(\mathcal{E}_{(v,\sigma)})}^{\|\cdot\|_2} \subset L^2_{s/a}(\Omega^N)$  for any  $\sigma \in [0, \infty]$ .

## Proposition 1

*Let  $d = 1$ , then any potential of the form  $v = v_{\sigma\text{-finite}} + v_{\text{meas.}} + c\delta_0$ , with  $c \in \{0, \infty\}$ , is allowed.*

# The Scattering Length

## Theorem 2

For  $B_R = \{0 \leq |x| < R\} \subset \mathbb{R}^d$  with  $R > R_0 := \text{range}(v)$ , let  $\phi \in H^1(B_R)$  satisfy

$$-\Delta\phi + \frac{1}{2}v\phi = 0, \quad \text{on } B_R, \quad (2)$$

with boundary condition  $\phi(x) = 1$  for  $|x| = R$ . Then  $\phi(x) = f(|x|)$  for some  $f : (0, R] \rightarrow [0, \infty)$ , and for  $\text{range}(v) < r < R$ , we have

$$f(r) = \begin{cases} (r - a)/(R - a) & \text{for } d = 1 \\ \ln(r/a)/\ln(R/a) & \text{for } d = 2 \\ (1 - ar^{2-d})/(1 - aR^{2-d}) & \text{for } d \geq 3, \end{cases} \quad (3)$$

with some constant  $a$  called the **(s-wave) scattering length**.

## 2D and 3D

For  $\Lambda_L = [0, L]^d$ , let  $e(\rho) := \lim_{\substack{L \rightarrow \infty \\ N/L^d \rightarrow \rho}} E(N, L)/L^d$ .

Theorem 3 ( $d = 3$  result, Lee-Huang-Yang 1957<sup>1</sup>)

$$e(\rho) = 4\pi\rho^2 a \left( 1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3} + o(\sqrt{\rho a^3}) \right). \quad (4)$$

Theorem 4 ( $d = 2$  result<sup>2</sup>)

$$e(\rho) = 4\pi\rho^2 Y \left( 1 - Y |\log Y| + \left( 2\Gamma + \frac{1}{2} + \ln(\pi) \right) Y \right) + o(\rho^2 Y^2), \quad (5)$$

$$Y = |\ln(\rho a^2)|^{-1}.$$

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<sup>a</sup>Upper bound: Yau-Yin 2009, Basti-Cenatiempo-Schlein 2021. Lower bound: Fournais-Solovej 2021

<sup>b</sup>Fournais-Girardot-Junge-Morin-Olivieri 2022

# Bosons Main Result

For the remainder of the presentation,  $d = 1$ .

## Theorem 5 (A., R. Reuvers, J. P. Solovej, 2022)

*Consider a Bose gas with repulsive interaction  $v = v_{\text{reg}} + v_{\text{h.c.}}$ . Define the density  $\rho = N/L$ . For  $\rho|a|$  and  $\rho R_0$  sufficiently small, the ground state energy can be expanded as*

$$E(N, L) = N \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a + \mathcal{O} \left( (\rho|a|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3} \right) \right), \quad (6)$$

*where  $a$  is the scattering length of  $v$ .*



# Examples

## The hard core gas

Energy behaves like free Fermi energy in volume  $L - NR$ , *i.e.*

$$\begin{aligned} E_{\text{hard core}}(N, L) &= N \frac{\pi^2}{3} \rho^2 (1 - NR/L)^{-2} \\ &= E_0 (1 + 2\rho R + \mathcal{O}((\rho R)^2)) . \end{aligned} \tag{7}$$

Scattering length is  $a = R$ .

## Lieb-Liniger model

Energy behaves asymptotically like

$$E_{LL}(N, L, c) = N \frac{\pi^2}{3} \rho^2 (1 - 4\rho/c + \mathcal{O}((\rho/c)^2)) , \tag{8}$$

with scattering length  $a = -\frac{2}{c}$ .

# Variational Principle

To obtain an upper bound, we use the variational principle, *i.e.*

$$E(N, L) \leq \frac{\mathcal{E}(\Psi)}{\|\Psi\|^2}, \quad \text{for any } \Psi \in \mathcal{D}(\mathcal{E}).$$

# Trial State

Trial state has to capture free Fermi energy, as well as corrections due to scattering processes. Hence we consider

$$\Psi(x) = \begin{cases} \omega(\mathcal{R}(x)) \frac{|\Psi_F(x)|}{\mathcal{R}(x)} & \text{if } \mathcal{R}(x) < b \\ |\Psi_F(x)| & \text{if } \mathcal{R}(x) \geq b, \end{cases}$$

where  $\omega$  is the suitably normalized solution to the two-body scattering equation,  $\Psi_F$  is the free Fermi ground state, and  $\mathcal{R}(x) := \min_{i < j} (|x_i - x_j|)$  is uniquely defined a.e.

# One-particle Reduced Density Matrix

For the free Fermi gas we have

$$\begin{aligned}\gamma^{(1)}(x, y) &= \frac{2}{L} \sum_{j=1}^N \sin\left(\frac{\pi}{L} jx\right) \sin\left(\frac{\pi}{L} jy\right) \\ &= \frac{\pi}{L} \left( D_N\left(\pi \frac{x-y}{L}\right) + D_N\left(\pi \frac{x+y}{L}\right) \right),\end{aligned}\tag{9}$$

where  $D_N(x) = \frac{1}{2\pi} \sum_{k=-N}^N e^{ikx} = \frac{\sin((N+1/2)x)}{2\pi \sin(x/2)}$  is the Dirichlet kernel.

By Wick's theorem all derivatives of reduced density matrices are bounded by a constant times an appropriate power of  $\rho$ .

# Some Useful Bounds

## Lemma 1

$$\rho^{(2)}(x_1, x_2) \leq \left( \frac{\pi^2}{3} \rho^4 + f(x_2) \right) (x_1 - x_2)^2 + \mathcal{O}(\rho^6 (x_1 - x_2)^4),$$

with  $\int f(x_2) dx_2 \leq \text{const. } \rho^3 \log(N)$ .

## Lemma 2

*We have the following bounds*

$$\rho^{(3)}(x_1, x_2, x_3) \leq \text{const. } \rho^9 (x_1 - x_2)^2 (x_2 - x_3)^2 (x_1 - x_3)^2,$$

$$\rho^{(4)}(x_1, x_2, x_3, x_4) \leq \text{const. } \rho^8 (x_1 - x_2)^2 (x_3 - x_4)^2,$$

$$\left| \sum_{i=1}^2 \partial_{y_i}^2 \gamma^{(2)}(x_1, x_2; y_1, y_2) \Big|_{y=x} \right| \leq \text{const. } \rho^6 (x_1 - x_2)^2,$$
$$\vdots$$

# Collecting Everything

## Upper bound

$$E \leq N \frac{\pi^2}{3} \rho^2 \frac{\left(1 + 2\rho a \frac{b}{b-a} + \text{const.} \left[ \frac{1}{N} + N(b\rho)^3 (1 + \rho b^2 \int v_{\text{reg}}) \right]\right)}{\|\Psi\|^2}, \quad (10)$$

where the finite measure  $v_{\text{reg}}$  is  $v$  with any hard core removed. By lemma 1 we know  $\|\Psi\|^2 \geq 1 - \text{const. } N(\rho b)^3$ .

## Localization

Divide into  $M$  smaller boxes with  $\tilde{N} = N/M$  particles in each, and make distance  $b$  between boxes (no interaction between boxes), and choose  $M$  such that  $\tilde{N} = (\rho b)^{-3/2} \gg 1$ .

# Upper Bound

After localization

$$E(N, L) \leq N \frac{\pi^2}{3} \rho^2 \frac{\left(1 + 2\rho a \frac{b}{b-a} + \text{const.} \frac{M}{N} + \text{const.} \tilde{N}(b\rho)^3 (1 + \rho b^2 \int v_{\text{reg}})\right)}{1 - \tilde{N}(\tilde{\rho}b)^3} \quad (11)$$

Choosing  $b = \max(\rho^{-1/5} |a|^{4/5}, R_0)$  we find

Proposition 2 (Upper bound Theorem 5)

*There exists a constant  $C_U > 0$  such that for  $\rho|a|$ ,  $\rho R_0 \leq C_U^{-1}$ , the ground state energy  $E^D(N, L)$  satisfies*

$$E^D(N, L) \leq N \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a + C_U \left((\rho|a|)^{6/5} + (\rho R_0)^{3/2} + N^{-1}\right)\right). \quad (12)$$

# Lower Bound

Proof of lower bound consists of the following steps:

- 1 Use Dyson's lemma to reduce to a nearest neighbor double delta-barrier potential.
- 2 Reduce to the Lieb Liniger model by discarding **a small part** of the wave function.
- 3 Use a known lower bound for the Lieb Liniger model.



# The Lieb-Liniger (LL) model

$$H_{LL} = - \sum_{i=1}^n \partial_i^2 + 2c \sum_{i < j} \delta(x_i - x_j). \quad (13)$$

Behavior in thermodynamic limit:  $\lim_{\substack{\ell \rightarrow \infty, \\ n/\ell \rightarrow \rho}} E_{LL}(n, \ell, c)/\ell = \rho^3 e(\gamma)$

with  $\gamma = c/\rho$ .

Lemma 3 (Lieb-Liniger lower bound)

Let  $\gamma > 0$ , then

$$e(\gamma) \geq \frac{\pi^2}{3} \left( \frac{\gamma}{\gamma + 2} \right)^2 \geq \frac{\pi^2}{3} \left( 1 - \frac{4}{\gamma} \right). \quad (14)$$

# Reducing to the LL Model

## Lemma 4 (Dyson)

Let  $R > R_0 = \text{range}(v)$  and  $\varphi \in H^1(\mathbb{R})$ , then for any interval  $\mathcal{I} \ni 0$

$$\int_{\mathcal{I}} |\partial \varphi|^2 + \frac{1}{2} v |\varphi|^2 \geq \int_{\mathcal{I}} \frac{1}{R-a} (\delta_R + \delta_{-R}) |\varphi|^2, \quad (15)$$

where  $a$  is the  $s$ -wave scattering length.

Hence we have, denoting  $\mathfrak{r}_i(x) = \min_j (|x_i - x_j|)$

$$\begin{aligned} \int \sum_i |\partial_i \Psi|^2 + \sum_{i \neq j} \frac{1}{2} v_{ij} |\Psi|^2 \geq \\ \int \sum_i |\partial_i \Psi|^2 \chi_{\mathfrak{r}_i(x) > R} + \sum_i \frac{1}{R-a} \delta(\mathfrak{r}_i(x) - R) |\Psi|^2. \end{aligned} \quad (16)$$

# Reducing to the LL Model

Define  $\psi \in L^2([0, \ell - (n-1)R]^n)$  by

$$\psi(x_1, x_2, \dots, x_n) = \Psi(x_1, R + x_2, \dots, (n-1)R + x_n),$$

for  $x_1 \leq x_2 \leq \dots \leq x_n$  and symmetrically extended.

Then

$$\begin{aligned} \mathcal{E}(\Psi) &\geq E_{LL}^N(n, \ell - (n-1)R, 2/(R-a)) \langle \psi | \psi \rangle \\ &\geq n \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho(a - R) + 2\rho R - \text{const.} \frac{1}{n^{2/3}} \right) \langle \psi | \psi \rangle. \end{aligned} \tag{17}$$

# Lower Bound for Mass of $\psi$

## Lemma 5

*Let  $\psi$  be defined as above, then*

$$1 - \langle \psi | \psi \rangle \leq 8 \left( R^2 \sum_{i < j} \int_{B_{ij}} |\partial_i \Psi|^2 + R(R - a) \sum_{i < j} \int v_{ij} |\Psi|^2 \right), \quad (18)$$

Combining lemmas 4 and 5 we have the following lemma:

## Lemma 6

*For  $n(\rho R)^2 \leq \frac{3}{16\pi^2} \frac{1}{8}$ ,  $\rho R \ll 1$  and  $R > 2|a|$  we have*

$$\langle \psi | \psi \rangle \geq 1 - \text{const.} \left( n(\rho R)^3 + n^{1/3}(\rho R)^2 \right). \quad (19)$$

# Lower Bound

By the reduction to the LL model we find

## Proposition 3

*For assumptions as in lemma 6 we have*

$$E^N(n, \ell) \geq n \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a + \text{const.} \left( \frac{1}{n^{2/3}} + n(\rho R)^3 + n^{1/3}(\rho R)^2 \right) \right). \quad (20)$$

## Corollary 1

*For  $n = \text{const.}$   $(\rho R)^{-9/5}$  we have*

$$E^N(n, \ell) \geq n \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a - \text{const.} \left( (\rho R)^{6/5} + (\rho R)^{7/5} \right) \right). \quad (21)$$

# Lower Bound Localization

To prove the lower bound, we localize, as in the upper bound, to smaller boxes.

## Lemma 7

Let  $\Xi \geq 4$  be fixed and let  $n = m\Xi\rho\ell + n_0$  with  $n_0 \in [0, \Xi\rho\ell)$  for some  $m \in \mathbb{N}$  with  $n^* := \rho\ell = \mathcal{O}(\rho R)^{-9/5}$ . Furthermore, assume that  $\rho R \ll 1$  and let  $\mu = \pi^2\rho^2 \left(1 + \frac{8}{3}\rho a\right)$ , then

$$E^N(n, \ell) - \mu n \geq E^N(n_0, \ell) - \mu n_0. \quad (22)$$

## Proposition 4 (Lower bound Theorem 5)

There exists a constant  $C_L > 0$  such that the ground state energy  $E^N(N, L)$  satisfies

$$E^N(N, L) \geq N \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a - C_L \left( (\rho|a|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3} \right) \right). \quad (23)$$

MATH

# Spinless/Spin-Polarized Fermions

Spinless fermions are unitarily equivalent to bosons with a zero b.c. at all planes of intersection, *i.e.* with an infinite delta potential. As a consequence we have the following corollary.

## Theorem 6 (Spin-polarized fermions)

*Consider a spin-polarized Fermi gas with repulsive interaction  $v = v_{\text{reg}} + v_{\text{h.c.}}$  as defined before. Let  $E_F(N, L)$  be its associated ground state energy. Write  $\rho = N/L$ . For  $\rho a_o$  and  $\rho R_0$  sufficiently small, the ground state energy can be expanded as*

$$E_F(N, L) = N \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a_o + \mathcal{O} \left( (\rho R_0)^{6/5} + N^{-2/3} \right) \right), \quad (24)$$

*where  $a_o \geq 0$  is the odd wave scattering length of  $v$ .*

This is consistent with lower bound  $E_F(N, L) \geq E_0$ , since  $a_o \geq 0$ .

# Two solvable model for spin-1/2 fermions

## The hard core gas

Ground state energy is independent of spin so

$$E_{\text{hard core}}(N, L) = N \frac{\pi^2}{3} \rho^2 (1 - NR/L)^{-2} \approx E_0(1 + 2\rho R). \quad (25)$$

Scattering length is  $a_e = a_o = R$ .

## Yang-Gaudin model

Is the spin-1/2 version of the LL model, *i.e.*  $H_{YG} = H_{LL}$ . Behaves asymptotically like

$$E_{YG}(N, L, c) = N \frac{\pi^2}{3} \rho^2 \left( 1 - 4\rho \ln(2)/c + \mathcal{O}((\rho/c)^2) \right), \quad (26)$$

with scattering length  $a_e = -\frac{2}{c}$ ,  $a_o = 0$ .



# A Conjecture for Spin-1/2 Fermions

Based on the two solvable cases, we expect

## Conjecture 1

*Let  $v \geq 0$  satisfy the assumption from above, then the ground state energy of the dilute spin-1/2 Fermi gas satisfies*

$$E = N \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho (\ln(2)a_e + (1 - \ln(2))a_o) + \mathcal{O}(\rho^2 \max(|a_e|, a_o)^2) \right). \quad (27)$$

$$E(N, L) = N \frac{\pi^2}{3} \rho^2 \left( 1 + 2 \ln(2) \rho a_e + 2(1 - \ln(2)) \rho a_o + \mathcal{O}((\rho \max(|a_e|, a_o))^2) \right) \quad (28)$$

# Spin-1/2 Fermions Main Result (Upper Bound)

## Theorem 7

*Let  $v \geq 0$  satisfy the assumption from above, then the ground state energy of the dilute spin-1/2 Fermi gas satisfies*

$$E \leq N \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho (\ln(2)a_e + (1 - \ln(2))a_o) + \mathcal{O}\left((\rho R)^{6/5} + N^{-1}\right) \right), \quad (29)$$

*with  $R = \max(|a_e|, a_o, R_0)$ .*

# Trial State

One the sector

$$\{1, 2, \dots, N\} = \{0 < x_1 < x_2 < \dots < x_N < L\}$$

we define the trial state by

$$\Psi_\chi = \begin{cases} \frac{\Psi_F}{\mathcal{R}} \left( (\eta \omega_e^{\mathcal{R}} + (1 - \eta) \omega_o^{\mathcal{R}}) P_s^{\mathcal{R}} + \omega_o^{\mathcal{R}} P_t^{\mathcal{R}} \right) \chi, & \mathcal{R}(x) < b \\ \Psi_F \chi, & \mathcal{R}(x) \geq b \end{cases}, \quad (30)$$

where  $\chi$  is some spin state,  $b > R_0$ ,  $\mathcal{R}(x) = \min_{i < j} |x_i - x_j|$ ,  
 $\omega_{e/o}^{\mathcal{R}}(x) := \omega_{e/o}(\mathcal{R}(x)) = b f_{e/o}(\mathcal{R}(x))$  and

$$\eta(x) := \begin{cases} 0, & \text{if } \mathcal{R}_2(x) \leq b \\ \left( \frac{\mathcal{R}_2(x)}{b} - 1 \right), & \text{if } b < \mathcal{R}_2(x) < 2b \\ 1, & \text{if } \mathcal{R}_2(x) \geq 2b. \end{cases} \quad (31)$$

with  $\mathcal{R}_2(x) = \min_{(i,j) \neq (k,l)} \max(|x_i - x_j|, |x_k - x_l|)$ .

Trial state energy is the free Fermi energy with a correction of the form

$$2\rho \left( (a_o - a_e) \left\langle \chi \left| \frac{1}{N} \sum_i S_i \cdot S_{i+1} \right| \chi \right\rangle + \frac{1}{4}a_e + \frac{3}{4}a_o \right) E_F.$$

*Proof:*

The image displays a grid of 12 small thumbnail images, each showing a page of mathematical derivations and proofs. The thumbnails are arranged in a 3x4 grid. The first row shows the initial setup and the definition of the trial state. The second row shows the calculation of the expectation value of the Hamiltonian. The third row shows the final result and the discussion of the energy correction. The thumbnails are labeled with numbers 1 through 12, indicating the sequence of the proof.

# Antiferromagnetic Heisenberg Chain

The (periodic) antiferromagnetic Heisenberg chain

$$H = \sum_{i=1}^N S_i \cdot S_{i+1}, \text{ with } S_{N+1} := S_1$$

Ground state energy per site of the infinite chain is known due to Hulthén

## Lemma 8

Let  $|\text{GS}_{\text{HAF}}\rangle$  denote the ground state of the periodic antiferromagnetic Heisenberg chain. Then

$$\lim_{N \rightarrow \infty} \left\langle \text{GS}_{\text{HAF}} \left| \frac{1}{N} \sum_{k=1}^N S_k \cdot S_{k+1} \right| \text{GS}_{\text{HAF}} \right\rangle = \frac{1}{4} - \ln(2) \quad (32)$$

## Control of the error for a finite chain

### Lemma 9

Let  $|\text{GS}_{\text{HAF}}\rangle$  denote the ground state of the periodic antiferromagnetic Heisenberg chain. Then

$$\left\langle \text{GS}_{\text{HAF}} \left| \frac{1}{N} \sum_{k=1}^N S_k \cdot S_{k+1} \right| \text{GS}_{\text{HAF}} \right\rangle = \frac{1}{4} - \ln(2) + \mathcal{O}(N^{-1}) \quad (33)$$

Proof.

Upper bound: Truncate longer of length  $M > N$  chain at length  $N$ . Lower bound: Construct trial state for longer chain of length  $mN$  by  $m$  copies of length  $N$  chain. Use translation invariance and uniqueness of the ground state:

$$\frac{1}{mN}(E_{mN} - m) \leq \frac{1}{N}E_N \leq \frac{1}{M}E_M + 1/N.$$



# Lower Bound in Terms of LLH Model

## Lemma 10 (Dyson's lemma spin-1/2 fermions)

Let  $R > R_0 = \text{range}(v)$  and

$\varphi \in \left( H_{\text{even}}^1(\mathbb{R}) \otimes P_s \left( (\mathbb{C}^2)^2 \right) \right) \oplus \left( H_{\text{odd}}^1(\mathbb{R}) \otimes P_t \left( (\mathbb{C}^2)^2 \right) \right)$ , then for any interval  $\mathcal{I} \ni 0$

$$\int_{\mathcal{I}} |\partial \varphi|^2 + \frac{1}{2} v |\varphi|^2 \geq \int_{\mathcal{I}} \bar{\varphi} \left( \frac{1}{R - a_e} P_s + \frac{1}{R - a_o} P_t \right) (\delta_R + \delta_{-R}) \varphi, \quad (34)$$

where  $a_{e/o}$  is the even/odd-wave scattering length.

The Lieb-Liniger-Heisenberg model:

$$H_{LLH} = - \sum_i \partial_i^2 + 2 \sum_{i < j} \left( c' \tilde{P}_s^{i,j} + c \tilde{P}_t^{i,j} \right) \delta(x_i - x_j), \quad (35)$$

where the spin projectors,  $\tilde{P}_{s/t}$  are defined on the sector  $\{\sigma\}$  to be

$$\tilde{P}_{s/t}^{ij} = P_{s/t}^{\sigma^{-1}(i)\sigma^{-1}(j)}.$$

## Proposition 5

For  $n(\rho R)^2 \leq \frac{3}{16\pi^2} \frac{1}{8}$ ,  $\rho R \leq \frac{1}{2}$  and  $R > 2 \max(|a_e|, a_o, R_0)$  we have

$$E^N(N, L) \geq E_{LLH}^N \left( N, \tilde{L}, \frac{2}{R - a_e}, \frac{2}{R - a_o} \right) \times \left( 1 - \text{const.} \left( n(\rho R)^3 + n^{1/3}(\rho R)^2 \right) \right). \quad (36)$$

## Remark 1

*The Lieb-Liniger-Heisenberg model is not exactly solvable. Thus no available good lower bound.*



# Conclusion and Outlook

We have shown that:

- Interaction pair-potential of the form  $v = v_{\sigma\text{-finite}} + v_{\text{meas.}} + c\delta_0$  give rise to a unique Hamiltonian.
- The ground state energy of the dilute Bose (spin polarized Fermi or anyon) gas in one dimension can be expanded in terms of the diluteness parameter to next-to-leading order (universality).
- The solvable models of spin-1/2 fermionic systems are consistent with a similar expansion.
- The ground state energy of the dilute spin-1/2 Fermi gas can be upper bounded by this expansion.
- The ground state energy of the dilute spin-1/2 Fermi gas can be lower bounded by the ground state energy of the Lieb-Liniger-Heisenberg model.



Thanks for your attention!