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Problem 1

(a) $T: X \to Y$ a linear map.

WTS $\|x\|_0 = \|x\|_X + \|Tx\|_Y$. is a norm on X

Remember to write $x, y \in X$
$\alpha \in \mathbb{K}$.

(1) $\|x+y\|_0 = \|x+y\|_X + \|T(x+y)\|_Y$

$\leq \|x\|_X + \|y\|_X + \|Tx + Ty\|_Y$

$\leq \|x\|_X + \|Tx\|_X + \|y\|_X + \|Ty\|_Y = \|x\|_0 + \|y\|_0.$ ✓

(2) $\|\alpha(x+y)\|_0 = \|\alpha(x+y)\|_X + \|T(\alpha(x+y))\|_Y$

$= |\alpha|\|x+y\|_X + |\alpha|\|T(x+y)\|_Y$ ✓

$= |\alpha|\,\|x+y\|_0.$

(3) $\|x\|_0 = 0 \iff \|x\|_X + \|Tx\|_Y = 0 \iff \|x\|_X = 0$, $\|Tx\|_Y = 0$. since $\|x\|_X \geq 0$, $\|Tx\|_Y \geq 0$ ✓

$\iff x = 0$.

To show $\|\cdot\|_X$. $\|\cdot\|_0$ equivalent $\iff$ T is bounded

"$\Rightarrow$" If $\|\cdot\|_X$. $\|\cdot\|_0$ are equivalent $\exists C > 0$.

$C\|\cdot\|_0 \leq \|\cdot\|_X \leq \|\cdot\|_0$ Then for all $x \in X$.

$C\|x\|_0 \leq \|x\|_X$.

$C\|x\|_X + C\|Tx\|_Y \leq \|x\|_X$.

$\|Tx\|_Y \leq \frac{1-C}{C}\|x\|_X$.

$\|T\| = \sup\{\|Tx\|_Y.\ \|x\|_X = 1\} \leq \frac{1-C}{C}.$

So T is bounded ✓

"$\Leftarrow$" If T is bounded then $\exists C$. for all $x \in X$.

$\|Tx\|_Y \leq C\|x\|_X$.

$\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq (1+C)\|x\|_X$.

$\frac{\|x\|_0}{1+C} \leq \|x\|_X \leq \|x\|_0$.

then $\|\cdot\|_X$. $\|\cdot\|_0$ are equivalent. ✓

(b). Show any linear map $T: X \to Y$ is bounded. if X is finite dimentional.

By theorem 1.6. any two norm are equivalent. Take $\|\cdot\|_\infty$ on X.

And choose a basis of X. $\{e_1 \dots e_n\}$

$Tx = \sum a_i Te_i$ Remember to write that you write $x = \sum_{i=1}^{n} a_i e_i$

$\|Tx\|_Y \le \|\sum_i a_i Te_i\| \le \sum_i |a_i| \|Te_i\|_Y$.

Let $C = \sup\{\|Te_i\|_Y\}$.

$\|Tx\|_Y \le \sum_i |a_i| \|Te_i\|_Y \le \sum_i |a_i| C \le n\max\{a_i \ i=1 \dots n.\} C$

$= nC\|x\|_\infty$ ✓

(c). Suppose that x is infinite dimentional.

WTS $\exists\ T: X \to Y$ which is not bounded.

Then we can take a Hamel basis for X. $\{e_i\}_{i \in I}$ where I is infinite. then we can take a countable subset $_M$ of I Missing an identification $M \cong \mathbb{N}$ here.

Choose $y \in Y$ define map $T: X \to Y$. $T(e_i) = iy$ $i \in \mathbb{N}$. What about $i \in I \setminus \mathbb{N}$?

What if $y=0$?

So $\nexists\, r$ $T(\overline{B(0,1)}_X) \subseteq \overline{B(0,r)}$ then T is not bounded. (✓)

(d) Suppose that X is infinite dimensional. By (a) and (c) we define $\|\cdot\|_0$ by $\|x\|_0 = \|x\|_X + \|Tx\|_Y$. where $\underline{T(e_i) = iy \quad y \in Y.\ i \in \mathbb{N}}$.

$\|x\|_X \le \|x\|_0 + \|Tx\|_Y$ since T is not bounded so $\nexists\, C$ Same issues as before

$\|Tx\|_Y \le C\|x\|_X$. then $\nexists\, M$ $M\|x\|_0 \le \|x\|_X$.

Moreover if $(X, \|\cdot\|_X)$ is a banach space. WTS $(X, \|\cdot\|_0)$ is not a banach space.

if $(x_i)_{i \in \mathbb{N}}$ is a cauchy sequence in $(X, \|\cdot\|_0)$ then for $\forall\, \varepsilon > 0$ $\exists\, N \in \mathbb{N}$ which $\forall\, m.n > N$ $\|x_m - x_n\|_X \le \|x_m - x_n\|_0 < \varepsilon$.

So $(x_i)_{i \in \mathbb{N}}$ is a cauchy sequence in $(X, \underline{\|\cdot\|_0})$

assume you mean $\|\cdot\|_X$ here?

If x is a limit point of the cauchy sequence $(x_i)_{i\in\mathbb{N}}$ in $(X, \|\cdot\|_0)$

then $\lim_{n\to\infty} \|x_n - x\|_X \leq \lim_{n\to\infty} \|x_n - x\|_0 = 0$.

So x is also the limit point of $(x_i)_{i\in\mathbb{N}}$ in $(X, \|\cdot\|_X)$.

Construct a cauchy sequence by $x_n = \frac{e_n}{n}$ where $\{e_i\}_{i\in\mathbb{N}}$ is the subset of Hamel basis of X with norm 1.

$$\begin{aligned}
\|x_m - x_n\|_0 &= \|x_m - x_n\|_X + \|T(x_m - x_n)\|_Y \\
&= \|x_m - x_n\|_X + \left\|\frac{my}{m} - \frac{ny}{n}\right\|_Y \\
&= \|x_m - x_n\|_X \\
&\leq \|x_m\|_X + \|x_n\|_X = \frac{1}{n} + \frac{1}{m}
\end{aligned}$$

$\forall \varepsilon > 0$. ~~take~~ $N > \frac{2}{\varepsilon}$ $N \in \mathbb{N}$ $\|x_m - x_n\|_0 \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \varepsilon$ $\forall m, n > N$.

So it is a cauchy sequence in both $(X, \|\cdot\|_0)$. $(X, \|\cdot\|_X)$.

In $(X, \|\cdot\|_X)$ we have $x_n \to 0$ $n \to \infty$

$\lim_{n\to\infty} \|x_n - 0\| = \lim_{n\to\infty} \left\|\frac{e_n}{n}\right\| = 0$ ← What norms?

So If $(X, \|\cdot\|_0)$ is complete then 0 is also the limit point of $(x_i)_{i\in\mathbb{N}}$ in $(X, \|\cdot\|_0)$

$\lim_{n\to\infty} \|x_n\|_0 = \lim_{n\to\infty} \left\|\frac{e_n}{n}\right\| + \lim_{n\to\infty} \left\|T\left(\frac{e_n}{n}\right)\right\|_Y = \|y\|_Y$.

$\|0\|_0 = 0 \neq \|y\|_Y$

So 0 is not the limit point of $(x_i)_{i\in\mathbb{N}}$. So $(X, \|\cdot\|_0)$ is not complete

(✓)

(e). Take $\|\cdot\|'$ in $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|_1)$. Why is $\|\cdot\|'$ a norm?

$\|\cdot\|' = \sum_n \frac{|x_n|}{n}$. $e_i = (0 \cdots, \underset{i\text{-th}}{1}, \cdots 0 \cdots)$.

Indeed. $\|\cdot\|' \leq \|\cdot\|_1$ $\forall c > 0$ $\exists i > \frac{1}{c}$ $\|e_i\|' \leq c\|e_i\|_1$

So $\|\cdot\|'$ is not equivalent to $\|\cdot\|_1$

How does that prove non-equivalence?

If x is the limit point of a cauchy sequence in $(\ell_1(\mathbb{N}), \|\cdot\|_1)$

$$\|x_m - x_n\|' \le \|x_m - x_n\|_1 \qquad \|x_n - x\|' \le \|x_n - x\|_1$$

x should also be the limit point of the same cauchy sequence in $(\ell_1(\mathbb{N}), \|\cdot\|')$ Missing the counterexample

2.

(a) $M = \{(a, b, 0 \cdots) \; a, b \in \mathbb{C}\} \subseteq (\ell_p(\mathbb{N}), \|\cdot\|_p)$

$f(a, b, 0 \cdots) = a+b$. let $b \neq 0$ $a > b$ $a, b \in \mathbb{C}$ and $\mathbb{C}$ does not have $>$

$$\|f\| = \sup\left\{\frac{\|f(x)\|}{\|x\|}\right\} = \sup \frac{|a+b|}{(|a|^p + |b|^p)^{\frac{1}{p}}}$$

By Hölders inequality we have What if $p=1$?

$$\sum_{k=1}^{n} |x_k y_k| \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}$$

$y_1 = 1 \quad y_2 = 1 \quad x_1 = a \quad x_2 = b$

$$|a| + |b| \le (|a|^p + |b|^p)^{\frac{1}{p}} (1+1)^{\frac{1}{q}}$$

So $\|f\| = \sup \frac{|a+b|}{(|a|^p + |b|^p)^{\frac{1}{p}}} \le \sup \frac{|a| + |b|}{2^{-\frac{1}{q}}(|a| + |b|)} \le (1+1)^{\frac{1}{q}} = 2^{1-\frac{1}{p}}$

$\|f\|$ is bounded

we can take $a = b$

$$\|f\| \ge \frac{|2a|}{(|a|^p + |a|^p)^{\frac{1}{p}}} = \frac{|2a|}{2^{\frac{1}{p}} |a|} = 2^{1-\frac{1}{p}}$$

So $\|f\| = 2^{1-\frac{1}{p}}$ (✓)

(2) $1 < p < \infty$ $f \in \mathcal{L}(M, K)$. by cor 2.6 $\exists$ F. on $\ell_p(\mathbb{N})$

St $F|_M = f$ $\|F\| = \|f\|$. Then we have to show the uniqueness we naturly have one $F(a, b, c \cdots) = a+b$.

Claim $F' = F + \alpha_3 \delta_3 + \alpha_4 \delta_4 + \cdots$ δ_i is the dual basis defined by $\delta_i(e_m) = \delta_{im} = \begin{cases} 1 & m = i \\ 0 & m \neq i \end{cases}$

$\|F'\| > \|f\|$ if $\exists \alpha_i \neq 0$. we just prove $\alpha_3 \neq 0$.

$\|F'\| = \|a\delta_1 + \overset{b}{}\delta_2 + \alpha_3 \cdot \delta_3 \quad\|$

$\alpha_3 = c$?

$$= \sup \frac{|a+b+\alpha_3|}{(|a|^p+|b|^p+| c |^p \quad)^{\frac{1}{p}}}$$

Unsure what happens here?

$$= \sup \frac{|a+b+\alpha_3 \ |}{(a^p+b^p+ \quad (^p)^{\frac{1}{p}}(\alpha_3^q+1^q+1^q)^{\frac{1}{q}}}$$

$$\leq \sup \left(\frac{|a|+|b|+|\alpha_3|}{(\frac{1}{2^q+\alpha_3^q})(|a|+|b|+|\alpha_3|}\right) = 2^q+\alpha_3^q \Leftarrow \text{by holder inequality.}$$

$q = \frac{p-1}{p}$ $\quad q = \frac{p}{p-1}$?

Should be in $\|\cdot\|_p$?

take $a = b = \alpha_3 \quad a+b+\alpha_3 = 1$

we get $\quad F'(a.b.\alpha_3 \ .0 \dots) = 2^q + \alpha_3^q$

$\|F'\| = 2^q + \alpha_3^q > \|f\|$.

In general $\|F'\|_p > \|f\|_p$ when $\exists \alpha_i \neq 0$.

This proves $\alpha_i = 0$ if $F' = F + \sum_{i\geq 3} \alpha_i \delta_i$ if $\|F'\| = \|F\|$. But what if $F' = F + \alpha x_i$ for $i = 1,2$?

(3)

F' is a functional so $\|\cdot\|_1$ is not defined for F'.

we know that when $p = 1 \quad \|F\| = 1$

$$\|F'\|_1 = \|a\delta_1 + b\delta_2 + \alpha_3\delta_3 + \dots + \alpha_n\delta_n + \dots\| \overset{?}{=} \|\delta_1 + \delta_2\|_1 = 1 \quad \forall \alpha_i \in \mathbb{C}$$

$$\|F'\| = \sup \frac{|a+b+\alpha_3+\dots|}{|a|+|b|+|\alpha_3|+\dots} \leq \sup \frac{|a|+\dots+}{|a|+\dots+} = 1$$

take $a = b = \alpha_3 = \dots$

we have $\|F'\| \geq 1$

so $\|F'\| = 1 = \|f\|$.

$\square$

3.

(a) X is an infinite dimensional normed vector space over $\mathbb{K}$.

We can take $(n+1)$ linearly independent elements $x_1 \cdots x_{n+1}$ from X. Let $S = \{x_1, \cdots x_{n+1}\}$ $M = \operatorname{Span} S$.

$T|_M : X \to \mathbb{K}^n$. Suppose T is injective. then $T|_M$ is also injective. M is finite dimensional. then we have.

$$\dim M = \dim \operatorname{Im} T|_M + \dim \ker T|_M$$

$$T|_M \text{ injective} \Rightarrow \dim \ker T|_M = 0.$$

$$\dim M = \dim \operatorname{Im} T|_M = n+1 > \dim \mathbb{K}^n = n.$$

There is no conclusion $(k=\mathbb{R})$ anywhere?

for $k = \mathbb{C}$ we just choose $2n+1$ elements for S.

No! $\mathbb{C}^n$ is viewed as complex vectorspace then.

(b) $f_1 \cdots f_n \in X^*$. WTS. $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$

Consider $F(x) = (f_1(x), f_2(x), \cdots, f_n(x))$. $x \in X$

X infinite dimension $\Rightarrow$ $F: X \to \mathbb{K}^n$ is not injective

So $\ker F \neq \{0\}$. $x \in \ker F$ $F(x) = (f_1(x) \cdots f_n(x)) = 0$.

$$x \in \bigcap_{i=1}^{n} \ker f_i$$

$$\Rightarrow \bigcap_{i=1}^{n} \ker f_i \neq \{0\}.$$

(c) Let $x_1 \cdots x_n \in X$.

For each x_i. By thm 2.7 (b).

we can find a map $f_i \in X^*$.

$\|f_i\| = 1$ $f_i(x) = \|x_i\|$ we have If $y \in \ker f_i$

$$\|y - x_i\| = \|f_i\| \, \|y - x_i\| \geq \|f_i(y - x_i)\| = \|\underset{0}{\underline{1}} - f_i(x_i)\| = \|x_i\|.$$

then we choose y from $\ker F$ (F is defined in (b) for f_i).

$y \in \ker F \Rightarrow y \in \bigcap_{i=1}^{n} \ker f_i \Rightarrow \|y - x_i\| \geq \|x_i\|$

$\ker F$ is a subspace of X.

So we can take $\|y\| = 1$.

(d). WTS One can not cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X s.t. none of the balls contain 0.

what is I?

Suppose $\exists$ finite closed ball $\{B(x_i, r_i), r_i < \|x_i\|\}_{i \in I}$ By (c)

we can find $y \in S$. $\|y - x_i\| \geq \|x_i\| > r_i$ $\quad y \notin B(x_i, r) \cap S$.

$y \notin \bigcup_i (B(x_i, r) \cap S)$ So $\{B(x_i, r), r_i < \|x_i\|\}_{i \in I}$ can not cover S

(e). WTS S is non-compact. We choose point step by step.

Take arbitrary $x_1 \in S$. use (c) find x_2. $x_2 \in S$. $\|x_2 - x_1\| \geq \|x_1\| = 1$.

And then choose $x_3, x_4 \ldots \in S$. for any finite sequence $x_1 \ldots x_n$ we can always choose x_{n+1}.

Therefore we get an infinite sequence $(x_i)_{i \in \mathbb{N}}$ which satisfies $\forall m, n \in \mathbb{N}$

$\|x_m - x_n\| \geq \|x_n\| = 1$

for all $x \in S$ take a neighborhood of x on S

$N_x(x, \frac{1}{2}) = B(x, \frac{1}{2}) \cap S$ if $\exists x_m \in (x_i)_{i \in \mathbb{N}}$ $x_m \in N_x(x, \frac{1}{2})$

for all $x_n \in (x_i)_{i \in \mathbb{N}}$ $n \neq m$.

$\|x_n - x\| \geq \|x_n - x_m\| - \|x_m - x\| \geq 1 - \frac{1}{2} = \frac{1}{2}$ So $x_n \notin N_x(x, \frac{1}{2})$

So $\forall x \in S$ is not the limit point of any subsequence of $(x_i)_{i \in \mathbb{N}}$.

$(x_i)_{i \in \mathbb{N}}$ is an infinite sequence that has not convergent subsequence.

So S is non-compact. sequential, should argue that metric space $\Rightarrow$ compact = seq. compact.

(Alternatively, conclude that $(x_i)_{i \in \mathbb{N}}$ has no convergent subnet)

For closed unit ball use the same construction of the sequence and choose neighborhood for $\forall x \in \overline{B(0,1)}$ $N_x(x, \frac{1}{2}) = B(x, \frac{1}{2}) \cap \overline{B(0,1)}$

we still have $\|x_n - x\| \geq \|x_n - x_m\| - \|x_m - x\| \geq \frac{1}{2}$

So $\overline{B(0,1)}$ is non-compact.

(4) $E_n = \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 \, dm \le n \right\}$.

(a) $n \ge 1$ Claim E_n is convex but not absorbing.

$f_1, f_2 \in E_n$ $\quad \|\alpha f_1 + (1-\alpha) f_2\|_p^p \le \left(\|\alpha f_1\|_p + \|(1-\alpha) f_2\|_p\right)^p$

$$\le \left(\alpha\, {}^p\!\sqrt{n} + (1-\alpha)^p \sqrt{n}\right)^p$$

$$= \left(\alpha^p + (1-\alpha)^p\right) n$$

$$= \left(\alpha^3 + (1-\alpha)^3\right) n \le n.$$

Suppose E_n is absorbing

$\exists t > 0.\ \forall x \in L_1([0,1], m)\ tx \in E_n$.

for $f \in L_1([0,1], m) \setminus E_n$ $\quad \int_{[0,1]} |f| \, dm < \infty$. $\int_{[0,1]} |f|^3 \, dm > n$.

We have $\int_{[0,1]} |tf|^3 \, dm \le n$. Let $f' = \frac{f}{t}$

No, need to show for t independent of f.

We find that $\int_{[0,1]} |f'| \, dm = \frac{1}{t} \int_{[0,1]} |f| \, dm < \infty$. $f' \in L_1$

But $\int_{[0,1]} |tf'|^3 \, dm = \int_{[0,1]} |f|^3 \, dm \ge n$ $\quad tf' \notin E_n$. but $t^2 f' \in E_n$

$\Rightarrow E_n$ is not absorbing.

(b) Show that E_n has empty interior in $L_1([0,1], m)$ for all $n \ge 1$.

It is equivalent to show that $\forall \varepsilon > 0$ $\forall f \in E_n$ $\exists f' \in L_1 \setminus E_n$.

$\|f - f'\| < \varepsilon$.

To construct f' let $f' - f = \begin{cases} \dfrac{6\sqrt{n}}{\sqrt{\varepsilon}} & \left[0, \dfrac{\sqrt[2]{\varepsilon^3}}{8\sqrt{n}}\right] \\ 0 & \left(\dfrac{\sqrt[2]{\varepsilon^3}}{8n}, 1\right] \end{cases}$

$\int |f' - f| \, dm = \frac{6\sqrt{n}}{\sqrt{\varepsilon}} \cdot \frac{\sqrt[2]{\varepsilon^3}}{8\sqrt{n}} = \frac{3}{4}\varepsilon < \varepsilon$.

$\int |f' - f|^3 \, dm = \left(\frac{6\sqrt{n}}{\sqrt{\varepsilon}}\right)^3 \cdot \frac{\sqrt[2]{\varepsilon^3}}{8\sqrt{n}} = 27n$.

$\|f' - f\|_3 = 3\sqrt[3]{n}$

$\|f\|_3 \le \sqrt[3]{n}$

$\|f'\|_3 \ge \|f' - f\|_3 - \|f\|_3 \ge 2\sqrt[3]{n}$

$\|f'\|_3^3 \ge 8n$.

So $f' \notin E_n$.

(C) Show that E_n is closed in $L_1([0,1].m)$ all $n \geq 1$

$f_n \in E_n$ $(x_n)_{n \in \mathbb{N}}$ is a convergent cauchy sequence in $L_1([0,1].m)$ converge to f $\{f_{n_k}\}$

not maybe?

take a subsequence, converge pointwise to f. a.e.

By fatou's lemma.

$$\int_{[0,1]} \liminf |f_{n_k}|^3 \, dm = \int_{[0,1]} |f|^3 \, dm \leq \liminf \int_{[0,1]} |f_{n_k}|^3 \, dm \leq n.$$

So f is in E_n.

(d) $L_3([0,1].m)$ is of the first category in $L_1([0,1].m)$

$E_n \subset L_3([0,1].m)$. $\bar{E}_n = E_n \Leftarrow E_n$ closed in L_1.

By (b) E_n has empty interior $\mathrm{Int}(\bar{E}_n) = \mathrm{Int}(E_n) = \phi$

$L_3([0,1].m) = \bigcup_{i=1}^{\infty} E_n$ $\forall f \in L_3$ $\exists M \in \mathbb{N} \, \|f\|_3 < M$.

$f \in E_{M^3}$ $L_3([0,1].m) \subseteq \bigcup_{i=1}^{\infty} E_n$ so $L_3([0,1].m)$ is of the first category in $L_1([0,1].m)$.

Problem 5.

H is an infinite dimentional separable Hilbert space.

(a) $x_n \to X$ in norm. it follows that $\|X_n\| \to \|X\|$.

as $n \to \infty$

Pf: $\lim_{n \to \infty} \|x_n - X\| = 0$ $\|x\| \leq \|x_n - X\| + \|x_n\|$.

$\|x_n\| \leq \|X\| + \|x_n - x\|$.

which implies $\lim_{n \to \infty} \|x_n\| \to \|x\|$. Be more explicit here, right now you've just written statements. (✓)

(b) $x_n \to x$ weakly. it does not follow that $\|x_n\| \to \|x\|$ as $n \to \infty$.

By Homework 4. Pr 2 and Pr 3.

$x_n \to x$ weakly $\Longleftrightarrow$.

$\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$, as $n \to \infty$

Lacks an argument of $\langle \cdot, y\rangle$, $y \in H$ being all linear functionals on H.

$\Longleftrightarrow$

$\langle x_n, e_i\rangle \to \langle x, e_i\rangle$ for all $e_i \in \{e_i\}_{i\in\mathbb{N}}$

Choose $x_i = e_i$, $x = 0$.

Why?

$\langle x_n, e_i\rangle \to 0 = \langle x, e_i\rangle$.

$\|x_n\| = \|e_i\| = 1$. which does not converge to $\|x\| = 0$.

(✓)

(c). It is true

By proposition 2.7 (b)

$0 \neq x \in X \; \exists \; f \in X^*$ $\quad f(x) = \|x\|$ $\quad \|f\| = 1$.

What if $x=0$?

Then the weak convergence of $x_n \to x$.

$$\|x\| = f(x) = \lim_{n\to\infty} f(x_i) = \lim_{n\to\infty} |f(x_i)| \leq \sup \|f\| \, \|x_i\| = \sup \|x_i\| \leq 1.$$

↑ weak conv. $\qquad$ ↑ Conv. $\Rightarrow$ abs. conv.

✓