Notes on 1D bosons

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We consider the dilute Bose gas in one dimension, where we seek to prove the formula for the ground state energy

$$\frac{E}{L} = \frac{\pi^2}{3}\rho^3\left(1 + 2\rho a + o\left(\rho\right)\right). \tag{0.1}$$

We assume that the interaction potential v has compact support, say in the ball of radius R_0, B_{R_0}.

1 Upper bound

1.1 Dirichlet boundary conditions

We provide the upper bound for (0.1), by using the variational principle with a suitable trial state. Since we are interested in an upper bound, we consider Dirichlet boundary conditions. For $b > R_0$, consider the trial state

$$\Psi(x) = \begin{cases} \omega(\mathcal{R}(x))\frac{\tilde{\Psi}_F(x)}{\mathcal{R}(x)} & \text{if } \mathcal{R}(x) < b, \\ \tilde{\Psi}_F(x) & \text{if } \mathcal{R}(x) \geq b, \end{cases} \tag{1.1}$$

where ω is the suitably normalized solution to the two-body scattering equation, *i.e.* $\omega(x) = f(x)\frac{b}{f(b)}$ where f is any solution of the two-body scattering equation, $\tilde{\Psi}_F(x) = |\Psi_F|$ is the absolute value of the free fermionic ground state, and $\mathcal{R}(x) = \min_{i<j}(|x_i - x_j|)$ is uniquely defined a.e.

The energy of this trial state is then

$$\mathcal{E}(\Psi) = \int \sum_{i=1}^{N} \left|\nabla_i \Psi\right|^2 + \sum_{i<j}^{N} v_{ij}\left|\Psi\right|^2, \tag{1.2}$$

where $v_{ij}(x) = v(x_i - x_j)$. Since v is supported in B_b and $\Psi = \tilde{\Psi}_F$ except in the region $B = \{x \in \mathbb{R}^N | \mathcal{R}(x) < b\}$, we may rewrite this as

$$\mathcal{E}(\Psi) = E_0 + \int_B \sum_{i=1}^{N} \left|\nabla_i \Psi\right|^2 + \sum_{i<j}^{N} v_{ij}\left|\Psi\right|^2 - \sum_{i=1}^{N}\left|\nabla_i \tilde{\Psi}_F\right|^2, \tag{1.3}$$

where $E_0 = N\frac{\pi^2}{3}\rho^2(1+\mathcal{O}(1/N))\|\Psi\|^2$ is the ground state energy of the free Fermi gas. Using that $v \geq 0$, symmetry of exchange of particles, and defining the set $B_{12} = \{x \in \mathbb{R}^N | \mathcal{R}(x) < b,\ \mathcal{R}(x) = |x_1 - x_2|\} \subset A_{12} = \{x \in \mathbb{R}^N |\, |x_1 - x_2| < b\}$ which up to a set of measure zero is the intersection of B and the set {"1 and 2 are closest"}, we find

$$
\begin{aligned}
\mathcal{E}(\Psi) &= E_0 + \binom{N}{2}\int_{B_{12}} \sum_{i=1}^N |\nabla_i \Psi|^2 + \sum_{i<j}^N v_{ij}|\Psi|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 \\
&= E_0 + \binom{N}{2}\int_{B_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}\right|^2 + \sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 \\
&= E_0 + \binom{N}{2}\int_{A_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}\right|^2 + \sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 \\
&\quad - \binom{N}{2}\int_{A_{12}\setminus B_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}\right|^2 + \sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 \\
&\leq E_0 + E_1 + \binom{N}{2}\int_{A_{12}\setminus B_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2
\end{aligned}
\tag{1.4}
$$

where we have defined

$$
\tilde{\Psi} = \begin{cases} \omega(x_1 - x_2)\frac{\tilde{\Psi}_F(x)}{|x_1 - x_2|} & \text{if } |x_1 - x_2| < b, \\ \tilde{\Psi}_F(x) & \text{if } |x_1 - x_2| \geq b, \end{cases}
$$

and $E_1 = \binom{N}{2}\int_{A_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}\right|^2 + \sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2 - \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2$.
We may estimate

$$
\begin{aligned}
\binom{N}{2}\int_{A_{12}\setminus B_{12}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 &= \binom{N}{2}\Bigg(2N\left[\int_{A_{12}\cap A_{13}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 - \int_{B_{12}\cap A_{13}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2\right] \\
&\qquad + \binom{N-2}{2}\left[\int_{A_{12}\cap A_{34}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 - \int_{B_{12}\cap A_{34}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2\right]\Bigg) \\
&\leq \binom{N}{2}\left[2N\int_{A_{12}\cap A_{13}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2 + \binom{N-2}{2}\int_{A_{12}\cap A_{34}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2\right]
\end{aligned}
\tag{1.5}
$$

Thus we find

$$
\mathcal{E}(\Psi) \leq E_0 + E_1 + E_2^{(1)} + E_2^{(2)} \tag{1.6}
$$

with $E_2^{(1)} = \binom{N}{2}2N\int_{A_{12}\cap A_{13}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2$ and $E_2^{(2)} = \binom{N}{2}\binom{N-2}{2}\int_{A_{12}\cap A_{34}} \sum_{i=1}^N \left|\nabla_i \tilde{\Psi}_F\right|^2$.
We notice that since $\tilde{\Psi}_F = |\Psi_F|$ so by the diamagnetic inequality we have $\left|\nabla_i \tilde{\Psi}_F\right|^2 \leq |\nabla_i \Psi_F|^2$, which implies that $\tilde{\Psi}_F$ is in $H^1(\Lambda_L)$. Furthermore, Ψ_F is $C^1(\Lambda_L)$ with a zero set $\{\Psi_F = 0\}$ of measure zero, so $\left|\nabla_i \tilde{\Psi}_F\right|^2$ and $|\nabla_i \Psi_F|^2$ are equal a.e. But then $\left|\nabla_i \tilde{\Psi}_F\right| = |\nabla_i \Psi_F|$ as $L^2(\Lambda_L)$ functions. Hence we may replace $\tilde{\Psi}_F$ with Ψ_F in all integrals above.

1.2 The free Fermi ground state

We now construct the free Fermi ground state. The Dirichlet eigenstates of the Laplacian are $\phi_j(x) = \sqrt{2/L}\sin(\pi j x/L)$. Thus the free Fermi ground state is

$$\Psi_F(x) = \det\left(\phi_j(x_i)\right)_{i,j=1}^N = \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \begin{vmatrix} e^{iy_1} - e^{-iy_1} & e^{i2y_1} - e^{-i2y_1} & \dots & e^{iNy_1} - e^{-iNy_1} \\ e^{iy_2} - e^{-iy_2} & e^{i2y_2} - e^{-i2y_2} & \dots & e^{iNy_2} - e^{-iNy_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{iy_N} - e^{-iy_N} & e^{i2y_N} - e^{-i2y_N} & \dots & e^{iNy_N} - e^{-iNy_N} \end{vmatrix}, \tag{1.7}$$

where we defined $y_i = \frac{\pi}{L}x_i$. Defining $z = e^{iy}$ and using the relation $(x^n - y^n)/(x-y) = \sum_{k=0}^{n-1} x^k y^{n-1-k}$ we find

$$\Psi_F(x) = \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \prod_{i=1}^N (z_i - z_i^{-1}) \begin{vmatrix} 1 & z_1 + z_1^{-1} & \dots & \sum_{k=0}^{N-1} z_1^{2k-N+1} \\ 1 & z_2 + z_2^{-1} & \dots & \sum_{k=0}^{N-1} z_2^{2k-N+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_N + z_N^{-1} & \dots & \sum_{k=0}^{N-1} z_N^{2k-N+1} \end{vmatrix}. \tag{1.8}$$

Notice now that $(z+z^{-1})^n = \sum_{k=0}^n \binom{n}{k} z^{2k-n}$. Now for i from 1 to $N-1$ we add $\left(\binom{N-1}{i} - \binom{N-1}{i-1}\right)$ times column $N-i$ to column N. This of course does not change the determinant, and we find

$$\Psi_F(x) = \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \prod_{i=1}^N (z_i - z_i^{-1}) \begin{vmatrix} 1 & z_1 + z_1^{-1} & \dots & \sum_{k=0}^{N-2} z_1^{2k-N+1} & (z_1 + z_1^{-1})^{N-1} \\ 1 & z_2 + z_2^{-1} & \dots & \sum_{k=0}^{N-2} z_2^{2k-N+1} & (z_2 + z_2^{-1})^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & z_N + z_N^{-1} & \dots & \sum_{k=0}^{N-2} z_N^{2k-N+1} & (z_N + z_N^{-1})^{N-1} \end{vmatrix}. \tag{1.9}$$

Now for $i = 1$ to $N-2$ we add $\left(\binom{N-2}{i} - \binom{N-2}{i-1}\right)$ times column $N-1-i$ to column $N-1$, continue this process, *i.e.* for $j = 3$ to N: for $i = 1$ to $N-j$ add $\left(\binom{N-j}{i} - \binom{N-j}{i-1}\right)$ times column $N-1-i$ to column $N-j+1$. Then we obtain

$$\Psi_F(x) = \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \prod_{i=1}^N (z_i - z_i^{-1}) \begin{vmatrix} 1 & z_1 + z_1^{-1} & (z_1 + z_1^{-1})^2 & \dots & (z_1 + z_1^{-1})^{N-1} \\ 1 & z_2 + z_2^{-1} & (z_2 + z_2^{-1})^2 & \dots & (z_2 + z_2^{-1})^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_N + z_N^{-1} & (z_N + z_N^{-1})^2 & \dots & (z_N + z_N^{-1})^{N-1} \end{vmatrix}. \tag{1.10}$$

The determinant is recognized as a Vandermonde determinant and thus we have

$$\begin{aligned}\Psi_F(x) &= \sqrt{\frac{2}{L}}^N \left(\frac{1}{2i}\right)^N \prod_{k=1}^N (z_k - z_k^{-1}) \prod_{i<j}^N \left((z_i + z_i^{-1}) - (z_j + z_j^{-1})\right) \\ &= 2^{\binom{N}{2}} \sqrt{\frac{2}{L}}^N \prod_{k=1}^N \sin\left(\frac{\pi}{L}x_k\right) \prod_{i<j}^N \left[\cos\left(\frac{\pi}{L}x_i\right) - \cos\left(\frac{\pi}{L}x_j\right)\right] \\ &= -2^{\binom{N}{2}+1} \sqrt{\frac{2}{L}}^N \prod_{k=1}^N \sin\left(\frac{\pi}{L}x_k\right) \prod_{i<j}^N \sin\left(\frac{\pi(x_i - x_j)}{2L}\right) \sin\left(\frac{\pi(x_i + x_j)}{2L}\right).\end{aligned} \tag{1.11}$$

1.2.1 Reduced density matrices

We compute the one-particle reduced density matrix of the free Fermi ground state with Dirichlet b.c. in the usual way

$$\begin{aligned}&\gamma^{(1)}(x,y) = \frac{2}{L}\sum_{j=1}^N \sin\left(\frac{\pi}{L}jx\right)\sin\left(\frac{\pi}{L}jy\right) \\ &= \frac{\cos\left(\pi\left[\left(\rho+\frac{1}{L}\right)x+\rho y\right]\right) - \cos\left(\pi\left[\left(\rho+\frac{1}{L}\right)x-\rho y\right]\right) - \cos\left(\pi\left[\left(\rho+\frac{1}{L}\right)y+\rho x\right]\right) + \cos\left(\pi\left[\left(\rho+\frac{1}{L}\right)y-\rho x\right]\right)}{4L\sin\left(\frac{\pi}{2L}(x-y)\right)\sin\left(\frac{\pi}{2L}(x+y)\right)} \\ &= \frac{\sin\left(\pi\left(\rho+\frac{1}{2L}\right)(x-y)\right)}{2L\sin\left(\frac{\pi}{2L}(x-y)\right)} - \frac{\sin\left(\pi\left(\rho+\frac{1}{2L}\right)(x+y)\right)}{2L\sin\left(\frac{\pi}{2L}(x+y)\right)}.\end{aligned} \tag{1.12}$$

Of course Wick's theorem applies to comupte a general n-particle reduced matrix.

1.2.2 Taylor's theorem

We notice now that

$$\gamma^{(1)}(x,y) = \frac{\pi}{L}\left(D_N\left(\pi\frac{x-y}{L}\right) + D_N\left(\pi\frac{x+y}{L}\right)\right), \tag{1.13}$$

where $D_N(x) = \frac{1}{2\pi}\sum_{k=-N}^N \mathrm{e}^{ikx} = \frac{\sin((N+1/2)x)}{2\pi\sin(x/2)}$ is the Dirichlet kernel. One obvious consequence is that $\left|\partial_x^{k_1}\partial_y^{k_2}\gamma^{(1)}(x,y)\right| \leq \frac{1}{\pi}(2N)^{k_1+k_2+1}\left(\frac{\pi}{L}\right)^{k_1+k_2+1} = \pi^{k_1+k_2}(2\rho)^{k_1+k_2+1}$. This bound will allow us to Taylor expand any $\gamma^{(k)}$, as all derivatives are uniformly bounded by a constant times some power of ρ. In fact the relevant power of ρ can be directly obtained from dimensional analysis. Alternatively Taylor expanding may be thought of a using the mean value theorem mulitple times.

1.3 Some useful bounds

Lemma 1. *$\rho^{(2)}(x_1,x_2) \leq \left(\frac{\pi^2}{3}\rho^4 + f(x_2)\right)(x_1-x_2)^2 + \mathcal{O}(\rho^6(x_1-x_2)^4)$ with $\int f(x_2)\,\mathrm{d}x_2 \leq$ const. $\rho^3\log(N)$.*

Proof. Notice that with periodic b.c. we have by translation invariance $\rho^{(2)}_{\text{per}}(x_1-x_2) = \frac{\pi^2}{3}\rho^4(x_1-x_2)^2 + \mathcal{O}(\rho^4(x_1-x_2)^4)$. Furhtermore, we have $\gamma^{(1)}(x_1,x_2) - \rho^{(1)}((x_1+x_2)/2) =$

$\gamma^{(1)}_{\text{per, }(\rho+1/(2L))}(x_1,x_2)-\rho$. Now by Wick's theorem we find

$$\rho^{(2)}(x_1,x_2)=\rho^{(1)}(x_1)\rho^{(1)}(x_2)-\gamma^{(1)}(x_1,x_2)\gamma^{(1)}(x_2,x_1). \tag{1.14}$$

Using that $\gamma^{(1)}$ is symmetric, and that

$$\rho^{(1)}(x_1)=\rho^{(1)}((x_1+x_2)/2)+\rho^{(1)\prime}((x_1+x_2)/2)\frac{x_1-x_2}{2}+\frac{1}{2}\rho^{(1)\prime\prime}((x_1+x_2)/2)\left(\frac{x_1-x_2}{2}\right)^2 +\mathcal{O}(\rho^4(x_1-x_2)^3) \tag{1.15}$$

$$\rho^{(1)}(x_2)=\rho^{(1)}((x_1+x_2)/2)+\rho^{(1)\prime}((x_1+x_2)/2)\frac{x_2-x_1}{2}+\frac{1}{2}\rho^{(1)\prime\prime}((x_1+x_2)/2)\left(\frac{x_1-x_2}{2}\right)^2 +\mathcal{O}(\rho^4(x_1-x_2)^3) \tag{1.16}$$

we see that

$$\begin{aligned}\rho^{(2)}(x_1,x_2)\le{}&\rho^{(1)}((x_1+x_2)/2)^2-\gamma^{(1)}(x_1,x_2)^2-\left[\rho^{(1)\prime}((x_1+x_2)/2)\right]^2\left(\frac{x_1-x_2}{2}\right)^2\\&+\rho^{(1)}((x_1+x_2)/2)\frac{1}{2}\rho^{(1)\prime\prime}((x_1+x_2)/2)\left(\frac{x_1-x_2}{2}\right)^2+\mathcal{O}(\rho^6(x_1-x_2)^4)\end{aligned} \tag{1.17}$$

Notice that $\mathcal{O}(\rho^5(x_1-x_2)^3)$ terms must cancel due to symmetry.
Now use the fact that $0\le\rho^{(1)}\le 2\rho$, and that $\rho^{(1)\prime}:[0,L]\to\mathbb{R}$, and that $\int_{[0,L]}|\rho^{(1)\prime\prime}|\le$ const. $\rho^2\log(N)$, which follows from the bound on Dirichlet's kernel $\left\|D_N^{(k)}\right\|_{L^1([0,2\pi])}\le$ const. $N^k\log(N)$, to conclude that

$$\rho^{(2)}(x_1,x_2)\le\rho^{(1)}((x_1+x_2)/2)^2-\gamma^{(1)}(x_1,x_2)^2+g_1(x_1+x_2)(x_1-x_2)^2+\mathcal{O}(\rho^6(x_1-x_2)^4), \tag{1.18}$$

for some function g_1 satisfying $\int_{[0,L]}g_1\le$ const. $\rho^3\log(N)$. Furthermore, notice that

$$\begin{aligned}\rho^{(1)}((x_1+x_2)/2)^2-\gamma^{(1)}(x_1,x_2)^2&=(\rho^{(1)}((x_1+x_2)/2)-\gamma^{(1)}(x_1,x_2))(\rho^{(1)}((x_1+x_2)/2)+\gamma^{(1)}(x_1,x_2))\\&=\left[\rho-\gamma^{(1)}_{\text{per, }(\rho+1/(2L))}(x_1,x_2)\right]\left[\rho-\gamma^{(1)}_{\text{per, }(\rho+1/(2L))}(x_1,x_2)+2\rho^{(1)}((x_1+x_2)/2)\right]\\&=\left[\rho-\gamma^{(1)}_{\text{per, }(\rho+1/(2L))}(x_1,x_2)\right]^2+2\left[\rho-\gamma^{(1)}_{\text{per, }(\rho+1/(2L))}(x_1,x_2)\right]\rho^{(1)}((x_1+x_2)/2)\\&=\mathcal{O}(\rho^6(x_1-x_2)^4)+2\left(\frac{\pi^2}{6}\rho^3(x_1-x_2)^2+\mathcal{O}(\rho^5(x_1-x_2)^4)\right)\left(\rho+\frac{1}{2L}-\frac{\pi}{L}D_N((x_1+x_2)/(2L))\right)\\&\le\frac{\pi^2}{3}\rho^4(x_1-x_2)^2+g_2(x_1-x_2)(x_1-x_2)^2+\mathcal{O}(\rho^6(x_1-x_2)^4)\end{aligned} \tag{1.19}$$

where we have choosen $g_2(x)=\frac{\pi^2}{3}\rho^3\left(\frac{1}{2L}+\left|\frac{\pi}{L}D_N(x/(2L))\right|\right)$ which clearly satifies $\int_{[0,L]}g_2\le$ const. $\rho^3\log(N)$. Thus we conclude that

$$\rho^{(2)}(x_1,x_2)\le\left(\frac{\pi^2}{3}\rho^4+f(x_2)\right)(x_1-x_2)^2+\mathcal{O}(\rho^6(x_1-x_2)^4) \tag{1.20}$$

with $f = g_1 + g_2$, satifying $\int_{[0,L]} f \leq \text{const. } \rho^3 \log(N)$ □

Lemma 2. *We have the following bounds*

$$\begin{aligned}
\rho^{(3)}(x_1,x_2,x_3) &\leq \text{const. } \rho^9 (x_1-x_2)^2(x_2-x_3)^2(x_1-x_3)^2, && \text{on } A_{12}\cap A_{23}.\\
\rho^{(4)}(x_1,x_2,x_3,x_4) &\leq \text{const. } \rho^8 (x_1-x_2)^2(x_3-x_4)^2, && \text{on } A_{12}\cap A_{34}.\\
\sum_{i=1}^{2} \partial_{y_i}^2 \gamma^{(2)}(x_1,x_2,y_1,y_2)|_{y=x} &\leq \text{const. } \rho^6 (x_1-x_2)^2, && \text{on } A_{12}.\\
\left| \partial_{y_1}^2 \left(\frac{\gamma^{(2)}(x_1,x_2,y_1,y_2)}{y_1-y_2} \right)\Bigg|_{y=x} \right| &\leq \text{const. } \rho^6 \left| x_1-x_2 \right|, && \text{on } A_{12}.\\
\sum_{i=1}^{2} (-1)^{i-1}\partial_{y_i} \left(\frac{\gamma^{(2)}(x_1,x_2,y_1,y_2)}{y_1-y_2} \right)\Bigg|_{y=x} &\leq \text{const. } \rho^6 (x_1-x_2)^2, && \text{on } A_{12}.
\end{aligned} \tag{1.21}$$

Proof. The bounds follows straightforwardly from Taylor's theorem and symmetries of the left-hand sides. As an example, consider $\sum_{i=1}^{2} \partial_{y_i}^2 \gamma^{(2)}(x_1,x_2,y_1,y_2)|_{y=x}$. Notice first that $\sum_{i=1}^{2} \partial_{y_i}^2 \gamma^{(2)}(x_1,x_2,y_1,y_2)$ is anti-symmetric in (x_1,x_2) and in (y_1,y_2). As we have previously argued that all derivatives of $\gamma^{(n)}$ are bounded by a constant times ρ^k for some $k\in\mathbb{N}$, we can clearly Taylor expand $\gamma^{(2)}$. Taylor expanding x_1 around x_2 and similarly y_1 around y_2 we see by the anti-symmetry that $\sum_{i=1}^{2} \partial_{y_i}^2 \gamma^{(2)}(x_1,x_2,y_1,y_2) \leq \text{const. } \rho^6 (x_1-x_2)(y_1-y_2)$, where the power of ρ can be found by simple dimensional analysis. □

Lemma 3. *We have the following bounds*

$$\begin{aligned}
\sum_{i=1}^{3} \left(\partial_{x_i}\partial_{y_i}\gamma^{(3)}(x_1,x_2,x_3;y_1,y_2,y_3) \right)\Bigg|_{y=x} &\leq \text{const. } \rho^9 (x_2-x_3)^2(x_1-x_2)^2, && \text{on } A_{12}\cap A_{23}.\\
\left| \sum_{i=1}^{3} \left(\partial_{y_i}^2\gamma^{(3)}(x_1,x_2,x_3;y_1,y_2,y_3) \right)\Bigg|_{y=x} \right| &\leq \text{const. } \rho^9 (x_1-x_2)^2(x_2-x_3)^2, && \text{on } A_{12}\cap A_{23}.\\
\left| \left[\partial_y \gamma^{(4)}(x_1,x_2,x_3,x_4;y,x_2,x_3,x_4)\Big|_{y=x_1} \right]_{x_1=x_2-b}^{x_1=x_2+b} \right| &\leq \text{const. } \rho^8 b (x_3-x_4)^2, && \text{on } A_{12}\cap A_{34}.
\end{aligned} \tag{1.22}$$

Proof. The proof follows straigthforwardly from Taylor's theorem and symmetries of the left-hand sides. □

Remark 1. *Notice that the lemmas 2 and 3 are not in any way optimal bounds, and many of them can indeed easily be improved by same proof technique using more symmetries. However, they are sufficient for our needs.*

1.4 Estimating E_1

Recall the definition

$$E_1 = \binom{N}{2} \int_{A_{12}} \sum_{i=1}^{N} \left|\nabla_i \tilde{\Psi}\right|^2 + \sum_{i<j}^{N} v_{ij} \left|\tilde{\Psi}\right|^2 - \sum_{i=1}^{N} \left|\nabla_i \tilde{\Psi}_F\right|^2 \tag{1.23}$$

We now prove the result

Lemma 4.

$$E_1 \leq E_0 \left(2\rho a + \text{const. } N(\rho b)^3 \left[1 + \rho b^2 \int v\right]\right) \tag{1.24}$$

Proof. We estimate E_1 by splitting it in four terms $E_1 = E_1^{(1)} + E_1^{(2)} + E_1^{(3)} + E_1^{(4)}$. First we have

$$\begin{aligned} E_1^{(1)} &= 2\binom{N}{2} \int_{A_{12}} \left|\nabla_1 \tilde{\Psi}\right|^2 \\ &= 2\binom{N}{2} \int_{A_{12}} \overline{\tilde{\Psi}} \left(-\Delta_1 \tilde{\Psi}\right) + 2\binom{N}{2} \int \left[\overline{\tilde{\Psi}} \nabla_1 \tilde{\Psi}\right]_{x_1 = x_2 - b}^{x_1 = x_2 + b} \mathrm{d}\bar{x}^1. \end{aligned} \tag{1.25}$$

The boundary term can be explicitly calculated and we find

$$\begin{aligned} 2\binom{N}{2} \int \left[\overline{\tilde{\Psi}} \nabla_1 \tilde{\Psi}\right]_{x_1 = x_2 - b}^{x_1 = x_2 + b} \mathrm{d}\bar{x}^1 = &\int \left[\frac{\omega(x_1 - x_2)}{(x_1 - x_2)} \partial_{x_1} \left(\frac{\omega(x_1 - x_2)}{(x_1 - x_2)}\right) \rho^{(2)}(x_1, x_2)\right]_{x_2 - b}^{x_2 + b} \mathrm{d}x_2 \\ &+ \int \left[\left(\frac{\omega(x_1 - x_2)}{(x_1 - x_2)}\right)^2 \partial_{x_1} \left(\gamma^{(2)}(x_1, x_2; y, x_2)\right)\Big|_{y = x_1}\right]_{x_2 - b}^{x_2 + b} \mathrm{d}x_2. \end{aligned} \tag{1.26}$$

Since the continuously differentiable function $\frac{\omega(x)}{(x_1 - x_2)} = \frac{|x_1 - x_2| - a}{b - a} \frac{b}{(x_1 - x_2)}$ for $|x_1 - x_2| > b$, we see that

$$\partial_{x_1} \left(\frac{\omega(x_1 - x_2)}{(x_1 - x_2)}\right)\Big|_{x = x_2 \pm b} = \pm b \frac{\frac{1}{b-a} - 1}{b} = \pm \frac{a}{b^2} \tag{1.27}$$

Using lemma 1, we see that

$$\int \left[\frac{\omega(x_1 - x_2)}{(x_1 - x_2)} \partial_{x_1} \left(\frac{\omega(x_1 - x_2)}{(x_1 - x_2)}\right) \gamma^{(2)}(x_1, x_2)\right]_{x_2 - b}^{x_2 + b} \mathrm{d}x_2 \leq 2aN \frac{\pi^2}{3} \rho^3 \left(1 + \text{const. } \frac{\log(N)}{N}\right) \tag{1.28}$$

Furthermore, we denote

$$\begin{aligned} &\int \left[\left(\frac{\omega(x_1 - x_2)}{(x_1 - x_2)}\right)^2 \partial_{x_1} \left(\gamma^{(2)}(x_1, x_2; y, x_2)\right)\Big|_{y = x_1}\right]_{x_2 - b}^{x_2 + b} \mathrm{d}x_2 \\ &= \int \left[\partial_{x_1} \left(\gamma^{(2)}(x_1, x_2; y, x_2)\right)\Big|_{y = x_1}\right]_{x_2 - b}^{x_2 + b} \mathrm{d}x_2 =: \kappa_1 \end{aligned} \tag{1.29}$$

Thus we have

$$E_1^{(1)} = \frac{\pi^2}{3} N \rho^3 (2a) + \kappa_1 + 2\binom{N}{2} \int_{A_{12}} \overline{\tilde{\Psi}} (-\Delta_1 \tilde{\Psi}) \tag{1.30}$$

Another contribution to E_1 is

$$\begin{aligned}
E_1^{(2)} &= -\binom{N}{2}\int_{A_{12}}\left(2\left|\nabla_1\Psi_F\right|^2+\sum_{i=3}^N\left|\nabla_i\Psi_F\right|^2\right)\\
&= -\binom{N}{2}\int_{A_{12}}\sum_{i=1}^N\overline{\Psi_F}(-\Delta_i\Psi_F)-2\binom{N}{2}\int\left[\overline{\Psi_F}\nabla_1\Psi_F\right]_{x_1=x_2-b}^{x_1=x_2+b} \qquad (1.31)\\
&= -E_0\binom{N}{2}\int_{A_{12}}\left|\Psi_F\right|^2-\underbrace{\int\left[\partial_y\gamma^{(2)}(x_1,x_2;y,x_2)|_{y=x_1}\right]_{x_2-b}^{x_2+b}\mathrm{d}x_2}_{\kappa_1}
\end{aligned}$$

Again using lemma 1 we find

$$E_1^{(2)} = -\text{const. } E_0N\rho^3b^3-\kappa_1. \qquad (1.32)$$

The last contributions are $E_1^{(3)}=\binom{N}{2}\int_{A_{12}}\sum_{i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2=\binom{N}{2}\int_{A_{12}}v_{12}\left|\tilde{\Psi}\right|^2+\binom{N}{2}\int_{A_{12}}\sum_{2\le i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2$ and $E_1^{(4)}=\int_{A_{12}}\sum_{i=3}^N\left|\nabla_i\tilde{\Psi}\right|^2$. First we notice that

$$\begin{aligned}
&\binom{N}{2}\int_{A_{12}}\sum_{2\le i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2\\
&\le \text{const. }\left(\int_{\{|x_1-x_2|<b\}\cap\operatorname{supp}(v_{34})}v(x_3-x_4)\rho^{(4)}(x_1,x_2,x_3,x_4)\right. \qquad (1.33)\\
&\qquad\left.+\int_{\{|x_1-x_2|<b\}\cap\operatorname{supp}(v_{23})}v(x_2-x_3)\rho^{(3)}(x_1,x_2,x_3)\right).
\end{aligned}$$

By lemma 2 we have

$$\begin{aligned}
&\binom{N}{2}\int_{A_{12}}\sum_{2\le i<j}^N v_{ij}\left|\tilde{\Psi}\right|^2\\
&\le \text{const. }\left(N^2(\rho b)^3\rho^3\int x^2v(x)\,\mathrm{d}x+N(\rho b)^3\rho^5\int x^4v(x)\,\mathrm{d}x+N(\rho b)^4\rho^4\int x^3v(x)\,\mathrm{d}x\right.\\
&\qquad\left.+N(\rho b)^5\rho^3\int x^2v(x)\,\mathrm{d}x\right)\\
&\le \text{const. }N^2(\rho b)^5\rho\int v=\text{const. }E_0N(\rho b)^3\left(\rho b^2\int v\right)
\end{aligned} \qquad (1.34)$$

and then we find that

$$\begin{aligned}
E_1 &= E_1^{(1)}+E_1^{(2)}+E_1^{(3)}+E_1^{(4)}\\
&\le \frac{2\pi^2}{3}N\rho^3a+2\binom{N}{2}\int_{A_{12}}\left(\overline{\tilde{\Psi}}(-\Delta_1)\tilde{\Psi}+\frac{1}{2}\sum_{i=3}^N\left|\nabla_i\tilde{\Psi}\right|^2+\frac{1}{2}v_{12}\left|\tilde{\Psi}\right|^2\right)+E_0N(\rho b)^3\text{const. }\left(1+\rho b^2\int v\right)
\end{aligned} \qquad (1.35)$$

Using the two body scattering equation this implies

$$
\begin{aligned}
E_1 \le{}& \frac{2\pi^2}{3} N\rho^3 a + 2\binom{N}{2}\int_{A_{12}} \bar{\tilde{\Psi}}\omega(-\Delta_1)\frac{\Psi_F}{(x_1-x_2)} \\
&+ 2\binom{N}{2}\int_{A_{12}} \bar{\tilde{\Psi}}(\nabla_1\omega)\nabla_1\frac{\Psi_F}{(x_1-x_2)} \\
&+ \binom{N}{2}\int_{A_{12}} \sum_{i=3}^{N} \bar{\tilde{\Psi}}\frac{\omega}{(x_1-x_2)}(-\Delta_i)\Psi_F \\
&+ \text{const.}\ E_0 N(\rho b)^3\left(1+\rho b^2\int v\right).
\end{aligned}
\tag{1.36}
$$

Furhtermore we have

$$
\begin{aligned}
&\binom{N}{2}\int_{A_{12}} \sum_{i=3}^{N} \bar{\tilde{\Psi}}\frac{\omega}{(x_1-x_2)}(-\Delta_i)\Psi_F \\
&= E_0\binom{N}{2}\int_{A_{12}} \left|\frac{\omega}{(x_1-x_2)}\Psi_F\right|^2 - 2\binom{N}{2}\int_{A_{12}} \bar{\tilde{\Psi}}\frac{\omega}{(x_1-x_2)}(-\Delta_1)\Psi_F,
\end{aligned}
\tag{1.37}
$$

so by lemma 1 it follows that

$$
\binom{N}{2}\int_{A_{12}} \left|\frac{\omega}{(x_1-x_2)}\tilde{\Psi}\right|^2 \le b^2\int_{\{|x_1-x_2|<b\}} \frac{\rho^{(2)}(x_1,x_2)}{|x_1-x_2|^2}\,\mathrm{d}x_1\,\mathrm{d}x_2 \le \text{const.}\ b^2\rho^4 L b = \text{const.}\ N\rho^3 b^3,
\tag{1.38}
$$

and by lemma 2 it follows that

$$
\begin{aligned}
2\binom{N}{2}\int_{A_{12}} \bar{\tilde{\Psi}}\frac{\omega}{(x_1-x_2)}(-\Delta_1)\Psi_F &= \frac{1}{2}\sum_{i=1}^{2}\int_{A_{12}} \left|\frac{\omega}{x_1-x_2}\right|^2 \left[\partial_{y_i}^2\gamma^{(2)}(x_1,x_2,y_1,y_2)\right]\Big|_{y=x} \\
&\le \text{const.}\ N\rho^2(\rho b)^3,
\end{aligned}
\tag{1.39}
$$

so we find

$$
\binom{N}{2}\int_{A_{12}} \sum_{i=3}^{N} \bar{\tilde{\Psi}}\frac{\omega}{(x_1-x_2)}(-\Delta_i)\Psi_F \le \text{const.}\ E_0 N(\rho b)^3.
\tag{1.40}
$$

Finally, again by lemma 2

$$
\begin{aligned}
2\binom{N}{2}\int_{A_{12}} \bar{\tilde{\Psi}}\omega(-\Delta_1)\frac{\Psi_F}{(x_1-x_2)} &= \int_{A_{12}} \left|\frac{\omega^2}{x_1-x_2}\right| \left[\partial_{y_1}^2\left(\frac{\gamma^{(2)}(x_1,x_2,y_1,y_2)}{(y_1-y_2)}\right)\right]\Big|_{y=x} \\
&\le \text{const.}\ N\rho^2(\rho b)^3,
\end{aligned}
\tag{1.41}
$$

and by using $\Delta\omega = \frac{1}{2}v\omega \ge 0$ which implies $0 \le \omega'(x) \le \omega'(b) = \frac{b}{b-a}$ for $|x|<b$, we find that

$$
\begin{aligned}
2\binom{N}{2}\int_{A_{12}} \bar{\tilde{\Psi}}(\nabla_1\omega)\nabla_1\left(\frac{\Psi_F}{(x_1-x_2)}\right) &\le \frac{1}{2}\sum_{i=1}^{2}\int_{A_{12}} \left|\frac{\omega}{x_1-x_2}\right| (-1)^{i-1}\omega'(x_1-x_2)\partial_{y_i}\left(\frac{\gamma^{(2)}(x_1,x_2,y_1,y_2)}{y_1-y_2}\right) \\
&\le \text{const.}\ \frac{b}{b-a}N\rho^2(\rho b)^3.
\end{aligned}
\tag{1.42}
$$

Combining everything we get

$$E_1 \le E_0 \left(2\rho a + \text{const. } N(\rho b)^3 \left[1 + \rho b^2 \int v\right]\right) \tag{1.43}$$

□

1.4.1 A remark about the hard core potential

Notice that it appears that we cannot deal with the hard core case. However, in the above calculation we threw away the term $\int_{A_{12}\setminus B_{12}} \sum_{2\le i<j}^N v_{ij} |\Psi|^2$. Adding this back in, we get the error $\binom{N}{2} \int_{B_{12}} \sum_{2\le i<j}^N v_{ij} \left|\tilde{\Psi}\right|^2$ instead of $\binom{N}{2} \int_{A_{12}} \sum_{2\le i<j}^N v_{ij} \left|\tilde{\Psi}\right|^2$. In doing so, we immediately see that in the presence of a hard core potential wall, $\tilde{\Psi}$ is zero, whenever two coordinates are within the hard core. Thus we may replace v_{ij} by $\tilde{v}_{ij}$ which is zero whenever $|x_i - x_j|$ is within the range of the hard core. Thus our result, generalizes to the case of a hard core, plus an integrable potential.

1.5 Estimating E_2

Recall that $E_2 = E_2^{(1)} + E_2^{(2)}$ with

$$\begin{aligned} E_2^{(1)} &= \binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=1}^N |\nabla_i \Psi_F|^2 \\ E_2^{(2)} &= \binom{N}{2}\binom{N-2}{2} \int_{A_{12}\cap A_{34}} \sum_{i=1}^N |\nabla_i \Psi_F|^2 . \end{aligned} \tag{1.44}$$

We now prove the result

Lemma 5.

$$E_2 \le E_0(N(\rho b)^4 + N^2(\rho b)^6) \tag{1.45}$$

Proof. To estimate $E_2^{(1)}$ and $E_2^{(2)}$, we first split them in two terms each and use partial integration. Consider first $E_2^{(1)}$:

$$\begin{aligned} E_2^{(1)} &= \binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=1}^N |\nabla_i \Psi_F|^2 \\ &= \binom{N}{2} 2N \left(\int_{A_{12}\cap A_{13}} |\nabla_1 \Psi_F|^2 + 2 \int_{A_{12}\cap A_{13}} |\nabla_2 \Psi_F|^2 \right) + \binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=4}^N |\nabla_i \Psi_F|^2 \end{aligned} \tag{1.46}$$

For the second term, we can perform partial integration directly, in order to obtain

$$
\begin{aligned}
&\binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=4}^{N} |\nabla_i \Psi_F|^2 = \binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=4}^{N} \overline{\Psi_F}(-\Delta_i \Psi_F) \\
&\quad \leq E_0 N^3 \int_{A_{12}\cap A_{23}} |\Psi_F|^2 - N^3 \int_{A_{12}\cap A_{23}} \sum_{i=1}^{3} \overline{\Psi_F}(-\Delta_i \Psi_F) \\
&\quad \leq 2E_0 \int_{[0,L]} \int_{[x_2-b,x_2+b]} \int_{[x_2-b,x_2+b]} \rho^{(3)}(x_1,x_2,x_3)\,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}x_2 - N^3 \int_{A_{12}\cap A_{23}} \sum_{i=1}^{3} \overline{\Psi_F}(-\Delta_i \Psi_F)
\end{aligned}
\tag{1.47}
$$

Using that lemma 2 we find

$$
2E_0 \int_{[0,L]} \int_{[x_2-b,x_2+b]} \int_{[x_2-b,x_2+b]} \rho^{(3)}(x_1,x_2,x_3)\,\mathrm{d}x_3\,\mathrm{d}x_1\,\mathrm{d}x_2 \leq N E_0 (\rho b)^6 \tag{1.48}
$$

Furthermore, we find by lemma 3

$$
\binom{N}{2} 2N \int_{A_{12}\cap A_{13}} \sum_{i=1}^{3} \left(|\nabla_i \Psi_F|^2 - \overline{\Psi_F}(-\Delta_i \Psi_F) \right) \leq \text{const. } \rho^9 L b^6 = \text{const. } E_0 (b\rho)^6. \tag{1.49}
$$

Collecting everything we find

$$
E_2^{(1)} \leq \text{const. } N E_0 (\rho b)^6. \tag{1.50}
$$

To estimate $E_2^{(2)}$, we use integration by parts

$$
\begin{aligned}
E_2^{(2)} &= \binom{N}{2}\binom{N-2}{2} \int_{A_{12}\cap A_{34}} \left(4\,|\nabla_1 \Psi_F|^2 + \sum_{i=5}^{N} |\nabla_i \Psi_F|^2 \right) \\
&= \binom{N}{2}\binom{N-2}{2} \left(4 \int_{|x_3-x_4|<b} \left[\overline{\Psi_F} \nabla_1 \Psi_F \right]_{x_1=x_2-b}^{x_1=x_2+b} + \int_{A_{12}\cap A_{34}} \sum_{i=1}^{N} \overline{\Psi_F}(-\Delta_i \Psi_F) \right) \\
&= 4 \int_{x_2\in[0,L]} \int_{|x_3-x_4|<b} \left[\partial_{y_1} \gamma^{(4)}(x_1,x_2,x_3,x_4;y_1,x_2,x_3,x_4) \Big|_{y_1=x_1} \right]_{x_1=x_2-b}^{x_1=x_2+b} + E_0 \int_{A_{12}\cap A_{34}} \rho^{(4)}(x_1,..,x_4).
\end{aligned}
\tag{1.51}
$$

By lemma 3 we get

$$
4 \int_{x_2\in[0,L]} \int_{|x_3-x_4|<b} \left[\partial_{y_1} \gamma^{(4)}(x_1,x_2,x_3,x_4;y_1,y_2,y_3,y_4) \Big|_{y_1=x_1} \right]_{x_1=x_2-b}^{x_1=x_2+b} = \text{const. } E_0 N (\rho b)^4 \tag{1.52}
$$

Furthermore, by lemma 3 again, it follows that

$$
E_0 \int_{A_{12}\cap A_{34}} \rho^{(4)}(x_1,..,x_4) \leq \text{const. } E_0 N^2 (\rho b)^6. \tag{1.53}
$$

□

1.6 Localization with Dirichlet b.c.

We will in this section localize in smaller boxes, in order to have better control on the error. The localization is straigthforward with Dirichlet boundary conditions, as gluing the wavefunctions for each box together is simple, since the wavefunctions vanish at the boundaries. Thus we consider the state $\Psi_{\text{full}} = \prod_{i=1}^{M} \Psi_\ell(x_1^i, ..., x_{\tilde{N}}^i)$, where $(x_1^i, ..., x_{\tilde{N}}^i)$ are the particles in box i and ℓ is the length of each box. Of course $\cup_{i=1}^{M}\{x_1^i, ..., x_{\tilde{N}}^i\} = \{x_1, ..., x_N\}$ and $\{x_1^i, ..., x_{\tilde{N}}^i\} \cap \{x_1^j, ..., x_{\tilde{N}}^j\} = \emptyset$ for $i \neq j$, such that $M\tilde{N} = N$. The boxes are of length $\ell = L/M - b$, and are equally spaced through out $[0, L]$ such that they are a distance of b from each other. This is to make sure that no particle interact between boxes. Combining lemmas 4 and 5, the full energy is then bounded by

$$E \leq Me_0 \left(1 + 2\tilde{\rho}a + \text{const. } \tilde{N}(b\tilde{\rho})^3 \left(1 + \rho b^2 \int v_{\text{reg}}\right)\right) / \|\Psi\|^2 \tag{1.54}$$

with $e_0 = \frac{\pi^2}{3}\tilde{N}\tilde{\rho}^2(1 + \text{const. } \frac{1}{\tilde{N}})$ and $\tilde{\rho} = \tilde{N}/\ell = \rho/(1 - \frac{bM}{L}) \simeq \rho(1 + bM/L)$. Clearly we have $\|\Psi\|^2 \geq 1 - \int_B |\Psi_F|^2 \geq 1 - \int_{|x_1 - x_2| < b} \rho^{(2)}(x_1, x_2) \geq 1 - \text{const. } \tilde{N}(\rho b)^3$, where the last inequality follows from lemma 1. Thus, choosing M such that $bM/L \ll 1$ we have

$$E \leq N\frac{\pi^2}{3}\rho^2 \frac{\left(1 + 2\rho a + \text{const. } \frac{M}{N} + \text{const. } 2\rho abM/L + \text{const. } \tilde{N}(b\rho)^3 \left(1 + \rho b^2 \int v_{\text{reg}}\right)\right)}{1 - \tilde{N}(\tilde{\rho}b)^3}. \tag{1.55}$$

Now in fact, we would choose $\tilde{N} = N/M = \rho L/M \gg 1$, *i.e.* $M/L \ll \rho$. It is clear that we minimize the error, by choosing $b = R_0$ the range of the potential. Furthermore, setting $x = M/N$ we see that the error is

$$\text{const. } \left[(1 + 2\rho^2 ab)x + x^{-1}(b\rho)^3 \left(1 + \rho b^2 \int v_{\text{reg}}\right)\right], \tag{1.56}$$

here we used that it will turn out to be the cases that $\tilde{N}(\rho b)^3 \leq 1/2$ so that we have $1/(1 - \tilde{N}(\rho b)^3) \leq 1 + 2\tilde{N}(\rho b)^3$. Optmizing in x we find $x = M/N = \frac{(b\rho)^{3/2}\left(1+\rho b^2 \int v_{\text{reg}}\right)^{1/2}}{1+2\rho^2 ab} \simeq (b\rho)^{3/2}\left(1 + \rho b^2 \int v_{\text{reg}}\right)^{1/2}$, which gives the error

$$\text{const. } (R_0\rho)^{3/2} \left(1 + R_0 \int v_{\text{reg}}\right)^{1/2} \tag{1.57}$$

Thus we arrive at the following result

Theorem 1. *Let the two-body potential $v \in L^1([0, L]) + \text{h.c.p.}$ be fixed, with two-body s-wave scattering length a. Then bosonic N-body ground state energy satisfies the upper bound*

$$E \leq E_0 \left(1 + 2\rho a + \mathcal{O}\left((R\rho)^{3/2}\left(1 + \rho R^2 \int v_{\text{reg}}\right)^{1/2}\right)\right), \tag{1.58}$$

where E_0 is the free fermionic ground state energy.

here h.c.p denotes the space of hard core potentials.
Notice that the result

$$E \leq N\frac{\pi^2}{3}\left(1+2\rho a+\mathcal{O}\left((R\rho)^{3/2}\left(1+\rho R^2\int v_{\text{reg}}\right)^{1/2}\right)\right) \tag{1.59}$$

holds only for $N \geq (\rho b)^{-3/2}$, but since the free Fermi, E_0 energy also contains a correction of order $\frac{1}{N}$, *i.e.* $E_0 = N\frac{\pi^2}{3}\rho^2(1+\text{const. } 1/N)$, the result remains true for $N < (\rho b)^{-3/2}$

1.7 Periodic boundary conditions

Another approach, would be to prove the result with periodic boundary conditions, and thus preserve translational invariance in all calculations above. Then one may use the bound [[2], lemma 4 or [3] lemma 2.1.12]

$$\left\langle\Psi_D|H_L^D|\Psi_D\right\rangle \leq \left\langle\Psi_P|H_{L-2d}^P|\Psi_P\right\rangle + \frac{4N}{d^2}\left\|\Psi\right\|^2. \tag{1.60}$$

In this case, we get errors from the periodic b.c. calculation plus errors $4N/d^2 + E_0\frac{d}{L}$, where the last one comes from change in the density assuming that $d/L \ll 1$. Optimizing in d this gives an error of const. $E_0\frac{1}{N^{2/3}}$. Thus the total result after localization is

$$E \leq E_0\left(1+2\rho a+\text{const. }\left(\frac{M}{N}\right)^{2/3}+\text{const. } 2\rho abM/L+\text{const. }\tilde{N}(b\rho)^3\left(1+\rho b^2\int v_{\text{reg}}\right)\right). \tag{1.61}$$

And optimizing in M and setting $b = R_0$ we find

$$E_0\left(1+2\rho a+\mathcal{O}\left((R\rho)^{6/5}\left(1+\rho R^2\int v_{\text{reg}}\right)^{1/2}\right)\right) \tag{1.62}$$

which is not quite as good, as the bound computed directly with Dirichlet b.c.

2 Lower bound

We will in this section provide a lower bound for the one dimensional dilute Bose gas. The proof is based on a reduction to a Lieb Liniger model, and thus we will first recall some known features about this model

2.1 The Lieb Liniger model

Recall that the energy in thermodynamic limit of the the Lieb Liniger model (with periodic boundary conditions), is determined by the sytem of equation ((3.3) and (3.18)–(3.20) in [1])

$$E_{LL}^{\rho,\ell,c=\gamma\rho} = N\rho^2 e(\gamma), \tag{2.1}$$

$$e(\gamma) = \frac{\gamma^3}{\lambda^3}\int_{-1}^{1} g(x)x^2\,\mathrm{d}x, \tag{2.2}$$

$$2\pi g(y) = 1 + 2\lambda\int_{-1}^{1}\frac{g(x)\,\mathrm{d}x}{\lambda^2+(x-y)^2}, \tag{2.3}$$

$$\lambda = \gamma\int_{-1}^{1} g(x)\,\mathrm{d}x. \tag{2.4}$$

The first lemma provides a rigorous lower bound for the thermodynamic Lieb Liniger energy density.

Lemma 6 (Lieb Liniger lower bound)**.** *Let $\gamma > 0$, then*

$$e(\gamma) \geq \frac{\pi^2}{3}\left(\frac{\gamma}{\gamma+2}\right)^2 \geq \frac{\pi^2}{3}\left(1-\frac{4}{\gamma}\right). \tag{2.5}$$

Proof. Neglecting $(x-y)^2$ in the denominator of (2.3), we see that $g \leq \frac{1}{2\pi} + 2\frac{1}{\lambda}\int_{-1}^{1} g(x)\,\mathrm{d}x$. On the other hand (2.4) shows that $e(\gamma) = \frac{\int_{-1}^{1} g(x)x^2\,\mathrm{d}x}{\left(\int_{-1}^{1} g(x)\,\mathrm{d}x\right)^3}$. Hence we denote $\int_{-1}^{1} g(x)\,\mathrm{d}x = M$, and notice that we have $g \leq \frac{1}{2\pi}\left(1+\frac{2M}{\lambda}\right)$. It is now easily verified that, $\int_{-1}^{1} g(x)x^2\,\mathrm{d}x$ with M fixed and $g \leq \frac{1}{2\pi}\left(1+\frac{2M}{\lambda}\right) = \frac{1}{2\pi}\left(1+2\gamma^{-1}\right)$ is mininmized by $g = K\chi_{[-M/(2K),M/(2K)]}$, with $K = \frac{1}{2\pi}\left(1+\frac{2}{\gamma}\right)$. This gives us $\int_{-1}^{1} g(x)x^2\,\mathrm{d}x = \frac{M^3}{3K^2}$ so that we have $e(\gamma) \geq \frac{1}{3K^2} = \frac{\pi^2}{3}\left(\frac{\gamma}{\gamma+2}\right)^2 \geq \frac{\pi^2}{3}(1-\frac{4}{\gamma})$ for $\gamma > 0$. ☐

The next result concerns finite volume correction to the thermodynamic limit. Since we are interested in a lower bound, we consider the Neumann boundary conditions case denoted by a superscript "N".

Lemma 7 (Finite volume corrections)**.**

$$E_{LL}^{N}(n,\ell,c) \geq \frac{\pi^2}{3}n\rho^2\left(1-4\rho/c-\text{const.}\,\frac{1}{n^{2/3}}\right) \tag{2.6}$$

Proof. By Robinsons bound [3], we have for any $b > 0$

$$E_{LL}^{N}(n,\ell,c) \geq E_{LL}^{D}(n,\ell+b,c) - \text{const.}\,\frac{n}{b^2}. \tag{2.7}$$

Since the range of the interaction in the Lieb-Liniger model is zero, we see that $e_{LL}^{D}(2^m n, 2^m\ell) = \frac{1}{2^m\ell}E_{LL}^{D}(2^m n, 2^m\ell)$ is a decreasing sequence. To see this, simply split the box of size $2^m\ell$ in two boxes of size $2^{m-1}\ell$, now by ignoring interactions between the boxes and using the the product state of the two $2^{m-1}n$-particle ground states in each box as a trial state, we see that

$E_{LL}^D(2^m n, 2^m\ell) \le 2E_{LL}^D(2^{m-1}n, 2^{m-1}\ell)$. Since we also have $e_{LL}^D(2^m n, 2^m\ell) \ge e_{LL}(2^m n, 2^m\ell) \to e_{LL}(n/\ell)$ as $m \to \infty$ [1], we see that

$$\begin{aligned} E_{LL}^N(n,\ell,c) &\ge e_{LL}(n/(\ell+b),c)(\ell+b) - \text{const.}\ \frac{n}{b^2} \\ &\ge \frac{\pi^2}{3} n\rho^2 \left(1 - 4\rho/c - \text{const.}\ \left(3b/\ell - \frac{1}{\rho^2 b^2}\right)\right), \end{aligned} \tag{2.8}$$

with $\rho = n/\ell$, where the second inequality follows from lemma 6. Optmizing in b we find

$$E_{LL}^N(n,\ell,c) \ge \frac{\pi^2}{3} n\rho^2 \left(1 - 4\rho/c - \text{const.}\ \frac{1}{n^{2/3}}\right). \tag{2.9}$$

□

2.2 Lieb Liniger reduction

We will in this subsection lower bound the dilute bose gas by a Lieb Liniger energy. The reduction is optained by constructing a trial state for a Lieb Liniger model i a smaller volume from the true ground state of the Bose gas.
Let Ψ be the ground state of $\mathcal{E}$, we then define $\psi \in L^2([0, \ell-(n-1)R]^n)$ by $\psi(x_1, x_2, ..., x_n) = \Psi(x_1, R+x_2, ..., (n-1)R+x_n)$ for $x_1 \le x_2 \le ... \le x_n$ and symmetrically extended.

Lemma 8. *For any function $\psi \in H^1(\mathbb{R})$ such that $\psi(0) = 0$ then we have*

$$\int_{[0,R]} |\partial\psi|^2 \ge \max_{[0,R]} |\psi|^2 / R \tag{2.10}$$

Proof. write $\psi(x) = \int_0^x \psi'(t)\,\mathrm{d}t$, and find that

$$|\psi(x)| \le \int_0^x |\psi'(t)|\,\mathrm{d}t. \tag{2.11}$$

Hence $\max_{x\in[0,R]} |\psi(x)| \le \int_0^R |\psi'(t)|\,\mathrm{d}t \le \sqrt{R}\left(\int |\psi'(t)|^2\,\mathrm{d}t\right)^{1/2}$ □

We can estimate the norm loss in the following way

$$\langle\psi|\psi\rangle = 1 - \int_B |\Psi|^2 \ge 1 - \sum_{i<j} \int_{B_{ij}} |\Psi|^2 \tag{2.12}$$

where $B = \{x \in \mathbb{R}^n |\, \min_{i,j} |x_i - x_j| < R\}$, and $B_{ij} = \{x \in \mathbb{R}^n | \mathfrak{r}_i(x) = |x_i - x_j| < R\}$. To give a good bound on the right-hand side, we need the following lemma

Lemma 9. *Let ψ be defined as above, then*

$$1 - \langle\psi|\psi\rangle \le \text{const.}\ \left(R^2 \sum_{i<j} \int_{B_{ij}} |\partial_i \Psi|^2 + R(R-a) \sum_{i<j} \int v_{ij} |\Psi|^2\right) \tag{2.13}$$

Proof. Notice that by (2.10) we have for any $\phi \in H^1$,

$$\big||\phi(x)| - |\phi(x')|\big|^2 \leq |\phi(x) - \phi(x')|^2 \leq R \left(\int_{[0,R]} |\partial\phi|^2 \right), \tag{2.14}$$

for $x, x' \in [0, R]$. Furhtermore,

$$|\phi(x)|^2 - |\phi(x')|^2 = \left(|\phi(x)| - |\phi(x')|\right)^2 + 2\left(|\phi(x)| - |\phi(x')|\right)|\phi(x')| \leq 2\left(|\phi(x)| - |\phi(x')|\right)^2 + |\phi(x')|^2 \tag{2.15}$$

So for It follows that

$$\max_{x\in[0,R]} |\phi|^2 \leq 2R \int_{[0,R]} |\partial\phi|^2 + 2 \min_{x'\in[0,R]} |\phi(x')|^2 \tag{2.16}$$

Viewing Ψ as a function of x_i we have

$$2 \min_{\mathfrak{r}_i(x)=|x_i-x_j|<R} |\Psi|^2 \geq \max_{\mathfrak{r}_i(x)=|x_i-x_j|<R} |\Psi|^2 - 4R \left(\int_{\mathfrak{r}_i(x)=|x_i-x_j|<R} |\partial_i\Psi|^2 \right). \tag{2.17}$$

Hence we find

$$\begin{aligned} 2\sum_{i<j} \int v_{ij} |\Psi|^2 &\geq 2\sum_{i<j} \int_{B_{ij}} v_{ij} |\Psi|^2 \\ &\geq \left(\int v\right) \sum_{i<j} \int \left(\max_{B'_{ij}} |\Psi|^2 - 4R \left(\int_{B'_{ij}} |\partial_i\Psi|^2 \, \mathrm{d}x_i \right) \right) \mathrm{d}\bar{x}^i \\ &\geq \frac{4}{R-a} \sum_{i<j} \left(\frac{1}{2R} \int_{B_{ij}} |\Psi|^2 - 4R \int_{B_{ij}} |\partial_i\Psi|^2 \right) \end{aligned} \tag{2.18}$$

where $B_{ij} = \{x \in \mathbb{R}^n | \mathfrak{r}_i(x) = |x_i - x_j| < R\}$ and $B'_{ij} = \{x_i \in \mathbb{R} | \mathfrak{r}_i(x) = |x_i - x_j| < R\}$. Now, by (2.12), we see that

$$1 - \langle\psi|\psi\rangle \leq \text{const.} \left(R^2 \sum_{i<j} \int_{B_{ij}} |\partial_i\Psi|^2 + R(R-a) \int \sum_{i<j} v_{ij} |\Psi|^2 \right) \tag{2.19}$$

□

Choosing $R \geq 2|a|$ we have $\langle\psi|\psi\rangle \geq 1 - \text{const. } R^2 E$.

The following lemma will also be useful

Lemma 10 (Dyson). *Let $R > R_0 = \mathrm{range}(v)$ and $\varphi \in H^1(\mathbb{R})$, then for any interval $\mathcal{B} \ni 0$*

$$\int_{\mathcal{B}} |\partial\varphi|^2 + \frac{1}{2} v |\varphi|^2 \geq \int_{\mathcal{B}} \frac{2}{R-a} (\delta_R + \delta_{-R}) \varphi \tag{2.20}$$

where a is the s-wave scattering length.

Proof. This follows from the variational scattering problem, by comparing left-hand side to the minimizer of the scattering functional. $\square$

This lemma will essentially allows us to replace the potential by a shell potential of range R and strength $\frac{2}{R-a}$.

Lemma 11. *Let ψ be defined as above with $R > \max(R_0, 2|a|)$ and let $\epsilon \in [0,1]$, then*

$$\int \sum_i |\partial_i \Psi|^2 + \sum_{i\neq j} \frac{1}{2} v_{ij} |\Psi|^2 \geq E_{LL}^N \left(n, \tilde{\ell}, \frac{2\epsilon}{R-a}\right) \langle\psi|\psi\rangle + \frac{(1-\epsilon)}{R^2} \text{const. } (1 - \langle\psi|\psi\rangle). \tag{2.21}$$

where $\tilde{\ell} = \ell - (n-1)R$.

Proof. Splitting the energy functional in two parts, and using lemma 9 on one term and Dyson's lemma on the other we find

$$\begin{aligned} &\int \sum_i |\partial_i \Psi|^2 + \sum_{i\neq j} \frac{1}{2} v_{ij} |\Psi|^2 \geq \\ &\int \sum_i |\partial_i \Psi|^2 \chi_{\mathfrak{r}_i(x) > R} + \epsilon \sum_i \frac{2}{R-a} \delta(\mathfrak{r}_i(x) - R) |\Psi|^2 \\ &\qquad + (1-\epsilon) \left(\sum_{i<j} \int_{B_{ij}} |\partial_i \Psi|^2 + \int \sum_{i<j} v_{ij} |\Psi|^2 \right) \end{aligned} \tag{2.22}$$

where $\mathfrak{r}_i(x) = \min_{j\neq i}(|x_i - x_j|)$. The nearest neighbor interaction is obtained from Dyson's lemma by dividing the integration domain into Voronoi cells, and restricting to the cell around particle i.

By use of lemma 9 with $R > 2|a|$ in the last term, and by realising that the first two terms can be obtained by using ψ as a trial state in the Lieb-Liniger model, we obtain

$$\int \sum_i |\partial_i \Psi|^2 + \sum_{i\neq j} \frac{1}{2} v_{ij} |\Psi|^2 \geq E_{LL}^N \left(n, \tilde{\ell}, \frac{2\epsilon}{R-a}\right) \langle\psi|\psi\rangle + \frac{(1-\epsilon)}{R^2} \text{const. } (1 - \langle\psi|\psi\rangle) \tag{2.23}$$

$\square$

The next lemma will bound how much mass is lost when going from the state Ψ of mass 1 to the state ψ

Lemma 12. *Let C denote the constant in lemma 9. For $n(\rho R)^2 \leq \frac{3}{16\pi^2} C$, $\rho R \ll 1$ and $R > 2|a|$ we have*

$$\langle\psi|\psi\rangle \geq 1 - \text{const. } \left(n(\rho R)^3 + n^{1/3} (\rho R)^2\right). \tag{2.24}$$

Proof. From the known upper bound, and by lemma 11 with $\epsilon = 1/2$, it follows that

$$n \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a + \text{const. } (\rho R)^{3/2}\right) \geq E_{LL}^N \left(n, \tilde{\ell}, \frac{1}{R-a}\right) \langle\psi|\psi\rangle + \frac{C}{2R^2} (1 - \langle\psi|\psi\rangle) \tag{2.25}$$

Subtracting $E_{LL}^N\left(n,\tilde{\ell},\frac{1}{R-a}\right)$ on both sides, and using lemma 7 on the right-hand side we find

$$\begin{aligned} &n\frac{\pi^2}{3}\rho^2\left(1+2\rho a+\text{const. }(\rho R)^{3/2}\right)-n\frac{\pi^2}{3}\tilde{\rho}^2\left(1+4\tilde{\rho}(R-a)-\text{const. }n^{-2/3}\right)\\ &\geq\left(\frac{C}{2R^2}-E_{LL}^N\left(n,\tilde{\ell},\frac{1}{R-a}\right)\right)(1-\langle\psi|\psi\rangle), \end{aligned} \tag{2.26}$$

with $\tilde{\rho}=n/\tilde{\ell}=\rho/(1-(\rho-1/\ell)R)$ Using now the upper bound $E_{LL}^N\left(n,\tilde{\ell},\frac{1}{R-a}\right)\leq n\frac{\pi^2}{3}\tilde{\rho}^2$ on the left-hand side, as well as $2\rho\geq\tilde{\rho}\geq\rho(1+\rho R)$ we find

$$\text{const. }n\rho^2R^2\left(\rho R+(\rho R)^{3/2}+n^{-2/3}\right)\geq\left(\frac{C}{2}-R^2n\frac{4\pi^2}{3}\rho^2\right)(1-\langle\psi|\psi\rangle) \tag{2.27}$$

It follows that we have

$$\langle\psi|\psi\rangle\geq 1-\text{const. }\left(n(\rho R)^3+n^{1/3}(\rho R)^2\right) \tag{2.28}$$

□

Remark: For $n=\mathcal{O}((\rho R)^{-9/5})$ we find

$$\langle\psi|\psi\rangle\geq 1-\text{const. }n(\rho R)^3=1-\text{const. }(\rho R)^{6/5} \tag{2.29}$$

It is now straightforward to show the result

Proposition 1. *For assumptions as in lemma 12 we have*

$$E^N(n,\ell)\geq n\frac{\pi^2}{3}\rho^2\left(1+2\rho a+\text{const. }\left(\frac{1}{n^{2/3}}+n(\rho R)^3+n^{1/3}(\rho R)^2\right)\right) \tag{2.30}$$

Proof. By lemma 11 with $\epsilon=1$, we reduce to a Lieb-Liniger model with volume $\tilde{\ell}$, density $\tilde{\rho}$, and coupling c, and we have $\tilde{\ell}=\ell-(n-1)R$, $\tilde{\rho}=\frac{n}{\tilde{\ell}}\approx\rho(1+\rho R)$ and $c=\frac{2}{R-a}$. Hence we have by Lemmas 7 and 12

$$\begin{aligned} E^N(n,\ell)&\geq E_{LL}^N(n,\tilde{\ell},c)\,\langle\psi|\psi\rangle\\ &\geq n\frac{\pi^2}{3}\rho^2\left(1+2\rho a-\text{const. }\frac{1}{n^{2/3}}\right)\left(1-\text{const. }\left(n(\rho R)^3+n^{1/3}(\rho R)^2\right)\right) \end{aligned} \tag{2.31}$$

□

Corollary 1. *For $n=\text{const. }(\rho R)^{-9/5}$ we have*

$$E^N(n,\ell)\geq n\frac{\pi^2}{3}\rho^2\left(1+2\rho a-\text{const. }\left((\rho R)^{6/5}+(\rho R)^{7/5}\right)\right). \tag{2.32}$$

2.3 Lower bound of the dilute Bose gas for general particle number

So far, we have shown the desired lower bound only for the case where the number of particles are of the order $(\rho R)^{-9/5}$. In this subsection, we generalize this to any number of particles. We do this, by performing a Legendre transformation in the particle number, *i.e.* going to the grand canonical ensemble. First we justify that only particle numbers of orders less than or equal to $(\rho R)^{-9/5}$ are relevant for a certain choice of μ.

Lemma 13. *Let $\Xi \geq 4$ be fixed and let $n = m\Xi\rho\ell + n_0$ with $n_0 \in [0, \Xi\rho\ell)$ for some $m \in \mathbb{N}$ with $n^* := \rho\ell = \mathcal{O}(\rho R)^{-9/5}$. Furhermore, assume that $\rho R \ll 1$ and let $\mu = \pi^2\rho^2\left(1 + \frac{8}{3}\rho a\right)$, then*

$$E^N(n, \ell) - \mu n \geq E^N(n_0, \ell) - \mu n_0. \tag{2.33}$$

Proof. By corollary 1 we have

$$E^N(\Xi\rho\ell, \ell) \geq \frac{\pi^2}{3}\Xi^3\ell\rho^3\left(1 + 2\Xi\rho a - \text{const. } (\rho R)^{6/5}\right). \tag{2.34}$$

By superadditivity (positive potential) we have

$$E^N(n, \ell) - \mu n \geq m\left(E^N(\Xi\rho\ell, \ell) - \mu\Xi\rho\ell\right) + E^N(n_0, \ell) - \mu n_0. \tag{2.35}$$

Thus the result follows from the fact that

$$\frac{\pi^2}{3}\Xi^3\ell\rho^3\left(1 + 2\Xi\rho a - \text{const. } (\rho R)^{6/5}\right) \geq \pi^2\rho^2\left(1 + \frac{8}{3}\rho a\right)\Xi\rho\ell \tag{2.36}$$

□

We are then ready to prove the lower bound for general particle numbers

Theorem 2 (Lower bound)**.** *Let $E^N(N, L)$ denote the ground state energy of $\mathcal{E}$ with Neumann boundary conditions. Then for $\rho R \ll 1$*

$$E^N(N, L) \geq N\frac{\pi^2}{3}\rho^2\left(1 + 2\rho a - \mathcal{O}\left((\rho R)^{6/5}\right)\right) \tag{2.37}$$

Proof. Notice that

$$E^N(N, L) \geq F^N(\mu, L) + \mu N \tag{2.38}$$

where $F^N(\mu, L) = \inf_{N'}\left(E^N(N', L) - \mu N'\right)$. Clearly we have

$$F^N(\mu, L) \geq MF^N(\mu, \ell) \tag{2.39}$$

with $\ell = L/M$ and $M \in \mathbb{N}_+$. Now we choose M such that $n^* := \rho\ell = \mathcal{O}\left((\rho R)^{-9/5}\right)$ and $\mu = \pi^2\rho^2\left(1 + \frac{8}{3}\rho a\right)$ (notice that $\mu = \frac{\mathrm{d}}{\mathrm{d}\rho}(\frac{\pi^2}{3}\rho^3(1 + 2\rho a))$). By lemma 13 we have that

$$F^N(\mu, \ell) := \inf_n\left(E^N(n, \ell) - \mu n\right) = \inf_{n < \Xi n^*}\left(E^N(n, \ell) - \mu n\right). \tag{2.40}$$

Now it is known from proposition 1 that for $n < \Xi n^*$ we have

$$\begin{aligned} E^N(n,\ell) &\geq n\frac{\pi^2}{3}\bar{\rho}^2\left(1+2\bar{\rho}a - \text{const.}\ \left(\frac{1}{n^{2/3}} + n(\bar{\rho}R)^3 + n^{1/3}(\bar{\rho}R)^2\right)\right) \\ &\geq \frac{\pi^2}{3}n\bar{\rho}^2\left(1+2\bar{\rho}a\right) - n^*\rho^2\mathcal{O}\left((\rho R)^{6/5}\right) \end{aligned} \tag{2.41}$$

where $\bar{\rho} = n/\ell$ (notice that now $\rho = N/L = n^*/\ell \neq n/\ell$) and where we used $\bar{\rho} < \Xi\rho$. Thus we have

$$F^N(\mu,\ell) \geq \inf_{\bar{\rho}<\Xi\rho}(g(\bar{\rho}) - \mu\bar{\rho})\ell - n^*\rho^2\mathcal{O}\left((\rho R)^{6/5}\right) \tag{2.42}$$

where $g(\bar{\rho}) = \frac{\pi^2}{3}\bar{\rho}^3\left(1+2\bar{\rho}a\right)$ for $\bar{\rho} < \Xi\rho$. g is a convex, C^1 function with invertible derivative for $\Xi\rho a \ll 1$. Hence we have

$$\begin{aligned} E^N(N,L) \geq M(F^N(\mu,\ell) + \mu n^*) &\geq Mn^*\frac{\pi^2}{3}\rho^2\left(1+2\rho a - \mathcal{O}\left((\rho R)^{6/5}\right)\right) \\ &= \frac{\pi^2}{3}N\rho^2\left(1+2\rho a - \mathcal{O}\left((\rho R)^{6/5}\right)\right) \end{aligned} \tag{2.43}$$

where the second inequality follows from the specific choice of $\mu = g'(\rho)$. □

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