Problem 1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be (non-zero) normed vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $T : X \to Y$ be a linear map. First we want to show that $\|x\|_0 := \|x\|_X + \|Tx\|_Y$ is a norm on X for all $x \in X$. So let $x_1, x_2 \in X$, then

$$\begin{aligned}
\|x_1 + x_2\|_0 &= \|x_1 + x_2\|_X + \|T(x_1 + x_2)\|_Y \\
&= \|x_1 + x_2\|_X + \|Tx_1 + Tx_2\|_Y \\
&\leq \|x_1\|_X + \|x_2\|_X + \|Tx_1\|_Y + \|Tx_2\|_Y \\
&= \|x_1\|_0 + \|x_2\|_0
\end{aligned}$$

Since T is linear and $\|\cdot\|_X$ and $\|\cdot\|_Y$ are both norms. For $x \in X$ and $\alpha \in \mathbb{K}$ we have

$$\|\alpha x\|_0 = \|\alpha x\|_X + \|T(\alpha x)\|_Y = \alpha\|x\|_X + \alpha\|Tx\|_Y = \alpha\|x\|_0.$$

Now for all $x \in X$ we have $\|x\|_0 = 0$ iff. $\|x\|_X + \|Tx\|_Y = 0$ iff. $\|x\|_X = 0$ and $\|Tx\|_Y = 0$ iff. $x = 0$. Hence it follows that $\|x\|_0$ is indeed a norm on X.

Now we want to show that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent iff. T is bounded. First notice that since $0 \leq \|Tx\|_Y$ we see that $\|x\|_X \leq \|x\|_0$ for all $x \in X$.
So assume now that T is bounded. In this case we know (Prop. 1.10) that there exists $C > 0$ s.t. $\|Tx\|_Y \leq C\|x\|_X$, which means that

$$\begin{aligned}
\|x\|_0 &= \|x\|_X + \|Tx\|_Y \\
&\leq \|x\|_X + C\|x\|_X \\
&= (C+1)\|x\|_X
\end{aligned}$$

So if T is bounded it follows that the norms are equivalent.
Now assume instead that the norms are equivalent. Since $\|x\|_X$ is always less than or equal to $\|x\|_0$ the equivalence here just means that there exists $C > 0$ s.t. $\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq C\|x\|_X$. In this case we have

$$\|Tx\|_Y \leq C\|x\|_X - \|x\|_X = (C-1)\|x\|_X.$$

So if we just let $C > 1$, it follows that T i bounded.

(b) We want to show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.
So let X be finite dimensional. Then any two norms on X are equivalent (Theorem 1.6).
This means that $\|\cdot\|_X$ and $\|\cdot\|_0$ from (a) are equivalent and hence it follows, from what we have already shown, that T is bounded.

Problem 1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be (non-zero) normed vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

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Now assume instead that the norms are equivalent. Since $\|x\|_X$ is always less than or equal to $\|x\|_0$ the equivalence here just means that there exists $C > 0$ s.t. $\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq C\|x\|_X$. In this case we have

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So let X be finite dimensional. Then any two norms on X are equivalent (Theorem 1.6).
This means that $\|\cdot\|_X$ and $\|\cdot\|_0$ from (a) are equivalent and hence it follows, from what we have already shown, that T is bounded.

This does not make sense as I is just some set, not IN...

(c) We want to show that if X is infinite dimensional, then there exists a linear map $T : X \to Y$ which is not bounded. So let X be infinite dimensional, then we know that X has a Hamel basis $(e_i)_{i\in I}$. Consider the family $(y_i)_{i\in I}$ in Y with $y_i = \alpha_i \cdot \|e_i\|_X \cdot c_y$ where $c_y \in Y$ is a (non-zero) constant and where $\alpha_i \in \mathbb{N}$ with $|\alpha_i| < |\alpha_{i+1}|$. From the assignment description we know that there exists precisely one linear map $T : X \to Y$ satisfying $T(e_i) = y_i$, for all $i \in I$. Now define the map $A : X \to X$ with $A(x) = x$ if x is a basis element, i.e. $x \in (e_i)_{i\in I}$, and zero otherwise. Then $T \circ A : X \to Y$ is a linear map which is bounded iff. $\|(T \circ A)(e_i)\|_Y = |\alpha_i| \cdot \|e_i\|_X \cdot \|c_y\|_Y \leq C\|e_i\|_X$ hence that $|\alpha_i|\|c_y\|_Y \leq C$ for every $i \in I$. But since the α_i's are increased by at least 1 for every term of the infinite basis, we conclude that there does not exist a C with this property, showing that $T \circ A$ is not bounded.

A is not well-defined. (or, at least, not linear)

(d) Suppose again that X is infinite dimensional. Then we know from (c) that there exist a linear map $T : X \to Y$ which is not bounded. So let $\|\cdot\|_0$ be defined as in (a) with this T. Since we have shown in (a) that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded, it follows that this norm is not equivalent to the given norm on X. Since $0 \leq \|Tx\|_Y$ we have $\|\cdot\|_X \leq \|\cdot\|_0$ for all $x \in X$. We know that $(X, \|\cdot\|_X)$ is a Banach space hence it is complete. From Homework 3, problem 1 we know that if $(X, \|\cdot\|_0)$ is also complete, then the two norms are equivalent, which we have just argued is not the case. We conclude that $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space. ✓

(e) Following the hint, we consider $l_1(\mathbb{N})$ with the norm $\|\cdot\|_1$. Define $\|\cdot\| := \sum_{n=1}^{\infty} \frac{|x_n|}{n}$. It is easily seen that this is also a norm on $l_1(\mathbb{N})$ and that $\|x\| \leq \|x\|_1$ for all $x \in l_1(\mathbb{N})$. Clearly the norms are inequivalent, since there is no $C > 0$ s.t. $\|x\|_1 \leq C\|x\|$ for all $x \in l_1(\mathbb{N})$. From the lecture notes (lecture 1, page 3, and An2) we know that $(l_1(\mathbb{N}), \|\cdot\|_1)$ is a Banach space and hence complete. Pr. the same argument as in (d) we conclude that $(l_1(\mathbb{N}), \|\cdot\|_n)$ can not be complete as desired.

Why?

Problem 2 Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(l_p(\mathbb{N}), \|\cdot\|_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, ...) : a, b \in \mathbb{C}\}$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, ...) = a + b$ for all $a, b \in \mathbb{C}$.

(a) First of we want to show that f is bounded on $(M, \|\cdot\|_p)$.
For $x \in M$ we have

$$\|f(x)\| = |a + b| \leq |a| + |b|$$

Why?

Note that f can be written as $f(x) = \sum_{n=1}^{\infty} x_n y_n$ for all $x \in M$ and some $y = (1, 1, x_3, x_4, ...) \in l_q(\mathbb{N})$ for some q satisfying $\frac{1}{q} = \frac{p-1}{p}$ so we can use Hölder's inequality, which gives us

What if p=1?

$$|a| + |b| \leq (|a|^p + |b|^p)^{\frac{1}{p}}(1+1)^{\frac{1}{q}} = 2^{\frac{1}{q}}\|x\|_p$$

Showing that f is bounded on $(M, \|\cdot\|_p)$.

Now we want to calculate $\|f\|$.
We have just shown that $f \in \mathcal{L}(M, \mathbb{C})$ and that $\|f(x)\| \leq 2^{\frac{1}{q}}\|x\|_p$
hence $2^{\frac{1}{q}} \in \{C > 0 : \|f(x)\| \leq C\|x\|_p, x \in M\}$
which means that $\|f\| = \inf\{C > 0 : \|f(x)\| \leq C\|x\|_p, x \in M\} \leq 2^{\frac{1}{q}}$.

Now let $x' = (\frac{1}{\sqrt[p]{2}}, \frac{1}{\sqrt[p]{2}}, 0, 0, ...) \in M$.
Then $\|x'\|_p = (|\frac{1}{\sqrt[p]{2}}|^p + |\frac{1}{\sqrt[p]{2}}|^p)^{\frac{1}{p}} = (1/2 + 1/2)^{\frac{1}{p}} = 1$.
Furthermore $\|f(x')\| = |\frac{1}{\sqrt[p]{2}} + \frac{1}{\sqrt[p]{2}}| = 2 \cdot \frac{1}{\sqrt[p]{2}} = 2^{\frac{p-1}{p}} = 2^{\frac{1}{q}}$.
This shows that $2^{\frac{1}{q}} \in \{\|f(x)\| : \|x\|_p = 1\}$ and therefore that
$2^{\frac{1}{q}} \leq \sup\{\|f(x)\| : \|x\|_p = 1\} = \|f\|$. We conclude that $\|f\| = 2^{\frac{1}{q}}$. (✓)

(b) We want to show that if $1 < p < \infty$, then there is a unique linear functional F on $l_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Since $f \in L(M, \mathbb{C})$ we know from Corollary 2.6 that there exists a linear functional $F \in (l_p(\mathbb{N}))^*$ such that $F_{|_M} = f$ and $\|F\| = \|f\|$. From homework 1, problem 5 we know that $(l_p(\mathbb{N}))^* \cong l_q(\mathbb{N})$ so we can write $F(x) = \sum_{n=1}^{\infty} x_n y_n$ for $y = (y_n)_{n\geq 1} \in l_q(\mathbb{N})$ and $x \in l_p(\mathbb{N})$. From (a) we know that $2^{\frac{1}{q}} = \|f\| = \|F\|$ and since F is represented by y we also have $\|F\| = \|y\|_q$ We see that $F_{|_M}(x) = f(x) = x_1 + x_2$ so we must have $y = (1, 1, y_3, y_4, ...)$, but since

$$\begin{aligned}\|y\|_q &= (\sum_{n=1}^{\infty} |y_n|^q)^{\frac{1}{q}} \\ &= (|1|^q + |1|^q + |y_3|^q + ...)^{\frac{1}{q}} \\ &= 2^{\frac{1}{q}}\end{aligned}$$

then $y_3, y_4, ...$ must all be zero, meaning $y = (1, 1, 0, 0, ...)$ whereas $F(x) = x_1 + x_2$ for all $x \in l_p(\mathbb{N})$.

It follows from our previous argumentation that F is unique - If there exists $F' \in (l_p(\mathbb{N}))^*$ satisfying $F'_{|_M} = f$ and $\|F'\| = \|f\|$ then $F'(x) = \sum_{n=1}^{\infty} x_n y_n$ for $y = (1, 1, 0, 0, ...) \in l_q(\mathbb{N}), x \in l_p(\mathbb{N})$ meaning that $F'(x) = x_1 + x_2 = F(x)$.
We conclude that a linear functional extending f which satisfies the desired properties is unique. ✓

(c) We want to show that if $p = 1$, then there are infinitely many linear functionals F on $l_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

You should also know this from (a).

For $p = 1$ and for $x \in M$ we have $\|f(x)\| = |a + b| \leq |a| + |b| = \|x\|_1$
Hence f is bounded on M by $C = 1$ and since $\|f\| = \sup\{\|f(x)\| : \|x\|_1 = 1\}$ clearly $\|f\| = 1$ (take for example $x = (\frac{1}{2}, \frac{1}{2}, 0, 0, ...)$.). Consider $F_i(x) = x_1 + x_2 + x_i$ for some $2 < x_i \leq n$. This F_i is obviously a linear functional on $l_1(\mathbb{N})$ and we see that $F_i|_M(x) = x_1 + x_2 = f(x)$ for $x \in M$. showing that F_i is an extension of f. Since F_i extends f we must have $\|F_i\| \geq \|f\| = 1$. At the same time

$$\begin{aligned}
\|F_i\| &= \sup\{\|F_i(x)\| : \|x\|_1 = 1\} \\
&= \sup\{|x_1 + x_2 + x_i| : \|x\|_1 = 1\} \\
&\leq \sup\{|x_1| + |x_2| + |x_i| : \|x\|_1 = 1\} \\
&\leq 1
\end{aligned}$$

So $\|F_i\| = \|f\| = 1$. This shows that any such $F_i \in l_1(\mathbb{N})$ is an extension of f satisfying the desired properties. But we can define F_i for every $\underbrace{i \in \mathbb{C} \text{ with } 2 < i}_{i \in \mathbb{N}?}$ hence we have infinitely many options.

Problem 3 Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $n \geq 1$ be an integer. We want to show that no linear map $F : X \to \mathbb{K}^n$ is injective.
Asumme that there exists a linear map $F : X \to \mathbb{K}^n$ which is injective. Let $x_1, x_2, ..., x_{n+1} \in X$ (all non-zero) be linearly independent. We know that this is possible since X is infinite dimensional (take for example $n+1$ elements in a basis for X). Since $\mathbb{K}^n$ is n-dimensional, we know that every linearly independent subset has at most n elements. Hence $F(x_1), ..., F(x_{n+1})$ can't all be linearly independent. This means that there must exist $\alpha_1, ..., \alpha_{n+1} \in \mathbb{K}$, not all zero s.t.

redundant

$$\sum_{i=1}^{n+1} \alpha_i F(x_i) = F(\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1}) = 0$$

Where the first equality follows from the linearity of F. It follows that $\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1} \in \ker(F) = \{x \in X \mid F(x) = 0\}$. Since F is injective we know that $\ker(F) = \{0\}$ but then $\alpha_1 x_1 + ... + \alpha_{n+1} x_{n+1} = 0$ which is a contradiction, since all the x_i's are linearly independent. We conclude that no such F can be injective.

(b) Let $n \geq 1$ be an integer and let $f_1, f_2, ..., f_n \in X^*$. We want to show that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}.$$

Following the hint, we consider the function $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), ..., f_n(x))$ for $x \in X$. In (a) we have just shown that F is not injective, which means that $\ker(F) \neq \{0\}$.

It follows that there exists at least one non-zero element in $x_0 \in X$ s.t. $x_0 \in \ker(F)$. This x_0 satisfies that $F(x_0) = (f_1(x_0), ..., f_n(x_0)) = 0$, hence that $f_j(x_0) = 0$ for all $j = 1, ..., n$ showing that $x_0 \in \bigcap_{j=1}^n \ker(f_j)$. This proves that $\bigcap_{j=1}^n \ker(f_j) \neq \{0\}$. ✓

(c) Let $x_1, x_2, ..., x_n \in X$. We want to show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, 2, ..., n$. Let $U := \{x_1, x_2, ..., x_n\}$. First note that if there is one (or several) element(s) $x_j \in U$ with $x_j = 0$ then obviously $\|y - x_j\| \geq \|x_j\| = 0$ so we can assume $0 \notin U$. Since all of the x_j's are non-zero, there exists $f_j \in X^*$ with $\|f_j\| = 1$ s.t. $f_j(x) = \|x_j\|$ for $j = 1, ..., n$ by theorem 2.7(b). It follows from (b) that we can take $0 \neq y \in \bigcap_{j=1}^n \ker(f_j)$ for any element y satisfying $\|y\| = 1$ (take for example $\frac{y_0}{\|y_0\|}$ for a non-zero element y_0 in the kernel). From the linearity of each f_j and by the definition of a norm, for each $x_j \in U$ we then have $\|y - x_j\| = \|f_j\|\|y - x_j\| \geq |f_j(y - x_j)| = |f_j(y) - f_j(x_j)| = |0 - \|x_j\|| = \|x_j\|$ for all $j = 1, ..., n$ as desired. ✓

(d) We want to show that the unit sphere $S = \{x \in X : \|x\| = 1\}$ cannot be covered with a finite family of closed balls in X s.t. none of the balls contains 0. Let $K_1(x_1, \epsilon_1), K_2(x_2, \epsilon_2), ..., K_n(x_n, \epsilon_n)$ be a finite family of closed balls, none of which contains 0 and where $K_j(x_j, \epsilon_j) = \{y \in X : \|y - x_j\| \leq \epsilon_j\}$ for $j = 1, ..., n$. Assume that S can be covered with a finite number of these balls, hence that $S \subseteq \bigcup_{j=1}^n K_j(x_j, \epsilon_j)$ for some $n \geq 1$. Let $y \in X$ be as defined in (c). Then we have $y \in S \subseteq \bigcup_{j=1}^n K_j(x_j, \epsilon_j)$ as $\|y\| = 1$. If $y \in K_j(x_j, \epsilon_j)$ we must have $\|y - x_j\| \leq \epsilon_j$, but we also know that $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, ..., n$. This means that $\|x_j - 0\| = \|x_j\| \leq \epsilon_j$ showing that $0 \in K_j(x_j, \epsilon_j)$ which contradicts the assumption that none of the balls contains 0. We conclude that S cannot be covered with a finite family of closed balls in X s.t. none of the balls contains 0. ✓

Not enough.

(e) First we want to show that S is non-compact.
Let $y_1, y_2, y_3, ... \in S$ be distinct. Define $B_i(y_i, \frac{1}{2}) = \{x \in X : \|x - y_i\| < \frac{1}{2}\}$ and let $\mathcal{B} = \bigcup_{i=1}^\infty B_i(y_i, \frac{1}{2})$. Obviously we have that $S \subseteq \mathcal{B}$ showing that $\mathcal{B}$ is an open cover. If S is compact, it means that every open cover has a finite subcover, so we want to show that no finite subset $\bigcup_{i=1}^n B_i(y_i, \frac{1}{2})$ covers S. From (c) we get that $\|y_i - y_j\| \geq 1$ showing that $y_i \notin B_j(y_j, \frac{1}{2})$ and $y_j \notin B_i(y_i, \frac{1}{2})$ for all $i, j \in \{1, 2, ..., n\}$ with $i \neq j$, which means that each of these open balls contains only 1 element of S. It follows that $\bigcup_{i=1}^n B_i(y_i, \frac{1}{2})$ contains exactly n elements of S hence it does not cover S, showing that S is non-compact. If we now let $B_i(x_i, \frac{1}{2}) = \{y \in K(0,1) : \|y - x_i\| < \frac{1}{2}\}$. Then $\mathcal{B}_x = \bigcup_{i=1}^\infty B_i(x_i, \frac{1}{2})$ covers the closed unit ball. If $u, v \in B_i(x_i, \frac{1}{2})$, then $\|u - v\| < 1$, showing as before that each open ball only contains a single element y with $\|y\| = 1$. Hence a finite number of these open balls cannot cover $K(0,1)$ showing that the unit ball in X is non-compact.

%

why? y_i is not specified yet?

$\{y_i\}_{i\geq 1} \neq S$

1/2

Many misunderstandings

Problem 4 Let $L_1([0,1], m)$ and $L_3([0,1], m)$ be the Lesbegue spaces on $[0,1]$.

For $n \geq 1$, define

$$E_n := \{f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm \leq n\}$$

(a) First let $n \geq 1$ and $f, g \in E_n$ be given. Then we know that $\|f\|_3^3 \leq n$ and $\|g\|_3^3 \leq n$. This means that for all $0 \leq \alpha \leq 1$ we get

$$\|\alpha f + (1-\alpha)g\|_3^3 \leq \alpha\|f\|_3^3 + (1-\alpha)\|g\|_3^3 \leq \alpha n + (1-\alpha)n = n$$

Showing that E_n is convex. Now assume that E_n is an absorbing subset of $L_1([0,1], m)$ for all $n \geq 1$. Then for all $0 \neq f \in L_1([0,1])$ there exists $t > 0$ such that $t^{-1}f \in E_n$. This means that $\|t^{-1}f\|_3^3 = t^{-3}\|f\|_3^3 \leq n$, hence that $\|f\|_3 \leq \frac{n^{\frac{1}{3}}}{(t^{-3})^{\frac{1}{3}}}$. This implies that the the 3-norm is bounded by $\frac{n^{\frac{1}{3}}}{(t^{-3})^{\frac{1}{3}}}$ for every $f \in L([0,1], m)$ which is not true. We conclude that E_n is not an absorbing subset of $L_1([0,1], m)$. ↖ t depends on f.

(b) Assume that E_n does not have empty interior in $L([0,1], m)$ for some $n \geq 1$. Then for $f \in E_n$ there exists $\epsilon > 0$ s.t.

$$B(f, \epsilon) := \{g \in L_1([0,1], m) : \|g - f\|_1 < \epsilon\} \subseteq E_n$$

h?

Let $0 \neq g \in L_1([0,1], m)$ and define $h := f + \frac{\epsilon}{2}\frac{g}{\|g\|_1}$. Then $\|h - f\|_1 = \frac{\epsilon}{2}$ showing that $z \in B(f, \epsilon) \subseteq E_n$. Note that since $f, h \in E_n$ we have $\|f\|_3^3 \leq n$ and $\|h\|_3^3 \leq n$ which means that $\|f\|_3 \leq n^{\frac{1}{3}} < \infty$ and $\|h\|_3 \leq n^{\frac{1}{3}} < \infty$ so $f, h \in L_3([0,1], m)$. We see that $g = (h - f)\frac{2\|g\|_1}{\epsilon}$ and since $L_3([0,1], m)$ is a subspace of $L_1([0,1], m)$ it follows that $g \in L_3([0,1], m)$. But since $g \in L_1([0,1], m)$ was chosen arbitrarily, this would now imply that $L_1([0,1], m) \subseteq L_3([0,1], m)$ which is a contradiction. We conclude that E_n has empty interior in $L([0,1], m)$.

(c) We want to show that E_n is closed in $L_1([0,1], m)$, for all $n \geq 1$.
So let $(f_n)_{n \geq 1} \in E_n$ be a sequence of functions converging to some $f \in L([0,1], m)$.
Since $|f_n|^3 \to |f|^3$, Fatou's lemma gives us that

L^1 convergence does not imply pointwise convergence.

$$\int_{[0,1]} |f|^3 dm \leq \liminf_{n\to\infty} \int_{[0,1]} |f_n|^3 dm.$$

Since each term $f_n \leq n$ the limit must be finite. This shows that $f \in E_n$ and hence that E_n is closed. $1/2$

(d) We have just shown in (b) and (c) that E_n is closed and has empty interior in $L_1([0,1], m)$ for every $n \geq 1$ so we have that $\text{int}(E_n) = \text{int}(\bar{E_n}) = \emptyset$ in $L_1([0,1], m)$. This means that each E_n is nowhere dense in $L_1([0,1], m)$. Since $f \in E_n$ means that $\|f\|_3^3 \leq n$ which in turn implies $\|f\|_3 \leq n^{\frac{1}{3}}$, we see that the infinite union $\bigcup_{n=1}^{\infty} E_n = L_3([0,1],m)$. We have now shown that $L_3([0,1],m)$ can be written as a union of nowhere dense sets in $L([0,1],m)$ so we conclude that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

Problem 5 Let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n\geq 1}$ be a sequence in H and let $x \in H$.

(a) Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$? Give a proof or a counterexample.

No, but it is a consequence of norm convergence.

Yes, this is the definition of strong convergence. Since $\lim_{n\to\infty}\|x_n - x\| = 0$, the triangle inequality gives us $\|x_n\| \leq \|x_n - x\| + \|x\|$ and $\|x\| \leq \|x_n - x\| + \|x_n\|$, which implies that $\lim_{n\to\infty}\|x_n\| = \|x\|$ since the norm is continuous.

This is what you aim to show.

(b) Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$, as $n \to \infty$? Give a proof or a counterexample.

weak convergence

It does not. Following the hint, we consider an orthonormal basis $(e_n)_{n\geq 1}$ in H. Since $\|e_n\| = 1$, $(e_n)_{n\geq 1}$ is bounded and since $\lim_{n\to\infty} e_i^{(n)} = 0$ for every $i \geq 1$ where $e_n = (e_1^{(n)}, e_2^{(n)}, \ldots)$, homework 4, problem 3 gives us that $\boxed{\lim_{n\to\infty} e_n = 0}$ hence $x = 0$. For any two terms e_i, e_j with $e_i \neq e_j$ we have $\|e_i - e_j\| > 1$ implying that $(e_n)_{n\geq 1}$ doesn't have any convergent subsequence. And since we have shown that the sequence is bounded by 1 we must have $\lim_{n\to\infty}\|e_n\| = 1 \neq 0$, hence it is a counterexample to the statement.

(c) Suppose that $\|x_n\| \leq 1$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Is it true that $\|x\| \leq 1$? Give a proof or a counterexample.

Why?

This is true. If x_n converges weakly to x, we have $|\langle x_n, \frac{x}{\|x\|}\rangle| \leq \|x_n\|$ and since $|\langle x_n, \frac{x}{\|x\|}\rangle| \to |\langle x, \frac{x}{\|x\|}\rangle| = \|x\|$ we get that $\|x\| \leq \liminf_{n\to\infty}\|x_n\| \leq 1$ by applying lim inf on both sides of the first inequality.