Problem 1.

(a) Clearly $\|\cdot\|_0$ is a function $X \to [0, \infty)$.

For $x, y \in X$ we have

$$\begin{aligned}\|x+y\|_0 &= \|x+y\|_X + \|T(x+y)\|_Y \\ &= \|x+y\|_X + \|T(x)+T(y)\|_Y \\ &\leq \|x\|_X + \|y\|_X + \|T(x)\|_Y + \|T(y)\|_Y \\ &= \|x\|_0 + \|y\|_0 .\end{aligned}$$

Here we used linearity of T in the second equality and the triangle inequality of $\|\cdot\|_X$ and $\|\cdot\|_Y$. This proves the triangle inequality for $\|\cdot\|_0$.

For $\alpha \in \mathbb{K}$ and $x \in X$ we have

$$\begin{aligned}\|\alpha x\|_0 &= \|\alpha x\|_X + \|T(\alpha x)\|_Y \\ &= \|\alpha x\|_X + \|\alpha T(x)\|_Y \\ &= |\alpha| \|x\|_X + |\alpha| \|T(x)\|_Y \\ &= |\alpha| \|x\|_0 .\end{aligned}$$

Here we used linearity of T in the second equality and the second property of norms.

Lastly if $\|x\|_0 = 0$ then $\|x\|_X + \|T(x)\|_Y = 0$ so since $\|x\|_X, \|T(x)\|_Y \geq 0$ we must have $\|x\|_X = \|T(x)\|_Y = 0$. In particular, as $\|\cdot\|_X$ is a norm we deduce that $x = 0$.

Taken together, this proves that $\|\cdot\|_0$ is a norm.

Suppose first that $\|\cdot\|_0$ and $\|\cdot\|_0$ are equivalent. Then there exists a $c > 0$ such that $\|x\|_0 \leq c \|x\|_X$ for all $x \in X$. From the definition of $\|\cdot\|_0$ we then get

$$\|T(x)\|_Y \leq (c-1) \|x\|_X$$

for all $x \in X$. Since X is non-zero there exists a $0 \neq x \in X$ which via the above inequality implies that $c - 1 \geq 0$. If $c = 1$ then we get $T(x) = 0$ for all $x \in X$ which implies that T is bounded. Otherwise we have $c - 1 > 0$ which also implies that T is bounded.

Conversely, if T is bounded, say $\|T(x)\|_Y \leq C \|x\|_X$ with $C > 0$ for all $x \in X$, then

$$\|x\|_X \leq \|x\|_0 \leq \|x\|_X + C \|x\|_X = (C+1) \|x\|_X$$

for all $x \in X$. As $C > 1 > 0$ this implies that $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent.

(b) If X is finite dimensional then any two norms on X are equivalent. In particular, for any linear map $T : X \to Y$ the norm $\|\cdot\|_0$ considered in (a) is equivalent to $\|\cdot\|_X$. By (a) this implies that any linear map T is bounded.

(c) Let $(e_i)_{i \in I}$ be a Hamel basis for X. Let $y \in Y$ be a non-zero vecter (such y exists as Y is non-zero). As X is infinite dimensional the index set I is infinite, so we may pick $\alpha_i \in \mathbb{K}$ with the property that for all $n > 0$ there exists $i \in I$ with $|\alpha_i| > n$. Then define a linear map $T : X \to Y$ by $T(e_i) = \alpha_i \|e_i\|_X y$.

Suppose T is bounded, i.e. $\|T(x)\|_Y \leq C \|x\|_X$ for all $x \in X$, for some $C > 0$. In particular,

$$C \|e_i\|_X \geq \|(\alpha_i \|e_i\|_X y)\|_Y = |\alpha_i| \|e_i\|_X \|y\|_Y$$

so $C \geq |\alpha_i| \|y\|_Y$ for all $i \in I$. By our construction we can find $i \in I$ with $|\alpha_i| > C / \|y\|_Y$, so this is a contradiction. We therefore conclude that T is not bounded.

(d) Using the linear map T from (c) we construct $\|\cdot\|_0$ as in (a). By the last part of (a) and (c) it follows that $\|\cdot\|_0$ and $\|\cdot\|_X$ are not equivalent. Also, from the construction we have $\|x\|_X \leq \|x\|_0$ because $\|T(x)\|_Y \geq 0$, for all $x \in X$. By problem 1 from HW3, it now follows that X cannot be equivalent with respect to both $\|\cdot\|_0$ and $\|\cdot\|_X$.

(e) We have $\ell_1(\mathbb{N}) \subseteq \ell_\infty(\mathbb{N})$ with $\|x\|_\infty \leq \|x\|_1$ for all $x \in \ell_1(\mathbb{N})$ (this was also part of HW2 problem 2). We know that $\ell_1(\mathbb{N})$ is complete wrt. $\|\cdot\|_1$. On the other hand, we show below that $\ell_1(\mathbb{N})$ is not complete wrt. $\|\cdot\|_\infty$, which in particular implies that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are inequivalent norms on $\ell_1(\mathbb{N})$. It follows from this that $(X, \|\cdot\|) = (\ell_1, \|\cdot\|_1)$ and $(X, \|\cdot\|') = (\ell_1, \|\cdot\|_\infty)$ have the desired properties.

Consider the sequence $x_n \in \ell_1(\mathbb{N})$ given by $x_n(k) = 1/k$ for $k \leq n$ and $x_n(k) = 0$ for $k > n$. In the larger space $(\ell_\infty(\mathbb{N}), \|\cdot\|_\infty)$ this sequence converges to $x(k) = 1/k$ for all $k \in \mathbb{N}$, so it follows that x_n is Cauchy wrt. $\|\cdot\|_\infty$. However since limits are unique and $x \notin \ell_1(\mathbb{N})$ it follows that x_n does not converge in $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$. Thus $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ contains a Cauchy sequence which is not convergent, so it is not complete.

Problem 2.

(a) First we consider the case $p > 1$. By Hölder's inequality applied to the sequences $(a, b, 0, 0, \ldots)$ and $(1, 1, 0, 0, \ldots)$ we have

$$|a| + |b| \leq (|a|^p + |b|^p)^{\frac{1}{p}} (1^q + 1^q)^{\frac{1}{q}} = (|a|^p + |b|^p)^{\frac{1}{p}} 2^{\frac{1}{q}}$$

where $1 < q < \infty$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Hence by the triangle inequality

$$|f(a, b, 0, 0, \ldots)| \leq |a| + |b| \leq 2^{\frac{1}{q}} \|(a, b, 0, 0, \ldots)\|_p$$

for all $a, b \in \mathbb{C}$, which shows that f is bounded on $(M, \|\cdot\|_p)$ with norm at most $2^{\frac{1}{q}}$. On the other hand, one has

$$|f(1, 1, 0, 0, \ldots)| = 2 = 2^{\frac{1}{p} + \frac{1}{q}} = 2^{\frac{1}{q}} \|(1, 1, 0, 0, \ldots)\|_p,$$

hence the norm of f on $(M, \|\cdot\|_p)$ must be equal to $2^{\frac{1}{q}}$.

Next we suppose that $p = 1$. In this case the triangle inequality implies $|f(a, b, 0, 0, \ldots)| \leq \|(a, b, 0, 0, \ldots)\|_1$ so f is bounded on $(M, \|\cdot\|_1)$ with norm at most 1. On the other hand $|f(1, 1, 0, 0, \ldots)| = 2 = \|(1, 1, 0, 0, \ldots)\|_1$ so the norm must be equal to 1.

(b) The existence of F follows from the Hahn-Banach extension theorem. To prove uniqueness, we recall from HW1 problem 5, that there exists an isometric isomorphism $\ell_q(\mathbb{N}) \cong \ell_p(\mathbb{N})^*$ given by sending $x \in \ell_q(\mathbb{N})$ to the functional given by $\ell_p(\mathbb{N}) \ni y \mapsto \sum_{n=1}^{\infty} x_n y_n \in \mathbb{C}$. Thus via this isomorphism F must correspond to an element $x \in \ell_q(\mathbb{N})$ with the property that $x_1 a + x_2 b = a + b$ for all $a, b \in \mathbb{C}$ and $\|x\|_q = 2^{\frac{1}{q}}$. Taking $(a, b) = (1, 0)$ implies $x_1 = 1$ and taking $(a, b) = (0, 1)$ implies $x_2 = 1$. We then have

$$2 = \|x\|_q^q = \sum_{k=1}^{\infty} |x_k|^q = 2 + \sum_{k=3}^{\infty} |x_k|^q$$

hence it follows that $x_k = 0$ for $k \geq 3$. Thus $x = (1, 1, 0, 0, \ldots)$ which is clearly unique, so F is also unique.

(c) Consider the subspace

$$N = \{(a, b, c, 0, 0, \ldots) \in \ell_1(\mathbb{N}) : a, b, c \in \mathbb{C}\}.$$

For every $\lambda \in \mathbb{C}$ we define a linear map $f_\lambda : N \to \mathbb{C}$ by $f_\lambda(a, b, c, 0, 0, \ldots) = a + b + \lambda c$. Clearly f_λ extends f. Moreover if $|\lambda| \leq 1$ then

$$|f_\lambda(a, b, c, 0, 0, \ldots)| \leq |a| + |b| + |\lambda||c| \leq \|(a, b, c, 0, 0, \ldots)\|_1 .$$

so f_λ is bounded with $\|f_\lambda\| \leq 1 = \|f\|$. On the other hand, as f_λ extends f we have $\|f_\lambda\| \geq \|f\|$, so we get $\|f_\lambda\| = \|f\|$.

Now for each λ we obtain by Hahn-Banach a linear functional F on $\ell_1(\mathbb{N})$ extending f_λ (and thus also extending f) with $\|F\| = \|f_\lambda\|$. For $|\lambda| \leq 1$ we thus get infinitely many extensions F which satisfy $\|F\| = \|f\|$, and we note that the F are all distinct as the f_λ are distinct.

Problem 3.

(a) Let $F : X \to \mathbb{K}^n$ be any linear map. Since X is of infinite dimension, we may find linearly independent vectors $x_1 \ldots, x_{n+1} \in X$. As $\dim(\mathbb{K}^n) = n$ the vectors $F(x_1), \ldots, F(x_{n+1})$ must be linearly independent, hence there exists $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{K}$ with

$$\alpha_1 F(x_1) + \cdots + \alpha_{n+1} F(x_{n+1}) = 0.$$

Then the vector $x = \alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1}$ is non-zero, as $\{x_1, \ldots, x_{n+1}\}$ is linearly independent, and we have $F(x) = 0$ using linearity of F. So F is not injective.

(b) The map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), \ldots, f_n(x))$ for $x \in X$ is linear, hence by (a) it is not injective. This means that there exists a non-zero vector y in the kernel of F. In that case $f_1(y) = \cdots = f_n(y) = 0$, so $y \in \ker(f_1) \cap \cdots \cap \ker(f_n)$. This shows that $\ker(f_1) \cap \cdots \cap \ker(f_n) \neq 0$.

(c) By Hahn-Banach (or more precisely theorem 2.7 (b)) there exists linear functionals $f_j \in X^*$ such that $\|f_j\| = 1$ and $f_j(x_j) = \|x_j\|$, for each $i = 1, \ldots, n$. By (b) there exists a $0 \neq y \in X$ with $f_1(y) = \cdots = f_n(y) = 0$, and by scaling we may assume that $\|y\| = 1$. It follows that

$$\|x_j\| = f_j(x_j) = |f_j(-x_j)| = |f_j(y - x_j)| \leq \|y - x_j\|$$

for $i = 1, \ldots, n$.

(d) Suppose the balls have centers $x_1, \ldots, x_n$ and radius $r_1, \ldots, r_n > 0$, respectively. As the balls do not contain 0 we must have $r_j < \|x_j\|$ for each $j = 1, \ldots, n$.

Now pick y as in (c). Then $\|y - x_j\| \geq \|x_j\| > r_j$ for each $j = 1, \ldots, n$, hence y is not contained in any of the balls. This contradicts the assumption that the balls cover S.

(e) Consider the collection of open balls whose closure does not contain 0. Every $0 \neq x \in X$ is contained in the open ball centered at x with radius $\frac{1}{2}\|x\|$, which is an open ball with closure not containing 0. Hence these open sets cover $X \backslash \{0\}$ and in particular they cover S. If S were compact, then this would yield a finite family of open balls covering S such that none of the balls contain 0 in their closure. In particular, the closures of these balls would yield a contradiction with (d), so we conclude that S must be non-compact.

As S is a closed subset of the closed unit ball, and any closed subset of a compact space is again compact, it follows that the closed unit ball is non-compact.

Problem 4.

(a) We recall the standard fact that the functions given by $f_\alpha(x) = x^\alpha$ for $x > 0$ satisfy $\int_{(0,1)} f dm < \infty$ if and only if $\alpha > -1$.

Now fix an α with $-1 < \alpha \leq -\frac{1}{3}$. After assigning some value to $x = 0$, the function f_α defines an element of $L_1([0,1], m)$, which does not depend on the choice of $f_\alpha(0)$. For any $t > 0$ we get

$$\int_{[0,1]} |t^{-1} f_\alpha|^3 dm = t^{-3} \int_{[0,1]} f_{3\alpha} dm = \infty,$$

since $3\alpha \leq -1$. Thus $t^{-1} f_\alpha \notin E_n$ for any $t > 0$, hence E_n is not absorbing.

(b) In fact, we show that $L_3([0,1], m)$ has empty interior in $L_1([0,1], m)$. As $E_n \subseteq L_3([0,1], m)$, any interior point of E_n would be an interior point of $L_3([0,1], m)$, so this will let us conclude that E_n has empty interior in $L_1([0,1], m)$.

Let $f \in L_3([0,1], m)$. We shall construct a sequence of elements not in $L_3([0,1], m)$ which converge to f, hence f cannot be an interior point of $L_3([0,1], m)$.

Let $f_\alpha \in L_1([0,1], m)$ be as in (a), i.e. with $-1 < \alpha \leq -\frac{1}{3}$, and put $f_n = f + \frac{1}{n} f_\alpha$. We have $f_n \notin L_3([0,1], m)$ as otherwise we would get $f_\alpha = n(f_n - f) \in L_3([0,1], m)$ by Minkowsky's inequality (or the fact that $L_3([0,1], m)$ is a vector space), and the calculation in (a) with $t = 1$ shows that this is not the case. On the other hand

$$\|f - f_n\|_1 = \frac{1}{n} \|f_\alpha\|_1 \to 0$$

for $n \to \infty$, so f_n converges to f as desired.

(c) We must show that if $(f_k)_{k \geq 1}$ is a sequence in E_n which converges to some $f \in L_1([0,1], m)$ wrt. the norm $\|\cdot\|_1$, then $f \in E_n$.

Following the proof of Riesz-Fischer we may find a subsequence which converges a.e. to f. Thus by considering this subsequence, we might as well assume that $(f_k)_{k \geq 1}$ converges a.e. to f. Also, as $\int_{[0,1]} |f_k|^3 dm$ does not change if we substitute for f_k a function which equals f_k a.e., we might as well assume that $(f_k)_{k \geq 1}$ converges to f everywhere.

Now $(|f_k|^3)_{k \geq 1}$ is a sequence of positive functions which converges pointwise to $|f|^3$, so by Fatou

$$\int_{[0,1]} |f|^3 dm \leq \liminf_{k \to \infty} \int_{[0,1]} |f_k|^3 dm \leq n.$$

This implies $f \in E_n$.

(d) The sets E_n are nowhere dense as they are closed by (c) and they have empty interior by (b). By definition $L_3([0,1], m) = \bigcup_{n \geq 1} E_n$, hence $L_3([0,1], m)$ is a countable union of nowhere dense subsets and is therefore of first category in $L_1([0,1], m)$.

Problem 5.

(a) Yes. If $x_n \to x$ in norm, as $n \to \infty$, then $\|x_n - x\| \to 0$, as $n \to \infty$. By the triangle inequality

$$0 \leq | \, \|x_n\| - \|x\| \, | \leq \|x_n - x\| \, .$$

It follows that $| \, \|x_n\| - \|x\| \, | \to 0$, as $n \to \infty$, which means that $\|x_n\| \to \|x_n\|$, as $n \to \infty$.

(b) No. Let $(e_n)_{n\geq 1}$ be an orthonormal basis. Then as H is separable there is an isometric isomorphism

$$\ell_2(\mathbb{N}) \to H \qquad (x_n)_{n\geq 1} \mapsto \sum_{n\geq 1} x_n e_n.$$

This induces also an homeomorphism with respect to the weak topologies. Thus it suffices to give a counterexample in the case $H = \ell_2(\mathbb{N})$.

By HW4 problem 3, a sequence x_n in $\ell_2(\mathbb{N})$ converges weakly to 0 if and only if the sequence is bounded, and it converges pointwise i.e. $x_n(k) \to 0$, as $n \to \infty$, for every $k \geq 1$. Consider the sequence given by $x_n(k) = 0$ if $k \neq n$ and $x_n(n) = 1$. This is bounded as $\|x_n\|_2 = 1$ for all $n \geq 1$, and it clearly converges pointwise to 0. Hence $x_n \to 0$ weakly. On the other hand $\|x_n\|_2 = 1$ does not converge to $\|0\|_2 = 0$.

(c) Yes. By Hahn-Banach (or more precisely theorem 2.7 (b)) one may find a linear functional $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\| = 1$. By HW4 problem 2(a) it follows that $f(x_n) \to f(x)$, as $n \to \infty$. Now we have $|f(x_n)| \leq \|x_n\| \leq 1$, since $\|f\| = 1$, hence it follows that

$$\|x\| = |f(x)| = \lim_{n\to\infty} |f(x_n)| \leq 1.$$