FunkAn Mandatory Assignment 1

Rune Thinggaard Hansen

December 2020

Problem 1

a)

We show the three criteria for norms: Triangle inequality, positive homogeneity and that the kernel is trivial. For each calculation we use the fact that $\|\cdot\|_X$ and $\|\cdot\|_Y$ have each of these properties as well.

$$\|x+y\|_0 = \|x+y\|_X + \|Tx+Ty\|_Y \leq \|x\|_X + \|Tx\|_Y + \|y\|_X + \|Ty\|_Y = \|x\|_0 + \|y\|_0$$
$$\|\alpha x\|_0 = \|\alpha x\|_X + \|T(\alpha x)\|_Y = |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y = |\alpha|\|x\|_0$$
$$0 = \|x_0\| = \|x\|_X + \|Tx\|_Y \Leftrightarrow \|x\|_X = 0 \wedge \|Tx\|_Y = 0 \Leftrightarrow x = 0$$

The last part comes from the fact that $\|x\|_X$ is a norm so x must be zero for that term to be zero. Therefore $\|\cdot\|_0$ is a norm.
If T is bounded we have that:

$$\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_0 \leq \|x\|_X + \|T\|\|x\|_X = (1+\|T\|)\|x\|_X$$

And thus, we have that the norms are equivalent.
If the norms are equivalent there exists a $C \in \mathbb{R}_+$ such that:

$$\|x\|_X + \|Tx\|_Y = \|x\|_0 \leq C\|x\|_X \Leftrightarrow \|Tx\|_Y \leq (C-1)\|x\|_X$$

Which shows that T is bounded. Thus we have the wanted result.

b)

Let $\{e_n\}_{n\in I}$ denote some basis on X of dimension M that is $I = \{1, 2, \ldots, M\}$.
Then for a vector $x = \sum_{n=1}^{M} x_n e_n \in X$ of norm 1, we have:

$$\|Tx\| = \left\| T \sum_{n=1}^{M} x_n e_n \right\| \leq \sum_{n=1}^{M} |x_n|\|Te_n\| \leq M \max_{n\in I}\{|x|_n \|Te_n\|\} \leq M \max_{n\in I}\{\|Te_n\|\} < \infty$$

Thus $\|Tx\|$ is bounded by something independent of x, so T is bounded.

c)

Choose an algebraic basis $\{e_n\}_{n\in I}$ of X. Pick a countable infinite subset $\tilde{J} \subset I$ and index it by $\mathbb{N}$. Now choose the family $\{f_n\}_{n\in\mathbb{N}}$ with $f_n = \frac{e_n}{\|e_n\|}$ each of norm 1. Choose $\{\lambda_n\}_{n\in I}$, $\lambda_n \in \mathbb{R}$ such that the subfamily $\{\lambda_n\}_{n\in\mathbb{N}}$ is $\lambda_n = n \cdot \|e_n\|$. Let T be the unique map $T : X \to \mathbb{R}$ such that $T(e_n) = \lambda_n$. Then $\|T\| = \sup\limits_{\|x\|\leq 1}\{|T(x_n)|\} \geq \sup\limits_{n\in\mathbb{N}}\{|T(f_n)|\} = \sup\limits_{n\in\mathbb{N}}\{\frac{|\lambda_n|}{\|e_n\|}\} = \sup\limits_{n\in\mathbb{N}}\{n\} = \infty$. There we are.

d)

Let T be an unbounded operator $T : X \to \mathbb{R}$. Define $\|x\|_0 = \|x\|_X + \|Tx\|_Y$. By part a) $\|\cdot\|_0$ is not equaivalent to $\|\cdot\|_X$ since T is not bounded. We have that $\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_0$ by the non-negativity of norms so the demand of the task is fulfilled.
By HW 3 prob 1, we have that if X is complete w.r.t the two norms and we have $\|\cdot\|_X \leq \|\cdot\|_0$ then the two norms are equivalent. Since the norms aren't equivalent and we have the upper bound-condition, X can't be complete with respect to both norms. If X is complete w.r.t $\|\cdot\|_X$ then it can't be complete w.r.t $\|\cdot\|_0$.

e)

Let $X = \ell_1(\mathbb{N})$, $\|x\| = \|x\|_1$ and $\|x\|' = \|x\|_\infty$. These norms aren't equivalent since otherwise that would mean that there existed $c > 0$ such that $\|x\|_\infty \leq c\|x\|_1$ which would imply $\ell_\infty(\mathbb{N}) \subset \ell_1(\mathbb{N})$ which isn't true (by page remark in the first Lecture 1).
For $x = \{x_n\}_{n\in\mathbb{N}}$ we have that:

$$\|x\|' = \max_{n\in\mathbb{N}}\{|x_n|\} \leq \sum_{n=1}^{\infty} |x_n| = \|x\|$$

So we have the claim of the task.
To show that $(X, \|\cdot\|_\infty)$ isn't complete, look at the sequence $y(m) = \{y(m)_n\}_{n\in\mathbb{N}} = (1, \frac{1}{2}, \dots, \frac{1}{m}, 0, \dots)$. We have that this converges in $(l_\infty(\mathbb{N}), \|\cdot\|_\infty)$ to the sequence with elements $y_n = \frac{1}{n}$. The series is furthermore Cauchy but $\{y_n\}_{n\in\mathbb{N}}$ doesn't lie in $\ell_1(\mathbb{N})$. Hence $(X, \|\cdot\|')$ isn't complete.

Problem 2

a)

For $(a, b, 0, \dots) \in M$ with norm less than or equal to 1, we have:

$$|f(a, b, 0, \dots)| = |a + b| \leq |a| + |b| = \sqrt[p]{|a|^p} + \sqrt[p]{|b|^p} \leq 2\|(a, b, 0, \dots)\|_p \leq 2$$

So f is bounded.
For $p = 1$ we have that: $|f(a,b,0,\dots)| \leq |a| + |b| = \|(a,b,0,\dots)\|_1$, but also $f(1,0,0,\dots) = 1$ so $\|f\| \geq 1$ and $\|f\| \leq 1$ so $\|f\| = 1$.
Assume $1 < p < \infty$. f will have its supremum on the boundary of the unit ball in $\mathbb{C}^2$ by remark 1.11 in the notes. The triangle inequality becomes an equality if a and b are real-linear dependant and since the norm on $\mathbb{C}$ is independent of a common phase of a and b, a and b can be assumed to be positive, real.
If we use the method of Lagrange Multipliers to maximize f under the condition that $g(a,b) = a^p + b^p - 1 = 0$, we get that there exists a λ such that:

$$(1,1) = \nabla f = \lambda \nabla g = \lambda(pa^{p-1}, pb^{p-1})$$

From which, by the assumption that a and b are positive, we get $a = b$. Therefore we have extremum at $a = b = |a| = \frac{1}{\sqrt[p]{2}}$:

$$|f(a,b,0,\dots)| = 2|a| = 2^{1-\frac{1}{p}}$$

Which is an local maximum, since for instance $f(1,0,0,\dots) = 1$. Thus, we find $\|f\| = 2^{1-\frac{1}{p}}$.

b)

We have that f is $\mathbb{C}$-linear on M and for $x \in M$, we have that:

$$|f(x)| = |a+b| \leq 2\|x\|_p$$

So f is upper bounded by a seminorm (actually just a norm). By the Hahn-Banach Extensions Theorem, there exists a linear functional F that extends f to $\ell_p(\mathbb{N})$ with $\|F\| = \|f\|$ for all $1 \leq p < \infty$.

Let $p > 1$ and $F : \ell_p(\mathbb{N}) \to \mathbb{C}$ be an extension of f i.e. $F|_M = f$. Our strategy for showing uniqueness will be to show that $F(\{\delta(m)_n\}_{n\in\mathbb{N}}) = 0$ where $\{\delta(m)_n\}_{n\in\mathbb{N}}$ is the Kronecker-delta sequence (zero in all entries except the m'th which has a 1) for $m \geq 2$. We do this by finding the a value of $|F((a,b,c,\dots)|$ that is strictly higher than $\|F\|\|(a,b,c,\dots)\|_p$ unless $F(0,0,1,0,\dots) = 0$.

We start by assuming $F(0,0,1,0,\dots) \neq 0$. Choose $\tilde{c} = c{\cdot}\exp(-i\mathrm{Arg}(F(0,0,1,0,\dots))$ where Arg is the function that maps a non-zero complex number to its unique argument between 0 and 2π. By linearity, we evaluate $F(a,b,\tilde{c})$:

$$F(a,b,\tilde{c},0,\dots) = F(a,b,0,\dots) + \tilde{c} \cdot F(0,0,1,\dots) = a + b + c \cdot |F(0,0,1,\dots)|$$

Let F_3 denote $|F(0,0,1,0,\dots)|$. I have used the Lagrange Multiplier method from above to find that we should evaluate $F(a,b,\tilde{c},0,\dots)$ at $a = b = \left(\left(2 + F_3^{\frac{p}{p-1}}\right)^{\frac{1}{p}}\right)^{-1}$

and $c = F_3^{\frac{1}{p-1}} \cdot a$. Notice that $\|(a, b, \tilde{c}, 0, \dots)\|_p = 1$. We get:

$$\begin{aligned}
F(a, b, \tilde{c}, 0, \dots) =& \frac{1}{\left(2 + F_3^{\frac{p}{p-1}}\right)^{\frac{1}{p}}} + \frac{1}{\left(2 + F_3^{\frac{p}{p-1}}\right)^{\frac{1}{p}}} + \frac{F_3^{\frac{1}{p-1}}}{\left(2 + F_3^{\frac{p}{p-1}}\right)^{\frac{1}{p}}} F_3 \\
=& \frac{2 + F_3^{\frac{p}{p-1}}}{\left(2 + F_3^{\frac{p}{p-1}}\right)^{\frac{1}{p}}} = \left(2 + F_3^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \geq 2^{1-\frac{1}{p}}
\end{aligned}$$

Here we only have equality for $F_3 = 0$. But since $\|F\| = \|f\| = 2^{1-\frac{1}{p}}$ we have that $|F(a, b, \tilde{c}, 0, \dots)| \leq 2^{1-\frac{1}{p}}$ and therefore we must have that $F_3 = 0$. By a similar argument, we get that $F(\{\delta(m)\}_{n\in\mathbb{N}} = 0$ for each $m \in \mathbb{N} \setminus \{1, 2\}$. Since we know the values of F on the basis $\{\delta(m)_n\}_{n\in\mathbb{N}}$, we have uniquely determined the linear mapping and thus there can only exist one extension of f to $\ell_p(\mathbb{N})$.

c)

We can define a family of functionals $F_i : \ell_1(\mathbb{N}) \to \mathbb{C}$, $i \in \mathbb{N}$ by:

$$F_i(x_1, x_2, \dots, x_i, \dots) = \sum_{j=1}^{i} x_j$$

We see that $F_i|_M = f$ and that F_i is $\mathbb{C}$-linear and continuous. Since F_i is an extension of f, we have that $\|f\| \leq \|F_i\|$ but also that:

$$|F_i(x)| \leq \sum_{j=1}^{i} |x_j| \leq \|x\|_1$$

And thus $\|F_i\| \leq 1 = \|f\|$, so $\|F_i\| = \|f\|$. Therefore, each F_i is an extension of f on $\ell_1(\mathbb{N})$ which preserves the norm.

Problem 3

a)

Let F be a given linear mapping $F : X \to \mathbb{K}^n$ for given n. Take n linearly independent vectors of X (which is possible since it's infinite dimensional) $\{x_i\}_{i\in I}$, $I = \{1, 2, \dots n\}$ and take their image $\{F(x_i)\}_{i\in I}$. Either these vectors are linearly independent or there exists a $j \in I$ so $F(x_j)$ is linearly dependent on the others with coefficients $\{a_i\}_{i\in I\setminus\{j\}}$. In that case we use linearity to get:

$$F(x_j) = \sum_{i\neq j} a_i F(x_i) = F\left(\sum_{i\neq j} a_i x_i\right)$$

Which shows that F isn't injective since $\sum_{i\neq j} a_i x_i \in X$ and $\sum_{i\neq j} a_i x_i \neq x_j$ by their linear independece.
If the image-vectors are linearly independent, take a new vector $x_{n+1} \in X$ linearly independent of $\{x_i\}_{i\in I}$. Since we have n linearly independent elements $\{F(x_i)\}_{i\in I}$ of a n-dimensionsal vector space, we have constructed a basis. So, if $F(x_{n+1})$ has coefficients $\{b_i\}_{i\in I}$ w.r.t the constructed basis we get by linearity:

$$F(x_{n+1}) = \sum_{i=1}^{n} b_i F(x_i) = F\left(\sum_{i=1}^{n} b_i x_i\right)$$

Which shows that F isn't injective, since $\sum_{i=1}^{n} b_i x_i \in X$ and $\sum_{i=1}^{n} b_i x_i \neq x_{n+1}$ by construction.

b)

Consider the linear mapping $F : X \to \mathbb{K}^n$, with $F(x) = (f_1(x), f_2(x), \dots, f_n(x))$. This mapping isn't injective by the last task. A linear map is injective if and only if its kernel is trivial. Thus, the kernel can't be trivial, so there must exist a non-zero element $x \in X$ with $F(x) = (0, 0, \dots, 0)$. This shows that there exists a non-zero $x \in \bigcap_{i=1}^{n} \ker(f_i)$ which was the statement.

c)

If one of the x_j's are zero, then $\|y - x_j\| \geq \|x_j\|$ holds trivially. Let's look at the each $x_j \neq 0$. By Theorem 2.7 we have that for each of these there exists $f_j \in X^*$ with:

$$\begin{aligned} f_j(x_j) &= \|x_j\| \\ \|f_j\| &= 1 \end{aligned}$$

By the previous task, we know that the intersection of their kernels must be nontrivial. Take a nonzero $a \in \bigcap_j \ker(f_j)$. Then by linearity of each map $y = \frac{a}{\|a\|}$ is also in the intersection of kernels and $\|y\| = 1$. By applying f_j to $y - x_j$, and using that y is in the kernel of each f_j and linearity, we get:

$$\|x_j\| = |f_j(x_j)| = |f(x_j) - f_j(y)| = |f(x_j - y)| \leq \|f_j\| \cdot \|x_j - y\| = \|y - x_j\|$$

There we are.

d)

If we have a finite union of balls centered at respectively $\{x_j\}_{j\in I}$, $I = \{1, 2, \dots, n\}$, each ball must have a radius r_j strictly less than $\|x_j\|$ if they aren't to contain 0. We have from task c) that there exists a y on the unit ball with $\|y - x_j\| \geq \|x_j\| > r_j$. Therefore we have an element which isn't in either of the balls and thus we can't cover the unit sphere.
Notice that the argument holds just as well with open balls. Here, we just have that $r_j \leq \|x_j\|$.

e)

Let $B_X(x,\epsilon)$ denote the ball of center x and radius ϵ for $0<\epsilon<1$. Then look at the union:

$$A = \bigcup_{\|x\|=1} B_X(x,\epsilon)$$

It is clear that $S \subset A$, since each point $x \in S$ is in the ball $B_X(x,\epsilon)$. If S should be compact then there should exist a finite subset K of $\subset \{x \in X | \|x\| = 1\}$ such that:

$$S \subset \bigcup_K B_X(x,\epsilon)$$

But we have from the previous task that this isn't possible. Therefore S isn't compact.
S is a closed subspace of the closed unit ball since if $(x_n)_{n\in\mathbb{N}} \to x$ and $x_n \in S$ for each n, we have by remark 1.2 in the notes that $\|x\| = \lim_{n\to\infty} \|x_n\| = 1$. If the closed unit ball in X would be compact then S would be compact since it would be a closed subset of a compact set. But S isn't compact so the closed unit ball isn't compact.

Problem 4

a)

For $t>0$, we have:

$$tE_n = \left\{ tf | \int_{[0,1]} |f|^3 dm \leq n \right\} = \left\{ f | \int_{[0,1]} |f|^3 \leq \frac{n}{t^3} \right\} \subset L_3([0,1],m)$$

Since we have that $tE_n \subset L_3([0,1],m)$ for all $t>0$ and $L_3([0,1],m)$ is a proper subset of $L_1([0,1],m)$, E_n is not an absorbing set.

b)

Let's assume that there exists some open subset of E_n (that the interior isn't empty). Then there exists some function $f \in E_n$ such that $B_{L_1([0,1],m)}(f,\epsilon) \subset E_n$ for some $\epsilon > 0$. Take that function $f \in E_n$. Look at the function $g(x) = f(x) + h(x)$ with $h(x) = \tau x^{\frac{-1+\beta}{3}} \exp(i\mathrm{Arg}(f(x)))$ where $\tau > 0$, $\beta > 0$ and $\mathrm{Arg}(f(x))$ is the argument-function which maps $f(x)$ to its unique argument between 0 and 2π (with $\mathrm{Arg}(0) = 0$).
Then we have:

$$\|f-g\|_1 = \int_{[0,1]} |f-g| dm = \int_{[0,1]} \tau x^{\frac{-1+\beta}{3}} dm = \tau \frac{3}{2+\beta}$$
$$\int_{[0,1]} |g|^3 dm = \int_{[0,1]} |f+h|^3 dm = \int_{[0,1]} (|f|+|h|)^3 dm \geq \int_{[0,1]} \tau^3 x^{-1+\beta} dm = \tau^3 \frac{1}{\beta}$$

We see that for fixed radius r, $g \in B_{\ell_1(\mathbb{N})}(f, r)$ if $\tau < \frac{2}{3}r$ but $g \notin E_n$ for $\beta < \frac{\tau^3}{n}$. Thus there exist no open ball that is a subset of E_n, so the interior of E_n is empty.

c)

Let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of functions in E_n that converge to f L_1-wise. Then by corollary 13.8 in Measures, Integrals and Martingales" by Schillings, we have that there exists a subsequence $\{f_{m_k}\}_{k\in\mathbb{N}}$ that converges pointwise to f Lebesgue-almost-surely. This means that $\{|f_{m_k}|\}_{k\in\mathbb{N}}$ converges to $|f|$ Lesbegue almost surely which again means that $\{|f_{m_k}|^3\}_{k\in\mathbb{N}}$ converges to $|f|^3$ Lesbegue almost surely (by continuity of absolute values and cubing). Therefore, by Fatou's Lemma:

$$\int_{[0,1]} |f|^3 dm = \int_{[0,1]} \liminf_{k\to\mathbb{N}} |f_{m_k}|^3 dm \le \liminf_{k\to\infty} \int_{[0,1]} |f_{m_k}|^3 dm \le n$$

So we have that $f \in E_n$ and thus E_n is closed.

d)

We have that

$$L_3([0,1], m) = \left\{f| \quad \|f\|_3^3 < \infty\right\} = \bigcup_{n=1}^{\infty} \left\{f| \quad \|f\|_3^3 \le n\right\} = \bigcup_{n\ge 1}^{\infty} E_n$$

And by definition 3.12 in the notes that means that $L_3([0,1], m)$ is first measurable in $L_1([0,1], m)$ since each $\bar{E}_n = E_n$ is nowhere dense.

Problem 5

a)

Let $\epsilon > 0$ be given, and $N \in \mathbb{N}$ be such that for all $n \ge N$ we have $\|x - x_n\| < \epsilon$. From the reverse triangle inequality we get:

$$\epsilon > \|x - x_n\| \ge \Big| \|x\| - \|x_n\| \Big| \ge 0$$

Which implies that the norms converge toward the norm of x.

b)

Let $\ell_2(\mathbb{N})$ be our Hilbert Space and look at the sequence indexed by m: $\delta_m = \{\delta(m)_n\}_{n\in\mathbb{N}}$, (The sequence with a 1 on the m'th entry and otherwise zeroes). We see from HW 4, task 2a that a sequence converges weakly if and only if all linear, continuous functionals evaluated at this sequence converges in $\mathbb{C}$. By

Riez-representation theorem we have that there exists $y \in \ell_2(\mathbb{N})$ such that for each $f_y \in X^*$, $f_y(x) = \langle x, y \rangle$. We have that for any $f_y \in X^*$:

$$\lim_{m\to\infty} f_y(\delta_m) = \lim_{m\to\infty} \langle \delta_m, y \rangle = \lim_{m\to\infty} y_m = 0$$

Where y_m denotes the m'th entry of y, which goes to zero since it's quadratic summable. We thus see that $\{\delta(m)_n\}_{n\in\mathbb{N}}$ converges weakly to zero, but since $\|\delta(m)_n\| = 1$ for each m, it doesn't converge towards zero in norm.

c)

If $\|x\| = 0$ we are trivially done. If not, let $\epsilon > 0$ and N be given such that for all $n \geq N$ we have: $|\langle x, x \rangle - \langle x_n, x \rangle| < \epsilon$. This can be done since the weak convergence of x_n implies that $f_x(x_n) = \langle x_n, x \rangle$ converges. Then by Cauchy-Schwartz:

$$\|x\|^2 = \langle x, x \rangle \leq |\langle x_n, x \rangle| + \epsilon \leq \|x_n\| \cdot \|x\| + \epsilon \Leftrightarrow \|x\| \leq \|x_n\| + \frac{\epsilon}{\|x\|} \leq 1 + \frac{\varepsilon}{\|x\|}$$

Since this holds for all $\epsilon > 0$, this shows that $\|x\| \leq 1$.