Problem 1.

(a) As H is an infinite dimensional Hilbert space we may identify it with $\ell_2(\mathbb{N})$. Then f_N is identified with the sequence $(f_N(n))_{n\geq 1}$ where $f_N(n) = N^{-1}$ if $n \leq N^2$ and $f_N(n) = 0$ if $n > N^2$.

We have $\|f_N\| = (\sum_{n=1}^{N^2} N^{-2})^{1/2} = 1$ for all $N \geq 1$.

To show that $f_N \to 0$ weakly, we use HW4 problem 3. Indeed the above shows that the sequence is bounded in norm. Since $|f_N(n)| \leq N^{-1}$ we have $f_N(n) \to 0$ as $N \to \infty$, for any fixed $n \geq 1$. Therefore the sequence also converges pointwise to 0. By the homework problem this shows that f_N converges to 0 weakly.

(b) Since the weak closure and norm closure agrees for convex sets, we see that K is also the weak closure and hence is closed in the weak topology. Also, by the triangle inequality we have $\left\|\sum_{N=1}^{M} \alpha_N f_N\right\| \leq \sum_{N=1}^{M} \alpha_N \|f_N\| = 1$ for any $\alpha_N \in [0,1]$ with sum 1. Therefore $C := \mathrm{co}\{\, f_N : N \geq 1 \,\}$ is contained in the closed unit ball, and hence its norm closure is also contained in the closed unit ball. By Alaoglu's theorem and the fact that $\ell_2(\mathbb{N})$ is reflexive, the closed unit ball is compact in the weak topology, so since closed subsets of compact sets are compact, it follows that K is compact in the weak topology.

Since K is closed in the weak topology, and 0 is the limit of a net in K by (a), it follows that $0 \in K$.

(c) We claim that elements $x = (x(n))_{n\geq 1} \in K$ satisfy the following properties, for all $n \geq 1$,

$$x(n) \in [0,\infty) \tag{1}$$
$$x(n^2) \leq n^{-1} \tag{2}$$
$$x(n) \geq x(n+1) \tag{3}$$
$$x(n^2+1) = x((n+1)^2) \tag{4}$$
$$\sum_{m=1}^{n} x(m^2) \leq 1. \tag{5}$$

Let us do the proof of (5) in detail. Then I hope that it is also clear to you, that one can easily adapt the same argument to prove (1) - (4).

First, observe that (5) holds for $x = f_N$ since

$$\sum_{m=1}^{n} f_N(m^2) \leq \sum_{m=1}^{\infty} f_N(m^2) = \sum_{m=1}^{N} N^{-1} = 1.$$

Next, we verify (5) for $x \in C$. We may write $x = \sum_{N=1}^{M} \alpha_N f_N$ with $\alpha_N \in [0,1]$ and $\sum_{N=1}^{M} \alpha_N = 1$. Then

$$\sum_{m=1}^{n} x(m^2) = \sum_{m=1}^{n}\sum_{N=1}^{M} \alpha_N f_N(m^2) = \sum_{N=1}^{M} \alpha_N \sum_{m=1}^{n} f_N(m^2) \leq \sum_{N=1}^{M} \alpha_N = 1.$$

Lastly, suppose $x \in K$. We may write x as the norm limit of a sequence $(x_k)_{k\geq 1}$ in C. In particular, $x(n)$ is the limit of $(x_k(n))_{k\geq 1}$ for every $n \geq 1$. Thus

$$\sum_{m=1}^{n} x(m^2) = \sum_{m=1}^{n} \lim_{k\to\infty} x_k(m^2) = \lim_{k\to\infty} \sum_{m=1}^{n} x_k(m^2) \leq 1.$$

Now we show that $0 \in K$ is an extreme point. Suppose $\alpha x + \beta y = 0$ for $x, y \in K$ where $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Then we have $\alpha x(n) + \beta y(n)) = 0$ for every $n \geq 1$, so by (1) we get $x(n) = y(n) = 0$. Thus $x = y = 0$, which proves that 0 is an extreme point.

Next we show that f_N is an extreme point for every $N \geq 1$. Suppose $\alpha x + \beta y = f_N$ for $x, y \in K$ where $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Using $\alpha x(N^2) + \beta y(N^2) = f_N(N^2) = N^{-1}$ and (2) applied to $n = N$, we get $x(N^2) = y(N^2) = N^{-1}$. From (3) we deduce that $x(n) \geq N^{-1}$ for all $1 \leq n \leq N^2$. Now by (5) we have

You also need (1).

$$1 \geq \sum_{n=1}^{N} x(n^2) \geq N \cdot N^{-1} = 1,$$

hence all the inequalities must be equalities, implying that $x(n^2) = N^{-1}$ for $n = 1, \dots, N$. In particular, $x(1) = x(N^2) = N^{-1}$ so from (3) we deduce that $x(n) = N^{-1}$ for $1 \leq n \leq N^2$. From (5) we now get $1 \geq \sum_{n=1}^{N+1} x(n^2) = 1 + x((N+1)^2)$ so $x((N+1)^2) = 0$. Hence by (4) we have $x(N^2+1) = 0$ and so $x(n) = 0$ for all $n > N^2$ by (3). We conclude that $x = f_N$ and similarly $y = f_N$, which proves that f_N is an extreme point.

(d) No. By the converse to Krein Milman the extreme points of K are contained in the weak closure of $\{ f_N : N \geq 1\}$, which we claim is just $\{ f_N : N \geq 1\} \cup \{0\}$. Since 0 is in the weak closure by (a) it suffices to show that the latter set is weakly closed. Furthermore as compact subsets of Hausdorff spaces are closed it suffices to show that the set is compact.

Now suppose we are given an open cover $(U)_{U \in \mathcal{U}}$. Then there exists some $U_0 \in \mathcal{U}$ such that $0 \in U_0$. By (a) there exists $N_0 \geq 1$ such that $f_N \in U_0$ whenever $N > N_0$. Also for each $N \leq N_0$ there exists $U_N \in \mathcal{U}$ such that $f_N \in U_N$. It follows that $\{U_0, U_1, \dots, U_{N_0}\}$ is a finite subcover. This proves that $\{ f_N : N \geq 1\} \cup \{0\}$ is compact and hence closed.

Problem 2.

(a) We apply HW4 problem 2(a). Suppose $f \in Y^*$. Then $f \circ T \in X^*$, hence as $x_n \to x$ weakly we get that $f(T(x_n)) \to f(T(x))$. As this holds for any $f \in Y^*$ we deduce that $T(x_n) \to T(x)$ weakly.

(b) We argue by contradiction: Suppose there exists a sequence $(x_n)_{n \geq 1}$ in X converging weakly to x, such that Tx_n does not converge to Tx in norm. Then for some $\epsilon > 0$ there exists a subsequence x_{n_k} such that

$$\|Tx_{n_k} - Tx)\| \geq \epsilon \qquad \text{for all } k \geq 1. \tag{$*$}$$

The subsequence x_{n_k} also converges weakly to x, so in particularly it is weakly bounded. As weakly bounded is equivalent to norm bounded, the subsequence x_{n_k} is therefore norm bounded. Now as T is compact we deduce that $T(x_{n_k})$ has a norm-convergent subsequence. Say that the limit of this subsequence is y. Then the subsequence also converges weakly to y, and by (a) it converges weakly to Tx, so $y = Tx$. Hence $T(x_{n_k})$ has a subsequence converging to Tx in norm, which contradicts $(*)$.

(c) Suppose that T is not compact. Then the image of the closed unit ball $B \subseteq H$ is not totally bounded which means that there exists a $\delta > 0$ such that $T(B)$ cannot be covered by finitely many open balls of radius δ. This implies that we can construct a sequence $(x_n)_{n \geq 1}$ inductively by choosing $x_n \in B$ such that Tx_n is not in the open ball of radius δ centered at Tx_m, for any $m < n$. Thus by construction we have $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$.

We identify H with $\ell_2(\mathbb{N})$ as in problem 1. Then we have just constructed a sequence $(x_n)_{\geq 1}$ inside the closed unit ball $B = \{\, x \in \ell_2(\mathbb{N}) : \sum_{k\geq 1} |x(k)|^2 \leq 1 \,\}$, and we shall now construct a subsequence which converges weakly to some $x \in \ell_2(\mathbb{N})$.

We start by constructing a subsequence which converges pointwise. Observe that for each $k \geq 1$ we have $|x_n(k)| \leq \|x_n\| \leq 1$. Taking $k = 1$ this lets us pick a subsequence $(x_{n_1(n)})_{n\geq 1}$ such that $(x_{n_1(n)}(1))_{n\geq 1}$ converges. Taking $k = 2$ we can pick a subsequence $(x_{n_1(n_2(n))})_{n\geq 1}$ (i.e. a subsequence of the last subsequence), which also converges at $k = 2$. We continue like this, obtaining for every $k \geq 1$ a subsequence $(x_{n_1(\cdots(n_k(n)))})_{n\geq 1}$ which converges at $1, \dots, k$.

?

Now we define

$$N(k) = n_1(n_2(\cdots(n_k(k)))).$$

Since n_k is strictly increasing, we have

$$n_k(k) \geq n_k(k-1) + 1 \geq \cdots \geq n_k(1) + k - 1 > k - 1.$$

Hence we get $N(k) > N(k-1)$ so $k \mapsto N(k)$ is strictly increasing.

Thus $(x_{N(n)})_{n\geq 1}$ is a subsequence of $(x_n)_{n\geq 1}$. We claim that it is convergent at every point $k \geq 1$. Indeed, by the same argument as above the sequence of natural numbers $k, n_{k+1}(k+1), n_{k+1}(n_{k+2}(k+2)), \dots$ is strictly increasing, hence $(x_{N(n)})_{n\geq k}$ is a subsequence of $(x_{n_1(\cdots(n_k(n)))})_{n\geq 1}$ which converges at k by construction.

The idea seems to be OK, but it is very difficult to follow.

Now let $x(k) = \lim_{n\to\infty} x_{N(n)}(k)$, for every $k \geq 1$. Then

$$\sum_{k=1}^{m} |x(k)|^2 = \lim_{n\to\infty} \sum_{k=1}^{m} |x_{N(n)}(k)|^2 \leq 1$$

so letting $m \to \infty$ we get $x = (x(k))_{k\geq 1} \in \ell_2(\mathbb{N})$. Now we show that $x_{N(n)} - x \to 0$ weakly, as $n \to \infty$, hence the sequence $(x_{N(n)})_{n\geq 1}$ converges weakly to x. This follows from HW4 problem 3. Indeed, the sequence is bounded in norm as $\|x_n\|_2 \leq 1$ for all $n \geq 1$, and by construction it converges pointwise to 0.

Finally we obtain a contradiction, as the sequence $(x_{N(n)})_{n\geq 1}$ converges weakly to x, but the sequence $(Tx_{N(n)})_{n\geq 1}$ does not converge to Tx in norm, as it is not even Cauchy. So we conclude that T must be compact.

(d) By (c) we must show that whenever $(x_n)_{n\geq 1}$ is a sequence in $\ell_2(\mathbb{N})$ which converges weakly to $x \in \ell_2(\mathbb{N})$, then the sequence $(Tx_n)_{n\geq 1}$ in $\ell_1(\mathbb{N})$ converges in norm to Tx.

By (a) we know that $(Tx_n)_{n\geq 1}$ converges weakly to Tx. By a remark in lectures every weakly convergent sequence in $\ell_1(\mathbb{N})$ is norm convergent, hence $(Tx_n)_{n\geq 1}$ is actually norm convergent.

(e) Suppose that T is onto, and we will derive a contradiction. Recall that every compact operator is bounded, so by the open mapping theorem we deduce that T is an open map. Hence $T(B_X(0,1))$ contains an open ball $B_Y(0,\epsilon)$ for some $\epsilon > 0$. By potentially making ϵ smaller we may assume that $\overline{B_Y(0,\epsilon)} \subset T(B_X(0,1))$. Since T is compact we know that $T(B_X(0,1))$ has compact closure, hence $\overline{B_Y(0,\epsilon)}$ is a closed subset of Y contained in a compact subset, so it must be compact itself. By scaling we deduce that the closed unit ball of Y is compact, however by the previous assignment we know that this is false since Y is infinite dimensional. This is the desired contradiction.

(f) M is self-adjoint since for any $f, g \in H$

$$\langle Mf, g\rangle = \int_{[0,1]} tf(t)\overline{g(t)}dm(t) = \int_{[0,1]} f(t)\overline{tg(t)}dm(t) = \langle f, Mg\rangle.$$ ✓

However, M is not compact as it has no eigenvalues by HW6 problem 3, and every compact self-adjoint operator has an eigenvalue by the spectral theorem for compact operators. ✓

Problem 3.

(a) K is continuous by the pasting lemma in topology. As $[0,1]$ is compact with finite measure and as T is the kernel operator associated to K it follows from a theorem in lectures that T is compact.

$T = T_{\tilde{K}}$ $\tilde{K}(s,t) = K(t,s)$ (+ you need to show $K \in L_2([0,1]^2)$)

(b) M is self-adjoint since for any $f, g \in H$ we have

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} (Tf)(s)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} f(t)\overline{K(s,t)g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(s)\overline{(Tg)(t)}dm(t) \\
&= \langle f, Tg\rangle.
\end{aligned}$$

← K real

← Here you need $K(s,t) = K(t,s)$

We only need to justify the change of integrals. This is justified by Fubini-Tonelli, since K and $f \otimes \overline{g}$ are L_2 functions on $[0,1]^2$, so their product is absolutely integrable by Cauchy-Schwartz. show this

(c) Since $1 = \mathbf{1}_{[0,s]}(t) + \mathbf{1}_{[s,1]}(t)$ a.e.

$$\begin{aligned}
(Tf)(s) &= \int_{[0,1]} (\mathbf{1}_{[0,s]}(t) + \mathbf{1}_{[s,1]}(t))K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t).
\end{aligned}$$

needs proof

Hence Tf is a product and sum of continuous or absolutely continuous functions, hence it is continuous. (Alternatively, one can use Lebesgue dominated convergence to see that the integral $\int_{[0,1]} \mathbf{1}_{[0,s]}(t)tf(t)dm(t)$ is continuous in s, and likewise for the other integral.) ← needs elaboration.

For $s = 0$ the first integral is 0 and the coefficient in the second term is 0, hence $(Tf)(0) = 0$. For $s = 1$ the coefficient in the first term is 0 and the second integral is 0, hence $(Tf)(1) = 0$. ✓

Problem 4.

(a) HW 7 problem 1. implies that $g_k \in \mathcal{S}(\mathbb{R})$. Specifically the first part states that e^{-x^2} is Schwartz. Then (d) implies that $e^{-x^2/2}$ is Schwartz, and finally (a) implies that g_k is Schwartz.

From lectures we have $\mathcal{F}(g_0)(\xi) = g_0(\xi)$. By properties of the Fourier transform we also have $(-i)\mathcal{F}(g_{k+1})(\xi) = (\frac{d}{d\xi}\mathcal{F}(g_k))(\xi)$. Thus

$$\begin{aligned}
\mathcal{F}(g_1)(\xi) &= ig_0'(\xi) = -i\xi e^{-\xi^2/2} \\
\mathcal{F}(g_2)(\xi) &= ig_1'(\xi) = e^{-\xi^2/2} - \xi^2 e^{-\xi^2/2} \\
\mathcal{F}(g_3)(\xi) &= ig_2'(\xi) = -3i\xi e^{-\xi^2/2} + i\xi^3 e^{-\xi^2/2}
\end{aligned}$$

(b) We can summarize the results of (a) by the equations $\mathcal{F}(g_0) = g_0$, $\mathcal{F}(g_1) = -ig_1$, $\mathcal{F}(g_2) = g_0 - g_2$, and $\mathcal{F}(g_3) = -3ig_1 + ig_3$.

We put $h_0 = g_0$, $h_1 = 3g_1 - 2g_3$, $h_2 = g_0 - 2g_2$, and $h_3 = g_1$. Then clearly $\mathcal{F}(h_0) = h_0$ and $\mathcal{F}(h_3) = -ih_3$. Furthermore, by linearity of the Fourier transform

$$\mathcal{F}(h_1) = 3\mathcal{F}(g_1) - 2\mathcal{F}(g_3) = 3(-ig_1) - 2(-3ig_1 + ig_3) = 3ig_1 - 2ig_3 = ih_1$$

and similarly

$$\mathcal{F}(h_2) = \mathcal{F}(g_0) - 2\mathcal{F}(g_2) = g_0 - 2(g_0 - g_2) = -g_0 + 2g_2 = -h_2.$$

for $f \in L_1$ (Although also true for $f \in L_2$) show this.

(c) It follows directly from the definitions that $\mathcal{F}(f) = S_{-1}\mathcal{F}^*(f)$, using the notation $(S_{-1}g)(x) = g(-x)$ from lectures. Thus $\mathcal{F}(\mathcal{F}(f)) = S_{-1}\mathcal{F}^*\mathcal{F}(f) = S_{-1}f$ by the Fourier inversion theorem. It follows that $\mathcal{F}^4 f = S_{-1}\mathcal{F}^2(f) = S_{-1}S_{-1}f = f$.

(d) If $\mathcal{F}(f) = \lambda f$ then by (c) we get $f = \mathcal{F}^4(f) = \lambda^4 f$. As $f \neq 0$ there exists $x \neq 0$ so that $f(x) \neq 0$. Then $f(x) = \lambda^4 f(x)$ implies that $\lambda^4 = 1$, which implies that $\lambda \in \{1, i, -1, -i\}$.

This proves that all eigenvalues are equal to ± 1 or $\pm i$. Conversely, (b) shows that these numbers are actually eigenvalues.

Problem 5.

1. Define $N \subset [0,1]$ to be the union of all open subsets $U \subset [0,1]$ with $\mu(U) = 0$. By definition $\text{supp}(\mu) = [0,1]\backslash N$ so we have to show that N is the empty set. This means that we have to show that for all non-empty open subset $U \subset [0,1]$ then $\mu(U) \neq 0$. Indeed, any non-empty open subset U will contain x_n for some n, as $(x_n)_{n\geq 1}$ is dense. But then

$$\mu(U) \geq 2^{-n}\delta_{x_n}(U) = 2^{-n} > 0.$$