Mandatory Assignment 2 for FunkAn

Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = \frac{1}{N}\sum_{n=1}^{N^2} e_n$ for all $N \geq 1$

(a) We want to show that $f_N \to 0$ weakly, as $N \to \infty$, while $\|f_N\| = 1$ for all $N \geq 1$.
First we show that $\|f_N\| = 1$. We have that $\|f_N\| = \sqrt{\langle f_N, f_N\rangle}$. So we compute the inner product of f_N.

$$\langle f_N, f_N\rangle = \left\langle \frac{1}{N}\sum_{n=1}^{N^2} e_n, \frac{1}{N}\sum_{k=1}^{N^2} e_k \right\rangle = \frac{1}{N^2}\sum_{n,k=1}^{N^2} \langle e_n, e_k\rangle$$

by bilinearity in both coordinates. Note that $\langle e_n, e_k\rangle = 1$ if $k = n$ otherwise 0, so we can just sum over the n's, then we have

$$\frac{1}{N^2}\sum_{n,k=1}^{N^2} \langle e_n, e_k\rangle = \frac{1}{N^2}\sum_{n=1}^{N^2} 1 = \frac{N^2}{N^2} = 1$$

This means that $\|f_N\| = \sqrt{\langle f_N, f_N\rangle} = \sqrt{1} = 1$.

Now we want to show that $f_N \to 0$ weakly as $N \to \infty$, $N \geq 1$. Since $f_N \to 0$ weakly as $N \to \infty$ we have by HW 4 Pb 2(a) that $\varphi(f_N) \to 0$ as $N \to \infty$ for all $\varphi \in H^*$. Then by Riesz representation theorem there exists a $y \in H$ such that $\varphi(f_N) = \langle f_N, y\rangle$, $f_N \in H$. Bessel's inequality says $\sum_{n=1}^{\infty} |\langle y, e_n\rangle|^2 \leq \|y\|^2$ where $(e_n)_{n\geq 1}$ is a orthonormal basis for H, and $y \in H$. This means that the series is bounded, so its tail goes to zero. Let $\epsilon > 0$ arbitrarily. Then for some $M \geq N_\epsilon$ we have $\sum_{n=M}^{\infty} |\langle y, e_n\rangle|^2 < \frac{\epsilon^2}{4}$. So since $y \in H$ then we can write $y = \sum_{n=1}^{\infty}\langle y, e_n\rangle e_n$ because $(e_n)_{n\geq 1}$ is a O.N.B. of H. Then

$$\begin{aligned}
|\langle f_N, y\rangle| &= \left|\left\langle f_N, \sum_{n=1}^{\infty}\langle y, e_n\rangle e_n \right\rangle\right| = \left|\left\langle f_N, \sum_{n=1}^{M-1}\langle y, e_n\rangle e_n, \sum_{n=M}^{\infty}\langle y, e_n\rangle e_n \right\rangle\right| \\
&\leq \left|\left\langle f_N, \sum_{n=1}^{M-1}\langle y, e_n\rangle e_n \right\rangle\right| + \left|\left\langle f_N, \sum_{n=M}^{\infty}\langle y, e_n\rangle e_n \right\rangle\right|
\end{aligned}$$

where we use biniliearity in the second coordinate and the triangle-inequality. First we look at the inner product with the finite sum.

$$\begin{aligned}
\left|\left\langle f_N, \sum_{n=1}^{M-1}\langle y, e_n\rangle e_n \right\rangle\right| &= \left|\sum_{n=1}^{M-1} \overline{\langle y, e_n\rangle}\langle f_N, e_n\rangle\right| = \left|\sum_{n=1}^{M-1} \overline{\langle y, e_n\rangle}\left\langle \frac{1}{N}\sum_{k=1}^{N^2} e_k, e_n \right\rangle\right| \\
&= \frac{1}{N}\left|\sum_{n=1}^{M-1} \overline{\langle y, e_n\rangle}\sum_{k=1}^{N^2}\langle e_k, e_n\rangle\right| = \frac{1}{N}\left|\sum_{n=1}^{\min(M-1,N^2)} \overline{\langle y, e_n\rangle}\right| \\
&< \frac{\epsilon}{2}
\end{aligned}$$

since $\sum_{k=1}^{N^2} \langle e_k, e_n\rangle = 1$ if $k = n$ and zero otherwise because n is fixed, this can happen if $n \in \{1, \dots, N^2\}$ if $N^2 \leq M-1$ or if $n \in \{1, \dots, M-1\}$ if $N^2 \geq M-1$. Now we look at the inner product with the infinite sum. First we use Cauchy-Schwarz inequality and get

$$\left|\left\langle f_N, \sum_{n=M}^{\infty} \langle y, e_n\rangle e_n \right\rangle\right| \leq \|f_N\| \left\| \sum_{n=M}^{\infty} \langle y, e_n\rangle e_n \right\| = \left\| \sum_{n=M}^{\infty} \langle y, e_n\rangle e_n \right\|$$

since $\|f_N\| = 1$ which we shown above. And because $\|y\| = \|\sum_{n=M}^{\infty} \langle y, e_n\rangle e_n\|$, so it's a convergent series. Then we have by Pythagorean theorem extended to series that

$$\begin{aligned}
\left\| \sum_{n=M}^{\infty} \langle y, e_n\rangle e_n \right\| &= \sqrt{\sum_{n=M}^{\infty} \|\langle y, e_n\rangle e_n\|^2} = \sqrt{\sum_{n=M}^{\infty} \langle (\langle y, e_n\rangle, e_n), (\langle y, e_n\rangle, e_n)\rangle} \\
&= \sqrt{\sum_{n=M}^{\infty} \langle y, e_n\rangle \overline{\langle y, e_n\rangle} \langle e_n, e_n\rangle} = \sqrt{\sum_{n=M}^{\infty} |\langle y, e_n\rangle|^2} \\
&< \sqrt{\frac{\epsilon^2}{4}} = \frac{\epsilon}{2}
\end{aligned}$$

for $M \geq N_\epsilon$. So now we have that $|\langle f_N, y\rangle < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, then $\langle f_N, y\rangle \to 0$ as $N \to \infty$, so $\varphi(f_N) \to 0$ as $N \to \infty$ and then by HW4 Pb 2(a) we have that $f_N \to 0$ weakly as $N \to \infty$.

(b) We want to argue that K is weakly compact and that $0 \in K$.
Since $co\{f_N : N \geq 1\}$ is convex we have by Theorem 5.7 norm closure is the same as weak closure so $K = \overline{co\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{co\{f_N : N \geq 1\}}^{\tau_w}$. And since by (a) we have that $f_N \to 0$ weakly as $N \to \infty$ we have that $0 \in K$.
We want to show that $K \subset \overline{B_H}(0,1)$. First we show that $co\{f_N : N \geq 1\} \subset \overline{B_H}(0,1)$. Let $x \in co\{f_N : N \geq 1\}$ then $x = \sum_{i=1}^{n} \alpha_i f_{N_i}$ where $\alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. So

$$\|x\| = \left\| \sum_{i=1}^{n} \alpha_i f_{N_i} \right\| \leq \sum_{i=1}^{n} |\alpha_i| \|f_{N_i}\| = \sum_{i=1}^{n} \alpha_i = 1$$

where use the triangle-inequality and we have that $\|f_{N_i}\| = 1$ from (a). So $x \in \overline{B_H}(0,1)$. Since K is the intersection of closed sets containing $co\{f_N : N \geq 1\}$ and $co\{f_N : N \geq 1\} \subset \overline{B_H}(0,1)$ then $K \subset \overline{B_H}(0,1)$. And because H is a Hilbert space it's a reflexive and Banach we have by Theorem 6.3 that $\overline{B_H}(0,1)$ is compact. Thus K is compact, since closed subspace of compact space is compact.

(c) We want to show that 0, as well as each f_N, $N \geq 1$, are extreme points in K. So we write $0 = \alpha k_1 + (1-\alpha)k_2$ where $0 < \alpha < 1$ and $k_1, k_2 \in K$. We want to show that $k_1 = k_2 = 0$.
We claim that $\langle k, e_m\rangle \in [0, \infty)$ for all $k \in K$ and e_m a orthonormal basis. This is so, since K is the norm closure of $co\{f_N : N \geq 1\}$ so there exists a sequence $(k_n)_{n\geq n} \subset co\{f_N : N \geq 1\}$ where $k_n \to k$ as $n \to \infty$(in the norm).

Then $k_n = \sum_{i=1}^t \alpha_i f_{N_i}$ where $\alpha_i > 0$ and $\sum_{i=1}^t \alpha_i = 1$ for $t \in \mathbb{N}$. We see that

$$\langle k_n, e_m\rangle = \left\langle \sum_{i=1}^t \alpha_i f_{N_i}, e_m \right\rangle = \sum_{i=1}^t \alpha_i \langle f_{N_i}, e_m\rangle = \sum_{i=1}^t \alpha_i \frac{1}{N_i} \sum_{l=1}^{N_i^2} \langle e_l, e_m\rangle \in [0,\infty)$$

since $\langle e_l, e_m\rangle$ are either 1 or 0 and $N_i \geq 1$, $\alpha_i > 0$.
We have that the function $x \mapsto \langle x, e_m\rangle$ is continuous so $\langle k_n, e_m\rangle \to \langle k, e_m\rangle$ as $n \to \infty$, by continuity we can pull the limit inside. So $\langle k_n, e_m\rangle \in [0,\infty)$ which is sequentially closed so $\langle k, e_m\rangle \in [0,\infty)$ for all $k \in K$, which prove our claim.
Then

$$0 = \langle 0, e_m\rangle = \langle \alpha k_1 + (1-\alpha)k_2, e_m\rangle = \alpha\langle k_1, e_m\rangle + (1-\alpha)\langle k_2, e_m\rangle$$

Then by our claim we have that $\langle k_1, e_m\rangle = \langle k_2, e_m\rangle = 0$ for all $m \in \mathbb{N}$.
And since e_m is a orthonormal basis then $k_1 = k_2 = 0$. So 0 is an extreme point.

Now we show that each of f_N, $N \geq$, are extreme points in K. First we set $F := \{f_N : N \geq 1\}$. Then we claim that weak closure of F is $\{0\} \cup \{f_n : N \geq 1\}$. To prove this let $(x_n)_{n\geq 1} \subset F$ We have that $f_N \to 0$ weakly so by HW 4 Pb 2(a) $\varphi(f_N) \to 0$ for $\varphi \in H^*$. So for $\epsilon > 0$ we have $|\varphi(f_N)| < \epsilon$ for $N \geq M$, we look at $N \geq M$ otherwise we may create a cluster point. But then there exists a n_ϵ such that $x_n \subset \{f_N : N \geq M\}$ for $n \geq n_\epsilon$. And so we have that $|\varphi(x_n)| < \epsilon$ for $n \geq n_\epsilon$. And then by HW 4 Pb 2(a) we have that $x_n \to 0$ weakly. This prove the claim.
We have that the weak topology makes H into a $LCTVS$. We have that K is non-empty, weakly compact by (b) and convex since the closure of a convex set is convex. And since $co(F)$ is convex we have by Theorem 5.7 that $\overline{co(F)}^{\|\cdot\|} = \overline{co(F)}^{\tau_w}$. So $K = \overline{co(F)}^{\tau_w}$. Then we can use Millman (Theorem 7.9) to say that $Ext(K) \subset \overline{F}^{\tau_w} = \{0\} \cup \{f_n : N \geq 1\}$ by our claim.
Again since K is non-empty, weakly compact by (b) and convex we have that by Krein-Millman that $K = \overline{co(Ext(K))}^{\tau_w} = \overline{co(Ext(K))}^{\|\cdot\|}$ again by theorem 5.7.
Assume for contradiction that there exists $N_0 \in N$ such that $f_{N_0} \in Ext(K)$ but $f_{N_0} \in K = \overline{co(Ext(K))}^{\|\cdot\|}$.
Let $(x_n)_{n\geq}$ be a sequence in $co(Ext(K)) \subset co(\{0\} \cup \{f_N : N \geq 1, N \neq N_0\})$ which holds because of what we have shown above. And $x_n \to f_{N_0}$ in the norm. Then $x_n = \sum_{i=1}^m \alpha_i f_{N_i}$, $N_i \neq N_0$, $\alpha_i > 0$ and α_i sums to 1. Then we have that

$$\begin{aligned}\langle x_n, e_{N_0^2}\rangle &= \left\langle \sum_{i=1}^m \alpha_i f_{N_i}, e_{N_0^2}\right\rangle = \left\langle \sum_{i=1}^m \alpha_i \frac{1}{N_i}\sum_{k=1}^{N_i^2} e_k, e_{N_0^2}\right\rangle \\ &= \sum_{i=1}^m \alpha_i \frac{1}{N_i}\sum_{k=1}^{N_i^2}\langle e_k, e_{N_0^2}\rangle = \sum_{\substack{i=1\\ N_i\geq N_0}}^m \alpha_i \frac{1}{N_i}\sum_{k=1}^{N_i^2}\langle e_k, e_{N_0^2}\rangle\end{aligned}$$

since if $N_i < N_0$ then $\sum_{k=1}^{N_i^2}\langle e_k, e_{N_0^2}\rangle$ is zero, and since $N_i \neq N_0$ we have that

$$\sum_{\substack{i=1\\ N_i\geq N_0}}^m \alpha_i \frac{1}{N_i}\sum_{k=1}^{N_i^2}\langle e_k, e_{N_0^2}\rangle = \sum_{\substack{i=1\\ N_i> N_0}}^m \alpha_i \frac{1}{N_i}\sum_{k=1}^{N_i^2}\langle e_k, e_{N_0^2}\rangle = \sum_{\substack{i=1\\ N_i> N_0}}^m \alpha_i \frac{1}{N_i}\sum_{k=1}^{N_i^2}\langle e_k, e_{N_0^2+1}\rangle$$

the last equality holds since both $\sum_{k=1}^{N_i^2}\langle e_k, e_{N_0^2}\rangle = 1$ and $\sum_{k=1}^{N_i^2}\langle e_k, e_{N_0^2+1}\rangle = 1$ when $N_i > N_0$. We then have that $\langle x_n, e_{N_0^2}\rangle = \langle x_n, e_{N_0^2+1}\rangle$.
This means that

$$\begin{aligned}\lim_{n\to\infty}\langle x_n, e_{N_0^2}\rangle &= \left\langle \lim_{n\to\infty} x_n, e_{N_0^2}\right\rangle = \langle f_{N_0}, e_{N_0^2}\rangle \\ &= \frac{1}{N_0}\left\langle \sum_{k=1}^{N_0^2} e_k, e_{N_0^2}\right\rangle = \frac{1}{N_0}\end{aligned}$$

since $\langle e_k, e_{N_0^2}\rangle$ is only one when $k = N_0^2$, and we can pull the limit inside by continuity. But $\lim_{n\to\infty}\langle x_n, e_{N_0^2+1}\rangle = \left\langle \lim_{n\to\infty} x_n, e_{N_0^2+1}\right\rangle = \langle f_{N_0}, e_{N_0^2+1}\rangle = 0$ since $\langle e_k, e_{N_0^2+1}\rangle = 0$ when $k < N_0^2+1$. This is a contradiction since $\langle x_n, e_{N_0^2}\rangle = \langle x_n, e_{N_0^2+1}\rangle$, so f_N, $N \geq 1$ are extreme points of K.

(d) There are no other extreme points in K, since in (c) we saw that $\overline{\{f_N \ : \ N \geq 1\}}^{\tau_w} = \{0\} \cup \{f_N \ : \ N \geq 1\}$. And then by Millman we have that $Ext(K) \subset \overline{\{f_N \ : \ N \geq 1\}}^{\tau_w} = \{0\} \cup \{f_N \ : \ N \geq 1\}$. And $Ext(K)$ denotes all extreme points of K.

Problem 2

Let X and Y be infinite dim Banach spaces.

(a) Let $T \in \mathcal{L}(X, Y)$. We want to show that for a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$ where $x_n \to x$ weakly, as $n \to \infty$ then $Tx_n \to Tx$ weakly as $n \to \infty$. Let $g : Y \to \mathbb{K}$ be a linear functional. Then $g \circ T : X \to \mathbb{K}$ is a functional since T is a continuous map. And then since $x_n \to x$ weakly, as $n \to \infty$ we have by HW 4 Pb 2(a) that $g \circ T(x_n) \to g \circ T(x)$ as $n \to \infty$. But this is the same as $g(Tx_n) \to g(Tx)$ as $n \to \infty$, and then again by HW € Pb 2(a) we have that $Tx_n \to T_x$ weakly, as $n \to \infty$.

(b) Let $T \in \mathcal{K}(X, Y)$. We want to show that for a sequence $(x_n)_{n\geq 1}$ in X if $x_n \to x$ weakly as $n \to \infty$ then $\|Tx_n - Tx\| \to 0$ as $n \to \infty$.
We claim that any subsequence of $(T(x_n))_{n\geq 1}$ has a subsequence which converge to Tx. To prove this claim let $(x_{n_k})_{k\geq 1}$ be a subsequence of $(x_n)_{n\geq 1}$ when by HW 4 2(b) the subsequence is bounded. Since T is compact we have by proposition 8.2 that $(x_{n_k})_{k\geq 1}$ contains a subsequence $(x_{n_{k_t}})_{t\geq 1}$ such that $(T(x_{n_{k_t}}))_{t\geq 1}$ converges. But it has to converge to Tx since norm convergence implies weak convergence and by (a) we have weak convergence.
Assume for contradiction that $(T(x_n))_{n\geq 1}$ does not converge to Tx in the norm. So there exists an $\epsilon > 0$ such for all m there exists $N(m) > m$ such that $\|Tx_n - Tx\| \geq \epsilon$. We now construct a subsequence where we set $n_1 = N(1)$ and then reflexive $n_k = N(n_{k-1})$ for $k = 2, \ldots$. And then this subsequence does not converges to Tx in the norm, but this is a contradiction of our claim. So $\|Tx_n - Tx\| \to 0$ as $n \to \infty$.

(c) Let H be a separable in infinite dimensional Hilbert space. We want to show that if $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$, whenever $(x_n)_{n\geq 1} \subset H$ converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Assume that T is non-compact with the property above. By proposition 8.2 we have that if T is non-compact then there exists a bounded sequence $(x_n)_{n\geq 1}$ such that $(T(x_n)_{n\geq 1})$ has no convergent subsequence. Since $(x_n)_{n\geq 1}$ is bounded we have that $(x_n)_{n\geq 1} \subset \overline{B_H}(0,1)$ by scaling. Then since H is reflexive (H it's a Hilbert space) we have by Theorem 6.3 that $\overline{B_H}(0,1)$ is weakly compact. And since H is reflexive we have by Theorem 5.9 that the weak and weak* topology is the same on H. And because H is separable we have by Theorem 5.13 that $\overline{B_H}(0,1)$ is metrizable in the weak topology. Then since $(x_n)_{n\geq 1} \subset \overline{B_H}(0,1)$ which is compact-metrizable so $(x_n)_{n\geq 1}$ has a convergent subsequence $(x_{n_k})_{k\geq 1}$. But then by property of T we have $(T(x_{n_k}))_{k\geq 1}$ is convergent in the norm which is a contradiction. So T is non-compact.

(d) We want to show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact. Let $(x_n)_{n\geq 1}$ be a sequence in $\ell_2(\mathbb{N})$ which converges weakly. Then by (a) we have that $Tx_n \to Tx$ weakly as $n \to \infty$. We know that in $\ell_1(\mathbb{N})$ weak converges is the same as norm converges. So we have that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. Note that $\ell_2(N)$ is a infinite dimensional Hilbert space, and the by (c) we have hat T is compact.

(e) We want to show that no $T \in \mathcal{K}(X,Y)$ is onto. Assume for contradiction that T is surjective. Then we have by the Open mapping theorem (Theorem 3.15) that T is open. So $T(B_X(0,1))$ is open, and $\overline{T(B_x(0,1))}$ is compact since T is compact. Then $0 \in T(B_X(0,1))$, so there exists a $r > 0$ such that $\overline{B_Y}(0,r) \subset T(B_X(0,1)) \subset \overline{T(B_X(0,1)}$. We have that closed subsets of compact sets are compact. This gives that $\overline{B_Y}(0,r)$ is compact. Since scalar multiplication is continuous and the image of compact set under continuous functions is compact we have that $\overline{B_Y}(0,1)$ is compact. But by the first mandatory assignment problem 3 we have that if the closed unit ball in Y is compact then Y is finite dimensional, but this is a contradiction. So T is not surjective.

(f) Let $H = L_2([0,1], m)$ and let $M \in \mathcal{L}(H,H)$ be the operator given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. We to justify that M is self-adjoint, but not compact. Let $f, g \in H$ and $t \in [0,1]$, so notice that $\bar{t} = t$, then

$$\begin{aligned} \langle Mf, g\rangle &= \int_{[0,1]} tf(t)\overline{g(t)}dm(t) \\ &= \int_{[0,1]} f(t)\overline{tg(t)}dm(t) \\ &= \langle f, Mg\rangle \end{aligned}$$

Thus M is self-adjoint.
Since H is a separable infinite Hilbert space and by example 9.15 we know that M has no eigenvalues. We have by The Spectral Theorem (Theorem 10.1) that M is non-compact, since M is self-adjoint.

Problem 3

(a) We want to justify that T is compact. First we want to show that $K \in L_2([0,1]\times[0,1], m\otimes m)$. We have that K is measurable. So we want to show that $\|K\|_2 < \infty$. Notice $|K(s,t)| \leq 1$. So we have that

$$\begin{aligned}
\|K\|_2^2 &= \int_{[0,1]\times[0,1]} |K(s,t)|^2 d(m\otimes m)(s,t) \\
&\leq \int_{[0,1]}\int_{[0,1]} 1 dm(s)dm(t) \\
&= \int_0^1\int_0^1 1 ds\ dt = 1 < \infty
\end{aligned}$$

And then by proposition 9.12 we have that T is a Hilbert-Schmidt operator, and then by proposition 9.11 T is compact.

(b) We want to show that $T = T^*$. Note that K is symmetric, so $K(s,t) = K(t,s)$. Let $f, g \in [0,1]$, then

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} (Tf)(s)\overline{g(s)}dm(s) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s)
\end{aligned}$$

We then use Fubini to change the integration order. We can do this since

$$\begin{aligned}
\int_{[0,1]}\int_{[0,1]} |K(s,t)f(t)\overline{g(t)}|dm(t)dm(s) &\leq \int_{[0,1]}\int_{[0,1]} |f(t)||\overline{g(t)}|dm(t)dm(s) \\
&= \int_{[0,1]} |\overline{g(t)}| \int_{[0,1]} |f(t)|dm(t)dm(s) \\
&< \infty
\end{aligned}$$

because as noted before $|K(s,t)| \leq 1$, and since by HW 2(a) we have that $L_2([0,1], m) \subsetneq L_1([0,1], m)$, so both $\int_{[0,1]} |\overline{g(s)}|dm(s)$, $\int_{[0,1]} |f(t)|dm(t) < \infty$. So get that

$$\begin{aligned}
\int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s) &= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t)\int_{[0,1]} \overline{K(t,s)g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t)\overline{\int_{[0,1]} K(t,s)g(s)dm(s)}dm(t) \\
&= \langle f, Tg\rangle
\end{aligned}$$

Thus T is self-adjoint.

(c) Let $s \in [0,1]$ and $f \in H$. Then

$$\begin{aligned}(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\ &= \int_{[0,1]} (1-s)tf(t)\mathbb{1}_{[0,s]} + (1-t)sf(t)\mathbb{1}_{[s,1]}dm(t) \\ &= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\ &= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)\end{aligned}$$

Now we want to show using this that Tf is continuous. We have that $f(t) \in L_1([0,1],m)$, since $L_2([0,1],m) \subsetneq L_1([0,1],m)$, so we see that $tf(t),(1-t)f(t) \in L_2([0,1],m)$. Then we note that $\int_{[0,s]} |(1-s)tf(t)|dm(t) \leq \int_{[0,s]} |tf(t)|dm(t) < \infty$, so $(1-s)tf(t) \in L_1([0,1],m)$ and that $(1-s)tf(t)$ is continuous for all fixed $s \in [0,1]$. And that $|(1-s)tf(t)| \leq tf(t) \in L_1([0,1],m)$. So by continuity lemma we have that $\int_{[0,s]}(1-s)tf(t)dm(t)$ is continuous on $[0,1]$.
We see that $\int_{[s,1]} |(1-t)sf(t)|dm(t) \leq \int_{[s,1]} |(1-t)f(t)|dm(t) < \infty$. So $(1-t)sf(t) \in L_1([0,1],m)$. And that $(1-t)sf(t)$ is continuous for all fixed $s \in [0,1]$. Also we have that $|(1-t)sf(t)| \leq (1-t)f(t)$. So again by the continuity lemma we have that $\int_{[s,1]}(1-t)sf(t)dm(t)$ is continuous. Therefor is Tf continuous.
When we integrate over singletons we get zero, so $(Tf)(0) = (Tf)(1) = 0$.

Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$

(a) For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$. We want to justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$. When take the derivatives of g_k we get a polynomial times $e^{-\frac{1}{2}x^2}$. So we have that g_k together with all its derivatives vanishes faster than any polynomial, since $e^{-\frac{1}{2}x^2}$ grows faster than any polynomial. So g_k is a Schwartz function.
Then we want to compute $\mathcal{F}(g_k)$, for $k = 0,1,2,3$. For $k = 0$ we see that by proposition 11.4 we have that

$$\hat{g_0}(\xi) = e^{-\frac{1}{2}\xi^2}$$

For $k = 1$. We notice that $g_1(x) = xg_0(x)$, and $g_k \in \mathscr{S}(\mathbb{R}$, for all integers $k \geq 0$ so specially $g_k \in L_1(\mathbb{R})$. So by Proposition 11.13(c) we have

$$\hat{g_1}(\xi) = \widehat{(xg_0)}(\xi) = i\left(\frac{\partial}{\partial\xi}\hat{g_0}\right)(\xi) = i\frac{\partial}{\partial\xi}e^{-\frac{1}{2}\xi^2} = -ie^{-\frac{1}{2}\xi^2}\xi$$

For $k = 2$. Here we notice that $g_2(x) = xg_1(x)$. And we use proposition 11.13(c) again and get

$$\begin{aligned}\hat{g_2}(\xi) &= \widehat{(xg_1)}(\xi) = i\left(\frac{\partial}{\partial\xi}\hat{g_1}\right)(\xi) = i\frac{\partial}{\partial\xi}(-ie^{-\frac{1}{2}\xi^2}\xi) \\ &= i(i\xi^2 e^{-\frac{1}{2}\xi^2}\xi - ie^{\frac{1}{2}\xi^2}) = i^2(\xi^2-1)e^{-\frac{1}{2}\xi^2}\end{aligned}$$

For k=3. We notice that $g_3(x) = xg_2(x)$ and so by proposition 11.13(c) we have that

$$\begin{aligned}
\hat{g_3}(\xi) &= \widehat{(xg_2)}(\xi) = i\left(\frac{\partial}{\partial\xi}\hat{g_2}\right)(\xi) \\
&= i\left(\frac{\partial}{\partial\xi}\left(i^2\xi^2 e^{-\frac{1}{2}\xi^2}\xi - i^2 e^{\frac{1}{2}\xi^2}\right)\right) \\
&= i\left(2\xi i^2 e^{-\frac{1}{2}\xi^2} - i^2\xi^3 e^{-\frac{1}{2}\xi^2} + i^2 e^{-\frac{1}{2}\xi^2}\right) \\
&= i^3(3\xi - \xi^3)e^{-\frac{1}{2}\xi^2}
\end{aligned}$$

(b) We want to find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F} = i^k h_k$, for $k = 0, 1, 2, 3$. From (a) we observe that

$$\begin{aligned}
\mathcal{F}(g_0) &= g_0 \\
\mathcal{F}(g_1) &= -ig_1 \\
\mathcal{F}(g_2) &= -g_2 + g_0 \\
\mathcal{F}(g_3) &= ig_3 - 3g_1
\end{aligned}$$

So for $k = 0, 3$ we see that $h_0 = g_0$ then $\mathcal{F}(h_0) = h_0$ and that $h_3 = g_1$ since $\mathcal{F}(h_3) = -ig_1 = i^3h_3$. For $k = 1$. Set $h_1 = 3ig_1 + 2g_3$. Remember that $\mathcal{F}$ is linear so we get that $\mathcal{F}(h_1) = \mathcal{F}(3ig_1 + 2g_3) = 3i\mathcal{F}(g_1) + 2\mathcal{F}(g_3) = 3g_i + 2ig_3 - 6g_1 = i(3ig_1 + 2g_3) = ih_1$. For $k = 2$. Set $h_2 = -g_0 + 2g_2$. Then we have that $\mathcal{F}(h_2) = -g_0 + 2(-g_2 + g_0) = g_0 - 2g_1 = -(-g_0 + 2g_2) = i^2h_2$.

(c) We want to show that $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$. Because $f \in \mathscr{S}(\mathbb{R})$ it has a inverse Fourier transform $\mathcal{F}^*(f)$. We first want to show that $\mathcal{F}(f) = -\mathcal{F}^*(f)$. We will do this by using change of variable in $\mathcal{F}(f)$ so we set $u = -x$ then $dx = -du$, and so

$$\begin{aligned}
\mathcal{F}(f) &= \int_{\mathbb{R}} f(\xi)e^{-i\xi x}dm(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(\xi)e^{-i\xi x}dx \\
&= -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(\xi)e^{i\xi u}du = -\int_{\mathbb{R}} f(\xi)e^{i\xi u}dm(u) \\
&= -\mathcal{F}^*(f)
\end{aligned}$$

We then have by corollary 12.12 (iii) that $\mathcal{F}^2(f) = \mathcal{F}(\mathcal{F}(f)) = -\mathcal{F}(\mathcal{F}^*(f)) = -f$. And so $\mathcal{F}^4(f) = \mathcal{F}^2(\mathcal{F}^2(f)) = \mathcal{F}^2(-f) = f$.

(d) We want to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, $\lambda \in \mathbb{C}$ then $\lambda \in \{1, i, -1, -i\}$. So $\mathcal{F}^4(f) = \lambda^4 f$, also by (c) we have that $\mathcal{F}^4(f) = f$, that means that $\lambda^4 = 1$, and so $\lambda \in \{1, i, -1, -i\}$. This together with (b) gives that we eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.

Problem 5

Let $(x_n)_{n\geq 1}$ be a dense subset og $[0,1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} \frac{1}{2^2}\delta_{x_n}$ on $[0,1]$. We want to show that $\operatorname{supp}(\mu) = [0,1]$. Since $(x_n)_{n\geq 1}$ is dense in $[0,1]$ then the closure of $(x_n)_{n\geq 1}$ is the whole $[0,1]$. But then the smallest closed set containing $(x_n)_{n\geq 1}$ are $[0,1]$. Hence the the largest open set in $[0,1]$ with $\mu(N) = 0$ is $N = \emptyset$. Therefore by definition we have that $\operatorname{supp}(\mu) = N^c = [0,1]$.