FunkAn - 2

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Problem 1

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space, H.

a

Note that $\langle f_N, h\rangle = \frac{1}{N}\langle\sum_{n=1}^{N^2} e_n, h\rangle = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, h\rangle$. By Parseval's identity, we have $\sum_{n=1}^{\infty}|\langle e_n, h\rangle|^2 < \infty$, for all $h \in H$. Hence, for all $\varepsilon > 0$ there exists $k \in \mathbb{N}$, such that $\sum_{n=k}^{\infty}|\langle e_n, h\rangle|^2 < \varepsilon$. As

$$\frac{1}{N}\sum_{n=1}^{N^2}|\langle e_n, h\rangle| \leq \frac{1}{N}\sum_{n=1}^{k-1}|\langle e_n, h\rangle| + \frac{1}{N}\sum_{n=k}^{N^2}|\langle e_n, h\rangle| \ (*),$$

and by the Cauchy-Schwartz inequality for sums, we have

$$\frac{1}{N}\sum_{n=k}^{N^2}|\langle e_n, h\rangle| = \frac{1}{N}\sum_{n=k}^{N^2}1\cdot|\langle e_n, h\rangle| \leq \frac{1}{N}\sqrt{N^2-k+1}\sqrt{\sum_{n=k}^{N^2}|\langle e_n, h\rangle|} < \sqrt{\frac{N^2-k+1}{N^2}}\sqrt{\varepsilon} \ (**).$$

Combining $(*)$ and $(**)$, it is easy to see that $\frac{1}{N}\sum_{n=1}^{N^2}|\langle e_n, h\rangle|$ goes to 0 as N goes to ∞. Hence by HW4 and Riesz representation theorem (for Hilbert spaces), we know that $f_N \to 0$ weakly.

By direct computation

$$\begin{aligned}
\|f_N\|^2 &= \langle f_N, f_N\rangle \\
&= \left\langle \frac{1}{N}\sum_{i=1}^{N^2} e_i, \frac{1}{N}\sum_{j=1}^{N^2} e_j \right\rangle \\
&= \frac{1}{N^2}\left\langle \sum_{i=1}^{N^2} e_i, \sum_{j=1}^{N^2} e_j \right\rangle \\
&= \frac{1}{N^2}\sum_{i=1}^{N^2}\sum_{j=1}^{N^2} \langle e_i, e_j\rangle \\
&= \frac{1}{N^2}\sum_{i=1}^{N^2} \langle e_i, e_i\rangle \\
&= \frac{1}{N^2} N^2 = 1.
\end{aligned}$$

b

Consider an element, f, in the convex hull $\mathrm{co}\{f_N : N \in \mathbb{N}\}$. Then $f = \sum_{i=1}^n a_i f_{N_i}$ with positive a_i's and $\sum_{i=1}^n a_i = 1$ and $N_j \in \mathbb{N}$ for all $j \in \mathbb{N}$. Hence $\|f\| = \|\sum_{i=1}^n a_i f_{N_i}\| \leq \sum_{i=1}^n a_i \|f_{N_i}\| = \sum_{i=1}^n a_i = 1$. Therefore, the convex hull is contained in the closed, norm unit ball, and so is the closed convex hull. By the Banach-Alaoglu theorem, the closed, norm unit ball in H^* is weak* compact. By the Riesz representation theorem, H^* is also a Hilbert space, hence the closed, norm unit ball is weakly compact. As Hilbert spaces are isometric isomorphic to its dual, the closed, norm unit ball in H is weakly compact. As weak and norm closures coincides for convex sets, (theorem 5.7), the norm closure of $\mathrm{co}\{f_N : N \in \mathbb{N}\}$ is weakly closed in H. As closed subsets of compact sets are compact, the norm closure of $\mathrm{co}\{f_N : N \in \mathbb{N}\}$ is weakly compact.

As f_N converges weakly to 0, as $N \to \infty$, 0 is the weak limit of elements in $\mathrm{co}\{f_N : N \in \mathbb{N}\}$, and so 0 is in the weak closure of $\mathrm{co}\{f_N : N \in \mathbb{N}\}$. We have just argued that the norm closure and the weak closure of $\mathrm{co}\{f_N : N \in \mathbb{N}\}$ coincides, hence $0 \in K$.

c

Let $N \in \mathbb{N}$, and assume $f_N = ag + (1-a)h$ is a non-trivial convex combination for some $a \in [0,1]$ and $g, h \in K$. Then g and h are limits of convex combinations of f_N's, Hence we have

$$f_N = a \sum_{i=1}^{\infty} b_i f_{N_i} + (1-a) \sum_{j=1}^{\infty} c_j f_{N_j},$$

where $\sum_{i=1}^{\infty} b_i = \sum_{j=1}^{\infty} c_j = 1$, and $b_i, c_j \geq 0$ for all $i, j \in \mathbb{N}$. Hence

$$\frac{1}{N} \sum_{n=1}^{N^2} e_n = a \sum_{i=1}^{\infty} b_i \frac{1}{N_i} \sum_{n=1}^{N_i^2} e_n + (1-a) \sum_{j=1}^{\infty} c_j \frac{1}{N_j} \sum_{n=1}^{N_j^2} e_n.$$

taking inner product with e_{N^2} yields

$$\frac{1}{N} = a \sum_{\{i : N_i^2 \geq N^2\}} \frac{b_i}{N_i} + (1-a) \sum_{\{j : N_j^2 \geq N^2\}} \frac{c_j}{N_j} \ (*).$$

As $\frac{1}{n}(a \sum_{i=1}^{\infty} b_i + (1-a) \sum_{j=1}^{\infty} c_j) = \frac{1}{n}$ for all $n \in \mathbb{N}$, and as the b- and c-sums are increasing, $(*)$ can only hold if $N_i = N_j = N$ for all $i, j \in \mathbb{N}$, hence $f_N = f_{N_i} = f_{N_j}$. Thus f_N is an extreme point for all $N \in \mathbb{N}$.

Now consider a convex combination for 0 of elements in K. Let $0 = ax + (1-a)y$ with $x, y \in K$. As x and y a limits of convex combinations of something ($\{f_N : N \in \mathbb{N}\}$) with

$$\langle f_N, e_n \rangle \geq 0 \text{ for all } n \in \mathbb{N}$$

we, by continuity of inner products, have that $\langle x, e_n \rangle \geq 0$, and $\langle y, e_n \rangle \geq 0$. As $0 = \langle 0, e_n \rangle = \langle x, e_n \rangle + \langle y, e_n \rangle$ for all $n \in \mathbb{N}$ along with our observation implies $0 = \langle x, e_n \rangle = \langle y, e_n \rangle$ for all $n \in \mathbb{N}$. As $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis, this implies that $x = y = 0$, hence 0 is an extreme point of K.

d

There does not. First, a small fact from general topology; Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence, i.e. $x_n \to x$, in a space, in which limits are unique. then $\{x\} \cup (x_n)_{n \in \mathbb{N}}$ is compact.

Proof. Let O be an open covering of $\{x\} \cup (x_n)_{n\in\mathbb{N}}$. Let O_x be an open set containing x. By convergence, there is only finitely many points in the sequence not contained in O_x. For each such point, x_i, choose open sets, O_i, from O, such that $x_i \in O_i$. This forms a finite subcovering of $\{x\} \cup (x_n)_{n\in\mathbb{N}}$. $\square$

As limits in the weak topology are unique, $\{f_N : N \in \mathbb{N}\} \cup \{0\}$ is weakly compact, hence weakly closed. By Milman's theorem, every extreme point of K is contained in $\{f_N : N \in \mathbb{N}\} \cup \{0\}$.

Problem 2

a

By HW4, we know that $x_n \to x$ weakly if and only if $f(x_n) \to f(x)$ for all $f \in X^*$. As T is linear and continuous, $g \circ T \in X^*$ for all $g \in Y^*$, hence $g(Tx_n)$ converges to $g(Tx)$ for all $g \in Y^*$, hence $Tx_n \to Tx$ weakly.

b

By HW4, we know that $(x_n)_{n\in\mathbb{N}}$ is bounded, hence every subsequence is also bounded. By proposition 8.2, every subsequence, x_{n_j} has a further subsequence $x_{n_{j_i}}$ such that $Tx_{n_{j_i}}$ converges in norm. As norm convergence implies weak convergence, each further subsequence has to satisfy that under T it converges to Tx, else it would contradict our assumption of weak convergence. From general metric space theory, we know that if every subsequence $(z_{n_j})_{j\in\mathbb{N}}$ of a sequence, $(z_n)_{n\in\mathbb{N}}$, in a metric, has a further subsequence, $(z_{n_{j_i}})_{i\in\mathbb{N}}$, that converges to the same point z, then $z_n \to z$ as $n \to \infty$. Applying this to $(Tx_n)_{n\in\mathbb{N}}$ shows that $Tx_n \to Tx$ as $n \in \mathbb{N}$ in the metric induced by the norm, in other words, in norm.

c

Assume for a contradiction that T is not compact. Then, by theorem 8.2, $T(\overline{B_H(0,1)})$ is not totally bounded, hence there exists $\delta > 0$, such that $T(\overline{B_H(0,1)})$ cannot be covered by finitely many balls of radius δ. In that spirit, choose $Tx \in T(\overline{B_H(0,1)})$, and for each $n \in \mathbb{N} \setminus \{1\}$, let be Tx_n such that

$$Tx_n \in T(\overline{B_H(0,1)}) \setminus \bigcup_{i=1}^{n-1} B_Y(Tx_{x-1}, \delta).$$

By the lack of total boundedness, this set is always non-empty. It is easy to see that

$$\|Tx_n - Tx_m\| > \delta, \text{ for } n \neq m.$$

Thus, $(x_n)_{n\in\mathbb{N}} \subset B_H(0,1)$ is a sequence in the Hilbert space H. As the weak topology on H^* and the weak-$*$ topology on H^* coincides, the respective topologies coincides on $\overline{B_{H^*}(0,1)}$. As H is seperable, so is H^*, hence $\overline{B_{H^*}(0,1)}$ is by, theorem 5.13 metrizable. As $H \cong H^*$ for all Hilbert spaces $\overline{B_H(0,1)}$ is also metrizable in the weak topology. As H is a Hilbert space, $\overline{B_H(0,1)}$ is by the Banach-Alaoglu weakly compact. Hence $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$. However, as, for $n_j \neq n_i$

$$\delta < \left\|Tx_{n_j} - Tx_{n_i}\right\|,$$

this subsequence is not a norm-convergent sequence under T. Hence we have shown the contrapositive of the desired result.

d

Let $(x_n)_\mathbb{N}$ be a weakly convergent sequence converging to x. Then by problem $2a$, $(Tx_n)_\mathbb{N}$ converges weakly to Tx in $\ell_1(\mathbb{N})$. As sequences in $\ell_1(\mathbb{N})$, by remark 5.3, converges weakly if and only if they converge in norm, $(Tx_n)_\mathbb{N}$ converges in norm to Tx. As $\ell_2(\mathbb{N})$ is a Hilbert space and $\ell_1(\mathbb{N})$ is a Banach space, we have, by problem $2c$, that T is compact.

e

Assume for a contradiction that T is a surjective, compact operator. By the open mapping theorem, T is also an open mapping. By compactness, $T(B_X(0,1))$ has compact closure, and as $T(B_X(0,1))$ is open, it contains the norm closure of $B(0,\varepsilon)$ for some $\varepsilon > 0$. Hence the norm closure of $B(0,\varepsilon)$ is a closed set in the compact set $\overline{T(B_X(0,1))}$, hence it is itself compact. This is a contradiction with problem 3, in the first mandatory assignment.

f

By direct computaion, as t is real-valued

$$\begin{aligned}\langle Mf, g\rangle &= \int_{[0,1]} tf(t)\overline{g(t)}\ dm(t)\\ &= \int_{[0,1]} f(t)\overline{tg(t)}\ dm(t)\\ &= \langle f, Mg\rangle.\end{aligned}$$

By HW6, M has no eigenvalues. As M is self-adjoint, if it was compact, the spectral theorem would provide several eigenvalues. As this is not the case, M cannot be compact

Problem 3

a

As K is continuous, $[0,1]$ is compact and the Lebesgue measure is finite on $[0,1]$, compactness of T follows from theorem 9.6, as T is the kernel operator of K.

b

By direct computation

$$\begin{aligned}
\langle T(f), g\rangle &= \int\int K(s,t)f(t)\ dm(t)\overline{g(s)}\ dm(s) \\
&= \int\int K(s,t)f(t)\overline{g(s)}\ dm(t)\ dm(s) \\
&\overset{(*)}{=} \int K(s,t)f(t)\overline{g(s)}\ dm\otimes m(s,t) \\
&\overset{(*)}{=} \int\int K(s,t)f(t)\overline{g(s)}\ dm(s)\ dm(t) \\
&= \int\int K(s,t)\overline{g(s)}\ dm(s)\ f(t)dm(t) \\
&\overset{(**)}{=} \int\int \overline{K(s,t)g(s)}\ dm(s)\ f(t)dm(t) \\
&= \int\overline{\int K(s,t)g(s)\ dm(s)}\ f(t)dm(t) \\
&= \int f(t)\overline{\int K(s,t)g(s)\ dm(s)}\ dm(t) \\
&= \langle f, T(g)\rangle,
\end{aligned}$$

where $(*)$ is due to Fubini's theorem, and $(**)$ is due to K being real-valued.

c

Let $f \in L_2([0,1], m)$. For every $s \in [0,1]$, by direct computation, we see

$$\begin{aligned}
Tf(s) &= \int_{[0,1]} K(s,t)f(t)\ dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)\ dm(t) + \int_{(s,1]} K(s,t)f(t)\ dm(t) \\
&\overset{(*)}{=} \int_{[0,s]} K(s,t)f(t)\ dm(t) + \int_{\{s\}} K(s,t)f(t)\ dm(t) + \int_{(s,1]} K(s,t)f(t)\ dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)\ dm(t) + \int_{[s,1]} K(s,t)f(t)\ dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)\ dm(t) + \int_{[s,1]} (1-t)sf(t)\ dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)\ dm(t) + s\int_{[s,1]} (1-t)f(t)\ dm(t),
\end{aligned}$$

where $(*)$ is due to the fact that $\{s\}$ is a null-set. At $s = 0$, the first term is an integral over a null-set, hence 0, and the second term is 0 times some number, hence zero. identically, at $s = 1$, the second term is an integral over a null-set, hence 0, and the first term is 0 times some number, hence zero. Therefore $Tf(0) = Tf(1) = 0$. Now, let $(s_n)_{n\in\mathbb{N}} \subseteq [0, 1]$ be a convergent sequence to some $s \in [0, 1]$. By the Cauchy-Shwartz inequality, we get

$$\begin{aligned}\left|\int_{[0,s]} tf(t)\ dm(t) - \int_{[0,s_n]} tf(t)\ dm(t)\right| &= \left|\int_{[\min s, s_n, \max s, s_n]} tf(t)\ dm(t)\right| \\ &\leq \left\|1_{[\min s, s_n, \max s, s_n]}\right\|_2 \|tf(t)\|_2 \\ &= \sqrt{|s - s_n|}\|tf(t)\|_2.\end{aligned}$$

And hence we see that the first function given by an integral is Hölder continuous, hence continuous. Note that $\int_{[s,1]}(1-t)f(t)\ dm(t) = \int_{[0,1]}(1-t)f(t)\ dm(t) - \int_{[0,s)}(1-t)f(t)\ dm(t)$, hence the second function given by an integral is also Hölder continuous. As all other components of Tf is trivially continuous, we conclude that Tf is continuous.

Problem 4

a

By HW7P1a $g_k \in \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$. We claim that, for non-negative Schwartz functions, f, the following statement hold:

If $\int_{\mathbb{R}} |x|^k f(x)\ dm(x) < \infty$, then, by proposition, the Fourier transform of f, $\mathcal{F}(f)$, is k times differentiable, and

$$i^k \frac{d^k}{d\xi^k}\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} x^k e^{-i\xi x} f(x)\ dm(x) \text{ for all } \xi \in \mathbb{R}.$$

We, by proposition 11.4, already know $\mathcal{F}(g_0)(\xi) = g_0(\xi) = e^{-\frac{\xi^2}{2}}$. By our observation, we can compute the rest of the desired Fourier transform by differentiating (and dividing by the appropriate scalar):

$$\begin{aligned}
\mathcal{F}(g_1)(\xi) &= i\frac{d}{d\xi}g_0(\xi) = -i\xi e^{-\frac{\xi^2}{2}} \\
\mathcal{F}(g_2)(\xi) &= -\frac{d^2}{d\xi^2}g_0(\xi) = -(\xi^2-1)e^{-\frac{\xi^2}{2}} = -\xi^2 e^{-\frac{\xi^2}{2}} + e^{-\frac{\xi^2}{2}} \\
\mathcal{F}(g_3)(\xi) &= -i\frac{d^3}{d\xi^3}g_0(\xi) = i\xi(\xi^2-3)e^{-\frac{\xi^2}{2}} = i\xi^3 e^{-\frac{\xi^2}{2}} - i3\xi e^{-\frac{\xi^2}{2}}
\end{aligned}$$

b

By the previous problem, we see that $h_0 := g_0$ works. Again, by the previous problem. We also see that $h_3 := g_1$ works. In the same spirit we see that. By direct computation, we see, for every $\xi \in \mathbb{R}$

$$\begin{aligned}
\mathcal{F}(g_2 - \frac{1}{2}g_0)(\xi) &= -\xi^2 e^{-\frac{\xi^2}{2}} + e^{-\frac{\xi^2}{2}} - \frac{1}{2}e^{-\frac{\xi^2}{2}} = i^2(g_2(\xi) - \frac{1}{2}g_0(\xi)) \\
\mathcal{F}(g_3 - \frac{3}{2}g_1)(\xi) &= i\xi^3 e^{-\frac{\xi^2}{2}} - i3\xi e^{-\frac{\xi^2}{2}} + i\frac{3}{2}\xi e^{-\frac{\xi^2}{2}} = -i\xi^3 e^{-\frac{\xi^2}{2}} - i\frac{3}{2}\xi e^{-\frac{\xi^2}{2}} = i(g_3 - \frac{3}{2}g_1).
\end{aligned}$$

c

Let $\mathcal{F}^*$ denote the inverse Fourier transform. Corollary 12.12 allows us to write $\mathcal{F}(\mathcal{F}^*(f)) = f$ for every Schwartz function, f. This implies that $\mathcal{F}^2(f^\vee) = \mathcal{F}(f)$, where $f^\vee(\alpha) = \int_{\mathbb{R}} f(x)e^{ix\alpha}\, dm(x)$ denotes the inverse fourier transform of f. Computing this yields

$$\mathcal{F}^2(f^\vee)(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\, dm(x) \text{ for all } \xi \in \mathbb{R}.$$

If we compare this to the inverse Fourier transform of f, we see that $f^\vee(-z) = \mathcal{F}^2(f^\vee)(z)$ for all $z \in \mathbb{R}$. So applying the Fourier transform twice yields a sign change in the argument. As the Fourier transform is a automorphism on the Schwartz space $\mathscr{S}(\mathbb{R})$, this implies that $f^\vee(z) = \mathcal{F}^4(f^\vee)(z)$ for all $z \in \mathbb{R}$. Computing the Fourier transform a last time, gives us

$$f(z) = \mathcal{F}^4(f)(z) \text{ for all } z \in \mathbb{R}$$

d

By the last problem, we have that any complex number, λ, satisfying $\mathcal{F}(f) = \lambda f$, must satisfy $\lambda^4 = 1$. The complex numbers satisfying this are exactly $1, -1, i, -i$. As all those are examples of eigenvalues, by problem 4.b, the eigenvalues are exactly those numbers.

Problem 5

From general topology, we know that dense sets intersect every non-empty open set. If that was not the case, then the complement of a disjoint open, non-empty set would contain the dense set, contradicting that the closure of the dense set is the entire space. In other words, every non-empty open set in $[0,1]$ contains at least one $x_i \in (x_n)_{n\in\mathbb{N}}$. As such we have, for every open set, U, that $\frac{1}{2^i} \leq \mu(U)$. Thus the only open set with μ-measure 0, is the empty set. Hence

$$\operatorname{supp}(\mu) = \left(\bigcup_{U \text{ open with } \mu(U)=0} U \right)^c = \emptyset^c = [0,1].$$