Mandatory Assignment 1
Functional Analysis

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Problem 1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces over K, where $K = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $T: X \to Y$ be a linear map. Set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$, for all $x \in X$. We wish to show, that $\|x\|_0$ is a norm on X.

If $\|x\|_0$ is a norm on X, then the following holds

a. $\|u + v\|_0 \leq \|u\|_0 + \|v\|_0$, $u, v \in X$.
b. $\|\alpha u\|_0 = |\alpha|\|u\|_0$, $\alpha \in K, u \in X$.
c. $\|u\|_0 = 0$ if and only if $u = 0$.

First, we check a) (the triangle inequality). For $u, v \in X$ we have

$$\|u + v\|_0 = \|u + v\|_X + \|T(u + v)\|_Y.$$

Since T is linear, we have

$$\|u + v\|_0 = \|u + v\|_X + \|Tu + Tv\|_Y.$$

Since we know, that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed vector spaces, we have

$$\|u + v\|_0 \leq \|u\|_X + \|v\|_X + \|Tu\|_Y + \|Tu\|_Y.$$

If we relocate the joints, we have

$$\|u + v\|_0 \leq \|u\|_X + \|Tu\|_Y + \|v\|_X + \|Tu\|_Y.$$

Thus, we have

$$\|u + v\|_0 \leq \|u\|_0 + \|v\|_0,$$

and the triangle inequality holds.

Now, we check b) (homogeneity). For $u \in X$ we have

$$\|\alpha u\|_0 = \|\alpha u\|_X + \|T(\alpha u)\|_Y.$$

Since T is linear, we have

$$\|\alpha u\|_0 = \|\alpha u\|_X + \|\alpha Tu\|_Y.$$

Since we know, that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed vector spaces, we have

$$\|\alpha u\|_0 = |\alpha|\|u\|_X + |\alpha|\|Tu\|_Y.$$

If we factorize, we have

$$\|\alpha u\|_0 = |\alpha|(\|u\|_X + \|Tu\|_Y).$$

Thus, we have

$$\|\alpha u\|_0 = |\alpha|\|u\|_0,$$

and the homogeneity holds.

At last, we check c) (positivity). For $X \ni u = 0$ we have

$$\|0\|_0 = \|0\|_X + \|T(0)\|_Y.$$

Since T is linear, we have

$$\|0\|_0 = \|0\|_X + \|0\|_Y.$$

Since we know, that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed vector spaces, we have

$$\|0\|_0 = 0 + 0.$$

Thus, we have

$$\|0\|_0 = 0.$$

For the converse, suppose that for $u \in X$, we have

$$\|u\|_0 = 0.$$

Hence,

$$\|u\|_X + \|Tu\|_Y = 0.$$

Since we know, that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed vector spaces, and T is linear, we have

$$u = 0,$$

and the positivity holds.

Now, we wish to show, that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded. Since $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ for all $x \in X$, we have

$$\|\cdot\|_X \leq \|\cdot\|_0.$$

Suppose that T is bounded. Thus, there exists C with

$$\|\cdot\|_0 \leq \mathrm{C}\|\cdot\|_x,$$

and the two norms are equivalent.

Suppose the two norms are equivalent. Then there exists C_1 and C_2 such that

$$C_1\|\cdot\|_X \leq \|\cdot\|_0 \leq \mathrm{C}_2\|\cdot\|_x.$$

If $\|\cdot\|_0 \leq \mathrm{C}_2\|\cdot\|_x$, then T is bounded.

(b) We wish to show that any linear map $T: X \to Y$ is bounded, if X is finite dimensional. Assume that X is finite dimensional, and that $\dim(X) = n$. Then there exists a basis $\{e_1, \dots, e_n\}$ of X such that every element of X is a linear combination of the form

$$x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n,$$

where $\alpha_1, \dots, \alpha_n \in K$.

For each $x \in X$ we have that

$$\begin{aligned}\|Tx\|_Y &= \|T(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n)\|_Y \\ &= \|\alpha_1 Te_1 + \alpha_2 Te_2 + \cdots + \alpha_n Te_n\|_Y \\ &\leq \sum_{k=1}^{n} |\alpha_k| \, \|Te_k\|_Y.\end{aligned}$$

Define M as follows

$$M = \left(\sum_{k=1}^{n} \|Te_k\|^2\right)^{\frac{1}{2}}.$$

Then by the Cauchy-Schwartz inequality we have that

$$\begin{aligned}\|Tx\|_Y &\leq \left(\sum_{k=1}^{n} |\alpha_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \|Te_k\|_Y^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^{n} |\alpha_k|^2\right)^{\frac{1}{2}} \cdot M.\end{aligned}$$

From Theorem 1.6 (lecture notes) we have

$$\|Tx\|_Y \leq M\|x\|_*.$$

Since $\|\cdot\|_X$ and $\|\cdot\|_*$ are equivalent norms, by Definition 1.4 (lecture notes) there exists $0 < C_1 \leq C_2 < \infty$ such that for every $x \in X$ we have

$$C_1\|x\|_X \leq \|x\|_* \leq C_2\|x\|_X.$$

Hence, for every $x \in X$ we have

$$\|Tx\|_Y \leq MC_2\|x\|_X,$$

thus, T is bounded.

(c) Suppose that X is infinite dimensional. We wish to show, that there exists a linear map $T: X \to Y$, which is not bounded.

Let $(e_i)_{i\in\mathbb{N}}$ be an infinite Hamel basis for X. Pick $y \neq 0$ in Y. Define $T\left(\frac{e_i}{\|e_i\|}\right) = i \cdot y$ where $T(X) = 0$ if $x \notin span\ (\{e_i\})$, and extend T linearly. This map is well-defined and linear since $\left\{\frac{e_i}{\|e_i\|}\right\}_{i\in\mathbb{N}}$ is a linearly independent subset of X.

Since

$$\left\{\frac{e_i}{\|e_i\|}\right\}_{i\in\mathbb{N}} \subseteq \{x \in X : \ \|x\| \leq 1\}$$

and

$$\sup_{\{x\in X:\ \|x\|\leq 1\}} \|T(x)\| \geq n\|y\| > 0$$

for each $n \in \mathbb{N}$, T is not bounded.

(d) Suppose again that X is infinite dimensional. We wish to argue that there exists a norm $\|\cdot\|_0$ on X that is not equivalent to the given norm $\|x\|_X$, and which satisfies $\|x\|_X \leq \|x\|_0$, for all $x \in X$.
Define $T: X \to Y$ as above. We have $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ for all $x \in X$, thus
$$\|\cdot\|_X \leq \|\cdot\|_0,$$
for all $x \in X$.
Since T is not bounded there exists no $0 < C < \infty$ such that $\|\cdot\|_0 \leq C\|\cdot\|_X$. Hence the two norms are not equivalent. This means, that the identity from $(X, \|\cdot\|_X)$ to $(X, \|\cdot\|_0)$ is not a homeomorphism. Thus, if they were both Banach spaces, the identity $(X, \|\cdot\|_0)$ to $(X, \|\cdot\|_X)$ would be open, by the open mapping theorem, but since the identity the other way around is not continuous, it is not. Hence, $(X, \|\cdot\|_0)$ is not a Banach space if $(X, \|\cdot\|_X)$ is.

(e) We wish to give an example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ satisfying $\|x\|' \leq \|x\|$ for all $x \in X$, such that $(X, \|\cdot\|)$ is complete, while $(X, \|\cdot\|')$ is not.
If we take $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|_1)$ and $(X, \|\cdot\|') = (\ell_1(\mathbb{N}), \|\cdot\|_\infty)$, where
$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$$
and
$$\|x\|_\infty = \sup\{|x_i| : i \geq 1\}.$$
For all $x \in X$ we have
$$\|x\|_\infty \leq \|x\|_1.$$
The two norms are inequivalent since there exists no $0 < C < \infty$ such that $\|\cdot\|_1 \leq C\|\cdot\|_\infty$.
$(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is a Banach space and $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ is not.

Problem 2 Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $((\ell_p(\mathbb{N}), \|\cdot\|_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, \dots) : a, b \in \mathbb{C}\}.$$

Let $f: M \to \mathbb{C}$ be given by $f(a, b, 0, 0, \dots) = a + b$, for all $a, b \in \mathbb{C}$.

(a) We wish to show that f is bounded on $(M, \|\cdot\|_p)$.
f is bounded if there exists some $C > 0$ such that $\|fx\|_p \leq C\|x\|_p$.
Let $x = (x_1, x_2, 0,0, \dots) \in M$. As $\frac{1}{p} + \frac{1}{\frac{p}{p-1}} = 1$ we get by Hölders inequality that

$$\begin{aligned} |fx| &\leq |x_1| + |x_2| \\ &= \sum_{i=1}^{2} |x_i \cdot 1| \\ &\leq \left(\sum_{i=1}^{2} |x_i|^{\frac{1}{p}}\right)\left(\sum_{i=1}^{2} |1|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^{2} |x_i|^{\frac{1}{p}}\right) \cdot 2^{1-\frac{1}{p}} \\ &= \|x\|_p \cdot 2^{1-\frac{1}{p}}. \end{aligned}$$

Thus, f is bounded on $(M, \|\cdot\|_p)$.

Now, we wish to compute $\|f\|$.
By the above we have for every $1 \leq p < \infty$ that

$$|fx| \leq 2^{1-\frac{1}{p}} \cdot \|x\|_p.$$

Thus, $2^{1-\frac{1}{p}} \in \{C > 0 : |fx| \leq C\|x\|_p\}$, hence

$$\|f\| = \inf\{C > 0 : |fx| \leq C\|x\|_p\} \leq 2^{1-\frac{1}{p}}.$$

Now let $x' = \left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0,0, \dots\right)$ then

$$\|x'\| = \left(\left|\frac{1}{2^{\frac{1}{p}}}\right|^p + \left|\frac{1}{2^{\frac{1}{p}}}\right|^p\right)^{\frac{1}{p}} = \left(\frac{1}{2} + \frac{1}{2}\right)^{\frac{1}{p}} = 1,$$

and since

$$|fx'| = \left|\frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}}\right| = 2\frac{1}{2^{\frac{1}{p}}} = 2^{1-\frac{1}{p}},$$

we have $2^{1-\frac{1}{p}} \in \{|fx| : \|x\|_p = 1\}$. Thus,

$$2^{1-\frac{1}{p}} \leq \sup\{|fx| : \|x\|_p = 1\} = \|f\|.$$

Hence, we can conclude $\|f\| = 2^{1-\frac{1}{p}}$.

(b) We wish to show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.
Let $1 < p < \infty$. Since $f \in M^*$, we know by Corollary 2.6 (lecture notes), that there exists a linear functional $F \in \left(\ell_p(\mathbb{N})\right)^*$, such that $F|_M = f$ and $\|F\| = \|f\|$.
By problem 5 in HW1, we know that if $\frac{1}{p} + \frac{1}{q} = 1$, then we have an isometric isomorphism $\left(\ell_p(\mathbb{N})\right)^* \cong \ell_q(\mathbb{N})$. Hence, we may write $F(x) = \sum_{n=1}^{\infty} x_n y_n$ for $y = (y_n)_{n\geq 1} \in \ell_q(\mathbb{N})$ and $x = (x_n)_{n\geq 1} \in \ell_p(\mathbb{N})$.
By (a) we know that $2^{\frac{1}{q}} = 2^{1-\frac{1}{p}} = \|f\| = \|F\|$, and as F is represented by $y \in \ell_q(\mathbb{N})$, we must also have $\|y\|_q = 2^{\frac{1}{q}}$.
We see that $F|_M(x) = f(x) = x_1 + x_2$ so $y = (1, 1, y_3, y_4, \dots)$. Furthermore we get that

$$\|y\|_q = \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{\frac{1}{q}} = (|1|^q + |1|^q + |y_3|^q + \cdots)^{\frac{1}{q}} = 2^{\frac{1}{q}}.$$

So this forces $y_3, y_4, \dots = 0$ and we may conclude $y = (1, 1, 0, 0, \dots)$, whereas $F(x) = x_1 + x_2$.
Now assume that $F' \in \left(\ell_p(\mathbb{N})\right)^*$ another linear functional, such that $F'|_M = f$ and $\|F'\| = \|f\|$. Then we would get $F'(x) = x_1 + x_2$ by the same argument as above. But this means $F(x) = F'(x)$ which shows that a linear functional extending f and having equal operator norm is unique.

(c) We wish to show that if $p = 1$, then there are infinitely many linear functionals F on $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.
Let $p = 1$ and define $F_i: \ell_1(\mathbb{N}) \to K$ given by $(x_1, x_2, x_3, \dots) \mapsto x_1 + x_2 + x_i$ for $i > 2$. This is clearly a linear functional on $\ell_1(\mathbb{N})$. Furthermore, we see that $F_i|_M(x) = x_1 + x_2 = f(x)$ for $x \in M$, hence an extension of f.

Now since F_i extends f, we must have that $\|F_i\| \geq \|f\| = 2^{1-\frac{1}{1}} = 1$, as supremum is true to inclusions. For the other inequality notice that per definition $\|\cdot\|_1$ we have

$$\begin{aligned}\|F_i\|_1 &= \sup\{|F_i x| : \|x\|_1 = 1\} \\ &= \sup\{|x_1 + x_2 + x_i| : \|x\|_1 = 1\} \\ &\leq \sup\{|x_1| + |x_2| + |x_i| : \|x\|_1 = 1\} \\ &\leq 1.\end{aligned}$$

Thus, we have $\|F_i\| = 1$.
Hence F_i is a linear functional extending f and having equal operator norm, and since we would define F_i for any $i > 2$, we can conclude that there are infinitely many functionals on $\ell_1(\mathbb{N})$ extending f and having equal operator norms.

Problem 3 Let X be an infinite dimensional normed vector space over K, where $K = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $n \geq 1$ be an integer. We wish to show that no linear map $F: X \to K^n$ is injective.
Suppose F is injective. Let $x_1, x_2, \dots, x_{n+1}$ be linearly independent in X.
$f(x_1), f(x_2), \dots, f(x_{n+1})$ linearly dependent in K^n. Thus, there exists $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ not all zero such that

$$\sum_{i=1}^{n+1} \alpha_i F(x_{n+1}) = 0.$$

But since F is linear, we have

$$\sum_{i=1}^{n+1} \alpha_i F(x_{n+1}) = \sum_{i=1}^{n+1} F(\alpha_i x_{n+1}),$$

and $x_1, x_2, \dots, x_{n+1}$ linearly independent in X, we have $\alpha_1, \alpha_2, \dots, \alpha_{n+1} = 0$, which is a contradiction.

(b) Let $n \geq 1$ be an integer and let $f_1, f_2, \dots, f_n \in X^*$. We wish to show that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}.$$

For $n \geq 1$ let $f_1, f_2, \dots, f_n \in X^*$.If we consider the map $F: X \to K^n$ given by

$$F(x) = \big(f_1(x), f_2(x), \dots, f_n(x)\big),$$

then

$$\ker f = \bigcap_{j=1}^{n} \ker f_j.$$

Thus, if $\bigcap_{j=1}^{n} \ker f_j = \{0\}$ we would have that f is an injective linear map from an infinite-dimensional space X to a finite dimensional space K^n. This is a contradiction, hence $\bigcap_{j=1}^{n} \ker f_j \neq \{0\}$.

(c) Let $x_1, x_2, \dots, x_n \in X$. We wish to show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1,2, \dots, n$.

For $1 \leq j \leq n$ let $f_j \in X^*$ be a bounded functional such that $f_j(x_j) = \|x_j\|$ and $\|f_j\| = 1$, which exists by Theorem 2.7 (b) (lecture notes). Then the intersection of kernels

$$\bigcap_{j=1}^{n} \ker(f_j)$$

is a non-trivial subspace of X. Define a linear map $f : X \to K^n$ as in (b). Then we know $\bigcap_{j=1}^{n} \ker f_j \neq \{0\}$.

Pick $y \in \bigcap_{j=1}^{n} \ker f_j$ suck that $\|y\| = 1$ and notice

$$\|y - x_j\| = \|f_j\|\|y - x_j\| \geq |f_j(y - x_j)| = |f_j(y) - f_j(x_j)| = |1 - \|x_j\|| = \|x_j\|.$$

Thus, $\|y - x_j\| \geq \|x_j\|$ for all $j = 1,2, \dots, n$.

(d) We wish to show that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains zero.

(e) We wish to show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

We can show, that S is non-compact, by constructing a sequence with Riesz's lemma, that has no convergent subsequence. Take the sequence of points $(1,0,0, \dots), (0,1,0, ..), (0,0,1, \dots) \dots$ on the unit sphere. This sequence has no convergent subsequence since the distance of any two points is $\sqrt{2}$. Hence, S is non-compact.

Problem 4 Let $L_1([0,1],m)$ and $L_3([0,1],m)$ be the Lebesgue spaces on $[0,1]$. We recall from HW2 that $L_3([0,1],m) \subsetneq L_1([0,1],m)$. For $n \geq 1$, define

$$E_n := \left\{ f \in L_1([0,1],m) : \int_{[0,1]} |f|^3 \, dm \leq n \right\}.$$

(a) Given $n \geq 1$, we wish to show, that the set $E_n \subset L_1([0,1],m)$ is not absorbing.
Let $f \in L_1([0,1],m) \backslash L_3([0,1],m)$ and let $t > 0$. Then

$$\int_{[0,1]} |t^{-1} f|^3 \, dm = t^{-3} \int_{[0,1]} |f|^3 \, dm = \infty.$$

Thus, $t^{-1} f \notin E_n$ for any $t > 0$, hence E_n is not absorbing.

(b) We wish to show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.
Assume for contradiction that there exists some $n \geq 1$ such that E_n does not have empty interior, and let $f_0 \in E_n$. Then there exists an open ball $B(f_o, r)$ around f_0 such that $B(f_0, r) \subseteq E_n$. Let $f \in L_1([0,1],m)$ be arbitrary and define $h := f_0 + \frac{r}{2} \frac{f}{\|f\|}$. Then

$$\|h - f_0\| = \left\| \frac{r}{2} \frac{f}{\|f\|} \right\| = \frac{r}{2},$$

hence $h \in B(f_0, r) \subseteq E_n$. Thus, we have $f_0, h \in E_n \subseteq L_3([0,1],m)$ and since we can write f as a linear combination of elements in $L_3([0,1],m)$, namely $f = \frac{2\|f\|(h - f_0)}{r}$, it follows by the fact that $L_3([0,1],m)$ is a vector space that $f \in L_3([0,1],m)$. Thus, we have shown that $L_3([0,1],m) = L_1([0,1],m)$ which is a contradiction, hence E_n must have empty interior.

(c) We wish to show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$.
Let $(f_k)_{k \geq 1} \subseteq E_n$ be a sequence such that $f_k \to f$ as $k \to \infty$ for some $f \in L_1([0,1],m)$. We wish to show that $f \in E_n$. By corollary 2.32 (Folland) there exists a convergent subsequence $\left(f_{k_q}\right)_{q \geq 1}$ such that $f_{k_q} \to f$ as $q \to \infty$ pointwise almost everywhere. It follows that $\left|f_{k_q}\right|^3 \to |f|^3$ pointwise almost everywhere since $|\cdot|^3$ is continuous. By corollary 2.19 (Folland) it follows that

$$\begin{aligned} \int_{[0,1]} |f|^3 \, dm &\leq \liminf_{q \to \infty} \int_{[0,1]} \left|f_{k_q}\right|^3 dm \\ &\leq \liminf_{q \to \infty} n \\ &= n. \end{aligned}$$

Thus, $f \in E_n$ and E_n is closed in $L_1([0,1],m)$.

(d) By (b) and (c) we have that $\overline{E_n} = E_n$ and E_n has empty interior, hence E_n is nowhere dense and since $L_3([0,1],m) = \bigcup_n E_n$, a countable union of nowhere dense sets, it follows that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

Problem 5 Let H be an infinite dimensional Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

(a) Suppose that $x_n \to x$ in norm, as $n \to \infty$. We wish to find out whether it follows that $\|x_n\| \to \|x\|$, as $n \to \infty$, or not.
By the triangle inequality, we have
$$\|x\| = \|x - x_n + x_n\| \leq \|x - x_n\| + \|x_n\|$$
and
$$\|x_n\| = \|x_n - x + x\| \leq \|x - x_n\| + \|x\|.$$
Thus, we have
$$|\|x\| - \|x_n\|| \leq \|x - x_n\|.$$
Let $\epsilon > 0$. By the fact that $x_n \to x$, there exists $n_\epsilon \in \mathbb{N}$ such that for $n \geq n_\epsilon$ we have
$$|\|x\| - \|x_n\|| \leq \|x - x_n\| \leq \epsilon.$$
Thus, we have
$$\|x_n\| \to \|x\|$$
ss $n \to \infty$.

(b) Suppose that $x_n \to x$ weakly, as $n \to \infty$. We wish to show that it does not follow, that $\|x_n\| \to \|x\|$, as $n \to \infty$.
Consider the sequence $(e_n)_{n\geq 1} \subseteq H$, where $(e_n)_{n\geq 1}$ is an orthonormal basis for H. Since we have $\dim H = \infty$, this basis exists, and we have $\|e_n\| = 1$. We wish to show that $e_n \to 0$ weakly, as $n \to \infty$, that is, we need to show that $f(e_n) \to f(0) = 0$ for all $f \in H^*$, as $n \to \infty$. Let $f \in H^*$. By Riesz's representation theorem there exists $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x \in H$. By Bessels inequality it follows that
$$\sum_{n\in\mathbb{N}} |\langle e_n, y\rangle| < \infty.$$
Hence $\sum_{n\in\mathbb{N}} |\langle y, e_n\rangle|$ converges and for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that
$$|f(e_n)| = |\langle e_n, y\rangle| < \epsilon$$
for all $n \geq N$. Thus $e_n \to 0$ weakly as $n \to \infty$ and $\|0\| = 0 \neq 1 = \|e_n\|$. Hence, it doesn't follow, that $\|x_n\| \to \|x\|$, as $n \to \infty$.

(c) Suppose that $\|x_n\| \leq 1$ for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Using Theorem 5.7 (lecture notes) from the lecture notes, we wish to show that $\|x\| \leq 1$. Let A be the set of $x_n \in H$ such that $\|x\| \leq 1$. Then A is convex and closed. Since $x_n \to x$ weakly, we have that $x \in \bar{A}^{\tau w}$. By Theorem 5.7 (lecture notes) we have that $\bar{A}^{\tau w} = \bar{A}^{\|\cdot\|} = A$. Thus, $x \in A$ and $\|x\| \leq 1$.