Notes on 1D anyons

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Let $\kappa \in [0,\pi]$ and define for each permuation $\sigma = (\sigma_1, ..., \sigma_N)$ the sector $\Sigma_\sigma = \{x_{\sigma_1} < x_{\sigma_2} < ... < x_{\sigma_N}\} \subset \mathbb{R}^N$, and consider the operator

$$H_N = -\sum_{i=1}^{N} \partial_{x_i}^2, \quad \text{on } \mathbb{R}^N \setminus \bigcup_{i<j} \{x_i = x_j\} \tag{0.1}$$

with domian

$$\begin{aligned}\mathcal{D}(H_N) = \Bigg\{ \varphi = \mathrm{e}^{-i\frac{\kappa}{2}\Lambda(x)} f(x) \;\Bigg|\; & f \in \big((\oplus_{\text{sym}})_{\sigma \in S_N} C^\infty(\overline{\Sigma_\sigma})\big) \cap C_0(\mathbb{R}^N), \\ & (\partial_i - \partial_j)\varphi|_{+}^{ij} - (\partial_i - \partial_j)\varphi|_{-}^{ij} = 2c\ \mathrm{e}^{-i\frac{\kappa}{2}\Lambda(x)} f|_0^{ij} \text{ for all } i \neq j \Bigg\}\end{aligned} \tag{0.2}$$

where $\Lambda(x) = \sum_{i<j} \epsilon(x_i - x_j)$ with $\epsilon(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases}$ and $|_{\pm,0}^{ij}$ means the function evaluated at $x_i = x_{j\pm,0}$. Then the following proposition holds

Proposition 1. *H_N is symmetric with corresponding quadratic form*

$$\mathcal{E}(\varphi) = \sum_{i=1}^{N} \int_{\mathbb{R}^N \setminus \bigcup_{i<j}\{x_i = x_j\}} |\partial_{x_i}\varphi(x)|^2 + \frac{2c}{\cos(\kappa/2)} \sum_{i<j} \delta(x_i - x_j)\, |\varphi(x)|^2 \,\mathrm{d}^N x \tag{0.3}$$

Proof. Let $\varphi, \vartheta \in \mathcal{D}(H_N)$, then by partial integration we have

$$\begin{aligned}\langle \vartheta | H_N \varphi \rangle &= -\sum_{i=1}^{N} \int_{\mathbb{R}^N \setminus \bigcup_{i<j}\{x_i = x_j\}} \overline{\vartheta} \partial_{x_i}^2 \varphi \\ &= \sum_{i=1}^{N} \int_{\mathbb{R}^N \setminus \bigcup_{i<j}\{x_i = x_j\}} \overline{\partial_{x_i}\vartheta} \partial_{x_i}\varphi - \int_{\mathbb{R}^{N-1} \setminus \bigcup_{i<j}\{x_i = x_j\}} \sum_{i \neq j} \Big(\overline{\vartheta} \partial_{x_i}\varphi|_{-}^{ij} - \overline{\vartheta}\partial_{x_i}\varphi|_{+}^{ij} \Big) \\ &= \sum_{i=1}^{N} \int_{\mathbb{R}^N \setminus \bigcup_{i<j}\{x_i = x_j\}} \overline{\partial_{x_i}\vartheta} \partial_{x_i}\varphi + \int_{\mathbb{R}^{N-1} \setminus \bigcup_{i<j}\{x_i = x_j\}} \sum_{i<j} \Big(\overline{\vartheta}(\partial_{x_i} - \partial_{x_j})\varphi|_{+}^{ij} - \overline{\vartheta}(\partial_{x_i} - \partial_{x_j})\varphi|_{-}^{ij} \Big) .\end{aligned} \tag{0.4}$$

Let $f, g \in C_0^\infty(\mathbb{R}^N)$ be the functions such that $\varphi = \mathrm{e}^{-i\frac{\kappa}{2}\Lambda} f$ and $\vartheta = \mathrm{e}^{-i\frac{\kappa}{2}\Lambda} g$. Then we have

$$
\begin{aligned}
\langle\vartheta|H_N\varphi\rangle &= \sum_{i=1}^{N}\int_{\mathbb{R}^N\setminus\bigcup_{i<j}\{x_i=x_j\}}\overline{\partial_{x_i}\vartheta}\partial_{x_i}\varphi + \int_{\mathbb{R}^{N-1}\setminus\bigcup_{i<j}\{x_i=x_j\}}\sum_{i<j}\left(\overline{g}(\partial_{x_i}-\partial_{x_j})f|_+^{ij} - \overline{g}(\partial_{x_i}-\partial_{x_j})f|_-^{ij}\right)\\
&= \sum_{i=1}^{N}\int_{\mathbb{R}^N\setminus\bigcup_{i<j}\{x_i=x_j\}}\overline{\partial_{x_i}\vartheta}\partial_{x_i}\varphi + \int_{\mathbb{R}^{N-1}\setminus\bigcup_{i<j}\{x_i=x_j\}} 2\sum_{i<j}\left(\overline{g}(\partial_{x_i}-\partial_{x_j})f|_+^{ij}\right)
\end{aligned}
\tag{0.5}
$$

where the last equality follows from symmetry of f. Notice that by the boundary condtion on $\mathcal{D}(H_N)$ we have

$$
(\partial_i-\partial_j)\varphi|_+^{ij}-(\partial_i-\partial_j)\varphi|_-^{ij} = \mathrm{e}^{-i\frac{\kappa}{2}(-1+S)}(\partial_i-\partial_j)f|_+^{ij}-\mathrm{e}^{-i\frac{\kappa}{2}(1+S)}(\partial_i-\partial_j)f|_-^{ij} = 2c\varphi|_0^{ij} = \mathrm{e}^{-i\frac{\kappa}{2}S}2cf|_0^{ij}
\tag{0.6}
$$

where $S = \Lambda - \epsilon(x_i - x_j)$. By symmetry of f it follows that

$$
\begin{aligned}
\mathrm{e}^{-i\frac{\kappa}{2}(-1+S)}(\partial_i-\partial_j)f|_+^{ij} - \mathrm{e}^{-i\frac{\kappa}{2}(1+S)}(\partial_i-\partial_j)f|_-^{ij} &= \mathrm{e}^{-i\frac{\kappa}{2}(-1+S)}(\partial_i-\partial_j)f|_+^{ij} + \mathrm{e}^{-i\frac{\kappa}{2}(1+S)}(\partial_i-\partial_j)f|_+^{ij}\\
&= \mathrm{e}^{-i\frac{\kappa}{2}S}2\cos(\kappa/2)(\partial_i-\partial_j)f|_+^{ij}\\
&= \mathrm{e}^{-i\frac{\kappa}{2}S}2cf|_0^{ij}.
\end{aligned}
\tag{0.7}
$$

so that

$$
2(\partial_i-\partial_j)f|_+^{ij} = \frac{2c}{\cos(\kappa/2)}f|_0^{ij}.
\tag{0.8}
$$

Hence it follows that

$$
\langle\vartheta|H_N\varphi\rangle = \sum_{i=1}^{N}\int_{\mathbb{R}^N\setminus\bigcup_{i<j}\{x_i=x_j\}}\overline{\partial_{x_i}\vartheta}\partial_{x_i}\varphi(x) + \frac{2c}{\cos(\kappa/2)}\sum_{i<j}\delta(x_i-x_j)\overline{\vartheta(x)}\varphi(x)\,\mathrm{d}^N x.
\tag{0.9}
$$

Now it is clear that starting from $\langle H_N\vartheta|\phi\rangle$, we can by the same steps arrive at (0.9), proving that H_N is symmetric. □

Remark 1. *Since $\mathcal{E}\geq 0$, H_N has a self-adjoint Friedrichs extension, $\tilde{H}_N$, which we regard as the Hamiltonian for the one dimensional anyon gas with statistical parameter, κ, and a zero-range interaction of strength, c.*

Remark 2. *By the quadratic form formulation, and the fact that the phase-factor is not contributing to the value of the quadratic form, it follows that $\tilde{H}_N$ is unitarily equivalent to the Lieb-Liniger Hamiltonian $H_{LL}(N, \frac{c}{\cos(\kappa/2)})$, with N particles and coupling $c/\cos(\kappa/2)$.*