FunAn assignment 1

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Problem 1

(a)

We start of by showing that $||x||_0$ is a norm. First we check the triangle inequality:

$$||x+y||_0 = ||x+y||_X + ||Tx+Ty||_Y \leq$$
$$||x||_X + ||y||_X + ||Tx||_Y + ||Ty||_Y = ||x||_0 + ||y||_0$$

Now we check if $||\lambda x||_X + ||T(\lambda x)||_Y = ||\lambda x||_0 = |\lambda| \cdot ||x||_0$

$$||\lambda x||_0 = ||\lambda x||_X + ||\lambda Tx||_Y = |\lambda| \cdot ||x||_X + |\lambda| \cdot ||Tx||_Y = |\lambda| \cdot ||x||_0$$

So it's a semi norm. Now we just have to check if $||x||_0 = 0 \Rightarrow x = 0$

$$||x||_0 = 0 \Rightarrow ||x||_X = -||Tx||_Y \Rightarrow ||x||_X = ||Tx||_Y = 0 \Rightarrow x = 0$$

So we have now shown that $||x||_0$ is a norm.

Next we show that $||x||_0$ and $||x||_X$ are equivalent iff T is bounded. Assume that the norms are equivalent then we have for $C > 1$:

$$||x||_0 = ||x||_X + ||Tx||_Y \leq C||x||_X \Rightarrow ||Tx||_Y \leq (C-1)||x||_X$$

But this means exactly that T is bounded.

Now assume that T is bounded: $||Tx|| \leq C||x||_X$ so we have:

$$||x||_0 = ||x||_X + ||Tx||_Y \leq ||x||_X + C||x||_X = ||x||_X(1+C) \Rightarrow \frac{1}{1+C}||x||_0 \leq ||x||_X$$

But we also have that $||x||_0 = ||x||_X + ||Tx||_Y$ and since the norms are either 0 or positive we get that $||x||_X \leq ||x||_0$. S we have shown that the norms are equivalent.

(b)

From Theorem 1.6[1]we have that if X is a finite dimensional vector space, then any two norms on X are equivalent. From (a) we then have that T is bounded, when X is finite dimensional.

[1]Every reference is to the FuncAn notes. If a referee to something else i note it.

(c)

we can choose a Hamel basis $(e_n)_{n\in I}$ s.t $||e_n|| = 1$. We then pick a family in Y s.t $||y_n|| \to \infty$ as $n \to \infty$. We then let $T : X \to Y$ be the unique map s.t $T(e_n) = y_n$. Let $N \in \mathbb{N}$ be given. Then we have that $\exists i \in I$ s.t $||y_n|| > N, \forall n \geq i$ and hence:

$$||Te_n|| = ||y_n|| > N = N||e_n||$$

But this means that T isn't bounded.

(d)

Since X is infinite dimensional, we have from (c) a linear map $T : X \to Y$ that isn't bounded. The norms $||\ \cdot\ ||_0$ and $||\ \cdot\ ||_X$ are not equivalent for this T. Furthermore we have that

$$||x||_0 = ||x||_X + ||Tx||_Y \geq ||x||_X$$

If $(X, ||\ \cdot\ ||_X)$ is a Banach space, we have that it's complete. Assume that $(X, ||\ \cdot\ ||_0)$ is also complete, then we have from Problem 1 from HW3, that the norms are equivalent, but this is a contradiction, so $(X, ||\ \cdot\ ||_0)$ isn't complete.

(e)

Let $(X, ||\ \cdot\ ||) = (\ell_i(\mathbb{N}, ||\ \cdot||\|_1)$ and $(X, ||\ \cdot\ ||') = (\ell_i(\mathbb{N}, ||\ \cdot||\|_\infty)$. From an2 we know that $(\ell_i(\mathbb{N}, ||\cdot||\|_1)$ is complete and from HW2 we have that $||\cdot||_1 \leq ||\cdot||_\infty$. We have to find a sequence of sequences in ℓ_1 that is Cauchy with $||\ \cdot\ ||_\infty$ but converges to something not in ℓ_1. We look at $(x_n)^i$ given by

$$x_n^i = \begin{cases} 1/n & n \leq i \\ 0 & n > i \end{cases}$$

Each x_n is in ℓ_1 Furthermore we have that it's Cauchy since for $j > i$ we get: $||(x_n)^k - (x_n)^i||_\infty = \frac{1}{i+1}$ and for any $\epsilon > 0$ we can pick $i > \frac{1-\epsilon}{\epsilon}$ thus $||(x_n)^k - (x_n)^i||_\infty = \frac{1}{i+1} < \epsilon$. But we have that $\lim_{i\to\infty}(x_n)^i = \frac{1}{n} \notin \ell_i$, hence $(\ell_1(\mathbb{N}, ||\ \cdot\ ||_\infty)$ is not complete.

Problem 2

(a)

We will show this for $p = 1$ and $p > 1$. Let $p = 1$, then by the triangle inequality we get:

$$|f(a, b, 0, 0, \ldots)| = |a + b| \leq |a| + |b| = ||(a, b, 0, 0, \ldots)||_1$$

So $||f|| \leq 1$ and since $|f(1, 1, 0, 0, \ldots)| = 2 = |1| + |1| = ||(1, 1, 0, 0, \ldots)||_1 \Rightarrow ||f|| = 1$ for $p = 1$.
Now we assume $p > 1$. We start of by noting that $\varphi : x \to |x|^p$ is convex since $\frac{d^2}{dx^x}\varphi(x) = p(p-1)x^{p-2} \geq 0$. We can then use Jensen's inequality (Schiling Thm 13.13):

$$\frac{1}{2^p}\varphi(a + b) = |\frac{a + b}{2}|^p = |\frac{1}{2}a + \frac{1}{2}b|^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p = \frac{1}{2}(|a|^p + |b|^p)$$

But this means that $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$. We now take the pth root and get:

$$|f(a,b,0,0,\dots)| = |a+b| \leq 2^{\frac{p-1}{p}}(|a|^p+|b|^p)^{\frac{1}{p}} = 2^{\frac{p-1}{p}}||(a,b,0,0,\dots)||_p$$

So we have $||f|| \leq 2^{\frac{p-1}{p}}$ and we also have

$$|f(1,1,0,0,\dots)| = 2 = 2^{\frac{p-1}{p}} \cdot 2^{\frac{1}{p}} = 2^{\frac{p-1}{p}}(|1|^p+|1|^p)^{\frac{1}{p}} = 2^{\frac{p-1}{p}}||(1,1,0,0,\dots)||_p$$

So we have that $||f|| = 2^{\frac{p-1}{p}}$

(b)

We know that a map F that extends f and with $||F|| = ||f||$ exists because of Corollary 2.6. We just have to show that it is unique. Assume that there exists two different extensions of f with this property; F_α, F_β. From HW1 Problem 5 we have that $(\ell_p)^*$ is isometrically isomorphic to ℓ_q where $1/q + 1/p = 1$. The isometry is given as $T : \ell_q \to (\ell_p)^*$, with $T(x) = g_x$, $g_x(y) = \sum_{n=1}^{\infty} x_n y_n$, for $y = (y_n)_{n\geq} \in \ell_p$ and $x = (x)_{n\geq 1} \in \ell_q$. Now let x, x' be the elements in ℓ_q that correspond to $F_\alpha, F_\beta \in (\ell_p)^*$. Since the two spaces are isometric isomorphic we have that:

$$||f|| = 2^{\frac{p-1}{p}} = ||F_\alpha|| = ||F_\beta|| = ||x||_q = ||x'||_q$$

We have that F_α, F_β and f are equal on M, so we let $(a,b,0,0,\dots)$ and by using the isometry we get:

$$a+b = F_\alpha(a,b,0,0,\dots) = T(x)(a,b,0,0,\dots) = g_x(a,b,0,0,\dots) = x_1 a + x_2 b$$
$$a+b = F_\beta(a,b,0,0,\dots) = T(x')(a,b,0,0,\dots) = g_{x'}(a,b,0,0,\dots) = x'_1 a + x'_2 b$$

Where $x_1 = g_x(1,0,0,\dots) = F(1,0,0,\dots) = f(1,0,0,\dots) = 1$ and the same argument for x_2 but with the first entry 0 and the second with a 1, this also holds for x'_1 and x'_2. But this means that $x_1 = x_2 = x'_1 = x'_2 = 1$. Furthermore we have that norm of x in ℓ_q is:

$$||x||_q = (1^q + 1^1 + \sum_{n=3}^{\infty} |x_n|^q)^{\frac{1}{q}} \geq (1^q + 1^q + 0) = 2^{\frac{1}{q}} = 2^{\frac{p-1}{p}}$$

But we have from 2 (a) that $||x|| = 2^{\frac{p-1}{p}}$, so we have $x_i = 0$ for $i \geq 3$. We also see by the same calculation and argument that all $x'_i = 0$ for $i \geq 3$; so we have that $x = x' \Rightarrow F_\alpha = F_\beta$, so F is unique.

(c)

We now have to show that there exists infinitely many linear functionals F on $\ell_1(\mathbb{N})$ extending f and with $||F|| = ||f||$. We construct infinitely many of these F. Let $(x_n)_{n\geq 1} \in \ell_1$, we then define $F_i = \sum_{n=1}^{i}$, $\forall i \geq 2$. The norm is given by:

$$||F_i(x)|| = |\sum_{n=1}^{i} x_n| \leq \sum_{n=1}^{\infty} |x_n| = ||x||_1 \Rightarrow ||Fi|| \leq 1$$

Given $\alpha_j \in \ell_1$ where $\alpha_j = (\alpha_1, \alpha_2, \ldots, \alpha_j, 0\ldots)$, with $\alpha_n = 1 \forall n \in \mathbb{N}$, we get

$$||F_i(\alpha_j)|| = |\sum_{n=1}^{i}| = \sum_{n=1}^{i}|1| + \sum_{n=j+1}^{\infty}|0| = ||\alpha_j||_1 \Rightarrow ||F_i|| \geq 1$$

So we get that $||F_i|| = 1 = ||f||$ for all i, and we also see that $F_i(a, b, 0, 0\ldots) = a + b = f(a, b, 0, 0, \ldots)$; so each F_i is a extension of f with the same norm.

Problem 3

(a)

We will use Lemma. 2.7 from Henrik Schlichtkrull's notes from AdVec. Let B be a basis for X, then we have that $\text{Span}(B) = X$. The Lemma then says that $T_{\text{Span}(B)} = T_X$ is injective iff T_B is injective and $T(B)$ is a linear independent set. But since $\#B > n$, we have that $T(B)$ can't be linear independent; so no linear map from X to $\mathbb{K}^n$ can be injective.

(b)

We follow the hint a consider $F : X \to \mathbb{K}^n$ given by:

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

We start of by showing, that F is a linear map:

$$\begin{aligned} F(\alpha x + y) &= (f_1(\alpha x + y), f_2(\alpha x + y), \ldots, f_n(\alpha x + y)) \\ &= (\alpha f_1(x) + f_1(y), f\alpha_2(x) + f_2(y), \ldots, f\alpha_n(x) + f_y) \\ &= \alpha F(x) + F(y) \end{aligned}$$

Since it's a linear map we have from (a), that it can't be injective. But this means:

$$\{0\} \neq \ker(F) = \bigcap_{j=1}^{n}(f_j)$$

since the kernel of F is exactly the kernel of $\cap_{j=1}^{n} f_j$, since each coordinate of F needs to be zero for it to be zero.

(c)

Let $y' \in \ker(F)$, where F is as in (b). We pick our y as $y = \frac{y'}{||y'||}$, so $||y|| = 1$. If $x_i = 0$ then the inequality is trivial. So we assume that $x_i \neq 0, \forall x_i$, and use theorem 2.7 (b) For each x_i we then get a f_i s.t $||f_i|| = 1$ and $f_i(x_i) = ||x_i||$. We then get:

$$||y - x_j|| \geq ||f_j(y - x_j)|| = ||f_j(y) - f_j(x_j)|| = || - f_j(x_j)|| = ||x_j||$$

Where the inequality follows because $||f_j x|| \leq ||x||$ for all f_j, since each f_j is bounded by $C = 1$.

(d)

Assume that we have a cover of closed balls which don't contain 0, and let $x_1, x_2, \dots, x_n \in X$ be the centers of these balls. From (c) we then have functionals f_i corresponding to each x_i. We now pick $0 \neq y \in \ker(F) = \bigcap_{j=1}^{n}(f_j)$, with $||y|| = 1$ (we can do this because of (c)). Without lose of generality we assume that $y \in \overline{B(x_j, r)}$, then we have $||y - x_j|| \geq ||x_j||$. But then we have that this $\overline{B(x_j, r)}$ have $r \geq ||x_j||$, and therefore $0 \in \overline{B(x_j, r)}$, but this is a contradiction with that none of the balls contains 0, and therefore such a cover doesn't exists.

(e)

Assume for contradiction that S is compact, therefore we can cover it with open balls, with radius r s.t 0 isn't in any of these balls, where the center of each balls is a point in S. Since S is compact we can take a finite subcover of open balls, where 0 isn't in any of these ball. We can then take the closure of these balls, where 0 still isn't in any of them. This is a contradiction with (d), hence S is non-compact.

Furthermore, since the unit-sphere is a closed subset of the closed unit-ball, and it's non-compact we must have that the closed unit-ball is non-compact, since every closed subset of compact space is compact.

Problem 4

(a)

The set E_n is convex because for $f, g \in E_n$ we have:

$$||\alpha f + (1-\alpha)g||^3 \leq (|\alpha|||f|| + (|1-\alpha|)||h||)^3 \leq (|\alpha|\sqrt[3]{n} + \sqrt[3]{n} - |\alpha|\sqrt[3]{n})^3 = n$$

but the set isn't absorbing because if $f \in L_1([0,1], m)$ but $f \notin L_3([0,1], m)$ then $||f||_3 \not< \infty \Rightarrow \int_{[0,1]} |f|^3 dm \not< \infty$. For any giving constant t^{-1} we have then have that

$$\int_{[0,1]} |t^{-1} f|^3 dm = t^{-3} \int_{[0,1]} |f^3| dm \not< \infty$$

But this means that E_n isn't absorbing.

(b)

Assume that E_n don't have an empty interior. Let $f \in E_n$ be given then there exists a ball around f i.e

$$B(f, r) = \{g \in L_1([0,1], m) : ||f - g||_1 < r\} \subseteq E_n$$

Since $|-f| = |f|$ we get that $B(-f, r) \in E_n$ but this implies that $B(0, r) \in E_n$, since E_n is convex. Let $g \in L_1([0,1]), m)$ then $g = ch$ for $h \in B(0, r)$ and $c \in \mathbb{R}$ since the ball is an absorbing set, and therefore we have that $g \in E_n$ since $||g||_3^3 = |c|^3 ||h||_3^3$. Since g was an arbitrary function in $L_1([0,1], m$ we get that $L_1([0,1], m) \subseteq L_3([0,1], m)$, but this is in contradiction with HW2 Problem 2, so we get that $\text{Int}(E_n) = \emptyset$

(c)

Let $(f_k)_{k\geq 1} \in E_n$ and assume that $f_k \to f$ in $L_1([0,1], m)$; i.e $||f_k - f||_1 \to 0$. So we have that $|f_n| \to |f|$ and thus $|f_n|^3 \to |f|^3$, from Corollary 13.8 Schilling we have that there exist a subsequence $|f_{n_j}|^3$ that converges almost everywhere to $|f|^3$, so by Fatou's lemma(9.11 Schilling) we have that:

$$\int_{[0,1]} |f|^3 dm = \int_{[0,1]} \liminf |f_k|^3 dm \leq \liminf \int_{[0,1]} |f_k|^3 dm \leq n$$

But this means that E_n contains all it's limits point so it's closed.

(d)

Since we have that the E_n is closed we have that $\bar{E}_n = E_n$, but this means that $\text{Int}(\bar{E}_n) = \emptyset$, but from Definition 3.12 (i) this means that E_n is nowhere dense. But we have that $L_3([0,1], m) = \bigcup_{n\geq}^{\infty} E_n$ so by Definition 3.12 (ii) we have that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$

Problem 5

(a)

From prop 5.21 from Folland we get that $\langle x_n, x_n \rangle \to \langle x, x \rangle$, since $x_n \to x$, but this also means that $||x_n|| = \sqrt{\langle x_n, x_n \rangle} \to \sqrt{\langle x, x \rangle} = ||x||$

(b)

We find a counterexample. We pick $(e_n)_{n\geq 1}$ as an orthonormal countable basis for H. We use the HW4.2a, and look at the $f \in H^*$. Let $f \in H^*$, then by HW2.1 we have $\exists! y \in H : f(e_n) = \langle y, e_n \rangle$, for each e_n in our basis. By Bessle's inequality (5.26 in Folland)[2] we then have that $\Sigma_n^{\infty} |\langle y, e_n \rangle|^2 \leq ||y||^2$, but this means that $|f(e_n)|^2 \to 0$ for $n \to \infty$ for $f \in H^*$, so e_n converges weakly to 0. But we have that $||e_n|| = 1 \not\to 0 = ||x||$, for $n \to \infty$

(c)

If $x_n \to x = 0$ the result is trivial. So assume $x \neq 0$ and $x_n \neq 0$, then by theorem 2.7 (b) we have that there $\exists f : ||f|| = 1$ and $f(x) = ||x|| \leq 1$, for each x. Since $x_n \to x$ weakly we have that $||x|| = |f(x)| = \lim_{n\to\infty} |f(x_n)|$. But we have that $|f(x_n)| \leq ||f|| \cdot ||x_n|| \leq 1$ for each x_n. But this means that we have $||x|| \leq 1$

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[2] we can also use the property from 5.27 b from Folland of an orthonormal Hilbert space basis