FunkAn Assignment 1

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December 2020

Problem 1

Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

a)

Remember to write $x, y \in X$, $\lambda \in \mathbb{K}$...

Note $||0||_0 = 0$.

Show that $||\cdot||_0$ is a norm on X.
(1) If $||x||_0 = 0 \Rightarrow ||x||_x + ||Tx||_Y = 0 \Rightarrow ||x||_X = 0$ and $||Tx|| = 0 \Rightarrow x = 0$ ✓
(2) $||\lambda x||_0 = ||\lambda x||_X + ||T(\lambda x)||_Y = |\lambda| ||x||_X + ||\lambda T(x)||_Y = |\lambda|(||x||_X + ||T(x)||_Y) = |\lambda| ||x||_0$ ✓
(3) $||x+y||_0 = ||x+y||_X + ||T(x+y)||_Y = ||x+y||_X + ||Tx+Ty||_Y \leq ||x||_X + ||Tx||_Y + ||y||_X + ||Ty||_Y = ||x||_0 + ||y||_0$ Thus we conclude $||\cdot||_0$ is a norm. ✓
Show next that the two norms$||\cdot||_X$ and $||\cdot||_0$ are equivalent if and only if T is bounded.
"$\Rightarrow$": If $||\cdot||_X$ and $||\cdot||_0$ are equivalent $C_1||x||_X \leq ||x||_0 = ||x||_X + ||Tx||_Y \leq C_2||x||_X$ therefore it follows $||Tx||_Y \leq C_2||x||_X - ||x||_X = ||x||_X(C_2 - 1)$. Which means $||T|| \leq C_2 - 1 < \infty$, thus T is bounded. ✓
"$\Leftarrow$": If T bounded $||Tx||_Y \leq ||T|| \cdot ||x||_X$. Inserting this equation in the next, one gets.

$$||x||_0 = ||x||_X + ||Tx||_Y \leq ||x||_X + ||T|| \cdot ||x||_X = ||x||_X(1 + ||T||) \Rightarrow \frac{1}{1+||T||}||x||_0 \leq ||x||_X$$ ✓

We also know that $||x||_0 = ||x||_X + ||Tx||_Y \Rightarrow ||x||_X \leq ||x||_0$ thus the norms are equivalent.

b)

Show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.
X is finite dimensional $\Rightarrow$ all norms are equivalent by Theorem 1.6 in the notes $\Rightarrow ||\cdot||_X, ||\cdot||_0$ are equivalent. Thus by (a) T must be bounded ✓

c)

Suppose that X is infinite dimensional. Show that there exists a linear map $T : X \to Y$, which is not bounded (= not continuous). [Hint: Take a Hamel basis for X(see below).]

what is T?

Let $(e'_i)_{i\in I}$ be a Hamel basis for X. One can chose the family $(e_i)_{i\in I}$ where $e_i = \frac{e'_i}{||e'_i||_X}$ by linearity of T, this family is again a Hamel basis where each element has norm 1. Now chose $y' \in Y$ where $y' \neq 0$ (Y is nonzero) and let $y = \frac{y'}{||y'||_Y}$ ($y \in Y$ as Y v.s.)
Define:

I is just some set, so why is $\mathbb{N} \subseteq I$?

$$T(e_i) = \begin{cases} i \cdot y & i \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

This T is unbounded as for any positive integer k chose e_i where $i = k + 1$. Then $||T(e_i)||_Y = ||y \cdot i||_Y = k + 1 \geq k = k||e_i||_X$. (✓)

d)

Suppose again that X is infinite dimensional. Argue that there exists a norm $||\cdot||_0$ on X, which is not equivalent to the given norm $||\cdot||_X$, and which satisfies $||x||_X \leq ||x||_0$,for all $x \in X$. Conclude that $(X, ||\cdot||_0)$ is not complete if $(X, ||\cdot||_X)$ is a Banach space.

Note that $||\cdot||_0$ is dependent on the choice of T!

Let $||\cdot||_0$ be the norm from (a). By (a) we also have $||x||_0 = ||x||_X + ||Tx||_Y \Rightarrow ||x||_X \leq ||x||_0$ $\forall x \in X$.
If the norms were equivalent one would have that $||x||_0 \leq ||x||_X \cdot C_1 \Rightarrow ||x||_X + ||Tx||_Y \leq ||x||_X \cdot C_1 \Rightarrow$ $||Tx||_Y \leq ||x||_X(C_1 - 1)$. But by (c) there exist a linear map $T : X \to Y$ which is not bounded. Thus for this T: $||Tx||_Y \not\leq ||x||_X(C_1 - 1)$. Therefore this norm is not equivalent with the $||\cdot||_X$ norm.
By HW3 Problem 1 we know: "If the norms are not equivalent X cannot be complete w.r.t both norms". Thus if $(X, ||\cdot||_X)$ Banach, $(X, ||\cdot||_0)$ cannot be complete

e)

Give an example of a vector space X equipped with two inequivalent norms $||\cdot||$ and $||\cdot||'$ satisfying $||x||' \leq ||x||$, for all $x \in X$ such that $(X, ||\cdot||)$ is complete, while $(X, ||\cdot||')$ is not.

My example is $(X, ||\cdot||) = (l_1(\mathbb{N}), ||\cdot||_1)$ and $(X, ||\cdot||') = (l_1(\mathbb{N}), ||\cdot||_\infty)$.

We know that $(l_1(\mathbb{N}), ||\cdot||_1)$ is complete (Analysis 2) and that $||\cdot||_\infty \leq ||\cdot||_1$ (if i have to give a reference: TA sessions). HW2Prb2
Further we know $||\cdot||_1 \not\leq C_1 \cdot ||\cdot||_\infty$ as for any C_1 we can construct a sequence in $l_1(\mathbb{N})$ such that $||(x_n)_{n\in\mathbb{N}}||_1 > C_1 \cdot ||(x_n)_{n\in\mathbb{N}}||_\infty$ (take for example the sequence of sequences i construct in a couple lines, for any C_1 there exists i big enough so $||(x_n)^i_{n\in\mathbb{N}}||_1 > C_1 \cdot ||(x_n)^i_{n\in\mathbb{N}}||_\infty$), therefore the norms are inequivalent.

Where does it converge?

To show that $(l_1(\mathbb{N}), ||\cdot||_\infty)$ is not complete i need to find a sequence of sequences in $l_1(\mathbb{N})$ that is Cauchy w.r.t $||\cdot||_\infty$ that "converges" to a sequence not in $l_1(\mathbb{N})$ (thus diverges in $l_1(\mathbb{N})$).
Let each term of the sequence $(x_n)^i_{n\in N}$ ($i \in \mathbb{N}$ is just an index not an exponent) be given by:

$$x^i_n = \begin{cases} \frac{1}{n} & n \leq i \\ 0 & n > i \end{cases}$$

in $||\cdot||_\infty$

Each $(x_n)^i_{n\in N}$ is a sequence in $l_1(\mathbb{N})$ as it has finite support and each element is finite (thus the absolute sum is finite). Furthermore the sequence is Cauchy as for integers $k > i$, $||(x_n)^k_{n\in N} - (x_n)^i_{n\in N}||_\infty = \frac{1}{i+1}$ thus given any $\epsilon > 0$ take $i > \frac{1-\epsilon}{\epsilon}$ then $||(x_n)^k_{n\in N} - (x_n)^i_{n\in N}||_\infty = \frac{1}{i+1} < \epsilon$ for all $k > i$.
But sadly $\lim_{i\to\infty}(x_n)^i_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$ which is not in $l_1(\mathbb{N})$ as $\sum_{n=1}^\infty |\frac{1}{n}| = \infty$ therefore we conclude that $(l_1(\mathbb{N}), ||\cdot||_\infty)$ is not complete.

Problem 2

Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(l_p(\mathbb{N}), ||\cdot||_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, ...) : a, b \in \mathbb{C}\}.$$

let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, ...) = a + b$, for all $a, b \in \mathbb{C}$

a)

Show that f is bounded on $(M, ||\cdot||_p)$ and compute $||f||$. (Answer depends on p.)
For $p = 1$:
Using the triangle inequality, which holds for all norms:

$$|f(a, b, 0, 0, ...)| = |a + b| \leq |a| + |b| = ||(a, b, 0, 0, ...)||_1$$

Thus $||f|| \leq 1$ and because $|f(1, 1, 0, ...)| = 2 = |1| + |1| = ||(1, 1, 0, ...)||_1$ then $||f|| = 1$ for $p = 1$.
Now assume $p > 1$:
We firstly note that $(x)^p$ is convex on the set $x \in \mathbb{R}^+$ and integer $p \geq 2$. This is shown with the double derivative test (from Matintro) $\frac{d^2}{dx^2}(x)^p = p \cdot (p-1)x^{p-2} \geq 0$. Thus $|x|^p$ is convex on $\mathbb{R}$.
By using Jensen's inequality (Thm 13.13 Schiling) one has

$$\frac{1}{2^p}|a+b|^p = \left|\frac{a+b}{2}\right|^p = \left|\frac{a}{2} + \frac{b}{2}\right|^p = \left|\frac{1}{2}a + \frac{1}{2}b\right|^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p = \frac{1}{2}(|a|^p + |b|^p) \Rightarrow |a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$$

By taking the pth root we get $|f(a,b,0,0,....)| = |a+b| \leq 2^{\frac{p-1}{p}}(|a|^p+|b|^p)^{\frac{1}{p}} = 2^{\frac{p-1}{p}}||(a,b,0,0,....)||_p$ Thus we conclude that $||f|| \leq 2^{\frac{p-1}{p}}$ and by noting that

$$|f(1,1,0,0,...)| = 1+1 = 2 = 2^{\frac{p-1}{p}} \cdot 2^{\frac{1}{p}} = 2^{\frac{p-1}{p}}(|1|^p+|1|^p)^{\frac{1}{p}} = 2^{\frac{p-1}{p}}||(1,1,0,0,...)||_p$$

we conclude $||f|| \geq 2^{\frac{p-1}{p}}$ and finally we get $||f|| = 2^{\frac{p-1}{p}}$ ✓

b)

Show that if $1 < p < \infty$, then there is a unique linear functional F on $l_p(\mathbb{N})$ extending f and satisfying $||F|| = ||f||$.
Existence:
As $f \in M^*$ and $(l_p(\mathbb{N}), ||\cdot||_p)$ is a normed vector space over $\mathbb{C}$ and M is a subspace of X by Corollary 2.6 in the lecture notes we know that there exists $F \in (l_p)^*$ such that F is an extension of f and $||F|| = ||f||$.
Uniqueness:
Assume there exists two different extensions of f, namely F, F'. By problem 5 week 1 we know that $(l_p)^*$ is isometrically isomorphic to (l_q) (where q satisfies $\frac{1}{q}+\frac{1}{p} = 1$) with the following isometry: $T: l_q \to (l_p)^*$ where $T(x) = f_x$ and $f_x(y) = \sum_{n=1}^{\infty} x_n y_n$ for $y = (y_n)_{n\geq 1} \in l_p$ and $x = (x_n)_{n\geq 1} \in l_q$. Let x, x' be the corresponding elements of F, F' in l_q. Because of the isometry we know $||f|| = 2^{\frac{p-1}{p}} = ||F|| = ||F'|| = ||x||_q = ||x'||_q$. As F, F' and f are equal on M let $(a,b,0,...) \in M$, using the isometry on x, x' we get

$$a+b = F(a,b,0,...) = (T(x))(a,b,0,...) = f_x(a,b,0,...) = x_1 a + x_2 b + \sum_{n=3}^{\infty} x_n \cdot 0$$

$$a+b = F'(a,b,0,...) = (T(x'))(a,b,0,...) = f_{x'}(a,b,0,...) = x_1' a + x_2' b + \sum_{n=3}^{\infty} x_n' \cdot 0$$

Thus we know that x and x' both start with two 1s. The norm of x is given by: $||x||_q = (1^q + 1^q + \sum_{n=3}^{\infty}|x_n|^q)^{\frac{1}{q}} \geq (1^q+1^q+0)^{\frac{1}{q}} = 2^{\frac{1}{q}} = 2^{\frac{p-1}{p}}$. But as we said before $||x|| = 2^{\frac{p-1}{p}}$ and this is only possible if all the remaining terms of x are equal to 0. By the exact same argument (the norm of x' is given by. $||x'||_q = (1^q + 1^q + \sum_{n=3}^{\infty}|x_n'|^q)^{\frac{1}{q}} \geq (1^q+1^q+0)^{\frac{1}{q}} = 2^{\frac{1}{q}} = 2^{\frac{p-1}{p}}$. But as we said before $||x'|| = 2^{\frac{p-1}{p}}$ and this is only possible of all the remaining terms of x' are equal to 0) we can conclude that $x = x'$ and thus $F = F'$, this shows uniqueness. ✓

c)

Show that if $p = 1$, then there are infinitely many linear functionals F on $l_1(N)$ extending f and satisfying $||F|| = ||f||$.
Let $x = (x_n)_{n\in\mathbb{N}} \in l_1(\mathbb{N})$, Define $F_i(x) = \sum_{n=1}^{i} x_n$ for all positive integers $i > 2$. We find the norm: $|F_i(x)| = |\sum_{n=1}^{i} x_n| \leq \sum_{n=1}^{\infty}|x_n| = ||x||_1$, thus $||F_i|| \leq 1$. Given the element α_i of $l_1(\mathbb{N})$ given by $(a_1, a_2, ..., a_i, 0, 0, ...)$ (where $a_n = 1$ $\forall n \in \mathbb{N}$) we see that $|F_i(\alpha_i)| = |\sum_{n=1}^{i} 1| = \sum_{n=1}^{i}|1| + \sum_{n=i+1}^{\infty}|0| = \sum_{n=1}^{i}|a_n| + \sum_{n=i+1}^{\infty}|a_n| = ||\alpha_i||_1$ thus showing $||F_i|| \geq 1$. Therefore we conclude that $||F_i|| = 1$ for all i. We also note that $F_i(a,b,0,...) = a+b$ for all i. Therefore each F_i is an extension of f on $l_1(\mathbb{N})$ that satisfies $||F_i|| = ||f|| = 1$, furthermore there are infinitely many of them. ✓

observe that F extends f.

Problem 3

Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

a)

Let $n \geq 1$ be an integer. Show that no linear map $F: X \to \mathbb{K}^n$ is injective.
Assume per contradiction that an such an injective map exists. This map would map bijectively onto a basis of a subspace of $\mathbb{K}^n$ (the subspace $Im(T)$), let this basis be (e_i) (consisting of j elements

where $0 < j \leq k$), further let the unique preimage elements of (e_i) be (x_i) $(Tx_i = e_i)$. The span of (x_i) will then be mapped by T in the following way:

$$T(\sum_{i=1}^{j} \alpha_i x_i) = \sum_{i=1}^{j} \alpha_i T(x_i) = \sum_{i=1}^{j} \alpha_i e_i$$

(Where we've used the linearity of T and $\alpha_i \in \mathbb{K}$). Thus we see that the span of (x_i) maps bijectively into $Im(T)$ but because X is infinite dimensional there exists an element $y \in X$ that is not in the span of the (x_i)s. As all elements in $Im(T)$ are mapped to by the span of the (x_i), y can only be mapped to an element already mapped to by the span of (x_i). Thus T cannot be injective.

b)

Let $n \geq 1$ be an integer and let $f_1, f_2, ..., f_n \in X^*$. Show that

$$\bigcap_{j=1}^{n} ker(f_j) \neq \{0\}$$

Consider $F(x) = (f_1(x), ..., f_n(x))$ this map is a linear map from X to $\mathbb{K}^n$ thus it cannot be injective (by (a)) and therefore the kernel cannot be only 0 as $\exists x_j, y_j \in X, x_j \neq y_j$ such that $F(x_j) = k = F(y_j)$ as F is linear we get $F(x_j - y_j) = 0$. This element is in the kernel of all f_j too, thus we conclude $\bigcap_{j=1}^{n} ker(f_j) \neq \{0\}$.

c)

Let $x_1, x_2, ..., x_n \in X$. Show that there exists $y \in X$ such that $||y|| = 1$ and$||y - x_j|| \geq ||x_j||$ for all $j = 1, 2, ...n$.

Using Thm 2.7(b) in the notes if $0 \neq x_i \in X, \exists f_i \in X^*$ s.t $||f_i|| = 1$ and $||f_i(x_i)|| = ||x_i||$. Consider the element (given by (b)) in the kernel of all the f_i that is non zero. We can pick it to be with norm $||y|| = 1$ as any if y' is in the kernel of all f_i then $y = \frac{y'}{||y'||}$ is too. Then for all x_j:

$$||y - x_j|| \geq ||f_j(y - x_j)|| = ||f_j(y) - f_j(x_j)|| = || - f_j(x_j)|| = ||x_j||$$

Thus showing what we wanted.

d)

Show that one cannot cover the unit sphere $S = \{x \in X : ||x|| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.
Assume that there exists such a cover. let $x_1, ..., x_n$ be the centers of the balls and let f_i be the corresponding functionals from (c). Let y be the element in the kernel of all the f_i also given by (b) and (c). As $||y|| = 1$ we have that $||y||$ must lie inside one of the balls. WLOG assume y is inside the ball centered around x_y. By (c) we have that $||y - x_y|| \geq ||x_y||$. But this means that the radius of the ball centered at x_y must be greater than $||x_y||$ and thus it must contain 0. This is a contradiction and therefore such a cover does not exist.

e)

Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

Firstly i note that the result for (d) also holds for open balls. Just exchange in the proof of (d) with "open" instead of "closed" and strict inequalities instead of weak.
Assume S is compact. Let an open cover be the family of open balls of a radius strictly less than 1 around each point $x \in S$. As S is compact then there exists a finite subcover for this cover. But as no ball contains 0 then by (d) we arrive at at contradiction. Thus we conclude S is not compact and as S is a closed subset of the closed unit ball and we know: "A closed subset of a compact space is compact". By contraposing that statement we get that as S is a closed subset of the unit ball and it is not compact then the closed unit ball cannot be compact either.

Problem 4

Let $L_1([0,1],m)$ and $L_3([0,1],m)$ be the Lebesgue spaces on $[0,1]$. Recall from HW2 that $L_3([0,1],m) \subsetneq L_1([0,1],m)$. For $n \geq 1$, define

$$E_n := \left\{ f \in L_1([0,1],m) : \int_{[0,1]} |f|^3 dm \leq n \right\}$$

a)

Given $n \geq 1$, is the set $E_n \subset L_1([0,1],m)$ absorbing? Justify.

I will show that it is convex for later but not absorbing.
Let $f, g \in E_n$ and $0 \leq \alpha \leq 1$

$$\int_{[0,1]} |\alpha f + (1-\alpha)g| dm \leq \alpha \int_{[0,1]} |f| dm + (1-\alpha) \int_{[0,1]} |g| dm < \infty$$

$$||\alpha f + (1-\alpha)g||_3^3 \leq (\alpha||f||_3 + (1-\alpha)||g||_3)^3 \leq (\sqrt[3]{n}(\alpha + 1 - \alpha))^3 = \sqrt[3]{n}^3 = n$$

Therefore $\alpha f + (1-\alpha)g \in E_n$ and thus the set is convex.
Let $f \in L_1([0,1],m)$ but $f \notin L_3([0,1],m)$. Then $||f||_3 \not< \infty \Rightarrow \int_{[0,1]} |f|^3 dm \not< \infty$. Thus given any positive constant t^{-1} we have $\int_{[0,1]} |t^{-1} f|^3 dm = t^{-3} \int_{[0,1]} |f|^3 dm \not< \infty$. Therefore there are functions in $L_1([0,1],m])$ that cannot be multiplied by a constant to "absorb" them into E_n thus E_n is not absorbing.

b)

Show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.
Suppose E_n didn't have empty interior, then there exists an open ball in E_n around an element $f \in E_n$, $B_r(f) = \{g \in L_1([0,1],m) \quad | \quad ||f-g||_1 < r\}$
As $||-g|| = ||g||$ there exists an open ball around $-f$, $B_r(-f) \subset E_n$. But as shown in (a) E_n is convex, using convexity we deduce that there exists an open ball $B_r(0)$ that is also in E_n. But we know (3.3 lecture notes) open/closed balls around 0 are absorbing, but this contradicts (a), as E_n is not absorbing then any subset in E_n cannot be absorbing either. Thus we conclude that E_n must have empty interior

c)

Show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$.
Let f_n be a sequence in E_n that converges to f w.r.t the $1-$norm, $||f_n - f||_1 \to 0$, then we also know that $|f_n| \to |f|$ and further $|f_n|^3 \to |f|^3$ (still w.r.t. the $1-$norm). By corollary 13.8 in Schilling there exists a subsequence $|f_{n_j}|^3$ that converges almost everywhere to $|f|^3$ Thus by Fatou's lemma (9.11 Schilling):

$$\int_{[0,1]} |f|^3 dm = \int_{[0,1]} \liminf_{n\to\infty} |f_{n_j}|^3 dm \leq \liminf_{n\to\infty} \int_{[0,1]} |f_{n_j}|^3 dm \leq n$$

Hence we have shown that any convergent (in $L_1([0,1],m)$) sequence in E_n converges to an element of E_n thus E_n is closed in $L_1([0,1],m)$.

d)

Conclude from (b) and (c) that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

As E_n is closed with empty interior (from (c) and (b) respectively) it follows that the interior of the closure of E_n is empty which means that E_n is nowhere dense.
Note that $L_3([0,1],m) = \cup_{n\in\mathbb{N}} E_n$, thus $L_3([0,1],m)$ is a countable union of nowhere dense sets and thus, by definition, it is of first category in $L_1([0,1],m)$.

Problem 5

Let H be an infinite dimensional separable Hilbert space with associated norm $||\cdot||$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

a)

Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $||x_n|| \to ||x||$, as $n \to \infty$? Give a proof or a counterexample.
By proposition 5.21 in Folland $\langle x_n, x_n\rangle \to \langle x, x\rangle$ but $||x_n|| = \sqrt{\langle x_n, x_n\rangle} \to \sqrt{\langle x, x\rangle} = ||x||$. Thus the assertion $||x_n|| \to ||x||$ follows.

b)

Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $||x_n|| \to ||x||$, as $n \to \infty$? Give a proof or a counterexample.

Pick $(e_n)_{n\geq 1}$ as a countable orthonormal basis for H, and let $f \in H^*$ then by the Riesz representation theorem (Theorem 5.25 Folland) there exists an unique $y \in H$ s.t. $f(e_n) = \langle e_n, y\rangle$. By Thm 5.26 Folland, $\sum_{n\in\mathbb{N}} |\langle e_n, y\rangle|^2 = \sum_{n\in\mathbb{N}} |\langle y, e_n\rangle|^2 \leq ||y||^2$ but this implies that $|f(e_n)|^2 \to 0$ for $n \to \infty$. As this holds for all $f \in H^*$, (e_n) converges weakly to 0. But $||e_n|| \to 1$ which is not 0 thus we have given a counterexample

c)

Suppose that $||x_n|| \leq 1$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Is it true that $||x|| \leq 1$? Give a proof or a counterexample.

If $x_n \to 0$ then $||0|| \leq 1$. Now suppose $x \neq 0$. By Theorem 2.7 (b) in the notes there exist $f \in H^*$ such that $||f|| = 1$ and $f(x) = ||x||$. As $x_n \to x$ weakly we have that (by problem 2 HMW4) $f(x_n) \to f(x)$. Then we have $|f(x_n)| \leq ||f|| \cdot ||x_n|| \leq 1$ for all n thus also for $\lim_{n\to\infty} |f(x_n)| = |f(x)| = ||x|| \leq 1$ as $[0, 1]$ is closed so any sequence will converge inside of it.