Assignment 1, Functional Analysis

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Problem 1. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be non-zero normed v.spaces over* $\mathbb{K}$.

(a) $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ is a norm because it is the sum of two norms, so all the axioms follow immediately. If $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent then there exists a constant $C > 0$ such that $\|\cdot\|_0 \leq C\|\cdot\|_X$, thus $\|Tx\|_Y \leq C\|x\|_X$ for every $x \in X$, so T is bounded. Conversely, we automatically have $\|\cdot\|_X \leq \|\cdot\|_0$, and if T is bounded there exists $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$, so $\|\cdot\|_0 \leq (1 + C)\|\cdot\|_X$. We conclude that the two norms are equivalent.

(b) Any linear map $T\colon X \longrightarrow Y$ is bounded if X is finite n-dimensional, because $C = \max\{C_i \mid 1 \leq i \leq n\}$ satisfies $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$, where each C_i is any constant such that $\|Te_i\|_Y \leq C_i\|e_i\|_X$, and $\{e_1, \ldots, e_n\}$ a basis for X.

(c) Consider a basis $\{e_i\}_{i\in I}$ for X, which we may assume each element of to have unit norm. Since X is infinite dimensional, we may assume $\mathbb{N} \subseteq I$. Let $y \in Y \setminus \{0\}$, which exists by hypothesis, and consider the family $(y_i)_{i\in I}$ in Y consisting of $y_i = ny$, if $i = n \in \mathbb{N}$, and $y_i = 0$ otherwise. Then there exists a (unique) linear map satisfying $Te_i = y_i$ for each $i \in I$. We have that $\|T\| \geq \sup\{\|Te_i\| \mid i \in I\} = \infty$.

(d) Since X is infinite dimensional, by part (c) there exists a linear map $T\colon X \longrightarrow Y$ which is not bounded. Then, by part (a), the corresponding norm $\|\cdot\|_0$ is not equivalent to $\|\cdot\|_X$, and satisfies $\|\cdot\|_X \leq \|\cdot\|_0$. If $(X, \|\cdot\|_X)$ is complete, using problem 1 in homework 3, we conclude that $(X, \|\cdot\|_0)$ is not complete.

(e) We know that $(l_1(\mathbb{N}), \|\cdot\|_1)$ is complete, that $\|\cdot\|_2 \leq \|\cdot\|_1$, and that, in fact, $l_1(\mathbb{N}) \subsetneq l_2(\mathbb{N})$. We claim that $(l_1(\mathbb{N}), \|\cdot\|_2)$ is not complete (in particular the two norms are not equivalent); this will provide the desired example. Indeed, for each $k \geq 1$, consider the following sequence in $l_1(\mathbb{N})$:

$$y_k(n) := \begin{cases} \frac{1}{n} & 1 \leq n \leq k \\ 0 & \text{else.} \end{cases}$$

The $\|\cdot\|_2$-norm of the difference of y_k and $y = (1/n)_{n\geq 1}$ is the square root of the tail of the series $\sum_{n\geq 1} 1/n^2$, so it converges to 0. In other words, $(y_k)_{k\geq 1}$ converges to y in $\|\cdot\|_2$, and in particular it is a Cauchy sequence in $(l_1(\mathbb{N}), \|\cdot\|_2)$. However, the limit point $y \notin l_1(\mathbb{N})$. We conclude that $(l_1(\mathbb{N}), \|\cdot\|_2)$ is not complete.

Problem 2. *Consider the subspace* $\{(a,b,0,0,0,\dots)|a,b\in\mathbb{C}\}$ *of* $(l_p(\mathbb{N}),\|\cdot\|_p)$ *over* $\mathbb{C}$*, and* f *the linear functional on it giving the sum of the first two terms.*

(a) Using the bound $\|\cdot\|_r \leq n^{\frac{1}{r}-\frac{1}{p}}\|\cdot\|_p$ (see proof below) on $\mathbb{K}^n$ for n finite and $0<r\leq p$ (in our case, $r=1$ and $n=2$), we get

$$\|f\| = \sup\{|a+b| \ : \ (|a|^p+|b|^p)^{\frac{1}{p}}=1\} \leq 2^{1-\frac{1}{p}},$$

because $|a+b|\leq|a|+|b|$. We can already say that f is bounded on $(M,\|\cdot\|_p)$. We now show that $\|f\|=2^{1-\frac{1}{p}}$ by showing that $|a+b|$ attains this value for appropiate $a,b\in\mathbb{C}$. It certainly attains it on $a=b=\frac{1}{2}2^{1-\frac{1}{p}}$, so it only remains to check that the p-norm in $\mathbb{C}^2$ is 1:

$$|a|^p+|b|^p = 2\frac{2^{p-1}}{2^p}=1.$$

Proof of bound: The case $r=p$ is trivial. Suppose $0<r<p$, and let $(x_1,\dots,x_n)\in\mathbb{K}^n$. The inequality is then obtained by taking the r^{-1}th power of the following, were we apply Hölder's inequality with $p/r>1$:

$$\sum_{1\leq i\leq n}|x_i|^r = \sum_{1\leq i\leq n}|x_i|^r\cdot 1 \leq \left(\sum_{1\leq i\leq n}(|x_i|^r)^{\frac{p}{r}}\right)^{\frac{r}{p}} n^{1-\frac{r}{p}}.$$

(b) Let $1<p<\infty$ and suppose there exists $F\in l_p(\mathbb{N})^*$ an extension of f satisfying $\|F\|=\|f\|$. We will prove that there is a unique such extension and the proof will also show that it exists. (The existence will be very easy, so we won't use Hahn-Banach. Also note that we are asked to prove that there exists a unique such linear functional, but it will automatically be bounded by the condition on the norm.) Recall the following isometric isomorphism from homework 1:

$$l_p(\mathbb{N})^* \xrightarrow{\simeq} l_q(\mathbb{N}), \qquad g\longmapsto (g(e_n))_{n\geq1},$$

where $e_n=(\delta^n_k)_{k\geq1}$ and $p^{-1}+q^{-1}=1$. Applying it to F we get $\|(Fe_n)_{n\geq1}\|_q = \|F\|=\|f\|=2^{1-\frac{1}{p}}$, by assumption and because the isomorphism is isometric. But let us look at

$$\|(Fe_n)_{n\geq1}\|_q = \left(\sum_{n\geq1}|Fe_n|^q\right)^{\frac{1}{q}} = \left(2+\sum_{n\geq3}|Fe_n|^q\right)^{1-\frac{1}{p}},$$

where we have used that F extends f, so $Fe_1=1=Fe_2$. Since the real number above must equal $2^{1-\frac{1}{p}}$, we deduce that it must be $Fe_n=0$ for all $n\geq3$. We obtain that the following (bounded) linear functional, arising from $(1,1,0,0,\dots)$ via the isomorphism, is the unique linear extension of f to $l_p(\mathbb{N})$ with norm $\|f\|$:

$$F\colon l_p(\mathbb{N})\longrightarrow\mathbb{C}, \qquad (x_n)_{n\geq1}\longmapsto x_1+x_2.$$

(c) Let $p=1$. Recall the following isometric isomorphism from homework 1:

$$l_1(\mathbb{N})^* \xrightarrow{\simeq} l_\infty(\mathbb{N}), \qquad g\longmapsto (g(e_n))_{n\geq1}.$$

There are infinitely many linear functionals on $l_1(\mathbb{N})$ with norm $\|f\| = 1$ and extending f. For example, for each $n \geq 3$, the following:

$$F_n\colon l_1(\mathbb{N}) \longrightarrow \mathbb{C}, \qquad (x_m)_{m\geq 1} \longmapsto x_1 + \ldots + x_n.$$

Via the isometric isomorphism, it arises from the sequence $(1, 1, \ldots, 1, 0, 0, \ldots)$ in wich every term after the nth position is zero, so it is (bounded) linear, and with norm $\|(1, \ldots, 1, 0 \ldots)\|_\infty = 1$, by isometry.

Problem 3. *Let X be an infinite dimensional normed vector space over $\mathbb{K}$.*

(a) If a linear map $F\colon X \longrightarrow \mathbb{K}^n$ were injective, it would then follow that $\dim X \leq \dim \mathbb{K}^n = n \in \mathbb{N}$, contradicting that X is infinite dimensional.

(b) Consider the map $F\colon X \longrightarrow \mathbb{K}^n$ defined by $F(x) = (f_1(x), \ldots, f_2(x))$, which is linear because each f_j is linear and because of the considered vector space structure on $\mathbb{K}^n$. By part (a), we conclude that there exists $0 \neq x \in X$ such that $f_j(x) = 0$ for each $1 \leq j \leq n$, so the intersection of their kernels is non-zero.

(c) If $x_j = 0$ then any $\|y\| = 1$ works, so we may now assume that all $x_j \neq 0$. By theorem 2.7(b), for each $1 \leq j \leq n$ there exists $f_j \in X^*$ such that $\|f_j\| = 1$ and $f(x_j) = \|x_j\|$. By part (b), there exists $0 \neq y \in X$ such that $f_j(y) = 0$ for every $1 \leq j \leq n$, and we may assume y has unit norm, by scaling. We have

$$\|x_j\| = f_j(x_j - y) \leq \|f_j\|\|x_j - y\| = \|x_j - y\| \qquad \text{for all } 1 \leq j \leq n,$$

so $y \in X$ is an element as desired.

(d) Let $x_1, \ldots, x_n \in X$ and consider open balls centered around them $B_j := B(x_j, r_j)$ such that $0 \notin \overline{B}_j$, i.e., $0 < r_j < \|x_j\|$. By part (c), there exists $y \in S$ such that $\|y - x_j\| \geq \|x_j\| > r_j$, that is $y \notin \overline{B}_j$, for every $1 \leq j \leq n$.

(e) The open cover $\{B(x, \frac{1}{2})\}_{x \in S}$ of S cannot be reduced to a finite subcover by part (d). We deduce that the closed unit ball in X is not compact because closed subsets of compact ones are compact, but S is a closed subset of the closed unit ball which is not compact.

Problem 4. *Consider the Lebesgue spaces $L_3([0,1], m) \subsetneq L_1([0,1], m)$.*

(a) There exists $f \in L_1([0,1], m) \setminus L_3([0,1], m)$, hence there doesn't exist $t > 0$ such that $\|tf\|_3 < \infty$, i.e., E_n is not absorbing for any $n \geq 1$.

(b) Let $n \geq 1$. We need to show that there is no open ball w.r.t. $\|\cdot\|_1$ centered at $0 \in E_n$ which is fully contained in E_n, and it follows that the same is true at any other point of E_n. Let $\epsilon > 0$ and consider $B_{\|\cdot\|_1}(0, \epsilon)$. Again, let $f \in L_1([0,1], m) \setminus L_3([0,1], m)$. Since $\|f\|_1 < \infty$, pick $t > 0$ such that $\|tf\|_1 < \epsilon$. Then $tf \in B_{\|\cdot\|_1}(0, \epsilon)$, but $tf \notin E_n$ because E_n is not absorbing. Finally, at any other point $g \in E_n$ consider $g - tf$, which is in $B_{\|\cdot\|_1}(g, \epsilon)$, but $g - tf \notin E_n$ because $\|g - tf\|_3 \geq |\|g\|_3 - \|tf\|_3| = \infty$ because $tf \notin L_3([0,1], m)$ and $\|g\|_3 \leq n$.

(c) To show that E_n is closed in $L_1([0,1], m)$, consider an arbitrary sequence $(f_k)_{k\geq 1}$ converging to some $f \in L_1([0,1], m)$ in $\|\cdot\|_1$. We want to show that f

is also in E_n. Since every convergent sequence in $\|\cdot\|_1$ admits a subsequence that converges pointwise almost everywhere, we may assume that our original sequence does. Thus we have that the sequence of positive measurable functions $(|f_k|^3)_{k\geq 1}$ converges pointwise a.e. to $|f|^3$. By Fatou's lemma we have

$$\int_{[0,1]} |f|^3 dm \leq \liminf_{k\to\infty} \int_{[0,1]} |f_k|^3 dm.$$

The right hand side is $\leq n$ because each f_k is in E_n. We conclude that $f \in E_n$.

(d) Clearly $L_3([0,1],m)$ is the union of the E_n for all $n \geq 1$, and each E_n is nowhere dense in $L_1([0,1],m)$ because $\mathrm{Int}(\bar{E}_n) = \mathrm{Int}(E_n) = \emptyset$, by parts (c) and (b) respectively. In other words, $L_3([0,1],m)$ is of the first category in $L_1([0,1],m)$.

Problem 5. *Let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$, $(x_n)_{n\geq 1}$ a sequence in H, and $x \in H$.*

(a) If $\|x_n - x\|$ converges to 0, then also does $|\|x_n\| - \|x\|| \leq \|x_n - x\|$, i.e. $\|x_n\|$ converges to $\|x\|$.

(b) We give a counterexample. Recall that H being separable Hilbert space is equivalent to it having a countable orthonormal basis, so we can consider $(e_n)_{n\geq 1}$, an orthonormal basis. We will show that $(e_n)_{n\geq 1}$ converges weakly to 0; however $\|e_n\| = 1$ doesn't converge to 0. We need to show that, for any $r > 0$ and any $f_1, \ldots, f_l \in H^*$, the sequence $(e_n)_{n\geq 1}$ is eventually in

$$B_H(0, f_1, \ldots, f_l, r) = \{y \in H \mid |f_i(y)| < r,\ 1 \leq i \leq l\}.$$

By the Riesz representation theorem, for each $1 \leq i \leq l$, there exists $y_i \in H$ such that $f_i = \langle \cdot, y_i \rangle$. Write $y_i = \sum_{k\geq 1} \lambda_{i,k} e_k$ as a finite sum with coefficients in $\mathbb{K}$. Let $N = \max\{k \mid \lambda_{i,k} \neq 0,\ 1 \leq i \leq l\}$. Then, for all $n > N$ we have $f_i(e_n) = \langle e_n, y_i \rangle = 0$ because the basis is orthonormal. We have shown that $(e_n)_{n\geq 1}$ is eventually in any given open set of the neighborhood base of 0, i.e., it converges weakly to it.

(c) We show that if $\|x_n\| \leq 1$ for all $n \geq 1$ and x_n converges weakly to x, then $\|x\| \leq 1$. Let $\epsilon > 0$ and consider the linear functional $\langle \cdot, x \rangle$ on H, which is bounded by the Cauchy-Scwharz inequality. By assumption, $(x_n)_{n\geq 1}$ is eventually in

$$B_H(x, \langle \cdot, x \rangle, \epsilon) = \{y \in H \mid |\langle y - x, x \rangle| < \epsilon\}.$$

That is, there exists $N \geq 1$ such that for all $n \geq N$ we have $|\langle x_n, x \rangle - \|x\|^2| < \epsilon$. By the reverse triangle inequality, $(|\langle x_n, x \rangle|)_{n\geq 1}$ converges to $\|x\|^2$, but also, by the Cauchy-Scwharz inequality $|\langle x_n, x \rangle| \leq \|x_n\|\|x\| \leq \|x\|$; for the second inequality we have used the hypothesis $\|x_n\| \leq 1$. Thus it must be $\|x\|^2 \leq \|x\|$, from which we deduce that $\|x\| \leq 1$.