CoCo - Exam 2021

Johannes Agerskov

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Question 1

Consider the following languages over the alphabet $\Sigma = \{a, b, c\}$:

$$
\begin{aligned}
L_1 &= \{w \in \Sigma^* : w \text{ contains an odd number of occurrences of the letter } a\} \\
L_2 &= \{a^m b^n c^{m+n} : m, n \in \mathbb{N}\} \\
L_3 &= \{a^m c^{m+n} b^n : m, n \in \mathbb{N}\} \\
L_4 &= \left\{a^m c^{m^2 n^2} b^n : m, n \in \mathbb{N}\right\}
\end{aligned}
$$

Part 1.1

We show that L_1 is regular, and it thus follows that it also is context-free by Corollary 2.32 in M. Sipser.

Proof. The DFA in Figure 1 recognizes L_1, it then follows from definition 1.16 that L_1 is regular. □

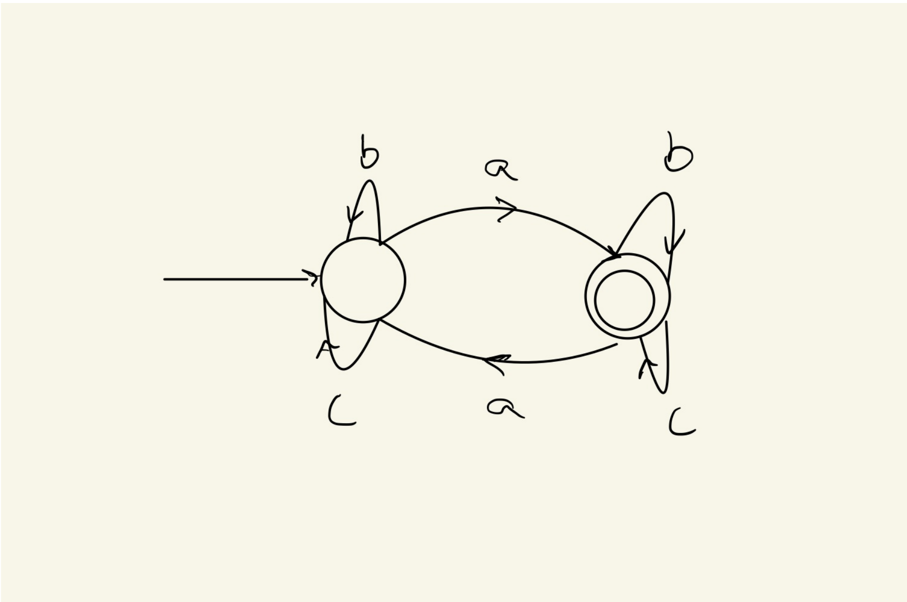

Figure 1: DFA that reckognizes L_1

Part 1.2

We show that L_2 is not regular, but it is context-free.

Proof. That is L_2 is not regular can be seen by the following proof by contradiction: Assume L_2 is regular, and let p denote the pumping length as given by the pumping lemma for regular languages Theorem 1.70. Consider now the string $w = a^p b^p c^{2p} \in L_2$, which clearly satisfies $|w| > p$. Thus by the pumping lemma we can split $w = xyz$ with $|xy| \leq p$ and $|y| > 0$. Clearly we then have $xy = a^k$ for some $0 < k \leq p$, and therefore $y = a^m$ for some $0 < m \leq p$. But then clearly $xy^i z = a^{p+(i-1)m} b^p c^{2p}$ which for $i \neq 1$ is not in L_2, contradicting the pumping lemma. Thus we conclude that L_2 is not regular.
That L_2 is context-free can be seen by the fact that it is generated by the CFG

$$\begin{aligned} S &\to aAc, \\ A &\to aAc \mid bBc, \\ B &\to bBc \mid \epsilon. \end{aligned}$$

Notice that it is not clear wether $\mathbb{N}$ contains 0 or not. Since the concention in M. Sipser is $\mathbb{N} = \{1, 2, 3, ...\}$ that is what is used here. But even if $\mathbb{N} = \{0, 1, 2, ...\}$ L_2 is still context-free since then if is generated by CFG

$$\begin{aligned} S &\to aAc \mid \epsilon, \\ A &\to aAc \mid B, \\ B &\to bBc \mid \epsilon. \end{aligned}$$

$\square$

Part 1.3

We show that L_3 is not regular, but it is context-free.

Proof. The proof essentially goes as that for L_2.
That is L_3 is not regular can be seen by the following proof by contradiction: Assume L_3 is regular, and let p denote the pumping length as given by the pumping lemma for regular languages Theorem 1.70. Consider now the string $w = a^p b^{2p} c^p \in L_3$, which clearly satisfies $|w| > p$. Thus by the pumping lemma we can split $w = xyz$ with $|xy| \leq p$ and $|y| > 0$. Clearly we then have $xy = a^k$ for some $0 < k \leq p$, and therefore $y = a^m$ for some $0 < m \leq p$. But then clearly $xy^i z = a^{p+(i-1)m} b^{2p} c^p$ which for $i \neq 1$ is not in L_3, contradicting the pumping lemma. Thus we conclude that L_3 is not regular.
That L_3 is context-free can be seen by the fact that it is generated by the CFG

$$\begin{aligned} S &\to aAc^2Cc, \\ A &\to aAc \mid \epsilon, \\ C &\to cCb \mid \epsilon. \end{aligned}$$

if $\mathbb{N} = \{1, 2, 3, ...\}$ and by CFG

$$S \to AC,$$
$$A \to aAc \mid \epsilon,$$
$$C \to cCb \mid \epsilon.$$

if $\mathbb{N} = \{0, 1, 2, 3, ...\}$ □

Part 1.4

We show that L_4 is not context-free, and it then clearly follows from Corollary 2.32 that L_4 is not regular.

Proof. Assume that L_4 is context-free, and let p be the pumping length as given by the pumping lemma for context-free languages Theorem 2.34. Consider then the string $w = a^p c^{p^2+1} b$. By the pumping lemma we may split $w = uvxyz$ where $|vy| > 0$ and $|vxy| \leq p$. Thus either $vxy = a^k$ for some $0 < k \leq p$ in which case $vy = a^m$ for some $0 < m \leq p$, but then $uv^ixy^iz = a^{p+(i-1)m}c^{p^2+1}b$ which is not in L_4 for $i \neq 1$, contradicting the pumping lemma. Or $vxy = a^k c^l$ for some $0 < k + l \leq p$, but then $vy = a^s c^t$ for some $0 < k + l \leq p$ and $uv^ixy^iz = a^{p+(i-1)s}c^{p^2+1+(i-1)t}b$, but clearly there exist for any s, t such that $s > 0$ an $i \in \{0, 1, 2, ...\}$ such that $(p + (i-1)s)^2 > p^2 + 1 + (i-1)t$, and thus for such an i, uv^ixy^iz is not in L_4 contradicting the pumping lemma and if $s = 0$ we obviously have $uv^ixy^iz = a^{p+(i-1)s}c^{p^2+1+(i-1)t}b \notin L_4$ also contradicting the pumping lemma. Or we may have $vxy = c^k$ for some $0 < k \leq p$, in which case $vy = c^m$ for some $0 < m \leq p$, but then $uv^ixy^iz = a^p c^{p^2+1+(i-1)m}b$ which is not in L_4 for $i \neq 1$ contradicting the pumping lemma. Finally we may have $vxy = c^k b$ in which case the contradicting is obtained by the same method as for $vxy = a^k c^l$. Thus a contradiction is unavoidable, and we conclude that L_4 is not context-free. □

Part 1.5

We show now that L_4 belongs to L.

Proof. We assume that $\mathbb{N} = \{1, 2, 3, ...\}$, but the proof works for $\mathbb{N} = \{0, 1, 2, ...\}$ with small modifications. Consider the log-space TM, $M =$"On input w

1. Scan the input, and compare neighboring letters (by storing and overwriting one letter on the worktape all the way). If ever substring ab, ca, ba or bc is found, *reject*.
2. Count the number of as, bs and cs with three counters, i, j, k respectively, on the worktape.
3. Check that $i^2 + j^2 = k$. If yes *accept*, if no, *reject*.

Evidently this M decides L_4. Furthermore, it is log-space, as the first step requires only one slot on the worktape, step 2 requires only tree counters (in binary) which take up only logarithmic space. And finally we may multiply $i \cdot i$ by adding i, i times, which can be done by having an extra counter keeping track of how many times we have added i. Of course we also need the obviuous seperator symbols, which can clearly be included in log space. Thus we see that M run in logarithmic space, and we conclude that L_4 is in L. $\square$

Question 2

Question 3

Question 4