Notes on stability of the Fermi gas with point interactions

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Contents

We study in these notes the Fermi gas, *i.e.* a many-body system of spin-$\frac{1}{2}$ fermions or more generally just two species of fermions. The specific gas we study is interacting *via.* point interactions *or* zero-range interactions. We will restrict to the case where the two species can have different mass, but all fermions in one species have equal mass. The relevant quantity in this case is the relative mass of the two. Thus by setting the mass of one species to 1 and the mass of the other to m we have en mass-ratio of m. Formally the system we are studying can thus be described by the Hamiltonian

$$H = -\frac{1}{2m}\sum_{j=1}^{M}\Delta_{y_j} - \frac{1}{2}\sum_{i=1}^{N}\Delta_{x_i} + \gamma\sum_{i=1}^{N}\sum_{j=1}^{M}\delta(x_i - y_j), \tag{0.1}$$

where $x_i \in \mathbb{R}^3$ for all $i \in \{1,...,N\}$ and $y_j \in \mathbb{R}^3$ for all $j \in \{1,...,M\}$. Notice that we also restrict to the case of equal coupling between all particles. These formally defined Hamiltonians are clearly ill-defined as the δ-function is a temperate distribution and thus is only defined on the Schwartz functions. However, restricting the domain to the Schwartz functions will not make the Laplacians self-adjoint, furthermore the codomain of δ is not in $L_2(\mathbb{R}^3)$. Thus no self-adjoint operator of this form exists. As a quadratic form $\langle\psi|H\psi\rangle$ might make sense. Since the δ-function only makes sense, at least on continuous functions, there exist no sensible domain of this quadratic form such that it is closed. If such a domain existed both the Laplacian would be closed on it, however, this is only true for $H_1(\mathbb{R}^3)$ which contains non-continuous functions (defined a.e.)

One way of rigorously studying such formal Hamiltonians is to consider self-adjoint extensions of more well defined Hamiltonians. While this approach is very successful in the $N = M = 1$ case, it becomes increasingly difficult as the number of self-adjoint extension become infinite already at the $N = 2$ and $M = 1$ case. In [2] quadratic forms where developed in order to describe systems of the form (0.1). These quadratic forms are generally more well defined, however their origin and connection the the formal Hamiltonian might be obscured as they need regularization and renormalization procedures to make sense of the point interactions. We will in these notes aim to construct the quadratic form corresponding to the formal expression in (0.1), and show that they can be reaches by considering a sequence of rank one perturbations. We aim at showing that operators corresponding to these rank one perturbations actually converge to the operator of the quadratic form given by [2]. Furthermore, it is our hope that this will shed light on the stability of these systems which has only been shown the cases of $(N, M) = (N, 1)$ and $(N, M) = (2, 2)$. We start out by considering the simpler case which is $(N, M) = (N, 1)$ also denoted the $N + 1$ case.

1 Formal Hamiltonian for the $N + 1$ case

The formal Hamiltonian of (0.1) can be rewritten in the $N + 1$ case by separating the centre of mass. Notice that this indeed already restricts the set of possible self-adjoint Hamiltonians mimicking (0.1) as this asserts the translational invariance of the Hamiltonian. Thus this separation of the centre of mass restricts to couplings that are independent of the centre of mass coordinate. Defining the centre of mass and the relative coordinates by

$$X = \frac{my + \sum_{i=1}^{N} x_i}{m + N}, \quad \tilde{x}_i = x_i - y, \tag{1.1}$$

we obtain that

$$\begin{aligned}\Delta_{x_i} = \sum_{j=1}^{3} \partial_{x_i^j}\partial_{x_i^j} &= \sum_{j=1}^{3} \left(\frac{\partial X^j}{\partial x_i^j}\partial_{X^j} + \frac{\partial \tilde{x}_i{}^j}{\partial x_i^j}\partial_{\tilde{x}_i^j}\right)\left(\frac{\partial X^j}{\partial x_i^j}\partial_{X^j} + \frac{\partial \tilde{x}_i{}^j}{\partial x_i^j}\partial_{\tilde{x}_i^j}\right) \\ &= \frac{1}{(m+N)^2}\Delta_X + \Delta_{\tilde{x}_i} + \frac{2}{m+N}\nabla_X \cdot \nabla_{\tilde{x}_i},\end{aligned} \tag{1.2}$$

$$\begin{aligned}\Delta_y &= \sum_{j=1}^{3} \partial_{y^j}\partial_{y^j} = \sum_{j=1}^{3} \left(\frac{\partial X^j}{\partial y^j}\partial_{X^j} + \sum_{i=1}^{N}\frac{\partial \tilde{x}_i{}^j}{\partial y^j}\partial_{\tilde{x}_i^j}\right)\left(\frac{\partial X^j}{\partial y^j}\partial_{X^j} + \sum_{i=1}^{N}\frac{\partial \tilde{x}_i{}^j}{\partial y^j}\partial_{\tilde{x}_i^j}\right) \\ &= \frac{m^2}{(m+N)^2}\Delta_X + \sum_{i=1}^{N}\Delta_{\tilde{x}_i} + 2\sum_{\substack{(i,j)=(1,1)\\ i<j}}^{(N,N)} \nabla_{\tilde{x}_i}\cdot\nabla_{\tilde{x}_j} - \frac{2m}{m+N}\sum_{i=1}^{N}\nabla_X\cdot\nabla_{\tilde{x}_i}.\end{aligned} \tag{1.3}$$

Thus we get the Hamiltonian

$$H = -\frac{1}{2(m+N)}\Delta_X - \frac{m+1}{2m}\sum_{i=1}^{N}\Delta_{\tilde{x}_i} + \frac{2}{2m}\sum_{\substack{(i,j)=(1,1)\\ i<j}}^{(N,N)} \nabla_{\tilde{x}_i}\cdot\nabla_{\tilde{x}_j} + \gamma\sum_{i=1}^{N}\delta(\tilde{x}_i), \tag{1.4}$$

which can be recast as

$$H = H_{\text{CM}} + \frac{m+1}{2m}H_{\text{rel}}, \tag{1.5}$$

with $H_{\text{CM}} = -\frac{1}{2(m+N)}\Delta_X$ the free centre of mass and the relative Hamiltonian given by

$$H_{\text{rel}} = \sum_{i=1}^{N}\Delta_{\tilde{x}_i} + \frac{2}{m+1}\sum_{\substack{(i,j)=(1,1)\\ i<j}}^{(N,N)} \nabla_{\tilde{x}_i}\cdot\nabla_{\tilde{x}_j} + \tilde{\gamma}\sum_{i=1}^{N}\delta(\tilde{x}_i), \tag{1.6}$$

where $\tilde{\gamma} = \frac{2m}{m+1}\gamma$. Notice that the problem has now been split in two independent parts and thus we recognize the centre of mass part as the free particle which is solved by the Laplacian being essentially self adjoint on $C_c^\infty(\mathbb{R}^3)$ functions with self-adjoint extension Δ on $H_2(\mathbb{R}^3)$ where Δ acts in the distributional sense. The relative Hamiltonian on the other hand will be the main focus in the first part of these notes.

2 The $1+1$ case

We are now going to study different ways of rigorously defining the relative Hamiltonian (1.6) in the case of $N=1$. The first method is easily implemented for $N=1$ but is hard to generalize.

2.1 Self-adjoint extension

The first method we are going to study is that of self-adjoint extension. We thus restrict the formal Hamiltonian to a domain in which it is well defined. This could for example be $C_c^\infty(\mathbb{R}^3\setminus\{0\})$. Notice since we have removed $\{0\}$ the δ-function has no support on this space and thus vanish. Therefore, we have the relative Hamiltonian

$$H_{\mathrm{rel}} = -\Delta|_{C_c^\infty(\mathbb{R}^3\setminus\{0\})}. \tag{2.1}$$

We now seek to extend this operator to a self-adjoint operator on a larger domain. This is possible since H_{rel} is symmetric and its closure, denoted $\dot{H}_{\mathrm{rel}}$ have deficiency indices $K_+ = K_- = 1$, with $K_\pm = \mathrm{Ran}(H_{\mathrm{rel}} \pm iI)^\perp = \ker(H_{\mathrm{rel}}^* \mp iI)$ where H_{rel}^* denotes the adjoint of H_{rel}.
By definition on the adjoint we have that $\mathscr{D}(H_{\mathrm{rel}}^*) = \{f\in L_2(\mathbb{R}^3) |\, \langle f|H_{\mathrm{rel}}\cdot\rangle \text{ is bounded on } \mathscr{D}(H_{\mathrm{rel}})\}$, where the adjoint of the Laplacian acts as the Laplacian in the distributional sense. We determine first the closure of H_{rel}. This can be done by taking the adjoint twice. Notice that the domain of the adjoint is $\mathscr{D}(H_{\mathrm{rel}}^*) = \{f\in L_2(\mathbb{R}^3) |\, \langle f|\Delta\cdot\rangle \text{ is bounded on } C_c^\infty(\mathbb{R}^3\setminus\{0\})\}$. This can be directly calculated to be

$$\mathscr{D}(H_{\mathrm{rel}}^*) = \{f\in H_{2,\mathrm{loc}}(\mathbb{R}^3\setminus\{0\})\cap L_2(\mathbb{R}^3) | \Delta f\in L_2(\mathbb{R}^3)\} \tag{2.2}$$

We emphasise that all elements in $\mathscr{D}(H_{\mathrm{rel}}^*)$ should be viewed as distributions in $H_{2,\mathrm{loc}}(\mathbb{R}^3\setminus\{0\})$. Therefore the requirement $\Delta f\in L_2(\mathbb{R}^3)$ does not simply restrict the domain to be $H_2(\mathbb{R}^3)$ as elements or their derivative (up to second order) can have singular behaviour at 0, e.g. δ-functions. Notice that $C_c^\infty(\mathbb{R}^3\setminus\{0\})$ is dense in $L_2(\mathbb{R}^3\setminus\{0\}) = L_2(\mathbb{R}^3)$ (only defined a.e). The domain of the double adjoint is then given by

$$\mathscr{D}(H_{\mathrm{rel}}^{**}) = \{f\in L_2(\mathbb{R}^3) |\, \langle H_{\mathrm{rel}}^*\cdot|f\rangle \text{ is bounded on } \mathscr{D}(H_{\mathrm{rel}}^*)\} = H_2^0(\mathbb{R}^3\setminus\{0\}), \tag{2.3}$$

where Δ acts in the distributional sense and we have defined

$$H_2^0(\mathbb{R}^3\setminus\{0\}) = \left\{u\in L_2(\mathbb{R}^3) \;\middle|\; \Delta u\in L_2(\mathbb{R}^3) \text{ and } u(x)\to 0 \wedge \nabla u(x)\to 0 \text{ for } |x|\to 0 \vee |x|\to\infty\right\}. \tag{2.4}$$

Thus we have

$$\dot{H}_{\mathrm{rel}} = -\Delta, \qquad \mathscr{D}\left(\dot{H}_{\mathrm{rel}}\right) = H_2^0(\mathbb{R}^3\setminus\{0\}). \tag{2.5}$$

The adjoint of $\dot{H}_{\mathrm{rel}}$ is simply given by $\dot{H}_{\mathrm{rel}}^* = H_{\mathrm{rel}}^*$, as the adjoint is already closed. Thus we are ready to find all self-adjoint extensions of H_{rel}. By the Krein theorem there exist self-adjoint extension if and only if $\dim\left(\mathrm{Ran}\left(H_{\mathrm{rel}} - iI\right)^\perp\right) = \dim\left(\mathrm{Ran}\left(H_{\mathrm{rel}} + iI\right)^\perp\right)$ or equivalently $\dim\left(\ker\left(H_{\mathrm{rel}}^* + iI\right)\right) = \dim\left(\ker\left(H_{\mathrm{rel}}^* - iI\right)\right)$ thus we seek solutions of the equation

$$H_{\mathrm{rel}}^*\psi_\pm = \pm i\psi_\pm, \quad \psi_\pm\in\mathscr{D}\left(H_{\mathrm{rel}}^*\right). \tag{2.6}$$

The equation $-\Delta\psi_\pm = \pm i\psi_\pm$ has the unique solution

$$\psi(x)_\pm = \frac{e^{i\sqrt{\pm i}|x-y|}}{|x-y|}, \qquad x\in\mathbb{R}^3\setminus\{y\}. \tag{2.7}$$

In order for this function to be in the domain of H_{rel}^* we need to choose $y = 0$. Thus we see that $\dim(\ker(H_{\text{rel}}^* + iI)) = \dim(\ker(H_{\text{rel}}^* - iI)) = 1$. By Krein's extension theorem for symmetric operators we have that there exist a one-parameter family of self-adjoint extensions of H_{rel}. Parametrizing the family by a complex phase we have the extensions

$$\mathscr{D}(H_{\text{rel},\theta}) = \left\{ h + c(\xi_+ + \mathrm{e}^{i\theta}\xi_-) \middle| h \in \mathscr{D}\left(\dot{H}_{\text{rel}}\right),\ c \in \mathbb{C} \right\}, \tag{2.8}$$

where $\theta \in [0, 2\pi)$, $\xi_+ \in \ker(H_{\text{rel}}^* + iI), \xi_- \in \ker(H_{\text{rel}}^* - iI)$ are fixed with $\|\xi_+\| = \|\xi_-\|$, and where

$$H_{\text{rel},\theta}(h + \xi_+ + \mathrm{e}^{i\theta}\xi_-) = H_{\text{rel}}^*(h + \xi_+ + \mathrm{e}^{i\theta}\xi_-) = h + i(\mathrm{e}^{i\theta}\xi_- - \xi_+) \tag{2.9}$$

Following the methods of [1], we however now that there is another characterization of these extensions. By decomposing the Hilbert space into spherical coordinates we obtain the decomposition

$$L_2(\mathbb{R}^3, d^3x) = L_2((0,\infty), r^2dr) \otimes L_2(S^2, d\Omega) \tag{2.10}$$

Furthermore by decomposing into spherical harmonics we have

$$L_2(\mathbb{R}^3, d^3x) = \bigoplus_{l=0}^{\infty} L_2((0,\infty), r^2dr) \otimes \langle Y_l^{-l}, Y_l^{-l+1}, ..., Y_l^0, ..., .Y_l^l \rangle \tag{2.11}$$

Now using the unitary transformation $U : L_2((0,\infty), r^2dr) \to L_2((0,\infty), dr)$, defined by $Uf(r) = rf(r)$

$$L_2(\mathbb{R}^3, d^3x) = \bigoplus_{l=1}^{\infty} U^{-1} L_2((0,\infty), r^2dr) \otimes \langle Y_l^{-l}, Y_l^{-l+1}, ..., Y_l^0, ..., .Y_l^l \rangle \tag{2.12}$$

where $\langle ... \rangle$ denotes the span. Using the Laplacian in spherical coordinates

$$\Delta\phi = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j(\phi)) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi\frac{\partial\phi}{\partial\varphi}) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\phi}{\partial^2\theta}, \tag{2.13}$$

with the usual notation g_{ij} the metric, g^{ij} the inverse metric, $g = \det(g_{ij})$ and where θ denotes the azimuthal angle and φ the zenith angle, it is straightforward to show that

$$\dot{H}_{\text{rel}} = \bigoplus_{l=0}^{\infty} U^{-1} h_l U \otimes \mathrm{Id}_\ell \tag{2.14}$$

with Id_ℓ being the identity on $\langle Y_l^{-l}, Y_l^{-l+1}, ..., Y_l^0, ..., .Y_l^l \rangle$. Here we have defined

$$h_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \tag{2.15}$$

with the domains

$$\mathscr{D}(h_0) = \left\{ u \in L_2((0,\infty), dr) | u, u' \in \mathrm{AC}_{\text{loc}}(0,\infty), u'' \in L_2((0,\infty), dr), u(0_+) = 0, u'(0_+) = 0 \right\} \tag{2.16}$$

$$\mathscr{D}(h_l) = \left\{ u \in L_2((0,\infty), dr) | u, u' \in \mathrm{AC}_{\text{loc}}(0,\infty), -u'' + l(l+1)r^{-2}u \in L_2((0,\infty), dr) \right\}, \quad l \geq 1 \tag{2.17}$$

Here $\mathrm{AC}(0,\infty)$ denotes the absolutely continuous functions on $(0,\infty)$, and $\mathrm{AC}_{\text{loc}}(0,\infty)$ denotes the locally absolutely continuous functions, *i.e.* AC on all compact intervals. Notice that for $l \geq 1$ the boundary conditions $u(0_+) = 0, u'(0_+) = 0$ are automatically satisfied by the requirement $-u'' + l(l+1)r^{-2}u \in L_2((0,\infty), dr)$ and continuity of u. According to [1], it is a standard result that h_l is self-adjoint for $l \geq 1$ However, it is not hard to see that h_0

has deficiency indices (1,1) and thus admits a one-parameter family of self-adjoint extensions. These extensions can all be characterized in terms of their self-adjoint boundary condition and are given by

$$h_{0,\alpha} = -\frac{d^2}{dr^2}, \tag{2.18}$$

with domain

$$\mathscr{D}(h_0) = \left\{ u \in L_2((0,\infty),dr) | u, u' \in \mathrm{AC}_{\mathrm{loc}}(0,\infty), u'' \in L_2((0,\infty),dr), -4\pi\alpha u(0_+) + u'(0_+) = 0 \right\}, \tag{2.19}$$

with $\alpha \in (-\infty, \infty]$. The case $\alpha = \infty$ simply corresponds to the boundary condition $u(0+) = 0$, which simply implies $\lim_{|x|\to 0} |x|\psi(x) = 0$ for all $\psi \in \mathscr{D}(H_{\mathrm{rel}}^{\infty})$. This is the usual Friedrich extension i.e. $H_{\mathrm{rel}}^{\infty} = -\Delta$ with $\mathscr{D}(H_{\mathrm{rel}}^{\infty}) = H_2(\mathbb{R}^3)$, *i.e.* the free particle. α can be related to θ from before by a simple computation: Let $f = h + c(\xi_+ + \mathrm{e}^{i\theta}\xi_-) \in \mathscr{D}(H_{\mathrm{rel},\theta})$ with $\xi_\pm = \frac{\mathrm{e}^{i\sqrt{\pm i}|x|}}{4\pi|x|} =$ then

$$\lim_{|x|\to 0} |x| f(x) = \frac{c}{4\pi}(1 + \mathrm{e}^{i\theta}), \qquad \lim_{|x|\to 0} \frac{d}{d|x|}(|x| f(x)) = \frac{ic}{4\pi}(\sqrt{i} + \mathrm{e}^{i\theta}\sqrt{-i}) \tag{2.20}$$

Thus we have

$$4\pi\alpha(1 + \mathrm{e}^{i\theta}) = i(\sqrt{i} + \mathrm{e}^{i\theta}\sqrt{-i}) = \sqrt{i}(i - \mathrm{e}^{i\theta}) = (\mathrm{e}^{i\frac{3}{4}} - \mathrm{e}^{i(\theta + \frac{1}{4})}) \tag{2.21}$$

from which it follows that

$$\begin{aligned} \alpha = \frac{(\mathrm{e}^{i\frac{3}{4}} - \mathrm{e}^{i(\theta+\frac{1}{4})})}{4\pi(1+\mathrm{e}^{i\theta})} &= \frac{1}{4\pi} \frac{\mathrm{e}^{i(\theta+1)/2}(\mathrm{e}^{-i(\theta-\frac{1}{2})/2} - \mathrm{e}^{i(\theta-\frac{1}{2})/2})}{\mathrm{e}^{i\theta/2}(\mathrm{e}^{-i\theta/2} + \mathrm{e}^{i\theta/2})} = \frac{1}{4\pi} \frac{i(\mathrm{e}^{-i(\theta-\frac{1}{2})/2} - \mathrm{e}^{i(\theta-\frac{1}{2})/2})}{(\mathrm{e}^{-i\theta/2} + \mathrm{e}^{i\theta/2})} \\ &= \frac{1}{4\pi} \frac{\sin((\theta - \frac{1}{2})/2)}{\cos(\theta/2)} = \frac{1}{4\pi} \frac{-\cos(\theta/2)\sin(\frac{1}{4}) + \sin(\theta/2)\cos(\frac{1}{4})}{\cos(\theta/2)} = \frac{1}{4\sqrt{2}\pi}\left(\tan(\theta/2) - 1\right) \end{aligned} \tag{2.22}$$

Thereby we see that $\alpha \in \mathbb{R}$ for $\theta \in [0,\pi) \cup (\pi, 2\pi)$ and that $\alpha \to \infty$ when $\theta \uparrow \pi$. Now we study the the resolvent of these extensions i.e. $H_{\mathrm{rel}}^{\alpha} = U^{-1}h_{0,\alpha}U \otimes \mathrm{Id}_0 \oplus \left(\bigoplus_{l=1}^{\infty} U^{-1}h_l U \otimes \mathrm{Id}_\ell\right)$. To do this let us briefly summarize Krein's formula. In the following $\rho(O)$ denotes the resolvent set of the operator O.

Theorem 1 (Krein's formula, A.2 in [1])**.** *Let B and C be self-adjoint extensions of the densely defined, closed, and symmetric operator A on the Hilbert space H with deficiency indices $(1,1)$. Then their resolvant are related by:*

$$(B - z)^{-1} - (C - z)^{-1} = \lambda(z) \langle \phi(\bar{z}), \cdot \rangle \phi(z), \qquad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.23}$$

where $\lambda(z) \neq 0$ for $z \in \rho(B) \cap \rho(C)$ and λ, ϕ may be chosen to be analytic functions in $z \in \rho(B) \cap \rho(C)$. In fact, ϕ may be taken as

$$\phi(z) = \phi(z_0) + (z - z_0)(C - z)^{-1}\phi(z_0), \quad z \in \rho(C) \tag{2.24}$$

with $\phi(z_0)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$ being a solution of

$$A^*\phi(z_0) = z_0\phi(z_0), \tag{2.25}$$

Choosing this ϕ, we furthermore have λ satisfying the equation

$$\lambda(z)^{-1} = \lambda(z')^{-1} - (z - z') \langle \phi(\bar{z}), \phi(z') \rangle, \qquad z, z' \in \rho(B) \cap \rho(C). \tag{2.26}$$

Proof. In order to prove this we remember the Krein extension theorem for densely defined, closed, symmetric operators. We have that the B and C are both of the form

$$\begin{aligned} \mathscr{D}(B) &= \{h + c(\xi_z + e^{i\theta}\xi_{\bar z}) \mid h \in \mathscr{D}(A), c \in \mathbb{C}\}, \\ B(h + c(\xi_z + e^{i\theta}\xi_{\bar z})) &= Ah + c(z\xi_z + e^{i\theta}\bar z \xi_{\bar z}), \qquad \mathrm{Im}(z) \neq 0, \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} \mathscr{D}(C) &= \{h + c(\xi_z + e^{i\omega}\xi_{\bar z}) \mid h \in \mathscr{D}(A),\ c \in \mathbb{C}\}, \\ C(h + c(\xi_z + e^{i\omega}\xi_{\bar z})) &= Ah + c(z\xi_z + e^{i\omega}\bar z \xi_{\bar z}). \qquad \mathrm{Im}(z) \neq 0 \end{aligned} \tag{2.28}$$

where $\xi_z \in \ker(A^* - z), \xi_{\bar z} \in \ker(A^* - \bar z), \omega \in [0, 2\pi)$ and $\theta \in [0, 2\pi)$ are fixed with $\|\xi_z\| = \|\xi_{\bar z}\| = 1$, Now for $z \in \rho(C)$, we know that $(C - z)$ has full range and thus for $x \in H$ we write $x = (C - z)y$. Assuming that $\mathrm{Im}(z) \neq 0$ we can write $y = h + \xi_z + e^{i\omega}\xi_{\bar z} \in \mathscr{D}(C)$ where we have absorbed the $c \in \mathbb{C}$ into the ξ_z and $\xi_{\bar z}$. Consider now

$$\left((B - z)^{-1} - (C - z)^{-1}\right) x = (B - z)^{-1}(C - z)(h + \xi_z + e^{i\omega}\xi_{\bar z}) - (h + \xi_z + e^{i\omega}\xi_{\bar z}). \tag{2.29}$$

Clearly $(B - z)^{-1}(C - z)h = h$ since $h \in \mathscr{D}(A)$, and C and B are both extension of A. Thus we obtain

$$\left((B - z)^{-1} - (C - z)^{-1}\right) x = (B - z)^{-1}((\bar z - z)e^{i\omega}\xi_{\bar z}) - (\xi_z + e^{i\omega}\xi_{\bar z}). \tag{2.30}$$

Since we have that $(B - z)(\xi_z + e^{i\theta}\xi_{\bar z}) = e^{i\theta}(\bar z - z)\xi_{\bar z}$, with $\xi_z + e^{i\theta}\xi_{\bar z} \in \mathscr{D}(B)$, we find that

$$\left((B - z)^{-1} - (C - z)^{-1}\right) x = (e^{i(\omega-\theta)} - 1)\xi_z, \qquad x = (C - z)(h + \xi_z + e^{i\omega}\xi_{\bar z}) = (C - z)h + (\bar z - z)e^{i\omega}\xi_{\bar z}, \tag{2.31}$$

and we conclude

$$\left((B - z)^{-1} - (C - z)^{-1}\right) x = \frac{(e^{-i\theta} - e^{-i\omega})}{\bar z - z}\frac{\langle \xi_{\bar z}, x\rangle}{\|\xi_{\bar z}\|^2}\xi_z = \lambda(z, \bar z)\langle \phi(\bar z), x\rangle \phi(z) \tag{2.32}$$

where we have used that $(C - z)h = (A - z)h \in \mathrm{Ran}\,(A - z) \subset \ker(A^* - \bar z)^\perp$, and the fact that by defining $\phi(z) = \phi(z_0) + (z - z_0)(C - z)^{-1}\phi(z_0)$, with $\phi(z_0)$ being a solution of $A^*\phi(z_0) = z_0\phi(z_0)$, we clearly have $\phi(z) \in \ker(A^* - z)$ such that $\phi(z) \parallel \xi_z$.
We have that λ is given by the formula

$$\lambda(z, \bar z) = \frac{(e^{-i\theta} - e^{-i\omega})}{\bar z - z}\frac{\langle \phi(z)\xi_z\rangle}{\langle \phi(\bar z), \xi_{\bar z}\rangle \|\phi(z)\|^2} \tag{2.33}$$

Notice that the above calculation is for fixed z. Thus if we want to vary z, we get that θ and ω might depend on z as we may choose ξ_z and $\xi_{\bar z}$ differently at each z making a fixed θ or ω correspond to different extensions for each z. The fact that λ is analytic stems from the fact that all matrix elements of the resolvents are analytic in their resolvent sets and that we have chosen $\phi(z)$ such that $\langle \phi(\bar z), x\rangle$ is analytic for all $x \in H$. To show that λ satisfies (2.26) is simply a long computation and we refer to appendix A for the computation. □

Now we have two self-adjoint extension of $\dot H_{\mathrm{rel}}$, namely H^∞_{rel} and H^α_{rel}. It is easily verified that by imposing $\alpha = \infty$ we obtain the Friedrich extension, given by (see above)

$$\mathscr{D}(H^\infty_{\mathrm{rel}}) = H_2(\mathbb{R}^3), \qquad H^\infty_{\mathrm{rel}} = -\Delta. \tag{2.34}$$

We have already found the solution of $H^*_{\mathrm{rel}}\phi(z) = z\phi(z)$ (although we only found it for $z = \pm i$) namely

$$\phi(z)(x) = \frac{e^{i\sqrt{z}|x|}}{4\pi|x|}, \qquad \mathrm{Im}(\sqrt{z}) > 0, \tag{2.35}$$

Furthermore it is a straightforward generalization of this result that the Green function of $(H^{\infty}_{\mathrm{rel}} - z)$, *i.e.* the integral kernel of the resolvent $(H^{\infty}_{\mathrm{rel}} - z)^{-1}$, is then

$$G_z(x,x') = \frac{\mathrm{e}^{i\sqrt{z}|x-x'|}}{4\pi|x-x'|}. \tag{2.36}$$

We immediately see that then

$$\langle \phi(\bar{z}), \phi(z') \rangle = \frac{1}{4\pi}\int_{(0,\infty)} dr \mathrm{e}^{i(\sqrt{z'}-\overline{\sqrt{\bar{z}}})r} = \frac{1}{4\pi}\frac{-i}{\sqrt{z}-\sqrt{z'}}, \qquad \mathrm{Im}(\sqrt{z'}), \mathrm{Im}(\sqrt{\bar{z}}) > 0 \tag{2.37}$$

Remember that $\overline{\sqrt{\bar{z}}}|_{\mathrm{Im}(\sqrt{\bar{z}})>0} = \sqrt{z}|_{\mathrm{Im}(\sqrt{z})<0} = -\sqrt{z}|_{\mathrm{Im}(\sqrt{z})>0}$ so we have

$$\lambda(z)^{-1} - \lambda(z')^{-1} = \frac{i}{4\pi}\frac{z'-z}{\sqrt{z}+\sqrt{z'}} = \frac{i}{4\pi}(\sqrt{z'}-\sqrt{z}), \qquad \mathrm{Im}(z), \mathrm{Im}(z') > 0 \tag{2.38}$$

From which it follows that $\lambda(z) = (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1}$ Furthermore we have from Krein's formula (Theorem 1) that

$$(H^{\alpha}_{\mathrm{rel}} - z)^{-1} = (H^{\infty}_{\mathrm{rel}} - z)^{-1} + (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1} \langle \phi(\bar{z}), \cdot \rangle \, \phi(z), \tag{2.39}$$

where we notice that by (2.24) we have

$$\begin{aligned}(\dot{H}^{*}_{\mathrm{rel}} - z)\phi(z) &= (\dot{H}^{*}_{\mathrm{rel}} - z)\phi(z_0) + (z-z_0)(\dot{H}^{*}_{\mathrm{rel}} - z)(H^{\infty}_{\mathrm{rel}} - z)^{-1}\phi(z_0) \\ &= (z_0 - z)\phi(z_0) + (z-z_0)\phi(z_0) = 0 \qquad z \in \rho(H^{\infty}_{\mathrm{rel}})\end{aligned} \tag{2.40}$$

where we used that $H^{\infty}_{\mathrm{rel}}$ is a restriction of $\dot{H}^{*}_{\mathrm{rel}}$ such that $(\dot{H}^{*}_{\mathrm{rel}} - z)(H^{\infty}_{\mathrm{rel}} - z)^{-1} = \mathrm{Id}$. However from this we conclude that $\phi(z) = G_z(x,0) = \frac{\mathrm{e}^{i\sqrt{z}|x|}}{|x|}$, $\mathrm{Im}(\sqrt{z}) > 0$. Thereby we have

$$(H^{\alpha}_{\mathrm{rel}} - z)^{-1} = (H^{\infty}_{\mathrm{rel}} - z)^{-1} + (\kappa + i\sqrt{z})^{-1} \langle G_{\bar{z}}(*,0), \cdot(*) \rangle \, G_z(\cdot,0), \qquad z \in \rho(H^{\alpha}_{\mathrm{rel}}) \cap \rho(H^{\infty}_{\mathrm{rel}}), \tag{2.41}$$

where the $*$ refers to the integrated variable in the inner product.
In order to determine κ, we perform a simple calculation. Let $u \in \mathscr{D}(h^{\alpha}_0)$ Then $\frac{1}{r}uY^0_0 \in \mathscr{D}(H^{\alpha}_{\mathrm{rel}})$ and we have

$$(H^{\alpha}_{\mathrm{rel}} - z)\frac{1}{r}uY^0_0 = \left(-\frac{1}{r}\frac{d^2u(r)}{dr^2} - z\frac{1}{r}u(r)\right)Y^0_0 \tag{2.42}$$

Thus we have

$$\begin{aligned}u(0) &= \lim_{r\to 0} r\left((H^{\alpha}_{\mathrm{rel}} - z)^{-1}(H^{\alpha}_{\mathrm{rel}} - z)\frac{1}{r}u\right)(r) \\ &= \lim_{r\to 0} r4\pi\left(\int_{(0,\infty)} drr\left(-\frac{d^2u}{dr^2} - zu\right)G_z(r,0) + (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1}G_z(r,0)\int drr\overline{G_{\bar{z}}(r,0)}\left(-\frac{d^2u}{dr^2} - zu\right)\right)\end{aligned} \tag{2.43}$$

Notice that $\overline{G_{\bar{z}}(r,0)} = G_z(r,0)$ and that $r4\pi G_z(r,0) = \mathrm{e}^{i\sqrt{z}r}$. By partial integration twice we have

$$\int_{(0,\infty)} dr\left(-\mathrm{e}^{i\sqrt{z}r}\frac{d^2}{dr^2}u + u\frac{d^2}{dr^2}\mathrm{e}^{i\sqrt{z}r}\right) = \frac{du}{dr}(0+) - i\sqrt{z}u(0+), \tag{2.44}$$

from which we get

$$u(0) = (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1}\frac{1}{4\pi}\left(\frac{du}{dr}(0+) - i\sqrt{z}u(0+)\right). \tag{2.45}$$

By imposing the boundary condition on u at 0 we obtain the equation for κ

$$1 = (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1}\frac{1}{4\pi}\left(4\pi\alpha - i\sqrt{z}\right), \tag{2.46}$$

Thus that $\kappa = \alpha$ and we have the resolvent

$$(H_{\text{rel}}^{\alpha} - z)^{-1} = (H_{\text{rel}}^{\infty} - z)^{-1} + (\alpha - \frac{i}{4\pi}\sqrt{z})^{-1}\langle G_{\bar{z}}(*,0), \cdot(*)\rangle G_z(\cdot,0), \tag{2.47}$$

We are now ready to study the spectrum of the the operators $(H_{\text{rel}}^{\alpha})_{\{\alpha\in(-\infty,\infty]\}}$. Clearly $z \in \sigma(H_{\text{rel}}^{\alpha})$ if $z \in \sigma(-\Delta|_{C_c^\infty(\mathbb{R}^3)}) = [0,\infty)$. On the other hand we see that if $\alpha < 0$ then $z = -(4\pi\alpha)^2 \in \sigma(H_{\text{rel}}^{\alpha})$. Therefore the spectrum can be characterized as

$$\sigma(H_{\text{rel}}^{\alpha}) = \begin{cases} [0,\infty) & \text{if } \alpha \geq 0, \\ \{-(4\pi\alpha)^2\} \cup [0,\infty) & \text{if } \alpha < 0. \end{cases} \tag{2.48}$$

It is of course an exercise to show that no other points are in the spectrum. We refer to [1] for a short proof, and further classification of different parts of the spectrum, *i.e.* point-, singular continuous-, and absolute continuous spectrum. We note that for $\alpha < 0$ the point in the spectrum $\{-(4\pi\alpha)^2\}$ is an eigenvalue (*i.e.* a part of the point spectrum). Furthermore, we can actually, in the $\alpha < 0$ case, determine the eigenfunction corresponding to the eigenvalue $-(4\pi\alpha^2)$. To do this, notice that the domain of H_{rel}^{α} can be written as
$\mathscr{D}(H_{\text{rel}}^{\alpha}) = \{w(x) + (\alpha - \frac{i}{4\pi}k)^{-1}w(0)G_{k^2}(x,0) \mid w \in H_2(\mathbb{R}^3),\ k^2 \in \rho(H_{\text{rel}}^{\alpha})\}$ for $k \in \rho(H_{\text{rel}}^{\alpha}) \cap (H_{\text{rel}}^{\alpha})$. This follows by the fact that

$$\mathscr{D}(H_{\text{rel}}^{\alpha}) = (H_{\text{rel}}^{\alpha} - k^2)^{-1}(H_{\text{rel}}^{\infty} - k^2)\mathscr{D}(H_{\text{rel}}^{\infty}), \tag{2.49}$$

where we have used that $\langle G_{\bar{z}}(x,0), (H_{\text{rel}}^{\infty} - z)w\rangle = \langle (H_{\text{rel}}^{\infty} - \bar{z})G_{\bar{z}}(x,0), w\rangle = \langle \delta_0, w\rangle = w(0)$. Notice that $w \in H_2(\mathbb{R}^3)$ is continuous, so $w(0)$ makes sense. We thus have the action of H_{rel}^{α}

$$(H_{\text{rel}}^{\alpha} - k^2)(w(x) + (\alpha - \frac{i}{4\pi}k)^{-1}w(0)G_{k^2}(x,0)) = (H_{\text{rel}}^{\infty} - k^2)w(x) = (-\Delta - k^2)w(x). \tag{2.50}$$

Now notice that if we fix $w \in H_2(\mathbb{R}^3)$ such that $w(0) = 1$ and we define $(x_n)_{(n\geq 1)}$ such that $x_n \to 0$ as $n \to \infty$, then $x_n w \to 0$ in $L_2(\mathbb{R}^3)$. Furthermore let $k_n^2 \to -(4\pi\alpha)^2$ such that $(\alpha - ik_n/(4\pi))\frac{1}{x_n} = 1$ for all $n \geq 1$, then we have

$$(H_{\text{rel}}^{\alpha} - k_n^2)(x_n w + (\alpha - \frac{i}{4\pi}k_n)^{-1}x_n G_{k_n^2}) = x_n(-\Delta - k^2)w \to 0 \qquad \text{as } n \to \infty, \tag{2.51}$$

with the notation G_{k^2} for $G_{k^2}(x,0)$. Thus we have that

$$\begin{aligned} \lim_{n\to\infty}\left(H_{\text{rel}}^{\alpha}(x_n w + (\alpha - \frac{i}{4\pi}k_n)^{-1}x_n G_{k_n^2})\right) &= \lim_{n\to\infty}\left(k_n^2(x_n w + (\alpha - \frac{i}{4\pi}k_n)^{-1}x_n G_{k_n^2})\right) \\ &= -(4\pi\alpha)^2 G_{-(4\pi\alpha)^2}. \end{aligned} \tag{2.52}$$

Where we have used that $k_n^2 I \to -(4\pi\alpha)^2 I$ in operator norm as $n \to \infty$ and that $(k_n^2 I)_{(n\geq 1)}$ is uniformly bounded. Furthermore, we used that $G_{k_n^2} \to G_{-(4\pi\alpha)^2}$ in $L_2(\mathbb{R}^3)$ for $\alpha < 0$. Thus we conclude that defining $\chi_n = x_n w + (\alpha - \frac{i}{4\pi}k_n)^{-1}x_n G_{k_n^2}$ we have that $(\chi_n)_{(n\geq 1)} \subset \mathscr{D}(H_{\text{rel}}^{\alpha})$ converges in $L_2(\mathbb{R}^3)$ to $G_{-(4\pi\alpha)^2}$ and that $(H_{\text{rel}}^{\alpha}\chi_n)_{(n\geq 1)}$ converges in $L_2(\mathbb{R}^3)$ to $-(4\pi\alpha)^2 G_{-(4\pi\alpha)^2}$. By closedness of the self-adjoint operator H_{rel}^{α} we conclude that $G_{-(4\pi\alpha)^2} \in \mathscr{D}(H_{\text{rel}}^{\alpha})$ and that

$$H_{\text{rel}}^{\alpha} G_{-(4\pi\alpha)^2} = -(4\pi\alpha)^2 G_{-(4\pi\alpha)^2}. \tag{2.53}$$

Thus $G_{-(4\pi\alpha)^2}$ is the eigenfunction corresponding to the bound state with energy $-(4\pi\alpha)^2$. We have thus constructed the Hamiltonian of the point-interaction in $3d$. We have seen that the parameter α, in some sense, controls the strength of the interaction, *i.e.* $\alpha = \infty$ is the free particle, and $\alpha < 0$ is the attractive point-interaction, since it has a bound state. By doing a analysis of the scattering theory of H_{rel}^{α} one finds that $-4\pi\alpha = \frac{1}{a}$, where a denotes the scattering length of the interaction.

2.2 Quadratic form

An alternative way of studying the point interaction is by the means of quadratic forms. In [2], the quadratic form, F_α describing a gas of point interacting fermions was obtained. It is a well-known result that if such a quadratic form is closed and bounded from below, then the corresponding operator is a bounded from below, self-adjoint operator. On the other hand it was proven in [2] that if the quadratic form F_α is not bounded from below, then the corresponding operator is not bounded from below and self-adjoint. The quadratic form was initially introduced by means of renormalization. On the other hand it is more clearly introduced as a rank-one perturbation of the free quadratic form. We write the rank-one perturbation as $\gamma \langle \phi, \cdot \rangle \phi$. Thus we imagine perturbing the Hamiltonian as $H = H_0 - \gamma \langle \phi, \cdot \rangle \phi$. For the point interaction we do this by simply projecting onto a ball $B_R(0)$, *i.e.* the ball of radius R centred at 0, in momentum space *i.e.*

$$\widehat{Hu} = \widehat{H_0 u} - \mathbb{1}_{B_R(0)} \frac{\gamma}{(2\pi)^3} \int_{B_R(0)} d^3p \ \hat{u}(p) = H_0 u - \mathbb{1}_{B_R(0)} \frac{\gamma}{(2\pi)^3} \int_{B_R(-k)} d^3p \ \hat{u}(k+p). \tag{2.54}$$

Thus we obtain the quadratic form

$$F_\gamma^R(\hat{u}) = \int_{\mathbb{R}^3} d^3k \ \left(\bar{\hat{u}}(k)(k^2)\hat{u}(k) - \frac{\gamma}{(2\pi)^3} \int_{B_R(-k)} d^3p \ \bar{\hat{u}}(k)\hat{u}(k+p) \right). \tag{2.55}$$

This can be rewritten in the following form

$$\begin{aligned} F_\gamma^R(u) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right) |\hat{u}(k) - \widehat{G\rho^R}(k)|^2 - \mu \|u\|^2_{L_2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} d^3k \ (k^2+\mu)|\widehat{G\rho^R}(k)|^2 \\ +2\mathrm{Re} \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)(k^2+\mu)\widehat{G\rho^R}(k) - \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)\hat{\rho}^R(k), \end{aligned} \tag{2.56}$$

where $\mu > 0$ and we have defined

$$\begin{aligned} \hat{G}(k) = \frac{1}{k^2+\mu}, \qquad \hat{\rho}^R(k) = \gamma_R \mathbb{1}_{B_R(0)}(k) \int_{B_R(-k)} d^3p \ \hat{u}(k+p) = \mathbb{1}_{B_R(0)}(k)\xi_R, \\ \xi_R = \gamma_R \int_{B_R(0)} d^3p \ \hat{u}(p), \end{aligned} \tag{2.57}$$

furthermore, $\widehat{G\rho^R}(k) = \hat{G}(k)\hat{\rho^R}(k)$ and $\gamma_R = \frac{\gamma}{(2\pi)^3}$, which we have allowed to depend on R, since it will need to be renormalized eventually. Now straightforward calculation shows that

$$\begin{aligned} \overline{\int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)(k^2+\mu)\widehat{G\rho^R}(k)} = \int_{\mathbb{R}^3} d^3k \ \overline{\bar{\hat{u}}(k)\hat{\rho}^R(k)} = \gamma_R \int_{B_R(0)} d^3k \ \hat{u}(k) \int_{B_R(0)} d^3p \ \bar{\hat{u}}(p) \\ = \gamma_R \int_{B_R(0)} d^3k \ \bar{\hat{u}}(k) \int_{B_R(0)} d^3p \ \hat{u}(p) = \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)\hat{\rho}^R(k) = \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)(k^2+\mu)\widehat{G\rho^R}(k), \end{aligned} \tag{2.58}$$

such that $2\mathrm{Re} \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)(k^2+\mu)\widehat{G\rho^R}(k) = 2 \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)\hat{\rho}^R(k)$. Thereby we find the quadratic form

$$\begin{aligned} F_\gamma^R(u) &= \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right) |\hat{u}(k) - \widehat{G\rho^R}(k)|^2 - \mu \|u\|^2_{L_2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} d^3k \ (k^2+\mu)|\widehat{G\rho^R}(k)|^2 \\ &\qquad + \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)\hat{\rho}^R(k) \\ &= \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right) |\hat{u}(k) - \widehat{G\rho^R}(k)|^2 - \mu \|u\|^2_{L_2(\mathbb{R}^3)} - |\xi_R|^2 \int_{B_R(0)} d^3k \hat{G}(k) + \gamma_R^{-1} |\xi_R|^2. \end{aligned} \tag{2.59}$$

Now by computing the

$$\int_{B_R(0)} \hat{G} = 4\pi \int_0^R dr\ \frac{r^2}{r^2+\mu} = 4\pi\sqrt{\mu}\int_0^{R/\sqrt{\mu}} dq\ \frac{q^2}{q^2+1} = 4\pi\sqrt{\mu}\left(\frac{R}{\sqrt{\mu}} - \int_0^{R/\sqrt{\mu}} dq\ \frac{1}{q^2+1}\right) = 4\pi\left(R-\sqrt{\mu}\arctan\left(\frac{R}{\sqrt{\mu}}\right)\right). \tag{2.60}$$

Thereby we have the quadratic form

$$\begin{aligned} F_\gamma^R(u) = \int_{\mathbb{R}^3} d^3k\ \left(k^2+\mu\right)|\hat{u}(k) - \widehat{G\rho^R}(k)|^2 - \mu\|u\|^2_{L_2(\mathbb{R}^3)} \\ -|\xi_R|^2\left(4\pi R - 4\pi\sqrt{\mu}\arctan\left(\frac{R}{\sqrt{\mu}}\right) - \gamma_R^{-1}\right). \end{aligned} \tag{2.61}$$

Since we are interested in the limit $R\to\infty$ (corresponding to localizing the interaction to a point) we choose γ_R such that the divergence in R disappears. Choosing the coupling $\gamma_R^{-1} = 4\pi R + \alpha$ we obtain the final quadratic form

$$F_\alpha^R(u) = \int_{\mathbb{R}^3} d^3k\ \left(k^2+\mu\right)|\hat{w}_R(k)|^2 - \mu\|u\|^2_{L_2(\mathbb{R}^3)} + |\xi_R|^2\left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{R}{\sqrt{\mu}}\right)\right), \tag{2.62}$$

with $\hat{w}_R = \hat{u} - \widehat{G\rho^R}$. Notice, that the domain of this quadratic form is

$$\mathscr{D}\left(F_\alpha^R\right) = \left\{u\in L_2(\mathbb{R}^3)\mid w_R\in H_1(\mathbb{R}^3)\right\} \tag{2.63}$$

Heuristically, we can take the limit $R\to\infty$. This is done by noticing that for $w\in H_1(\mathbb{R}^3)$ we have that $\hat{w}(k) = \frac{\hat{f}(k)}{(|k|^2+1)^{\frac{1}{2}}}$ for some $f\in L_2(\mathbb{R}^3)$. By Hölder's inequality $(2,2)$ we thus see

$$\int_{B_R(0)} \hat{w} \le \left(\int_{B_R(0)}\left|\frac{1}{|k|^2+1}\right|^2\right)^{\frac{1}{2}}\|f\|_2 \lesssim \sqrt{R} \tag{2.64}$$

where by $f(x)\lesssim g(x)$ we mean that there exist some constant $C\in\mathbb{R}$ such that $f(x)\le Cg(x)$. Thus we see that the equation for ξ_R

$$\xi_R = \frac{1}{(4\pi R+\alpha)}\int_{B_R(0)}\left(\hat{w} + \widehat{G\rho^R}\right) = \frac{1}{(4\pi R+\alpha)}\int_{B_R(0)}\left(\hat{w}+\xi_R\hat{G}\right), \tag{2.65}$$

becomes in the limit $R\to\infty$ the equation $\xi_R = \xi_R$. Thus any choice of ξ_R is consistent and simply let it be a free parameter $\xi_R = \xi\in\mathbb{C}$. We also see that in the limit $R\to\infty$ we have that $\arctan\left(R/\sqrt{\mu}\right)\to\frac{\pi}{2}$. Thus we get the quadratic form

$$F_\alpha(u) = \int_{\mathbb{R}^3} d^3k\ \left(k^2+\mu\right)|\hat{w}(k)|^2 - \mu\|u\|^2_{L_2(\mathbb{R}^3)} + |\xi|^2\left(\alpha+2\pi^2\sqrt{\mu}\right), \tag{2.66}$$

with domain

$$\mathscr{D}\left(F_\alpha(u)\right) = \left\{u\in L_2(\mathbb{R}^3)\mid \hat{u} = \hat{w}+\xi\hat{G},\ w\in H_1(\mathbb{R}^3),\ \xi\in\mathbb{C}\right\}, \tag{2.67}$$

which matches the expression of [6] in the $N=M=1$ case.

2.3 Hamiltonian from quadratic form

We are in this subsection going to construct the Hamiltonian for the point interactions from the quadratic form. This will serve both as an example of how obtain the Hamiltonian given its quadratic form, but also as a motivation that the quadratic form given in the previous section is indeed equivalent to the the self-adjoint extension H^{α}_{rel}.
First we need to define a few properties of quadratic forms.

Definition 1. *We say a quadratic form on some Banach space $q : \mathscr{D}(q) \to \mathbb{R}$ is bounded from below if $q(v) \geq -c\|v\|^2$ for some $c > 0$.*

Definition 2. *We say that a quadratic form, $q : \mathscr{D}(q) \to \mathbb{R}$, which is bounded from below, $q(v, v) \geq -c\|v\|^2$, is closed if its domain, $\mathscr{D}(q)$, is a Banach space when equipped with the norm $\|v\|_q^2 = q(v, v) + C\|v\|^2$, where $C > c$.*

Notice that given a quadratic form $q : \mathscr{D}(q) \to \mathbb{R}$ we can always construct a symmetric sesquilinear from by

$$\begin{aligned}\text{Re}(q(u, v)) &= \frac{1}{2}(q(u - v) - q(u) - q(v)),\\ \text{Im}(q(u, v)) &= \frac{1}{2i}(q(u + iv) - q(u) - q(v)).\end{aligned} \tag{2.68}$$

where we abuse notation and use the symbol q for both the quadratic form and the sesquilinear form. This motivates the following proposition

Proposition 1. *A quadratic form on some Hilbert space H, $q : \mathscr{D}(q) \to \mathbb{R}$, which is bounded from below, $q(v) \geq -c\|v\|^2$, is closed if and only if its domain, $\mathscr{D}(q)$, is a Hilbert space when equipped with the inner product $\langle u, v\rangle_q = q(u, v) + C\langle u, v\rangle$, where $C > c$.*

Proof. Since the norm $\|\cdot\|_q$ is generated by $\langle\cdot,\cdot\rangle_q$ it is clear that this follows if we can show that $\langle u, v\rangle_q$ is in fact an inner product. Sesquilinearity is obvious by construction. Furthermore $\langle v, v\rangle_q = \|v\|_q^2 \geq 0$ since q is bounded from below, $q(v) \geq -c\|v\|^2$, and $\langle v, v\rangle_q = 0$ if and only if $v = 0$ follows from the fact that $\langle v, v\rangle_q \geq (C - c)\|v\|^2$. □

Notice that it follows from the fact that $\langle\cdot,\cdot\rangle_q$ is an inner product that $\|\cdot\|_q$ is in fact a norm. We start out by the quadratic form from then previous section

$$F_\alpha(u) = \int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right)|\hat{w}(k)|^2 - \mu\|u\|^2_{L_2(\mathbb{R}^3)} + |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right), \tag{2.69}$$

with domain

$$\mathscr{D}(F_\alpha(u)) = \left\{u \in L_2(\mathbb{R}^3) \mid \hat{u} = \hat{w} + \xi\hat{G},\ w \in H_1(\mathbb{R}^3),\ \xi \in \mathbb{C}\right\}. \tag{2.70}$$

This quadratic for is closed and bounded from below, it is also clear that this quadratic form has a corresponding symmetric sesquilinear form, with domain
$\mathscr{D}(F_\alpha(\cdot,\cdot)) = \mathscr{D}(F_\alpha(\cdot)) \times \mathscr{D}(F_\alpha(\cdot))$, given by

$$F_\alpha(u, v) = \int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right)\overline{\hat{w}(k)}\hat{h}(k) - \mu\langle u, v\rangle_{L_2(\mathbb{R}^3)} + \bar{\xi}\chi\left(\alpha + 2\pi^2\sqrt{\mu}\right), \tag{2.71}$$

where $u = w + \xi G$ and $v = h + \chi G$. The domain of the corresponding operator H_α is defined by

$$\mathscr{D}(H_\alpha) = \{u \in \mathscr{D}(F_\alpha) \mid F_\alpha(u, \cdot) \text{ is an } L_2(\mathbb{R}^3) \text{ bounded linear functional on } \mathscr{D}(F_\alpha)\}. \tag{2.72}$$

By density of $\mathscr{D}(F_\alpha)$ in $L_2(\mathbb{R}^3)$ and Riez representation theorem, we known that if $h \in \mathscr{D}(H_\alpha)$ then $F_\alpha(u, \cdot) = \langle x, \cdot\rangle$ and we define $H_\alpha u = x$. Clearly H_α is linear and symmetric by the very construction

$$\langle H_\alpha u, v\rangle = F_\alpha(u, v) = \overline{F_\alpha(v, u)} = \overline{\langle H_\alpha v, u\rangle} = \langle u, H_\alpha v\rangle \tag{2.73}$$

For $u, v \in \mathscr{D}(H_\alpha)$. Notice that

$$\begin{aligned}\mathscr{D}(H_\alpha^*) &= \{v \in \mathscr{D}(F_\alpha) \mid \langle H_\alpha \cdot, v\rangle \text{ is bounded on } \mathscr{D}(H_\alpha)\} \\ &= \{v \in L_2(\mathbb{R}^3) \mid F_\alpha(\cdot, v) \text{ is bounded on } \mathscr{D}(H_\alpha)\}.\end{aligned} \tag{2.74}$$

Assuming that $\mathscr{D}(H_\alpha)$ is dense in $\mathscr{D}(F_\alpha)$ we then have $\mathscr{D}(H_\alpha^*) = \mathscr{D}(H_\alpha)$ and the operator, H_α, is self-adjoint. It is a general fact that $\mathscr{D}(H_\alpha)$ is dense whenever F_α is closed ([4], Thm 12.18). Thus we are now ready to calculate the Hamiltonian of the quadratic form F_α. Notice that by the definition of $\mathscr{D}(H_\alpha)$ we must have that for $u \in \mathscr{D}(H_\alpha)$ and $(v_n)_{n\geq 1} \subset \mathscr{D}(F_\alpha)$ such that $v_n \to 0$ in $L_2(\mathbb{R}^3)$ it holds that $F_\alpha(v, u_n) \to 0$. Thus by writing $u = w + \xi G$ with $w \in H_1(\mathbb{R}^3)$ and $\xi \in \mathbb{C}$ and $v_n = h_n + \chi G \in \mathscr{D}(F_\alpha)$, with $h_n \in H_1(\mathbb{R}^3)$ such that $h_n \to -\chi G$ in $L_2(\mathbb{R}^3)$ we have

$$F(u, v_n) = \int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) \overline{\hat{w}(k)} \hat{h}_n(k) - \mu \left\langle u, v_n\right\rangle_{L_2(\mathbb{R}^3)} + \bar{\xi}\chi\left(\alpha + 2\pi^2\sqrt{\mu}\right). \tag{2.75}$$

We immediately see that for the first term to be $L_2(\mathbb{R}^3)$ bounded in h_n we must have that $w \in H_2(\mathbb{R}^3)$. Secondly since the first term is $L_2(\mathbb{R}^3)$ bounded we must have

$$\int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) \overline{\hat{w}(k)} \hat{h}_n(k) \to -\int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) \overline{\hat{w}(k)} \chi \hat{G}(k) = -\chi \int_{\mathbb{R}^3} d^3k \ \overline{\hat{w}(k)} \quad \text{as } n \to \infty. \tag{2.76}$$

The second term obviously goes to zero by continuity of the inner product. Thus we need to estimate $\int_{\mathbb{R}^3} \hat{h}$. This is done in the following lemma.

Lemma 1. *Let $w \in H_2(\mathbb{R}^3)$, then w is continuous and $\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3k \ \hat{w}(k) e^{ik\cdot x} = w(x)$.*

Proof. First that w is continuous follows from Sobolev's embedding theorem. Next notice that since $w \in H_2(\mathbb{R}^3)$ we must have that

$$\hat{w}(k) = \frac{\hat{f}(k)}{|k|^2 + 1}, \tag{2.77}$$

for some $\hat{f} \in L_2(\mathbb{R}^3)$. Thus we also see that $\hat{w}$ is clearly in $L_1(\mathbb{R}^3)$ by Hölder's inequality and the fact that $\frac{1}{|k|^2+1} \in L_2(\mathbb{R}^3)$ and therefore $\check{\hat{w}}$ is bounded and continuous. By Fourier's inversion theorem we have that $\check{\hat{w}} = w$ a.e, and by continuity of w, we conclude that $\check{\hat{w}} = w$. Since $\hat{w} \in L_1(\mathbb{R}^3)$ this amounts to

$$\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3k \ \hat{w}(k) e^{ik\cdot x} = w(x) \tag{2.78}$$

□

Given the above lemma we clearly see that

$$\chi \int_{\mathbb{R}^3} d^3k \ \overline{\hat{w}(k)} = (2\pi)^{3/2} \chi \overline{w(0)}. \tag{2.79}$$

Thereby we find the condition on the domain

$$-(2\pi)^{3/2} \chi \overline{w(0)} + \bar{\xi}\chi(\alpha + 2\pi^2\sqrt{\mu}) = 0 \tag{2.80}$$

correpsponding to the boundary condition $w(0) = (2\pi)^{-3/2}\xi(\alpha + 2\pi^2\sqrt{\mu})$. Now turning to the action of the operator H_α it is easier to consider $H_\alpha + \mu$ since we then have

$$\begin{aligned}
\langle (H_\alpha+\mu)u, v\rangle = F_\alpha(u,v) + \mu\langle u,v\rangle &= \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)\overline{\hat{w}(k)}\hat{h}(k) + \bar{\xi}\chi\left(\alpha + 2\pi^2\sqrt{\mu}\right) \\
&= \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)\overline{\hat{w}(k)}\hat{h}(k) + \chi\int_{\mathbb{R}^3} d^3k\overline{\hat{w}(k)} \\
&= \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)\overline{\hat{w}(k)}\hat{h}(k) + \chi\int_{\mathbb{R}^3} d^3k(k^2+\mu)\overline{\hat{w}(k)}\hat{G}(k) \\
&= \langle(-\Delta+\mu)w, v\rangle\,,
\end{aligned} \tag{2.81}$$

with $u = w + \xi G$ and $v = h + \chi G$ and where we used the boundary condition we found above in line 2.

We can therefore write down the Hamiltonian in the following manner:

$$\begin{gathered}
\mathscr{D}(H_\alpha) = \left\{u \in L_2(\mathbb{R}^3) \mid u = w + \xi G,\ w \in H_2(\mathbb{R}^3),\ w(0) = (2\pi)^{-3/2}\xi(\alpha+2\pi^2\sqrt{\mu}),\ \xi\in\mathbb{C}\right\} \\
(H_\alpha+\mu)u = (-\Delta+\mu)w
\end{gathered} \tag{2.82}$$

This concludes how to obtain the Hamiltonian given the quadratic form. Notice that this expression also matches the one we found by self-adjoint extension. To see this, we have to notice that there is a bit a mismatch between the normalizations of the Green functions used in the two methods. We see this by computing

$$\begin{aligned}
\hat{G}_{-\mu}(k) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} d^3x G_{-\mu}(x,0)\mathrm{e}^{-ik\cdot x} &= \frac{2\pi}{4\pi(2\pi)^{3/2}}\int_{-1}^{1} d[\cos(\varphi)]\int_0^\infty dr r\mathrm{e}^{-\sqrt{\mu}r}\mathrm{e}^{-i|k|r\cos(\varphi)} \\
= \frac{1}{(2\pi)^{3/2}}\int_0^\infty drr = \frac{1}{(2\pi)^{3/2}}\frac{\mathrm{e}^{-\sqrt{\mu}r}\sin(|k|r)}{|k|r} &= \frac{1}{(2\pi)^{3/2}}\frac{1}{2i|k|}\left[\frac{1}{i(i\sqrt{\mu}-|k|)} - \frac{1}{i(i\sqrt{\mu}+|k|)}\right] \\
= \frac{1}{(2\pi)^{3/2}}\frac{1}{2|k|}\left[\frac{i\sqrt{\mu}-|k|-i\sqrt{\mu}-|k|}{-\mu-|p^2|}\right] &= \frac{1}{(2\pi)^{3/2}}\frac{1}{\mu+|k|^2} = \frac{1}{(2\pi)^{3/2}}\frac{1}{\mu+|k|^2}
\end{aligned} \tag{2.83}$$

Thus we see that $G_{-\mu} = \frac{1}{(2\pi)^{3/2}}G$, and we get

$$\begin{aligned}
\mathscr{D}(H_\alpha) &= \left\{u \in L_2(\mathbb{R}^3) \mid u = w + \xi G,\ w \in H_2(\mathbb{R}^3),\ w(0) = (2\pi)^{-3/2}\xi(\alpha+2\pi^2\sqrt{\mu}),\ \xi\in\mathbb{C}\right\} \\
&= \left\{u \in L_2(\mathbb{R}^3) \mid u = w + (2\pi)^{3/2}(\alpha+2\pi^2\sqrt{\mu})^{-1}(2\pi)^{3/2}w(0)G_{-\mu},\ w\in H_2(\mathbb{R}^3)\right\} \\
&= \left\{u \in L_2(\mathbb{R}^3) \mid u = w + \left(\frac{\alpha}{(2\pi)^3} + \frac{\sqrt{\mu}}{4\pi}\right)^{-1} w(0)G_{-\mu},\ w\in H_2(\mathbb{R}^3)\right\}
\end{aligned} \tag{2.84}$$

We see that this exactly equal to the domain found in the previous section except for the fact that α has been replaced by $\alpha/(2\pi)^3$

$$\mathscr{D}(H^\alpha_{\mathrm{rel}}) = \left\{u \in L_2(\mathbb{R}^3) \mid u = w + \left(\alpha - i\frac{i\sqrt{\mu}}{4\pi}\right)^{-1} w(0)G_{-\mu},\ w\in H_2(\mathbb{R}^3)\right\}. \tag{2.85}$$

Therefore we conclude that $H_\alpha = H_{\mathrm{rel}}^{\frac{\alpha}{(2\pi)^3}}$.

3 Γ-convergence

In this section we are going to study the notion of Γ-convergence and apply it to the convergence problem of the quadratic form constructed in the previous section.

3.1 Introducing Γ-convergence

Let us first introduce a few definitions in order to study the Γ-convergence properties of F_α^R.

Definition 3 (Lower semicontinuity)**.** *Let $F : X \to \mathbb{R}$ be a functional on some topological space X. We say that F is lower semicontinuous at $x \in X$ if for every $t < F(x)$ there exist a neighbourhood $N_{x,t}$ such that $F(y) > t$ for all $y \in N_{x,t}$. We furthermore say that F is lower semicontinuous if it is lower semicontinuous at every point $x \in X$. In particular, if X is a normed space, we say that $F : X \to \mathbb{R}$ is norm lower semicontinuous if it is lower semicontinuous with respect to the norm topology.*

An equivalent formulation of lower semicontinuity is given by the following proposition

Proposition 2. *Let F be a functional on a topological vector space X. Then F is lower semicontinuous at $x \in X$ if and only if*

$$\liminf_{z \to x} F(z) := \sup_{U \in \mathcal{N}_x} \inf_{z \in U} F(z) \geq F(x), \tag{3.1}$$

where $\mathcal{N}_x$ denotes the set of all open neighbourhoods of x.

Proof. " $\implies$ ": Choose any $t < F(x)$, then by lower semicontinuity of F there exist a neighbourhood U' of x such that $F(y) > t$ for all $y \in U'$ but then $\inf_{y \in U'} F(y) > t$. Thus the existence of such a U' implies that $\sup_{U \in \mathcal{N}_x} \inf_{z \in U} F(z) \geq t$. Since this was true for any $t < F(x)$ we conclude that $\sup_{U \in \mathcal{N}_x} \inf_{z \in U} F(z) \geq F(x)$
" $\impliedby$ ": Assume that there exists a $t < F(x)$ such that for all $U \in \mathcal{N}_x$ there exist a $y \in U$ with $F(y) \leq t$. Then $\inf_{z \in U} F(z) \leq t$ for all $U \in \mathcal{N}_x$ which implies that $\sup_{U \in \mathcal{N}_x} \inf_{z \in U} F(z) \leq t < F(x)$. Thus we have proven the contrapositive. □

Definition 4 (Γ-upper/-lower limit)**.** *Given a topological space X, and some sequence of functional F_n on X, we define the Γ-upper and Γ-lower limits by*

$$\begin{aligned} \Gamma\text{-}\limsup_{n \to \infty} F_n(x) &= \sup_{N_x \in \mathcal{N}_x} \limsup_{n \to \infty} \inf_{y \in N_x} F_n(y), \\ \Gamma\text{-}\liminf_{n \to \infty} F_n(x) &= \sup_{N_x \in \mathcal{N}_x} \liminf_{n \to \infty} \inf_{y \in N_x} F_n(y), \end{aligned} \tag{3.2}$$

where $\mathcal{N}_x$ denotes the collection of open neighbourhoods of x.

Definition 5. *(Γ-limit) Given a topological space X, and some sequence of functional F_n on X, we say that F_n Γ-converges to the Γ-limit F if*

$$\Gamma\text{-}\liminf_{n \to \infty} F_n(x) = \Gamma\text{-}\limsup_{n \to \infty} F_n(x) = F(x), \quad \textit{for all } x \in X. \tag{3.3}$$

One immediate result of having Γ-convergence is lower semicontinuity of the limit, which is the following proposition

Proposition 3. *Let F_n be a sequence of functionals on a topological space X. Then the Γ-lower and Γ-upper limits are both lower semicontinuous in the topology of X.*

Proof. We proof this for the Γ-lower limit, but the proof is equally valid for the upper limit by exchanging all lim inf by lim sup. For the lower-limit this follows immediately by observing that For any $z \in X$ and $U \in \mathcal{N}_z$ we have that

$$\Gamma\text{-}\liminf_{n \to \infty} F_n(z) \geq \liminf_{n \to \infty} \inf_{y \in U} F_n(y).$$

It follows by the fact that U is an open neighbourhood of all its points that we then have

$$\inf_{z \in U} \Gamma\text{-}\liminf_{n \to \infty} F_n(z) \geq \liminf_{n \to \infty} \inf_{y \in U} F_n(y). \tag{3.4}$$

Now taking supremum on both sides gives us

$$\sup_{U\in\mathcal{N}_x}\inf_{z\in U}\Gamma\text{-}\liminf_{n\to\infty}F_n(z)\geq\sup_{U\in\mathcal{N}_x}\liminf_{n\to\infty}\inf_{y\in U}F_n(y)=\Gamma\text{-}\liminf_{n\to\infty}F_n(x). \tag{3.5}$$

By using Proposition 2 and the fact that the above inequality was for all $x\in X$ we conclude that $\Gamma\text{-}\liminf_{n\to\infty}F_n(x)$ is lower semicontinuous. □

As an obvious consequence of the above proposition we get the following corollary.

Corollary 1. *Let F_n be a sequence of functionals on a topological space X, such that F_n Γ-converge to F. Then F is lower semicontinuous in the topology of X.*

Now we state an interesting result relating lower semicontinuity of quadratic form to them being bounded from below. A tool that will be of great importance when applying tools of Γ-convergence to prove stability of quantum mechanical systems. The following proposition and theorem is from lecture notes by Jan Philip Solovej.

Proposition 4 (Prop. 7.5 in [7])**.** *Let $F_D\to\mathbb{R}\cup\{\infty\}$ be a norm lower semicontinuous functional on a subspace D of a Hilbert space, H. If F satisfies $F(\alpha\phi)=|\alpha|^2F(\phi)$ for all $\alpha\in\mathbb{R}$ and all $\phi\in D$,* i.e. *F is homogenous of degree 2, then F is bounded from below in the sence that there exist $M<\infty$ such that $F(\phi)\geq -M\|\phi\|^2$ for all $\phi\in D$.*

Proof. We consider the the set $S=\{h\in D\mid F(\phi)>-1\}$. By lower semi continuity of F we have that for every $h\in S$ there exist a neighbourhood $N_h\in\mathcal{N}_h$ such that $F(h')>-1$ for all $h'\in N_h$, *i.e.* all points of S are interior points. Thus S is open. By observing that $0\in S$ we conclude that there exist some $\epsilon>0$ such that the ball in D, $B_\epsilon(0)\subset D$ is contained in S. Thereby we know that for all $x\in D$ with $\|x\|<\epsilon$, we have $F(x)>-1$. For all $h\in D$ we therefore have $F(\epsilon h/2\|h\|)>-1$ which is equivalent to

$$F(h)>-\frac{4}{\epsilon^2}\|h\|^2 \tag{3.6}$$

□

Theorem 2 (Thm. 7.6 in [7])**.** *A quadratic form Q defined on a subspace $\mathscr{D}(Q)$ of a Hilbert space H is closable and bounded from below if and only if it is norm lower semicontinuous.*

Proof. Assume first that Q is bounded from below and closable. Then we can extend Q to a closed quadratic form on the set (Def. 7.4 in [7])

$$\bar{\mathscr{D}}(Q)=\{\phi\in H\mid \text{there exist a Cauchy sequence in } \mathscr{D}(Q) \text{ converging to } \phi\}.$$

Thus we may take Q to be closed. In that case we know that $\mathscr{D}(Q)$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_Q=Q(\cdot,\cdot)+M\langle\cdot,\cdot\rangle$, for some $M>0$ such that $Q(v)>-M\|v\|$ for all $v\in H$, and where $Q(\cdot,\cdot)$ denotes the symmetric sesquilinear form associated to Q. Thus the norm on the $\mathscr{D}(Q)$ is $\|\cdot\|_Q=Q(\cdot)+M\|\cdot\|^2$. Assume now that we have some sequence $(x_n)_{(n\geq1)}\subset\mathscr{D}(Q)$ converging to $x\in\mathscr{D}(Q)$. Let x_{n_j} denotes a subsequence such that

$$\liminf_{n\to\infty}Q(x_n)=\lim_{j\to\infty}Q(x_{n_j}).$$

Now if $\liminf_{n\to\infty}Q(x_n)=\infty$ we trivially have $\liminf_{n\to\infty}Q(x_n)\geq Q(x)$. On the other hand, by assuming $\liminf_{n\to\infty}Q(x_n)<\infty$ we may conclude that $Q(x_{n_j})$ is bounded. Thus $(x_{n_j})_{j\geq1}$ is bounded in $\|\cdot\|_Q$ norm. By Alaoglu's theorem and the fact that $\mathscr{D}(Q)$ is a Hilbert space, we may conclude that there any further subsequence $x_{n_{j_k}}$ contain a subsequence $x_{n_{j_{k_l}}}$ converging weakly in $\mathscr{D}(Q)$. However the limit is then bound to be x by the fact that the weak limit in

$\mathscr{D}(Q)$ is also the weak limit in H and the weak limit in H is necessarily equal to the norm limit whenever it exists. Thus all subsequences, $x_{n_{j_k}}$, of x_{n_j}, contain a further subsequence $x_{n_{j_{k_l}}}$ converging to weakly to x in $\mathscr{D}(Q)$. Therefore we may conclude that x_{n_j} converge weakly to x. However, it is a known fact that $\liminf_{j\to\infty}\|v_j\| \geq \|v\|$ for any $v_j \rightharpoonup v$, as it follows directly from Cauchy Schwartz. Using this we obtain

$$\begin{aligned}\liminf_{n\to\infty} Q(x_n) = \lim_{j\to\infty} Q(x_{n_j}) = \lim_{j\to\infty}\left(\|x_{n_j}\|_Q^2 - M\|x_{n_j}\|\right) &\geq \liminf_{j\to\infty}\|x_{n_j}\|_Q^2 - M\|x\|^2 \\ &\geq \||x\|_Q^2 - M\|x\|^2 = Q(x)\end{aligned} \tag{3.7}$$

Showing that Q is indeed norm lower semicontinuous (Proposition 2).
Now assume instead that Q is lower semicontinuous. Then we now from Proposition 4 that Q is bounded from below. On the other hand let $(x_n)_{n\geq 1}$ be a sequence such that $Q(x_n - x_m) \to 0$ for $n, m \to \infty$ and $x_n \to 0$. Then $Q(x_n) \leq \liminf_{m\to\infty} Q(x_n - x_m) \to 0$ for $n \to \infty$. So Q is closable by definition of closability (Def. 7.2, [7]). □

See also [7], Thm 7.6 for a possibly even stronger result.
Finally in this subsection we state some important result from the book by Gianni Dal Maso [5].

Theorem 3 (Thm. 13.6(a), in [5]). *Let F_n be a sequence of lower semicontinuous positive semi-definite quadratic forms on a Hilbert space H and let F be a lower semicontinuous positive semi-definite quadratic form also on H. Let A_n and A denote the corresponding operators of F_n and F respectively. If F_n Γ-converges to F in the strong topology (norm topology), and in addition*

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n), \tag{3.8}$$

for all sequences x_n converging weakly to x. Then A_n converges to A in the strong resolvent sense, i.e. *$R_z(A_n)x \to R_z(A)x$ in norm for all $x \in H$ and all $z \in \rho(A)$, where $R_z(A) = (A - z)^{-1}$ denotes the resolvent of A.*

Notice that we can construct a positive semi-definite quadratic form from any bounded from below quadratic form, $q(v) \geq -m\|v\|^2$ by $Q = q + M\|\cdot\|^2$ where $M \geq m$. Thereby the result in Theorem 3 extends to bounded from below operators: Consider a bounded from below operator $F_n(v) \geq -m\|v\|^2$, and construct the positive semi-definite quadratic forms $\tilde{F}_n = F_n + m\|\cdot\|^2$ and $\tilde{F} = F + m\|\cdot\|^2$. If F_n and F satisfies the assumptions of Theorem 3, then $\tilde{F}_n$ and $\tilde{F}$ also satisfies them. To see this, notice that $m\|\cdot\|^2$ is a constant sequence and thus it is pointwise (continuously) convergent therefore by Prop. 6.20 in [5] we know that

$$\Gamma\text{-}\lim_{n\to\infty} \tilde{F}_n = \Gamma\text{-}\lim_{n\to\infty} F_n + m\|\cdot\| \tag{3.9}$$

Furthmore, if $x_n \to x$ weakly, it also holds that

$$\liminf_{n\to\infty} \tilde{F}_n(x_n) \geq \liminf_{n\to\infty} F_n(x_n) + m \liminf_{n\to\infty}\|x_n\|^2 \geq \liminf_{n\to\infty} F_n(x_n) + m\|x\|^2 \geq \tilde{F}(x), \tag{3.10}$$

where we have used the basic property of lim inf that $\liminf_n(a_n+b_n) \geq \liminf_n a_n + \liminf_n b_n$, and that $\liminf_{n\to\infty}\|x_n\| \geq \|x\|$ for $x_n \to x$ weakly. The last property follows from the Cauchy-Schwartz inequality: If $x_n \rightharpoonup x$ (converges weakly), then $|\langle x_n, x\rangle| \to \|x^2\|$, however $|\langle x_n, x\rangle| \leq \|x_n\|\|x\|$, so ultimately we have for all $\epsilon > 0$ there exist $k \geq 1$ such that $\|x_n\| \geq \|x\| - \epsilon$, but then $\liminf_{n\to\infty}\|x_n\| \geq \|x\|$. Therefore we use Theorem 3 on $\tilde{F}$ and conclude that its corresponding operators $\tilde{A}_n$ converges to $\tilde{A}$ in the norm resolvent sense. Now using that $\tilde{A}_n = A_n + mI$ and $\tilde{A} = A + mI$, where A and A_n denotes the operators corresponding to F and F_n respectively, and the fact that if $R_z(A_n + mI)x \to R_z(A + mI)x$ in norm as $n \to \infty$, then $R_z(A_n)x = R_{z+m}(A_n + Im)x \to R_{z+m}(A + Im)x = R_z(A)$ in norm as $n \to \infty$, we get the following corollary

Corollary 2. *Let F_n be a sequence of lower semicontinuous bounded from below quadratic forms with a common lower bound, $F_n(v) \geq -m\|v\|^2$ on a Hilbert space H and let F be a lower semicontinuous bounded from below quadratic form with the same lower bound $F(v) \geq -m\|v\|^2$, on H. Let A_n and A denote the corresponding operators of F_n and F respectively. If F_n Γ-converges to F in the strong topology (norm topology), and in addition*

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n), \tag{3.11}$$

for all sequences x_n converging weakly to x. Then A_n converges to A in the strong resolvent sense, i.e. *$R_z(A_n)x \to R_z(A)x$ in norm for all $x \in H$ and all $z \in \rho(A)$, where $R_z(A) = (A-z)^{-1}$ denotes the resolvent of A.*

Another important result concerning Γ-convergence, it the classification of Γ-convergence in first countable spaces, such as for example normed spaces. The result can be characterized by the following proposition

Proposition 5 (Prop. 8.1 (e) and (f), in [5])**.** *Let X be a first countable space and F_n be a sequence of functions on X. Then F_n Γ-converges to the function F if and only the two following requirements are satisfied:*
(i) For all $x \in X$ and for every sequence x_n converging to x in X we have

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n); \tag{3.12}$$

(ii) For all $x \in X$ there exist a sequence $(x_n)_(n \geq 1) \subset X$ such that

$$F(x) = \lim_{n\to\infty} F_n(x_n). \tag{3.13}$$

Other important result concerning Γ-convergence are preservation of minimizers under Γ-convergence. These can be found in chapter 7 of [5].

3.2 Γ-convergence of F_α^R

We are in this subsection going to show that the quadratic forms, F_α^R that we discussed earlier, have some Γ-convergence properties, that will allow us to conclude on the convergence properties of their corresponding operators.
In the following we will apply the lemma:

Lemma 2. *Let $w \in H_1(\mathbb{R}^3)$, then*

$$\frac{1}{\sqrt{n}} \int_{B_n(0)} \hat{w} \to 0, \quad \text{as } n \to \infty, \tag{3.14}$$

where $\hat{w}$ denotes the Fourier transform of w. In other notation

$$\left| \int_{B_n(0)} \hat{w} \right| = \epsilon(n)\sqrt{n}, \tag{3.15}$$

for some ϵ-function, $\epsilon(n) \to 0$ as $n \to \infty$.

Proof. Notice that we have already seen that for $w \in H_1(\mathbb{R}^3)$ we have

$$\int_{B_n(0)} \hat{w} \lesssim \sqrt{n}, \tag{3.16}$$

which follows from Cauchy-Schwartz inequality. To improve the bound, we split the integral. Clearly $\hat{w}(k) = \frac{f(k)}{(k^2+1)^{1/2}}$ for some $f \in L_2(\mathbb{R}^3)$. Let $\epsilon_n = (\ln(n))^2/n^{3/2}$ and define the sets

$B_n^{>} = B_n(0) \cap \{|f| > \epsilon_n\}$ and $B_n^{\leq} = B_n(0) \cap \{|f| \leq \epsilon_n\}$. Notice that $\epsilon_n \to 0$ as $n \to \infty$. By the usual triangle inequality we have

$$\left| \int_{B_n(0)} \hat{w} \right| \leq \left| \int_{B_n^{>}} \hat{w} \right| + \left| \int_{B_n^{\leq}} \hat{w} \right|. \tag{3.17}$$

We see that

$$\left| \int_{B_n^{\leq}} \hat{w} \right| \leq \| \mathbb{1}_{|f| \leq \epsilon_n} f \|_2 \sqrt{n} \tag{3.18}$$

By dominated convergence theorem we now that $\| \mathbb{1}_{|f| \leq \epsilon_n} f \|_2 \to 0$, since $|\mathbb{1}_{|f| \leq \epsilon_n} f|^2 \to 0$ pointwise ($\epsilon_n \to 0$), and is dominated by integrable function $|f|^2$. Thereby we have

$$\left| \int_{B_n^{\leq}} \hat{w} \right| = \epsilon_1(n) \sqrt{n} \tag{3.19}$$

for $\epsilon_1(n) \to 0$ as $n \to \infty$. On the other hand we use Hölder's inequality $(3/2, 3)$ on the other integral and obtain

$$\left| \int_{B_n^{>}} \hat{w} \right| \leq \int_{B_n^{>}} |\hat{w}| \leq \| \mathbb{1}_{|f| > \epsilon_n} f \|_{3/2} \| \mathbb{1}_{B_n(0)} (1 + k^2)^{-1/2} \|_3. \tag{3.20}$$

Now $\| \mathbb{1}_{B_n(0)} (1 + k^2)^{-1/2} \|_3 \sim (\ln(n))^{1/3}$. Furthermore, we have

$$\| \mathbb{1}_{|f| > \epsilon_n} f \|_{3/2} \| = \left(\int_{f > \epsilon_n} |f|^{3/2} \right)^{2/3} < \left(\frac{1}{\sqrt{\epsilon_n}} \int_{f > \epsilon_n} |f|^2 \right)^{2/3} = \frac{1}{\epsilon^{1/3}} \|f\|_2^{4/3}. \tag{3.21}$$

Thus we have

$$\left| \int_{B_n^{>}} \hat{w} \right| \lesssim \frac{1}{(\ln(n))^{1/3}} \|f\|_2^{4/3} \sqrt{n} = \epsilon_2(n) \sqrt{n} \tag{3.22}$$

for some $\epsilon_2(n) \to 0$ as $n \to \infty$. Combining our two estimates gives us

$$\left| \int_{B_n(0)} \hat{w} \right| \lesssim (\epsilon_1(n) + \epsilon_2(n)) \sqrt{n} \implies \left| \int_{B_n(0)} \hat{w} \right| = \epsilon(n) \sqrt{n}, \tag{3.23}$$

for some $\epsilon(n) \to 0$ as $n \to \infty$. This concludes the proof. □

We show that F_α^n where $n \in \mathbb{N}$ satisfies the assumptions of Theorem 3. By Proposition 5 we see that it suffices to show that:

1. For all $u \in \mathscr{D}(F_\alpha)$ there exist a sequence $(u_n)_{n \geq 1}$, such that $u_n \in \mathscr{D}(F_\alpha^n)$, u_n converges in $L_2(\mathbb{R}^3)$ to u and
$$F_\alpha(u) = \lim_{n \to \infty} F_\alpha^n(u_n). \tag{3.24}$$
2. For all $u \in \mathscr{D}(F_\alpha)$ and all $(u_n)_{n \geq 1}$, with $u_n \in \mathscr{D}(F_\alpha^n)$, converging weakly to u, we have
$$F_\alpha(u) \leq \liminf_{n \to \infty} F_\alpha^n(u_n). \tag{3.25}$$

Notice then that (2.) include both the weak lower bound on F_α as required by Theorem 3 but also the strong lower bound on F_α as required by Proposition 5, since all strongly converging sequences also are weakly convergent. Let us briefly remind ourselves that

$$F_\alpha^R(u) = \int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) |\hat{w}_R(k)|^2 - \mu \|u\|_{L_2(\mathbb{R}^3)}^2 + |\xi_R|^2 \left(\alpha + 4\pi\sqrt{\mu} \arctan\left(\frac{R}{\sqrt{\mu}} \right) \right), \tag{3.26}$$

with $\hat{w}_R = \hat{u} - \widehat{G\rho^R}$, $\hat{\rho}^R = \xi_R \mathbb{1}_{B_R(0)}$, $\xi_R = (4\pi R + \alpha)^{-1} \int_{B_R(0)} \hat{u}$, and domain

$$\mathscr{D}\left(F_\alpha^R\right) = \left\{u \in L_2(\mathbb{R}^3) \mid w_R \in H_1(\mathbb{R}^3)\right\} = H_1(\mathbb{R}^3). \tag{3.27}$$

And

$$F_\alpha(u) = \int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) |\hat{w}(k)|^2 - \mu \|u\|^2_{L_2(\mathbb{R}^3)} + |\xi|^2 \left(\alpha + 2\pi^2 \sqrt{\mu}\right), \tag{3.28}$$

with domain

$$\mathscr{D}\left(F_\alpha(u)\right) = \left\{u \in L_2(\mathbb{R}^3) \mid \hat{u} = \hat{w} + \xi \hat{G}, \ w \in H_1(\mathbb{R}^3), \ \xi \in \mathbb{C}\right\}, \tag{3.29}$$

Proposition 6. *Let F_α^n and F_α be defined as earlier. Then for all $u \in \mathscr{D}(F_\alpha)$ there exist $(u_n)_{(n\geq 1)}$ with $u_n \in \mathscr{D}(F_n)$ such that*

$$F_\alpha(u) = \lim_{n\to\infty} F_\alpha^n(u_n). \tag{3.30}$$

Proof. We prove this result by simply constructing the correct sequence u_n. For $u = w + \xi G$, with $w \in H_1(\mathbb{R}^3)$, let $u_n = w + \xi G_n$, where $\widehat{G_n} = \mathbb{1}_{B_n(0)} \hat{G}$. Then we clearly have

$$F_\alpha^n(u_n) = \int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) |\hat{w}_n(k)|^2 - \mu \|u_n\|^2_{L_2(\mathbb{R}^3)} + |\xi_n|^2 \left(\alpha + 4\pi\sqrt{\mu} \arctan\left(\frac{n}{\sqrt{\mu}}\right)\right), \tag{3.31}$$

with $\hat{w}_n = \hat{w} - (\xi_n - \xi)\mathbb{1}_{B_n(0)}\hat{G}$. Now it is obvious that $u_n \in H_1(\mathbb{R}^3)$ for all n and that $u_n \to u$ in $L_2(\mathbb{R}^3)$ as $n \to \infty$. Furthermore, we see that

$$\xi_n = \frac{1}{4\pi n + \alpha} \int_{B_n(0)} \hat{w} + \xi \hat{G}, \tag{3.32}$$

and since $\int_{B_n(0)} \hat{w} \lesssim \sqrt{n}$, and $\int_{B_n(0)} \hat{G} \sim 4\pi n$ we see that $\xi_n \to \xi$ as $n \to \infty$. We can even say that $(\xi - \xi_n) \sim \frac{1}{4\pi n} \int_{B_n(0)} \hat{w}$. By lemma 2, we know that $\left|\frac{1}{4\pi n} \int_{B_n(0)} \hat{w}\right| = \epsilon(n) \frac{1}{\sqrt{n}}$ for some ϵ-function $\epsilon(n) \to 0$ as $n \to \infty$. Now expanding the expression for $F_\alpha^n(u_n)$ above we have

$$\begin{aligned} F_\alpha^n(u_n) = \int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) |\hat{w}(k)|^2 + |\xi_n - \xi|^2 \int_{B_n(0)} d^3k \ \left(k^2 + \mu\right) |\hat{G}(k)|^2 \\ - 2\mathrm{Re}(\xi_n - \xi) \int_{B_n(0)} d^3k \ \left(k^2 + \mu\right) \overline{\hat{w}(k)} \hat{G} \\ - \mu \|u_n\|^2_{L_2(\mathbb{R}^3)} + |\xi_n|^2 \left(\alpha + 4\pi\sqrt{\mu} \arctan\left(\frac{n}{\sqrt{\mu}}\right)\right) \end{aligned} \tag{3.33}$$

Using that $\hat{G}(k) = \frac{1}{k^2+\mu}$ we have

$$|\xi_n - \xi|^2 \int_{B_n(0)} d^3k \ \left(k^2 + \mu\right) |\hat{G}(k)|^2 \sim \frac{\epsilon(n)^2}{n} \int_{B_n(0)} d^3k \ \hat{G}(k) \sim \epsilon(n)^2 \to 0 \quad \text{as } n \to \infty. \tag{3.34}$$

Furthermore we know that

$$\left|(\xi_n - \xi) \int_{B_n(0)} d^3k \ \left(k^2 + \mu\right) \overline{\hat{w}(k)} \hat{G}\right| = \left|(\xi_n - \xi) \int_{B_n(0)} d^3k \ \overline{\hat{w}(k)}\right| \sim \epsilon(n)^2 \to 0 \quad \text{as } n \to \infty. \tag{3.35}$$

Finally we have that $\|u_n\| \to \|u\|$ as $n \to \infty$ since $u_n \to u$ in $L_2(\mathbb{R}^3)$ and that

$$|\xi_n|^2 \left(\alpha + 4\pi\sqrt{\mu} \arctan\left(\frac{n}{\sqrt{\mu}}\right)\right) \to |\xi|^2 \left(\alpha + 2\pi^2 \sqrt{\mu}\right), \quad \text{as } n \to \infty, \tag{3.36}$$

where we have used that the limit of a product is the product of the limits whenever both limits exist. Thus we have in total

$$F_\alpha^n(u_n) \to \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{w}(k)|^2 - \mu\|u\|^2_{L_2(\mathbb{R}^3)} + |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right) = F_\alpha(u), \tag{3.37}$$

as we wanted. □

Proposition 7. *Let $(u_n)_{(n\geq 1)}$ be any sequence such that $u_n \in \mathscr{D}(F_\alpha^n) = H_1(\mathbb{R}^3)$ for all $n \geq 1$ and such that u_n converges weakly to u ($u_n \rightharpoonup u$) in $L_2(\mathbb{R}^3)$ as $n \to \infty$. Then we have the following inequality*

$$F_\alpha(u) \leq \liminf_{n\to\infty} F_\alpha^n(u_n). \tag{3.38}$$

Proof. Let u_n be as in the proposition above. Since the weak limit, u, is in $\mathscr{D}(F_\alpha)$ it is of the form $u = w + \xi G$ with $w \in H_1(\mathbb{R}^3)$ and $\xi \in \mathbb{C}$. On the other hand we then may without loss of generality take u_n to be of the form $u_n = w + \xi g_n$, with $g_n \in H_1(\mathbb{R}^3)$ such that $g_n \rightharpoonup G$ in $L_2(\mathbb{R}^3)$ as $n \to \infty$. We then have

$$F_\alpha^n(u_n) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{w}_n(k)|^2 - \mu\|u_n\|^2_{L_2(\mathbb{R}^3)} + |\xi_n|^2\left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{n}{\sqrt{\mu}}\right)\right), \tag{3.39}$$

where we have defined $\hat{w}_n(k) = \hat{w}(k) + \xi\hat{g}_n - \xi_n\hat{G}_n$, with $\hat{G}_n = \mathbb{1}_{B_n(0)}\hat{G}$, and $\xi_n = (4\pi n+\alpha)^{-1}\int_{B_n(0)} \hat{w} + \xi\hat{g}_n$. Now there exist a subsequence $F_\alpha^{n_j}(u_{n_j})$ that converges to $\liminf_{n\to\infty} F_\alpha^n(u_n)$. Expanding $F_\alpha^{n_j}(u_{n_j})$ we have

$$\begin{aligned} F_\alpha^{n_j}(u_{n_j}) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)\left\{|\hat{w}(k)|^2 + |\xi\hat{g}_{n_j}(k) - \xi_{n_j}\hat{G}_{n_j}(k)|^2 + 2\mathrm{Re}\left[(\xi\hat{g}_{n_j}(k) - \xi_{n_j}\hat{G}_{n_j}(k))\overline{\hat{w}(k)}\right]\right\} \\ -\mu\|u_{n_j}\|^2_{L_2(\mathbb{R}^3)} + |\xi_{n_j}|^2\left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{n_j}{\sqrt{\mu}}\right)\right) \end{aligned} \tag{3.40}$$

Since $\int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{f}(k)|^2 \cong \|f\|^2_{H_1(\mathbb{R}^3)}$ we see that either $\liminf F_\alpha^n(u_n) = \infty$ or $(\xi g_{n_j} - \xi_{n_j}G_{n_j})$ is $H_1(\mathbb{R}^3)$ norm bounded. In the first case the desired result is trivially true. In the second case we observe that if $h_j := (\xi g_{n_j} - \xi_{n_j}G_{n_j})_{j\geq 1}$ is $H_1(\mathbb{R}^3)$ norm bounded such that $(h_j)_{j\geq 1} \subset B(0,M)$ for some $M > 0$. By Alaoglu's theorem and the fact that $H_1(\mathbb{R}^3)$ is a Hilbert space we then known that all subsequences h_{j_k} have a further subsequence $h_{j_{k_l}}$ that converge weakly in $H_1(\mathbb{R}^3)$ to some $h \in H_1(\mathbb{R}^3)$. However, if $h_{j_{k_l}}$ converge weakly in $H_1(\mathbb{R}^3)$ it also converges weakly in $L_2(\mathbb{R}^3)$ to the same limit, since the dual space of $H_1(\mathbb{R}^3) \subset L_2(\mathbb{R}^3)$ is $H_{-1}(\mathbb{R}^3) \supset L_2(\mathbb{R}^3)$. Thus we know that $h_{j_{k_l}}$ converges weakly in $L_2(\mathbb{R}^3)$. However, as we know that $g_n \rightharpoonup G$ in $L_2(\mathbb{R}^3)$ and $G_n \xrightarrow{\|\cdot\|_2} G$ so $G_n \rightharpoonup G$ in $L_2(\mathbb{R}^3)$ we conclude that in order for $h_{j_{k_l}} \rightharpoonup h$ in $L_2(\mathbb{R}^3)$ we must have that $\xi_{n_{j_{k_l}}} \to \chi$ for some χ and then $h = (\xi - \chi)G$. However notice that $G \notin H_1(\mathbb{R}^3)$. So $h \in H_1(\mathbb{R}^3)$ implies that $\chi = \xi$, such that $h_{j_{k_l}} \rightharpoonup 0$ in $H_1(\mathbb{R}^3)$ and $L_2(\mathbb{R}^3)$. But then we have shown that for all subsequences $(h_{j_k})_{k\geq 1}$ of $(h_j)_{j\geq 1}$, there exist a further subsequence, $h_{j_{k_l}}$ converging weakly to 0. Thereby $h_j \rightharpoonup 0$ in $H_1(\mathbb{R}^3)$ and $L_2(\mathbb{R}^3)$ and $\xi_{n_j} \to \xi$ as $j \to \infty$. Thus we have

$$\begin{aligned} \lim_{j\to\infty} F_\alpha^{n_j}(u_{n_j}) &= \lim_{j\to\infty}\left(\int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)\left\{|\hat{w}(k)|^2 + |\xi\hat{g}_{n_j}(k) - \xi_{n_j}\hat{G}_{n_j}(k)|^2\right\} - \mu\|u_{n_j}\|^2_{L_2(\mathbb{R}^3)}\right) \\ &\quad + |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right) \\ &\geq \lim_{j\to\infty}\left(\int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{w}(k)|^2 + \mu(\|\xi g_{n_j} - \xi_{n_j}G_{n_j}\|^2_{L_2(\mathbb{R}^3)} - \|u_{n_j}\|^2_{L_2(\mathbb{R}^3)})\right) \\ &\quad + |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right) \end{aligned} \tag{3.41}$$

where we in the second line threw away the positive term $\int_{\mathbb{R}^3} d^3k\; k^2|\xi\hat{g}_{n_j}(k)-\xi_{n_j}\hat{G}_{n_j}(k)|^2$. Now we see that

$$\begin{aligned}\|\xi g_{n_j}-\xi_{n_j}G_{n_j}\|^2-\|u_{n_j}\|^2=\|\xi g_{n_j}\|^2+\|\xi_{n_j}G_{n_j}\|^2-2\xi_{n_j}\bar{\xi}\mathrm{Re}\left\langle g_{n_j},G_{n_j}\right\rangle\\ -\|w\|^2-\|\xi g_{n_j}\|^2-2\xi\mathrm{Re}\left\langle w,g_{n_j}\right\rangle,\end{aligned} \tag{3.42}$$

from which it follows that

$$\begin{aligned}\lim_{j\to\infty}\left(\|\xi g_{n_j}-\xi_{n_j}G_{n_j}\|^2-\|u_{n_j}\|^2\right)&=\|\xi G\|^2-\|w\|^2-2|\xi|^2\left\langle G,G\right\rangle-2\xi\mathrm{Re}\left\langle w,G\right\rangle\\ &=-\|w\|^2-\|\xi G\|^2-2\xi\mathrm{Re}\left\langle w,G\right\rangle=-\|w+\xi G\|^2\\ &=-\|u\|^2\end{aligned} \tag{3.43}$$

where $\|\cdot\|=\|\cdot\|_{L_2(\mathbb{R}^3)}$, $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{L_2(\mathbb{R}^3)}$ and we have used that for $f_n\xrightarrow{\|\cdot\|}f$ and $g_n\rightharpoonup g$ as $n\to\infty$ we have $\langle g_n,f_n\rangle\to\langle g,f\rangle$, which follows directly from norm boundedness of weakly convergent sequences. Collecting all the above we have

$$\lim_{j\to\infty}F_\alpha^{n_j}(u_{n_j})\geq\int_{\mathbb{R}^3}d^3k\;\left(k^2+\mu\right)|\hat{w}(k)|^2-\mu\|u\|^2_{L_2(\mathbb{R}^3)}+|\xi|^2\left(\alpha+2\pi^2\sqrt{\mu}\right)=F_\alpha(u), \tag{3.44}$$

which was the desired result. □

Now collecting the above results in Propositions 6 and 7 we find that F_α^n and F_α indeed satisfies the assumptions of Theorem 3, and thus we may conclude that the corresponding operator H_α^n and H_α fulfil that $H_\alpha^n\to H_\alpha$ in the strong resolvent sense. This tells us that points in the spectrum cannot suddenly emerge under the convergence as the following proposition will show.

Proposition 8. *Let $(A_n)_{n\geq1}$ be a sequence of operators in a Hilbert space H and let A be an operator on H. Assume that $A_n\to A$ in the strong resolvent sense. If $\lambda\notin\sigma(A_n)$ for all $n\geq M_\lambda$ for some $M_\lambda>0$, then $\lambda\notin\sigma(A)$, where $\sigma(\cdot)$ denotes the spectrum.*

Proof. Let $R_\lambda(A_n)=(A_n-\lambda I)^{-1}$ denote the resolvents of A_n at λ. If there exist M_λ such that $\lambda\notin\sigma(A_n)$ for all $n\geq M_\lambda$ we know that $R_\lambda(A_n)\in\mathcal{B}(H)$ (bounded operators on H) for all $n\geq M_\lambda$. By the strong resolvent convergence we know that $R_\lambda(A_n)f$ converge to $R_\lambda(A)f$ as $n\to\infty$ for all $f\in H$. Thus we conclude that

$$\sup_{n\geq1}\left(\|R_\lambda(A_n)f\|\right)<\infty,\qquad\text{for all } f\in H. \tag{3.45}$$

By the uniform boundedness principle ([3], 5.13) we conclude that

$$\sup_{n\geq1}\left(||R_\lambda(A_n)||\right)<\infty. \tag{3.46}$$

Now estimating the operator norm of $R_\lambda(A)$ we find

$$\|R_\lambda(A)f\|<\|R_\lambda(A_{n_\epsilon(f)})f\|+\epsilon \tag{3.47}$$

for some $n_\epsilon(f)\geq1$ depending on ϵ and f. Thus taking supremum on both sides over all $f\in H$ with $\|f\|\leq1$ we find

$$\begin{aligned}\|R_\lambda(A)\|=\sup\left(\|R_\lambda(A)f\|\mid f\in H,\ \|f\|\leq1\right)&\leq\sup\left(\|R_\lambda(A_{n_\epsilon(f)})f\|\mid f\in H,\ \|f\|\leq1\right)+\epsilon\\ &\leq\sup_{n\geq1}\left(||R_\lambda(A_n)||\right)+\epsilon<\infty.\end{aligned} \tag{3.48}$$

Thereby $R_\lambda(A)\in\mathcal{B}(H)$ and $\lambda\notin\sigma(A)$. □

Another immediate consequence of Propositions 6 and 7 is that F_α^n Γ-converge to F_α in the norm topology. Thus we may conclude that F_α is a norm lower semicontinuous quadratic form, an by Theorem 2 we may conclude that F_α is closable and bounded from below.

A λ relation in Krein formula (2.26)

To show the relation of $\lambda(z,\bar{z})$ we use the following properties established in the proof in the main text. We have that

$$(B-z)^{-1}-(C-z)^{-1}=\lambda(z)\,\langle\phi(\bar{z}),\cdot\rangle\,\phi(z), \tag{A.1}$$

where we have already used that $\lambda(z,\bar{z})=\lambda(z)$, and where we have defined

$$\phi(z)=\phi(z_0)+(z-z_0)(C-z)^{-1}\phi(z_0), \tag{A.2}$$

for some $\phi(z_0)$ satisfying $A^*\phi(z_0)=z_0\phi(z_0)$.
In the following we will switch to bra-ket notation to simplify the calculations and ease the notation. Now let us consider the relation

$$(B-z)^{-1}=(C-z)^{-1}+\lambda(z)\,|\phi(z)\rangle\,\langle\phi(\bar{z})|\,. \tag{A.3}$$

By multiplying this relation with itself, but with z' instead of z we get

$$\begin{aligned}(B-z)^{-1}(B-z')^{-1}&=(C-z)^{-1}(C-z')^{-1}+\lambda(z)\,|\phi(z)\rangle\,\langle\phi(\bar{z})|\,(C-z')^{-1}\\&+\lambda(z')(C-z)^{-1}\,|\phi(z')\rangle\,\langle\phi(\bar{z}')|+\lambda(z)\lambda(z')\,|\phi(z)\rangle\,\langle\phi(\bar{z}),\phi(z')\rangle\,\langle\phi(\bar{z}')|\,.\end{aligned} \tag{A.4}$$

Now using that $(B-z)^{-1}-(B-z')^{-1}=(z-z')(B-z)^{-1}(B-z')^{-1}$ and the same relation for C we get, by multiplying through with $(z-z')$ that

$$\begin{aligned}&(B-z)^{-1}-(B-z')^{-1}-(C-z)^{-1}+(C-z')^{-1}=\\&\quad(z-z')\lambda(z)\,|\phi(z)\rangle\,\langle\phi(\bar{z})|\,(C-z')^{-1}+(z-z')\lambda(z')(C-z)^{-1}\,|\phi(z')\rangle\,\langle\phi(\bar{z}')|\\&\qquad+(z-z')\lambda(z)\lambda(z')\,|\phi(z)\rangle\,\langle\phi(\bar{z}),\phi(z)'\rangle\,\langle\phi(\bar{z}')|\,.\end{aligned} \tag{A.5}$$

Using again the relation (A.1) we obtain

$$\begin{aligned}&\lambda(z)\,|\phi(z)\rangle\,\langle\phi(\bar{z})|-\lambda(z')\,|\phi(z')\rangle\,\langle\phi(\bar{z}')|=\\&\quad(z-z')\lambda(z)\,|\phi(z)\rangle\,\langle\phi(\bar{z})|\,(C-z')^{-1}+(z-z')\lambda(z')(C-z)^{-1}\,|\phi(z')\rangle\,\langle\phi(\bar{z}')|\\&\qquad+(z-z')\lambda(z)\lambda(z')\,|\phi(z)\rangle\,\langle\phi(\bar{z}),\phi(z)'\rangle\,\langle\phi(\bar{z}')|\end{aligned} \tag{A.6}$$

from which we obtain by simple rearrangement

$$\begin{aligned}&\lambda(z)\,|\phi(z)\rangle\,\langle\phi(\bar{z})|\,\big(I-(z-z')(C-z')^{-1}\big)-\lambda(z')\,\big(I-(z'-z)(C-z)^{-1}\big)\,|\phi(z')\rangle\,\langle\phi(\bar{z}')|=\\&\qquad(z-z')\lambda(z)\lambda(z')\,|\phi(z)\rangle\,\langle\phi(\bar{z}),\phi(z)'\rangle\,\langle\phi(\bar{z}')|\end{aligned} \tag{A.7}$$

Notice now that

$$\big(I-(z-z')(C-z')^{-1}\big)=(C-z')^{-1}(C-z),\quad\text{and}\quad\big(I-(z'-z)(C-z)^{-1}\big)=(C-z')(C-z)^{-1}. \tag{A.8}$$

Now clearly by (A.2) we have

$$(C-z')(C-z)^{-1}\phi(z')=(C-z')(C-z)^{-1}\phi(z_0)+(z'-z_0)(C-z)^{-1}\phi(z_0) \tag{A.9}$$

where we have used that $(C-z')(C-z)^{-1}h=(C-z)^{-1}(C-z')h$ whenever $h\in\mathscr{D}\left((C-z')\right)$.
By using that $(C-z')(C-z)^{-1}=I+(z-z')(C-z)^{-1}$ we obtain

$$(C-z')(C-z)^{-1}\phi(z')=\phi(z_0)+(z-z_0)(C-z)^{-1}\phi(z_0)=\phi(z). \tag{A.10}$$

By a similar computation we have

$$(C-z)(C-z')^{-1}\phi(z)=\phi(z_0)+(z'-z_0)(C-z')^{-1}\phi(z_0)=\phi(z'), \tag{A.11}$$

and thus we obtain

$$\begin{aligned}&\lambda(z)\left|\phi(z)\right\rangle\left\langle\phi(\bar z)\right|\left(I-(z-z')(C-z')^{-1}\right)-\lambda(z')\left(I-(z'-z)(C-z)^{-1}\right)\left|\phi(z')\right\rangle\left\langle\phi(\bar z')\right|=\\&\lambda(z)\left|\phi(z)\right\rangle\left\langle\left(I-(\bar z-\bar z')(C-\bar z')^{-1}\right)\phi(\bar z)\right|-\lambda(z')\left|\left(I-(z'-z)(C-z)^{-1}\right)\phi(z')\right\rangle\left\langle\phi(\bar z')\right|=\\&\lambda(z)\left|\phi(z)\right\rangle\left\langle\phi(\bar z')\right|-\lambda(z')\left|\phi(z)\right\rangle\left\langle\phi(\bar z')\right|=(z-z')\lambda(z)\lambda(z')\left|\phi(z)\right\rangle\left\langle\phi(\bar z),\phi(z)'\right\rangle\left\langle\phi(\bar z')\right|.\end{aligned} \tag{A.12}$$

Observing the last line in the above calculation we observe that we have the relation

$$\lambda(z)-\lambda(z')=\lambda(z)\lambda(z')(z-z')\left\langle\phi(\bar z),\phi(z')\right\rangle, \tag{A.13}$$

which is equivalent to the relation

$$\lambda(z)^{-1}-\lambda(z')^{-1}=-(z-z')\left\langle\phi(\bar z),\phi(z')\right\rangle. \tag{A.14}$$

This proves equation (2.26)

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