Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.

(a)

Let $x \in H$ then, we can write $x = \sum_{k=1}^{m} x_k e_{n(k)}$ because $(e_n)_{n\geq 1}$ is a basis. We have that

$$|\langle f_N, x\rangle - \langle 0, x\rangle| = |\langle f_N, x\rangle| = |\langle N^{-1}\sum_{n=1}^{N^2} e_n, \sum_{k=1}^{m} x_k e_{n(k)}\rangle| \leq N^{-1}|\sum_{k=1}^{m}\overline{x_k}\langle e_{n(k)}, e_{n(k)}\rangle|$$
$$= N^{-1}\sum_{k=1}^{m}|\overline{x_k}| = N^{-1}\sum_{k=1}^{m}|x_k|$$

If $x = 0$ then $|\langle f_N, x\rangle - \langle 0, x\rangle| = 0$ for all $N \geq 1$. So $\langle f_N, 0\rangle \to \langle 0, x\rangle$ as $N \to \infty$. If $x \neq 0$ then $0 < \sum_{k=1}^{m}|x_k| < \infty$. Let $\epsilon > 0$ and choose $N' \geq \epsilon^{-1}\sum_{k=1}^{m}|x_k|$ then for $N \geq N'$ we have that:

$$|\langle f_N, x\rangle - \langle 0, x\rangle| \leq N^{-1}\sum_{k=1}^{m}|x_k| \leq (\epsilon^{-1}\sum_{k=1}^{m}|x_k|)^{-1}\sum_{k=1}^{m}|x_k| = \epsilon$$

So $\langle f_N, x\rangle \to \langle 0, x\rangle$ as $N \to \infty$. By Riesz representation theorem (HW2 problem 1) any linear functional in H most have the form $\langle \cdot, x\rangle$ for some $x \in H$. So for any linear functional $f \in H^*$, we have $f(f_N) \to f(0)$ as $N \to \infty$. So by HW4 problem 2 (a) we have that $f_N \to 0$ weakly as $N \to \infty$ because $f(f_N) \to f(0)$ as $N \to \infty$ for every $f \in H^*$.

We have that

$$\|f_N\| = \sqrt{\langle f_N, f_N\rangle} = \sqrt{\langle N^{-1}\sum_{n=1}^{N^2} e_n, N^{-1}\sum_{n=1}^{N^2} e_n\rangle} = \sqrt{N^{-1}\overline{N^{-1}}\sum_{n=1}^{N^2}\langle e_n, e_n\rangle}$$
$$= \sqrt{N^{-1}N^{-1}N^2} = 1$$

(b)

Let K be the norm closure of $\text{co}\{f_N : N \geq 1\}$. By definition $\text{co}\{f_N : N \geq 1\}$ is convex.

We show the closure is also convex: Because $\text{co}\{f_N : N \geq 1\}$ is non-empty there exists points $x, y \in K$. Because H is a metric space there exist sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ inside $\text{co}\{f_N : N \geq 1\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. So we have $\alpha x_n + (1-\alpha)y_n$ is in $\text{co}\{f_N : N \geq 1\}$ for every $n \geq 1$ and $0 \leq \alpha \leq 1$. So $\alpha x + (1-\alpha)y = \alpha\lim_{n\to\infty} x_n + (1-\alpha)\lim_{n\to\infty} y_n = \lim_{n\to\infty}(\alpha x_n + (1-\alpha)y_n)$ is inside K for every $0 \leq \alpha \leq 1$, hence K is convex.

Because $\text{co}\{f_N : N \geq 1\}$ is a convex subset of H, we have by Theorem 5.7 that $K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w}$. So K is closed in the weak topology.

H is a reflexive Banach space because Hilbert spaces are reflexive Banach spaces by Proposition 2.10. $\overline{B_H(0,1)}$ is compact in the weak topology on H because this is the case in reflexive Banach spaces, cf. Theorem 6.3.

An element in $\text{co}\{f_N : N \geq 1\}$ has the form $\sum_{i=1}^n \alpha_i x_i$ for $x_i \in \{f_N : N \geq 1\}$, $n \in \mathbb{N}$, $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$. We must have $x_i = f_{N_i}$ for some $N_i \geq 1$. So

$$\|\sum_{i=1}^n \alpha_i f_{N_i}\| \leq \sum_{i=1}^n \alpha_i \|F_{N_i}\| = \sum_{i=1}^n \alpha_i = 1$$

So we have that $\text{co}\{f_N : N \geq 1\} \subset \overline{B_H(0,1)}$ and therefore also the closure because $\overline{B_H(0,1)}$ is closed i.e. $K \subset \overline{B_H(0,1)}$.

Now because K is closed in the weak topology and $\overline{B_H(0,1)}$ is is compact in the weak topology and $K \subset \overline{B_H(0,1)}$, we have that K is weakly compact.

We see that $0 \in K$: Define in $\text{co}\{f_N : N \geq 1\}$ a sequence $(x_n)_{n\geq 1}$ by $x_n := \sum_{i=1}^n \frac{1}{n} f_{2^i}$. For any $N \geq 1$ and $k \geq 1$:

$$\langle f_N, f_{2^k N}\rangle = \langle \frac{1}{N}\sum_{i=1}^{N^2} e_i, \frac{1}{2^k N}\sum_{j=1}^{(2^k N)^2} e_j\rangle = \frac{1}{2^k N^2}\sum_{i=1}^{N^2}\langle e_i, e_i\rangle = \frac{N^2}{2^k N^2} = \frac{1}{2^k}$$

Now

$$\begin{aligned}
\langle x_n, x_n\rangle &= \langle \sum_{i=1}^n \frac{1}{n} f_{2^i}, \sum_{j=1}^n \frac{1}{n} f_{2^j}\rangle = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \langle f_{2^i}, f_{2^j}\rangle = \frac{1}{n^2}\left(n + \sum_{i,j=1, i\neq j}^n \langle f_{2^i}, f_{2^j}\rangle\right) \\
&= \frac{1}{n^2}\left(n + \sum_{i=1, i<j}^n \langle f_{2^i}, f_{2^j}\rangle + \overline{\langle f_{2^i}, f_{2^j}\rangle}\right) = \frac{1}{n^2}\left(n + \sum_{i=1,i<j}^n 2\langle f_{2^i}, f_{2^j}\rangle\right) \\
&= \frac{1}{n^2}\left(n + 2\sum_{i=1}^n\sum_{j=1}^{n-i}\frac{1}{2^j}\right) \leq \frac{1}{n^2}\left(n + 2\sum_{i=1}^n 1\right)\frac{3n}{n^2} = \frac{3}{n}
\end{aligned}$$

Let $\epsilon > 0$ and choose $m \geq 3\epsilon^{-2}$. Then for $n \geq m$

$$\|x_n - 0\| = \sqrt{\langle x_n, x_n\rangle} = \sqrt{\frac{3}{n}} \leq \sqrt{\frac{3}{m}} \leq \sqrt{\frac{3}{3\epsilon^{-2}}} = \epsilon$$

So $x_n \to 0$ as $n \to \infty$ so therefore $0 \in \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = K$.

(c)

We show 0 is an extreme point in K:

Let $x, y \in K$ and $0 < \alpha < 1$ such that $0 = \alpha x + (1-\alpha)y$. We have that $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$ for some sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ in $\text{co}\{f_N : N \geq 1\}$. We have that $x_n = \sum_{i_n=1}^{m_n} \alpha_{i_n} f_{N_{i_n}}$ and $y_n = \sum_{j=1}^{m_n'} \alpha_{j_n} f_{N_{j_n}}$ where $\sum_{i_n=1}^{m_n} \alpha_{i_n} = \sum_{j_n=1}^{m_n'} \alpha_{j_n} = 1$ and $\alpha_{i_n}, \alpha_{j_n} > 0$ for all $n \geq 1$ and all i and j. By linearity of the limit and because two things in a vector space are equal if and only if they have the same coordinates with respect to some basis, we must also have

$$
\begin{aligned}
0 = \pi_k(0) &= \alpha \lim_{n\to\infty} \pi_k(x_n) + (1-\alpha)\lim_{n\to\infty} \pi_k(y_n) \\
&= \alpha \lim_{n\to\infty} \sum_{i_n=1, k\leq N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} e_k + (1-\alpha) \lim_{n\to\infty} \sum_{j=1, k\leq N_{j_n}^2}^{m_n'} \alpha_{j_n} \frac{1}{N_{j_n}} e_k
\end{aligned}
$$

Where π_k is the restriction to the k'th factor of the basis (The restriction is continuous viewed as the projection, but this is really not needed even though the notation is suggestive of such a need, but linearity of the limit is enough. But of course the argument also works by using continuity of π_k). So we must have

$$
0 = \alpha \lim_{n\to\infty} \sum_{i_n=1, k\leq N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + (1-\alpha) \lim_{n\to\infty} \sum_{j=1, k\leq N_{j_n}^2}^{m_n'} \alpha_{j_n} \frac{1}{N_{j_n}}
$$

This is a expression is in $\mathbb{R}$ and the terms in the limits are non-negative (i.e a positive real number or zero), so therefore the limits must both (both because $0 < \alpha < 1$) be zero because otherwise the right-hand-side of the equation would be bigger than 0. So because 0 and $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$ have the same coordinates with respect to the basis $(e_n)_{n\geq 1}$, we must have that $0 = x = y$. So 0 is an extreme point of K.

We show F_N is an extreme point in K: Now let $x, y \in K$ and $0 < \alpha < 1$ such that $f_N = \alpha x + (1-\alpha)y$. We have that $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$ for some sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ in $\mathrm{co}\{f_N : N \geq 1\}$. By restricting to the k'th element in the basis we get

$$
\begin{aligned}
\pi_k(f_N) &= \alpha \lim_{n\to\infty} \pi_k(x_n) + (1-\alpha)\lim_{n\to\infty} \pi_k(y_n) \\
&= \alpha \lim_{n\to\infty} \sum_{i_n=1, k\leq N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} e_k + (1-\alpha) \lim_{n\to\infty} \sum_{j=1, k\leq N_{j_n}^2}^{m_n'} \alpha_{j_n} \frac{1}{N_{j_n}} e_k
\end{aligned}
$$

So if $N^2 < k$ then

$$
\pi_k(f_N) = 0 = \alpha \lim_{n\to\infty} \sum_{i_n=1, k\leq N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + (1-\alpha) \lim_{n\to\infty} \sum_{j=1, k\leq N_{j_n}^2}^{m_n'} \alpha_{j_n} \frac{1}{N_{j_n}}
$$

Then like we have seen before because the terms in both of the limits are greater than or equal to zero, we must have that the limits are zero. Restricting to

$k = N^2$ we get:

$$\frac{1}{N} = \pi_k(f_N) = \alpha \lim_{n\to\infty} \sum_{i_n=1, N^2 \leq N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + (1-\alpha) \lim_{n\to\infty} \sum_{j=1, N^2 \leq N_{j_n}^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}}$$
$$= \alpha \left(\lim_{n\to\infty} \sum_{i_n=1, N^2 = N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + \lim_{n\to\infty} \sum_{i_n=1, N^2 < N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} \right)$$
$$+ (1-\alpha) \left(\lim_{n\to\infty} \sum_{j=1, N^2 = N_{j_n}^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}} + \lim_{n\to\infty} \sum_{j=1, N^2 < N_{j_n}^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}} \right)$$
$$= \alpha \left(\lim_{n\to\infty} \sum_{i_n=1, N^2 = N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + 0 \right) + (1-\alpha) \left(\lim_{n\to\infty} \sum_{j=1, N^2 = N_{j_n}^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}} + 0 \right)$$
$$= \alpha \lim_{n\to\infty} \sum_{i_n=1, N^2 = N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + (1-\alpha) \lim_{n\to\infty} \sum_{j=1, N^2 = N_{j_n}^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}}$$

So we must have $1 = \lim_{n\to\infty} \sum_{i_n=1, N^2 = N_{i_n}^2}^{m_n} \alpha_{i_n} = \lim_{n\to\infty} \sum_{j=1, N^2 = N_{j_n}^2}^{m'_n} \alpha_{j_n}$ because $0 < \alpha < 1$ and the terms in booth limits are non-negative real number and therefore the limits are non-negative real numbers. Now for $k < N^2$:

$$\frac{1}{N} = \pi_k(f_N) = \alpha \lim_{n\to\infty} \sum_{i_n=1, k \leq N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + (1-\alpha) \lim_{n\to\infty} \sum_{j=1, k \leq N_{j_n}^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}}$$
$$= \alpha \left(\lim_{n\to\infty} \sum_{i_n=1, k < N_{i_n}^2 < N^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + \lim_{n\to\infty} \sum_{i_n=1, N^2 = N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + \lim_{n\to\infty} \sum_{i_n=1, N^2 < N_{i_n}^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} \right)$$
$$+ (1-\alpha) \left(\lim_{n\to\infty} \sum_{j=1, k \leq N_{j_n}^2 < N^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}} + \lim_{n\to\infty} \sum_{j=1, N^2 = N_{j_n}^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}} + \lim_{n\to\infty} \sum_{j=1, N^2 < N_{j_n}^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}} \right)$$
$$= \alpha \left(\lim_{n\to\infty} \sum_{i_n=1, k < N_{i_n}^2 < N^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + \frac{1}{N} + 0 \right) + (1-\alpha) \left(\lim_{n\to\infty} \sum_{j=1, k \leq N_{j_n}^2 < N^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}} + \frac{1}{N} + 0 \right)$$

So

$$0 = \alpha \lim_{n\to\infty} \sum_{i_n=1, k < N_{i_n}^2 < N^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} + (1-\alpha) \lim_{n\to\infty} \sum_{j=1, k \leq N_{j_n}^2 < N^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}}$$

So as we have seen when we checked that 0 is an extreme point, $0 = \lim_{n\to\infty} \sum_{i_n=1, k < N_{i_n}^2 < N^2}^{m_n} \alpha_{i_n} \frac{1}{N_{i_n}} = \lim_{n\to\infty} \sum_{j=1, k \leq N_{j_n}^2 < N^2}^{m'_n} \alpha_{j_n} \frac{1}{N_{j_n}}$ because the terms in the sequences are non-negative real numbers and $0 < \alpha < 1$. So we have that f_N and x and y have the same coordinates with respect to the basis $(e_n)_{n \geq 1}$, hence $f_N = x = y$. So f_N is an extreme point of K for $N \geq 1$.

(d)

Let $F = \{f_N : N \geq 1\} \cup \{0\}$. Because K is a nonempty compact (as seen in (b)), convex (as seen in (b)) subset of the locally convex Hausdorff topological vector space (H, τ_w), we have $K = \overline{\text{co}\{\text{Ext}(K)\}}^{\tau_w}$ by Theorem 7.8. We have seen in (c) that $F \subset \text{Ext}(K)$ and therefore $\overline{\text{co}(F)}^{\tau_w} \subset \overline{\text{co}\{\text{Ext}(K)\}}^{\tau_w} = K$. Because $\text{co}\{f_N : N \geq 1\}$ is a convex subset of H, we have by Theorem 5.7 that $K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w}$. Because $\{f_N : N \geq 1\} \subset F$ there, we have $K = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w} \subset \overline{\text{co}(F)}^{\tau_w}$. So $K = \overline{\text{co}(F)}^{\tau_w}$. Because (H, τ_w) is a LCTVS, K is a nonempty compact, convex subset of H, and F is a subset of K such that $K = \overline{\text{co}(F)}^{\tau_w}$, we have by Theorem 7.9 that $\text{Ext}(K) \subset \overline{\text{co}(F)}^{\tau_w}$.

We calculate $\overline{F}^{\tau_w}$: Let $(x_\alpha)_{\alpha \in A}$ be a net in F that converges to x in the weak topology on H. Then by HW4 Problem 2 (a) and $(\langle x_\alpha, e_1 \rangle)_{\alpha \in A}$ must converge to $\langle x, e_1 \rangle$. Because $x_\alpha \in F$ we must have $\langle x_\alpha, e_1 \rangle = \frac{1}{N}$ for some $N \geq 1$ or $\langle x_\alpha, e_1 \rangle = 0$. So if $(\langle x_\alpha, e_1 \rangle)_{\alpha \in A}$ converges it must converge to 0 or $\frac{1}{N}$ for some $N \geq 1$:

$(\langle x_\alpha \rangle)_{\alpha \in A}$ can not converge to anything in $\mathbb{C} \setminus \mathbb{R}$ because $(\langle x_\alpha \rangle)_{\alpha \in A}$ takes values in $\mathbb{R}$ and because the distance between a specific number $c \in \mathbb{C} \setminus \mathbb{R}$ and $\mathbb{R}$ is larger than δ for some $\delta > 0$. Suppose $(\langle x_\alpha \rangle)_{\alpha \in A}$ converges to some number $c \in \mathbb{R}$. If $c \notin \{\frac{1}{N} : N \geq 1\} \cup \{0\}$ then then either $c > 0$ or $c < 0$. If $c < 0$ then $\text{dist}(c, \{\frac{1}{N} : N \geq 1\} \cup \{0\}) = \text{dist}(c, 0) > 0$. If $c > 0$ then there exists an $N \geq 1$ such that $\frac{1}{N} > c > \frac{1}{N+1}$ (note $c \notin \{\frac{1}{N} : N \geq 1\} \cup \{0\}$ is assumed) so $\text{dist}(c, \{\frac{1}{N} : N \geq 1\} \cup \{0\}) = \min\{\text{dist}(c, \frac{1}{N}), \text{dist}(c, \frac{1}{N+1})\} > 0$.

If $(\langle x_\alpha, e_1 \rangle)_{\alpha \in A}$ converges to 0 then $(\langle x_\alpha, e_n \rangle)_{\alpha \in A}$ must converge to 0 because $|\langle x_\alpha, e_n \rangle| \leq \langle x_\alpha, e_1 \rangle$. If $(\langle x_\alpha, e_1 \rangle)_{\alpha \in A}$ converges to $\frac{1}{N}$ then $(x_\alpha)_{\alpha \in A}$ must be equal f_N eventual because the distance between f_N and $x_\alpha \neq f_N$ must be bigger or equal than $\min\{|f_N - f_{N+1}|, |f_N - f_{N-1}|\} = \min\{\frac{1}{N(N+1)}, \frac{1}{N(N-1)}\}$. Because $(x_\alpha)_{\alpha \in A}$ is eventually f_N it must converge to f_N. So $\overline{F}^{\tau_w} = F$. Remember $F \subset \text{Ext}(K)$.

So $\text{Ext}(K) = F$. So there are no other extreme points than 0 and f_N for $N \geq 1$.

Problem 2

Let X and Y infinite dimensional Banach spaces.

(a)

Let $T \in \mathcal{L}(X, Y)$. Let $(x_n)_{n \geq 1}$ be a sequence such that $x_n \to x$ weakly as $n \to \infty$ for some $x \in X$. The sequence $(Tx_n)_{n \geq 1}$ converges weakly to Tx if for all $f \in Y^*$, $(f(Tx_n))_{n \geq 1}$ converges to $f(Tx)$ by HW4 Problem 2 (a). But $f(T) \in X^*$ for $f \in Y^*$ and therefore $(f(Tx_n))_{n \geq 1}$ converges to Tx by HW4 Problem 2 (a). So because $(f(Tx_n))_{n \geq 1}$ converges to $f(Tx)$ for all $f^* \in Y$, we have by HW 4 Problem 2 (a) that $Tx_n \to T_x$ as $n \to \infty$.

(b)

Let $T \in \mathcal{K}(X, Y)$. Let $(x_n)_{n \geq 1}$ be a sequence such that $x_n \to x$ weakly as $n \to \infty$ for some $x \in X$. If we do not have that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$

then there must exist an $\epsilon > 0$ such that for every $N \geq 1$ there exist a $k' \geq N$ such that $\|Tx_{n(k')} - Tx\| > \epsilon$. Now let $(Tx_{n(k)})_{k\geq 1}$ be a subsequence of such k's. Because $(x_{n(k)})_{k\geq 1}$ is a subsequence of a weakly convergent sequence and therefore weakly convergent, it must be bounded by HW4 Problem 2 (b).

Because T is compact and because $(x_{n(k)})_{k\geq 1}$ is bounded, there exists a subsequence $(x_{n(k(i))})_{i\geq 1}$ such that $(Tx_{n(k(i))})_{i\geq 1}$ is norm convergent by Proposition 8.2. Since $(Tx_{n(k(i))})_{i\geq 1}$ can't converge to Tx (because it is atleast ϵ away) it must converge to something else and therefore also converge weakly to something else. But by (a) $T(x_n) \to Tx$ weakly as $n \to \infty$ and therefore also $Tx_{n(k(i))} \to Tx$ weakly as $n \to \infty$. So $T(x_n) \to Tx$ is weakly convergent to two different points, which is a contradiction because Y is Banach space and therefore a Hausdorff space with the weak topology. So if $(x_n)_{n\geq 1}$ is such that $x_n \to x$ weakly as $n \to \infty$ for some $x \in X$ then $\|Tx_n - Tx\| \to 0$ as $n \to \infty$.

(c)

Let H be a separable infinite dimensional Hilbert space and let $T \in \mathcal{L}(H, Y)$ be such that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence converging weakly to $x \in H$. Assume for contradiction that $T \notin \mathcal{K}(H, Y)$. Because T is a linear map between Banach space that is not compact, we must have by Proposition 8.2 that $T(\overline{B_H(0,1)})$ is not totally bounded. So there must exist an $\delta > 0$ such that $T(\overline{B_H(0,1)})$ cannot be covered by finitely many balls with radius δ.

$\overline{B_H(0,1)}$ is non-empty, so choose $x_1 \in \overline{B_H(0,1)}$. Choose some $x_{n+1} \in \overline{B_H(0,1)}$ such that $Tx_{n+1} \in T(\overline{B_H(0,1)}) \setminus \bigcup_{i=1}^{n} B_Y(Tx_i, \delta)$. This can be done because otherwise $T(\overline{B_H(0,1)})$ could be covered by n balls of radius δ. By choosing x_n like this, we get a sequence $(x_n)_{n\geq 1}$ in $\overline{B_H(0,1)}$ such that $\|Tx_n - Tx_m\| \geq \delta$ for $m \neq n$.

Because H is a Hilbert space it is a reflexive Banach space by Theorem 2.10 and therefore by Theorem 6.3 $\overline{B_H(0,1)}$ is compact in the weak topology. So there exists a weakly convergent subsequence $(x_{n(k)})_{k\geq 1}$ of $(x_n)_{n\geq 1}$. So by assumption $(Tx_{n(k)})_{k\geq 1}$ would be convergent in norm because $(x_{n(k)})_{k\geq 1}$ is weakly convergent, but because $(Tx_{n(k)})_{k\geq 1}$ is not Cauchy it can't be convergent which is a contradiction. So $T \in \mathcal{K}(X, Y)$.

(d)

Let $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$. By (a) any $(x_n)_{n\geq 1}$ such that $x_n \to x$ weakly as $n \to \infty$ means that $Tx_n \to Tx$ weakly as $n \to \infty$. By Remark 5.3 we have that $Tx_n \to T_x$ in norm as $n \to \infty$ because it is in $\ell_1(\mathbb{N})$. So we have that whenever a sequence $(x_n)_{n\geq 1}$ in $\ell_2(\mathbb{N})$ converges weakly then $\|Tx_n \to Tx\| \to 0$ as $n \to \infty$ and because $\ell_2(\mathbb{N})$ is a separable Hilbert space, we have by (c) that T is compact.

(e)

Let $T \in \mathcal{K}(X, Y)$. Assume for contradiction that T is onto. Then by the Open Mapping Theorem (Theorem 3.15) T is an open map because it is a surjective map between Banach spaces. So $T(B_X(0,1))$ is an open set around 0 and therefore there exists a ball $B_Y(0, r) \subset T(B_X(0,1))$ where $r > 0$. $\overline{T(B_X(0,1))}$

is compact because T is compact. So $\overline{B_Y(0,r)}$ is compact because it is a closed subset of a compact set. The map that multiplies by $\frac{1}{r}$ is continuous and therefore $\frac{1}{r}\overline{B_Y(0,r)} = \overline{B_Y(0,1)}$ is compact. But $\overline{B_Y(0,1)}$ is only compact if Y is finite dimensional by Problem 3 Mandatory assignment 1. But this is a contradiction with the assumption that Y is infinite dimensional Banach space. So there exist no maps $T \in \mathcal{K}(X,Y)$ that are onto.

(f)

Let $H = L_2([0,1],m)$ and $M \in \mathcal{L}(H,H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. M is self-adjoint: Let $f, g \in H$ then:

$$\begin{aligned}\langle f(t), Mg(t)\rangle &= \int_{[0,1]} f(t)\overline{Mg(t)}dm(t) = \int_{[0,1]} f(t)\overline{tg(t)}dm(t) = \int_{[0,1]} tf(t)\overline{g(t)}dm(t) \\ &= \int_{[0,1]} Mf(t)\overline{g(t)}dm(t) = \langle Mf(t), g(t)\rangle\end{aligned}$$

So M is self-adjoint.

Assume for contradiction that M is compact. Because $L_2([0,1],m)$ is a infinite dimensional separable Hilbert space and M is self-adjoint and assumed to be compact, we have by the Theorem 10.1 (The Spectral Theorem for self-adjoint compact operators) that $L_2([0,1],m)$ would have an orthonormal basis of consisting of eigenvectors. But this is impossible because by HW6 Problem 3 (a) M has no eigenvalues. So M cannot be compact.

Problem 3

Let $H = L_2([0,1],m)$, where m is the Lebesgue measure. Define $K(s,t)$: $[0,1] \times [0,1] \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t & 0 \leq t \leq s \leq 1 \\ (1-t)s & 0 \leq s \leq t \leq 1 \end{cases}$$

and let $T\mathcal{L}(H,H)$ be defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t)$$

(a)

$K(s,t)$ can bee seen as the "pasting" of the two function continuous functions $(1-s)t$ and $(1-t)s$ from the two closed $\{(t,s)|0 \leq t \leq s \leq 1\}$ and $\{(t,s)|0 \leq t \leq s\}$ that are equal on the overlap $\{(t,s)|0 \leq t \leq s \leq 1\} \cap \{(t,s)|0 \leq t \leq s\} = \{(s,t)|0 \leq s = t \leq 1\}$, and therefore by the pasting lemma $K(s,t)$ is continuous. Because [0,1] is a compact Hausdorff space and m is a finite Borel measure on $[0,1]$ and $K(s,t) \in C([0,1] \times [0,1])$ and because $T = T_K$ where T_K is given by

$$T(f) = T_K(f) = \int_{[0,1]} K(s,t)f(t) \cdot dm(t)$$

We have by Theorem 9.6 that T is compact.

(b)

Let $f, g \in H$. Then

$$\langle Tf(s), g(s)\rangle = \int_{[0,1]} Tf(s)\overline{g(s)}dm(s) = \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s)$$

$K(s,t)$ is measurable with respect to the product measure because it is continuous. $f(t)$ is measurable because $f \in L_2([0,1], M)$ and $\overline{g(s)}$ is measurable because $g \in L_2([0,1], m)$. f and g are also measurable with respect to the product measure because they are measurable with respect to one of the measure. A product of measurable functions is measurable so $K(s,t)f(t)\overline{g(s)}$ is measurable with respect to the product measures. We have that $K(s,t) \le 1$ and $\int_{[0,1]} |f(t)|dm(t) = x$ for some $x \in \mathbb{K}$ and $\int_{[0,1]} |\overline{g(s)}|dm(s) < \infty$ because $f, g \in L_2([0,1], m) \subset L_1([0,1], m)$ (where $L_2([0,1], m) \subset L_1([0,1], m)$ by HW2 Problem 2 (b)). So

$$\begin{aligned}\int_{[0,1]}\int_{[0,1]} |K(s,t)f(t)\overline{g(s)}|dm(t)dm(s) &\le \int_{[0,1]}\int_{[0,1]} |f(t)\overline{g(s)}|dm(t)dm(s)\\ = \int_{[0,1]}\int_{[0,1]} |f(t)|dm(t)|\overline{g(s)}|dm(s) &= \int_{[0,1]} x|\overline{g(s)}|dm(s) < \infty\end{aligned}$$

So by Corrolary 14.9 (Funini's Theorem) Measures, integrals and martingales, we have that:

$$\int_{[0,1]}\int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s) = \int_{[0,1]} f(t)\int_{[0,1]} K(s,t)\overline{g(s)}dm(s)dm(t)$$
$$= \int_{[0,1]} f(t)\int_{[0,1]} \overline{K(s,t)g(s)}dm(s)dm(t) = \int_{[0,1]} f(t)\overline{Tg(t)}dm(t) = \langle f(t), \overline{Tg(t)}\rangle$$

So $T = T^*$.

(c)

We have that

$$\begin{aligned}(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) = \int_{[0,1]} K(s,t)f(t)\mathbb{1}_{[0,1]}(t)dm(t)\\ &= \int_{[0,1]} K(s,t)f(t)(\mathbb{1}_{[0,s]}(t) + \mathbb{1}_{[s,1]}(t))dm(t)\\ &= \int_{[0,1]} (1-s)tf(t)\mathbb{1}_{[0,s]}(t)dm(t) + \int_{[0,1]} (1-t)sK(s,t)f(t)\mathbb{1}_{[s,1]}(t)dm(t)\\ &= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)\end{aligned}$$

Let $(s_n)_{n\ge1}$ be a sequence in [0,1] converging to $s \in [0,1]$. Then

$$\lim_{n\to\infty}\int_{[0,s_n]} tf(t)dm(t) = \lim_{n\to\infty}\int_{[0,1]} tf(t)\mathbb{1}_{[0,s_n]}(t)dm(t)$$

Because $t, \mathbb{1}_{[0,s_n]} \in L_1([0,1], m)$ and $f \in L_2([0,1], m) \subset L_1([0,1], m)$. We have that $|tf(t)\mathbb{1}_{[0,s_n]}| \le |tf(t)|$ for all $n \ge 1$, $t \in [0,1]$ and because the limit

$\lim_{n\to\infty} tf(t)\mathbb{1}_{[0,s_n]}(t)$ exists for almost every $t \in [0,1]$ (for every t except $t = s$), we have

$$\lim_{n\to\infty} \int_{[0,1]} tf(t)\mathbb{1}_{[0,s_n]}(t)dm(t) = \int_{[0,1]} tf(t) \lim_{n\to\infty} \mathbb{1}_{[0,s_n]}(t)dm(t)$$

By Theorem 12.2 and Remark 12.3 Measures, Integrals and Martingales. Because $\lim_{n\to\infty} \mathbb{1}_{[0,s_n]}(t) = \mathbb{1}_{[0,\lim_{n\to\infty} s_n]}(t)$ can only differ or not be defined in the point $t = s$ i.e. a zero set, we have that

$$\int_{[0,1]} tf(t) \lim_{n\to\infty} \mathbb{1}_{[0,s_n]}(t)dm(t) = \int_{[0,1]} tf(t)\mathbb{1}_{[0,\lim_{n\to\infty} s_n]}(t)dm(t)$$

So the integral $\int_{[0,s]} tf(t)dm(t)$ is continuous in s and likewise $\int_{[s,1]}(1-t)f(t)dm(t)$ is continuous in s. So because it is a product and sum of continuous functions, $Tf(s)$ is continuous.

We have that $\mathbb{1}_{[0,0]}(t)$ and $\mathbb{1}_{[1,1]}(t)$ are zero except in 0 and 1 respectively i.e. zero sets. So

$$\begin{aligned}
Tf(0) &= (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]}(1-t)f(t)dm(t) \\
&= \int_{[0,1]} tf(t)\mathbb{1}_{[0,0]}dm(t) = \int_{[0,1]} 0dm(t) = 0 \\
Tf(1) &= (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]}(1-t)f(t)dm(t) \\
&= \int_{[0,1]}(1-t)f(t)\mathbb{1}_{[1,1]}dm(t) = \int_{[0,1]} 0dm(t) = 0
\end{aligned}$$

Problem 4

(a)

For each integer $k \geq 0$, set $g_k(x) = x^k e^{-\frac{x^2}{2}}$, for $x \in \mathbb{R}$. Let $p_1(x)$ be a polynomial. Then $(p_1(x)e^{-\frac{x^2}{2}})' = p_1(x)'e^{-\frac{x^2}{2}} - xp_1(x)e^{-\frac{x^2}{2}} = (p_1(x)' - xp_1(x))e^{-\frac{x^2}{2}}$. $p_1(x)' - xp_1(x)$ is also a polynomial. So $p_2(x)\frac{d^n}{dx^n}x^k e^{-\frac{x^2}{2}} = \frac{p_2(x)p_3(x)}{e^{\frac{x^2}{2}}}$ and this goes to 0 as $\|x\| \to \infty$ because $e^{\frac{x^2}{2}}$ grows faster than all polynomials so faster than $|p_2(x)p_3(x)|$ and we have this for all $k \geq 0$. So $g_k \in \mathscr{S}(\mathbb{R})$ for $k \geq 0$

By Proposition 11.4 $\mathcal{F}(g_0) = \mathcal{F}(e^{-\frac{x^2}{2}}) = \mathcal{F}(e^{-\frac{\|x\|^2}{2}}) = e^{-\frac{\|\xi\|^2}{2}} = e^{-\frac{\xi^2}{2}} = g_0$. Because g_k and xg_k are in $\mathscr{S}(\mathbb{R})$ for $k \geq 0$, we have by HW7 Problem 1 (c) that $g_k, xg_k \in L_1(\mathbb{R})$ for $k \geq 0$ and therefore by Proposition 11.13

$$g_1\hat{}(\xi) = i\left(\frac{d}{dx}\hat{g_0}\right)(\xi) = -i\xi e^{-\frac{\xi^2}{2}} = -ig_1$$

$$g_2\hat{}(\xi) = i\left(\frac{d}{dx}\hat{g_1}\right)(\xi) = e^{-\frac{\xi^2}{2}} - \xi^2 e^{-\frac{\xi^2}{2}} = (1-\xi^2)e^{-\frac{\xi^2}{2}} = g_0 - g_2$$

$$g_3\hat{}(\xi) = i\left(\frac{d}{dx}\hat{g_2}\right)(\xi) = -2\xi ie^{-\frac{\xi^2}{2}} + i(1-\xi^2)(-\xi)e^{-\frac{\xi^2}{2}} = (\xi^2-3)i\xi e^{-\frac{\xi^2}{2}} = ig_3 - i3g_1$$

(b)

Define $h_0 := g_0$, $h_1 := 2g_3 - 3g_1$, $h_2 := 2g_2 - g_0$ and $h_3 := g_1$. $h_0, h_1, h_2, h_3 \neq 0$ and because $\mathscr{S}(\mathbb{R})$ is a vector space $h_0, h_1, h_2, h_3 \in \mathscr{S}(\mathbb{R})$. We have because $\mathcal{F}$ is linear by Proposition 11.5:

$$\mathcal{F}(h_0) = \mathcal{F}(g_0) = g_0 = h_0$$
$$\mathcal{F}(h_1) = \mathcal{F}(2g_3 - 3g_1) = 2\mathcal{F}(g_3) - 3\mathcal{F}(g_1) = 2(ig_3 - i3g_1) - 3(-ig_1) = i(2g_3 - 3g_1) = ih_1$$
$$\mathcal{F}(h_2) = \mathcal{F}(2g_2 - g_0) = 2\mathcal{F}(g_2) - \mathcal{F}(g_0) = 2(g_0 - g_2) - g_0 = 2g_2 - g_0 = -h_2$$
$$\mathcal{F}(h_3) = \mathcal{F}(g_1) = -ig_1 = -ih_3$$

So $h_0, h_1, h_2, h_3 \in \mathscr{S}(\mathbb{R})$ are non-zero functions such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

(c)

Define $R(f(x)) := f(-x)$ for $f \in \mathscr{S}(\mathbb{R})$. We have:

$$R\mathcal{F}(f) = R(\hat{f}(\xi))R = \hat{f}(-\xi) = \int_{\mathbb{R}} f(x)e^{-i\langle x,-\xi\rangle}dm(x) = \int_{\mathbb{R}} f(x)e^{i\langle x,\xi\rangle}dm(x) = \mathcal{F}^*(f)$$

For $f \in \mathscr{S}(\mathbb{R})$ we have $Rf = R\mathcal{F}^*\mathcal{F} = RR\mathcal{F}\mathcal{F}f = \mathcal{F}\mathcal{F}f$ by Corollary 12.12. So $\mathcal{F}^2(f) = R(f)$ for $f \in \mathscr{S}(\mathbb{R})$ and therefore $\mathcal{F}^4(f) = R^2 f = f$ for $f \in \mathscr{S}(\mathbb{R})$.

(d)

If $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$ then by linearity of $\mathcal{F}$ and (c) $\lambda^4 f = \mathcal{F}^4(f) = f$. So $\lambda^4 = 1$ because f is non-zero. The solutions to this equation are $\{e^{i\frac{2\pi n}{4}} | n = 0, 1, 2, 3\}$ so $\lambda \in \{1, i, -1, -i\}$. so the only possible eigenvalues are $\{1, i, -1, -i\}$. By (b) h_k for $k = 0, 1, 2, 3$ are nonzero functions such that $\mathcal{F}(h_k) = i^k h_k$, hence $\{1, i, -1, -i\}$ are eigenvalues. So the eigenvalues for $\mathcal{F}$ are precisely $\{1, i, -1, -i\}$.

Problem 5

Let $(x_n)_{n\geq 1}$ be as dense subset of $[0, 1]$ and define the radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{n_x}$ on $[0, 1]$. By HW8 problem 3 $\text{supp}(\mu)$ can be written as N^c where $N = \bigcup_{U\in I} U$ where $I = \{U \subset [0, 1] | \mu(U) = 0, U \text{ is open}\}$. Let $U \in I$ then if $U \neq \emptyset$ then there exists some $x \in U \subset [0, 1]$, but because $(x_n)_{n\geq 1}$ is dense in $[0, 1]$, x is a limit point of $(x_n)_{n\geq 1}$ and therefore there does not exist a open neighborhood V of x such that $V \cap (x_n)_{n\geq 1} = \emptyset$. So $U \cap (x_n)_{n\geq 1} \neq \emptyset$ so there exist some $k \geq 1$ such that $x_k \in U$ and therefore $\mu(U) = \sum_{n=1}^{\infty} 2^{-n}\delta_{n_x}(U) \geq 2^{-k}\delta_{n_k}(U) = 2^{-k} \neq 0$. So if $U \in I$ then $U = \emptyset$. So $\text{supp}(\mu) = N^c = \emptyset^c = [0, 1]$