Problem 1

a)

To show $||\cdot||_0$ is a norm, we just need to show the three conditions from **definition 1.1**, as we clearly see its a function defined on vektor space, into the positive real line:

a) For $x, y \in X$ we have

$$||x+y||_0 = ||x+y||_X + ||T(x+y)||_Y = ||x+y||_X + ||T(x)+T(y)||_Y \leq$$
$$||x||_X + ||y||_X + ||T(x)||_Y + ||T(y)||_Y = ||x||_0 + ||y||_0$$

where we have used that T is linear and that $||\cdot||_X$ and $||\cdot||_Y$, are norms and therefore satisfies the triangle inequality.

b) For $x \in X$ and $a \in \mathbb{K}$ we have

$$||ax||_0 = ||ax||_X + ||T(ax)||_Y = |a|||x||_X + |a|||T(x)||_Y = |a|||x||_0$$

where we again have used the T is linear, and that $||\cdot||_X$ and $||\cdot||_Y$ are norms.

c) For $x \in X$ we have

$$||x||_0 = ||x||_X + ||T(x)||_Y \Leftrightarrow x = 0$$

where we have used that $T(0) = 0$ as it is linear, and that $||\cdot||_X$ and $||\cdot||_Y$ are norms.

Assume first that the two norms $||\cdot||_X$ and $||\cdot||_0$ are equivalent, and let us show T is bounded. From assumption we have that there exists $C > 0$ such that:

$$||x||_0 \leq C||x||_X \Leftrightarrow ||x||_X + ||T(x)||_Y \leq C||x||_X \Leftrightarrow ||T(x)||_Y \leq (C-1)||x||_X$$

But actually we have that $C > 1$, since we know $1 \cdot ||x||_X \leq ||x||_0$ as $||T(x)||_Y \geq 0$. Hence $C - 1 > 0$ and T fulfills the definition of being bounded.
Assume now that T is bounded with the intent to prove that $||\cdot||_X$ and $||\cdot||_0$ are equivalent. Notice as before that $1 \cdot ||x||_X \leq ||x||_0$, as $||T(x)||_Y \geq 0$. Further as T is bounded we can find $C > 0$ such that $||T(x)||_Y \leq C||x||_X$, and have for $x \in X$:

$$1 \cdot ||x||_X \leq ||x||_0 = ||x||_X + ||T(x)||_Y \leq ||x||_X + C||x||_X = (1+C)||x||_X$$

hence the norms are equivalent.

b)

Given any linear map $T : X \to Y$, we can define a norm on X by: $||x||_0 = ||x||_X + ||T(x)||_Y$, for any $x \in X$. As X is finite dimensional it follows from **theorem 1.6** that any two norms om X are equivalent. Hence $||\cdot||_X$ and $||x||_0$ are equivalent, and it follows from a) that T is bounded.

c)

Note that for any infinite index set I, we can find a surjective function $f : I \to \mathbb{N}$, as $\text{card}(I) \geq \text{card}(\mathbb{N})$. Now for a fixed non-zero $y \in Y$ consider a family $(y_{f(i)})_{i\in I}$ in Y, with the property that $y_{f(i)} = f(i) \cdot y$ for all $i \in I$. We can now use the hint to say X admits a Hamel basis, meaning, there exists a family $(e_i)_{i\in I}$ in X, with $||e_i||_X = 1$ for all $i \in I$, and a linear map $T : X \to Y$, satisfying $T(e_i) = y_{f(i)} = f(i) \cdot y$ for all $i \in I$. But we see that $||T(e_i)||_Y = f(i) \cdot ||y||_Y$, can be made arbitrarily large as $f(i)$ is surjective into $\mathbb{N}$ and $y \neq 0$. But $||e_i||$ is alway equal to 1, and hence T cannot be bounded.

d)

Take a linear map $T : X \to Y$ which it not bounded, as we know such map exists from c). Define again the now well know norm on X: $||x||_0 = ||x||_X + ||T(x)||_Y$, for any $x \in X$. Since T is not bounded we know these norms are not equivalent from a), and further $|| \cdot ||_X \leq || \cdot ||_0$ as $|| \cdot ||_Y \geq 0$.
From **problem 1 HW3** it follows that if both $(X, || \cdot ||_X)$ and $(X, || \cdot ||_0$ are complete, and $|| \cdot ||_X \leq || \cdot ||_0$ then the norms are equivalent. But since we know $|| \cdot ||_X \leq || \cdot ||_0$ and that they are *not* equivalent, we can not have $(X, || \cdot ||_0)$ being complete, if $(X, || \cdot ||_X)$ is.

e)

Take the space $\ell_1(\mathbb{N})$, equipped with the two norms $|| \cdot ||_1$ and $|| \cdot ||_\infty$. From An2 we know that $(\ell_1(\mathbb{N}), || \cdot ||_1)$ is complete, and further we know $|| \cdot ||_\infty \leq || \cdot ||_1$. These two norms are not equivalent. Consider for an example the sequence $(x_n)_{n\in\mathbb{N}} \subset \ell_1(\mathbb{N})$, where:

$$x_n(k) = \begin{cases} \frac{1}{k} & \text{if } k \leq n \\ 0 & \text{else} \end{cases}$$

We see $||x_n||_\infty = 1$ for all $n \in \mathbb{N}$, but $||x||_1$ can be arbitrarily large, as $\sum_{k=1}^{\infty} \frac{1}{k} \to \infty$ as $n \to \infty$, hence they are not equivalent norms.
We now wish to show that $(\ell_1(\mathbb{N}), || \cdot ||_\infty)$ is not complete. Consider again the same $(x_n)_{n\in\mathbb{N}} \subset \ell_1(\mathbb{N})$ sequence as before. We see that this sequence is cauchy, as for any $n, m \in \mathbb{N}$ we have
$\lim_{n,m\to\infty} ||x_n - x_m||_\infty = \lim_{n,m\to\infty} \frac{1}{n+1} = 0$, where we without loss of generality have asummed $n < m$. But for the sequence $(a_k)_{k\in\mathbb{N}} = \frac{1}{k}$ (which is not in $\ell_1(\mathbb{N})$), we have $\lim_{n\to\infty} ||x_n - a_k||_\infty = \lim_{n\to\infty} \frac{1}{n+1} = 0$, meaning $(x_n)_{n\in\mathbb{N}})$ is not convergent in $\ell_1(\mathbb{N}$.

Problem 2

a)

We want to show f satisfies **Proposition 1.10 (3)**. Observe that for $m = (a, b, 0, 0, ...) \in M$ we have

$$||f(m)||^p = |a + b|^p \leq 2^p \max\{|a|^p, |b|^p\} \leq 2^p(|a|^p + |b|^p) = 2^p||m||_p^p$$

hence we have for all $m \in M$ that

$$||f(m)|| \leq 2||m||_p$$

and f is bounded.

In order to computer $||f||$ observe first that for $m = ((\frac{1}{2})^{1/p}, (\frac{1}{2})^{1/p}, 0, 0, ...) \in M$ we have $||m||_p = (\frac{1}{2} + \frac{1}{2})^{1/p} = 1$ and $||f(x)|| = (\frac{1}{2})^{1/p} + (\frac{1}{2})^{1/p} = 2 \cdot (\frac{1}{2})^{1/p}$. Hence $||f|| \geq 2 \cdot (\frac{1}{2})^{1/p}$.
In order to prove the reverse inequality, observe first that for any element $m = (a, b, 0, ...) \in M \subset \ell_p(\mathbb{N})$, and $x = (1, 1, 0, ...) \in \ell_q(\mathbb{N})$, where $\frac{1}{p} + \frac{1}{q} = 1$, then we have by Hölders inequality (**Schilling Theorem 13.2**)

$$|a+b| = \sum_{n=1}^{\infty} |m_n x_n| \leq ||m||_p \cdot ||x||_q = ||m||_p \cdot (1+1)^{\frac{1}{q}} = ||m||_p \cdot 2^{\frac{p-1}{p}} = ||m||_p \cdot 2 \cdot (\frac{1}{2})^{\frac{1}{p}}$$

and actually this holds even when $p = 1$, where we then use norm ∞-norm in place of our q-norm, and get the same inequality.
But now it follows

$$||f|| = \sup\{|a+b| : ||m||_p = 1\} \leq 2 \cdot (\frac{1}{2})^{\frac{1}{p}}$$

and we can conclude $||f|| = 2 \cdot (\frac{1}{2})^{\frac{1}{p}}$

b)

Note that whenever $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then we know from **Problem 5 HW1** that $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$. This means that we bijectively can identify every element $x \in \ell_q(\mathbb{N})$, with a functional $f_x : \ell_p(\mathbb{N}) \to \mathbb{C}$ given by

$$f_x(y) = \sum_{n=1}^{\infty} x_n y_n, \qquad \text{for fixed } (x_n)_{n\in\mathbb{N}} \in \ell_q(\mathbb{N}), \text{and for all } (y_n)_{n\in\mathbb{N}} \in \ell_p(\mathbb{N})$$

and further $||f_x|| = ||x||_p$, for every $x \in \ell_p(\mathbb{N})$. Now let us choose an x, such that f_x extends f and has $||f_x|| = ||f||$, (This exists from Hahn-Banach extension). Consider $x = (1, 1, 0, 0, ...) \in \ell_q(\mathbb{N})$, and observe that for any $m \in M$ we have $f_x(m) = a + b = f(m)$, and further $||f_x|| = ||x||_q = 2^{\frac{1}{q}} = 2 \cdot (\frac{1}{2})^{\frac{1}{p}} = ||f||$.
But actually this is the only $x \in \ell_q(\mathbb{N})$ satisfying this construction. Since in order for f_x to extend f the first two terms of the x sequence must be 1. But the rest of the sequence must be 0's, as otherwise $||f_x|| = ||x||_q > ||1 + 1||_q = ||f||$. Hence our $x = (1, 1, 0, 0, ...)$ was the unique $x \in \ell_q(\mathbb{N})$ giving a linear functional on $\ell_p(\mathbb{N})$, with the desired properties, hence this functional is unique.

c)

Similarly as in b) we know from **Problem 5 HW1** that $(\ell_1(\mathbb{N}))^* \cong \ell_\infty(\mathbb{N})$. Consider now the sequence $(y_n)_{n\in\mathbb{N}} \subset \ell_\infty(\mathbb{N})$, where $y_n = (1, 1, 0, ..., 0, 1, 0, ...)$ has two 1's in the beginning and then a 1 on the n'th place. Each of the functionals f_{y_n} on $\ell_1(\mathbb{N})$, extends f on M from similarly arguments as in b), and further for all $n \in \mathbb{N}$ we have: $||f_{y_n}|| = ||y_n||_\infty = 1 = ||f||$ (as here $p = 1$). Hence there exists infinitely many functionals on $\ell_1(\mathbb{N})$ with the desired properties.

Problem 3

a)

Given $n \geq 1$ consider a Hamel base for F $(e_i)_{i\in\mathbb{N}}$, and the restriction of F to $\text{span}\{e_1, e_2, ..., e_{n+1}\}$ given by $F' : \text{span}\{e_1, e_2, ..., e_{n+1}\} \to \mathbb{K}^n$, where $F' = F$. We know the generel finite dimension formula:

$\dim(f) = \dim(\ker(f)) + \dim(\mathrm{im}(f))$, which in our case of F' amounts to

$$n + 1 = \dim(F') = \dim(\ker(F') + \dim(\mathrm{im}(F')$$

and as $\dim(\mathrm{im}(F') \leq n$, this means $\dim(\ker(F') \geq 1$, hence F' and not injective and therefore F cannot be either.

b)

Consider the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), ..., f_n(x))$ for $x \in X$. This map is linear, as each of $f_j, i = 1, ..., n$ is linear. Hence we know from a) that F is not injective, meaning its kernel contains an element different from 0, and we get

$$\bigcap_{j=1}^{n} \ker(f_j) = \{x \in X | f_1(x) = f_2(x) = ... = f_n(x) = 0\} = \ker F \neq \{0\}$$

as desired.

c)

The claim holds for any $y \in X$ with $||y|| = 1$, if $x_j = 0$, so for a start assume none of $x_1, x_2, ..., x_n \in X$ are zero. From 2.7(b) we get that there exists $f_1, ..., f_n \in X^*$ such that for any $j = 1, ...n$ we have $||f_j|| = 1$ and $f_j(x_j) = ||x_j||$.
From b) we know we can take $y' \in \bigcap_{j=1}^{n} \ker f_j$ where $y' \neq 0$, and observe now that $y = \frac{y'}{||y'||}$ is in $\bigcap_{j=1}^{n} \ker f_j$ and has $||y|| = 1$. We can now use that f_j is bounded to see:

$$||y - x_j|| = ||y - x_j||||f|| \geq ||f(y - x_J)|| = ||f(y) - f(x_j)|| = || - f(x_j)|| = ||x_j||$$

for any $j = 1, ..., n$.

d)

Assume for contradiction that we have a family of closed balls in X covering S, and not containing 0: $(\overline{B(x_i, r_i)})_{i \in I}$, where $x_i \in X, i \in I$ and $r_i \in \mathbb{K}, i \in I$, for some finite index set I. As none of the balls contain 0, it must hold that $||x_i|| > r_i$ for all $i \in I$. But if we take a $y \in X$ as in c), we know that $||y - x_i|| \geq ||x_i| > r_i$ for all $i \in I$, but this means that y is not contained in any of the balls in our fsamily. This is a contradiction as $||y|| = 1 \Rightarrow y \in S$, which the family of balls is supposed to cover. We conclude the unit phere cannot be covered with a finite family of closed balls not containing 0.

e)

Assume for contradiction that S was compact. Then we can take an open subcover of S, given by: $\bigcup_{x \in S} B(x, \frac{1}{2})$, and since S is compact, this can be thinned to a finite open cover $\bigcup_{x \in I} B(x, \frac{1}{2})$, which still contains S and where I is some finite set. But then we must have that $S \subset \bigcup_{x \in I} \overline{B(x, \frac{1}{2})}$, which we know from d) is a contradiction as $\bigcup_{x \in I} \overline{B(x, \frac{1}{2})}$ does not contain 0. Hence S cannot be compact.
Now it follows from **Folland proposition 4.22**, that the closed unit ball in X cannot be compact, as S is a closed subset of the closed unit ball, and hence S would be compact if the closed unit ball was.

Problem 4

a)

Consider $f : [0, 1] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} x^{-1/3} & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

this is an element in $L_1([0, 1], m)$ as $\int_{[0,1]} f(x)dm(x) = 3/2 < \infty$.
But for any $a > 0$ we have

$$\int_{[0,1]} |af(x)|^3 dm(x) = a^3 \int_{[0,1]} |f(x)|^3 dm(x) = \infty$$

so given an n, we can never scale f to be in E_n, hence E_n is not absorbing.

b)

We start off by noticing that for any $n \geq 1$ we have $E_n \subset L_3([0, 1], m)$, hence it suffices to show that $L_3([0, 1], m)$ has empty interior in $L_1([0, 1], m)$. Assume for contradiction that $L_3([0, 1], m)$ has non-empty interior in $L_1([0, 1], m)$. Then there exists an element $g \in L_3([0, 1], m)$, where we can choose $\epsilon > 0$ so small that $B(g, \epsilon) \subset L_3([0, 1], m)$. Now for any $f \in L_1([0, 1], m)$ we have $h = g + \frac{\epsilon}{2} \frac{f}{||f||_1} \in B(g, \epsilon)$, since $||g - h||_1 = \frac{\epsilon}{2} || \frac{f}{||f||_1} ||_1 = \frac{\epsilon}{2} < \epsilon$.
But since $L_3([0, 1], m)$ is a subspace, this implies $f = \frac{2}{\epsilon ||f||_1}(h - g) \in L_3([0, 1], m)$. But if this holds for any $f \in L_1([0, 1], m)$ it contradicts the fact that $L_3([0, 1], m)$ is a proper subset of $L_1([0, 1], m)$. Hence we conclude $L_3([0, 1], m)$ must have empty interior, and further E_n must have empty interior.

c)

We want to show E_n is closed in $L_1([0, 1], m)$, so vi start off be taking a sequence $(f_k)_{k\in\mathbb{N}} \subset E_n$, where $\lim_{k\to\infty} ||f_k - f||_1 = 0$, and wish to show that $f \in E_n$. Note first that $\lim_{k\to\infty} ||f_k - f||_1 = 0$ implies there is a subsequence $(f_{k_i})_{i\in\mathbb{N}}$ which converges pointwise almost everywhere to f **(Schilling Corollary 13.8)**. Then we get by Fatou's lemma **(Schilling Theorem 9.11)**

$$\int_{[0,1]} |f|^3 dm = \int_{[0,1]} \liminf_{i\to\infty} |f_{k_i}|^3 dm \leq \liminf_{i\to\infty} \int_{[0,1]} |f_{k_i}|^3 dm \leq n$$

hence $f \in E_n$, hence A_n is closed.

d)

We have shown in b) combined with c), that for any $n \geq 1$ E_n is nowhere dense, as it has no interior points and is equal to its closure. Further we see that $\bigcup_{n=1}^{\infty} E_n = L_3([0, 1], m)$, as for any $f \in L_3([0, 1], m)$, we can find an n large enough that $f \in E_n$. Now we can conclude from **definition 3.12 (ii)** that $L_3([0, 1], m)$ is of first category in $L_1([0, 1], m)$.

Problem 5

a)

The statement is *true*. As $x_n \to x$ it follows from the reverse triangle inequality that

$$0 \leq |\ ||x_n|| - ||x||\ | \leq ||x_n - x|| \to 0$$

as $n \to \infty$, implying $||x_n|| \to ||x||$.

b)

The statement is *false*, counterexample: Consider an orthnormal basis $(e_n)_{n\in\mathbb{N}}$, which we know exists since H is seperable. We see that $||e_n|| = 1$ for all $n \in \mathbb{N}$ and hence $||e_n|| \to 1$ as $n \to \infty$. We now wish to show that $e_n \to 0$ weakly. From **Problem 2 HW4** we know that this is equivalent to showing $f(e_n) \to f(0) = 0$ for every $f \in H^*$. Recall **Problem 1 HW2** - Riesz representation theorem, which gives us that for every $f \in H^*$ there exists $y \in H$ such that $f(x) = \langle x, y\rangle$ for any $x \in H$. Recall further that from Bessels inequality (**Schilling Theorem 26.19)**, that for any $y \in H$, and a orthnormal basis $(e_n)_{n\in\mathbb{N}}$

$$\sum_{n=1}^{\infty} |\langle y, e_n\rangle|^2 \leq ||y||$$

and since $||y|| < \infty$ this means $|\langle y, e_n\rangle|^2 \to 0$, and further $\overline{\langle y, e_n\rangle} \to 0$ as $n \to \infty$. We now see that for any $f \in H*$

$$f(e_n) = \langle e_n, y\rangle = \overline{\langle y, e_n\rangle} \to 0 = f(0),$$

as we set out to proof.

c)

The statement is *true*. If $x = 0$ the statement holds trivially, so assume first that $x \neq 0$. Then from **Theorem 2.7 (b)** we know there exists $f \in H^*$ such that $||f|| = 1$ and $f(x) = ||x||$. Further we know from **Problem 2 HW4**, that $x_n \to x$ weakly, implies $f(x_n) \to f(x)$ for any $f \in H^*$, and inparticular for our chosen f. Now we see that

$$||x|| = |f(x)| = \liminf_{n\to\infty} |f(x_n)| \leq \liminf_{n\to\infty} ||f||||x_n|| = \liminf_{n\to\infty} ||x_n|| \leq \liminf_{n\to\infty} 1 = 1$$

where we have used the standard inequality for bounded linear maps (equation (1.8) in the notes).