FunkAn - 1

Milos Mathias Koch - skg675

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Problem 1

(a)

As T is linear, we know that $T(0) = 0$, hence

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y = 0 \iff (\|x\|_X = 0 \wedge \|Tx\|_Y = 0) \iff x = 0.$$

Thus $\|\cdot\|_0$ is positive definite. Let $x, y \in X$ and $a \in \mathbb{K}$. By direct computation, we see

$$\begin{aligned}
\|ax\|_0 &= \|ax\|_X + \|T(ax)\|_Y \\
&= |a|\|x\|_X + |a|\|Tx\|_Y \\
&= |a|(\|x\|_X + \|Tx\|_Y) \\
&= |a|\|x\|_0
\end{aligned}$$

and

$$\begin{aligned}
\|x+y\|_0 &= \|x+y\|_X + \|T(x+y)\|_Y \\
&= \|x+y\|_X + \|Tx+Ty\|_Y \\
&\leq \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y \\
&\leq \|x\|_X + \|Tx\| + \|y\|_X + \|Ty\|_Y \\
&= \|x\|_0 + \|y\|_0.
\end{aligned}$$

Thus we have shown that $\|\cdot\|_0$ is indeed a norm on X.

Now assume that $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent. Then there exists $c, C \in (0, \infty)$ such that $c\|x\|_0 \leq \|x\|_X \leq C\|x\|_0$ for all $x \in X$. Hence

$$\|Tx\|_Y = \|x\|_0 - \|x\|_X \leq C\|x\|_X - \|x\|_X = (C-1)\|x\|_X.$$

Thus T is bounded with $\|T\|_{\mathcal{L}(X,Y)} \leq C - 1$. For the converse implication, assume that T is bounded. We can establish the first inequality by noting that, since $0 \leq \|Tx\|_Y$, we have $\|x\|_X \leq \|x\|_0$. Hence by setting $c = 1$, we have $c\|x\|_X \leq \|x\|_0$. As T is bounded, we have $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$ and some $C > 0$. This immediatly gives us that

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq \|x\|_X + C\|x\|_X = (1 + C)\|x\|_X,$$

thus the two norms are equivalent.

(b)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be finite-dimensional normed spaces and let $T : X \to Y$ a linear map. Define $\|\cdot\|_0$ as in problem 1 (a). By theorem 1.6, $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent, hence T is bounded by the result of problem problem 1 (a). As T was chosen arbitrarily, any linear map between $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ will be bounded.

(c)

Choose a Hamel basis of non-zero elements, $(e_i)_{i\in I}$, for X. As X is infinite dimensional, I must be at least countably infinite. Let $J \subseteq I$ be countably infinite subset of I. Choose a enumeration $(j_n)_{n\in\mathbb{N}}$ of J. Let $(e_n)_{n\in\mathbb{N}}$ be a subset of $(e_i)_{i\in I}$, defined as $e_n := e_{j_n}$ for each $n \in \mathbb{N}$. As $Y \neq \{0\}$, we can choose a non-zero $y \in Y$ with $\|y\|_Y = 1$. Define the set $(y_i)_{i\in I}$ as $y_i := n\|e_n\|y_X$, for $i \in J$, and 0 otherwise. As $(e_i)_{i\in I}$ is a Hamel basis, there exists a unique linear map $T : X \to Y$ such that $T(e_i) = y_i$ for all $i \in I$.

If T is bounded, then $\sup(\|Tx\|_Y | \|x\|_X = 1) < \infty$. However, by direct computation, we see

$$\begin{aligned}
\sup(\|Tx\|_Y \| \|x\|_X = 1) &\geq \sup_{n\in\mathbb{N}} \left(\left\| T\left(\frac{e_n}{\|e_n\|}\right) \right\|_Y \right) \\
&= \sup_{n\in\mathbb{N}} \left(\frac{1}{\|e_n\|} \|Te_n\|_Y \right) \\
&= \sup_{n\in\mathbb{N}} \left(\frac{1}{\|e_n\|} \|y_i\|_Y \right) \\
&= \sup_{n\in\mathbb{N}} \left(\frac{1}{\|e_n\|} \|n\|e_n\|_X y\|_Y \right) \\
&= \sup_{n\in\mathbb{N}} \left(\frac{1}{\|e_n\|} n\|e_n\|_X \|y\|_Y \right) \\
&= \sup_{n\in\mathbb{N}} (n) \\
&= \infty,
\end{aligned}$$

¨ hence T is unbounded,

(d)

Let $(X, \|\cdot\|_X)$ be a Banach space. Let $T : X \to X$ be an unbounded linear map. Define $\|\cdot\|_0$ as in problem 1 (a). We saw in problem 1 (a) that $\|x\|_X \leq \|x\|_0$ for all $x \in X$. By problem 1 (a), we have that $\|x\|_X$ and $\|x\|_0$ are not equivalent. By HW3 problem 1, or rather the contraposition of HW3 problem 1, we have that if $(X, \|\cdot\|_X)$ and $(X, \|\cdot\|_0)$ are normed spaces, with $\|x\|_X \leq \|x\|_0$ for all $x \in X$, and $\|x\|_X$ and $\|x\|_0$ are not equivalent, then either $(X, \|\cdot\|_X)$ or $(X, \|\cdot\|_0)$ are not complete. By assumption $(X, \|\cdot\|_X)$ is a Banach space, hence $(X, \|\cdot\|_0)$ is not.

(e)

Let $(X, \|\cdot\|_X) = (\ell_1(N), \|\cdot\|_1)$. And consider $(\ell_1(N), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the uniform norm. If an element $x \in \ell_1(\mathbb{N})$ has at most 1 non-zero entries, then $\|x\|_\infty = \|x\|_1$, and if x has more than 1 non-zero entries, then $\|x\|_\infty < \|x\|_1$. Hence $\|x\|_\infty \leq \|x\|_1$ for all $x \in X$.

Consider the sequence of sequences $(x_n)_{n\in\mathbb{N}}$, defined as

$$x_n(k) \begin{cases} \frac{1}{k} & k \leq n \\ 0 & \text{else} \end{cases}$$

As each x_n has compact support, $(x_n)_{n\in\mathbb{N}} \subseteq \ell_1(\mathbb{N})$. Furthermore, for $x := (\frac{1}{n})_{n\in\mathbb{N}} \in \ell_\infty(\mathbb{N})$, we have

$$\|x - x_n\|_\infty = \frac{1}{n+1} \to 0,$$

hence x_n converges to x in the uniform norm. Furthermore, it is a Cauchy sequence in $(\ell_1(N), \|\cdot\|_\infty)$, as, for $m, n > N$

$$\|x_m - x_n\|_\infty \leq \frac{1}{N} \to 0.$$

But as $x \notin \ell_1(\mathbb{N})$, (the harmonic series diverges) $(\ell_1(N), \|\cdot\|_\infty)$ can not be complete.

Assume for a contradiction that there exists $C > 0$ such that $\|x\|_1 \leq C\|x\|_\infty$ for all n. Let $N > C$ Consider the sequence

$$z(k) \begin{cases} C & k \leq N \\ 0 & \text{else} \end{cases}.$$

Once again, z is compactly supported, hence $z \in \ell_1(\mathbb{N})$. By direct computation we see

$$\|z\|_1 = N \cdot C = N\|z\|_\infty > C\|z\|_\infty,$$

contradicting $\|x\|_1 \leq C\|x\|_\infty$, hence the two norms are not equivalent.

Problem 2

Throughout this problem, we will suppress norm subscripts, so they will have to be inferred from context.

(a)

Firstly, let $p \in (1, \infty)$. We, by HW1 problem 5, know that the mapping $\Phi : \ell_q(\mathbb{N}) \to (\ell_p(\mathbb{N}))^*$, where $q = \frac{p}{p-1}$, given by $x \longmapsto f_x(\cdot) = \sum_{k=1}^{\infty} (\cdot)(k)x(k)$, is a well-defined isometric isopmorphism. Hence so is its inverse $\Phi^{-1} : (\ell_q(\mathbb{N}))^* \to \ell_p(\mathbb{N})$. This implies that $\|f_x\| = \|x\|$. Now let $x = (1, 1, 0 \ldots)$. It is immediate that $f_{x|M} = f$, and as $M \subseteq \ell_p(\mathbb{N})$, we have

$$
\begin{aligned}
\sup_{x \in M}(\|f(y)\| \mid \|y\| \leq 1) &= \sup_{x \in M}(\|f_x(y)\| \mid \|y\| \leq 1) \\
&\leq \sup_{x \in X}(\|f_x(y)\| \mid \|y\| \leq 1) \\
&= \|f_x\| = \|x\|,
\end{aligned}
$$

hence $\|f\| \leq \|f_x\|$ which shows that f is bounded, and we are only one inequality away from computing $\|f\|$. Now let $y \in \ell_p(\mathbb{N})$, and let $y_M := (y(1), y(2), 0, 0, \ldots)$. We see

$$
\|y_M\| = (|y(1)|^p + |y(2)|^p)^{\frac{1}{p}} \leq (\sum_{n=1}^{\infty} |y(n)|^p)^{\frac{1}{p}} = \|y\|.
$$

Hence

$$
\begin{aligned}
|f_x(y)| &= |f_x(y_M)| \\
&= |f(y_M)| \\
&\leq \|f\|\|y_M\| \\
&\leq \|f\|\|y\|.
\end{aligned}
$$

Thus we have $\|f_x\| \leq \|f\|$, and so $\|f_x\| \leq \|f\| = \|x\|_p$. Hence $\|f\| = \|x\|_q = \|(1, 1, 0 \ldots)\|_q$ for $p(1, \infty)$.

As the above argument is essentially an application of the isometric isopmorphic relation $\ell_p(\mathbb{N}) \cong (\ell_q(\mathbb{N}))^*$ shown in HW1 problem 5, we will use the corresponding result for $p = 1$, that states that $\ell_\infty(\mathbb{N}) \cong (\ell_1(\mathbb{N}))^*$. Let $p = 1$ and $x = (1, 1, 0 \ldots)$. A step for step copy of the norm-consideration above shows that $\|f\| = \|f_x\| = \|x\|_\infty = 1$. And so we have seen computed the norm of f for $1 \in [1, \infty)$.

(b)

Existence is showed in problem 2 (a). Let $x = (1, 1, 0 \ldots)$, and let $f_x(\cdot) = \sum_{k=1}^{\infty}(\cdot)(k)x(k)$. We also saw that $\|f_x\| = \|x\| = \|f\|$. Assume for a contradiction that there exist another linear functional $f_{x'}$ that extends f to $\ell_p(\mathbb{N})$ with $f_x \neq f_{x'}$ and $\|f_x\| = \|f_x'\| = \|f\|$. As $z \longmapsto f_z$ is bijective, so is $f_z \longmapsto z$, hence $x \neq x'$. As $f_{x|M} = f_{x'|M}$, x and x' must agree on the first and second entries. Hence there exists $k > 2$ such that $x'(k) \neq x(k) = 0$, hence $|x'(k)| > 0$. By direct computation we see

$$\begin{aligned}
\|f_x'\|^q &= \sum_{n=1}^{\infty} |x'(n)|^q \\
&\geq |x'(1)|^q + |x'(2)|^q + |x'(k)|^q \\
&= \|x\|^p + |x'(k)|^q \\
&> \|x\|^p = \|f_x\|,
\end{aligned}$$

which contradicts our assumption of equal norms.

(c)

We know that $\Phi : \ell_\infty(\mathbb{N}) \to (\ell_1(\mathbb{N}))^*$ given by $x \longmapsto f_x$, is an isometric isomorphism. Hence $\|x\|_\infty = \|f_x\| = 1$. However choose some natural number $k > 2$ and non-zero complex number α with $|\alpha| \leq 1$, and let x' be defined as

$$x'(n) = \begin{cases} x(n) = 1 & \text{for } n \in \{1, 2\} \\ \alpha & \text{for } n = k \\ 0 & \text{else.} \end{cases}$$

Clearly $\|x\|_\infty = \|x'\|_\infty$, hence $\|f_x\| = \|f_{x'}\|$. f_x and $f_{x'}$ agree on M. Indeed, let $y \in M$, and compute

$$f_{x'}(y) = \sum_{n=1}^{\infty} x(n)y(n) = |y(1)| + |y(2)| = f(y),$$

Hence $f_{x'|M} = f$. Since $f_{x|M} = f$, we see that $f_{x'}$ actually is an extension of f. As Φ is bijective, and so $x \neq x'$ implies $f_x \neq f_{x'}$. As there are uncountable many $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$

(and countably infinite entries to place those α), we see that there exist infinitely many linear extension of f for $p = 1$.

Problem 3

(a)

Assume for a contradiction that there exists a linear injection $F : X \to \mathbb{K}^n$. As X is infinite dimensional, we can find $n+1$ distinct, non-zero linearly independent elements in X, denote these with $(x_i)_{1\leq i\leq n+1}$. By elementary linear algebra, we know that injective linear mappings preserve linear independence. Hence $(F(x_i))_{1\leq i\leq n+1}$ is also a set of linear independent elements in $\mathbb{K}^n$, a clear contradiction. Hence there does not exists a linear injection $F : X \to \mathbb{K}^n$.

(b)

Consider the map $F : X \to \mathbb{K}^n$, given by $x \longmapsto (f_1(x), f_2(x)\dots, f_n(x))$. It is clear that $F(x) = 0_n \in \mathbb{K}^n$ if and only if $f_j(x) = 0 \in \mathbb{K}$ for all $j \in \{1, \dots n\}$. By Problem 3 (a), F is not injective, hence $\ker F \neq \{0\}$. Let $x_0 \in \ker F$ be a non-zero element in the kernel of F. Hence x_0 is also a non-zero element of the kernel of f_j for each $j \in \{1, \dots n\}$. Thus we have shown the desired result.

(c)

For $x_j = 0$ for any $j \in \{1 \dots n\}$, the assertion is trivial, indeed we can simply choose any $y \in X$ with norm 1. Hence, assume that $x_j \neq 0$ for all $j \in \{1 \dots n\}$. By theorem 2.7(b) there exist for $f_1, \dots f_n \in X^*$ such that $f_j(x_j) = \|x_j\|_X$ and $\|f_j\|_{X^*} = 1$ for all $j \in \{1 \dots n\}$. By problem 3 (b), we have that $\bigcap_{i=1}^n \ker(f_i) \neq \{0\}$. As a finite intersection of subspaces are again a subspace, we can find a $y \in \bigcap_{i=1}^n \ker(f_i)$ with $\|y\| = 1$. Thus the following computations hold for all $i \in \{1 \dots n\}$

$$
\begin{aligned}
\|x_i - y\|_X &= \|f_i\|_{X^*} \|x_i - y\|_X \\
&\geq |f_i(x_i - y)| \\
&= |f_i(x_i) - f_i(y)| \\
&= |f_i(x_i)| \\
&= \|x_i\|,
\end{aligned}
$$

thus we have derived the desired result.

(d)

Let y be the element, whose existence we showed in problem 3 (c). Any finite collection of closed ball covering S would necessarily include at least one ball containing y. Let B_y denote a closed ball containing y. As $d(0, x_i) = \|x_i\|_X \leq \|x_i - y\|_X = d(x_i, y)$, whre d is associated norm-metric on X, such a ball would also contain 0.

(e)

Assume for a contradiction that S is compact. Then consider the open cover $\mathcal{B} = \{B(x, \frac{1}{2}) | x \in S\}$ of open ball with radius $\frac{1}{2}$. As S is compact, there exist a finite subcover $\mathcal{B}_n = \{B(x_i, \frac{1}{2})\}_{i \in \{1, \dots n\}}$. Let $\bar{B}(c, r)$ denote the closure of the open ball with center in c and radius r. As the closure of a open ball with center in c and radius r a closed ball with center in c and radius r, $\bar{B}(x_i, \frac{1}{2})$ is a closed ball with center in $x_i \in S$ and radius $\frac{1}{2}$. As $B(x_i, \frac{1}{2}) \subset \bar{B}(x_i, \frac{1}{2})$, the collection of the closure of each open ball in $\mathcal{B}_n$, denoted by $\mathcal{CB}_n$, is a (closed) cover of S. As $d(s, 0) = 1$ for each $s \in S$, no balls in $\mathcal{CB}_n$ contain 0. This a contradiction with problem 3 (d). Hence S cannot be compact.

Assume for a contradiction that the unit ball is compact. As S is, by analysis 1, a closed subset of the unit ball, S is compact, but as we have just shown, S is not compact, hence the unit ball cannot be compact.

Problem 4

(a)

It is not the case. By HW2 problem 2, we know that $L_3 := L_3([0,1], m)$ is a proper subset of $L_1 : L_1([0,1], m)$, hence we can choose $f \in L_1 \setminus L_3$. As L_3 consists of all complex-valued functions, g, such that $\int_{[0,1]} |g|^3 \, dm < \infty$, we know that $\int_{[0,1]} |f|^3 \, dm = \infty$. If E_n was absorbing for some $n \in \mathbb{N}$, there would have to exist $t > 0$, such that $\int_{[0,1]} |tf|^3 \, dm < n$, but by linearity of integrals, we have

$$\int_{[0,1]} |tf|^3 \, dm = t^3 \int_{[0,1]} |f|^3 \, dm = \infty,$$

for all $t > 0$. Hence there does not exist $t > 0$, such that $tf \in E_n$ for any $n \in \mathbb{N}$.

(b)

Let $n \in \mathbb{N}$ and let $f \in E_n$. Let $f' \in L_1 \setminus L_3$, and let $(f_k)_{k \in \mathbb{N}}$ be the sequence defined as $f_k = f + \frac{f'}{k}$. Assume for a contradiction that there exist $m \in \mathbb{N}$ such that $f_m = f - \frac{f'}{m} \in E_n$. Then it would in particular also be in L_3. As L_3 is a vector space, this implies that $k(f - f_m) = f'$ would also be in L_3, which is a contradiction. Hence $(f_k)_{k \in \mathbb{N}}$ is entirely outside of E_n. We note that, since $f' \in L_1$ implies that $\|f'\|_{L_1} < \infty$

$$\|f - f_k\|_{L_1} = \left\| \frac{f'}{k} \right\| = \frac{1}{k} \|f'\| \to 0,$$

for $k \to \infty$. Hence f is not in the interior of E_n. As f was chosen arbitrarily, E_n has empty interior. As n was chosen arbitrarily, E_n has empty interior for all $n \in \mathbb{N}$.

(c)

Let $(f_k)_{k \in \mathbb{N}} \subseteq E_n$ for some $n \in \mathbb{N}$. Assume that $(f_k)_{k \in \mathbb{N}}$ converges in L_1 to some function $f \in L_1$. By corollary 13.8 of "Measures, Integrals and Martingales" by René Schilling, there exists a subsequence $(f_{k_j})_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} f_{k_j}(x) \to f(x)$ almost surely. As $|\cdot|^3$ is continuous, we have $\lim_{j \to \infty} \left| f_{k_j}(x) \right|^3 \to |f(x)|^3$ almost surely, note that $\liminf_{j \to \infty} \left| f_{k_j}(x) \right|^3 = f(x)$ almost surely. Hence, as $(\left|f_{k_j}\right|^3)_{j\in\mathbb{N}}$ is a sequence of positive measurable functions, by Fatou's lemma, we have

$$\int_{[0,1]} |f|^3 \, dm = \int_{[0,1]} \liminf_{j\to\infty} \left|f_{k_j}\right|^3 \, dm \leq \liminf_{j\to\infty} \int_{[0,1]} \left|f_{k_j}\right|^3 \, dm \overset{(*)}{\leq} n,$$

where $(*)$ is due to the fact that $\int_{[0,1]} \left|f_{k_j}\right|^3 \, dm \leq n$ for all $j \in \mathbb{N}$. Hence E_n is closed for all $n \in \mathbb{N}$.

(d)

As E_n is closed, we have $E_n = \bar{E_n}$, where $\bar{E_n}$ denotes the closure of E_n. Thus, as E_n has empty interior, so does its closure, so E_n is a nowhere dense set, for all $n \in \mathbb{N}$. As

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \left\{ f \in L_1 | \int_{[0,1]} |f|^3 \, dm \leq n \right\} = \left\{ f \in L_1 | \int_{[0,1]} |f|^3 \, dm < \infty \right\} = L_3,$$

we have that L_3 is a countable union of nowhere dense sets in L_1, hence it is of first category.

Problem 5

(a)

By the inverse triangle inequality, we have

$$|\|x_n\| - \|x\|| \leq \|x_n - x\|.$$

As $\|x_n - x\| \to 0$ as $n \to \infty$, we have that $\|x_n\| \to \|x\|$.

(b)

It is not the case. Consider the following counterexample.

Let $(e_n)_{n\in\mathbb{N}}$ be a countable orthonormal basis of $\mathcal{H}$. For all $f \in \mathcal{H}^*$, we have $f(e_n) = \langle e_n, x_f\rangle$ for some unique $x_f \in \mathcal{H}^*$ by the Riesz representation theorem (proved on 332 in Schilling). Then by the equivalent definitions (listed and proved on page 335-336 in "Measures, Integrals and Martingales" by René Schilling), we have that $\langle e_n, x_f \rangle \to 0$ as $n \to \infty$ for all $x_f \in \mathcal{H}^*$. Hence, by HW4, e_n converges weakly to 0 as $n \to \infty$. But as $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis, $\|e_n\| = 1$ for all $n \in \mathbb{N}$, hence $\|e_n\| \to 1 \neq 0$ as $n \to \infty$.

(c)

Let $x_n \to x$ weakly as $n \to \infty$, then by HW4, we know that $f(x_n) \to f(x)$ for all $f \in \mathcal{H}^*$. By theorem 2.7(b) there exist a functional $\phi \in \mathcal{H}^*$, such that $\|\phi\|_{\mathcal{H}^*} = 1$ and $\phi(x) = |\phi(x)| = \|x\|_{\mathcal{H}}$. Hence we have

$$\begin{aligned}
\|x\|_{\mathcal{H}} &= |\phi(x)| \\
&= \lim_{n\to\infty} |\phi(x_n)| \\
&= \liminf_{n\to\infty} |\phi(x_n)| \\
&\leq \liminf_{n\to\infty} \|\phi\|_{\mathcal{H}^*} \|x_n\|_{\mathcal{H}} \\
&= \liminf_{n\to\infty} \|x_n\|_{\mathcal{H}} \\
&\leq 1.
\end{aligned}$$

Hence we see that the statement is true.

Merry Christmas and happy new years!