Problem 1a

We have that:

$$\|f_N\|^2 = \left\| \frac{1}{N} \sum_{n=1}^{N^2} e_n \right\|^2 = \frac{1}{N^2} \left\| \sum_{n=1}^{N^2} e_n \right\|^2 = \frac{1}{N^2} \left\langle \sum_{n=1}^{N^2} e_n, \sum_{k=1}^{N^2} e_k \right\rangle$$

$$= \frac{1}{N^2} \sum_{n,k=1}^{N^2} \langle e_n, e_k \rangle = \frac{1}{N^2} \sum_{n=1}^{N^2} \langle e_n, e_n \rangle = \frac{1}{N^2} \sum_{n=1}^{N^2} \|e_n\|^2 = \frac{1}{N^2} \sum_{n=1}^{N^2} 1 = \frac{1}{N^2} N^2 = 1$$ ✓

using that $\langle e_n, e_k \rangle = 0$ if $n \neq k$. As such the sequence $(f_N)_{N\in\mathbb{N}}$ is contained in $\overline{B}(0,1)$, which is weakly compact as H is reflexive and weakly metrizable as H is separable by theorem 5.13 (using that $H \cong H^{**}$ and that the weak and weak* topologies coincide). Hence $\overline{B}(0,1)$ is weakly sequentially compact and hence has a weakly convergent subsequence. Note that for $N^2 > k$: $(f_N)_{N\in\mathbb{N}}$ has that!

$$\langle f_N, e_k \rangle = N^{-1} \sum_{n=1}^{N^2} \langle e_n, e_k \rangle = \frac{1}{N} \to 0 \text{ for } N \to \infty$$

so by homework 4 problem 2a, the convergent subsequence must converge to 0. Taking any subsequence of $(f_N)_{N\in\mathbb{N}}$, we can apply the same argument to any subsequence of such subsequence. As all subsequences of $(f_N)_{N\in\mathbb{N}}$ has a subsequence that weakly converges to 0, $(f_N)_{N\in\mathbb{N}}$ also weakly converges to 0. weakly ✓

Problem 1b

Note that as $\text{co}\{f_N \mid N \geq 1\} = \{\sum_{i=1}^n \alpha_i f_i \mid \alpha_i \geq 0, \ \sum_{i=1}^n \alpha_i = 1\}$, we have as $\|f_i\| = 1$:

$$\left\| \sum_{i=1}^n \alpha_i f_i \right\| \leq \sum_{i=1}^n \alpha_i \|f_i\| = \sum_{i=1}^n \alpha_i = 1$$

so $\text{co}\{f_N \mid N \geq 1\} \subset \overline{B}_H(0,1)$, so in particular $K = \overline{\text{co}\{f_N \mid N \geq 1\}}^{\|-\|} \subset \overline{B}_H(0,1)$. As $\text{co}\{f_N \mid N \geq 1\}$ is a convex set $K = \overline{\text{co}\{f_N \mid N \geq 1\}}^{\|-\|} = \overline{\text{co}\{f_N \mid N \geq 1\}}^{\tau_w}$ by theorem 5.7 in the notes. As H is a Hilbert space it's reflexive and by theorem 6.3 in the notes $\overline{B}_H(0,1)$ is weakly compact. As K is a weakly closed subset it's also weakly compact. Note that as $f_N \to 0$ weakly: ✓

$$0 \in \overline{\{f_N \mid N \geq 1\}}^{\tau_w} \subset \overline{\text{co}\{f_N \mid N \geq 1\}}^{\tau_w} = K$$

Problem 1c

As H is a separable infinite-dimensional Hilbert space it's isometrically isomorphic to $l_2(\mathbb{N})$. By homework 5 problem 3b, we know that the elements of norm 1 are extreme points of the closed unit ball in $l_2(\mathbb{N})$ and hence also in the closed unit ball of H. As $\|f_N\| = 1$ these are extreme points of $\overline{B}_H(0,1)$ and hence also extreme points of $K \subset \overline{B}_H(0,1)$ (as the lines and points in K is contained in $\overline{B}_H(0,1)$, hence $\mathrm{Ext}(\overline{B}_H(0,1)) \subset \mathrm{Ext}(K)$ by definition). We have that:

No, but extreme points in $\overline{B}_H(0,1)$ that also live in K are extreme points of K.

$$\mathrm{co}\{f_N \mid N \geq 1\} = \left\{ \sum_{i=1}^{n} \alpha_i f_i \mid \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1 \right\} = \left\{ \sum_{i=1}^{n} \sum_{k=1}^{i^2} \frac{\alpha_i}{i} e_k \mid \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

for all $m \in \mathbb{N}$ and $\alpha \in \mathrm{co}\{f_N \mid N \geq 1\}$. We then have:

$$\langle \alpha, e_m \rangle = \sum_{i=1}^{n} \sum_{k=1}^{i^2} \frac{\alpha_i}{i} \langle e_k, e_m \rangle \geq 0$$

hence we must also have $\langle \beta, e_m \rangle \geq 0$ for all $\beta \in \overline{\mathrm{co}\{f_N \mid N \geq 1\}} = K$ and $m \in \mathbb{N}$. Hence if $0 = \alpha u + (1-\alpha)v$, with $0 < \alpha < 1$ and $u, v \in K$, then for all m:

$$0 = \langle \alpha u + (1-\alpha)v, e_m \rangle = \alpha \langle u, e_m \rangle + (1-\alpha) \langle v, e_m \rangle$$

and as $\langle u, e_m \rangle, \langle v, e_m \rangle \geq 0$ we must have $\langle u, e_m \rangle = \langle v, e_m \rangle = 0$ for all $m \in \mathbb{N}$. Hence $u = v = 0$, showing that $0 \in \mathrm{Ext}(K)$.

Problem 1d

Letting $F = \{f_N \mid N \geq 1\}$, then $K = \overline{\mathrm{co}(F)}^{\|-\|} = \overline{\mathrm{co}(F)}^{\tau_w}$ (using that $\mathrm{co}(F)$ is convex and theorem 5.7 in the notes) and $\overline{F}^{\tau_w} = F \cup \{0\}$ as all nets with infinitely many different points of F (and taking out finitely many points) is a subnet of $(f_N)_{N \in \mathbb{N}}$ and hence converge to 0. As H equipped with the weak topology is in particular a LCTVS and K is weakly compact, we have by theorem 7.9 in the notes that $\mathrm{Ext}(K) \subset \overline{F}^{\tau_w} = F \cup \{0\}$. Combining this with 1c this means $\mathrm{Ext}(K) = \{f_N \mid N \geq 1\} \cup \{0\}$.

Problem 2a

Weak convergence of $x_n \to x$ is equivalent to $f(x_n) \to f(x)$ for all $f \in X^*$ by homework 4 problem 2a. We have $g \circ T \in X^*$ for all $g \in Y^*$, as continuity and linearity is preserved under composition. So by assumption $g \circ T(x_n) \to g \circ T(x)$ for all $g \in Y^*$, which is equivalent to $Tx_n \to Tx$ weakly.

Problem 2b

As $(x_n)_{n \in \mathbb{N}}$ converges weakly to x it's a bounded sequence by homework 4 problem 2b, by proposition 8.2 $(Tx_n)_{n \in \mathbb{N}}$ has a strong convergent subsequence. Such a subsequence must necessarily converge to Tx as by 2a $Tx_n \to Tx$ weakly and hence this is true for any subsequence. As the same argument can be applied to any subsequence of $(Tx_n)_{n \in \mathbb{N}}$, we have that all subsequences of $(Tx_n)_{n \in \mathbb{N}}$ has a further subsequence converging strongly to Tx, meaning that Tx_n converges to Tx.

Problem 2c

Assume $T \in \mathcal{L}(H, Y)$ is not compact. By proposition 8.2 in the notes there then exists a bounded sequence, $(x_n)_{n\in\mathbb{N}}$, in H which has **no** subsequence, $(x_{n_k})_{k\in\mathbb{N}}$, so that the sequence $(Tx_{n_k})_{k\in\mathbb{N}}$ converges strongly in Y. By dividing the terms in the sequence by $c := \sup\{x_n \mid n \in \mathbb{N}\}$, we can assume that it's contained in the closed unit ball, while not changing the facts mentioned above. This means that for $\delta > 0$ small enough, then $\|Tx_n - Tx_m\| \geq \delta$ for infinitely many $n \neq m$ and we can assume this is true for all $n \neq m$ by considering the subsequence of $(x_n)_{n\in\mathbb{N}}$ for which this is true, which doesn't change the facts above as a subsequence of a subsequence is a subsequence. As H is a Hilbert space, it's reflexive and hence $\overline{B}_H(0,1)$ is compact, and since H is separable by theorem 5.13 in the notes $(\overline{B}_H(0,1), \tau_w)$ is also metrizable (as the weak and weak* topologies coincide on $H = H^{**}$) and hence is sequintially compact. Hence there exists a weakly convergent subsequence, $(x_{n_k})_{k\in\mathbb{N}}$, of $(x_{n_k})_{k\in\mathbb{N}}$, but by assumption on the sequence $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$ and hence $\|Tx_{n_k} - Tx_{n_l}\| \geq \delta$ for all $k \neq l$. Combining this with 2b, this shows that $T \in \mathcal{K}(H, Y)$ if and only if $(Tx_n)_{n\in\mathbb{N}}$ is strongly convergent whenever $(x_n)_{n\in\mathbb{N}}$ is weakly convergent.

Problem 2d

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $l_2(\mathbb{N})$ which converges weakly to $x \in l_2(\mathbb{N})$. By 2a, $Tx_n \to Tx$ weakly for all $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$, then $Tx_n \to Tx$ in norm as weak convergence of a sequence is the same as norm convergence of a sequence in $l_1(\mathbb{N})$ by remark 5.3 in the notes. As $\{x_n\}_{n\in\mathbb{N}}$ was an arbitrary weakly convergent sequence, by 2c (using that $l_2(\mathbb{N})$ is a separable Hilbert space) $T \in \mathcal{K}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ and as T was arbitrary $\mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N})) = \mathcal{K}(l_2(\mathbb{N}), l_1(\mathbb{N}))$.

Problem 2e

If T was surjective it would be an open map by the open mapping theorem (as X and Y are Banach spaces), so $T(B_X(0,1))$ would be open and totally bounded (by compactness of T) and contains 0 (by linearity of T). As $T(B_X(0,1))$ is open and contains 0 there exists a $a \in \mathbb{K} \setminus \{0\}$ so that $aB_Y(0,1) \subset T(B_X(0,1))$, but then $\overline{aB_Y(0,1)} \subset \overline{T(B_X(0,1))}$ where the latter is compact and hence $\overline{aB_Y(0,1)}$ is compact and hence so is $\overline{B_Y(0,1)}$ (since there is a bijection of open covers by $\{U_\alpha\}_{\alpha\in I} \mapsto \{a^{-1}U_\alpha\}_{\alpha\in I}$). But from assignment 1 problem 3 we know that since Y is infinite-dimensional $\overline{B_Y(0,1)}$ cannot be compact, hence by contradiction T is not onto.

Problem 2f

We have that for all $f, g \in H$ that:

⟨Mf,g⟩

$$\langle Mf(t), g(t)\rangle = \int_{[0,1]} Mf(t)\overline{g(t)}dm(t) = \int_{[0,1]} tf(t)\overline{g(t)}dm(t) = \int_{[0,1]} f(t)\overline{tg(t)}dm(t)$$

$$= \int_{[0,1]} f(t)\overline{tg(t)}dm(t) = \int_{[0,1]} f(t)\overline{Mg(t)}dm(t) = \langle f(t), Mg(t)\rangle$$

⟨f,Mg⟩

Noting that $L^2([0,1],m)$ is separable and infinite-dimensional.

So $M = M^*$. If M was in addition compact it would satisfy the conditions for the spectral theorm for self-adjoint compact operators and would therefore have infinitely many eigenvalues. However by homework 6 problem 3a, M has no eigenvalues, and hence M can't be compact.

Problem 3a

$T = T_{\tilde{k}}$ $\tilde{k}(s,t) = k(t,s)$

Note that T is the associated operator of K. As $[0,1]$ is compact and Hausdorff, the Lebesgue measure is finite on $[0,1]$ ($m([0,1]) = 1$), and clearly $K \in C([0,1] \times [0,1])$ (as $(1-s)t = (1-t)s$ when $s = t$), by theorem 9.6 in the notes T is compact.

Problem 3b

For all $1 \geq a \geq b \geq 0$ we have $K(a,b) = (1-a)b = K(b,a)$, from this we have that:

$$\langle Tf(s), g(s) \rangle = \int_{[0,1]} Tf(s)\overline{g(s)}ds = \int_{[0,1]} \int_{[0,1]} K(s,t)f(t)\overline{g(s)}dt\ ds$$

$$= \int_{[0,1]} \int_{[0,1]} f(t)K(t,s)\overline{g(s)}ds\ dt = \int_{[0,1]} f(t) \int_{[0,1]} \overline{K(s,t)g(s)}ds\ dt = \int_{[0,1]} f(t)\overline{Tg(s)}dt$$
$$= \langle f(t), Tg(t) \rangle$$

why is Fubini justified?

using Fubini's theorem, $K(s,t) = K(t,s)$ and that complex conjugation commutes with integrals (as s is a real variable). This shows that $T = T^*$.

Problem 3c

We have:

$$T(f)(s) = \int_{[0,1]} K(s,t)f(t)dm(t) = \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{(s,1]} K(s,t)f(t)dm(t)$$

$$\int_{[0,s]} (1-s)tf(t)dm(t) + \int_{(s,1]} s(1-t)f(t)dm(t) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)$$

as taking the integral on $(s,1]$ and $[s,1]$ is the same. Clearly:

$$T(f)(0) = \int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]} (1-t)f(t)dm(t) = 0 + 0 = 0$$

$$T(f)(1) = 0\int_{[0,1]} tf(t)dm(t) + \int_{[1,1]} (1-t)f(t)dm(t) = 0 + 0 = 0$$

By homework 2 problem 2b we have that $f \in L_2([0,1],m) \subset L_1([0,1],m)$, so $C = \int_{[0,1]} |f(t)|dm(t) < \infty$. Given $s_1 \in [0,1]$ and $\epsilon > 0$, choose $s_0 \in [0,1]$ (say $s_0 < s_1$), so that $|s_1 - s_0| < \frac{\epsilon}{4C}$ and $\int_{[s_0,s_1]} |f(t)|dm(t) < \frac{\epsilon}{4}$. We get:

why is this possible? At this point you assume $s_0 \mapsto \int_{[s_0,s_1]} |f(t)|dm(t)$ continuous.

$$|T(f)(s_1) - T(f)(s_0)| = \left|(1-s_1)\int_{[0,s_1]} tf(t)dm(t) + s_1\int_{[s_1,1]} (1-t)f(t)dm(t)\right.$$

$$-(1-s_0)\int_{[0,s_0]} tf(t)dm(t) - s_0\int_{[s_0,1]}(1-t)f(t)dm(t)\Bigg|$$

Using the identities $\int_{[0,s_1]} tf(t)dm(t) = \int_{[0,s_0]} tf(t)dm(t) + \int_{[s_0,s_1]} tf(t)dm(t)$ and $\int_{[s_0,1]}(1-t)f(t)dm(t) = \int_{[s_0,s_1]}(1-t)f(t)dm(t) + \int_{[s_1,1]}(1-t)f(t)dm(t)$ we get:

$$= \Bigg|(s_0-s_1)\int_{[0,s_0]} tf(t)dm(t) + (1-s_1)\int_{[s_0,s_1]} tf(t)dm(t)$$

$$+(s_1-s_0)\int_{[s_1,1]}(1-t)f(t)dm(t) - s_0\int_{[s_0,s_1]}(1-t)f(t)dm(t)\Bigg|$$

4/0

Using that $|(1-t)f(t)|, |tf(t)| \leq |f(t)|$ for $t \in [0,1]$, we get:

$$\leq |s_1-s_0|\int_{[0,s_0]}|f(t)|dm(t) + (1-s_1)\int_{[s_0,s_1]}|f(t)|dm(t)$$

$$+|s_1-s_0|\int_{[s_1,1]}|f(t)|dm(t) + s_0\int_{[s_0,s_1]}|f(t)|dm(t)$$

$$\leq 2C|s_1-s_0| + 2\int_{[s_0,s_1]}|f(t)|dm(t) < \epsilon$$

showing that $T(f)$ is continuous.

Problem 4a

By the chain rule and product rule we have $\frac{d^n}{dx^n}e^{-\frac{x^2}{2}} = p_n(x)e^{-\frac{x^2}{2}}$, where $p_n(x)$ is a polynomial. As all polynomials are small o of $e^{\frac{x^2}{2}}$ we have $x^k\frac{d^n}{dx^n}e^{-\frac{x^2}{2}} = x^k p_n(x)e^{-\frac{x^2}{2}} \to 0$ for $x \to \infty$ for all $k \geq 0$, so by remark 11.12(a): $g_k(x) = x^k e^{-\frac{x^2}{2}} \in \mathscr{S}(\mathbb{R})$ for all $k \geq 0$. We have by proposition 11.4 that $\mathcal{F}(g_0(x)) = \mathcal{F}(e^{-\frac{x^2}{2}}) = e^{-\frac{\xi^2}{2}} = g_0(\xi)$. By proposition 11.13(d) (using that $\mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$):

Maybe just write a formula

$$\mathcal{F}(g_1(x)) = \mathcal{F}(xe^{-\frac{x^2}{2}}) = i\frac{d}{d\xi}\mathcal{F}(e^{-\frac{x^2}{2}})(\xi) = i\frac{d}{d\xi}e^{-\frac{\xi^2}{2}} = -i\xi e^{-\frac{\xi^2}{2}}$$

$$\mathcal{F}(g_2(x)) = \mathcal{F}(x^2e^{-\frac{x^2}{2}}) = i^2\frac{d^2}{d\xi^2}\mathcal{F}(e^{-\frac{x^2}{2}})(\xi) = -\frac{d^2}{d\xi^2}e^{-\frac{\xi^2}{2}} = e^{-\frac{\xi^2}{2}} - \xi^2e^{-\frac{\xi^2}{2}}$$

$$\mathcal{F}(g_3(x)) = \mathcal{F}(x^3e^{-\frac{x}{2}}) = i^3\frac{d^3}{d\xi^3}\mathcal{F}(e^{-\frac{x^2}{2}})(\xi) = -i\frac{d^3}{d\xi^3}e^{-\frac{\xi^2}{2}} = i\xi^3e^{-\frac{\xi^2}{2}} - 3i\xi e^{-\frac{\xi^2}{2}}$$

Problem 4b

Clearly by setting $h_0 = g_0$, we have by what we showed in 4a that $\mathcal{F}(h_0) = \mathcal{F}(g_0) = g_0 = i^0h_0$. Clearly also for $h_3 = g_1$, we have:

$$\mathcal{F}(h_3(x)) = \mathcal{F}(g_1(x)) = -i\xi e^{-\frac{\xi^2}{2}} = -ig_1(\xi) = i^3h_3(\xi)$$

Setting $h_2 = 2g_2 - g_0$, then:

$$\mathcal{F}(h_2(x)) = \mathcal{F}(2g_2(x) - g_0(x)) = 2\mathcal{F}(x^2e^{-\frac{x^2}{2}}) - \mathcal{F}(e^{-\frac{x^2}{2}}) = 2e^{-\frac{\xi^2}{2}} - 2\xi^2e^{-\frac{\xi^2}{2}} - e^{-\frac{\xi^2}{2}}$$

$$= e^{-\frac{\xi^2}{2}} - 2\xi^2 e^{-\frac{\xi^2}{2}} = -\left(2\xi^2 e^{-\frac{\xi^2}{2}} - e^{-\frac{\xi^2}{2}}\right) = -(2g_2(\xi) - g_0(\xi)) = i^2 h_2(\xi)$$

Setting $h_1 = 2g_3 - 3g_1$, then:

$$\mathcal{F}(h_1(x)) = \mathcal{F}(2g_3(x) - 3g_1(x)) = 2\mathcal{F}(x^3 e^{-\frac{x^2}{2}}) - 3\mathcal{F}(xe^{-\frac{x^2}{2}}) = 2i\xi^3 e^{-\frac{\xi^2}{2}} - 6i\xi e^{-\frac{\xi^2}{2}} + 3i\xi e^{-\frac{\xi^2}{2}}$$

$$= 2i\xi^3 e^{-\frac{\xi^2}{2}} - 3i\xi e^{-\frac{\xi^2}{2}} = i\left(2\xi^3 e^{-\frac{\xi^2}{2}} - 3\xi e^{-\frac{\xi^2}{2}}\right) = i(2g_3(\xi) - 3g_1(\xi)) = i^1 h_1(\xi)$$

Problem 4c

We have:

$$\mathcal{F}(f(x)) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dm(x) = \int_{\mathbb{R}} f(-x) e^{ix\xi} dm(x) = \mathcal{F}^*(f(-x))$$

as $\mathcal{F}$ is invertible on $\mathscr{S}(\mathbb{R})$ with inverse $\mathcal{F}^*$, we have $\mathcal{F}^2(f) = f \circ -\mathrm{id}$, so that $\mathcal{F}^4(f) = (f \circ -\mathrm{id}) \circ -\mathrm{id} = f$.

Problem 4d

As $\mathcal{F}^4(f) = f$, if $\mathcal{F}(f) = \lambda f$ then we must have $\lambda^4 = 1$, therefore $\lambda \in \{1, i, -i, -1\}$. By 4b $\{1, i, -i, -1\}$ are all eigenvalues of $\mathcal{F}$ and hence are exactly the eigenvalues of $\mathcal{F}$.

Notation looks wrong

Problem 5

Letting N_μ and $N_{\delta_{2^{-n}x_n}}$ be the complement of their respective supports. As $2^{-n}\delta_{x_n} \leq \mu$ which means, for some open U, $\mu(U) = 0$ implies $2^{-n}\delta_{x_n}(U) = 0$, so:

$$N_\mu \subset N_{\delta_{2^{-n}x_n}}$$

So clearly by homework 8 problem 3c

$$\{x_n\} = \mathrm{supp}(\delta_{x_n}) = \mathrm{supp}(2^{-n}\delta_{x_n}) \subset \mathrm{supp}(\mu)$$

for all n. So $\cup_{n=1}^{\infty}\{x_n\} \subset \mathrm{supp}(\mu)$ and as by definition the support is closed:

$$[0,1] = \overline{\cup_{n=1}^{\infty}\{x_n\}} \subset \mathrm{supp}(\mu)$$

by density of $\{x_n\}_{n\in\mathbb{N}}$. Hence $\mathrm{supp}(\mu) = [0,1]$.