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a) Positivity of $\|\cdot\|_0$ follows from positivity of $\|\cdot\|_X, \|\cdot\|_Y$. We have $\|x\|_0 = 0 \implies \|x\|_X = 0 \implies x = 0$ since $\|\cdot\|_X$ is a norm. Since T is linear $Tx = 0 \implies \|Tx\|_Y = 0$ which implies $\|x\|_0 = 0$. So $\|x\|_0 = 0 \implies x = 0 \implies \|x\|_0 = 0$. This means $\|x\|_0 = 0 \iff x = 0$ and we can move on to the triangle inequality. By linearity of T, $\|x+x'\|_0 = \|x+x'\|_X+\|T(x+x')\|_Y \leq \|x\|_X+\|Tx\|_Y+\|x'\|_X+\|Tx'\|_Y = \|x\|_0+\|x'\|_0$, where we also use that $\|\cdot\|_X, \|\cdot\|_Y$ are norms. Linearity of T also means that $\|T(\alpha x)\|_Y = \|\alpha Tx\|_Y = |\alpha|\|Tx\|_Y$, so the last property follows as well since $\|\alpha x\|_0 = |\alpha| \cdot (\|x\|_X + \|Tx\|_Y) = |\alpha|\|x\|_0$.

Assume T is bounded. That means $\|Tx\|_Y \leq C\|x\|_X, \forall x \in X$. So $\|x\|_0 \leq (C+1)\|x\|_X$. $\|x\|_X \leq \|x\|_0$ is trivial, so they are equivalent. On the other hand if they are equivalent we have $\|x\|_0 \leq C\|x\|_X$ and subtracting $\|x\|_X$ yields $\|Tx\|_Y \leq (C-1)\|x\|_X$ so T is bounded.

b) We proved this in AdVec, and this is here my strategy comes from. Theorem 1.6 in the lecture notes states that any two norms on a finite dimensional vector space are equivalent, so we will prove that any T is bounded with a particular norm. It follows then it is bounded with any norm on X. Let $e_1, .., e_n$ be a basis on X and define a norm on X by $\|\sum_{i=1}^n a_i e_i\|_X = \max\{|a_i| : i = 1, .., n\}$. This is well defined since the coordinates of $x \in X$ wrt a basis exists and are unique. Positivity is immediate and $\|x\|_X = 0 \iff a_1, .., a_n = 0 \iff x = 0$. The triangle inequality holds since $\max\{|a_i + b_i|, i = 1, ..., n\} \leq \max\{|a_i| + |b_i|, i = 1, ..., n\} \leq \max\{|a_i|, i = 1, ..., n\} + \max\{|b_i|, i = 1, ..., n\}$ and $|\alpha a_i| = |\alpha| \cdot |a_i|$ proves the last property required to be a norm.
For any $x = \sum_{i=1}^n a_i e_i$ with $\|x\|_X \leq 1$ we have $\|Tx\|_Y \leq \sum_{i=1}^n |a_i|\|Te_i\|_Y \leq \sum_{i=1}^n \|Te_i\|_Y < \infty$, where we use $|a_i| \leq \|x\|_X \leq 1$ to get the second inequality. So T is bounded with this and therefore any norm on X.

c) Let $(e_i)_{i\in I}$ be a basis for X. We can define another basis $\hat{e_i} = e_i/\|e_i\|_X$ and may therefore assume that every e_i has norm one. Choose an injection $\varphi : \mathbb{N} \to I$ and let $(y_i)_{i\in I}$ be any family in Y so $\|y_{\varphi(n)}\|_Y = n$. For example we could let $y_{\varphi(n)} = n \cdot y_{\varphi(1)}$ and $y_{\varphi(1)}$ be some unit vector. Then there exits a linear map T so $Te_i = y_i$ (by the comments below the exercise). This map is unbounded since $\|e_{\varphi(n)}\|_X = 1$ but $\|Te_{\varphi(n)}\|_Y = \|y_{\varphi(n)}\|_Y = n$ can be arbitrarily large. Since it is not bounded it is not continuous.

d) Choose $T : X \to Y$ linear but not continuous, this is possible by c). Then $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ is not equivalent to $\|x\|_X$ by a) and clearly satisfies $\|\cdot\|_0 \geq \|\cdot\|_X$. By problem 1 in Homework 3, if X is complete wrt both norms they are equivalent. So if $(X, \|\cdot\|_X)$ is a Banach space, X is complete wrt $\|\cdot\|_X$ and since they are not equivalent, X cannot be complete wrt $\|\cdot\|_0$.

e) $X = \ell_1(\mathbb{N})$ is complete with the $\|\cdot\|_1$ norm which is stronger than the $\|\cdot\|_\infty$ norm. Clearly if $x = 0$ we have $\|x\|_1 \geq \|x\|_\infty = 0$. Since $x \in \ell_1$ it must have limit zero, so after a finite number of elements all the subsequent elements in x have smaller norm than the first non-zero element. Therefore $\|x\|_\infty$ is really a max over a finite set of real numbers, which is bounded by their sum which is bounded by $\|x\|_1$ as there are more terms in this sum.

To see X is not complete with this norm consider the sequence of sequences $x_i(n) = 1_{\{1,...,i\}}(n)/n$ that converges to $1/n$ in $\ell_\infty(\mathbb{N})$. But since the limit is not in X, here it is a Cauchy sequence without a limit proving X is not complete with the ∞-norm.

2

a) Notice $\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p} \leq 1 \iff |x_1|^p + |x_2|^p \leq 1$ so we must have $|x_1|, |x_2| \leq 1$ so clearly $|f(x)| \leq |x_1| + |x_2| \leq 2$. So f is bounded. Notice there is no reason to consider other values of x_1, x_2 than those that are real and positive. If $x = (x_1, x_2, 0, ...)$ has norm less than 1 then so too $x' = (|x_1|, |x_2|, 0, ...)$. And $|f(x)| \leq |x_1| + |x_2| = f(x')$.

This reduces the problem to finding $x_1, x_2 \geq 0$ that maximizes $x_1 + x_2$ subject to the restriction $x_1^p + x_2^p \leq 1$. Notice that $f(x)$ increases as x_1, x_2 increases so if x has norm strictly less than 1 we can find x' with $f(x) < f(x')$ by increasing x_1 a little, but not so much than $\|x\|_p \leq 1$ is no longer true. Therefore the problem reduces further to finding x_1, x_2 maximizing $x_1 + x_2$ subject to the restriction $|x_1|^p + |x_2|^p = 1$. Now notice if $x_1 > x_2$ then $px_1^{p-1} \geq px_2^{p-1}$, since y^{p-1} is an increasing function since $p - 1 \geq 0$. So the derivative of $\|x\|_p^p$ with respect to x_1 is bigger than that with respect to x_2. So if we decrease x_1 a little we can increase x_2 slightly more, getting a bigger value of f while still satisfying the boundary condition. The opposite is true when $x_2 > x_1$, so therefore choosing $x_1 = x_2 > 0$ such that $x_1^p + x_2^p = 1$ is optimal. This means that $x_1 = \sqrt[p]{1/2}$ and so $\|f\| = f(x) = 2\sqrt[p]{1/2} = \sqrt[p]{2^{p-1}} = 2^{(p-1)/p}$.

b) First note that $F(x) = x_1 + x_2$ is an extension of f with the same operator norm. There is no point in considering $x \notin M$ since the terms past x_2 contribute to $\|x\|_p$ without contributing to $|F(x)|$. When considering $x \in M$ all the above apply and we conclude $\|f\| = \|F\|$.

I will prove uniqueness by showing any other extension F' of f will have strictly greater norm. Since $F \neq F'$ they must disagree on some x' and since they agree on M, $x \notin M$. Since they agree on M, they must also disagree on $x = x' - (x_1', x_2', 0, ...) \in \ker F$ so $F'(x) \neq 0$ and scaling and then multiplication with a unit we may assume that $\|x\|_p = 1$ and $F'(x) = y > 0$. The idea now is to find a vector in $\text{Span}\{1_{\{1\}}, 1_{\{2\}}, x\}$ proving that $\|F'\| > \|f\|$. We already know that $x_1 = x_2 = \sqrt[p]{1/2}$ satisfies the boundary condition and $F(x_1, x_2, 0...) = F(y) = 2^{(p-1)/p}$. For very small ε, if we decrease them both by ε then $\|y\|_p^p$ decreases approximately $2\varepsilon p 2^{(1-p)/p} = \varepsilon 2^{1/p}$ (since $\Delta f(x) \approx f'(x)\Delta x$) while $F'(y)$ of course decreases by 2ε. So adding $(\sqrt[p]{\varepsilon 2^{1/p}}) \cdot x$ gives us a vector that still satifies the boundary condition. Clearly $F'((\sqrt[p]{\varepsilon 2^{1/p}}) \cdot x) = y\sqrt[p]{\varepsilon 2^{1/p}}$ which goes to zero slower than 2ε as $\varepsilon \to 0$, since the same is true for $\sqrt[p]{\varepsilon}$ and ε. So for any given y the contribution from adding a small multiple of x is eventually bigger than the 2ε we loose for making room for it. Therefore we have produced a vector so $|F'(y)| > \|f\|$ proving F is unique. In the preceding argument it is absolutely critical that $p > 1$ since otherwise $\sqrt[p]{\varepsilon}$ only goes to zero as fast as ε.

c) Consider $F'(x) = \sum_{i=1}^{3} x_i$, which is clearly an extension of f. We have $\|f\| = 2^{(1-1)/1} = 1$, and for any x with norm 1 $|F(x)| \leq |x_1| + |x_2| + |x_3| \leq \|x\|_1 \leq 1$ which means $\|f\| = \|F'\|$ since $\|f\| \leq \|F'\|$ is trivial since it is an extension. Notice this argument also works for $F'(x) = \sum_{i=1}^{N} x_i$ and since there are infinitely many natural numbers there are infinitely many generalizations. We could also multiply x_3 by $\alpha \leq 1$ instead of or in addition to adding more terms.

3

a) Let Y be an $n+1$ dimensional subspace of X with basis y_i. In AdVec (lemma 2.7) we learned that a linear map $A : Y \to \mathbb{K}^n$ being injective is equivalent with $A(y_i) \neq A(y_j), i \neq j$ and linear independence of $A(\{y_1, ..., y_{n+1}\})$ which in particular means the existence of n+1 linearly independent vectors in $\mathbb{K}^n$. This is impossible and therefore there cannot exist a linear, injective map $F : Y \to \mathbb{K}^n$. A linear, injective map from X to $\mathbb{K}^n$ would restrict to a linear, injective map from Y to $\mathbb{K}^n$ so such a map can also not exist.

b) Define $F : X \to \mathbb{K}^n$ by $F(x) = (f_1(x), ..., f_n(x))$. By a) it is not injective so it has a nontrivial kernel. Since $\ker F$ is precisely $\cap_{i=1}^n \ker f_i$ this intersection is not zero.

c) Note that if some x_i is zero, then it is trivially true for any $y \in X$ that $\|y - x_i\| \geq \|x_i\| = 0$ since norms are positive. Therefore we can safely ignore those x's that are zero, and we will now assume none of them are zero. By theorem 2.7 b there is $f_i \in X^*$ with norm 1 and $f_i(x_i) = \|x\|$. By b) we know that $\cap_{i=1}^n \ker f_i$ is not zero, and from AdVec we know the kernel of a linear function (F as defined above) is a subspace. Therefore we can take $0 \neq y \in \cap_{i=1}^n \ker f_i$ and scale it to have norm 1. For every x_j we have

$$\|x_j\| = f_j(x_j) = f_j(x_j - y) \leq \|f\|\|x_j - y\| = \|y - x_j\|$$

where the second equality is $y \in \ker f_j$ and the last one is using the operator norm of f is 1.

d) Suppose we have n closed balls that cover S and let $x_1, ..., x_n$ denote their centres. By c) we can find y with norm 1 so $\|y - x_j\| \geq \|x_j\|$ for all j. Since it has norm 1, $y \in S$ so one of balls must contain y. But $y \in \overline{B(x_j, r_j)} \implies \|y - x_j\| \leq r_j$ and since $\|x_j\| \leq \|y - x_j\|$ we must have $0 \in \overline{B(x_j, r_j)}$.
So whenever we have a finite family of closed balls covering S, one of them will contain 0. Therefore covering the unit sphere with a finite family of closed ball without one of them containing 0 cannot be done.

e) Consider the open covering consisting of the sets $U_x = B(x, 1/2), x \in S$. Since $0 \notin \overline{B(x, 1/2)}, x \in S$, for any finite subset of the open covering $U_1, ..., U_n$ we have $S \not\subseteq \cup_{i=1}^n \overline{B(x_i, 1/2)}$ by d) since it would constitute a finite covering of S by closed balls, none of which contain 0. And since $B(x, 1/2) \subseteq \overline{B(x, 1/2)}$ it follows $S \not\subseteq \cup_{i=1}^n B(x_i, 1/2)$. So since this particular open covering cannot be "thinned" to a finite open covering, S is not compact. If we add the open unit ball to this open covering, we get an open covering of the closed unit ball. Since $B(0, 1) \cap S = \emptyset$ this added set does not help us cover S. So since we could not thin the previous open covering to a finite open covering of S, we cannot thin this open covering to a finite covering of the closed unit ball. Therefore it is not compact.

4

Let $I = [0,1], X = L_1(I,\lambda), Y = L_3(I,\lambda)$ where λ is the Lebesgue measure.

a) Absorbing implies that for every $f \in X$ and some $t > 0$ we have $tf \in E_n$. This is impossible since for any $t > 0, f \in X \setminus Y$ we have $\int_I |tf|^3 d\lambda = t^3 \int_I |f|^3 d\lambda = \infty$. Note that $E_n \subseteq Y \subsetneq X$, so such an f definitely exists.

b) Let $f(x) = \frac{2}{3}x^{-1/3}$. It is integrable over I with integral 1 (its anti-derivative is $x^{2/3}$). Let $g \in E_n$ and notice that $g + \varepsilon f \in B(g, 2\varepsilon) : \|g - g - \varepsilon f\|_X = \varepsilon\|f\|_X = \varepsilon$. So if $g + \varepsilon f \notin E_n$ for any ε we see that arbitrarily close to any $g \in E_n$ there is a function not in E_n and then we are done, since any g cannot be an interior point. Using the reverse triangle inequality we see $|\|\varepsilon f\|_Y - \|g\|_Y| \leq \|\varepsilon f + g\|_Y$. But $\int_I |\varepsilon f|^3 d\lambda = \varepsilon^3(2/3)^3 \int_I x^{-1} d\lambda(x) = \infty$, so $|\|\varepsilon f\|_Y - \|g\|_Y| = \infty$ and the same holds for $\|\varepsilon f + g\|_Y$. So since $\|\varepsilon f + g\|_Y^3 = \int_I |\varepsilon f + g|^3 d\lambda = \infty > n, \varepsilon f + g \notin E_n$.

c) After trying to prove E_n is closed for more than a day and in my desperation typing up an argument involving a proof of a weaker version of Egorov's theorem, I find it is much easier proving that $X \setminus E_n$ is open. Such is life.
Take $f \notin E_n, g \in E_n$. I will bound $\|f - g\|_1$ in a way that does not depend on g. Assume there exists $\tilde{g} \in E_n : |f - \tilde{g}| \leq \min\{1, |f - g|\}$ almost everywhere. This is justified since it makes the problem harder. Since $|f - \tilde{g}| \leq 1, |f - \tilde{g}|^3 \leq |f - \tilde{g}|$ and we have

$$\int_I |f - g| d\lambda \geq \int_I |f - \tilde{g}| d\lambda \geq \int_I |f - \tilde{g}|^3 d\lambda = \|f - \tilde{g}\|_3^3 \geq |\|f\|_3 - \|\tilde{g}\|_3|^3,$$

where we use the reverse triangle inequality in the last step. Since $\tilde{g} \in E_n$ and $f \notin E_n, \|f\|_3 > \sqrt[3]{n} \geq \|\tilde{g}\|_3$. So $|\|f\|_3 - \|\tilde{g}\|_3| \geq \|f\|_3 - \sqrt[3]{n}$ and we can use $(\|f\|_3 - \sqrt[3]{n})^3 > 0$ as a lower bound on $\|f - g\|_1$ for any$g \in E_n$. So in a ball around any $f \in X \setminus E_n$ there are no elements from E_n and so E_n^c is open which means E_n is closed in X.

d) By c) and then b) $\mathrm{Int}(\overline{E_n}) = \mathrm{Int}(E_n) = \emptyset$. So $Y = \cup_{n=1}^{\infty} E_n$ is a countable union of nowhere dense sets. So Y is of the first category in X.

5

a) By the reverse triangle inequality $|\|x_n\|_X - \|x\|_X| \leq \|x_n - x\|_X \to 0$ by convergence in norm. But this means $\|x_n\|_X \to \|x\|_X$.

b) I will find a counterexample. Let X be $\ell_2(\mathbb{N})$ and $x_n = 1_{\{n\}}$. By HW4 weak convergence is equivalent with $f(x_n) \to f(x), \forall f \in X^*$, and I will use this to show weak convergence to 0. Let $f \in X^*$ by RF representation theorem $f(x) = \langle x, y \rangle$ for some $y \in X$. But note $\langle x_n, y \rangle = \overline{y(n)}$ (the conjugate of the n'th term in the sequence y). Since $y \in \ell_2(\mathbb{N})$ we must have $y(n) \to 0$. So for every $f \in X^*$ we have

$f(x_n) \to f(0)$, proving weak convergence to 0. But clearly we do not have convergence in norm, every x_n has norm one, and 0 has norm 0.

c) Given x (and Choice!) we can define an orthonormal basis $(e_i)_{i\in I}$ for X where $x = \alpha e_1$. Then consider the functional $f(y) = \langle y, e_1\rangle$ which is bounded since it is given by an inner product. By thm 5.27 in Folland we have $1 \geq \|x_n\|_X = \sqrt{\sum_{i\in I} |\langle x_n, e_i\rangle|^2} \geq |\langle x_n, e_1\rangle|$. And since $f(x - x_n) \to 0$ we have $|\alpha - \langle x_n, e_1\rangle| \to 0$ so by the reverse triangle inequality $||\alpha| - |\langle x_n, e_1\rangle|| \to 0$. So if $\|x\|_X = |\alpha| > 1$ we could not have convergence since $|\alpha| - |\langle x_n, e_1\rangle| \geq |\alpha| - \|x_n\|_X = |\alpha| - 1 > 0$ would not go to zero. I follows x must have norm less than 1.