FuncAn assignment 2

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a

We will start of by showing that $||f_N|| = 1$ for all $N \geq 1$. Since $(e_n)_{n\geq 1}$ is a orthonormal basis we have that $\langle e_j, e_k \rangle$ for $j \neq k$, so we get that:

$$||f_N||^2 = \langle N^{-1}\sum_{j=1}^{N^2} e_j, N^{-1}\sum_{k=1}^{N^2} e_k\rangle = N^{-2}\sum_{j,k=1}^{N^2}\langle e_j, e_k\rangle = N^{-2}\sum_{k=1}^{N^2}\langle e_k, e_k\rangle = \frac{N^2}{N^2} = 1$$

So we have that $||f_N||^2 = 1 = ||f_N||$.

By Folland page 169 we have that $f_N \to 0$ weakly iff $F(f_N) \to F(0)$, $\forall F \in H^*$. Theorem 5.25 from Folland states that there exists a unique $y \in H$ s.t $F(f_N) = \langle f_N, y\rangle$ (This also gives us that $F(0) = 0$). Since H has an ONB we can write $y = \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i$ and since $||y|| < \infty$ we get that there for any ϵ exists a K s.t $||\sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i|| < \epsilon$ hence:

$$|F(f_N)| = |\langle f_N, y\rangle| = |\langle f_N, \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i\rangle| = |\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle + \langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle|$$

By the triangle inequality we then get:

$$|\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle + \langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle| \leq |\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle| + |\langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle|$$

As H is a Hilbert space we can use the Schwartz Inequality on the expression on the right side of the addition sign:

$$|\langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle| \leq ||f_N|| \cdot ||\sum_{i=K+1}^{\infty}\langle y, e_i\rangle|| \leq 1 \cdot \epsilon$$

For the left side we get:

$$|\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i| = N^{-1}|\sum_{n=1}^{N^2}\langle e_n, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle| = N^{-1}|\sum_{n=1}^{N^2}\overline{\sum_{i=1}^{K}\langle y, e_i\rangle\langle e_n, e_i\rangle}| \leq N^{-1}|\sum_{i=1}^{K}\overline{\langle y, e_i\rangle}| < \epsilon$$

for $N \to \infty$. Thus we have shown that $\forall F \in H^*$ that $F(f_N) \to 0 = F(0)$ for $N \to \infty$ and hence that $f_N \to 0$ weakly.

b

Be explicit in saying H is dual of H^* by reflexivity when talking about w*-top.

Since H is a Hilbert space we have that it's reflexive, and thus that the weak topology is equal to the weak star topology, so $\overline{\text{co}(K)} = \overline{\text{co}(K)}^{w} = \overline{\text{co}(K)}^{w^*}$ and that $\overline{\text{co}(K)} \subset \overline{B(0,1)}$, and this also holds in the weak star topology. By Alaoglu's theorem we have that the closed unit ball is compact in the weak star topology, and since $\overline{\text{co}(K)}$ is a closed subset of this ball it's also compact w.r.t the weak star topology, and therefore also in the weak topology.

It's clear that the sequence $(f_N)_{N\geq 1}$ is in K since each $f_N \in K$. From a) it's known that the sequence converges weakly to 0 and therefore $0 \in \overline{\text{co}\{f_N | N \geq 1\}}^{w}$ and therefore also in norm closure; K.

c

We will start of by showing that 0 is an extreme point. We start of by noting that over element in co$\{f_N | N \geq 1\}$ has an positive inner product with e_n, since $\langle f_N, e_n\rangle \geq 1$. Now let take a sequence $(x_n)_{n\geq 1}$ in co$\{f_N | N \geq 1\}$ that converges to x, and let $\gamma_n(x) = \langle x, e_n\rangle$ for $\gamma_n \in H^*$. Since $x_n \to x$ we have that there are continuous functions s.t $\langle, x_n, e_n\rangle \to \langle x, e_n\rangle, \forall n$, and since all $\langle x_n, e_n\rangle \geq$ we have that $\langle x, e_n\rangle \geq 0$. So we have shown that all elements in $K = $ co$\{f_N | N \geq 1\}$ have positive inner product with e_n.

Let us write $0 = \alpha x + (1-\alpha)y$, and this also means that $0 = \alpha\langle x, e_n\rangle + (1-\alpha)\langle y, e_n\rangle$, $\forall n \geq 0$. We know that 0 is a extreme point of the positive real line and hence for each n we must have that $\langle x, e_n\rangle = \langle y, e_n\rangle = 0$ and then by theorem 5.27 a) from Folland we have that $x = y = 0$ thus 0 is an extreme point of K. ✓

Now we will show that each f_N is an extreme point of K. Let $f_N = \alpha x + (1-\alpha)y$ be a convex combination in K, s.t x is a limit point of $(x_n)_{n\geq 1}$ and y is a limit point of $(y_n)_{n\geq 1}$ where both sequences is in co$\{f_N | N \geq 1\}$. This means that $\alpha(x_n)_{n\geq 1} + (1-\alpha)(y_n)_{n\geq 1} \to f_N$. Define $g_{N^2}(x) = \langle x, e_{N^2}\rangle$(This fct is clearly cts) and apply it to the proceeding formula:

Write out the words

$$g_{N^2}(\alpha(x_n) + (1-\alpha)(y_n)) = \alpha g_{N^2}(x_n) + (1-\alpha)g_{N^2}(y_n) \to g_{N^2}(f_N) = \frac{1}{N}$$

Next it will be shown that $g_{N^2}(x_n) \leq \frac{1}{N}$. We start of by noting that if $j < N \Rightarrow g_{N^2}(f_j) = 0$ and iff $j \geq N \Rightarrow g_{N^2}(f_j) = \frac{1}{j} \leq \frac{1}{N}$. We will also write the elements $x_n \in K$ as their convex combination: $x_n = \sum_{k=1}^{\infty} \alpha_{n_k} f(k)$, here the sums of the α_{n_k} is one and thus there is only a finite part the elements of the sum that are different from zero, so we can write $x_n = \sum_{k=1}^{M_n} \alpha_{n_k} f_k$. We will now calculate $g_{N^2}(x_n)$:

$$g_{N^2}x_n = \sum_{k=1}^{M_n} \alpha_{n_k} g_{N^2}(f_k) \leq \sum_{k=1}^{M_n} \alpha_{n_k}\frac{1}{N} = \frac{1}{N}$$

It's also clear that the same argument holds for (y_n). This means that only way that it's possible that $\alpha g_{N^2}(x_n) + (1-\alpha)g_{N^2}(y_n) \to \frac{1}{N}$ is if both $g_{N^2}(x_n) \to \frac{1}{N}$ and $g_{N^2}(y_n) \to \frac{1}{N}$. It is known that $(x_n)_{n\geq 1}$ converges to a f_j if the sequence (β_{n_j}) of the j'th coefficent of the sequence $(x_n)_{n\geq 1}$ converges to 1. So we will show that this sequence converges to 1. Assume for contradiction that β_{n_j} doesn't converge to 1, i.e there exists an $\epsilon > 0$ s.t for every M there exists an $n > M$ where $|1 - \beta_{n_j}| > \epsilon$. Furthermore as each $\beta_{n_j} \leq 1$ we have that $r_n = 1 - \beta_{n_j} > \epsilon$. This gives us the following: From where?

$$\begin{aligned}
|\frac{1}{N} - g_{N^2}(\alpha(x_n) + (1+\alpha)(y_n))| &= \frac{1}{N} - (\alpha g_{N^2}(x_n) + (1-\alpha)g_{N^2}(y_n)) \\
&\leq \frac{1}{N} - (\alpha g_{N^2}(x_n) + (1-\alpha)\frac{1}{N}) \\
&\leq \alpha\frac{1}{N} - (\alpha \sum_{k=1}^{M_n} \beta_{n_k} g_{N^2}(f_k)) \\
&= \alpha\frac{1}{N}(1 - \beta_{n_j}) - (\alpha \sum_{k=1, k\neq N}^{M_n} \beta_{n_k} a g_{N^2}(f_k))
\end{aligned}$$

These inequalities are not compatible.

And this inequality is $\geq$ as $g_{N^2}(y_n) \leq 1/N \Rightarrow -g_{N^2}(y_n) \geq -1/N$.

And this is =

We now use that $\sum_{i=1, i\neq N}^{M_n} \beta_{n_k} = 1 - \beta_{n_N} = r_n$ and for $k \neq N$ we have that $g_{N^2}(f_k) \leq \frac{1}{N+1}$, so we have that:

$$= \alpha\frac{1}{N}(1 - \beta_{n_j}) - (\alpha \sum_{k=1, k\neq N}^{M_n} \beta_{n_k} a g_{N^2}(f_k)) \leq \alpha(\frac{r_n}{N} - \frac{r_n}{N+1}) \leq \epsilon \cdot \alpha(\frac{1}{N} - \frac{1}{N+1})$$

Be explicit.

This contradicts that $g_{N^2}(x_N) \to \frac{1}{N}$ hence we can now conclude that β_{n_j} converges to 1 and thus $(x_n)_{n\geq 1} \to x = f_N$ and by the exact same argument we also get that $(y_n) \to x = f_N$.

We have now shown that for every convex combination in K s.t $f_N = \alpha x + (1-\alpha)y$ we have that $x = y = f_n$ and thus f_N is an extreme point in K. 0/0

d

We will start of by showing that $\overline{\{f_N\}}^w = \{f_N, N \geq 1\} \cup \{0\} = A$. Since 0 is a weak limit point of f_N we have that $A \subseteq \overline{\{f_N\}}^w$. We just have to show the other inclusion, i.e a sequence of $\{f_N\}$ only has 0 or f_N as weakly limit point. Assume that x is such a weak limit point, then we have $\forall g \in H^*$ that is $g(x)$ is the limit of some sequence in $\{f_N\}$. Now let $g_1(x) := \langle x, e_i\rangle$ for $g_1(x) \in H^*$. We now see that $g_1(\{f_N\}) = \{N^{-1} | \forall N \in \mathbb{N}\}$;

no, $g(x)$ will be limit of $g(x_n)$ for some $(x_n)_{n\in\mathbb{N}} \subseteq \{f_N | N \geq 1\}$.

the only accumulation points of this set are 0 and N^{-1}. If N^{-1} is an accumulation point we would have that, since $\{N^{-1}\}$ is discrete set that for any sequence $(f_{N_j})_{j\in\mathbb{N}} \in \{f_N\}$ s.t $g_1(f_{N_j}) = N_j^{-1} \to N^{-1}$ for $j \to \infty$ will converge weakly to f_N. if 0 is an accumulation point for this setup, we will have that N_j will go to infinity for $j \to \infty$, hence (f_{N_j}) has a subsequence s.t each $N_{jk} < N_{jl}$ for $k < l$. This subsequence is also a subsequence of (f_N) so it must converge weakly to 0, thus (f_{N_j}) must converge weakly to 0. So we have shown that each sequence in (f_n) have weak limit points in A.

Furtheremore we have that $K = \overline{\text{co}\{f_N\}}^{||\cdot||} = \overline{\text{co}\{f_N\}}^{w}$ and that H is a LCTVS with the weak topology (notes page 27), so by Milman we have that

$$\text{Ext}(K) \subset \overline{\{f_N\}}^{w} = A$$

So the only extreme point of K is contained in the set A which consist of points we have already have shown are extreme points, i.e these are the only extreme points of K.

2

a

T^t

We use the definition of weak convergence from Folland page 169 and theorem 7.13 from the notes that states that there exists a Banach space adjoint; T^*, and that each $T^*g(x_n) \in X^*$. We then have:

$$x_n \to x \text{ weakly} \Leftrightarrow f(x_n) \to f(x), \forall f \in X^* \Rightarrow T^*g(x_n) \to T^*g(x)$$
$$\Leftrightarrow g(Tx_n) \to g(Tx) \Leftrightarrow T(x_n) \to T(x) \text{ weakly}$$

b

with limit y

From problem 2 HW4 we have that $\sup\{||x_n||\} < \infty$, i.e $A = \{x_1, x_2, \ldots\}$ is bounded and since T is compact we have that $\overline{T(A)}$ is compact. Furtherer we know from Analysis 1 that if all subsequences of a sequence have a convergent subsequence then the original sequence is convergent. with limit y.

Now let (Ty_n) be a sub-seq of $(T(x_n - x)$. Since T is a compact operator we have that $\overline{T(\{y_1, y_2, \ldots\}}$ is compact hence there exists a converging subsequence $(Ty_{n_j}$ of $TY_n)$ s.t $Ty_{n_j} \to \lambda$. It will now be shown that $\lambda = 0$. From a) we have that $g(T(x_n)) \to g(T(x))$, $\forall g \in Y^*$ and hence $g(T(y_{n_j})) \to g(T(x - x)) = 0$ so $\lambda = 0$. Using norm convergence -> weak convergence.

Hence we have that Tx_{n_j} converges to Tx and therefore each subsequence of Tx_n has a convergent subsequence that converges to Tx and therefore the sequence itself must converge to Tx i.e $||Tx_n - Tx|| \to 0$.

c

Elaborate.

We will assume that $T \notin \mathcal{K}(H, Y)$, from the notes we then have that $T(\overline{B_H(0,1)})$ is not totally bounded, and therefore from prop 8.2 (4) we have that there exists at least one sequence $(y_n)_{n\geq 1}$ in $T(\overline{B_H(0,1)})$ which has no convergent sub-sequences. We can pick an $x_n \in \overline{(B_h(0,1)}$ s.t $Tx_n = y_n$ for some y_n from the sequence with no converging subsequence. We can see $(x_n)_{n\geq 1}$ as sequence in the closed unit ball of H. From theorem 6.3 we have that the closed unit ball of H is weakly compact, so $(x_n)_{x\geq 1}$ have a weakly convergent subsequence (x_{n_j}) and from 2.b) we have that $T(x_{n_j}$ is a strongly convergent sequence that is a subsequence of (y_n), and that is a contradiction, so $T \in \mathcal{K}(H, Y)$.

d

From a) we have that if a sequence $(x_n)_{n\geq 1}$ converges weakly to x then $Tx_n \to Tx$ weakly. From remark 5.3 we then get that a sequence in $\ell_1(\mathbb{N})$ converges weakly iff it converges in norm, so we have $||Tx_n - Tx|| \to 0$. From 2.c) (since $\ell_2(\mathbb{N})$ is a Hilbert space) we then get that all $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

e

Assume for contradiction that $T \in \mathcal{K}(X, Y)$ is surjective, then by the open mapping theorem (Folland 5.10) we have that it's an open map. This means that there exist a ball centered at 0 in Y that is contained in $T(B_X(0,1))$, i.e $rB_Y(0,1) \subset T(B_X(0,1))$. We now take the closure on both sides: $\overline{rB_Y(0,1)} \subset \overline{T(B_X(0,1))}$, and since T is compact the right side is compact, and the left side is compact since it is a closed subset of a compact space. But we know from the last assignment (problem 3.e) that the closed unit ball can't be compact in a infinite dimensional vector space. So we get a contradiction and hence no $T \in \mathcal{K}(X, Y)$ can be surjective.

How does this extend from $B_Y(0,1)$ to $rB_Y(0,1)$?

f

We start of by showing that M is self-adjoint:

What are f, g?

$$\langle Mf, g\rangle = \int_{[0,1]} Mf \cdot \bar{g} dm = \int_{[0,1]} t \cdot f \cdot \bar{g} dm = \int_{[0,1]} f \cdot \overline{t \cdot g} dm = \int_{[0,1]} f \cdot \bar{M} g dm = \langle f, Mg\rangle$$

Hence $M = M^*$

We will use theorem 10.1 in notes to state that M is not compact. We know from the notes that M has no eigenvalues (example 9.15, HW6 Problem 3.a). It would contradict theorem 10.1 if it was compact, since this theorem then states the M would have eigenvalues. (✓)

Which is applicable as $L^2([0,1], m)$ is separable and infinite-dimensional.

3

Check $s \to t$ limit.

a

We will use theorem 9.6 to prove this. We have that $[0,1]$ is a compact Hausdorff topological space and the Lebesgue measure on this set is a finite Borel measure. It's clear that $K \in C([0,1] \times [0,1])$ and that T is the associated operator, hence from theorem 9.6 it's closed ?

$T = T_{\tilde{k}}$ $\tilde{k}(s,t) = k(t,s)$.

b

We will show that $\langle Tf, g\rangle = \langle f, Tg\rangle$. By the definition of the inner product we have that:

$$\begin{aligned}\langle Tf, g\rangle = \int_{[0,1]} Tf\bar{g} dm &= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)dm(t)\bar{g}(s)dm(s) \\ &= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\bar{g}(s)dm(t)dm(s)\end{aligned}$$

only for $k \in L_2$.

We can use Tonelli-Fubini (we know from lecture 9, that the integral is finite), so we have:

So show that.

$$\begin{aligned}\int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\bar{g}(s)dm(t)dm(s) &= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\bar{g}(s)dm(s)dm(t) \\ &= \int_{[0,1]} f(t)\int_{[0,1]} K(s,t)\bar{g}(s)dm(s)dm(t) \\ &= \int_{[0,1]} f(t)\int_{[0,1]} K(s,t)\bar{g}(s)dm(s)dm(t) = \langle f, Tg\rangle\end{aligned}$$

Hence we have shown that T is self-adjoint

only for $k(s,t) = k(t,s)$

c

We know from MI that we can split the integral of a piecewise function up as follows:

$$\begin{aligned}(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t) &= \int_{[0,s]} (1-s)tft(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\ &= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)\end{aligned}$$

where $s \in [0,1]$ and $f \in H$.

To show that Tf is continuous we use Lemma 12.4 from Schilling, and note that the lemma also can be used for function into $\mathbb{C}$, when we set the function as $f(x) = \alpha(x) + i\beta(x)$ where $\alpha(x)$ and $\beta(x)$ are real valued functions. It's clear that the set $[0,1]$ is nondegenrate closed. Set $u(s,t) = K(s,t)f(t)$, we now check the conditions for the lemma. a) $t \to u(s,t) \in L_1([0,1], m)$ for every $s \in [0,1]$ as we were shwon in lecture 9. b) it is clear that $s \to u(s,t)$ is continuous for every fixed $t \in [0,1]$. c) $|u(s,t)| = |K(s,t)f(t)| \leq w(t) = |f(t)|$ for all $(s,t) \in [0,1] \times [0,1]$, where we know that $|f(t)| \in L_1$ from Problem 2b in HW2. We can now use the lemma to state that $\int u(s,t)dm = \int k(s,t)f(t)dm$ is continuous on $[0,1]$. ✓

We now calculate at $(Tf)(0)$ and $(Tf)(1)$:

$$(Tf)(0) = (1-0)\int_{[0,0]} tf(t)dm(t) + 0 \cdot \int_{[0,1]} (1-t)f(t)dm(t) = \int_{[0,0]} tf(t)dm(t) = 0$$

$$(Tf)(1) = (1-1)\int_{[0,1]} tf(t)dm(t) + 1 \cdot \int_{[1,1]} (1-tf(t)dm(t) = \int_{[1,1]} (1-t)f(t)dm(t) = 0$$

✓

4

a

We start of by showing that g_k is in the Schwartz of the real numbers by showing that $\lim_{||x||\to\infty} x^\beta \partial\alpha g(x) = 0$. This is clear since we know from analysis 0 that e^{-x} goes faster to zero for $x \to \infty$ and any polynomial goes to infinity. what about C^∞? Maybe add to compare speed of convergence to ∞ vs 0

We will now use Proppostion 11.13 d) from the notes to compute $\mathcal{F}(g_k)$; we will calculate $i^{|k|}(\partial^k \hat{f})(\xi)$ for $k \in \{0, 1, 2, 3\}$, and where $f = e^{-1x^2/2}$. Furthere we know from the prof of Prop 11.4 in the notes that $\hat{f}(\xi) = e^{-\xi^2/2}$. We can now calculate:

$$\begin{aligned}
i^0(\partial^0 \hat{f})(\xi) &= \hat{f}(\xi) = e^{-\xi^2/2} \\
i^1(\partial^1 \hat{f})(\xi) &= i(\partial e^{-x^2/2})(\xi) = i\xi e^{-\xi^2/2} \\
i^2(\partial^2 \hat{f})(\xi) &= -1 \cdot (\partial^2 e^{-x^2/2})(\xi) = -1(\xi^2 - 1)e^{-\xi^2/2} = (1 - \xi^2)e^{-\xi^2/2} \\
i^3(\partial^3 \hat{f})(\xi) &= -i \cdot (\partial^3 e^{-x^2/2})(\xi) = -i\xi(\xi^2 - 3)e^{-\xi^2/2} = i(3\xi e^{-\xi^2/2} - \xi^3 e^{-\xi^2/2})
\end{aligned}$$

b

For $k = 0$ we set $h_0 = g_0$ and see that we get $\mathcal{F}(h_0) = \mathcal{F}(g_0) = \mathcal{F}(e^{-x^2/2}) = e^{-\xi^2/2} = i^0 h_0$. For $k = 3$ we set $h_3 = g_1$ and get that $\mathcal{F}(h_3) = \mathcal{F}(g_1) = \mathcal{F}(xe^{-x^2/2}) = -ie^{-\xi^2/2} = i^3 h_0$. For $k = 1$ we choose the following combination: $h_1 = g_3 - \frac{3}{2}g_1$ and see that we get:

$$\begin{aligned}
\mathcal{F}(h_1) = \mathcal{F}(g_3 - \frac{3}{2}) &= \mathcal{F}(x^3 e^{-x^2/2} - \frac{3}{2}xe^{-x^2/2}) \\
&= i3\xi e^{-\xi^2/2} - i\xi^3 e^{-\xi^2/2} - \frac{3}{2}i\xi e^{-\xi^2/2} = i(\frac{3}{2}\xi e^{-\xi^2/2} - \xi^3 e^{-\xi^2/2}) = i^1 h_i
\end{aligned}$$

For $k = 2$ we set $h_2 = g_2 - \frac{1}{2}g_0$ and see that we get:

$$\begin{aligned}
\mathcal{F}(h_2) = \mathcal{F}(g_2 - \frac{1}{2}g_0) &= \mathcal{F}(x^2 e^{-x^2/2} - \frac{1}{2}e^{-x^2/2}) \\
&= (1 - \xi^2)e^{-\xi^2/2} - \frac{1}{2}e^{-\xi^2/2} = -1(\xi^2 e^{-\xi^2/2} - \frac{1}{2}e^{\xi^2/2}) = i^2 h_2
\end{aligned}$$

c

We have that

$$\mathcal{F}^2(f(x)) = \mathcal{F}(\mathcal{F}(f)) = \mathcal{F}(\hat{f}(\xi)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{-ix\xi} dx$$

Since f is in the Schwartz space of $\mathbb{R}$ we get the following from Corollary 12.12 iii) and the definition of the inverse Fourier transformation:

$$f(x) = \mathcal{F}^*(\hat{f}(\xi)) = \int_{\hat{\mathbb{R}}} \hat{f}(\xi) e^{ix\xi} dx$$

We now see from these to equations that $\mathcal{F}^2(f(x)) = f(-x)$, so we have that $\mathcal{F}^4(f(x)) = \mathcal{F}^2(\mathcal{F}^2(f(x))) = \mathcal{F}^2(f(-x)) = f(x)$ for all f in the Schwartz space of $\mathbb{R}$

d

We have from c) that $\mathcal{F}^4(f(x)) = f(x) = \lambda^4 f \Rightarrow \lambda^4 = 1$, since $\lambda \in \mathbb{C}$. Since the eigenvalues are given as $\mathcal{F}f = \lambda f$ and from the fundamental theorem of algebra we have that there are exactly 4 solutions to this equation: $\lambda = \{1, i, -1, -i\}$. This does not show that $\lambda \in \{1, i, -1, -i\}$ is an eigenvalue. This follows from (b).

5

Let U be the open subsets of $[0, 1]$ s.t $\mu(U) =$ and let N be the union of all those. From Problem 3 HW 8 we have that $\text{supp}(\mu) = N^c$. We now have to show that $N = \emptyset$, i.e all the sets U are the empty-set.

Assume for contradiction, that there is a $U \neq \emptyset$ with $\mu(U) = 0$. For each element of $(x_n)_{n\geq}$ we have that $x_n \notin U$, by the definition of our Radon measure μ. Since U is non empty and open we have that there exists an open ball of radius ϵ around an element $x \in U$ s.t $B(x, \epsilon) \in U$. But since $(x_n)_{x\geq 1}$ is a dense subset of $[0, 1]$ we have that this ball contains an element of $(x_n)_{n\geq 1}$, which is a contradiction, so $N = \emptyset$ and therefore $N^c = [0, 1]$ i,e $\text{supp}(\mu) = [0, 1]$.