1.

a) By HW 4. Problem 2, (a), a net $(x_\alpha)_{\alpha\in A}$ in X converges to x in the weak topology on X if and only if the net $(f(x_\alpha))_{\alpha\in A}$ converges to $f(x)$, for every $f\in X^*$.

In the case of the sequence $(f_N)_{N\geq1}$, we have $(f_N)_{N\geq1}$ converges weakly to 0 if and only if the sequence $(\langle f_N, v\rangle)_{N\geq1}$, converges to $\langle 0, v\rangle = 0$ for every $v\in H$, since every $f\in H^*$ can be represented by inner product $\langle \cdot, v\rangle$ for some $v\in H$.

For any $v\in H$, write v as the unique decomposition:

$$v=\sum_{i\geq1} a_i e_i.$$

It is clear that $\{a_i\}_{i\geq1}\in l_2(\mathbb{N})$, in fact, H is isomorphic to $l_2(\mathbb{N})$.

Now since $\langle f_N, v\rangle = N^{-1}(1\cdot a_1+\cdots+1\cdot a_{N^2})$, it suffices to show that the sequence $(N^{-1}(\sum_{n=1}^{N^2} a_n))_{N\geq1}$ converges to 0 for any $(a_1, \cdots)\in l_2(\mathbb{N})$.

Since $(N^{-1}(\sum_{i=1}^{N^2}|a_i|))_{N\geq1}$ converging to 0 implies $(N^{-1}(\sum_{i=1}^{N^2} a_i))_{N\geq1}$ converging to 0, we may assume $a_i=|a_i|\geq0$, for any $i\in\mathbb{N}$.

Since $N^{-1}(\sum_{i=1}^{N^2}|a_i|\cdot a)_{N\geq0}$ converging to 0 implies $(N^{-1}(\sum_{i=1}^{N^2} a_i))_{N\geq1}$ converging to 0, we may further assume.

$a_i = |a_i| \geq 0$, and $\Sigma_{i\geq 1} a_i^2 = 1$.

Now we have the inequality from the fact $x_1 x_2 \leq \frac{x_1^2 + x_2^2}{2}$,

$(\Sigma_{i=1}^{n} a_i)^2 \leq n (\Sigma_{i=1}^{n} a_i^2)$, (*)

for any $\varepsilon > 0$, we have $m > 0$, such that $\Sigma_{i\geq 1} a_{m+i}^2 < \frac{\varepsilon^2}{4}$,

then $\Sigma_{i=1}^{N^2} a_i \leq \Sigma_{i=1}^{m} a_i + \Sigma_{j=1}^{N^2-m} a_{m+j}$

$< \Sigma_{i=1}^{m} a_i + (N^2 \cdot \frac{\varepsilon^2}{4})^{\frac{1}{2}}$ by (*).

$= \Sigma_{i=1}^{m} a_i + N \cdot \frac{\varepsilon}{2}$. , for any N, $N^2 > m$.

Hence $N^{-1} \Sigma_{i=1}^{N^2} a_i < \frac{\Sigma_{i=1}^{m} a_i}{N} + \frac{\varepsilon}{2}$, for any N, $N^2 > m$.

We have $M > 0$ such that $\frac{\Sigma_{i=1}^{m} a_i}{N} < \frac{\varepsilon}{2}$ when $N > M$,

then $N^{-1} \Sigma_{i=1}^{N^2} a_i < \varepsilon$ when $N > M$, $N^2 > m$, therefore

$N^{-1} \Sigma_{i=1}^{N^2} a_i$ converges to 0.

From the ~~abov~~ above we proved $f_N \to 0$ weakly. as $N \to \infty$, and it is easy to verify:

$\| f_N \| = N^{-1} \langle \Sigma_{n=1}^{N^2} e_n, \Sigma_{n=1}^{N^2} e_n \rangle^{\frac{1}{2}}$

$= N^{-1} \cdot N = 1$, for all $N \geq 1$.

(b). $co\{f_N : N \geq 1\} = \{\sum_{i=1}^{n} a_i f_i : \sum_{i=1}^{n} a_i = 1, n \in \mathbb{N}\}$.

Since $\|\sum_{i=1}^{n} a_i f_i\| \leq \sum_{i=1}^{n} \|a_i f_i\| = \sum_{i=1}^{n} |a_i| \|f_i\| = 1$, $co\{f_N : N \geq 1\} \subseteq \overline{B_H(0,1)}$, and hence $\overline{co\{f_N : N \geq 1\}} \subseteq \overline{B_H(0,1)}$.

Let $F_v \in H^*$ denote the linear functional $\langle \cdot, v \rangle$, and consider the map $F : v \mapsto F(v)$, from lecture 2 we know that when F is regarded as a map $H \to H^*$ with norm topologies, it is an isometry, therefore $\overline{B_{H^*}(0,1)} = F(\overline{B_H(0,1)})$.

When F is regarded as a map : $(H, \mathcal{T}_w) \to (H^*, \mathcal{T}_{w^*})$, since $\mathcal{T}_{w^*}$ is the coarsest topology on H^* making all seminorms induced by linear functionals induced by H continuous. we can prove F continuous by verifying $|v \circ F|$ continous for every $v \in H \subseteq H^{**}$.

For any $v \in H$, $|v \circ F(v_0)| = |\langle v, v_0 \rangle| = |\langle v_0, v \rangle|$, since $v_0 \mapsto \langle v_0, v \rangle$ is continuous linear functional on $(H, \mathcal{T}_w)$, $|v \circ F(v_0)|$ is continuous, and ~~thef~~ therefore F is continuous.

Similarly we can prove that $F^{-1} : (H^*, \mathcal{T}_{w^*}) \to (H, \mathcal{T}_w)$ is continuous. therefore $(H^*, \mathcal{T}_{w^*})$ and $(H, \mathcal{T}_w)$ are homeomorphic.

Therefore, in order to prove K is compact in $(H, \mathcal{T}_w)$, it suffices to prove. $F(K)$ is compact in $\mathcal{T}_{w^*}$.

Since K is norm closure of $co\{f_N : N \geq 1\}$, which is a convex subset of H, then $K = \overline{co\{f_N : N \geq 1\}}^{\mathcal{T}_w}$, by theorem 5.7, hence K is $\mathcal{T}_w$ - closed, and $F(K)$ is $\mathcal{T}_{w^*}$ closed.

Remember that $K \subseteq \overline{B_H(0,1)}$, $F(K) \subseteq F(\overline{B_H(0,1)}) = \overline{B_{H^*}(0,1)}$, hence $F(K)$ is a τ_{w^*}-closed subset of $\overline{B_{H^*}(0,1)}$.

Recall from Alaoglu's theorem, $\overline{B_{H^*}(0,1)}$ is compact in w^*-topology, and the fact that a closed subset of a compact subset is compact. $F(K)$ is τ_{w^*}-compact.

From the above we proved K is weakly compact, to show that $0 \in K$, just regard K as the weak closure of $co\{f_N : N \geq 1\}$, cf theorem 5.7. and since $f_N \to 0$ weakly, 0 is clearly a limit point of $co\{f_N : N \geq 1\}$ in the weak topology, therefore $0 \in K$.

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(1). Note that since f_N has non-negative coefficient for all $N \geq 1$, and that norm convergence implies coefficient convergence with regard to the orthonormal basis $(e_n)_{n \geq 1}$.

Therefore, if $x_1, x_2 \in K$, $0 < \alpha < 1$, and $0 = \alpha x_1 + (1-\alpha)x_2$, let $x_1 = a_1^i e_1 + \cdots$, $x_2 = a_2^i e_1 + \cdots$, then $\alpha a_1^i + (1-\alpha) a_2^i = 0$, and since a_1^i and $a_2^i \geq 0$, $a_1^i = a_2^i = 0$, for any $i \geq 1$. Hence $x_1 = x_2 = 0$, so 0 is an extreme point in K.

To prove that for any $m \geq 1$, f_m is an extreme point in K, we consider the coefficient of e_{m^2}, then for all $N \geq 1$, the coeffient of e_{m^2} in f_N is $\frac{1}{m}, \frac{1}{m+1}, \cdots$, and 0, therefore f_m has the largest coefficient of e_{m^2} and it is the only such element that have coefficient $\frac{1}{m}$.

Since $\mathrm{Co}\{f_N : N \geq 1\} = \{\sum_{i=1}^{n} \alpha_i f_i : \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N}\}$, we have f_m is the only element in $\mathrm{Co}\{f_N : N \geq 1\}$ such that it has the largest e_{m^2} coefficient $\frac{1}{m}$.

If $f \in K$ has e_{m^2} coefficient larger or equal to $\frac{1}{m}$, there is $(y_i)_{i \geq 1}$ in $\mathrm{Co}\{f_N : N \geq 1\}$ converging to f, therefore the e_{m^2} coefficient of $(y_i)_{i \geq 1}$ converges to $y \geq \frac{1}{m}$, but since e_{m^2} coefficients of f_N are discrete, it follows if we write $y_i = \sum_{i=1}^{n} \alpha_i f_i$, then α_m converges to 1, and then $(y_i)_{i \geq 1}$ converges to f_m, and hence f_m is the only element in K that have largest e_{m^2} coefficient $\frac{1}{m}$.

If $x_1, x_2 \in K$, $0<\alpha<1$, $f_m = \alpha x_1 + (1-\alpha) x_2$, then

$e_{m^2}(f_m) = \alpha e_{m^2}(x_1) + (1-\alpha) e_{m^2}(x_2)$, so

$\frac{1}{m} = \alpha e_{m^2}(x_1) + (1-\alpha) e_{m^2}(x_2)$, with

$e_{m^2}(x_1) \leq \frac{1}{m}$, $e_{m^2}(x_2) \leq \frac{1}{m}$, so

$e_{m^2}(x_1) = e_{m^2}(x_2) = \frac{1}{m}$, therefore

$x_1 = x_2 = f_m$, so for any $m \geq 1$, f_m is

an extreme point in K.

Therefore each f_N, $N \geq 1$, is an extreme point of K.

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d). $(H, \mathcal{T}_w)$ is a LCTVS, K is non-empty compact, convex because it is the $\mathcal{T}_w$-closure of a convex subset, and $K = \overline{co(\{f_N : N \geq 1\})}^{\mathcal{T}_w}$, so by theorem 7.9,

$Ext(K) \subset \overline{\{f_N : N \geq 1\}}^{\mathcal{T}_w}$.

Now since $f_N \to 0$ weakly (in $\mathcal{T}_w$), any infinite subset of $\{f_N : N \geq 1\}$ has the only limit point, 0, since any subsequence converges to 0, therefore

$\overline{\{f_N : N \geq 1\}}^{\mathcal{T}_w} = \{f_N : N \geq 1\} \cup \{0\}$.

Therefore there is no other extreme points in K.

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2、

a). From HW 4, problem 2, (a), we have $Tx_n \to Tx$ weakly if and only if $f(Tx_n) \to f(Tx)$ as $n \to \infty$ for any $f \in Y^*$.

Note that $f \circ T$ is a linear functional on X and it is bounded, with $\|f \circ T\| \le \|f\| \cdot \|T\|$, therefore $f \circ T \in X^*$.

As $x_n \to x$ weakly, we have $g(x_n) \to g(x)$ for any $g \in X^*$, hence $(f \circ T)(x_n) \to (f \circ T)(x)$ as $n \to \infty$, hence

$f(T(x_n)) \longrightarrow f(T(x))$ as $n \to \infty$, so

$Tx_n \to Tx$ weakly.

b). If $\|Tx_n - Tx\| \not\to 0$ as $n \to \infty$, there exists $\varepsilon > 0$ and a subsequence $(x_{n_k})_{k \ge 1}$ such that $\|Tx_{n_k} - Tx\| > \varepsilon$ for any $k \ge 1$.

From HW 4, problem 2. (b), $(x_n)_{n \ge 1}$ converges weakly imply that $(x_n)_{n \ge 1}$ is bounded, so $(x_{n_k})_{k \ge 1}$ is bounded, then from proposition 8.2, since T is compact, we have a subsequence of $(x_{n_k})_{k \ge 1}$, denoted by $(x_{k_l})_{l \ge 1}$, such that $T(x_{k_l}) \to y \in Y$, as $l \to \infty$. Obviously $y \ne Tx$. Also since $\|Tx_{k_l} - y\| \to 0$, as $l \to \infty$, we have $f(Tx_{k_l}) \to f(y)$ for any $f \in Y^*$ as $l \to \infty$. However, we also have $f(Tx_{k_l}) \to f(Tx)$, as proved in (a). By theorem 3.6 there exists $g \in Y^*$ such that $g(y) \ne g(Tx)$, which gives a contradiction since a sequence in $\mathbb{R}$ or $\mathbb{C}$ cannot converge to two different numbers. Hence $\|Tx_n - Tx\| \to 0$ as $n \to \infty$.

c). If T is not compact, $T(\overline{B_H(0,1)})$ is not totally bounded, that is, there is $\delta > 0$, such that there does not exist any finite cover of $T(\overline{B_H(0,1)})$ by open balls of radius δ.

Therefore if we have the first k elements of a sequence, $x_1, \cdots, x_k$, such that $x_1, \cdots, x_k \in \overline{B_H(0,1)}$, and $\|Tx_m - Tx_l\| \geq \delta$ for all $m \neq l$, $m, l \leq k$, since $\{B_Y(Tx_i, \delta), 1 \leq i \leq k\}$ does not cover $T(\overline{B_H(0,1)})$, we may find an element in $T(\overline{B_H(0,1)}) \setminus \bigcup_{1 \leq i \leq k} B_Y(Tx_i, \delta)$, put an inverse element of this element in $\overline{B_H(0,1)}$, as x_{k+1}. (It exists because T is from $\overline{B_H(0,1)}$ onto $T(\overline{B_H(0,1)})$.

So $\|Tx_{k+1} - Tx_i\| \geq \delta$ for all $1 \leq i \leq k$, thus $\|Tx_m - Tx_l\| \geq \delta$ for all $m \neq l$, $1 \leq m, l \leq k+1$.

Continuing this process we construct an infinite sequence $(x_n)_{n \geq 1}$ in $\overline{B_H(0,1)}$ such that $\|Tx_n - Tx_m\| \geq \delta$, for all $n \neq m$.

Recall in the proof of Problem 1 we proved that the map $F: v \mapsto \langle \cdot, v \rangle$, $(H, \mathcal{J}_w) \to (H^*, \mathcal{J}_{w^*})$ gives a homeomorphism from $\overline{B_H(0,1)}$ onto $\overline{B_{H^*}(0,1)}$, so since $\overline{B_{H^*}(0,1)}$ is compact in w^*-topology, c.f. Theorem 6.1, $\overline{B_H(0,1)}$ is weakly compact. From (c) Theorem 4.29, Folland, $\overline{B_H(0,1)}$ being compact means every net in $\overline{B_H(0,1)}$ has a convergent subnet.

In the case of $(x_n)_{n\geq 1}$, in weak topology of $\overline{B_H(0,1)}$, this means that $(x_n)_{n\geq 1}$ has a weakly convergent subsequence, denoted by $(x_{n_k})_{k\geq 1}$.

Then we see: since $(x_{n_k})_{k\geq 1}$ is a sequence in H converging to $x \in H$ weakly, but $\|Tx_{n_{k_1}} - Tx_{n_{k_2}}\| \geq \delta$, for all $k_1 \neq k_2$, $(Tx_{n_k})_{k\geq 1}$ is not Cauchy, therefore not convergent to Tx in the norm topology, which gives a ~~et~~ contradiction to the hypothesis.

Therefore T must be compact, $T \in K(H, Y)$.

d). Clearly $l_2(\mathbb{N})$ is seperable infinite dimensional Hilbert space with orthonormal basis $e_i = (0, 0, \cdots, 1, 0, \cdots)$ with the only non-zero coefficient 1 in the i-th position, and $l_1(\mathbb{N})$ is an infinite dimentional Banach space.

If $(x_n)_{n\geq 1}$ is a sequence in $l_2(\mathbb{N})$ coverging weakly to $x \in H$, since $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$, by (a) we have

$$Tx_n \to Tx \text{ weakly in } l_1(\mathbb{N}), \text{ as } n \to \infty,$$

then from Remark 5.3 in Lecture 5, a sequence converges weakly in $l_1(\mathbb{N})$ if and only if it converges in norm.

Therefore $Tx_n \to Tx$ in norm, as $n \to \infty$, that is,

$$\| Tx_n - Tx \| \to 0, \text{ as } n \to \infty.$$

From (c), we conclude that $T \in K(l_2(\mathbb{N}), l_1(\mathbb{N}))$, then each $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ is compact.

e). From HW 6, problem 4, (b), if $K(X,Y)$ contains an invertible operator, then X and Y are finite dimensional. Here X and Y are Banach spaces.

Now if $T \in K(X,Y)$, where X and Y are infinite dimensional Banach spaces, then it is trivial from linear algebra that

$X/\ker(T)$ is linearly isomorphic to Y.

For the vector space $X/\ker(T)$, we give the quotient norm defined as following:

$\|[x]\| = \inf\{\|x+y\|,\ y \in \ker T\}$, $[x]$ means the quotient class represented by x.

Clearly this is well defined on $X/\ker T$, since if $[x]=[x']$, then $[x]=[x'+y]$ for some $y \in \ker T$, then $\|[x]\| = \|[x']\|$.

Now we verify that this is a norm.

$$\|[x]+[y]\| = \|[x+y]\| = \inf\{\|x+y+z\|,\ z \in \ker T\}$$
$$\le \inf\{\|x+z_1\|,\ z_1 \in \ker T\} + \inf\{\|y+z_2\|,\ z_2 \in \ker T\}$$
since $\|x+y+z_1+z_2\| \le \|x+z_1\| + \|y+z_2\|$.

If $\alpha \neq 0$, then $\|\alpha[x]\| = \|[\alpha x]\| = \inf\{\|\alpha x+y\|,\ y \in \ker T\}$

(since when $\alpha \neq 0$, $y \in \ker T \Leftrightarrow \alpha y \in \ker T$) $= \inf\{\|\alpha x+\alpha y\|,\ y \in \ker T\}$
$$= |\alpha| \inf\{\|x+y\|,\ y \in \ker T\}$$
$$= |\alpha|\,\|[x]\|$$

If $\alpha = 0$, then clearly $\|\alpha[x]\| = 0 = |\alpha|\,\|[x]\|$.

$\|[x]\| = 0 \Leftrightarrow \inf\{\|x+y\|,\ y \in \ker T\} = 0$.

$\Leftrightarrow$ exists $y \in \ker T$ such that $x = -y$, since $\ker T$ is closed subspace of X,

$\Leftrightarrow x \in \ker T$.

$\Leftrightarrow [x] = [0]$,

So $\|[x]\| = 0$ if and only if $[x] = [0]$.

Hence $(X/\ker T, \|\cdot\|)$ is a normed space with the quotient norm.

To see that $X/\ker T$ is Banach space, we apply Theorem 1.7. Recall that $\ker T$ is closed, so $\{\|x+y\|,\ y \in \ker T\}$ attains its infimum. For an absolute convergent sequence $\{[x_n]\}_{n \geq 1}$, we set $\|x_i + y_i\| = \|[x_i]\|$ with $y_i \in \ker T$ for any $i \geq 1$. then $\{\|x_n + y_n\|\}_{n \geq 1}$ converges absolutely, then by Theorem 1.7 we have $\{x_n + y_n\}_{n \geq 1}$ converges in X, as X is complete. Let $\{x_n + y_n\}_{n \geq 1}$ converge to x in X, then $\|x_n + y_n - x\| \to 0$ as $n \to \infty$. therefore $\|[x_n] - [x]\| = \|[x_n - x]\| \to 0$, so $\{[x_n]\}_{n \geq 1}$ converges to $[x]$ in $X/\ker T$, therefore $X/\ker T$ is Banach space.

Denote by T' the isomorphism from $X/\ker T$ to Y induced by T. Note that $x \in B_X(0,1) \Rightarrow [x] \in B_{X/\ker T}(0,1)$, and $[x] \in B_{X/\ker T}(0,1) \Rightarrow x = x' + y$, where $x' \in B_X(0,1)$ and $y \in \ker T$. Therefore $T(B_X(0,1)) = T'(B_{X/\ker T}(0,1))$, and then

$\overline{T(B_X(0,1))} = \overline{T'(B_{X/\ker T}(0,1))}$, therefore

T' is compact, and from the result in the beginning of this proof, $X/\ker T$ and Y are both finite ~~dimensio~~ dimensional, contradicting to the fact that Y is infinite dimensional.

Therefore no $T \in K(X,Y)$ is onto.

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f). $\langle f, g\rangle = \int_{[0,1]} f\cdot\bar{g}\, dm$, for any $f, g \in H$,

then $\langle Mf, g\rangle = \int_{[0,1]} t f(t)\cdot \overline{g(t)}\, dm(t)$.

$= \int_{[0,1]} f(t)\cdot t\,\overline{g(t)}\, dm(t)$ then $t = \bar{t}$.

$= \int_{[0,1]} f(t)\cdot \overline{t\cdot g(t)}\, dm(t)$ since $t\in[0,1]$.

$= \langle f, Mg\rangle$, for any $f, g\in H$.

therefore ~~M~~ M is self-adjoint.

To see that M is not compact, consider the sequence: $(f_n)_{n\geq 1}$, defined as:

$$f_n = \begin{cases} \sqrt{n(n+1)}. & x\in\left(1-\frac{1}{n}, 1-\frac{1}{n+1}\right). \\ 0 & \text{otherwise}, \end{cases}$$

then clearly ~~$\langle f, f\rangle =$~~

$$\int_{[0,1]} f_n^2\, dm = 1. \quad \text{for any } n\geq 1,$$

therefore $\|f_n\| = 1^{\frac{1}{2}} = 1$, for any $n\geq 1$.

Note that for $n\geq 2$, $Mf_n(x) \geq \frac{1}{2} f_n(x)$, $0\leq x\leq 1$, therefore $|Mf_n(x)| \geq \frac{1}{2}|f_n(x)|$, hence $\|Mf_n\| \geq \frac{1}{2}$.

Also since the support of each pair of f_n intersect at only a point or does not intersect, so is each pair of Mf_n, therefore $\|Mf_n - Mf_m\|^2 = \|Mf_n\|^2 + \|Mf_m\|^2 \geq \frac{1}{2}$, whenever $m\neq n$, $m, n\geq 2$, therefore $\|Mf_n - Mf_m\| \geq \frac{1}{\sqrt{2}}$.

This implies $(Mf_n)_{n\geq 1}$ does not contain a converging ~~subseq~~ subsequence, because such a sequence is cauchy. but

$\|Mf_n - Mf_m\| \geq \frac{1}{\sqrt{2}}$ for any $m \neq n$, $m, n \geq 2$.

However, since $(f_n)_{n\geq 1}$ is bounded, from proposition 8.2, (4) we deduce that M can't be compact.

3、

a) If $0 \le s = t \le 1$, $(1-s)t = (1-t)s$. therefore

$$K(s,t) = \begin{cases} (1-s)t. & \text{if } 0 \le t \le s \le 1 \\ (1-t)s. & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Therefore if we write $[0,1] \times [0,1]$ as $A \cup B$, where $A = \{(s,t),\ 0 \le t \le s \le 1\}$, $B = \{(s,t),\ 0 \le s \le t \le 1\}$, then A and B are both closed subsets of $[0,1] \times [0,1]$. therefore since $K(s,t)$ is continuous on A and B, and agree on $A \cap B = \{(s,t),\ 0 \le s = t \le 1\}$, by the glueing lemma, we conclude that $K(s,t)$ is continuous on $[0,1] \times [0,1]$ to $\mathbb{R}$, and hence continuous from $[0,1] \times [0,1]$ to $\mathbb{C}$.

Now we verify a property of K: $K(s,t) = K(t,s)$, when $0 \le t \le 1$, $0 \le s \le 1$.

If $s \le t$, $K(s,t) = (1-t)s$, $K(t,s) = (1-t)s$,

then $K(s,t) = K(t,s)$.

If $t \le s$, similarly we verify that $K(s,t) = K(t,s)$.

Therefore we conclude that $K(s,t) = K(t,s)$, $0 \le t \le 1$, $0 \le s \le 1$.

We know that $[0,1]$ is compact Hausdorff topological space, and m is finite Borel measure on $[0,1]$, and $K \in C([0,1] \times [0,1])$, then from theorem 9.6, the associated operator:

$T_K : L_2([0,1], m) \to L_2([0,1], m)$:

$$(T_K f)(t) = \int_{[0,1]} K(s,t) f(s)\, dm(s), \quad f \in L_2([0,1], m).$$

is compact.

Since $K(s,t) = K(t,s)$, we have:

$$\int_{[0,1]} K(s,t) f(s)\, dm(s) = \int_{[0,1]} K(t,s) f(s)\, dm(s)$$
$$= (Tf)(t), \quad t \in [0,1], \ f \in H.$$

Therefore T and T_K are the same operator from H to H. since T_K is compact, T is compact.

b). Note that it suffices to prove $\langle Tf, g\rangle = \langle f, Tg\rangle$ for any $f, g \in H$.

We have:

$$\langle Tf, g\rangle = \int_{[0,1]} \int_{[0,1]} K(s,t) f(t)\, dm(t)\, \overline{g(s)}\, dm(s)$$
$$= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \overline{g(s)}\, dm(t)\, dm(s)$$
$$= \int_{[0,1]\times[0,1]} K(s,t) f(t) \bar{g}(s)\, dm(t)\, dm(s).$$

Since $f \in H$, $g \in H$, $\bar{g} \in H$, $\bar{g} \otimes f \in L_2([0,1]\times[0,1], m\otimes m)$ and $\|\bar{g}\otimes f\|_2 = \|\bar{g}\|_2 \|f\|_2 = \|g\|_2 \|f\|_2$ from lecture 9, and $|\bar{g}\otimes f| \in L_2([0,1]\times[0,1], m\otimes m)$, $\||\bar{g}\otimes f|\|_2 = \|\bar{g}\otimes f\|_2$.

Therefore

$$\int_{[0,1]\times[0,1]} |K(s,t)|\, |\bar{g}(s) f(t)|\, dm(t)\, dm(s)$$
$$= \int_{[0,1]\times[0,1]} K(s,t)\, |\bar{g}\otimes f(s,t)|\, dm(t)\, dm(s), \quad \text{since } K(s,t) \geq 0.$$
$$= \int_{[0,1]\times[0,1]} K \cdot |\bar{g}\otimes f|\, d\, m\otimes m = \langle K, |\bar{g}\otimes f|\rangle.$$

here the inner product is taken in $L_2([0,1]\times[0,1], m\otimes m)$, and the last equation from $\overline{|\bar{g}\otimes f|} = |\bar{g}\otimes f|$.

Since $\langle K, |\bar{g}\otimes f|\rangle \le \|K\|_2 \, \||\bar{g}\otimes f|\| < \infty$, by Fubini's theorem, we have

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]}\int_{[0,1]} K(s,t)\overline{g(s)} f(t)\,dm(s)\,dm(t) \\
&= \int_{[0,1]}\int_{[0,1]} K(s,t)\overline{g(s)}\,dm(s) f(t)\,dm(t) \\
&= \int_{[0,1]}\underline{\int_{[0,1]} \overline{K(s,t)g(s)}\,dm(s)} f(t)\,dm(t), \text{ since } K(s,t)\in\mathbb{R}. \\
&= \int_{[0,1]}\int_{[0,1]} K(t,s)g(s)\,dm(s)\, f(t)\,dm(t), \text{ since } K(s,t)=K(t,s) \\
&= \int_{[0,1]} \overline{T(g)(t)} f(t)\,dm(t) \\
&= \langle f, Tg\rangle, \text{ the product taken in } H.
\end{aligned}$$

Therefore, we have $\langle Tf, g\rangle = \langle f, Tg\rangle$ for any $f, g\in H$, hence $T = T^*$.

$c)$. When $t\in[0,s]$, that is, $0\le t\le s\le 1$, we have $k(s,t)=(1-s)t$, and when $t\in[s,1]$, $0\le s\le t\le 1$, we have $k(s,t)=s(1-t)$. Therefore we may decompose $(Tf)(s)$ as:

$$
\begin{aligned}
(Tf)(s) &= \int_{[0,1]} k(s,t)f(t)\,dm(t) \\
&= \int_{[0,s]} k(s,t)f(t)\,dm(t) + \int_{[s,1]} k(s,t)f(t)\,dm(t) \\
&= (1-s)\int_{[0,s]} t f(t)\,dm(t) + s\int_{[s,1]}(1-t)f(t)\,dm(t).
\end{aligned}
$$

For $h>0$, such that $s+h\in[0,1]$, then

$$
\begin{aligned}
(Tf)(s+h)-Tf(s) &= (1-s-h)\int_{[0,s+h]} t f(t)\,dm(t) + (s+h)\int_{[s+h,1]}(1-t)f(t)\,dm(t) \\
&\quad - (1-s)\int_{[0,s]} t f(t)\,dm(t) - s\int_{[s,1]}(1-t)f(t)\,dm(t) \\
&= -h\cdot\int_{[0,s]} t f(t)\,dm(t) + (1-s-h)\int_{[s,s+h]} t f(t)\,dm(t) \\
&\quad + h\int_{[s+h,1]}(1-t)f(t)\,dm(t) - s\int_{[s,s+h]}(1-t)f(t)\,dm(t)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|Tf(s+h)-Tf(s)| &\le h\cdot\Big(\int_{[0,s]}|t f(t)|\,dm(t) \\
&\quad + \int_{[s+h,1]}|(1-t)f(t)|\,dm(t)\Big) \\
&\quad + \int_{[s,s+h]}|t f(t)|\,dm(t) \\
&\quad + \int_{[s,s+h]}|(1-t)f(t)|\,dm(t), \text{ since } |s|\le 1,\ |1-s-h|\le 1. \\
&\le h\cdot\int_{[0,1]}|f(t)|\,dm(t) \\
&\quad + \int_{[s,s+h]}|f(t)|\,dm(t),
\end{aligned}
$$

since $t\cdot|f(t)|\ge 0$, $(1-t)\cdot|f(t)|\ge 0$,

and $t\cdot|f(t)|+(1-t)\cdot|f(t)| = |f(t)|$.

Note that $|f|\in H$, the map F_1 defined as $F_1(t)=1$, $0\le t\le 1$, is also in H, and $\|F_1\|_2=1$, then

$\int_{[0,1]} |f(t)|\, dm(t) = \int_{[0,1]} |f(t)| \cdot 1 \, dm(t)$

$= \langle f, F_1 \rangle \leq \|f\|_2 \|F_1\|_2 < \infty$,

Let $N \in \mathbb{R}$ such that $N > \int_{[0,1]} |f(t)|\, dm(t)$.

Then we claim: $\int_{[s,s+h]} |f(t)|\, dm(t) \to 0$ as $h \to 0$.

proof: If not, then exists a $\varepsilon > 0$, such that

$\int_{[s,s+h]} |f(t)|\, dm(t) \geq \varepsilon$, whenever $h > 0$, since as

$h \searrow 0$, $\int_{[s,s+h]} |f(t)|\, dm(t)$ decreases.

However, for $h = \delta$, $\delta > 0$, $\int_{[s,s+\delta]} |f(t)|\, dm(t) \geq \varepsilon$

means that $|f(t)| \geq \frac{\varepsilon}{\delta}$, when $t \in [s, s+\delta]$.

Hence $\int_{[s,s+h]} |f(t)|^2 dm(t) \geq \frac{\varepsilon}{\delta^2} \cdot \delta = \frac{\varepsilon^2}{\delta}$,

therefore $\int_{[0,1]} |f(t)|^2 dm(t) \geq \int_{[s,s+h]} |f(t)|^2 dm(t)$

$\geq \frac{\varepsilon^2}{\delta}$, for any $\delta > 0$.

Hence $\int_{[0,1]} |f(t)|^2 dm(t)$ is unlimited, contradicting to the fact that $\|f\|_2 < \infty$.

Hence the claim is proved.

Now for any $\varepsilon > 0$, we may choose $h < \frac{\varepsilon}{2N}$, and also h is small enough that $\int_{[s,s+h]} |f(t)|\, dm(t) < \frac{\varepsilon}{2}$, then

$|Tf(s+h) - Tf(s)| \leq h \cdot \int_{[0,1]} |f(t)|\, dm(t) + \int_{[s,s+h]} |f(t)|\, dm(t)$.

$< \frac{\varepsilon}{2N} \cdot N + \frac{\varepsilon}{2} = \varepsilon$.

Hence $|Tf(s+h) - Tf(s)| \to 0$ as $h \to 0$, therefore Tf is right continuous.

By similar argument, i.e. considering $|Tf(s-h) - Tf(s)|$ for $h>0$, ~~s~~ $s-h \in [0,1]$, we can prove Tf is also left-continuous.

Therefore Tf is continuous on $[0,1]$.

And also:

$$(Tf)(0) = \int_{[0,0]} tf(t)\,dm(t) + 0 \cdot \int_{[0,1]} (1-t)f(t)\,dm(t).$$

$$= 0.$$

$$(Tf)(1) = 0 \cdot \int_{[0,1]} tf(t)\,dm(t) + \int_{[1,1]} (1-t)f(t)\,dm(t).$$

$$= 0.$$

4.

a). We first verify that $g_k \in \mathcal{S}(\mathbb{R})$, for all integers $k \geq 0$.

From HW7, problem 1, (a), it suffices to prove that $g_0 \in \mathcal{S}(\mathbb{R})$.

By using L'Hospital's rule repeatedly, we can prove that $\frac{p(x)}{e^{\frac{x^2}{2}}} \to 0$ as $|x| \to \infty$, so $p(x)e^{-\frac{x^2}{2}} \to 0$ as $|x| \to \infty$, for any polynomial function $p(x)$. However, since $(e^{-\frac{x^2}{2}})' = -x e^{-\frac{x^2}{2}}$, it is easy to see that $x^\beta \partial^\alpha e^{-\frac{x^2}{2}}$ is always in the form of $p(x)e^{-\frac{x^2}{2}}$ for some polynomial function $p(x)$, for any $\alpha, \beta \in \mathbb{N}$.

Therefore $\lim_{|x| \to \infty} x^\beta \partial^\alpha e^{-\frac{x^2}{2}} = 0$, hence $g_0 \in \mathcal{S}(\mathbb{R})$, and it follows from HW7, problem 1, (a) that $g_k \in \mathcal{S}(\mathbb{R})$ for all integers $k \geq 0$.

Now we calculate $F(g_k)$, for $k = 0, 1, 2, 3$. From proposition 11.4, in the case of $n=1$, we have $F(g_0) = g_0$, that is,

$$F(g_0)(\xi) = e^{-\frac{\xi^2}{2}}.$$

From proposition 11.13, (c), we have, since g_0 and $g_1 = x g_0$ are in $\mathcal{S}(\mathbb{R})$:

$$\widehat{x g_0}(\xi) = i\left(\frac{\partial}{\partial \xi} \hat{g_0}\right)(\xi)$$

$$= i \cdot -\xi e^{-\frac{\xi^2}{2}} = -i\xi e^{-\frac{\xi^2}{2}}.$$

That is, $F(g_1)(\xi) = -i\xi e^{-\frac{\xi^2}{2}} = -i g_1(\xi)$.

Similarly we have:

$$(x g_1)^\wedge(\xi) = i\left(\frac{\partial}{\partial \xi}\hat{g}_1\right)(\xi)$$
$$= e^{-\frac{\xi^2}{2}} - \xi^2 e^{-\frac{\xi^2}{2}}$$

That is, $F(g_2)(\xi) = e^{-\frac{\xi^2}{2}} - \xi^2 e^{-\frac{\xi^2}{2}} = -g_2(\xi) + g_0(\xi)$

$$(x g_2)^\wedge(\xi) = i\left(\frac{\partial}{\partial \xi}\hat{g}_2\right)(\xi)$$
$$= i\left(\xi^3 e^{-\frac{\xi^2}{2}} - 3\xi e^{-\frac{\xi^2}{2}}\right).$$

That is, $F(g_3)(\xi) = i\left(\xi^3 e^{-\frac{\xi^2}{2}} - 3\xi e^{-\frac{\xi^2}{2}}\right) = i g_3(\xi) - 3 g_1(\xi)$.

b). For $k=0$, $F(h_0) = h_0$, we can take $h_0 = g_0$.

For $k=1$, $F(h_1) = i h_1$, we can take $h_1 = g_3 + \frac{3}{2} i g_1$,

then since F is linear operator, we have

$$F(h_1) = F(g_3 + \tfrac{3}{2} i g_1) = F(g_3) + F(\tfrac{3}{2} i g_1)$$
$$= i g_3 - 3 g_1 + \tfrac{3}{2} i \cdot (-i g_1)$$
$$= i g_3 - \tfrac{3}{2} g_1$$
$$= i(g_3 + \tfrac{3}{2} i g_1)$$
$$= i h_1.$$

For $k=2$, $F(h_2) = -h_2$, we can take $h_2 = g_2 - \frac{1}{2} g_0$,

since $F(h_2) = F(g_2) - \frac{1}{2} F(g_0)$

$$= -g_2 + g_0 - \tfrac{1}{2} g_0$$
$$= -g_2 + \tfrac{1}{2} g_0 = -(g_2 - \tfrac{1}{2} g_0)$$
$$= -h_2.$$

For $k=3$, $F(h_3) = -i h_3$, we can take $h_3 = g_1$.

c). Define the function f_- as $f_-(t) := -f(t)$, it is obvious that $f_- \in \mathcal{S}(\mathbb{R})$, and $(f_-)_-(t) = f_-(-t) = f(t)$.

Now since $f \in \mathcal{S}(\mathbb{R})$, it is Riemann integrable, and we can thus calculate:

$$\overline{\int_{\mathbb{R}} f(x) e^{-ix\xi}\, dm(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x) e^{-ix\xi}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)} \cdot e^{ix\xi}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f_-(-x)} \cdot e^{ix\xi}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f_-(-x)} \cdot e^{-i(-x)\xi}\, dx$$

change to $y = -x$. $$= \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f_-(y)} \cdot e^{-iy\xi}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f_-(y)} \cdot e^{-iy\xi}\, dy$$

$$= \int_{\mathbb{R}} \overline{f_-(y)} \cdot e^{-iy\xi}\, dm(y)$$

$$= F(\overline{f_-})(\xi)$$

Hence we have $\overline{(Ff)(\xi)} = F(\overline{f_-})(\xi)$.

Recall from HW 7, problem 6, we have

$$\overline{F(f)} = F^*(\overline{f}) \Rightarrow F(f) = \overline{F^*(\overline{f})}.$$

Then $$F \circ F(f) = \overline{F^*(\overline{F(f)})}$$

$$= \overline{F^*(F(\overline{f_-}))}$$

$$= \overline{\overline{f_-}} \quad \text{since } F^*F(g) = g, \text{ for any } g \in \mathcal{S}(\mathbb{R}).$$

$$= f_-.$$

Therefore $F^2(f) = f_-$, hence $F^4(f) = F^2(F^2(f)) = F^2(f_-)$
$= (f_-)_- = f$.

d). If $f \in \mathcal{S}(\mathbb{R})$, f non-zero, and $F(f) = \lambda f$ for some $\lambda \in \mathbb{C}$, then since F is linear operator on $\mathcal{S}(\mathbb{R})$, we have

$$F^4(f) = \lambda^4 f.$$

however, from (c) we have $F^4(f) = f$, for any $f \in \mathcal{S}(\mathbb{R})$, therefore $\lambda^4 = 1$, $\lambda \in \mathbb{C}$.

The equation $\lambda^4 - 1 = 0$ has exactly 4 solutions in $\mathbb{C}$, namely, $1, i, -1, -i$, then $\lambda \in \{1, i, -1, -i\}$.

Therefore the eigenvalues of F, belong to the set $\{1, i, -1, -i\}$. In (b) we have found an eigenvectors $\{h_k\}$ for eigenvalue $1, i, -1, -i$, therefore the eigenvalues of F are precisely $\{1, i, -1, -i\}$.

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5.

Recall in HW 8. problem 3 (a) we defined the support of μ as the complement of the union of all μ-null open sets.

Therefore to prove $supp(\mu) = [0,1]$, it suffices to show that the union of all μ-null open sets is empty set, this is equivalent to say that the only μ-null open set is ϕ.

Consider any open subset $U \subseteq [0,1]$, $U \neq \phi$, from definition 14.7 we have $\mu(U) = \sup\{\mu(K): K \text{ compact}, K \subset U\}$, from that fact that $(x_n)_{n\geq 1}$ is dense in $[0,1]$, we know that exists x_i, $i \geq 1$, $x_i \in (x_n)_{n\geq 1}$, and $x_i \in U$. Since $\{x_i\}$ as a single point is compact in $[0,1]$, $\mu(\{x_i\}) = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}(x_i)$

$$= 2^{-i} \delta_{x_i}(x_i)$$

$$= 2^{-i} > 0.$$

Therefore $\mu(U) > 0$, since $\mu(U) \geq \mu(\{x_i\})$.

Therefore any non-empty open subset is not μ-null, hence the only μ-null open subset is ϕ, $supp(\mu) = [0,1]$.