Problem 1a

We have that $x = 0$ if and only if $\|x\|_X = 0$, hence if $x = 0$:

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y = 0 + \|0\|_Y = 0$$

by also using linearity of T, so $x = 0$ implies $Tx = 0$. If $x \neq 0$ then:

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y \geq \|x\|_X \neq 0$$

so $\|x\|_0 = 0$ if and only if $x = 0$. In addition for $\alpha \in \mathbb{K}$ and $x, y \in X$, we have by using that T is linear and $\|-\|_X$ and $\|-\|_Y$ are norms:

$$\|\alpha x\|_0 = \|\alpha x\|_X + \|T\alpha x\|_Y = |\alpha|\|x\|_X + \|\alpha Tx\|_Y = |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y$$

$$= |\alpha|(\|x\|_X + \|Tx\|_Y) = |\alpha|\|x\|_0$$

and

$$\|x + y\|_0 = \|x + y\|_X + \|T(x + y)\|_Y = \|x + y\|_X + \|Tx + Ty\|_Y$$

$$\leq \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y = \|x\|_0 + \|y\|_0$$

showing that $\|-\|_0$ is a norm. Now assume that T is not bounded, then we have a sequence $x_1, x_2, ... \in X$, where $\|x_k\|_X = 1$ for all k, such that $\|Tx_k\|_Y \to \infty$ for $k \to \infty$. If we assume by contradiction that $\|-\|_0$ and $\|-\|_X$ are equivalent, we will in particular have that there exist a $C \in \mathbb{R}$, so that $\|x\|_0 \leq C\|x\|_X$ for all $x \in X$, but then we have for all $k \geq 1$:

$$C = C\|x_k\|_X \geq \|x_k\|_0 = \|x_k\|_X + \|Tx_k\|_Y = 1 + \|Tx_k\|_Y$$

which is a contradiction as $\|Tx_k\|_Y \to \infty$ for $k \to \infty$, so $\|-\|_0$ and $\|-\|_X$ are not equivalent. Assume now that T is bounded, we have that:

$$\|x\|_X \leq \|x_k\|_X + \|Tx\|_Y = \|x\|_0$$

for all $x \in X$ (even if T is not bounded). And since T is bounded:

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq \|x\|_X + \|T\|\|x\|_X = (\|T\| + 1)\|x\|_X$$

showing that $\|-\|_0$ and $\|-\|_X$ are equivalent.

Problem 1b

Let $x_1, x_2, ... \in X$ be a sequence so $x_k \to 0$ for $k \to \infty$. As X is finite dimensional we can find a basis, $e_1, ..., e_n \in X$, so the above sequence becomes $x_k = \sum_{i=1}^n a_{i,k}e_i$, where $a_{i,k} \to 0$ for $k \to \infty$ for all $i \in \{1, ..., n\}$. By linearity of T, we have:

$$Tx_k = T\left(\sum_{i=1}^n a_{i,k}e_i\right) = \sum_{i=1}^n a_{i,k}T(e_i) \to \sum_{i=1}^n 0T(e_i) = 0, \text{ for } k \to \infty$$

so T is continuous at 0 and is hence bounded by proposition 1.10.

Problem 1c

Let $(e_i)_{i\in I}$ be an Hamel basis of X. Pick a countable infinite subset $(x_n)_{n\in\mathbb{N}} \subset (e_i)_{i\in I}$, which exist as X is infinite-dimensional, and pick a non zero element $e \in Y$, which exists as Y is non-zero. We define $T : X \to Y$ by $T(x_n) = n\|x_n\|e$ and $T(e_k) = 0$ for all $e_k \in (e_i)_{i\in I} \setminus (x_n)_{n\in\mathbb{N}}$. Note that $z_n := \frac{x_n}{\|x_n\|}$ has norm 1 and $T(z_n) = ne$ by linearity of T and hence for all $n \in \mathbb{N}$:

$$\|T\| = \sup\{x \in X \mid \|x_n\| = 1\} \geq \|Tz_n\| = n\|e\|$$

and as $\|e\| \neq 0$, we must have $\|T\| = \infty$, so T is unbounded.

Problem 1d

By problem 1c we can find a linear map $T : X \to Y$ for some non-zero normed vector space Y, such that T is not bounded. By $1a$, the norm $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ is not equivalent to $\|x\|_X$ as T is not bounded and clearly $\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_0$ for all $x \in X$. If $(X, \|-\|_X)$ is complete, then if $(X, \|-\|_0)$ also was complete by Homework 3 problem 1 $\|-\|_0$ and $\|-\|_X$ would then be equivalent as $\|-\|_X \leq \|-\|_0$, as this is not the case $(X, \|-\|_0)$ can't be complete.

Problem 1e

Note that for any element $(x_n)_{n\in\mathbb{N}} \in l_1(\mathbb{N})$, we have $\lim_{n\to\infty} x_n = 0$ as $\sum_{n=1}^{\infty} |x_n| < \infty$, so $\|(x_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}}\{|x_n|\} = \max_{n\in\mathbb{N}}\{|x_n|\}$. In particular we have:

$$\|(x_n)_{n\in\mathbb{N}}\|_\infty = \max_{n\in\mathbb{N}}\{|x_n|\} \leq \sum_{n=1}^{\infty} |x_n| = \|(x_n)_{n\in\mathbb{N}}\|_1$$

for all $(x_n)_{n\in\mathbb{N}} \in l_1(\mathbb{N})$.

Let:

$$(x_k)_{k\in\mathbb{N}} = \left(\left(\frac{1}{n^{1+\frac{1}{k}}}\right)_{n\in\mathbb{N}}\right)_{k\in\mathbb{N}} \subset l_1(\mathbb{N}) \subset l_\infty(\mathbb{N})$$

this sequence converges in $l_\infty(\mathbb{N})$ and is hence a Cauchy sequence (and hence also in $(l_1(\mathbb{N}), \|-\|_\infty)$), but it doesn't converge in $l_1(\mathbb{N})$ as $x_k \to \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ for $k \to \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. So $(l_1(\mathbb{N}), \|-\|_\infty)$ is not a Banach space while $(l_1(\mathbb{N}), \|-\|_1)$ is and $\|-\|_\infty \leq \|-\|_1$.

Problem 2a

By proposition 2.1 in the notes:

$$\|f\| = \|\mathrm{Re}(f)\| = \sup\{|a_1 + b_1| \mid \|(a_1 + ia_2, b_1 + ib_2, 0, 0, ...)\|_p = 1\}$$

$$= \sup\left\{ |a_1 + b_1| \;\middle|\; \left(\left(\sqrt{a_1^2 + a_2^2}\right)^p + \left(\sqrt{b_1^2 + b_2^2}\right)^p\right)^{1/p} = 1 \right\}$$

$$= \sup\left\{|a_1+b_1| \;\middle|\; \left(\sqrt{a_1^2+a_2^2}\right)^p + \left(\sqrt{b_1^2+b_2^2}\right)^p = 1\right\}$$

Clearly this is the same as taking the supremum where a_1 and b_1 has the same sign, we can hence assume $a_1, b_1 \geq 0$. In addition this is also clearly the same as taking the supremum where $a_2, b_2 = 0$ (as otherwise the values of a_1 and b_1 wille be lower), hence:

$$\|f\| = \sup\{a_1+b_1 \mid a_1, b_1 \geq 0,\ a_1^p + b_1^p = 1\}$$

We see that $a_1^p + b_1^p = 1$ implies $b_1 = (1-a_1^p)^{1/p}$, and hence:

$$\|f\| = \sup\left\{a_1 + (1-a_1^p)^{1/p} \;\middle|\; 0 \leq a_1 \leq \frac{1}{2^{1/p}}\right\}$$

So we are trying to find the maximal value of $g(x) = x + (1-x^p)^{1/p}$, for $0 \leq x \leq \frac{1}{2^{1/p}}$. We have:

$$g'(x) = 1 - px^{p-1}\frac{1}{p}(1-x^p)^{1/p-1} = 1 - x^{p-1}\frac{(1-x^p)^{1/p}}{1-x^p}$$

so in particular $g'(0) = 1$, so it's increasing at the start. We have:

$$g'(x) = 1 - x^{p-1}\frac{(1-x^p)^{1/p}}{1-x^p} = 0 \quad \Rightarrow \quad x^{p-1}(1-x^p)^{1/p} = 1 - x^p$$

$$\Rightarrow \quad x^{p-1} = (1-x^p)^{(p-1)/p} \quad \Rightarrow \quad x = (1-x^p)^{1/p} \quad \Rightarrow \quad x^p = 1 - x^p$$

$$\Rightarrow \quad x = \frac{1}{2^{1/p}}$$

so g is increasing on the segment $0 \leq x \leq \frac{1}{2^{1/p}}$, and hence:

$$\|f\| = \frac{1}{2^{1/p}} + \frac{1}{2^{1/p}} = \frac{2}{2^{1/p}} = 2^{\frac{p-1}{p}}$$

on $(l_p(\mathbb{N}), \|-\|_p)$ and in particular f is bounded.

Problem 2b

Note that $F : l_p(\mathbb{N}) \to \mathbb{C}$, defined by $F(z_1, z_2, ...) = z_1 + z_2$ is clearly linear and an extension of f. We have for $z_k = x_k + iy_k$:

$$\|F\| = \|\mathrm{Re}(F)\| = \sup\left\{|x_1+x_2| \;\middle|\; \sum_{k=1}^{\infty}\sqrt{x_k^2+y_k^2} = 1\right\}$$

Clearly this is the same as taking the supremum where x_1 and x_2 have the same sign and $y_k = 0$ for all k and $x_j = 0$ for all $j \geq n+1$, so:

$$\|F\| = \sup\{x_1 + x_2 \mid x_1, x_2 \geq 0,\ x_1^p + x_2^p = 1\} = \|f\|$$

Now let $F' : l_p(\mathbb{N}) \to \mathbb{C}$ be a linear extension of f, so that $F' \neq F$. By linearity we can find an element $(0, 0, z_3, z_4, ...,)$ so that $F'(0, 0, z_3, z_4, ...,) \neq 0$. We can in addition assume that there exists a finite sequence so that $F'(0, 0, ..., z_k, z_{k+1}, ..., z_l, 0, 0, ...) \neq$

0, since otherwise we would have $F'(0, 0, z_3, z_4, ..., z_k, 0, 0, ...) = 0$ for all k. But then, for the sequence $v_k = (0, 0, z_3, z_4, ..., z_k, 0, 0, ...)$ we would have:

$$F'(v_k) = 0 \to 0 \neq F'(0, 0, z_3, z_4, ...,)$$

so F' would not be continuous and hence not bounded. So assume $F'(0, 0, ..., z_k, z_{k+1}, ..., z_l, 0, 0, ...) \neq 0$ for some finite sequence. By linearity of F' we must have $F'(0, 0, ..., 0, z_k, 0, ...) \neq 0$ for some k (we can assume $k = 3$ without loss of generality). As F' is in particular linear on the 1-dimensional subspace spanned by $(0, 0, 1, 0, ...)$, it's must be of the form $F'(0, 0, z_3, 0, ..) = \alpha z_3$ for some $\alpha \in \mathbb{C}^*$. So we have $F'(z_1, z_2, z_3, 0, ...) = z_1 + z_2 + \alpha z_3$ and in particular $\mathrm{Re}(F'(z_1, z_2, z_3, 0, ...)) = x_1 + x_2 + \mathrm{Re}(\alpha)x_3 - \mathrm{Im}(\alpha)y_3$. Let $\beta = \mathrm{Re}(\alpha)$ and assume $\beta > 0$ and set:

$$z_1 = z_2 = \left(\frac{1}{2 + \beta^{\frac{p}{p-1}}}\right)^{1/p}$$

$$z_3 = \left(1 - \frac{2}{2 + \beta^{\frac{p}{p-1}}}\right)^{1/p}$$

so that:

$$|z_1|^p + |z_2|^p + |z_3|^p = \frac{1}{2 + \beta^{\frac{p}{p-1}}} + \frac{1}{2 + \beta^{\frac{p}{p-1}}} + 1 - \frac{2}{2 + \beta^{\frac{p}{p-1}}} = 1$$

and:

$$|z_1 + z_2 + \beta z_3| = \left| \left(\frac{1}{2 + \beta^{\frac{p}{p-1}}}\right)^{1/p} + \left(\frac{1}{2 + \beta^{\frac{p}{p-1}}}\right)^{1/p} + \beta\left(1 - \frac{2}{2 + \beta^{\frac{p}{p-1}}}\right)^{1/p} \right|$$

$$= \left| \frac{2}{(2 + \beta^{\frac{p}{p-1}})^{1/p}} + \beta\left(\frac{\beta^{\frac{p}{p-1}}}{2 + \beta^{\frac{p}{p-1}}}\right)^{1/p} \right| = \left| \frac{2}{(2 + \beta^{\frac{p}{p-1}})^{1/p}} + \frac{\beta \cdot \beta^{\frac{1}{p-1}}}{(2 + \beta^{\frac{p}{p-1}})^{1/p}} \right|$$

$$= \frac{2 + \beta^{\frac{p}{p-1}}}{(2 + \beta^{\frac{p}{p-1}})^{1/p}} = (2 + \beta^{\frac{p}{p-1}})^{\frac{p-1}{p}}$$

If $\beta < 0$ we simply change the sign of z_1 and z_2 and gain the same result by substituting β with $-\beta$ in the formula above. All in all:

$$\|F'\| = \|\mathrm{Re}(F')\| \geq (2 + \beta^{\frac{p}{p-1}})^{\frac{p-1}{p}} > 2^{\frac{p-1}{p}} = \|f\|$$

So F is the unique linear extension for which $\|F\| = \|f\|$.

Problem 2c

Note that $F_n : l_1(\mathbb{N}) \to \mathbb{C}$, defined by $F_n(z_1, z_2, ...) = z_1 + z_2 + ... + z_n$ is clearly linear and an extension of f for all $n \geq 2$. We have for $z_k = x_k + iy_k$:

$$\|F_n\| = \|\mathrm{Re}(F_n)\| = \sup\left\{ \left|\sum_{k=1}^{n} x_k\right| \,\middle|\, \sum_{k=1}^{\infty} \sqrt{x_k^2 + y_k^2} = 1 \right\}$$

Clearly this is the same as taking the supremum where all x_k have the same sign and $y_k = 0$ for all k and $x_j = 0$ for all $j \geq n+1$, so:

$$\|F_n\| = \sup\left\{\sum_{k=1}^{n} x_k \;\middle|\; \sum_{k=1}^{n} x_k = 1\right\} = 1 = \|f\|$$

showing that there are infinite linear extensions of f with $\|F_n\| = \|f\|$.

Problem 3a

Let $e_1, ..., e_{k+1} \in X$ be $k+1$ linearly independent elements, which exists since otherwise X would be spanned by $\leq k$ elements and hence not infinite dimensional. Let M be the subspace spanned by $e_1, ..., e_{k+1}$, for a linear map $F : X \to \mathbb{R}^k$, the restriction $F|_M : M \to \mathbb{R}^k$ is also linear and can't be injective as $\dim M = k+1$, hence neither is F.

Problem 3b

Define $F : X \to \mathbb{R}^k$ by $F(x) = (f_1(x), ..., f_k(x))$, which is linear as all the entries are. By problem 3a F is not injective hence there exists distinct elements $x, y \in X$, so $F(x) = F(y)$ and hence $0 = F(x-y) = (f_1(x-y), ..., f_k(x-y))$, so $0 \neq x - y \in \bigcap_{i=1}^k \ker(f_i)$ and therefore $\bigcap_{i=1}^k \ker(f_i) \neq \{0\}$.

Problem 3c

By theorem 2.7(b) in the lecture notes we can find $f_1, ..., f_k \in X^*$, so that $f_i(x_i) = \|x_i\|$ and $\|f_i\| = 1$ for all i. By problem 3b the vector space $\bigcap_{i=1}^k \ker(f_i)$ is nonzero and hence we can find an element $y \in \bigcap_{i=1}^k \ker(f_i)$ which we can assume $\|y\| = 1$ by acting on it with $\frac{1}{\|y\|}$. As $f_i(y) = 0$ for all i, we get by linearity of all the f_i's:

$$\|x_i\| = f_i(x_i) = f_i(x_i) - f_i(y) = f_i(x_i - y) \leq \|f_i\|\|x_i - y\| = \|x_i - y\|$$

for all i, showing the result.

Problem 3d

Let $\{\overline{B}(x_i, r_i)\}_{i \in \{1,...,n\}}$ be closed balls such that $S \subset \bigcup_{i=1}^n \overline{B}(x_i, r_i)$. By problem 3c we have a $y \in S$, so that $\|x_i - y\| \geq \|x_i\|$ for all i. But y has to be contained in one such ball, let's say $y \in \overline{B}(x_k, r_k)$, but then:

$$\|x_k - 0\| = \|x_k\| \leq \|x_k - y\| \leq r_k$$

and hence $0 \in \overline{B}(x_k, r_k)$.

Problem 3e

If S was compact the open cover $S \subset \bigcup_{y \in S} B(y, \varepsilon)$ would give a finite subcover $S \subset \bigcup_{i=1}^{n} B(y_i, \varepsilon) \subset \bigcup_{i=1}^{n} \overline{B}(y_i, \varepsilon)$ for any $\varepsilon > 0$. However assuming $\varepsilon < 1$, we get from problem 3d that $0 \in: \overline{B}(y_k, \varepsilon)$ for some $k \in \{1, ..., n\}$, so that:

$$\varepsilon \geq \|y_k - 0\| = \|y_k\| = 1$$

which is a contradiction and hence S is not compact.

Note that S is closed in X, since if $x_1, x_2, ... \in S$ with $x_k \to x$ for $k \to \infty$, we in particular have $1 = \|x_k\| \to \|x\|$, so we must have $\|x_k\| = 1$, so $x \in S$, showing S contains all it's limit points and is hence closed in X. It's in particular closed in $S \subset \overline{B}(0,1)$ and if $\overline{B}(0,1)$ was compact then so would S have to be, as all closed supspaces of a compact space are compact, but as S is not compact this means that $\overline{B}(0,1)$ is not compact.

Problem 4a

As $L_3([0,1], m) \subsetneq L_1([0,1], m)$, there exist a $f \in L_1([0,1], m)$, so that $\int_{[0,1]} |f|^3 dm = \infty$. Clearly:

$$\int_{[0,1]} |\lambda f|^3 dm = \int_{[0,1]} |\lambda|^3 |f|^3 dm = |\lambda|^3 \int_{[0,1]} |f|^3 dm = \infty$$

for all $\lambda \in \mathbb{K}$, so $\lambda f \notin L_3([0,1], m)$ for all $\lambda \in \mathbb{K}$ and therefore in particular $\lambda f \notin E_n$ for all $\lambda \in \mathbb{K}$ and all n. So E_n is not absorbing in $L_1([0,1], m)$ for all n.

Problem 4b

Let $f \in E_n$, and let $B(f, r) \subset L_1([0,1], m)$ be the open ball of radius $r > 0$, i.e all elements $g \in L_1([0,1], m)$ for which $\int_0^1 |f - g| dm < r$. Let $g = f + \frac{r}{4} x^{-1/2} \in L_1([0,1], m)$ (which lies in $L_1([0,1], m)$ since both summands does), then:

$$\int_0^1 |f - g| dm = \int_0^1 \frac{r}{4} x^{-1/2} dm = \frac{r}{4} [2\sqrt{x}]_0^1 = \frac{r}{4}(2 - 0) = \frac{r}{2} < r$$

so $g \in B(f, r)$. Now if $g \in E_n$, then in particular $g \in L_3([0,1], m)$ and hence $f - g \in L_3([0,1], m)$ (as it's a vector space and therefore closed under sum), but:

$$\int_0^1 |f - g|^3 dm = \int_0^1 \frac{r^3}{64} x^{-3/2} dm = \frac{r^3}{64} [-2x^{-1/2}]_0^1 = \frac{r^3}{64}(-2 + \infty) = \infty$$

so $f - g \notin L_3([0,1], m)$ and hence $g \notin L_3([0,1], m)$ and in particular $g \notin E_n$. So $B(f, r) \not\subset E_n$ for all $f \in E_n$ and all $r > 0$ and hence E_n contains no interior points.

Problem 4c

Let $(f_k)_{k \in \mathbb{N}} \subset E_n$ be a sequence, which converges $f_k \to f$ in $L_1([0,1], m)$. From Analysis 2 we know that there exists a subsequence $(f_{k_j})_{j \in \mathbb{N}}$ that converges pointwise to a function $\tilde{f}$, so that $\tilde{f} = f$ almost everywhere. Note that $|f_{k_j}|^3$ is a positive measurable function, for all k_j, as $f_{k_j} \in L_1([0,1], m)$ and $|-|^3$ is continuous, the latter comment also means that $|f_{k_j}|^3 \to |\tilde{f}|^3$ as $k \to \infty$. By Fatou's lemma:

$$\int_0^1 |f|^3 dm = \int_0^1 |\tilde{f}|^3 dm = \int_0^1 \lim_{j\to\infty} |f_{k_j}|^3 dm \leq \lim_{j\to\infty} \int_0^1 |f_{k_j}|^3 dm \leq \lim_{j\to\infty} n = n$$

so $f \in E_n$. So E_n contains all it's limit points, which is a sufficient condition for it to be closed in a metric space.

Problem 4d

By problem 4c: $\overline{E}_n = E_n$ for all n and hence by problem 4b $\text{int}(\overline{E}_n) = \text{int}(E_n) = \emptyset$ for all n, showing that E_n is nowhere dense for all n. And as:

$$L_3([0,1], m) = \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm < \infty \right\}$$

$$= \bigcup_{n=1}^{\infty} \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm \leq n \right\} = \bigcup_{n=1}^{\infty} E_n$$

showing that $L_3([0,1], m)$ is of the first category when considered as a subspace of $L_1([0,1], m)$.

Problem 5a

The norm is a continuous map in $(H, \|-\|)$, which is equivalent to $\|x_k\| \to \|x\|$ for $k \to \infty$.

Problem 5b

Let $e_1 = (1, 0, 0, ...), e_2 = (0, 1, 0, 0, ...), ... \in l_2(\mathbb{N})$ (which is a seperable Hilbert space by homework 4 problem 4), then $\|e_n\|_2 = 1$, so the sequence is bounded. Meanwhile the sequence (here $e_n(i)$ denotes the i'th term of e_n):

$$(e_1(i) = 0, e_2(i) = 0, ..., e_{i-1}(i) = 0, e_i(i) = 1, e_{i+1}(i) = 0, ...)$$

clearly converges to 0 for all $i \in \mathbb{N}$. By Homework 4 problem 3, $e_n \to 0$ as $n \to \infty$ weakly, however $\|e_n\|_2 = 1 \to 1 \neq 0 = \|0\|_2$ as $n \to \infty$.

Problem 5c

As H is a Hilbert space it's in particular reflexive by proposition 2.10 in the lecture notes and hence $\overline{B}(0,1)$ is compact in the weak topology by theorem 6.3 in the lecture notes. In particular $\overline{B}(0,1)$ is closed in the weak topology and hence contains all it's limit points in the weak topology, so if $x_1, x_2, ... \in \overline{B}(0,1)$ and $x_n \to x$ weakly for $n \to \infty$, we must have $x \in \overline{B}(0,1)$, i.e. $\|x\| \leq 1$.