Solutions for Mandatory Assignment 2 for FunkAn 2020
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Problem 1

(a) Show that $f_N \to 0$ weakly, as $N \to \infty$, while $\|f_N\| = 1$, for all $N \geq 1$.

Proof Since $\langle f_N, e_n\rangle = N^{-1}$ for all $N \geq \sqrt{n}$, we have $\langle f_N, e_n\rangle \xrightarrow{N\to\infty} 0$. Because of the linearity of the inner product, for all finite sum of the basis vectors $v = \sum_{i=1}^{k}\langle v, e_i\rangle e_i$, we have

$$\langle f_N, v\rangle \xrightarrow{N\to\infty} 0 \tag{1}$$

For all $v \in H$, there exists $\varepsilon > 0$, $0 < k \in \mathbb{N}$, such that

$$\sum_{i=k+1}^{\infty} |\langle v, e_i\rangle|^2 < (\frac{\varepsilon}{2})^2. \tag{2}$$

By (1), we have $\langle f_N, \sum_{i=1}^{k}\langle v, e_i\rangle e_i\rangle \xrightarrow{N\to\infty} 0$. Hence, there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$ we have

$$|\langle f_N, \sum_{i=1}^{k}\langle v, e_i\rangle e_i - 0| = |\langle f_N, \sum_{i=1}^{k}\langle v, e_i\rangle e_i\rangle| < \frac{\varepsilon}{2} \tag{3}$$

and

$$\begin{aligned}
|\langle f_N, v\rangle - 0| &= |\langle f_N, \sum_{i=1}^{k}\langle v, e_i\rangle e_i\rangle + \langle f_N, \sum_{i=k+1}^{\infty}\langle v, e_i\rangle e_i\rangle| \\
&< \frac{\varepsilon}{2} + \|f_N\|\sqrt{\sum_{i=k+1}^{\infty}|\langle v, e_i\rangle|^2} && \text{(by(2))} \\
&< \frac{\varepsilon}{2} + 1\cdot\frac{\varepsilon}{2} && \text{(by(3))} \\
&= \varepsilon
\end{aligned}$$

Hence, $|\langle f_N, v\rangle - 0| < \varepsilon$. Namely, $f_N \to 0$ weakly, as $N \to \infty$. $\square$

Let K be the norm closure of $\mathrm{co}\{f_N : N \geq 1\}$.

(b) Argue that K is weakly compact, and that $0 \in K$.

Proof Claim that any closed convex bounded set is weakly compact in a reflexive Banach space. First we prove this claim.

Let X be a separable Banach space and let $Y \subset X^*$. Assume that Y is bounded and weakly closed. Choose $c > 0$ such that $\|x^*\| \leq c$ for all $x^* \in Y$. Since the set

$$c\overline{B}_{X^*}(0,1) = \{x^* \in X^* | \|x^*\| \leq c\}$$

is weakly compact in the w^*-topology by Theorem 6.1 (Lecture6_FunkAn20-21.pdf) and $Y \subset c\overline{B}_{X^*}(0,1)$ is weakly closed, it follows that Y is weakly compact.
Now in this case, noticed that Hilbert spaces are reflexive. Also, by Theorem 5.7 (Lecture5_FunkAn20-21.pdf), the norm and weak closures of A coincide if A is a convex subset of X. It follows that K is weakly closed and bounded. Hence K is weakly compact.
By problem (a), $f_N \to 0 \in H$ weakly. By Problem 1 HW5, there exists a sequence $9y_n)_{n\geq 1} \subseteq \text{co}\{f_N : N \geq 1\}$ such that $(y_n)_{n\geq 1}$ converges to 0 in norm. □

Another solution:

Proof Noticed that Hilbert spaces are reflexive. By Theorem 6.3 (Lecture6_FunkAn20-21.pdf), $\overline{B_X(0,1)}$ is compact with respect to the weak topology. For all $f \in \text{co}\{f_N : N \geq 1\}$, with $\sum_{i=1}^n a_i = 1$,

$$\begin{aligned}\|f\| &= \|a_1 f_{N_1} + \cdots + a_n f_{N_n}\| \\ &\leq a_1\|f_{N_1}\| + \cdots + a_n\|f_{N_n}\| \\ &\leq a_1 + \cdots + a_n = 1.\end{aligned}$$

Since $f \in \overline{B_X(0,1)}$, we have $\text{co}\{f_N : N \geq 1\} \subseteq \overline{B_X(0,1)}$.
On the other hand, by Theorem 5.7 (Lecture5_FunkAn20-21.pdf), the norm and weak closures of A coincide if A is a convex subset of X. It follows that $\overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau w}$. Thus, $\overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} \subseteq \overline{B_X(0,1)}$. Therefore, $K \subseteq \overline{B^*_X(0,1)}$ and K is a closed subset of a compact set. It deduces that K is weakly compact.
By problem (a), $f_N \to 0 \in H$ weakly. By Problem 1 HW5, there exists a sequence $9y_n)_{n\geq 1} \subseteq \text{co}\{f_N : N \geq 1\}$ such that $(y_n)_{n\geq 1}$ converges to 0 in norm. □

(c) Show that 0, as well as each f_N, $N \geq 1$, are extreme points in K.

Proof Suppose that there exists $x_1, x_2 \in K$, and $\alpha \in (0,1)$ such that $\alpha x_1 + (1-\alpha)x_2 = 0$. $x_1, x_2 \in \overline{\text{co}\{f_N, N \geq 1\}}$. We want to show that $\langle x_1, e_i\rangle \leq 0$ and $\langle x_2, e_i\rangle \leq 0$ for all $i \leq 0$.
By (b) we have $\overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau w}$. By HW5 we have if $x_1 \in K$, then there exists $f_{N_i}, f_{N_{i'}} \in \{f_N\}$ such that $\alpha f_{N_i} + (1-\alpha) f_{N_{i'}} \to x_1$ both in weak and norm. Hence,

$$\alpha_i\langle f_{N_i}, e_m\rangle + (1-\alpha)\langle f_{N_{i'}}, e_m\rangle \to \langle x_1, e_m\rangle where$$

$\langle f_{N_i}, e_m\rangle \geq 0$, $\langle f_{N_{i'}}, e_m\rangle \geq 0$, and $\langle x_1, e_m\rangle \geq 0$ for all $m \geq 1$.

$$\begin{aligned}&\langle \alpha x_1 + (1-\alpha)x_2, e_m\rangle = \langle 0, 3_m\rangle = 0,\ m \geq 1 \\ &\alpha\langle x_1, e_m\rangle + (1-\alpha)\langle x_2, e_m\rangle = 0,\ m \geq 1 \\ &\langle x_1, e_m\rangle = \langle x_2, e_m\rangle = 0,\ m \geq 1\end{aligned}$$

It follows that $x_1 = x_2 = 0$. Thus, 0 is extreme point. □

(d) Are there any other extreme points in K? Justify your answer.

Proof There are no other extreme points in K.
(H, τ) is a LCTVS, K is a non-empty compact, convex subset of H, and $F = \{f_N\} \cup \{0\}$ is a subset of K such that $K = \overline{co(F)}^{\tau}$, according to Theorem 7.9 (Lecture7_FunkAn20-21.pdf) we have $\text{ext}(K) \subset \overline{F}^{\tau}$. Therefore, it cannot have any other extreme points in K. $\square$

Problem 2 Let X and Y be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

Proof We want to show that $Tx_n \to Tx$ weakly. Since $x_n \to x$ weakly if and only if $f(x_n) \to f(x)$ weakly for every $f \in X^*$. Now what we need to show is that $g(Tx_n) \to g(Tx)$ weakly for every $g \in Y^*$. Noticed that for every $g \in Y^*$ we have $gT \in X^*$. Therefore,

$$\begin{aligned} Tx_n \to Tx \text{ weakly} &\Leftrightarrow g(Tx_n) \to g(Tx) \text{ weakly} \\ &\Leftrightarrow (gT)x_n \to (gT)x \text{ weakly} \\ &\Leftrightarrow x_n \to x \text{ weakly.} \end{aligned}$$

Whence, $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$. $\square$

(b) Let $T \in \mathcal{K}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$.

Proof Assume that $\|Tx_n - Tx\| \nrightarrow 0$, as $n \to \infty$, while $x_n \to x$ weakly, as $n \to \infty$. Then there exists a subsequence Tx_{n_k} such that $\|Tx_{n_k} - Tx\| > \varepsilon$ for some $\varepsilon > 0$. Hence, there exists a norm-convergent subsequence $Tx_{n_{k_l}} \subset Tx_{n_k}$ such that $Tx_{n_{k_l}} \nrightarrow Tx$ weakly, as $l \to \infty$. This contradicts to problem (a). Therefore, $x_n \to x$ weakly, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$. $\square$

(c) Let H be separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Proof Noted that Hilbert spaces are reflexive, by Theorem 6.1 (Lecture6_FunkAn20-21.pdf) the unit ball of H is weakly compact. Hence, for a bounded sequence $(x_n)_{n\geq 1}$ contained in a weak compact ball, there exists a subsequence $(x_{n_j})_{j\geq 1}$ such that converges weakly in H to some x. Since $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, we have $\|Tx_{n_j} - Tx\| \to 0$. It follows that T is a compact operator. That is, $T \in \mathcal{K}(H, Y)$. $\square$

(d) Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

Proof Noticed that $\ell_2(\mathbb{N})$ is an infinite dimensional Banach space. Let $(x_n)_{n\geq 1}$ be a sequence in $\ell_2(\mathbb{N})$ and $x \in \ell_2(\mathbb{N})$ such that $x_n \to x$ weakly, as $n \to \infty$. Since $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ and by (a) we have $Tx_n \to Tx$ weakly, as $n \to \infty$. According to Remark 5.3 (Lecture5_FunkAn20-21.pdf), a sequence converges weakly in $\ell_1(\mathbb{N})$ if and only if it converges in norm, we get $\|Tx_n - Tx\| \to 0$, as $n \to \infty$. Therefore, use (c) to deduce that $T \in \mathcal{K}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$. Namely, each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact. $\square$

(e) Show that no $T \in \mathcal{K}(X, Y)$ is onto.

Proof Assume that $T \in \mathcal{K}(X, Y)$ is onto. By the open mapping theorem, surjective linear maps in Banach spaces are open. Namely, T is open. Then $B_Y(0, r) \subseteq T(B_X(0, 1)) \subseteq \overline{T(B_X(0,1))}$ and $T(B_X(0, 1))$ is open. Since T is compact, $\overline{T(B_X(0,1))}$ is compact. It follows that $\overline{B}_Y(0, r)$ is a closed subset of a compact set and hence is compact. However, the closed unit ball is not compact in Y, neither is $\overline{B}_Y(0, r)$. This is a contradiction. Therefore, no $T \in \mathcal{K}(X, Y)$ is onto. $\square$

(f) Let $H = L_2([0, 1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0, 1]$. Justify that M is self-adjoint, but not compact.

Proof M is bounded and let M^* denote the adjoint of M. For $f \in H$ and $t \in [0, 1]$, we have

$$\langle Mf, g\rangle = \int_{[0,1]} (Mf)(t)\bar{g}(t)\mathrm{d}m(t) = \int_{[0,1]} tf(t)\bar{g}(t)\mathrm{d}m(t)$$

and

$$\langle f, Mg\rangle = \int_{[0,1]} f(t)\overline{(Mg)(t)}\mathrm{d}m(t) = \int_{[0,1]} f(t)\overline{tg(t)}\mathrm{d}m(t) = \int_{[0,1]} f(t)t\bar{g}(t)\mathrm{d}m(t)$$

Hence, $\langle f, Mg\rangle = \langle Mf, g\rangle = \langle f, M^*g\rangle$. It follows that $M = M^*$ and M is self-adjoint.
By the Spectral Theorem for self-adjoint compact operators (Lecture10_FunkAn20-21.pdf, Theorem 10.1), if M is a compact self-adjoint operator on Hilbert space then it has either finitely many eigenvalues or a sequence of eigenvalues $\lambda_n \to 0$ as $n \to 0$. However, by Problem 3(a) (HW6_FunkAn20-21.pdf), we know that M has no eigenvalues. Hence M can not be compact. $\square$

Problem 3

(a) Justify that T is compact.

Proof Let $(f_n)_{n=1}^{\infty}$ be a bounded sequence in $L_2([0, 1], m)$ with $\|f_n\| \leq M$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|K(s, t) - K(s', t')| < \varepsilon$$

where $|s-s'|+|t-t'|<\delta$. Therefore, (Tf_n) is a sequence of continuous functions for which

$$\begin{aligned}
|Tf_n(s)-Tf_n(s')| &\leq \int_{[0,1]} |K(s,t)-K(s',t)||f_n(t)|\mathrm{d}m(t) \\
&\leq \varepsilon \int_{[0,1]} |f_n(t)|\mathrm{d}m(t) \\
&\leq \varepsilon \|1\| \|f_n\| \\
&\leq M\varepsilon,
\end{aligned}$$

where $|x-x'|<\delta$. It follows that (Tf_n) is an equicontinuous family of continuous functions on $[0,1]$. Hence, there exists a subsequence (Tf_{n_k}) that converges uniformly to a continuous function g. Since uniform convergence implies convergence in $L_2([0,1],m)$, it follows that (Tf_{n_k}) converges in $L_2([0,1],m)$. Since the image of a bounded sequence always contains a convergent subsequence, T is compact. □

Another solution:

Proof First, we show that T is bounded. By the Cauchy-Schwarz inequality

$$\begin{aligned}
|(Tf)(s)| &= \left| \int_{[0,1]} K(s,t)f(t)\mathrm{d}m(t) \right| \\
&\leq \int_{[0,1]} |K(s,t)||f(t)|\mathrm{d}m(t) \\
&\leq \left(\int_{[0,1]} |K(s,t)|^2 \mathrm{d}m(t) \right)^{\frac{1}{2}} \left(\int_{[0,1]} |f(t)|^2 \mathrm{d}m(t) \right)^{\frac{1}{2}} \\
&= \left(\int_{[0,1]} |K(s,t)|^2 \mathrm{d}m(t) \right)^{\frac{1}{2}} \|f\|.
\end{aligned}$$

Then

$$|(Tf)(s)|^2 \leq \left(\int_{[0,1]} |K(s,t)|^2 \mathrm{d}m(t) \right) \|f\|^2$$

and

$$\|Tf\|^2 = \int_{[0,1]} |(Tf)(s)|^2 \mathrm{d}m(t) \leq \left(\int_{[0,1]} \left(\int_{[0,1]} |K(s,t)|^2 \mathrm{d}m(t) \right) \mathrm{d}m(s) \right) \|f\|^2 = \|T\|^2 \|f\|^2,$$

that is

$$\|Tf\| \leq \|T\| \|f\|.$$

Next, let $\{e_n | n \in \mathbb{N}\}$ be an orthonormal basis for $L_2([0,1],m)$. Then $\phi_{m,n}(s,t) = e_n(s)e_m(t)$ for all $s,t \in [0,1]$ and for all $m,n \in \mathbb{N}$ forms an orthonormal basis for $L_2([0,1]\times[0,1],m)$.

Hence

$$K(s,t) = \sum_{m,n=1}^{\infty} \langle K(s,t), \phi_{m,n}(s,t) \rangle \phi_{m,n}(s,t).$$

Let

$$K_N(s,t) = \sum_{m,n=1}^{N} \langle K(s,t), \phi_{m,n}(s,t) \rangle \phi_{m,n}(s,t).$$

Now we define $T_N : H \to H$ $(H = L_2([0,1], m))$ by

$$(T_N f)(s) = \int_{[0,1]} K_N(s,t) f(t) \mathrm{d}m(t)$$

for all $f \in H$. Note that T_N is a finite rank operator and $T_N \to T$ as $N \to \infty$. Hence, by Theorem 9.11 (Lecture9_FunkAn20-21.pdf) and every compact operator on a separable Hilbert space H is a norm limit of a sequence of finite rank operators, T is compact. $\square$

(b) Show that $T = T^*$.

Proof By definition of adjoint and Fubini's theorem ($\int \bar{f} = \overline{\int f}$), we have

$$\begin{aligned}
\langle Tf, g \rangle &= \int_{[0,1]} (Kf)(s) \bar{g}(s) \mathrm{d}m(s) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(s,t) f(t) \mathrm{d}m(t) \right) \bar{g}(s) \mathrm{d}m(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \bar{g}(s) \mathrm{d}m(s) \mathrm{d}m(t) \\
&= \int_{[0,1]} \int_{[0,1]} f(t) \overline{K(t,s) g(s)} \mathrm{d}m(s) \mathrm{d}m(t) \\
&= \int_{[0,1]} f(t) \left(\overline{\int_{[0,1]} K(t,s) g(s) \mathrm{d}m(s)} \right) \mathrm{d}m(t) \\
&= \langle f, T^* g \rangle
\end{aligned}$$

It follows that

$$(T^* g)(t) = \int_{[0,1]} K(t,s) f(s) \mathrm{d}m(s).$$

Change the position of s and t in the above equation, we have

$$(T^* g)(s) = \int_{[0,1]} K(s,t) f(t) \mathrm{d}m(t) = (Tg)(s).$$

This holds for arbitrary $g \in H$, hence $T = T^*$. $\square$

(c) Show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)\mathrm{d}m(t) + s\int_{[s,1]} (1-t)f(t)\mathrm{d}m(t), s \in [0,1], \quad f \in H.$$

Use this to show that Tf is continuous on $[0,1]$, and that $(Tf)(0) = (Tf)(1) = 0$.

Proof Since

$$K(s,t) = \begin{cases} (1-s)t \text{ if } 0 \le t \le s \le 1 \\ (1-t)s \text{ if } 0 \le s < t \le 1 \end{cases},$$

we have

$$\begin{aligned} (Tf)(s) &= \int_{[0,1]} K(s,t)f(t)\mathrm{d}m(t) \\ &= \int_{[0,s]} K(s,t)f(t)\mathrm{d}m(t) + \int_{[s,1]} K(s,t)f(t)\mathrm{d}m(t) \\ &= \int_{[0,s]} (1-s)tf(t)\mathrm{d}m(t) + \int_{[s,1]} (1-t)sf(t)\mathrm{d}m(t) \\ &= (1-s)\int_{[0,s]} tf(t)\mathrm{d}m(t) + s\int_{[s,1]} (1-t)f(t)\mathrm{d}m(t), \end{aligned}$$

where $s \in [0,1]$ and $f \in H$.
Next we show that Tf is continuous on $[0,1]$.

$$\begin{aligned} |Tf(s) - Tf(s')| &= \left| \int_{[0,1]} (K(s,t) - K(s',t))f(t)\mathrm{d}m(t) \right| \\ &\le \int_{[0,1]} |(K(s,t) - K(s',t))||f(t)|\mathrm{d}m(t) \\ &\le \|(K(s,\cdot) - K(s',\cdot)\|_{L_2}\|f\|_{L_2} \\ &\le \max_{t\in[0,1]} |K(s,t) - K(s',t)|(1-0)^{\frac{1}{2}}\|f\|_{L^2} \end{aligned}$$

Continuity now follows from the continuity of K.
Let $s = 0$ and $s = 1$, respectively.

$$\begin{aligned} (Tf)(0) &= (1-0)\int_{[0,0]} tf(t)\mathrm{d}m(t) + 0 \cdot \int_{[0,1]} (1-t)f(t)\mathrm{d}m(t) \\ &= 1 \cdot 0 + 0 = 0, \end{aligned}$$

and

$$\begin{aligned} (Tf)(1) &= (1-1)\int_{[0,1]} tf(t)\mathrm{d}m(t) + 1 \cdot \int_{[1,1]} (1-t)f(t)\mathrm{d}m(t) \\ &= 0 + 1 \cdot 0 = 0. \end{aligned}$$

Tf is continuous on $[0,1]$, and that $(Tf)(0) = (Tf)(1) = 0$ as desired. $\square$

Problem 4

(a) For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$. Justify that $g_k \in S(\mathbb{R})$, for all integers $k \geq 0$. Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

Proof Since $e^{-x^2/2}$ is a composition of $f = \frac{x^2}{2}$ and $g = e^{-y}$, and $f, g \in \mathcal{C}^\infty(\mathbb{R})$, then $e^{-x^2/2}$ $\in$C$^\infty(\mathbb{R})$. We have

$$\begin{aligned}
\partial^\beta e^{-\|x\|^2} &= Pol_{|\beta|}(x) e^{-\frac{\|x\|^2}{x}} \\
x^\alpha \partial^\beta e^{-\|x\|^2} &= Pol_{|\alpha|+|\beta|}(x) e^{-\frac{\|x\|^2}{x}} \xrightarrow{\|x\|\to\infty} 0
\end{aligned}$$

By HW7, we have $f \in S(\mathbb{R}) \Rightarrow x^\alpha f \in S(\mathbb{R})$. Hence, $g_k \in S(\mathbb{R})$.
Case $k = 0$:

$$\begin{aligned}
F(g_0(x)) &= \int_{\mathbb{R}} e^{-\frac{x^2}{2}} e^{-ix\xi} \mathrm{d}m \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2} - ix\xi} \mathrm{d}m \\
&\overset{\psi = x+i\xi}{=\!=\!=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+i\xi} e^{-\frac{\psi^2+\xi^2}{2}} \mathrm{d}\psi \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{\mathbb{R}+i\xi} e^{-\frac{\psi^2}{2}} \mathrm{d}\psi
\end{aligned}$$

By Proposition 11.4. (Lecture11_FunkAn20-21.pdf), we have

$$\int_{C_1} e^{-\frac{\psi^2}{2}} \mathrm{d}\psi = \int_{C_2} e^{-\frac{\psi^2}{2}} \mathrm{d}\psi = 0$$

Hence

$$\int_{\mathbb{R}+i\xi} e^{-\frac{\psi^2}{2}} \mathrm{d}\psi = \int_{\mathbb{R}} e^{-x^2/2} \mathrm{d}x = \sqrt{2\pi},$$

and

$$\mathcal{F}(g_0) = e^{-\frac{x^2}{2}}.$$

Case $k = 1$:

$$\begin{aligned}
\mathcal{F}(g_1)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{-\frac{x^2}{2}} e^{-ix\xi} \mathrm{d}x \\
&\overset{\psi = x+i\xi}{=\!=\!=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+i\xi} (\psi - i\xi) e^{-\frac{\psi^2+\xi^2}{2}} \mathrm{d}\psi \\
&= \frac{1}{\sqrt{2\pi}} \left[e^{-\frac{\xi^2}{2}} \int_{\mathbb{R}+i\xi} \psi e^{-\frac{\psi^2}{2}} \mathrm{d}\psi + \left(-i\xi e^{-\frac{\xi^2}{2}} \right) \int_{\mathbb{R}+i\xi} e^{-\frac{\psi^2}{2}} \mathrm{d}\psi \right]
\end{aligned}$$

where

$$\begin{aligned}
\int_{\mathbb{R}+i\xi} \psi e^{-\frac{\psi^2}{2}} \mathrm{d}\psi &= \int_{\mathbb{R}} x e^{-x^2/2} \mathrm{d}x \\
&= \int_{\mathbb{R}} e^{-x^2/2} \mathrm{d}(\frac{x^2}{2}) \quad t = \frac{x^2}{2} \\
&= \int_0^{\infty} e^{-t} \mathrm{d}t \\
&= -e^{-t}|_0^{\infty} \\
&= 1.
\end{aligned}$$

Hence, $\mathcal{F}(g_1)(\xi) = (\frac{1}{\sqrt{2\pi}} - i\xi)e^{-\frac{\xi^2}{2}}$.
Case $k = 2$:

$$\begin{aligned}
\mathcal{F}(g_2)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} e^{-ix\xi} \mathrm{d}x \\
&\xlongequal{\psi = x + i\xi} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+i\xi} (\psi^2 - 2i\xi - \xi^2) e^{-\frac{\psi^2+\xi^2}{2}} \mathrm{d}\psi \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \left[\int_{\mathbb{R}+i\xi} \psi^2 e^{-\frac{\psi^2}{2}} \mathrm{d}\psi + (-2i\xi - \xi^2) \int_{\mathbb{R}+i\xi} e^{-\frac{\psi^2}{2}} \mathrm{d}\psi \right]
\end{aligned}$$

where

$$\begin{aligned}
\int_{\mathbb{R}+i\xi} \psi^2 e^{-\frac{\psi^2}{2}} \mathrm{d}\psi &= \int_{\mathbb{R}} x^2 e^{-x^2/2} \mathrm{d}x \\
&= 2 \int_0^{\infty} x^2 e^{-x^2/2} \mathrm{d}x
\end{aligned}$$

Let $u = x^2/2$ and then $\mathrm{d}u = x\mathrm{d}x$, hence

$$\begin{aligned}
\int x^2 e^{-x^2/2} \, \mathrm{d}x &= \int \frac{x^2 e^{-u}}{x} \, \mathrm{d}u \\
&= \int \sqrt{2u} e^{-u} \, \mathrm{d}u \\
&= \sqrt{2} \int u^{1/2} e^{-u} \, \mathrm{d}u \\
\int_0^{\infty} x^2 e^{-x^2/2} \, \mathrm{d}x &= \sqrt{2} \, \Gamma\left(\frac{1}{2} + 1\right) \\
&= \sqrt{2} \frac{\sqrt{\pi}}{2}
\end{aligned}$$

where $\Gamma(z)$ is the gamma function $\int_0^{\infty} u^{z-1} e^u \mathrm{d}u$. Hence,

$$\int_{\mathbb{R}+i\xi} \psi^2 e^{-\frac{\psi^2}{2}} \mathrm{d}\psi = 2 \int_0^{\infty} x^2 e^{-x^2/2} \mathrm{d}x = \sqrt{2\pi}.$$

Therefore, $\mathcal{F}(g_2)(\xi) = (1 - 2i\xi - \xi^2)e^{-\frac{\xi^2}{2}}$.
Case $k = 3$:

$$\begin{aligned}
\mathcal{F}(g_3)(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^3 e^{-\frac{x^2}{2}} e^{-ix\xi} \mathrm{d}x \\
&\xlongequal{\psi = x + i\xi} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+i\xi} (\psi - i\xi)^3 e^{-\frac{\psi^2+\xi^2}{2}} \mathrm{d}\psi \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{\mathbb{R}+i\xi} (\psi^3 - 3i\psi^2\xi - 3\xi^2\psi + i\xi^3) e^{-\frac{\psi^2}{2}} \mathrm{d}\psi \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{\mathbb{R}} (^3 - 3ix^2\xi - 3\xi^2 x + i\xi^3) e^{-\frac{x^2}{2}} \mathrm{d}x
\end{aligned}$$

where

$$\begin{aligned}
\int_{\mathbb{R}} x^3 e^{-x^2/2} \mathrm{d}x &= \int_{\mathbb{R}} -x^2 \mathrm{d}(e^{-x^2/2}) \\
&= -x^2 e^{-x^2/2}|_{-\infty}^{+\infty} + 2 \int_{\mathbb{R}} x e^{-x^2/2} \mathrm{d}x \\
&= -2.
\end{aligned}$$

Therefore, $\mathcal{F}(g_3)(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} (-2 - 3i\xi\sqrt{2\pi} - 3\xi^2 + \sqrt{2\pi} i\xi^3)$. $\square$

(b)

Proof Case $k = 0$: $F(h_0) = h_0 = e^{-\frac{x^2}{2}}$.
Case $k = 3$: Let $h_3 = ae^{-\frac{x^2}{2}} - xe^{-\frac{x^2}{2}}$

$$\begin{aligned}
&F(h_3) = ae^{-\frac{x^2}{2}} - (\frac{1}{\sqrt{2\pi}} - i\xi)e^{-\frac{\xi^2}{2}} \\
&(a - \frac{1}{\sqrt{2\pi}}) = -ia \\
&a = \frac{1}{1+i} \frac{1}{\sqrt{2\pi}} = \frac{1-i}{2\sqrt{2\pi}}
\end{aligned}$$

Hence, $h_3 = \frac{1-i}{2\sqrt{2\pi}} e^{-\frac{x^2}{2}} - xe^{-\frac{x^2}{2}}$.

Case $k=2$: Let $h_2 = a_1 e^{-\frac{x^2}{2}} + a_2 x e^{-\frac{x^2}{2}} + x^2 e^{-\frac{x^2}{2}}$

$$
\begin{aligned}
&F(h_2) = a_1 e^{-\frac{\xi^2}{2}} + a_2(\frac{1}{\sqrt{2\pi}} - i\xi)e^{-\frac{\xi^2}{2}} + \quad (1-2i\xi-\xi^2)e^{-\frac{\xi^2}{2}} \\
&a_1 + a_2\frac{1}{\sqrt{2\pi}} = -a_1 \\
&-ia_2 - 2i = -a_2 \\
&a_2 = i-1 \\
&a_1 = -\frac{1}{2}\left(\frac{i-1}{\sqrt{2\pi}} + 1\right)
\end{aligned}
$$

Hence, $h_2 = -\frac{1}{2}\left(\frac{i-1}{\sqrt{2\pi}}+1\right)e^{-\frac{x^2}{2}} + (i-1)xe^{-\frac{x^2}{2}} + x^2 e^{-\frac{x^2}{2}}$. □

(c)

Proof

$$
\begin{aligned}
F^2(f) &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)e^{-i\xi_1 x}\mathrm{d}x e^{-i\xi_2\xi_1}\mathrm{d}\xi_2 \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)e^{-i\xi_1(x+\xi_2)}\mathrm{d}x\mathrm{d}\xi_2 \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)e^{-i\xi_1(x+\xi_2)}\mathrm{d}\xi_2\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\int_{\mathbb{R}} e^{-i\xi_1(x+\xi_2)}\mathrm{d}\xi_2\mathrm{d}x \\
&= \int_{\mathbb{R}} f(x)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi_1(x+\xi_2)}\mathrm{d}\xi_2\mathrm{d}x
\end{aligned}
$$

Note that $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ip(x-\alpha)}\mathrm{d}p = \delta(x-2)$ (Dirac function), we have

$$
\begin{aligned}
F^2(f) &= \int_{\mathbb{R}} f(x)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi_1(x+\xi_2)}\mathrm{d}\xi_2\mathrm{d}x \\
&= \int_{\mathbb{R}} f(x)\delta(-x-\xi_2)\mathrm{d}x \\
&= f(-\xi_2).
\end{aligned}
$$

Therefore, $F^4(f(x)) = F^2(f(-x)) = f(x)$ as desired. □

(d)

Proof

$$
\begin{aligned}
F(f) &= \lambda f \\
F^4(f) &= \lambda^4 f = f
\end{aligned}
$$

Note that $\lambda^4 = 1$ has four roots in $\mathbb{C}$ and they are precisely $\{1, -1, i, -i\}$. $\square$

Problem 5

Proof Let N be the union of all open subsets U of $[0, 1]$. Since $(x_n)_{n\geq 1}$ is a dense subset of $[0, 1]$, for all open subset U we have

$$U \cap (x_n)_{n\geq 1} \neq \emptyset.$$

That is, there always exists some x_n such that $\delta_{x_n}(U) = 1$. For all U we have

$$\mu(U) = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}(U) \neq 0.$$

By Problem 3(a) HW8, it follows that $N = \emptyset$ and $\text{supp}(\mu) = [0, 1]$ as desired. $\square$