Problem 1 Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $T : X \to Y$ be a linear map. Set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$, for all $x \in X$. Show that $\|\cdot\|_0$ is a norm on X. Show next that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded.

To show that $\|\cdot\|_0$ is a norm on X we check the three relevant requirements:

(i) (Subadditivity) Let $x, y \in X$. By the linearity of T and the triangle inequality for the norms $\|\cdot\|_X, \|\cdot\|_Y$ we get

$$\begin{aligned}\|x+y\|_0 &= \|x+y\|_X + \|T(x+y)\|_Y = \|x+y\|_X + \|Tx+Ty\|_Y \\ &\leq \|x\|_X + \|y\|_Y + \|Tx\|_Y + \|Ty\|_Y = \|x\|_0 + \|y\|_0.\end{aligned}$$

(ii) (Scalable) Let $\alpha \in \mathbb{K}, x \in X$. Now use the linearity of T and the absolute homogeneity of $\|\cdot\|_X$ and $\|\cdot\|_Y$ to conclude that

$$\begin{aligned}\|\alpha x\|_0 &= \|\alpha x\|_X + \|T(\alpha x)\|_Y = \|\alpha x\|_X + \|\alpha Tx\|_Y = |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y \\ &= |\alpha|(\|x\|_X + \|Tx\|_Y) = |\alpha|\|x\|_0.\end{aligned}$$

(iii) (Positive definite) Assume $x = 0$. Linear functions map zero to zero, and combining this with the positive definiteness of $\|\cdot\|_X, \|\cdot\|_y$ yields

$$\|x\|_0 = \|0\|_0 = \|0\|_X + \|T(0)\|_Y = \|0\|_X + \|0\|_Y = 0.$$

Now assume $0 = \|x\|_0 = \|x\|_X + \|Tx\|_Y$. Since $\|\cdot\|_X$ and $\|\cdot\|_Y$ are both norms they cannot map to negative values; in particular $\|x\|_X$ must equal zero, which happens if and only if $x = 0$ due to the positive definiteness of $\|\cdot\|_X$. Thus we conclude that $\|x\|_0 = 0$ if and only if $x = 0$.

Since $\|\cdot\|_0$ satisfies (i), (ii) and (iii), we may conclude that it is a norm on X.

Let T be bounded. Then there exists some $C > 0$ such that

$$\|Tx\|_Y \leq C\|x\|_X \quad \text{for all } x \in X,$$

which implies

$$\|x\|_0 = \|x\|_X + \|Tx\|_Y \leq \|x\|_X + C\|x\|_X = (1+C)\|x\|_X.$$

And clearly $\|\cdot\|_0$ is bounded by $\|\cdot\|_X$ from below:

$$\|x_0\| = \|x\|_X + \|Tx\|_Y \geq \|x\|_X.$$

Combining the above with $C_1 := 1, C_2 := 1 + C$ we get the desired result:

$$C_1\|x\|_X \leq \|x\|_0 \leq C_2\|x\|_X \quad \text{for all } x \in X, \tag{1}$$

i.e. $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent (see Definition 1.4, Lecture 1).

Assume instead that $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent. Then there exist $0 < C_1 \leq C_2 < \infty$ such that the relation (1) is satisfied. Rearranging the inequality we get

$$\|Tx\|_Y \leq C_2\|x\|_X - \|x\|_X = (C_2 - 1)\|x\|_X.$$

Note that unless T maps everything to zero (in which case it is trivially bounded), C_2 must be *strictly* larger than 1, because otherwise we would have $\|x\|_0 = \|x\|_X + \|Tx\|_Y > C_2\|x\|_X$ for some x such that $Tx \neq 0$, contradicting the assumption. So the constant $C = C_2 - 1 > 0$ is valid and we conclude that T is bounded (see Proposition 1.10, Lecture 1). ✓

(b) Show that any linear map $T : X \to Y$ is bounded if X is finite dimensional.

Any two norms on a finite dimensional vector space are equivalent (see Theorem 1.6, Lecture 1), so in particular $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent. Then it follows from (a) that T is bounded. ✓

(c) Suppose that X is infinite dimensional. Show that there exists a linear map $T : X \to Y$, which is not bounded.

Let $(e_i')_{i\in I}$ be a Hamel basis for X and normalize it if necessary, i.e. construct a new Hamel basis $(e_i)_{i\in I}$ by defining

$$e_i = \frac{e_i'}{\|e_i'\|} \quad \text{for all } i \in I.$$

Since the indexing set I is infinite (because X is infinite dimensional) it has cardinality greater than or equal to the cardinality of $\mathbb{N}$, hence there exists a surjective function $g : I \to \mathbb{N}$. Now take some non-zero $y \in Y$ (recall that Y was assumed to be non-zero) and define

$$y_i = g(i)\frac{y}{\|y\|} \quad \text{for all } i \in I.$$

Then there exists precisely one map $T : X \to Y$ satisfying

$$T(e_i) = y_i \quad \text{for all } i \in I.$$

Assume by contradiction that T *is* bounded. Then there exists some real number $C > 0$ such that

$$\|Tx\| \leq C\|x\| \quad \text{for all } x \in X. \tag{2}$$

Now take any natural number $N > C$ and consider an $i \in g^{-1}(\{N\}) \subset I$ (note that $g^{-1}(\{N\})$ is nonempty since g is surjective). For this particular i we have

$$\|T(e_i)\| = \|g(i)\frac{y}{\|y\|}\| = |N|\frac{\|y\|}{\|y\|} = N > C \cdot 1 = C\|e_i\|, \tag{3}$$

where we in the last equality used that our Hamel basis had been normalized. Clearly (3) contradicts the assumption (2), so we have found a linear map $T : X \to Y$ which is not bounded. ✓

(d) Suppose again that X is infinite dimensional. Argue that there exists a norm $\|\cdot\|_0$ on X, which is *not* equivalent to the given norm $\|\cdot\|_X$, and which satisfies $\|x\|_X \leq \|x\|_0$, for all $x \in X$. Conclude that $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space.

Let $T : X \to Y$ be an unbounded linear map, whose existence follows from (c). Define $\|\cdot\|_0$ as before, i.e.

$$\|x\|_0 := \|x\|_X + \|Tx\|_Y \quad \text{for all } x \in X,$$

which we know from (a) is a norm. It furthermore follows from (a) that $\|\cdot\|_0$ and $\|\cdot\|_X$ are *not* equivalent because T is not bounded, and clearly we still have

$$\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_0 \quad \text{for all } x \in X.$$

Now recall the conclusion from Problem 1, Homework Week 3:

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on a vector space X such that $\|\cdot\|_1 \leq \|\cdot\|_2$, and X is complete with respect to both norms, then the norms are equivalent. (4)

Let $(X, \|\cdot\|_X)$ be a Banach space and assume by contradiction that $(X, \|\cdot\|_0)$ is also complete. Then X is complete with respect to *both* norms, and hence it follows from (4) that the two norms must be equivalent. This is a contradiction, as we have just shown that they are *not* equivalent, hence $(X, \|\cdot\|_0)$ *cannot* be a Banach space. ✓

(e) Give an example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ satisfying $\|x\|' \leq \|x\|$ for all $x \in X$, such that $(X, \|\cdot\|)$ is complete, while $(X, \|\cdot\|')$ is not.

Take $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|_1)$ and $(X, \|\cdot\|') = (\ell_1(\mathbb{N}), \|\cdot\|_\infty)$. We know from Lecture 1 (also shown in Analysis 2, and mentioned in Problem 2 in the Mandatory Assignment) that $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is complete. Clearly the infinity norm is bounded by the 1-norm: Let $x \in \ell_1(\mathbb{N})$ and let $x(k)$ denote the kth term in the sequence. Then

$$\|x\|_\infty = \sup_{k\in\mathbb{N}} |x(k)| \leq \sum_{k\in\mathbb{N}} |x(k)| = \|x\|_1 \quad \text{for all } x \in X.$$

Now define a 'sequence of sequences' $(x_n)_{n\in\mathbb{N}} \subset \ell_1(\mathbb{N})$ by setting the kth term in the nth sequence to

$$x_n(k) = \begin{cases} \frac{1}{k}, & k \leq n \\ 0, & k > n. \end{cases}$$

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$, as for any $\epsilon > 0$ we can choose an $N = \lceil \frac{1}{\epsilon} \rceil \in \mathbb{N}$ such that for all $n > m > N$ we have

$$\begin{aligned}
\|x_n - x_m\|_\infty &= \|(\frac{1}{1}, \frac{1}{2}, \dots, \frac{1}{n-1}, \frac{1}{n}, 0, 0, \dots) - (\frac{1}{1}, \frac{1}{2}, \dots, \frac{1}{m-1}, \frac{1}{m}, 0, 0, \dots)\|_\infty \\
&= \|(0, 0, \dots, \frac{1}{m+1}, \frac{1}{m+2}, \dots, \frac{1}{n-1}, \frac{1}{n}, 0, 0, \dots)\|_\infty \\
&= \sup\{0, 0, \dots, \frac{1}{m+1}, \frac{1}{m+2}, \dots, \frac{1}{n-1}, \frac{1}{n}, 0, 0, \dots\} \\
&= \frac{1}{m+1} \leq \frac{1}{N} = \frac{1}{\lceil \frac{1}{\epsilon} \rceil} \leq \epsilon, \qquad (5)
\end{aligned}$$

where we assumed without loss of generality that $n > m$ to reduce unnecessary bookkeeping (alternatively you could populate the above calculation with a myriad of $\min\{n, m\}$ and $\max\{n, m\}$ expressions and keep track of the absolute value of the terms). But notice that x_n converges to $x = (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots)$ in $\|\cdot\|_\infty$, since

$$\|x_n - x\|_\infty = \dots = \frac{1}{n+1} \to 0 \quad \text{for } n \to \infty,$$

where we skipped the intermediate steps of the calculation because it would effectively just be a repetition of (5). Thus we have found a Cauchy sequence in $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ that converges to an $x \notin \ell_1(\mathbb{N})$ (note that $\sum_{k\in\mathbb{N}} \frac{1}{k} \not< \infty$) and hence the space is not complete.

It remains to be shown that the two norms are not equivalent. Assume by contradiction that they *are* equivalent. Then there exists some real number $C > 0$ such that

$$\|x\|_1 \leq C\|x\|_\infty \quad \text{for all } x \in X.$$

But we can construct sequences with arbitrarily large absolute sums without increasing the supremum of the terms – simply set the first $\lceil C \rceil + 1$ terms equal to 1, and zero otherwise:

$$x(k) = \begin{cases} 1, & k \leq \lceil C \rceil + 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{k\in\mathbb{N}} |x(k)| < \infty$ so $x \in l_1(\mathbb{N})$, and

$$\|x\|_1 = \sum_{k\in\mathbb{N}} |x(k)| = \sum_{k=1}^{\lceil C \rceil+1} 1 = \lceil C \rceil + 1 > C = C \cdot \sup\{1, \dots, 1, 0, 0, \dots\} = C\|x\|_\infty,$$

thus the norms are *not* equivalent.

Problem 2 Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), \|\cdot\|_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, \dots) \ : \ a, b \in \mathbb{C}\}.$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, 0, \dots) = a + b$, for all $a, b \in \mathbb{C}$.

(a) Show that f is bounded on $(M, \|\cdot\|_p)$ and compute $\|f\|$.

Observe that for any $m = (a, b, 0, 0, \dots) \in M$ we have

$$|f(m)|^p = |a + b|^p \leq (|a| + |b|)^p \leq 2^p \max\{|a|^p, |b|^p\} \leq 2^p(|a|^p + |b|^p) = 2^p\|m\|_p^p$$

which implies

$$|f(m)| \leq 2\|m\|_p.$$

So by the above calculations (heavily inspired by the proof Corollary 13.4 in Schilling) we get that f is bounded (see Proposition 1.10, Lecture 1).

Should be $q = \frac{p}{p-1}$.

To calculate the operator norm of f we start by assuming that $p > 1$. Let $q = \frac{p-1}{p}$ such that $\frac{1}{q} + \frac{1}{p} = 1$ and define $x = (1, 1, 0, 0, \dots)$, which is an element of $\ell_q(\mathbb{N})$ since $(|1|^q + |1|^q)^{\frac{1}{q}} = 2^{\frac{1}{q}} < \infty$. Then for any $m = (a, b, 0, 0, \dots) \in M$ we get by Holder's inequality (see for example Problem 5 from Homework Week 1) that

$$|f(m)| = |a + b| \leq |a| + |b| = \sum_{k=1}^{\infty} |m_k x_k| \leq \|m\|_p \|x\|_q = 2^{\frac{1}{q}} \|m\|_p = 2^{\frac{p}{p-1}} \|m\|_p,$$

and so we have $\|f\| = \sup\{|f(m)| : \|m\|_p = 1\} \leq 2^{\frac{p}{p-1}}$. On the other hand we note that the specific element $m' = (\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, 0, \dots) \in M$ has norm

These are diffent numbers.

$$\|m'\|_p = \left(\left(\frac{1}{2^{\frac{1}{p}}} \right)^p + \left(\frac{1}{2^{\frac{1}{p}}} \right)^p \right)^{\frac{1}{p}} = \left(\frac{1}{2} + \frac{1}{2} \right)^{\frac{1}{q}} = 1,$$

and we further have

$$|f(m')| = \left| \frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}} \right| = 2 \frac{1}{2^{\frac{1}{p}}} = 2^{1-\frac{1}{p}} = 2^{\frac{p-1}{p}},$$

and thus $\|f\| = \sup\{|f(m)| : \|m\|_p = 1\} \geq |f(m')| = 2^{\frac{p-1}{p}}$. Combining the two inequalities yields $\|f\| = 2^{\frac{p-1}{p}}$ in the case of $p > 1$.

If $p = 1$ then $|f(m)| = |a + b| \leq |a| + |b| = \|m\|_1$ for any $m \in M$ and thus we clearly have $\|f\| = \sup\{|f(m)| : \|m\|_1 = 1\} \leq 1$. On the other hand, if we take the element $m' = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots) \in M$, which has norm

$$\|m\|_1 = |\frac{1}{2}| + |\frac{1}{2}| = 1,$$

then we notice that

$$|f(m)| = |\frac{1}{2} + \frac{1}{2}| = 1.$$

This implies that $\|f\| = \sup\{|f(m)| : \|m\|_1 = 1\} \geq 1$. We conclude that $\|f\| = 1$, in which case the equality

$$\|f\| = 2^{\frac{p-1}{p}}$$

actually holds for all $1 \leq p < \infty$.

Some authors (e.g. Schilling) consider $p = 1, q = \infty$ to be conjugate numbers, in which case Holder's inequality covers both cases. Folland does not use this convention (see page 182), hence the case $p = 1$ was covered separately. ✓

(b) Show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Note that f is a linear functional on $M \subset (\ell_p(\mathbb{N}), \|\cdot\|_p)$, since for $\alpha, \beta \in \mathbb{C}, x, y \in M$ we have

$$f(\alpha x + \beta y) = (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) = \alpha(x_1 + x_2) + \beta(y_1 + y_2) = \alpha f(x) + \beta f(y).$$

By the complex Hahn-Banach extension theorem (or rather, Corollary 2.6, Lecture 2) it actually follows that there exists an $F \in (\ell_p(\mathbb{N}))^*$ such that F extends f and $\|F\| = \|f\|$. Our challenge is to show that such an extension is unique.

Since $p \in (1, \infty)$ we know from Problem 5 in Homework Week 1 that $(\ell_p(\mathbb{N}))^*$ and $\ell_q(\mathbb{N})$ are isometrically isomorphic for a conjugate number $q = \frac{p-1}{p}$ via the mapping $T : \ell_q(\mathbb{N}) \to (\ell_p(\mathbb{N}))^*$, $T(x) = F_x$, where $F_x : \ell_p(\mathbb{N}) \to \mathbb{C}$ is defined by

$$F_x(y) = \sum_{n=1}^{\infty} x_n y_n, \quad \text{for all } y \in \ell_p(\mathbb{N}),$$

for some $x \in \ell_q(\mathbb{N})$. So we can identify any element F_x in the dual with an element x in $\ell_q(\mathbb{N})$. Note that any possible extension F_x of f must satisfy $x_1 = x_2 = 1$, since

$$\begin{aligned} x_1 &= x_1 \cdot 1 + x_2 \cdot 0 + x_2 \cdot 0 + \ldots = F_x(1, 0, 0, \ldots) = f(1, 0, 0, \ldots) = 1 + 0 = 1, \\ x_2 &= x_1 \cdot 0 + x_2 \cdot 1 + x_2 \cdot 0 + \ldots = F_x(0, 1, 0, \ldots) = f(0, 1, 0, \ldots) = 0 + 1 = 1, \end{aligned}$$

where we used that $(1, 0, 0, \ldots), (0, 1, 0, 0, \ldots) \in M$ and that F_x must equal f on M. Clearly the functional $F_{x'}$ induced by $x' = (1, 1, 0, 0, \ldots)$ extends f, since for all elements $m = (a, b, 0, 0, \ldots) \in M$ we have

$$F_{x'}(m) = 1 \cdot a + 1 \cdot b + 0 \cdot 0 + 0 \cdot 0 + \ldots = a + b = f(m).$$

Since T is norm preserving (isometric) we can also show that $F_{x'}$ satisfies the desired norm property:

$$\|F_{x'}\| = \|Tx'\| = \|x'\|_q = (1^q + 1^q)^{\frac{1}{q}} = 2^{\frac{1}{q}} = 2^{\frac{p-1}{p}} = \|f\|. \tag{6}$$

And in fact, there can exist no *other* extension such that (6) holds: Assume by contradiction that there *does* exists some $x'' \neq x'$ such that $\|F_{x'}\| = \|f\| = 2^{\frac{p-1}{p}}$ and where $F_{x'}$ extends f. We showed before that the first two coordinates of x'' must be $x''_1 = x''_2 = 1$, and there must exist some $x''_n \neq 0$ for some $n > 2$ (otherwise $x'' = x'$). But then

$$\|F_{x''}\| = \|x''\|_q = (1^q + 1^q + |x''_3|^q + \ldots + |x''_n|^q + \ldots)^{\frac{1}{q}} > (2 + |x''_n|^q)^{\frac{1}{q}} > 2^{\frac{1}{q}} = \|f\|,$$

which is a contradiction. We conclude that the extension defined by $F_{x'}$ is unique.

(c) Show that if $p = 1$, then there are infinitely many linear functionals F on $\ell_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

For $p = 1$ we have $\|f\| = 1$, and we know from Problem 5 in Homework Week 1 that $\ell_1(\mathbb{N})$ and $\ell_\infty(\mathbb{N})$ are isometrically isomorphic, so by the same line of reasoning as in (b) we can uniquely identify any $F_x \in (\ell_1(\mathbb{N}))^*$ with some $x \in \ell_\infty(\mathbb{N})$ through an analogous mapping $T : \ell_\infty(\mathbb{N}) \to (\ell_1(\mathbb{N}))^*$. We again know from Corollary 2.6 that there exists at least one extension $F : \ell_1(\mathbb{N}) \to \mathbb{C}$ of f such that $\|F\| = \|f\|$. Our challenge is to find *infinitely* many extensions with the desired properties.

Define a sequence $(x^k)_{k \in \mathbb{N}} \subset \ell_\infty(\mathbb{N})$ by

$$x^k_n = \begin{cases} 1, & n \leq k+1 \\ 0, & n > k+1. \end{cases}$$

For example, $x^2 = (x_1^2, x_2^2, x_3^2, x_4^2, \ldots) = (1, 1, 1, 0, \ldots)$. Then the functionals $T(x^k) = F_{x^k}$ all extend f, since for any $m = (a, b, 0, 0, \ldots) \in M$ we have

$$F_{x^k}(m) = \sum_{n=1}^{k+1} m_n = a + b = f(m),$$

where we used that $x_1^k = x_2^k = 1$ all $k \in \mathbb{N}$. And the operator norm satisfies

$$\|F_{x^k}\| = \|Tx^k\| = \|x^k\|_\infty = \sup\{1, 1, \ldots, 1, 0, 0, \ldots\} = 1 = \|f\|.$$

for all $k \in \mathbb{N}$. So we have in fact found infinitely many extensions $(F_{x^k})_{k\in\mathbb{N}}$ of f such that $\|F_{x^k}\| = \|f\|$. ✓

Problem 3 Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $n \in \mathbb{N}$. Show that no linear map $F : X \to \mathbb{K}^n$ is injective. I need not contain $\mathbb{N}$.

Consider a Hamel basis $(e_i)_{i\in I}$ for X, and let $(e_1, \ldots, e_{n+1})$ be a finite subset of $(e_i)_{i\in I}$ consisting of $n+1$ basis vectors. Then $E := \operatorname{span}(e_1, \ldots, e_{n+1})$ is an $n+1$ dimensional vector space. Let the linear map $F_E : E \to \mathbb{K}^n$ denote the restriction of F to E. Then it follows from the rank-nullity theorem on linear maps in finite dimensional vector spaces that

$$\begin{aligned}\dim(\operatorname{Im}(F_E)) + \dim(\ker(F_E)) = \dim(E) = n + 1 \Leftrightarrow\\ \dim(\ker(F_E)) = n + 1 - \dim(\operatorname{Im}(F_E)),\end{aligned}$$

and because the image of F_E can *at most* be n dimensional (since $\mathbb{K}^n$ is n dimensional), we conclude that the kernel of F_E must *at least* be one-dimensional, which implies that F_E cannot be injective (more than one element must map to zero). If F_E is not injective, then clearly F cannot be injective either: simply take two points $x, y \in E, x \neq y$ such that $F_E(x) = F_E(y)$. Then $F(x) = F_E(x) = F_E(y) = F(y)$ for $x, y \in X, x \neq y$. ✓

(b) Let $n \in \mathbb{N}$ and let $f_1, \ldots, f_n \in X^*$. Show that

$$\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}.$$

Consider the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), \ldots, f_n(x))$. Since the f_j's are linear it follows that F itself is linear: For $\alpha, \beta \in \mathbb{K}$ and $x, x' \in X$ we have

$$\begin{aligned}F(\alpha x + \beta x') &= (f_1(\alpha x + \beta x'), \ldots, f_n(\alpha x + \beta x')) = (\alpha f_1(x) + \beta f_2(x'), \ldots, \alpha f_n(x) + \beta f_n(x'))\\ &= (\alpha f_1(x), \ldots, \alpha f_n(x)) + (\beta f_2(x'), \ldots, \beta f_n(x')) = \alpha F(x) + \beta F(x').\end{aligned}$$

It then follows from (a) that F cannot be injective, i.e. there exist $x, x' \in X, x \neq x'$ such that $F(x) = F(x')$, and by the linearity of F we get

$$F(x) - F(x') = 0 \Leftrightarrow F(x - x') = 0 \Leftrightarrow (f_1(x - x'), \ldots, f_n(x - x')) = (0, \ldots, 0),$$

i.e. we have found an element $x - x' \in X \setminus \{0\}$ such that $f_j(x - x') = 0$ for all $j = 1, \ldots, n$, and thus $0 \neq x - x' \in \bigcap_{j=1}^{n} \ker(f_j)$. We conclude that the shared kernel consists of more than singleton zero, which is what we set out to prove. ✓

(c) Let $x_1, \dots, x_n \in X$. Show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, \dots, n$.

If $x_j = 0$ then the inequality $\|y - x_j\| = \|y\| \geq 0 = \|x_j\|$ holds for all $y \in X$, so assume without loss of generality that the x_j's are non-zero for all $j = 1, \dots, n$. Then it follows from Theorem 2.7 (b) (Lecture 2) that there exist linear functionals $f_1, \dots, f_n \in X^*$ such that $\|f_j\| = 1$ and $f_j(x_j) = \|x_j\|$ for all $j = 1, \dots, n$. Part (b) implies the existence of an $x \in X \setminus \{0\}$ such that $f_j(x) = 0$ for all $j = 1, \dots, n$; now define an element $y \in X$ from such an x as follows:

$$y = \frac{x}{\|x\|}.$$

Clearly $\|y\| = \|x\|/\|x\| = 1$, and it satisfies the desired inequality:

$$\begin{aligned} \|y - x_j\| &= \|f_j\|\|y - x_j\| \geq \|f_j(y - x_j)\| \\ &= \|f_j(y) - f_j(x_j)\| = \|f_j(x_j)\| = \|\|x_j\|\| = \|x_j\|, \end{aligned} \tag{7}$$

for all $j = 1, \dots, n$. Let us unpack the details in (7): The first step used that $\|f_j\| = 1$. The following inequality applied equation (1.8) from Lecture 1, which states that for any $T \in \mathcal{L}(X, Y)$, the inequality $\|Tx\| \leq \|T\|\|x\|$ holds for all $x \in X$. The third step simply follows from the linearity of f_j, while the fourth step used that the kernel is a subspace (simply pull the constant out: $f_j(y) = \frac{1}{\|x\|} f_j(x) = 0$). Finally we used the fact that $f_j(x_j) = \|x_j\|$.

(It should be noted that there are two separate norms at play in (7) – one on X and one on $\mathbb{K}$ – but that it should be clear from the context which is which).

(d) Show that one cannot cover the unit sphere $S = \{x \in X \ : \ \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contain 0.

Let

$$C := \bigcup_{j=1}^{n} \bar{B}_j = \bigcup_{j=1}^{n} \bar{B}(x_j, r_j)$$

be an arbitrary finite union of closed balls in X such that none of the balls contain 0, where x_j are the centers and r_j are the radii of the balls. It follows from (c) that there exists a $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, \dots, n$. Since none of the balls contain 0 we have

$$\|x_j - 0\| > r_j, \quad \text{for all } j = 1, \dots, n,$$

and hence

$$\|y - x_j\| \geq \|x_j\| = \|x_j - 0\| > r_j, \quad \text{for all } j = 1, \dots, n,$$

which implies that $y \notin \bar{B}(x_j, r_j)$ for all $j = 1, \dots, n$, i.e. $y \notin C$. But clearly $y \in S$ since $\|y\| = 1$, so we have found an element in S that is *not* covered by C, and since C was chosen arbitrarily we conclude that we cannot cover S with such a family.

(e) Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

Let $r \in (0,1)$ and consider the *specific* open cover

$$O := \bigcup_{x \in S} B(x,r)$$

consisting of all open balls with centers in S with some shared radius $r < 1$. Clearly O covers S since any $x \in S$ is contained within the ball $B(x,r)$, and it is a union of open sets, so it is open. And by construction $0 \notin O$ since for any $x \in S$, $\|x - 0\| = \|x\| = 1 > r$, i.e. 0 is not contained in any of the balls. Now assume by contradiction that we can construct a finite subcover from the above open balls, i.e. that there exist $x_1, \ldots, x_n \in S$ such that

$$S \subset \bigcup_{j=1}^{n} B(x_j, r).$$

By taking the the closures of each $B(x_j, r)$ the inclusion still holds:

$$S \subset \bigcup_{j=1}^{n} B(x_j, r) \subset \bigcup_{j=1}^{n} \bar{B}(x_j, r).$$

and since r was *strictly* less than 1 we still have $0 \notin \bar{B}(x_j, r)$ for all $j = 1, \ldots, n$. Thus we have found a finite family of *closed* balls in X that cover the unit sphere S such that none of the balls contain 0, but this obviously contradicts our conclusion in (d), and hence our assumption that there exists a finite subcover cannot hold true. We conclude that S is non-compact.

The sphere $S = \{x \in X : \|x\| = 1\}$ is clearly a closed subset of the closed unit ball $\{x \in X : \|x\| \leq 1\}$. Every closed subset of a compact space is itself compact, but since we just showed that S is non-compact it follows that the closed unit ball cannot be compact.

Problem 4 Let $L_1([0,1], m)$ and $L_3([0,1], m)$ be the Lebesgue spaces on $[0,1]$. Recall that $L_3([0,1], m) \subsetneq L_1([0,1], m)$. For $n \geq 1$, define

$$E_n := \Big\{ f \in L_1([0,1], m) \ : \ \int_{[0,1]} |f|^3 dm \leq n \Big\}.$$

(a) Given $n \geq 1$, is the set $E_n \subset L_1([0,1], m)$ absorbing?

Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^{-\frac{1}{3}}, & x \in (0,1] \\ 0, & x = 0. \end{cases}$$

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This function is measurable and integrable:

$$\int_{[0,1]} |f| dm = \left[\frac{3}{2} x^{\frac{2}{3}} \right]_0^1 = \frac{3}{2} < \infty,$$

so $f \in L_1([0,1], m)$, but for any $t > 0$,

$$\int_{[0,1]} |tf|^3 dm = t^3 \int_{(0,1]} x^{-1} dm = \infty \not< n.$$

So we cannot find a constant $t > 0$ such that $tf \in E_n$, i.e. E_n is not absorbing.

Alternatively we could argue that if E_n *was* absorbing then any function $f \in L_1([0,1], m)$ could be scaled by some constant $t > 0$ such that $tf \in E_n \subset L_3([0,1], m)$. But since $L_3([0,1], m)$ is a subspace, we could simply multiply tf by t^{-1} to conclude that f itself must also be in $L_3([0,1], m)$, from which it follows that $L_1([0,1], m) = L_3([0,1], m)$ – which we know to be false – hence E_n cannot be absorbing.

(b) Show that E_n has empty interior in $L_1([0,1], m)$ for all $n \geq 1$.

Assume by contradiction that E_n has non-empty interior in $L_1([0,1], m)$, i.e. that there exists an element $f \in (E_n)^o$. The interior is of course open, so it must also contain an open ball $B(f, r)$ centered at f for a sufficiently small radius $r > 0$. Now consider some $f' \in L_1([0,1], m)$ and note that

$$\|f - (f + \frac{r}{2}\frac{f'}{\|f'\|})\| = \frac{r}{2} < r,$$

from which we can conclude that $f + \frac{r}{2}\frac{f'}{\|f'\|} \in B(f, r)$. Furthermore, since $B(f, r)$ is a subset of $(E_n)^o \subset E_n$, which itself (practically by definition) is a subset of $L_3([0,1], m)$, we have

$$f \in L_3([0,1], m), \text{ and } f + \frac{r}{2}\frac{f'}{\|f'\|} \in L_3([0,1], m). \tag{8}$$

But we know that $L_3([0,1], m)$ is a subspace, so by recognizing that f' can be written as a linear combination of the two $L_3([0,1], m)$ functions in (8) we see that

$$f' = \frac{2}{r}\|f'\|(f + \frac{r}{2}\frac{f'}{\|f'\|} - f) = \frac{2}{r}\|f\|(f + \frac{r}{2}\frac{f'}{\|f'\|}) - \frac{2}{r}\|f'\|f \in L_3([0,1], m), \tag{9}$$

thus we have shown that *any* given element $f' \in L_1([0,1], m)$ is also an element of $L_3([0,1], m)$ – however this contradicts the conclusion from HW2, which stated that $L_3([0,1], m)$ was a *proper* subset of $L_1([0,1], m)$. Thus our assumption must be wrong, i.e. the interior of E_n is empty.

(c) Show that E_n is closed in $L_1([0,1], m)$ for all $n \geq 1$.

We know from topology that in metric spaces (induced by our norm) it suffices to show that the set contains the limit points of convergent sequences (which, as an aside, follows from the fact that metric spaces are first countable).

Consider a sequence $(f_k)_{k\in\mathbb{N}} \subset E_n$ and assume that $f_k \overset{\|\cdot\|_1}{\to} f$ for some $f \in L_1([0,1], m)$, where we have emphasized that we are talking about L_1 convergence. We wish to show that $f \in E_n$. By Corollary 13.8 in *Measures, Integrals and Martingales* (René

L. Schilling), there exists a subsequence $(f_{k_l})_{l\in\mathbb{N}}$ such that $f_{k_l} \to f$ *pointwise* almost everywhere (and hence we also have $|f_{k_l}|^3 \to |f|^3$ pointwise almost everywhere). So by Fatou's Lemma (Theorem 9.11, René L. Schilling) we get

$$\int_{[0,1]} |f|^3 dm = \int_{[0,1]} \liminf_{l\to\infty} |f_{k_l}|^3 dm \leq \liminf_{l\to\infty} \int_{[0,1]} |f_{k_l}|^3 dm \leq \liminf_{l\to\infty} n = n,$$

where the last inequality used that each f_{k_l} was in E_n. We conclude that $f \in E_n$, and hence E_n is closed in $L_1([0,1], m)$.

(d) Conclude from (b) and (c) that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$.

We showed in (b) and (c) that $\text{Int}(E_n) = \emptyset$ and $\bar{E}_n = E_n$, respectively. Combining these we see that

$$\text{Int}(\bar{E}_n) = \text{Int}(E) = \emptyset,$$

which implies that every E_n is nowhere dense in $L_1([0,1], m)$ (see Definition 3.13, Lecture 3). Now consider some $f \in L_3([0,1], m)$, i.e. a measurable function such that

$$\alpha =: \int_{[0,1]} |f|^3 dm < \infty.$$

Since the integral is finite, we can of course find a natural number $N \in \mathbb{N}$ such that $N > \alpha$. Referring back to the definition of the E_n's to see that $f \in E_N$, i.e. any $L_3([0,1], m)$ function is in *some* E_n. We conclude that the Lebesgue space can be written as countable union of nowhere dense sets:

$$L_3([0,1], m) = \bigcup_{n=1}^{\infty} E_n,$$

which means that it is of first category in $L_1([0,1], m)$ (see Definition 3.12, Lecture 3).

Problem 5 Let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$. Let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

(a) Suppose that $x_n \to x$ in norm as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$ as $n \to \infty$?

To converge in norm means that $\|x_n - x\| \to 0$ as $n \to \infty$. Then a simple application of the reverse triangle inequality yields

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| \to 0,$$

which, of course, implies that $\|x_n\| \to \|x\|$ as $n \to \infty$.

(b) Suppose that $x_n \to x$ weakly as $x \to \infty$. Does it follow that $\|x_n\| \to \|x\|$ as $n \to \infty$?

Let us start by proving a well known property of Hilbert spaces. *Lemma: x_n converges weakly to x if and only if $\langle x_n, y\rangle$ converges to $\langle x, y\rangle$ for every $y \in \mathbb{H}$, where $\langle\cdot,\cdot\rangle$ is the associated inner product.*

Proof: A sequence is a special case of a net (see example in Lecture 6), so by Problem 2 in Homework Week 4 it follows that x_n converges weakly to x if and only if $f(x_n)$ converges to $f(x)$ for every $f \in H^*$. Recall that by the Riesz representation theorem (see Problem 1, Homework Week 2) we can identify each $f \in H^*$ with an element $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x \in H$ (and note in particular that the mapping $x \mapsto \langle x, y\rangle$ is itself an element of H^* since inner products are linear in the first argument). Thus it follows that x_n converges weakly to x if and only if $\langle x_n, y\rangle = f(x_n) \to f(x) = \langle x, y\rangle$ for all $y \in H$, which concludes the proof.

Given that H is separable we can take a *countable* orthonormal basis $\{e_n\}$ for H and consider the sequence $(x_n)_{n\in\mathbb{N}} = (e_n)_{n\in\mathbb{N}}$ (in any particular order). By Bessel's inequality (see Theorem 26.19, Schilling) it holds that

$$\sum_{n=1}^{\infty} |\overline{\langle e_n, y\rangle}|^2 = \sum_{n=1}^{\infty} |\langle y, e_n\rangle|^2 \leq \|y\|^2 \tag{10}$$

for all $y \in H$. Since the sum in (10) is bounded (i.e. convergent) it follows that its terms $|\overline{\langle e_n, y\rangle}|^2$ converge to zero, which of course implies that $\langle e_n, y\rangle \to 0$. But clearly the null element $0 \in H$ satisfies $\langle 0, y\rangle = 0$ for all $y \in H$, so by the lemma above it follows that e_n converges weakly to 0. However, since the basis was chosen to be *orthonormal* we get

$$\|e_n\| = 1 \underset{n\to\infty}{\to} 1 \neq \|0\|,$$

so we have found a counterexample where a sequence x_n converges weakly to $x = 0$ but where $\|x_n\| \nrightarrow \|x\| = \|0\|$.

(c) Suppose that $\|x_n\| \leq 1$ for all $n \geq 1$, and that $x_n \to x$ weakly as $n \to \infty$. Does it follow that $\|x\| \leq 1$?

If $x = 0$ then trivially $\|x\| \leq 1$, so assume that $x \neq 0$. Then it follows from Theorem 2.7 (b) (Lecture 2) that there exists $f \in H^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$, and by Problem 2 (a) in Homework Week 4 we know that $f(x_n)$ converges to $f(x)$, which implies that $\|f(x_n)\|$ converges to $\|f(x)\|$ (by a similar line of argument as in (a)). Therefore

$$\begin{aligned}\|x\| = \|f(x)\| &= \lim_{n\to\infty} \|f(x_n)\| = \liminf_{n\to\infty} \|f(x_n)\| \\ &\leq \liminf_{n\to\infty} \|f\|\|x_n\| = \liminf_{n\to\infty} \|x_n\| \leq \liminf_{n\to\infty} 1 = 1,\end{aligned}$$

where we used the classic $\|Tx\| \leq \|T\|\|x\|$ inequality for linear operators, and where we used *liminf* instead of *lim* to avoid having to worry about the limit of $\|x_n\|$.

We conclude that it indeed follows that $\|x\| \leq 1$.