FunkAn Mandatory 2

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Problem 1

(a) We begin by showing that $\|f_N\| = 1$ for every $N \geq 1$. Since $(e_n)_{n\in\mathbb{N}}$ is an orthonormal set, then for each $N \geq 1$ we get by The Pythagorean Theorem (5.23 in Folland):

$$\|f_N\|^2 = \left\| \frac{1}{N} \sum_{n=1}^{N^2} e_n \right\|^2 = \frac{1}{N^2} \left\| \sum_{n=1}^{N^2} e_n \right\|^2 = \frac{1}{N^2} \sum_{n=1}^{N^2} \|e_n\|^2 = \frac{1}{N^2} \sum_{n=1}^{N^2} 1 = \frac{N^2}{N^2} = 1.$$

Hence $\|f_N\| = 1$ for every $N \geq 1$.

Next, we show that $f_N \to 0$ weakly, as $N \to \infty$. Let $g \in H^*$. By Problem 1 HW2, there exists $y \in H$ such that $g(x) = \langle x, y \rangle$ for all $x \in H$. If we can show that $g(f_N) \to g(0) = 0$ in norm, as $N \to \infty$, then the conclusion follows from Problem 2(a) HW4. First, let $\varepsilon > 0$ be given. By Bessel's inequality

$$\sum_{n=1}^{\infty} |\langle y, e_n \rangle|^2 \leq \|y\|^2,$$

so there exists $M \geq 1$ such that

$$\sum_{n=M}^{\infty} |\langle y, e_n \rangle|^2 < \frac{\varepsilon^2}{4}.$$

By Thm 5.27 in Folland, we can write $y = \sum_{n=1}^{\infty} \langle y, e_n \rangle e_n$. Now consider $|g(f_N)| = |\langle f_N, y \rangle|$. We have

$$\begin{aligned} |\langle f_N, y \rangle| &= \left| \langle f_N, \sum_{n=1}^{\infty} \langle y, e_n \rangle e_n \rangle \right| \\ &= \left| \langle f_N, \sum_{n=1}^{M-1} \langle y, e_n \rangle e_n \rangle + \langle f_N, \sum_{n=M}^{\infty} \langle y, e_n \rangle e_n \rangle \right| \\ &\leq \left| \langle f_N, \sum_{n=1}^{M-1} \langle y, e_n \rangle e_n \rangle \right| + \left| \langle f_N, \sum_{n=M}^{\infty} \langle y, e_n \rangle e_n \rangle \right| \end{aligned}$$

Consider the second term of the sum. By the Cauchy-Schwarz' inequality, the above proven fact, that

$\|f_N\| = 1$, and the properties of the orthonormal basis $(e_n)_{n\geq 1}$, we get:

$$\begin{aligned}
\left|\langle f_N, \sum_{n=M}^{\infty}\langle y,e_n\rangle e_n\rangle\right| &\leq \|f_N\|\left\|\sum_{n=M}^{\infty}\langle y,e_n\rangle e_n\right\| \\
&= \left\|\sum_{n=M}^{\infty}\langle y,e_n\rangle e_n\right\| \\
&= \langle \sum_{n=M}^{\infty}\langle y,e_n\rangle e_n, \sum_{n=M}^{\infty}\langle y,e_n\rangle e_n\rangle^{1/2} \\
&= \left(\sum_{n=M}^{\infty}\sum_{i=M}^{\infty}|\langle y,e_n\rangle||\langle y,e_i\rangle|\langle e_n,e_i\rangle\right)^{1/2} \\
&= \left(\sum_{n=M}^{\infty}|\langle y,e_n\rangle|^2\right)^{1/2} < (\frac{\varepsilon^2}{4})^{1/2} = \frac{\varepsilon}{2}
\end{aligned}$$

Should be $\langle y,e_n\rangle\overline{\langle y,e_i\rangle}\langle e_n,e_i\rangle$ $\rightarrow$

Now, consider the other term in sum:

$$\begin{aligned}
\left|\langle f_N, \sum_{n=1}^{M-1}\langle y,e_n\rangle e_n\rangle\right| &= \left|\sum_{n=1}^{M-1}|\langle y,e_n\rangle|\langle f_N,e_n\rangle\right| \\
&= \left|\sum_{n=1}^{M-1}|\langle y,e_n\rangle|\langle\frac{1}{N}\sum_{i=1}^{N^2}e_i,e_n\rangle\right| \\
&= \left|\sum_{n=1}^{M-1}\sum_{i=1}^{N^2}\frac{1}{N}|\langle y,e_n\rangle|\langle e_i,e_n\rangle\right| \\
&\leq \sum_{n=1}^{M-1}\sum_{i=1}^{N^2}\frac{1}{N}|\langle y,e_n\rangle||\langle e_i,e_n\rangle|
\end{aligned}$$

$\leftarrow \overline{\langle y, e_n\rangle}$

For all $N \in \mathbb{N}$ with $N^2 > M-1$, we have

$$\sum_{n=1}^{M-1}\sum_{i=1}^{N^2}\frac{1}{N}|\langle y,e_n\rangle||\langle e_i,e_n\rangle| = \sum_{n=1}^{M-1}\frac{1}{N}|\langle y,e_n\rangle|.$$

Set $c = \sum_{n=1}^{M-1}|\langle y,e_n\rangle|$ and choose $N' \in \mathbb{N}$ such that $(N')^2 > M-1$ and such that $c/N < \varepsilon/2$ for all $N \geq N'$. Then we have for all $N > N'$

$$\begin{aligned}
|\langle f_N, y\rangle| &\leq \left|\langle f_N, \sum_{n=1}^{M-1}\langle y,e_n\rangle e_n\rangle\right| + \left|\langle f_N, \sum_{n=M}^{\infty}\langle y,e_n\rangle e_n\rangle\right| \\
&\leq \frac{c}{N} + \left|\langle f_N, \sum_{n=M}^{\infty}\langle y,e_n\rangle e_n\rangle\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

This proves, that $g(f_N) \to 0$ in norm, as $N \to \infty$. And thus, $f_N \to 0$ weakly, as $N \to \infty$.

(b) First, let's show that $K \subset \overline{B}_H(0,1)$. Let $x \in \text{co}\{f_N : N \geq 1\}$, i.e. there exists $n \in \mathbb{N}$, f_{N_i} and $\alpha_i > 0$, for $i = 1,\dots,n$, with $\sum_{i=1}^{n}\alpha_i = 1$ such that

$$x = \sum_{i=1}^{n}\alpha_i f_{N_i}.$$

Then the norm

$$\|x\| = \left\|\sum_{i=1}^{n}\alpha_i f_{N_i}\right\| \leq \sum_{i=1}^{n}\alpha_i\|f_{N_i}\| = \sum_{i=1}^{n}\alpha_i = 1,$$

where we used that $\|f_N\| = 1$ for all $N \geq 1$ (from part (a)). Hence $\text{co}\{f_N : N \geq 1\} \subset \overline{B}_H(0,1)$, so also the norm closure is contained in the closed unit ball, i.e.

$$K = \overline{\text{co}\{f_N : n \geq 1\}}^{\|\cdot\|} \subset \overline{B}_H(0,1).$$

Now, by definition, the set $\text{co}\{f_N : N \geq 1\}$ is convex. So by Thm. 5.7,

$$K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w},$$

where τ_w denotes the weak topology on H. Every Hilbert space is reflexive by Prop. 2.10, so by Thm 6.3, the closed unit ball $\overline{B}_H(0,1)$ is compact in the weak topology on H. Hence K is a weakly closed subset of a weakly compact set, which implies that K itself is weakly compact.

Furthermore, since K is weakly closed, K contains all its limit points with respect to the weak topology. And since $f_N \in K$ for all $N \geq 1$ and $f_N \to 0$ weakly, as $N \to \infty$ (by part (a)), also the limit point $0 \in K$.

(c) We first show that $0 \in K$ is an extreme point.
Let $m \in \mathbb{N}$ and $N \in \mathbb{N}$. Then

$$\langle e_m, f_N\rangle = \langle e_m, \frac{1}{N}\sum_{n=1}^{N^2} e_n\rangle = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_m, e_n\rangle \geq 0,$$

by the properties of the orthonormal basis $(e_n)_{n\geq 1}$.

Next, let $x \in \text{co}\{f_N : N \geq 1\}$, i.e. there exists $n \in \mathbb{N}$, f_{N_i} and $\alpha_i > 0$, for $i = 1, \dots, n$, with $\sum_{i=1}^n \alpha_i = 1$ such that

$$x = \sum_{i=1}^n \alpha_i f_{N_i}.$$

Then

$$\langle x, e_m\rangle = \langle \sum_{i=1}^n \alpha_i f_{N_i}, e_m\rangle = \sum_{i=1}^n \alpha_i\langle f_{N_i}, e_m\rangle \geq 0.$$

Finally, let $y \in K = \overline{\text{co}\{f_N : N \geq 1\}}$, i.e. there is a sequence $(y_n)_{n\geq 1} \subset \text{co}\{f_N : N \geq 1\}$ such that $\|y_n - y\| \to 0$ as $n \to \infty$. By Prop. 5.21 in Folland, we then have $\langle y_n, e_m\rangle \to \langle y, e_m\rangle$ as $n \to \infty$. Since $\langle y_n, e_m\rangle \geq 0$ for every $n \in \mathbb{N}$, we must have that also the limit $\langle y, e_m\rangle \geq 0$.

Suppose now that $0 = \alpha x_1 + (1-\alpha)x_2$ for some $x_1, x_2 \in K$ and $0 < \alpha < 1$. Let $m \in \mathbb{N}$. Then

$$0 = \langle 0, e_m\rangle = \langle \alpha x_1 + (1-\alpha)x_2, e_m\rangle = \alpha\langle x_1, e_m\rangle + (1-\alpha)\langle x_2, e_m\rangle.$$

Since we have proved above, that $\langle x_1, e_m\rangle \geq 0$ and $\langle x_2, e_m\rangle \geq 0$, this equality implies that $\langle x_1, e_m\rangle = \langle x_2, e_m\rangle = 0$. I.e. both x_1 and x_2 are orthogonal to every element in the basis $(e_n)_{n\geq 1}$, since $m \in \mathbb{N}$ was arbitrary, but that can only happen, if $x_1 = x_2 = 0$. This proves that 0 is an extreme point in K.

We now prove, that f_N is an extreme point in K for every $N \geq 1$. First, let $M, N \in \mathbb{N}$ and assume that $M \leq N$. Then

$$\langle f_N, f_M\rangle = \langle \frac{1}{N}\sum_{n=1}^{N^2} e_n, \frac{1}{M}\sum_{m=1}^{M^2} e_m\rangle = \frac{1}{NM}\sum_{n=1}^{N^2}\sum_{m=1}^{M^2}\langle e_n, e_m\rangle = \frac{1}{NM}M^2 = \frac{M}{N},$$

since $\langle e_n, e_m\rangle = \delta_{n,m}$. Since M, N are positive numbers and $M \leq N$, this proves that $\langle f_N, f_M\rangle \in (0,1]$. This implies that $\langle f_N, f_M\rangle = \overline{\langle f_M, f_N\rangle} = \langle f_M, f_N\rangle$, so the case when $N \leq M$ is automatically covered.

Let $x \in \text{co}\{f_N : N \geq 1\}$, i.e. there exists $n \in \mathbb{N}$, f_{N_i} and $\alpha_i > 0$, for $i = 1, \dots, n$, with $\sum_{i=1}^n \alpha_i = 1$ such that

$$x = \sum_{i=1}^n \alpha_i f_{N_i}.$$

Then, for any $M \geq 1$,

$$\langle x, f_M\rangle = \langle \sum_{i=1}^n \alpha_i f_{N_i}, f_M\rangle = \sum_{i=1}^n \alpha_i\langle f_{N_i}, f_M\rangle \in (0,1].$$

This follows by the above, since $\langle f_{N_i}, f_M\rangle \in (0,1]$.

Let $y \in K = \overline{\text{co}}\{f_N : N \geq 1\}$, i.e. there is a sequence $(y_n)_{n\geq 1} \subset \text{co}\{f_N : N \geq 1\}$ such that $\|y_n - y\| \to 0$ as $n \to \infty$. By Prop. 5.21 in Folland, we then have $\langle y_n, f_M\rangle \to \langle y, f_M\rangle$, as $n \to \infty$. Since $\langle y_n, f_M\rangle \in (0,1]$ for all $n \in \mathbb{N}$, then $\langle y, f_M\rangle \in [0,1]$.

Now, let $N \geq 1$ and assume that f_N can be written as a convex combination of two elements $x_1, x_2 \in K$, i.e. $f_N = \alpha x_1 + (1-\alpha)x_2$ for some $0 < \alpha < 1$. Then by part (a),

$$1 = \langle f_N, f_N\rangle = \langle \alpha x_1 + (1-\alpha)x_2, f_N\rangle = \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle.$$

We proved above, that $\langle x_1, f_n\rangle, \langle x_2, f_N\rangle \in [0,1]$, and since $0 < \alpha < 1$, the equation implies that $\langle x_1, f_n\rangle = \langle x_2, f_n\rangle = 1$. By Cauchy-Schwartz and the fact from part (b) that $K \subset \overline{B}_H(0,1)$, we see that

$$1 = |\langle x_1, f_N\rangle|^2 \leq \|x_1\|\|f_N\| = \|x_1\| \leq 1.$$

So we have equality in Cauchy-Schwartz, which implies that $f_N = \gamma x_1$ for some $\gamma \in \mathbb{C}$. Hence

$$1 = \langle f_N, f_N\rangle = \langle \gamma x_1, f_N\rangle = \gamma\langle x_1, f_N\rangle = \gamma,$$

which proves that $x_1 = f_N$. The exact same argument proves that $x_2 = f_N = x_1$. Thus we have proved that f_N is an extreme point in K for every $N \geq 1$. ✓

(d) We argued in part (b) that $K = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w}$ is weakly compact, and since it is also convex by construction, the theorem of Milman (Thm. 7.9) implies that $\text{Ext}(K) \subset \overline{\{f_N : N \geq 1\}}^{\tau_w}$. The weak closure of $\{f_N : N \geq 1\}$ is exactly the set itself together with all its limit points (w.r.t. the weak topology). Furthermore, every sequence in $\{f_N : N \geq 1\}$ is a subsequence of $(f_N)_{N\in\mathbb{N}}$, and we proved in part (a) that $f_N \to 0$ weakly, as $N \to \infty$, hence also every subsequence must converge weakly to 0. So 0 is the only limit point, hence

No, what about $f_1, f_1, f_1, \dots$?

$$\text{Ext}(K) \subset \overline{\{f_N : N \geq 1\}}^{\tau_w} = \{f_N : N \geq 1\} \cup \{0\}.$$

Only nontrivial limit point.

In part (c) we proved the opposite inclusion, namely that

$$\{f_N : N \geq 1\} \cup \{0\} \subset \text{Ext}(K).$$

(✓)

Hence the extreme points of K are precisely 0 and f_N for every $N \geq 1$.

Problem 2

(a) Assume that $x_n \to x$ weakly, as $n \to \infty$. By Problem 2(a) HW4, this happens if and only if $f(x_n) \to f(x)$ in norm, as $n \to \infty$, for every $f \in X^*$. Now, let $g \in Y^*$. Then the composition $g \circ T : X \to \mathbb{K}$ is linear and bounded, since both g and T are assumed to be. So $g \circ T \in X^*$. Thus, this means that $g \circ T(x_n) \to g \circ T(x)$ in norm (in the scalar field), as $n \to \infty$. Using Problem 2(a) HW4 again, we see that, since $g \in Y^*$ was arbitrary, it follows that $T(x_n) \to T(x)$ weakly, as $n \to \infty$. ✓

(b) First, we prove that a sequence $(y_n)_{n\geq 1}$ in any normed space Z converges to $y \in Z$ if every subsequence has a further subsequence converging to y. Assume by contrapostion that the sequence $(y_n)_{n\geq 1}$ does not converge to y. Then there exists $\varepsilon > 0$ such that for all $k \geq 1$ there is an $n_k > k$ such that $\|y_{n_k} - y\| \geq \varepsilon$. If not, then we would have that $\|y_{n_k} - y\| < \varepsilon$ for all $k \geq n_k$, and y_n would converge to y. But this means that the subsequence $(y_{n_k})_{k\geq 1}$ cannot have any converging further subsequences. This proves the lemma.

Assume now that $(x_n)_{n\geq 1}$ is a sequence in X such that $x_n \to x$ weakly, as $n \to \infty$. By Problem 2(b) HW4, $(x_n)_{n\geq 1}$ is a bounded sequence. Let $(x_{n_k})_{k\geq 1}$ be a subsequence. This is also necessarily bounded, since

$$\sup\{\|x_{n_k}\| : k \geq 1\} \leq \sup\{\|x_n\| : n \geq 1\} < \infty.$$

By Prop. 8.2, since $T \in \mathcal{K}(X,Y)$, there is a further subsequence $(x_{n_{k_l}})_{l\geq 1}$ such that $(Tx_{n_{k_l}})_{l\geq 1}$ converges in norm in Y. By part (a), $Tx_n \to Tx$ weakly, as $n \to \infty$, so also $Tx_{n_{k_l}} \to Tx$ weakly. Convergence in norm implies weak converge (since the weak topology is contained in the topology induced by the norm), and the limit is unique, hence we must have that $Tx_{n_{k_l}} \to Tx$ in norm, as $l \to \infty$. Hence every subsequence $(Tx_{n_k})_{k\geq 1}$ of $(Tx_n)_{n\geq 1}$ has a subsequence converging to $Tx \in Y$. So by the lemma, $\|Tx_n - Tx\| \to 0$, as $n \to \infty$. ✓

(c) Assume by contraposition that $T \in \mathcal{L}(H,Y)$ is not compact. By Prop. 8.2, there exists a bounded sequence $(y_n)_{n\geq 1}$ in H with the property that for every subsequence $(y_{n_k})_{k\geq 1}$, the sequence $(Ty_{n_k})_{k\geq 1}$ in Y is not convergent. Set $x_n = y_n/\|y_n\|$ for every $n \geq 1$. Then the sequence $(x_n)_{n\geq 1}$ is contained in $\overline{B}_H(0,1)$ and satisfies the same property as $(y_n)_{n\geq 1}$.

On the other hand, H is a reflexive Banach space, since every Hilbert space is reflexive by Prop. 2.10, so by Thm 6.3, the closed unit ball $\overline{B}_H(0,1)$ is compact in the weak topology on H. The sequence $(x_n)_{n\geq 1}$ was chosen to be inside $\overline{B}_H(0,1)$, so compactness implies, that it has a weakly convergent subsequence $(x_{n_k})_{k\geq 1}$. But by the above, the sequence $(Tx_{n_k})_{k\geq 1}$ in Y is not convergent. Hence we have found a sequence $(x_{n_k})_{k\geq 1}$ in H converging weakly to some $x \in H$, but which does not converge to Tx in norm. Thus, it holds that if $T \in \mathcal{L}(H,Y)$ satisfies that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$.

Generally, it has a weakly convergent subnet, not a subseq. (✓)

(d) Let $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$. We want to show that T is compact by using part (c). This we may do, since $l_2(\mathbb{N})$ is an infinite dimensional separable Hilbert space (by Problem 4(a) HW4). Let $(x_n)_{n\geq 1}$ be a sequence in $l_2(\mathbb{N})$ converging weakly to $x \in l_2(\mathbb{N})$. Since T is linear and bounded, part (a) implies that $Tx_n \to Tx$ weakly in $l_1(\mathbb{N})$, as $n \to \infty$. By Remark 5.3, a sequence in $l_1(\mathbb{N})$ converges weakly if and only if it converges in norm. Hence we also have that $Tx_n \to Tx$ in norm, as $n \to \infty$. Now, by part (c), this exactly proves that T is compact. ✓

(e) Assume by contradiction that $T \in \mathcal{K}(X,Y)$ is surjective. By the Open Mapping Theorem (Thm. 3.15), T is an open map. Hence the image of the open unit ball $B_X(0,1)$ under T is open in Y. This means that for some $c > 0$, $B_Y(0,c) \subset T(B_X(0,1))$. Then also

$$\overline{B}_Y(0,1) \subset \overline{T(B_X(0,1))}.$$

Since T is compact, the set $\overline{T(B_X(0,1))}$ is compact by definition, so since $\overline{B}_Y(0,1)$ is closed, it must also be compact. But this gives rise to a contradiction, because we proved in Problem 3 of Mandatory 1, that the closed unit ball in an infinite dimensional Banach space is not compact, so in particular the closed ball scaled by a factor c cannot be compact. In this problem we assumed Y to be infinite dimensional, so we conclude that T is not surjective. ✓

(f) First, let's see that M is self-adjoint. Using that $t \in [0,1] \subset \mathbb{R}$, we have for $f, g \in H = L_2([0,1], m)$:

$$\langle Mf, g\rangle = \int_{[0,1]} (Mf)(t)\overline{g(t)}dt = \int_{[0,1]} tf(t)\overline{g(t)}dt = \int_{[0,1]} f(t)\overline{tg(t)}dt = \int_{[0,1]} f(t)\overline{(Mg)(t)}dt = \langle f, Mg\rangle.$$

✓

So $M = M^*$, by definition of the Hilbert space adjoint.

But M is not a compact operator. Because if it was, the Spectral Theorem for self-adjoint compact operators on a separable, infinite dimensional Hilbert space (Thm. 10.1) would apply ($L_2([0,1], m)$ is infinite dimensional and separable by Problem 4(a) HW4). The Spectral theorem would then imply, that there is an orthonormal basis for $L_2([0,1], m)$ consisting of eigenvectors for M. But we proved in Problem 3(a) HW6, that M has no eigenvalues. Hence M cannot be compact. ✓

Problem 3

(a) We begin by showing that $K \in L_2([0,1]^2, m^2)$, where m^2 is the product Lebesgue measure on $[0,1]$. First, fix $t \in [0,1]$. Then

$$\int_{[0,1]} |K(s,t)|^2 \, dm(s) \leq \int_{[0,1]} 1 \, dm(s) = 1$$

since $K(s,t) \leq 1$ for all $s,t \in [0,1]$. So by Tonelli's theorem

$$\begin{aligned}\int_{[0,1]^2} |K(s,t)|^2 \, dm(s,t) &= \int_{[0,1]} \int_{[0,1]} |K(s,t)|^2 \, dm(s) \, dm(t) \\ &\leq \int_{[0,1]} 1 \, dm(t) \\ &= 1 < \infty\end{aligned}$$

✓

Hence $K \in L_2([0,1]^2, m^2)$. T is now recognized as the associated kernel operator of K, so by Prop. 9.12, T is Hilbert-Schmidt, and this implies, by Prop. 9.11, that T is compact.

$T = T_{\tilde{k}}$, $\tilde{k}(s,t) = k(t,s)$ so only if you show $k(s,t) = k(t,s)$

(b) We want to prove that $T = T^*$. Let $f, g \in H$ and consider $\langle Tf, g\rangle$.

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} (Tf)(s)\overline{g(s)}\, dm(s) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(s,t) f(t)\, dm(t) \right) \overline{g(s)}\, dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t)\overline{g(s)} f(t)\, dm(t)\, dm(s) \\
&\overset{(*)}{=} \int_{[0,1]} \int_{[0,1]} K(s,t)\overline{g(s)} f(t)\, dm(s)\, dm(t) \\
&= \int_{[0,1]} f(t) \left(\int_{[0,1]} K(s,t)\overline{g(s)}\, dm(s) \right) dm(t) \\
&\overset{(**)}{=} \int_{[0,1]} f(t) \overline{\left(\int_{[0,1]} K(s,t) g(s)\, dm(s) \right)}\, dm(t) \\
&= \int_{[0,1]} f(t)\overline{(Tg)(t)}\, dm(t) \\
&= \langle f, Tg\rangle
\end{aligned}$$

only if $k(s,t) = k(t,s)$

At $(**)$, we used that $K(s,t) \in \mathbb{R}$ for all $(s,t) \in [0,1]$. And at $(*)$, we used Fubini's theorem. We may do this, because $f, g \in L_2([0,1], m)$, so also $\overline{g} \in L_2([0,1], m)$. In part (a) we proved that also $K \in L_2([0,1]^2, m^2)$. By Problem 2(b) HW2, $L_2([0,1], m) \subset L_1([0,1], m)$, so $f, \overline{g}$ and K as a function of both one and two variables are all integrable, hence also the product $Kf\overline{g}$. Thus, by definition of the Hilbert space adjoint, we have proved that T is self-adjoint.

Product of integrable is not necessarily integrable. e.g. $\frac{1}{\sqrt{x}}$

Measurability in product top. is not obvious for f and g.

(c) Let $f \in H$ and $s \in [0,1]$. Then

$$\begin{aligned}
(Tf)(s) &= \int_{[0,1]} K(s,t) f(t)\, dm(t) \\
&= \int_{[0,s]} (1-s)t f(t)\, dm(t) + \int_{[0,1]} (1-t)s f(t)\, dm(t) \\
&= (1-s) \int_{[0,s]} t f(t)\, dm(t) + s \int_{[s,1]} (1-t) f(t)\, dm(t)
\end{aligned}$$

Note that the functions $t \mapsto tf(t)$ and $t \mapsto (1-t)f(t)$ are integrable on $[0,1]$, since $|tf(t)| \le |f(t)|$ and $|(1-t)f(t)| \le |f(t)|$ for all $t \in [0,1]$ and $f \in L_1([0,1], m)$ (since $L_2([0,1], m) \subset L_1([0,1], m)$ by Problem 2(b) HW2). This ensures that the functions $s \mapsto \int_{[0,s]} tf(t)\, dm(t)$ and $s \mapsto \int_{[s,1]} (1-t) f(t)\, dm(t)$ are continuous. why? Hence Tf is composed of continuous functions on $[0,1]$, and is itself continuous.

Furthermore, $(Tf)(0) = (Tf)(1) = 0$ as seen in the following computation:

$$\begin{aligned}
(Tf)(0) &= (1-0) \int_{[0,0]} t f(t)\, dm(t) + 0 \int_{[0,1]} (1-t) f(t)\, dm(t) = 0 \\
(Tf)(1) &= (1-1) \int_{[0,1]} t f(t)\, dm(t) + \int_{[1,1]} (1-t) f(t)\, dm(t) = 0
\end{aligned}$$

Problem 4

(a) Notice first that the map $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(x) = e^{-x^2/2}$ is smooth, i.e. belongs to $C^\infty(\mathbb{R})$, and that $\varphi \in L_1(\mathbb{R})$, since

$$\int_{\mathbb{R}} \varphi(x) dx = \int_{\mathbb{R}} e^{-x^2/2} dx = \sqrt{2\pi}.$$

Next, notice that for any multi-index, in this case any $\alpha \in \mathbb{N}_0$, we have

$$\partial^\alpha e^{-x^2/2} = \frac{\partial^\alpha}{\partial x^\alpha} e^{-x^2/2} = \mathrm{Pol}_\alpha(x) e^{-x^2/2},$$

where $\mathrm{Pol}_\alpha(x)$ is a polynomial in x of degree α. So for every other index $\beta \in \mathbb{N}_0$, we have

$$x^\beta \partial^\alpha e^{-x^2/2} = \mathrm{Pol}_{\alpha+\beta}(x) e^{-x^2/2}.$$

By repetetive use of l'Hôpital's rule, we get that

$$x^\beta \partial^\alpha e^{-x^2/2} \to 0, \qquad \text{as } \ x \to \infty.$$

This shows, that φ is a Schwartz function. By Problem 1(a) HW7, then also $g_k(x) = x^k \varphi(x)$ is Schwartz, for every integer $k \geq 0$.

We want to compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$. Observe first, that for $k = 0$, we have $g_0(x) = e^{-x^2/2} = \varphi(x)$. Then Prop. 11.4 tells us, that $\hat{g_0}(\xi) = e^{-\xi^2/2} = g_0(\xi)$, $\xi \in \hat{\mathbb{R}}$. So $\mathcal{F}(g_0) = g_0$.

Note that $\varphi \in L_1(\mathbb{R})$ and $g_k \in L_1(\mathbb{R})$; indeed, we proved above that $g_k \in \mathcal{S}(\mathbb{R})$ for every $k \geq 0$ and, by Problem 1(c) HW7, $\mathcal{S}(\mathbb{R}) \subset L_1(\mathbb{R})$. Now we can use Prop. 11.13(d) to compute $\mathcal{F}(g_k)$ for $k = 1, 2, 3$.

For $\xi \in \mathbb{R}$,

$$\begin{aligned}
\hat{g_1}(\xi) &= (x\varphi(x))\hat{}(\xi) \\
&= i\left(\frac{\partial}{\partial x}\hat{\varphi}(x)\right)(\xi) \\
&= i\left(\frac{\partial}{\partial x}\varphi(x)\right)(\xi) \\
&= i\left(\frac{\partial}{\partial x}e^{-x^2/2}\right)(\xi) \\
&= i(-xe^{-x^2/2})(\xi) \\
&= -i\xi e^{-\xi^2/2}
\end{aligned}$$

$$\begin{aligned}
\hat{g_2}(\xi) &= (x^2\varphi(x))\hat{}(\xi) \\
&= i^2\left(\frac{\partial^2}{\partial x^2}\hat{\varphi}(x)\right)(\xi) \\
&= -\left(\frac{\partial}{\partial x}(-xe^{-x^2/2})\right)(\xi) \\
&= -(-e^{-\xi^2/2} + (-\xi)(-\xi)e^{-\xi^2/2}) \\
&= (1-\xi^2)e^{-\xi^2/2}
\end{aligned}$$

$$\begin{aligned}
\hat{g_3}(\xi) &= (x^3\varphi(x))\hat{}(\xi) \\
&= i^3\left(\frac{\partial^3}{\partial x^3}\hat{\varphi}(x)\right)(\xi) \\
&= -i\left(\frac{\partial}{\partial x}(x^2-1)e^{-x^2/2}\right)(\xi) \\
&= -i(2\xi e^{-\xi^2/2} + (\xi^2-1)(-\xi)e^{-\xi^2/2}) \\
&= i(\xi^3 - 3\xi)e^{-\xi^2/2}
\end{aligned}$$

To conclude, we have shown that for $\xi \in \mathbb{R}$:

$$\begin{aligned}
\hat{g}_0(\xi) &= e^{-\xi^2/2} \\
\hat{g}_1(\xi) &= -i\xi e^{-\xi^2/2} \\
\hat{g}_2(\xi) &= (1-\xi^2)e^{-\xi^2/2} \\
\hat{g}_3(\xi) &= i(\xi^3 - 3\xi)e^{-\xi^2/2}.
\end{aligned}$$

We can also write the result like this:

$$\begin{aligned}
\mathcal{F}(g_0) &= g_0 \\
\mathcal{F}(g_1) &= -ig_1 \\
\mathcal{F}(g_2) &= g_0 - g_2 \\
\mathcal{F}(g_3) &= -3ig_1 + ig_3.
\end{aligned}$$

(b) Let

$$\begin{aligned}
h_0 &= g_0 \\
h_1 &= -\frac{3}{2}g_1 + g_3 \\
h_2 &= 2g_2 - g_0 \\
h_3 &= g_1
\end{aligned}$$

Note that linear combinations of Schwartz functions are Schwartz. If $f, g \in \mathcal{S}(\mathbb{R})$ and $c, d \in \mathbb{R}$, then $cf + dg \in \mathcal{S}(\mathbb{R})$, since for all multi-indices α, β:

$$\lim_{\|x\|\to\infty} x^\beta \partial^\alpha (cf+dg)(x) = c\left(\lim_{\|x\|\to\infty} x^\beta \partial^\alpha f(x)\right) + d\left(\lim_{\|x\|\to\infty} x^\beta \partial^\alpha g(x)\right) = c\cdot 0 + d \cdot 0 = 0.$$

So since $g_k \in \mathcal{S}(\mathbb{R})$ for every $k = 0,1,2,3$ by part (a), also $h_k \in \mathcal{S}(\mathbb{R})$ (as defined above), for every $k = 0,1,2,3$. Also, clearly, the h_k are non-zero. Furthermore, by part (a) and linearity of $\mathcal{F}$ (cf. Prop. 11.5), we have

$$\begin{aligned}
\mathcal{F}(h_0) &= \mathcal{F}(g_0) = g_0 = h_0 \\
\mathcal{F}(h_1) &= \mathcal{F}(-\frac{3}{2}g_1 + g_3) = -\frac{3}{2}\mathcal{F}(g_1) + \mathcal{F}(g_3) = -\frac{3}{2}(-ig_1) - 3ig_1 + ig_3 = ih_1 \\
\mathcal{F}(h_2) &= \mathcal{F}(2g_2 - g_0) = 2\mathcal{F}(g_2) - \mathcal{F}(g_0) = 2(g_0 - g_2) - g_0 = g_0 - 2g_2 = -h_2 \\
\mathcal{F}(h_3) &= \mathcal{F}(g_1) = -ig_1 = -ih_3
\end{aligned}$$

This proves that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0,1,2,3$.

(c) Let $f \in \mathcal{S}(\mathbb{R})$. By Cor. 12.14, the restriction of $\mathcal{F}$ to $\mathcal{S}(\mathbb{R})$ is a an isomorphism onto $\mathcal{S}(\mathbb{R})$ with inverse $\mathcal{F}^*$. We can therefore let $g \in \mathcal{S}(\mathbb{R})$ be such that $\mathcal{F}(f) = g$, i.e. $\mathcal{F}^*(g) = f$. Now consider $\mathcal{F}^2(f)$. We then have, for $\xi \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{F}^2(f)(\xi) &= \mathcal{F}^2(\mathcal{F}^*(g))(\xi) \\
&= \mathcal{F}(g)(\xi) \\
&= \int_{\mathbb{R}} g(x)e^{-ix\xi}dm(x) \\
&= \mathcal{F}^*(g)(-\xi) \\
&= f(-\xi),
\end{aligned}$$

by the definition of $\mathcal{F}$ and the Fourier transform inverse $\mathcal{F}^*$. Hence we see that

$$\mathcal{F}^4(f)(\xi) = \mathcal{F}^2(\mathcal{F}^2(f))(\xi) = \mathcal{F}^2(f(-\xi)) = f(\xi).$$

This proves that $\mathcal{F}^4(f) = f$, for all $f \in \mathcal{S}(\mathbb{R})$.

(d) Note first, that if $\lambda = 0$ for some $f \in \mathcal{S}(\mathbb{R})$, then $\mathcal{F}(f) = 0$. By Cor. 12.13 ($f \in \mathcal{S}(\mathbb{R}) \subset L_1(\mathbb{R})$), this means that $f = 0$ almost everywhere. So we can forget this case, when we assume that f is non-zero. Now, combine the fact from (c), namely that $\mathcal{F}^4(f) = f$ for all $f \in \mathcal{S}(\mathbb{R})$, with the assumption that $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$ (and use linearity of the Fourier transform $\mathcal{F}$). Let $f \in \mathcal{S}(\mathbb{R})$, then

$$f = \mathcal{F}^4(f) = \mathcal{F}^3(\lambda f) = \mathcal{F}^2(\lambda \mathcal{F}(f)) = \mathcal{F}(\lambda^2 \mathcal{F}(f)) = \lambda^3 \mathcal{F}(f) = \lambda^4 f.$$

Hence $\lambda \in \{1, -1, i, -i\}$, the set of complex 4-roots of 1. This means that the eigenvalues of $\mathcal{F}$ is contained in the set $\{1, -1, i, -i\}$. In part (b), we constructed Schwartz functions h_k, for $k = 0, 1, 2, 3$, such that $\mathcal{F}(h_k) = i^k h_k$. This means exactly, that $\lambda = 1$ is an eigenvalue with eigenvector h_0, $\lambda = i$ is an eigenvalue with eigenvector h_1, $\lambda = -1$ is an eigenvalue with eigenvector h_2 and $\lambda = -i$ is an eigenvalue with eigenvector h_3, since

$$\begin{aligned}
\mathcal{F}(h_0) &= i^0 h_0 = h_0 \\
\mathcal{F}(h_1) &= i^1 h_1 = i h_1 \\
\mathcal{F}(h_2) &= i^2 h_2 = -h_2 \\
\mathcal{F}(h_3) &= i^3 h_3 = -i h_3
\end{aligned}$$

Hence the eigenvalues for $\mathcal{F}$ are precisely $1, -1, i$ and $-i$.

Problem 5

Let U be an open non-empty subset of $[0, 1]$. Then there exists an $n \geq 1$ such that $x_n \in U$. Otherwise we would have

$$\{x_n : n \geq 1\} \subset [0, 1] \setminus U,$$

so also

$$\overline{\{x_n : n \geq 1\}} \subset \overline{[0, 1] \setminus U} = [0, 1] \setminus U.$$

But $[0, 1] \setminus U$ is a proper subset of $[0, 1]$, so this would contradict the assumption that

$$\overline{\{x_n : n \geq 1\}} = [0, 1].$$

Hence, taking the Dirac measure centred at x_n of the Borel set U yields $\delta_{x_n}(U) = 1$. So we have that

$$\mu(U) = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}(U) \underline{\geq 1}.$$

Should be something like $\geq 2^{-n}$ (but be careful with the indices).

But this means that there exists no open sets U of $[0, 1]$ such that $\mu(U) = 0$. The union of all such sets is therefore empty, so by definition, we have

$$\text{supp}(\mu) = \emptyset^c = [0, 1].$$