# Notes on the scattering length of rank one perturbation

Johannes Agerskov

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We are in these notes going to study a bosonic system consisting of bosons of equal mass, $m$, interacting via rank one perturbations. For simplicity we assume $m = 1$. The $N$-body Hamiltonian act in the following way

$$H_N\psi(x) = -\frac{1}{2}\sum_{i=1}^{N}\Delta_i\psi(x)+\gamma_R\sum_{i<j}\mathbb{1}_{B_R}(x_i-x_j)\int_{B_R}\psi(x_1,...,(x_{ij}+\bar{x}_{ij})/2,...,(-x_{ij}+\bar{x}_{ij})/2,...,x_N)\,\mathrm{d}x_{ij}. \tag{0.1}$$

The two-body $s$-wave scattering length is found by the usual variational. principle. Notice first, that for the two-body problem, we may split the Hamiltonian in a centre of mass part, and a relative motion part obtaining

$$H_2 = H_{CM} + H_{\mathrm{rel}} \tag{0.2}$$

with $H_{CM} = -\frac{1}{4}\Delta_{X_{CM}}$ and $H_{\mathrm{rel}} = -\Delta_y + \gamma_R\left|\phi_R\right\rangle\left\langle\phi_R\right|$, with $\phi_R = \mathbb{1}_{B_R}$ and $y = x_1 - x_2$ being the relative coordinate. The scattering legth is defined by the assymtotics of the radial zero energy solution to the scattering equation

$$H_{\mathrm{rel}}\psi = 0 \tag{0.3}$$

It is clear that for $r > R$ we have $\psi(x) = 1 - a/\left|x\right|$, for some number $a$, called the scattering length, since this is just the usual Laplace's equation.
writing $\psi = 1-\omega$, we have $-\Delta\omega = A\mathbb{1}_{B_R}$ with $\omega(x)\to 0$ as $|x|\to\infty$, and we see by Gauss' law that $a = \frac{R^3}{3}A$ and that $\omega(x) = -\frac{|x|^2}{6}A + k$ for $|x| < R$. Continuity gives implies $k = AR^2/2$. Now, self-consistency demands $4\pi\gamma_R\int_0^R(1 - A(-\frac{1}{6}r^2 + \frac{1}{2}R^2))r^2\,\mathrm{d}r = A$, or equivalently $A = 4\pi\gamma_R\left(\frac{1}{3}R^3 + \frac{A}{30}R^5 - \frac{A}{6}R^5\right)$. Hence we find $A\left(1 + \frac{2}{15}4\pi\gamma_R R^5\right) = \frac{4\pi\gamma_R R^3}{3}$. Combining with the above we find

$$\frac{a}{R} = \frac{4\pi\gamma_R R^5}{3^2(1 + \frac{2}{15}4\pi\gamma_R R^5)} \tag{0.4}$$