Sibo Chen urn648.

Pr1

H be an infinite hilbert space

$f_N = N^{-1}\sum_{n=1}^{N^2} e_n$. for all $N \geq 1$.

(a) Show that $f_N \to 0$ weak. while $\|f_N\|=1$. for all $N\geq 1$

Pf: $\{\langle \cdot, e_n\rangle\}_{n\in\mathbb{N}}$ is a dual basis of H^*.

when $N\to\infty$. for an fixed e_i.

$\langle f_N, e_i\rangle = \frac{\langle e_i, e_i\rangle}{N} = \frac{1}{N} \to 0$ when $N\to\infty$.

$\langle f_N, e_i\rangle \to \langle 0, e_i\rangle$ for $\forall e_i$ (✓)

So $f_N \to 0$ weak Why does it suffice to consider $\langle f_N, e_i\rangle$?

$\|f_N\| = \langle f_N, f_N\rangle = \frac{1}{N^2}\sum_{i=1}^{N^2}\langle e_i, e_i\rangle = 1$

[What is happening here?

Do the calculations!

(b) $K = \overline{co\{f_N : N\geq 1\}}$

WTS. K is weakly compact.

We know that H is reflexive as a Hilbert. By theorem 6.3. $\overline{B_X(0,1)}$ is compact respect to the weak topology. H?

$\forall f\in co\{f_N : N\geq 1\}$.

$\|f\| = \|a_1 f_{N_1} + \cdots + a_n f_{N_n}\|$ $\quad \sum_{i=1}^n a_i = 1$

$\leq a_1\|f_{N_1}\| + \cdots + a_n\|f_{N_n}\|$

$\leq a_1 + \cdots + a_n = 1$.

$f\in \overline{B_X(0,1)} \Rightarrow co\{f_N : N\geq 1\} \subseteq \overline{B_X(0,1)}$

By theorem 5.7 A convex subset of X.

$\overline{A}^{\|\cdot\|} = \overline{A}^{\tau_w}$ $\quad \overline{co\{f_N : N\geq 1\}}^{\|\cdot\|} = \overline{co\{f_N : N\geq 1\}}^{\tau_w}$

$\overline{co\{f_N : N\geq 1\}}^{\|\cdot\|} \subseteq \overline{B_X(0,1)}$

$K \subseteq \overline{B_X(0,1)}$ H? K is closed subset of a compact set. [weakly

K is weak compact.

By (a). $f_N \to 0$ weakly.

From HW5. (1). we know that $(f_N)_{N\geq 1}$ is a sequence in X converging to $0\in X$ in weak topology. then there exists a sequence $(y_n)_{n\geq 1} \subseteq co\{f_N, N\geq 1\}$ $y_n \to 0$ converges in norm.

$0\in K$. ✓

(c). Show that 0. as well as each f_N. $N\geq 1$ are extreme points in K.

$\alpha x_1 + (1-\alpha)x_2 = 0$. $x_1, x_2\in K$. $\alpha\in(0,1)$

$x_1, x_2 \in \overline{co\{f_N, N\geq 1\}}$.

we claim $\langle x_1, e_i\rangle \geq 0$ for all $i\geq 0$.

$\langle x_2, e_i\rangle \geq 0$,

By (b) we have norm closure of $co\{f_N\}$ is equal to the weak closure of $co\{f_N\}$.

Not sure what you mean here? The convex combinations of the sequence in HW5 Pb1 does not have to be the same combination in the entire seq.

By HW5. (1). we have if $x_1\in K$. $\exists$ $\alpha f_{N_i} + (1-\alpha)f_{N_i'} \to x_1$ both in weak and norm. ($f_{N_i}, f_{N_i'}\in\{f_N\}$).

$\alpha_i\langle f_{N_i}, e_m\rangle + (1-\alpha_i)\langle f_{N_i'}, e_m\rangle \to \langle x_1, e_m\rangle$

$\langle f_{N_i}, e_m\rangle \geq 0$ $\langle f_{N_i'}, e_m\rangle \geq 0$.

$\langle x_1, e_m\rangle \geq 0$, $m\geq 1$

$\langle \alpha x_1 + (1-\alpha)x_2, e_m\rangle = \langle 0, e_m\rangle = 0$. $m\geq 1$

$\alpha\langle x_1, e_m\rangle + (1-\alpha)\langle x_2, e_m\rangle = 0$ $m\geq 1$

$\langle x_1, e_m\rangle = \langle x_2, e_m\rangle = 0$ for all $m\geq 1$. You can obtain this from $\langle f_n, e_m\rangle \geq 0$ $\forall n, m$.

$\Rightarrow x_1 = x_2 = 0$. 0 is extreme point. (✓)

Want to show f_N is extreme point.

Suppose $\alpha x_1 + (1-\alpha)x_2 = f_N$ $x_1, x_2 \in K$

$\alpha\langle x_1, e_m\rangle + (1-\alpha)\langle x_2, e_m\rangle = \langle f_N, e_m\rangle$

for all $m\geq 1$

y_i can be a convex combination of several f_N's.

$\exists\ \|\underbrace{\alpha_1 f_{N_1} + (1-\alpha_1)f_{N_2}}_{=y_1} - x_1\| < \frac{\varepsilon}{2\alpha}$

$\|\underbrace{\alpha_2 f_{N_3} + (1-\alpha_2)f_{N_4}}_{=y_2} - x_2\| < \frac{\varepsilon}{2(1-\alpha)}$

$\|\alpha y_1 + (1-\alpha)y_2 - f_N\|$

$\leq \alpha\|y_1 - x_1\| + (1-\alpha)\|y_2 - x_2\|$

$< \varepsilon$.

$\forall\varepsilon$ $\exists f_{N_1}, f_{N_2}, f_{N_3}, f_{N_4}$ WLOG $N_1\leq N_2\leq N_3\leq N_4$

$t = \|a_1 f_{N_1} + a_2 f_{N_2} + a_3 f_{N_3} + a_4 f_{N_4} - f_N\| < \varepsilon.$

$a_1 + a_2 + a_3 + a_4 = 1$

for small ε we have $f_{N_1} = f_{N_2} = f_{N_3} = f_{N_4} = f_N$

Elaborate! Why is this the case? Because now you've only shown that f_N is ε-close to something in $co(\{f_k \mid k \in \mathbb{N}\})$...

then $x_1 = x_2 = f_N$.

f_N is extreme point

□

(d) there aren't any other extreme point in K.

Pf: (H, τ_w) is a LCTVS.

$\tau_w{}^*$ only makes sense on dual spaces so be explicit in why H reflexive means it is a dual space

H reflexive $\tau_w = \tau_{w^*}$ By Alaoglu's theorem $\overline{B(0,1)}$ is compact in the w^*-top.

(w^*)?

By Theorem 7.9.

$K = \overline{Co(\{f_N\})}^{\tau_w}$ is a compact, convex subset of $\mathrm{Ext}(K) \subset \overline{\{f_N\}}^{\tau_w}$

And we have $\overline{\{f_N\}}^{\tau_w} = \{f_N\} \cup \{0\}$

[Why?]

so there is not any other extreme points

(✓)

□

2. Let X and Y be infinite dimentional Banach spaces.

(a). Let $T \in \mathcal{L}(X, Y)$. $(x_n)_{n\geq 1}$ in X $x \in X$.

$x_n \to x$ weakly. WTS $Tx_n \to Tx$ weakly. I.e $\forall y \in Y^*$. $y(Tx_n) \to y(Tx)$.

we have $\forall f \in X^*$ $f(x_n) \to f(x)$.

$(yT)(x_n) = y(T(x_n))$

$(yT)(\alpha x_1 + \beta x_2) = y\,T(\alpha x_1 + \beta x_2)$

$= \alpha yT(x_1) + \beta\, yT(x_2)$ $\quad yT \in X^*$

$yTx_n \to yTx$ Elaborate more.

□

(b)

$T \in K(X, Y)$. $(x_n)_{n\geq 1}$ in X. $x \in X$.

$x_n \to x$ weakly.

as $T \in K(X,Y) \subseteq L(X,Y)$.

we just need to consider $x_n \to 0$ weakly.

What is ε?

By contradiction. If $Tx_n \not\to T0 = 0$ in norm then $\exists$ a subsequence $\{Tx_{n_k}\}$

$\|Tx_{n_k} - 0\| > \varepsilon$.. By T is compact.

$\overline{TB_X(0,1)}$ is compact. Tx_{n_k} has a

But $(x_n)_n$ need not live in $B(0,1)$.

norm-convergent subsequence Tx_{n_L}

$Tx_{n_L} \to a \neq 0$. by (a) we know $Tx_{n_L} \to 0$

Why?

weakly. contradict. so $Tx_n \to 0$ in norm

Where lies the contradiction? Elaborate!

In general $Tx_n \to Tx$ in norm whenever $x_n \to x$ weakly.

(✓)

□.

(c) H. seperable infinite hilbert space.

If $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$.

WTS $T \in K(H, Y)$.

Pf: Suppose T is not compact.

By Prop 8.2 $A = T(\overline{B_X(0,1)})$ is not totally bounded. which means

$\exists\, \varepsilon > 0$. $\forall N \in \mathbb{N}$ $A \not\subset \bigcup_{i=1}^N U_i$

U_i are open balls with radius ε.

we take points by induction

arbitrary take $x_1 \in B_X(0,1)$. take

$Tx_2 \in A \setminus A \cap B_{Tx_1}(\varepsilon)$.

What is the center of these balls?

take $Tx_{n+1} \in A \setminus A \cap \left(\bigcup_{i=1}^{n} B_{Tx_n}(\varepsilon)\right)$.

Then we get a sequence $(x_n)_{n\geq 1}$.

$\forall n \neq m$ $\|Tx_n - Tx_m\| \geq \varepsilon$.

By Thm 6.3. A banach space X is reflexive $\Leftrightarrow$ $\overline{B_X(0,1)}$ is compact with respect to the weak topology on X.

H is a reflexive banach space.

Then x_n has a weakly convergent subsequence $x_{n_k} \to x$ then $Tx_{n_k} \to Tx$ in norm. contradict

How do you know it is a subsequence and not a subnet? Contradicting what? Be explicit!

So T is compact □

(d). Show that $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact

$\ell_2(\mathbb{N})$ is a hilbert space if $T \in \mathcal{L}(\ell^2(\mathbb{N}), \ell^1(\mathbb{N}))$.

$(x_n)_{n\geq 1}$ is a sequence in $\ell^2(\mathbb{N})$ converging weakly to $x \in \ell^2(\mathbb{N})$.

by (a).

we have $Tx_n \to Tx$ weak in $\ell^1(\mathbb{N})$

By remark 5.3 $Tx_n \to Tx$ weak $\Leftrightarrow$ $Tx_n \to Tx$ norm

By (c). we have $T \in K(\ell^2(\mathbb{N}), \ell^1(\mathbb{N}))$. ✓

(e). Show that no $T \in K(X, Y)$ is onto.

By contradiction suppose $T \in K(X,Y)$ is onto.

we have $B_Y(0,r) \subseteq T(B_X(0,1))$. Why?

$\overline{B_Y(0,r)} \subseteq \overline{T(B_X(0,1))}$ we know that $\overline{T(B_X(0,1))}$ is compact.

$\overline{B_Y(0,r)}$ is a closed subset of compact set. then $\overline{B_Y(0,r)}$ is compact. By assignment 1. Problem (3) e. closed unit ball in an infinite dimentional normed vector space is non-compact. contradict.

How does this extend to the ball $B(0,r)$?

So no T is onto □. (✓)

(f). $H = L_2([0,1], m)$ $M \in \mathcal{L}(H, H)$.

$Mf(t) = t f(t)$. $t \in [0,1]$. $\forall f, g \in H$.

$$\langle Mf, g \rangle = \int_{[0,1]} (Mf)(t)\bar{g}(t)\, dm$$

$$= \int_{[0,1]} t f(t)\, \bar{g}(t)\, dm(t)$$

$$= \int_{[0,1]} f(t)\, \overline{t g(t)}\, dm$$

$$= \langle f, Mg \rangle$$

$M = M^*$. M is self-adjiont. ✓

To prove M is not compact.

Notice $L_2([0,1], m)$ is separable Because $L_2([0,1], m)$ have a Schauder basis? So $L_2([0,1], m)$ is a inf. sep. Hilbert space.

Assume M compact by thm 10.1 (Spectrum theorem). M has eigenvalues $\{\lambda_i\}$. $\lambda_n \to 0$ as $n \to \infty$

by Hw 6. M has no eigen-values. contradict

then M is not compact ✓

□

Pr 3.

$H = L_2([0,1], m)$. $k: [0,1] \times [0,1] \to \mathbb{R}$.

$$k(s,t) = \begin{cases} (1-s)t & 0 \leq t \leq s \leq 1 \\ (1-t)s & 0 \leq s < t \leq 1 \end{cases}.$$

$$(Tf)(s) = \int_{[0,1]} k(s,t) f(t)\, dm(t) \quad s \in [0,1].$$

$f \in H$.

(a). Justify that T is compact.

Find a bounded sequences $(f_n)_{n\geq 1}$ in $L_2([0,1])$ $\|f\| \leq M$.

$$k(s,t) = \begin{cases} (1-s)t & 0 \leq t \leq s \leq 1 \\ (1-t)s & 0 \leq s < t \leq 1 \end{cases}$$

$k(s,t)$ is bounded and cont. further more $k(s,t)$ is equi-cont. w.r.t s. if $|s_1 - s_2| < \delta$. we have

$$|k(s_1,t) - k(s_2,t)| \leq \delta \max(1-t, t) \leq \delta$$

Then we will check the equi-cont. for each $Tf_n(s)$.

$|s-s'|<\delta$

$$|Tf_n(s)-Tf_n(s')| \leq \int_{[0,1]} |k(s,t)-k(s',t)|\,|f_n(t)|\,dm(t)$$

$$\leq \delta \cdot \int_{[0,1]} |f_n(t)|\,dm(t)$$

I don't see where this is going.

$\frac{1}{2}$

(b). $T=T^*$.

Use fubini's thm. why justified?

$$\langle Tf, g\rangle = \int_{[0,1]} (kf)(s)\,\bar{g}(s)\,dm(s)$$

$$= \int_{[0,1]} \left(\int_{[0,1]} k(s,t) f(t)\,dm(t) \right) \bar{g}(s)\,dm(s)$$

k real

$$= \int_{[0,1]} \int_{[0,1]} f(t)\,\overline{k(t,s)}\,\overline{g(s)}\,dm(s)\,dm(t)$$

$$= \int_{[0,1]} f(t) \left(\overline{\int_{[0,1]} k(t,s) g(s)\,dm(s)} \right) dm(t)$$

$$= \langle f, T^* g\rangle.$$

$$(T^*g)(t) = \int_{[0,1]} k(t,s) f(s)\,dm(s)$$

By the symmetric of s.t.

$$(T^*g)(t) = \int_{[0,1]} k(s,t) f(t)\,dm(t)$$

$$= (Tg)(s)$$

$T^* = T$.

(3) $(Tf)(s) = \int_{[0,1]} k(s,t) f(t)\,dm(t)$.

$$= \int_{[0,s]} k(s,t) f(t)\,dm(t) + \int_{[s,1]} k(s,t) f(t)\,dm(t)$$

$$= (1-s) \int_{[0,s]} t f(t)\,dm(t)$$

$$+ s\int_{[s,1]} (1-t) f(t)\,dm(t).$$

WTS Tf is cont. on $[0,1]$

$\frac{1}{2}$

we have $Tf(0) = (1-0)\int_{[0,0]} t f(t)\,dm(t) +$

$0 \cdot \int_{[0,1]} (1-t) f(t)\,dm(t) = 0.$

$(Tf)(1) = (1-1)\int_{[0,1]} t f(t)\,dm(t) +$

$1 \cdot \int_{[1,1]} (1-t) f(t)\,dm(t) = 0$

Pr4.

(a). $k\ge 0$. $g_k(x)=x^k e^{-\frac{x^2}{2}}$

$x \mapsto e^{-\frac{x^2}{2}}$

notice $e^{-\frac{x^2}{2}}$ is a composition of $f=\frac{x^2}{2}$ and $g=e^{-y}$ $f, g\in C^\infty(\mathbb{R})$. then $e^{-\frac{x^2}{2}}$ is $C^\infty(\mathbb{R})$ $\partial^\beta e^{-\|x\|^2} = Pol_{|\beta|}(x) e^{-\frac{\|x\|^2}{2}}$

$x^\alpha \partial^\beta e^{-\|x\|^2} = Pol_{|\alpha|+|\beta|}(x) e^{-\frac{\|x\|^2}{2}} \to 0$

as $\|x\|\to\infty$.

By HW7. we know $f\in \mathcal{S}(\mathbb{R}) \Rightarrow x^\alpha f \in \mathcal{S}(\mathbb{R})$

so $g_k(x)\in\mathcal{S}(\mathbb{R})$.

$g_0(x)=e^{-\frac{x^2}{2}}$

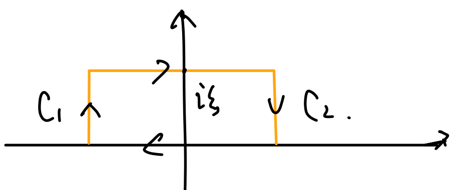
$\mathcal{F}(g_0(x)) = \int e^{-\frac{x^2}{2}} e^{-ix\xi}\, dm$

$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{x^2}{2}-ix\xi}\, dm$.

$z = x+i\xi$

$=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}+i\xi} e^{-\frac{z^2+\xi^2}{2}}\, dz$.

$=\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\int_{\mathbb{R}+i\xi} e^{-\frac{z^2}{2}}\,dz$



by prop 11.4 we have known that

$\int_{C_1} e^{-\frac{z^2}{2}}dz = \int_{C_2} e^{-\frac{z^2}{2}}dz = 0.$

so $\int_{\mathbb{R}+i\xi} e^{-\frac{z^2}{2}}dz = \int_{\mathbb{R}} e^{-\frac{x^2}{2}}dx = \sqrt{2\pi}$

$\mathcal{F}(g_0(x)) = e^{-\frac{x^2}{2}}$. ✓

$F(g_1)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x e^{-\frac{x^2}{2}} e^{-i\xi x}\, dx$.

$=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}+i\xi} (z-i\xi) e^{-\frac{z^2+\xi^2}{2}}\, dz$. $z=x+i\xi$.

$=\frac{1}{\sqrt{2\pi}}\left[e^{-\frac{\xi^2}{2}}\int_{\mathbb{R}+i\xi} z e^{-\frac{z^2}{2}}dz + (-i\xi e^{-\frac{\xi^2}{2}})\int_{\mathbb{R}+i\xi} e^{-\frac{z^2}{2}}dz\right]$

Use the same complex analysis method in prop 11.4. to compute $\int_{\mathbb{R}+i\xi} z e^{-\frac{z^2}{2}}dz$.

$\int_{\mathbb{R}+i\xi} z e^{-\frac{z^2}{2}}dz = \int_{\mathbb{R}} x e^{-\frac{x^2}{2}}dx = 0$

$=\int_{[0,+\infty]} e^{-y}dy = -e^{-y}\Big|_0^\infty = 1$. $y=\frac{x^2}{2}$

$F(g_1)(\xi) = \left(\frac{1}{\sqrt{2\pi}} - i\xi\right) e^{-\frac{\xi^2}{2}}$

not injective on $\mathbb{R}$

For $F(g_2)$.

$F(g_2) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} \cdot e^{-ix\xi}\, dx$. $z = x+i\xi$.

$=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}+i\xi}(z^2 - 2i\xi - \xi^2) e^{-\frac{z^2+\xi^2}{2}}\,dz$

$=\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\left(\int_{\mathbb{R}+i\xi} z^2 e^{-\frac{z^2}{2}}dz + (-2i\xi-\xi^2)\int_{\mathbb{R}+i\xi} e^{-\frac{z^2}{2}}dz\right)$

compute $\int_{\mathbb{R}+i\xi} z^2 e^{-\frac{z^2}{2}}dz$ in same process

$\int_{\mathbb{R}+i\xi} z^2 e^{-\frac{z^2}{2}}dz = \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}}dx$.

$= 2\int_0^{\mathbb{R}} e^{-\frac{x^2}{2}}dx$?

$x=\sqrt{2t}$

$=2\sqrt{2}\int_0^\infty t^{1/2} e^{-t}dt = 2\sqrt{2}\,\Gamma(\tfrac{3}{2}) = \sqrt{2\pi}$.

$F(g_2) = e^{-\frac{\xi^2}{2}}(1-2i\xi-\xi^2)$.

For $F(g_3)$.

$F(g_3) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^3 e^{-\frac{x^2}{2}} e^{-ix\xi}\,dx$ $z=x+i\xi$

$=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}+i\xi}(z-i\xi)^3 e^{-\frac{z^2+\xi^2}{2}}\,dz$

$=\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\int_{\mathbb{R}+i\xi}(z^3 - 3iz^2\xi - 3\xi^2 z + i\xi^3) e^{-\frac{z^2}{2}}\,dz$

$=\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}\int_{\mathbb{R}}(x^3 - 3ix^2\xi - 3\xi^2 x + i\xi^3) e^{-\frac{x^2}{2}}\,dx$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{\mathbb{R}} (x^3 - 3ix^2\xi - 3\xi^2 x + i\xi^3) e^{-\frac{x^2}{2}} dx$$

$$\int_{\mathbb{R}} x^3 e^{-\frac{x^2}{2}} dx \overset{y=\frac{x^2}{2}}{=} \int_0^\infty 2y e^{-y} dy.$$

(integration by part.)

$$= \int_0^\infty -2y \, de^{-y}.$$

$$= -2ye^{-y}\Big|_0^\infty + \int_0^\infty 2e^{-y} dy$$

$$= -2e^{-y}\Big|_0^\infty$$

$$= -2.$$

$$F(\varphi_3)(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} (-2 - 3i\xi\sqrt{2\pi} - 3\xi^2 + i\xi^3).$$

$\sqrt{2\pi}$.

(2). $F(h_0) = h_0$. For.

$k=0$ $h_0 = e^{-\frac{x^2}{2}}$

$k=3$ Let $h_3 = ae^{-\frac{x^2}{2}} - xe^{-\frac{x^2}{2}}$

$$F(h_3) = ae^{-\frac{\xi^2}{2}} - \left(\frac{1}{\sqrt{2\pi}} - i\xi\right)e^{-\frac{\xi^2}{2}}$$

$$\left(a - \frac{1}{\sqrt{2\pi}}\right) = -ia.$$

$$a = \frac{1}{1+i}\frac{1}{\sqrt{2\pi}} = \frac{1-i}{2\sqrt{2\pi}}$$

$$h_3 = \frac{1-i}{2\sqrt{2\pi}} e^{-\frac{x^2}{2}} - xe^{-\frac{x^2}{2}}$$

$k=2$ $h_2 = a_1 e^{-\frac{x^2}{2}} + a_2 x e^{-\frac{x^2}{2}} + x^2 e^{-\frac{x^2}{2}}$

$$F(h_2) = a_1 e^{-\frac{\xi^2}{2}} + a_2\left(\frac{1}{\sqrt{2\pi}} - i\xi\right)e^{-\frac{\xi^2}{2}} + (1 - 2i\xi - \xi^2)e^{-\frac{\xi^2}{2}}$$

$$a_1 + a_2\frac{1}{\sqrt{2\pi}} + 1 = -a_1$$

$$-ia_2 - 2i = -a_2.$$

$$a_2 = \frac{2i}{1-i} = i-1.$$

$$a_1 = -\frac{1}{2}\left(\frac{i-1}{\sqrt{2\pi}} + 1\right)$$

$$h_2 = -\frac{1}{2}\left(\frac{i-1}{\sqrt{2\pi}} + 1\right)e^{-\frac{x^2}{2}} + (i-1)xe^{-\frac{x^2}{2}} + x^2 e^{-\frac{x^2}{2}}$$

For h_1

$$F(h_1) = ih_1$$

$$h_1 = \varphi_3 + b_2\varphi_2 + b_1\varphi_1 + b_0\varphi_0.$$

$$F(h_1) = ih_1$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}(-2 - 3i\xi\sqrt{2\pi} - 3\xi^2 + i\sqrt{2\pi}\,\xi^3)$$
$$+ b_2(1 - 2i\xi - \xi^2)e^{-\frac{\xi^2}{2}}$$
$$+ b_1\left(\frac{1}{\sqrt{2\pi}} - i\xi\right)e^{-\frac{\xi^2}{2}}$$
$$+ b_0 e^{-\frac{\xi^2}{2}}$$

$$-b_2 - \frac{3}{\sqrt{2\pi}} = ib_2$$

$$b_2 = -\frac{3}{\sqrt{2\pi}(1+i)} = -\frac{3\sqrt{2}(1-i)}{4\pi}.$$

$$-2ib_2 - 3i - ib_1 = ib_1$$

$$b_1 = -\frac{2ib_2 + 3i}{2i} = -b_2 + \frac{3}{2}$$

$$= \frac{3\sqrt{2}(1-i)}{4\pi} + \frac{3}{2}$$

$$b_0 + \frac{b_1}{\sqrt{2\pi}} + b_2 - \frac{\sqrt{2}}{\sqrt{\pi}} = ib_0.$$

$$b_0 = \frac{1}{i-1}\left(\frac{3\sqrt{2}(1-i)}{4\sqrt{2}\,\pi^{\frac{3}{2}}} - \frac{3\sqrt{2}(1-i)}{4\pi} - \frac{\sqrt{2}}{\sqrt{\pi}}\right)$$

$$= -\frac{3}{4\pi^{\frac{3}{2}}} + \frac{3\sqrt{2}}{4\pi} + \frac{1+i}{\sqrt{2\pi}}$$

Propagation of mistakes. But okay.

(e). Show that $F^4(f) = f$ for all $f \in \mathcal{S}(\mathbb{R})$

$$F^2(f) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x) e^{-i\xi_1 x} dx \cdot e^{-i\xi_2\xi_1} d\xi_2.$$

(3)

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x) e^{-i\xi_1(x+\xi_2)} dx\, d\xi_1$$

Fubini not allowed!

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x) e^{-i\xi_1(x+\xi_2)} d\xi_1\, dx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} f(x) \int_{\mathbb{R}} e^{-i\xi_1(x+\xi_2)} d\xi_1\, dx.$$

$$= \int_{\mathbb{R}} f(x) \cdot \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi_1(x+\xi_2)} d\xi_1\, dx$$

we know that
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-\alpha)}\,dp = \delta(x-\alpha) \text{ (Dirac function)}$$

Not rigorous! One can define Fourier transform of temperate distribution. But integral formula exist.

$$F^2(f) = \int_{\mathbb{R}} f(x)\cdot\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi_1(x+\xi_2)}\,d\xi_1\,dx$$
$$= \int_{\mathbb{R}} f(x)\cdot\delta(-x-\xi_2)\,dx.$$
$$= f(-\xi_2).$$
$$F^4(f(x)) = F^2(f(-x)) = f(x).$$

(d) $F(f) = \lambda f$

$F^4(f) = \lambda^4 f = f.$

$\lambda^4 = 1$ has four roots in $\mathbb{C}$.

Precisely they are $\{1, i, -1, -i\}$. Not shown that they are eigenvalues!

Prf. Let $(x_n)_{n\geq 1}$ be a dense subset of

$\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ is the radon measure

on $[0,1]$ $supp(\mu) = [0,1]$

Proof: By HW8. Pr3. (a)

N is the union of all open subsets U of X

such that $\mu(N)=0$ You probably mean $\mu(U)$. But it is true that $\mu(N)=0$ as well.

$supp(\mu) = N^c$

we know N is open (non-empty) open sets.

In $\mathbb{R}$ open interval can be expressed as

the union of open intervals (a_i, b_i).
we work in $[0,1)$, so be careful.

Suppose $N \neq \phi$. $\exists\ (a,b) \subseteq N \subseteq [0,1]$

Since $(x_n)_{n\geq 1}$ is a dense subset of

$[0,1]$. then $(x_n)_{n\geq 1} \cap (a,b) \neq \phi$

$\exists\ x_i \in (a,b)$.

$$\mu((a,b)) \geq 2^{-i}\delta_{x_i}(a,b) = 2^{-i} > 0.$$

$$\mu(N) \geq \mu((a,b)) > 0.$$

which contradicts to $\mu(N)=0$ ✓

So $N = \phi$ $supp(\mu) = N^c = [0,1]$ □