Mandatory assignment 2, FunkAn 2020

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If not specified otherwise, references are to the Lecture Notes.
When refering to 'Schilling', I refer to the book 'Measures, Integrals and Martingales' by René L. Schilling, Second Edition, 2017.

Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.

(a) Show that $f_N \to 0$ weakly, as $N \to \infty$, while $\|f_N\| = 1$, for all $N \geq 1$.

Solution First we show that $\|f_N\| = 1$ for all $N \geq 1$. Since e_i and e_j are orthogonal for $i \neq j$, we use Pythagoras' theorem (Theorem 26.19(ii) of Schilling) repeatedly to get

$$\|f_N\|^2 = N^{-2}\left\|\sum_{n=1}^{N^2} e_n\right\|^2 = N^{-2}\sum_{n=1}^{N^2}\|e\|_n^2 = N^{-2}\sum_{n=1}^{N^2} 1 = 1.$$

We conclude that $\|f_N\| = 1$ for all $N \geq 1$.

Now we show that $f_N \to 0$ weakly as $N \to \infty$, by using Homework 4, Problem 3. Since H is a separable Hilbert space, consider the canonical isometric isomorphism $\Phi : H \to l_2(\mathbb{N})$ defined by mapping $h \in H$ to $(\langle h, e_n\rangle)_{n\geq 1} \in l_2(\mathbb{N})$ (see Example 9.14 in the Lecture notes or Remark 26.25(i) in Schilling). We use the notation $x_N := (\langle f_N, e_n\rangle)_{n\geq 1}$ and $x_N(n) = \langle f_N, e_n\rangle$ for $N, n \geq 1$, hence the sequence $(f_N)_{N\geq 1}$ in H is mapped by Φ to $(x_N)_{N\geq 1}$ in $l_2(\mathbb{N})$.

Since Φ is an isometry, we have $\|x_N\|_2 = \|f_N\| = 1 < \infty$ for all $N \geq 1$. Therefore, the sequence $(x_N)_{N\geq 1}$ is bounded in $\|\cdot\|_2$. Furthermore, for a fixed $n \geq 1$, we see that

$$|x_N(n)| = |\langle f_N, e_n\rangle| = \left|N^{-1}\sum_{i=1}^{N^2}\langle e_i, e_n\rangle\right| \leq N^{-1} \to 0$$

as $N \to \infty$, that is, $x_N(n) \to 0$ as $N \to \infty$.
Thus, the two conditions in Homework 4, Problem 3 are satisfied for $(x_N)_{N\geq 1} \subset l_2(\mathbb{N})$, so we conclude that $x_N \to 0$ weakly as $N \to \infty$. Φ^{-1} maps $(x_N)_{N\geq 1}$ isometrically isomorphically to $(f_N)_{N\geq 1}$, so we deduce that $f_N \to 0$ weakly as $N \to \infty$.

(b) Let K be the norm closure of $\text{co}\{f_N : N \geq 1\}$, that is, $K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|}$. Argue that K is weakly compact, and that $0 \in K$.

Solution Since H is a Hilbert space, it is reflexive (Proposition 2.10), hence it is the dual space of an isometrically isomorphic copy of H^*. Therefore we can apply Alaoglu's theorem to H to deduce that the closed unit ball $\overline{B}_H(0,1) = \{f \in H \mid \|f\| \leq 1\}$ is compact in the w^*-topology. And since H is reflexive, the weak and weak* topologies coincide on H (Theorem 5.9). So $\overline{B}_H(0,1)$ is compact in the weak topology. This is a bit imprecise.
Note now that $\{f_N : N \geq 1\} \subset \overline{B}_H(0,1)$, since $\|f_N\| = 1$ for all $N \geq 1$. We know that the closed unit ball in H is convex, so we obtain

$$\mathrm{co}\{f_N : N \geq 1\} \subseteq \overline{B}_H(0,1),$$

since the convex hull is the minimal convex set. We deduce now that

$$K = \overline{\mathrm{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\mathrm{co}\{f_N : N \geq 1\}}^{\tau_w} \subset \overline{B}_H(0,1);$$

the second equality is due to Theorem 5.7, and the last inclusion holds because $\overline{B}_H(0,1)$ is weakly closed (since it is weakly compact).
Now, since K is weakly closed and it is contained in a weakly compact set, K itself is weakly compact. ✓
Finally, we know that $f_N \to 0$ weakly as $N \to \infty$. Furthermore we know that $\{f_N : N \geq 1\} \subset K$ and that K is weakly closed. Therefore the limit point 0 also lies in K.

(c) Show that 0, as well as each f_N, $N \geq 1$, are extreme points in K.

Solution First, we prove that f_N are extreme points for all $N \geq 1$. Since K is convex and weakly compact, the Krein-Milman theorem states that $\overline{\mathrm{co}\{f_N : N \geq 1\}} = \overline{\mathrm{co}(\mathrm{Ext}(K))}$, where the closures are either norm or weak since the norm and weak closures of convex sets are identical. From part (d), we know that $\mathrm{Ext}(K) \subseteq \{f_N : N \geq 1\} \cup \{0\}$ (we will use this even though we haven't proven it yet; you may read the solution to part (d) first).
Fix some $N \geq 1$, and assume now that f_N is not an extreme point. Let's show that $f_N \notin \overline{\mathrm{co}(\mathrm{Ext}(K))}$ which is a contradiction. We have the following characterisation.

$$\begin{aligned}
\mathrm{co}(\mathrm{Ext}(K)) &\subseteq \mathrm{co}\left(\{0\} \cup \{f_i : i \geq 1, i \neq N\}\right) \\
&= \left\{\sum_{i=1}^{n} \alpha_i x_i \mid x_i \in \{0\} \cup \{f_i : i \geq 1, i \neq N\}, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1\right\} \\
&= \left\{\sum_{i=1}^{n} \alpha_i f_i \mid \alpha_N = 0, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i \leq 1, n \in \mathbb{N}\right\}
\end{aligned}$$

Now, take some $x \in \mathrm{co}(\mathrm{Ext}(K))$ of the form $x = \sum_{i=1}^{n} \alpha_i f_i$ with the above mentioned restrictions. Define $\beta_i := \alpha_i$ for $i \neq N$ and $\beta_N := -1$. Hence, we have $x - f_N = \sum_{i=1}^{n} \beta_i f_i$; if at first $n > N$, we simply increase n until $n = N$ and fill in with $\alpha_i = 0$ where appropriate.

We obtain the following.

What happens here?

$$
\begin{aligned}
\|x - f_N\| &= \left\| \sum_{i=1}^{n} \beta_i f_i \right\| \\
&= \left\| \sum_{i=1}^{n} \beta_i i^{-1} \sum_{j=1}^{i^2} e_j \right\| \\
&= \left\| \sum_{i=1}^{n^2} \left(\sum_{j=\lceil \sqrt{i} \rceil}^{n} \beta_j j^{-1} \right) e_i \right\| \\
&= \sum_{i=1}^{n^2} \left\| \left(\sum_{j=\lceil \sqrt{i} \rceil}^{n} \beta_j j^{-1} \right) e_i \right\| \\
&= \sum_{i=1}^{n^2} \left| \sum_{j=\lceil \sqrt{i} \rceil}^{n} \beta_j j^{-1} \right| \\
&\geq \left| \sum_{j=N}^{n} \beta_j j^{-1} \right| \quad (*) \\
&= \left| -N^{-1} + \beta_{N+1}(N+1)^{-1} + \ldots + \beta_n n^{-1} \right| \\
&\geq N^{-1} - (N+1)^{-1} \\
&= \frac{1}{N(N+1)}
\end{aligned}
$$

I think there might be an issue with this inequality?

If $n=N$, isn't $(*)=0$?

The fourth equality is due to Pythagoras' theorem. In the last inequality, we use the reverse triangle inequality and the fact that $\beta_j \geq 0$ for $j = N+1, ..., n$, and $\sum_{j=N+1}^{n} \beta_j \leq 1$. If $n = N$, the last couple of computations might be a bit odd, but the bound of $1/N(N+1)$ still holds.

What we have shown is that the distance between f_N and some arbitrary $x \in \mathrm{co}(\mathrm{Ext}(K))$ is larger than or equal to the constant $\frac{1}{N(N+1)}$. Therefore, no sequence from $\mathrm{co}(\mathrm{Ext}(K))$ converges to f_N (in norm) and thus we conclude that $f_N \notin \overline{\mathrm{co}(\mathrm{Ext}(K))}$. This is a contradiction since $\overline{\mathrm{co}\{f_N : N \geq 1\}} = \overline{\mathrm{co}(\mathrm{Ext}(K))}$. Therefore, f_N is an extreme point of K.

The idea is OK, but there may be some problems with the calculations

Now we show that 0 is also an extreme point of K. The same procedure will not work, but we do something of the same flavor. Let $y \in H$. Since $(e_n)_{n\geq 1}$ is an ONB of H, we can write $y = \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i = \sum_{i=1}^{\infty} y_i e_i$ with $y_i = \langle y, e_i \rangle$. Assume that for some fixed $N \geq 1$, $y_N \notin [0, \infty)$. We have the following characterisation.

$$
\begin{aligned}
\operatorname{co}\{f_N : N \geq 1\} &= \left\{ \sum_{i=1}^{n} \alpha_i x_i \mid x_i \in \{f_N : N \geq 1\}, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N} \right\} \\
&= \left\{ \sum_{i=1}^{n} \alpha_i f_i \mid \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N} \right\}
\end{aligned}
$$

Now, let $x \in \operatorname{co}\{f_N : N \geq 1\}$, that is,

$$
x = \sum_{i=1}^{n} \alpha_i f_i = \sum_{i=1}^{n^2} \left(\sum_{j=\lceil \sqrt{i} \rceil}^{n} \alpha_j j^{-1} \right) e_i
$$

with the above mentioned restrictions (and we have used the same rewriting of the sum as above). Denote

$$
\beta_i := \begin{cases} \sum_{j=\lceil \sqrt{i} \rceil}^{n} \alpha_j j^{-1}, & i \leq n^2 \\ 0, & i > n^2 \end{cases}
$$

We note that $\beta_j \geq 0$ for all $j \geq 1$. Now we obtain the following.

$$
\begin{aligned}
\|y - x\| &= \left\| \sum_{i=1}^{\infty} (y_i - \beta_i) e_i \right\| \\
&= \sum_{i=1}^{\infty} \left\| (y_i - \beta_i) e_i \right\| \\
&= \sum_{i=1}^{\infty} |y_i - \beta_i| \\
&\geq |y_N - \beta_N|
\end{aligned}
$$

Consider two cases: If y_N is a real number, it must be negative by the previous assumption. Therefore, $|y_N - \beta_N| \geq |y_N| > 0$. If y_N is complex, write $y_N = a + ib$ with $b \neq 0$. Then $|y_N - \beta_N| \geq |b| > 0$. In both cases we conclude that $\|y - x\|$ is greater than or equal to some fixed constant (depending only on y). Therefore, no sequence in $\operatorname{co}\{f_N : N \geq 1\}$ converge in norm to y, so y is not in K. In other words, if $z = \sum_{i=1}^{\infty} z_i e_i$ lies in K, then $z_i \in [0, \infty)$ for all $i \geq 1$.

Finally we show that 0 is an extreme point: Assume that $0 = \lambda x + (1-\lambda) y$ for some $0 < \lambda < 1$ and $x, y \in K$. We write $x = \sum_{i=1}^{\infty} x_i e_i$ and $y = \sum_{i=1}^{\infty} y_i e_i$. From the above, we know that $x_i, y_i \geq 0$ for all $i \geq 1$. We have

$$
0 = \lambda x + (1 - \lambda) y = \sum_{i=1}^{\infty} (\lambda x_i + (1 - \lambda) y_i) e_i.
$$

We deduce that $\lambda x_i + (1-\lambda)y_i = 0$ for all $i \geq 1$ and further that $x_i, y_i = 0$ for all $i \geq 1$. Thus, $x = y = 0$ and we conclude that 0 is an extreme point.

(d) Justify whether there are other extreme points in K.

Solution We justify that there are no other extreme points in K. Denote $F = \{f_N : N \geq 1\} \subset K$. By definition, F satisfies $K = \overline{\mathrm{co}(F)}$ (note that norm closure and weak closure are equal). Since K is weakly compact and convex (taking closure of a convex set yields a convex set), Theorem 7.9 (Milman) states that $\mathrm{Ext}(K) \subset \overline{F}^{\tau_w} = \overline{\{f_N : N \geq 1\}}^{\tau_w}$.

Let's argue that $\overline{F}^{\tau_w} = F \cup \{0\}$. Since $f_N \to 0$ weakly as $N \to \infty$ and $\overline{F}^{\tau_w}$ is weakly closed, we obtain $0 \in \overline{F}^{\tau_w}$. On the other hand, any (weakly) convergent sequence together with its limit point is a (weakly) compact set. If this needs justification, let $(U)_{i\in\Lambda}$ be an open covering of $F \cup \{0\}$. Pick i_0 such that $0 \in U_{i_0}$. Since $f_N \to 0$, there exists $M \geq 1$ such that $f_N \in U_{i_0}$ for all $N \geq M$. Now for all $j = 1, ..., M-1$, let $f_j \in U_{i_j}$. We conclude that $U_{i_0}, U_{i_1}, ..., U_{i_{M-1}}$ is a finite open subcover of $F \cup \{0\}$, hence $F \cup \{0\}$ is compact. weakly! weakly!
Since $F \cup \{0\}$ is compact, the set is also closed. So we conclude that $\mathrm{Ext}(K) \subset \overline{F}^{\tau_w} = F \cup \{0\}$. Therefore, 0 together with f_N for all $N \geq 1$ are the only extreme points in K.

Be more explicit in why this implies $\overline{F}^{\tau_w} = F \cup \{0\}$.

Problem 2

Let X and Y be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

Solution To show that $Tx_n \to Tx$ as $n \to \infty$, we show equivalently that $g(Tx_n) \to g(Tx)$ as $n \to \infty$ for all $g \in Y^*$ (Homework 4, Problem 2(a)). Since both g and T are linear and bounded, $g \circ T$ is also linear and bounded, hence $g \circ T \in X^*$. And since $x_n \to x$ weakly as $n \to \infty$, we know that $f(x_n) \to f(x)$ for all $f \in X^*$. In particular, we have $(g \circ T)(x_n) \to (g \circ T)(x)$ weakly as $n \to \infty$, or in other words, $g(Tx_n) \to g(Tx)$ weakly as $n \to \infty$. So we conclude that $Tx_n \to Tx$ weakly as $n \to \infty$.

in norm!

(b) Let $T \in \mathcal{K}(X,Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$.

Solution We will appeal to the fact that a sequence converges to some fixed point if any subsequence has a further subsequence that converges to the same fixed point (this has been used in the exercises, for example in the proof of Homework 4, Problem 3).

So let $(Tx_{n_k})_{k\geq 1}$ be a subsequence of $(Tx_n)_{n\geq 1}$. Since $x_n \to x$ weakly, we know from Homework 4, Problem 2(b) that $(x_n)_{n\geq 1}$ is bounded in norm, and therefore, $(x_{n_k})_{k\geq 1}$ is bounded in norm. Since T is compact, by Proposition 8.2 there exists a further subsequence $(x_{n_{k_i}})_{i\geq 1}$ such that the sequence $(Tx_{n_{k_i}})_{i\geq 1}$ converges in norm in Y to some point $y \in Y$. And since the weak topology is coarser than the norm topology, this also means that $Tx_{n_{k_i}} \to y$ weakly as $i \to \infty$.

From part (a) we know that since $x_n \to x$ weakly, we have $Tx_n \to Tx$ weakly as $n \to \infty$. Therefore, the subsubsequence $(Tx_{n_{k_i}})_{i\geq 1}$ also converges weakly to Tx. From uniqueness of limit points, we get that $y = Tx$. So we see that the subsubsequence $(Tx_{n_{k_i}})_{i\geq 1}$ converges in norm to Tx. By the above mentioned fact, we conclude that $(Tx_n)_{n\geq 1}$ converges to Tx in norm, or in other words, $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. ✓

(c) Let H be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Solution Assume for contradiction that T is not compact. We will construct a sequence $(x_n)_{n\geq 1}$ in the closed unit ball of H that satisfy contradicting properties. Since T is not compact, $T(B_H(0,1))$ is not totally bounded. This means that there exists an $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ and open balls $U_1, ..., U_N$ of radius ε, $T(B_H(0,1))$ cannot be covered by these open balls.

Now, choose $x_1 \in B_H(0,1)$ arbitrarily. Recursively for $n \geq 2$, choose $x_n \in B_H(0,1)$ such that $\|Tx_i - Tx_n\| \geq \varepsilon$ for all $1 \leq i \leq n-1$. This is possible since $T(B_H(0,1))$ cannot be covered by any finite number of open balls of radius ε. We conclude that $(x_n)_{n\geq 1}$ is a sequence in the closed unit ball of H such that $\|Tx_n - Tx_m\| \geq \varepsilon$ for all $n \neq m$. Elaborate on this.

How do you know it is a subseq and not a subnet?

Note that since H is reflexive, it is a dual space of some other space, namely an isometrically isomorphic copy of H^*. So Alaoglu's theorem states that $\overline{B}_H(0,1)$ is weak*-compact and hence also weakly compact since weak and weak* coincide on reflexive spaces. Since $(x_n)_{n\geq 1}$ is in $\overline{B}_H(0,1)$, there exists a weakly convergent subsequence $(x_{n_k})_{k\geq 1}$ converging to some point $x \in H$. By the assumption of the problem, we obtain that $\|Tx_{n_k} - Tx\| \to 0$ as $k \to \infty$. In particular, there exist $k, l \in \mathbb{N}$ such that

$$\|Tx_{n_k} - Tx_{n_l}\| \leq \|Tx_{n_k} - Tx\| + \|Tx - Tx_{n_l}\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This contradicts the fact that $\|Tx_n - Tx_m\| \geq \varepsilon$ for all $n \neq m$. We conclude that $T \in \mathcal{K}(H, Y)$. (✓)

(d) Show that each $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ is compact.

Solution Let $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ be arbitrary. We show that the assumption in part (c) is satisfied. So let $(x_n)_{n\geq 1}$ be a sequence in the Hilbert space $l_2(\mathbb{N})$ converging weakly to $x \in l_2(\mathbb{N})$. By part (a), we obtain that $Tx_n \to Tx$ weakly as $n \to \infty$. So $(Tx_n)_{n\geq 1}$ is a weakly convergent sequence in $l_1(\mathbb{N})$. By Remark 5.3, we know that $(Tx_n)_{n\geq 1}$ converges in norm to Tx, that is, $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. Hence the assumption in part (c) is fulfilled, and we conclude that $T \in \mathcal{K}(l_2(\mathbb{N}), l_1(\mathbb{N}))$.

(e) Show that no $T \in \mathcal{K}(X, Y)$ is onto.

Solution Assume for contradiction that T is a compact and surjective operator. From the open mapping theorem (Theorem 3.15) we know that T is open. Therefore the set $T(B_X(0,1))$ is open. We know that $T(0) = 0$, so $0 \in T(B_X(0,1))$, so $B_Y(0,r) \subseteq T(B_X(0,1))$ for some (small) $r > 0$. Therefore, we also have $\overline{B}_Y(0,r) = \overline{B_Y(0,r)} \subseteq \overline{T(B_X(0,1))}$. Since T is compact, $\overline{T(B_X(0,1))}$ is compact. And since $\overline{B}_Y(0,r)$ is a closed subset of a compact set, it is also compact. Note that $\overline{B}_Y(0,r)$ is a scaling of the closed unit ball by a factor of r, and since scaling is continuous, the closed unit ball is also compact. But this is a contradiction: In Problem 3 of Mandatory Assignment 1, we showed that $\overline{B}_Y(0,1)$ is non-compact. So we conclude that no compact operator between Banach spaces is onto.

(f) Let $H = L_2([0,1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. Justify that M is self-adjoint, but not compact.

Solution Consider the following computations for $f, g \in H$.

$$\begin{aligned}
\langle Mf, g\rangle &= \int_{[0,1]} (Mf)(t)\overline{g(t)}dm(t) \\
&= \int_{[0,1]} tf(t)\overline{g(t)}dm(t) \\
&= \int_{[0,1]} f(t)\overline{tg(t)}dm(t) \\
&= \int_{[0,1]} f(t)\overline{(Mg)(t)}dm(t) = \langle f, Mg\rangle
\end{aligned}$$

We deduce that $\langle f, Mg\rangle = \langle Mf, g\rangle = \langle f, M^*g\rangle$ for all $f, g \in H$. Here, M^* is the adjoint of M. Since the adjoint satisfies the above property uniquely, we conclude that $M = M^*$, that is, M is self-adjoint.

Lastly, M cannot be compact; because if M was compact, it would satisfy the conditions of the spectral theorem for self-adjoint compact operators (Theorem 10.1). That would imply that H has an ONB consisting of eigenvectors for M. In particular, M would have atleast one eigenvector. But we showed in Homework 6, Problem 3(a) that M has no eigenvalues and hence no eigenvectors. Therefore, M is not compact.

Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$, where m is the Lebesgue measure. Define $K : [0,1] \times [0,1] \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t, & 0 \le t \le s \le 1, \\ (1-t)s, & 0 \le s < t \le 1, \end{cases}$$

and consider $T \in \mathcal{L}(H,H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t) f(t) dm(t), \quad s \in [0,1], \ f \in H.$$

(a) Justify that T is compact.

Solution Note that $[0,1]$ with the usual topology is a compact Hausdorff space and m is a finite Borel measure on $[0,1]$. We also see that $K(s,t)$ is continuous: Both $(s,t) \mapsto (1-s)t$ and $(s,t) \mapsto (1-t)s$ are continuous, and these two functions agree on the diagonal $s = t$ of $(s,t) \in [0,1] \times [0,1]$, which is exactly the line where $K(s,t) = (1-s)t$ changes to $K(s,t) = (1-t)s$ and vice versa. Theorem 9.6 now states that the operator $T_K \in \mathcal{L}(H,H)$ given by

$$(T_K f)(s) = \int_{[0,1]} K(t,s) f(t) dm(t), \quad s \in [0,1], \ f \in H$$

is compact. We see that if $K(s,t) = K(t,s)$ then T and T_K are exactly the same. Since we will also use this result later on, we will prove it here.

So let $s,t \in [0,1]$ be fixed. If $s = t$, then $K(s,t) = K(t,s)$ is clear. If $s < t$ then $K(s,t) = (1-t)s$ and $K(t,s) = (1-t)s$ by the definition of K. Similarly if $t < s$ we get $K(s,t) = (1-s)t$ and $K(t,s) = (1-s)t$. So we always have $K(s,t) = K(t,s)$. Hence $T = T_K$ and T is compact.

✓

(b) Show that $T = T^*$.

Solution We must show that $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in H$. From this and the uniqueness of the adjoint, it follows that $T = T^*$. In the computations below we wish to use Fubini, so let us show that $K(s,t)f(t)\overline{g(s)} \in \mathcal{L}_1([0,1], m)$: We use Tonelli on the following integral since the integrand is positive, and note also that $K(s,t) \le 1$.

$$\begin{aligned} \int_{[0,1]^2} |K(s,t)f(t)\overline{g(s)}| dm_2(s,t) &\le \int_{[0,1]^2} |f(t)||g(s)| dm_2(s,t) \\ &= \int_{[0,1]} \int_{[0,1]} |f(t)||g(s)| dm(s) dm(t) \\ &= \int_{[0,1]} |f(t)| dm(t) \int_{[0,1]} |g(s)| dm(s) < \infty \end{aligned}$$

Note that the two integrals on the last line are finite, since $f, g \in H = L_2([0,1], m) \subset L_1([0,1], m)$ by Homework 2, Problem 2(b).

Now we are ready for the following computations in which we use Fubini and $K(s,t) = K(t,s)$ as well as the fact that the complex conjugate of an integral is the integral of the complex conjugate of the integrand. For $f, g \in H$, we have

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} (Tf)(s)\overline{g(s)}dm(s) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(s,t)f(t)dm(t) \right) \overline{g(s)}dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t) \left(\int_{[0,1]} \overline{K(t,s)g(s)}dm(s) \right) dm(t) \\
&= \int_{[0,1]} f(t) \left(\overline{\int_{[0,1]} K(t,s)g(s)dm(s)} \right) dm(t) \\
&= \int_{[0,1]} f(t)\overline{(Tg)(t)}dm(t) \\
&= \langle f, Tg\rangle
\end{aligned}$$

K real

Since the adjoint is unique, we conclude that $T = T^*$.

(c) Show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t), \quad s \in [0,1], \;\; f \in H.$$

Use this to show that Tf is continuous, and that $(Tf)(0) = (Tf)(1) = 0$.

Solution We simply use the definition of K to obtain the formula that we are asked to show. Let $f \in H$ and $s \in [0,1]$.

$$
\begin{aligned}
(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)
\end{aligned}
$$

To show that Tf is continuous for some $f \in H$, we first consider the term

$$
\lim_{s\to s_0} \int_{[0,s]} tf(t)dm(t) = \lim_{s\to s_0} \int_{[0,1]} 1_{[0,s]}tf(t)dm(t)
$$

for some fixed s_0. Note that $f \in H = L_2([0,1],m) \subset L_1([0,1],m)$ is a dominating function of $1_{[0,s]}tf(t)$ for all $s \in [0,1]$, and that the limit $\lim_{s\to s_0} 1_{[0,s]}tf(t) = 1_{[0,s_0]}tf(t)$ exists in $\overline{\mathbb{R}}$ for almost all $t \in [0,1]$. Now we can use the dominated convergence theorem (Theorem 12.2 of Schilling) to deduce that

$$
\lim_{s\to s_0} \int_{[0,s]} tf(t)dm(t) = \lim_{s\to s_0} \int_{[0,1]} 1_{[0,s]}tf(t)dm(t) = \int_{[0,1]} \lim_{s\to s_0} 1_{[0,s]}tf(t)dm(t) = \int_{[0,s_0]} tf(t)dm(t).
$$

Completely analogous we use the dominated convergence theorem to deduce that

$$
\lim_{s\to s_0} \int_{[s,1]} tf(t)dm(t) = \int_{[0,1]} \lim_{s\to s_0} 1_{[s,1]}tf(t)dm(t) = \int_{[s_0,1]} tf(t)dm(t).
$$

Of course, the dominated convergence theorem applies to sequences of functions. However, we could transform the limit $s \to s_0$ into a limit $s_n \to s_0$ for $n \in \mathbb{N}$. We omit this like we have done in the lectures.
To show that Tf is continuous, let s_0 be fixed and let us show that $\lim_{s\to s_0}(Tf)(s) = (Tf)(s_0)$. We get

$$
\begin{aligned}
\lim_{s\to s_0}(Tf)(s) &= (1-s_0)\lim_{s\to s_0}\int_{[0,s]} tf(t)dm(t) + s_0 \lim_{s\to s_0}\int_{[s,1]} (1-t)f(t)dm(t) \\
&= (1-s_0)\int_{[0,s_0]} tf(t)dm(t) + s_0\int_{[s_0,1]} (1-t)f(t)dm(t) \\
&= (Tf)(s_0)
\end{aligned}
$$

As $s_0 \in [0,1]$ was arbitrary, this shows that Tf is continuous for any $f \in H$.

Lastly, note that the formula for $(Tf)(s)$ also holds for $s \in \{0,1\}$. The interval $[0,0]$ and $[1,1]$ should be interpreted as the singletons $\{0\}$ and $\{1\}$ respectively. We get

$$
(Tf)(0) = 1\cdot\int_{[0,0]} tf(t)dm(t) + 0\cdot\int_{[0,1]} (1-t)f(t)dm(t) = 1\cdot 0 + 0\cdot\int_{[0,1]} (1-t)f(t)dm(t) = 0.
$$

The integral $\int_{[0,1]}(1-t)f(t)dm(t)$ is finite, since $f \in H \subseteq L_1([0,1],m)$, and $|1-t| \leq 1$. Similarly, we obtain

$$(Tf)(1) = 0 \cdot \int_{[0,1]} tf(t)dm(t) + 1 \cdot \int_{[1,1]} (1-t)f(t)dm(t) = 0 \cdot \int_{[0,1]} tf(t)dm(t) + 1 \cdot 0 = 0.$$

This is what we wanted.

Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

(a) For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$. Justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$. Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

Not shown explicitly only e^{-x^2}

Solution From Homework 7, Problem 1, we know that $x \mapsto e^{-|x|^2/2} = e^{-x^2/2}$ belongs to $\mathscr{S}(\mathbb{R})$, that is, $g_0 \in \mathscr{S}(\mathbb{R})$. Also from Homework 7, Problem 1(a), we know that since $g_0 \in \mathscr{S}(\mathbb{R})$, we get that $x \mapsto x^k e^{-x^2/2}$ belongs to $\mathscr{S}(\mathbb{R})$, that is, $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$.

First, we compute the following derived functions:

$$\left(e^{-x^2/2}\right)' = -xe^{-x^2/2}$$
$$\left(e^{-x^2/2}\right)'' = -e^{-x^2/2} + x^2 e^{-x^2/2}$$
$$\left(e^{-x^2/2}\right)''' = 3xe^{-x^2/2} - x^3 e^{-x^2/2}$$

In the following, we use repeatedly that $\mathcal{F}$ is linear. Also we use Proposition 11.13(b) which states that $\left(f^{(n)}\right)^{\wedge}(\xi) = i^n \xi^n \hat{f}(\xi),\ \ \xi \in \mathbb{R}$. We get the following expressions for $\mathcal{F}(g_k)(\xi)$ for $\xi \in \mathbb{R}$ and $k = 0, 1, 2, 3$.

$$\mathcal{F}(g_0)(\xi) = \mathcal{F}(e^{-x^2/2})(\xi) = e^{-\xi^2/2}$$
$$\mathcal{F}(g_1)(\xi) = \mathcal{F}(xe^{-x^2/2})(\xi) = -\mathcal{F}\left(\left(e^{-x^2/2}\right)'\right)(\xi) = -i\xi e^{-\xi^2/2}$$
$$\mathcal{F}(g_2)(\xi) = \mathcal{F}(x^2 e^{-x^2/2})(\xi) = \mathcal{F}\left(\left(e^{-x^2/2}\right)'' + e^{-x^2/2}\right)(\xi) = -\xi^2 e^{-\xi^2/2} + e^{-\xi^2/2}$$
$$\mathcal{F}(g_3)(\xi) = \mathcal{F}(x^3 e^{-x^2/2})(\xi) = \mathcal{F}\left(3xe^{-x^2/2} - \left(e^{-x^2/2}\right)'''\right)(\xi) = -3i\xi e^{-\xi^2/2} + i\xi^3 e^{-\xi^2/2}$$

Note that in the above, $x^k e^{-x^2/2}$ should be understood as the function $x \to x^k e^{-x^2/2}$.

(b) Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

Solution Consider the following functions:

$$\begin{aligned}
h_0(x) &:= g_0(x) = e^{-x^2/2} \\
h_1(x) &:= 2g_3(x) - 3g_1(x) = 2x^3 e^{-x^2/2} - 3xe^{-x^2/2} \\
h_2(x) &:= 2g_2(x) - g_0(x) = 2x^2 e^{-x^2/2} - e^{-x^2/2} \\
h_3(x) &:= g_1(x) = xe^{-x^2/2}
\end{aligned}$$

The functions h_k are non-zero Schwartz functions since $g_k \in \mathscr{S}(\mathbb{R})$ for $k = 0, 1, 2, 3$. We compute the Fourier transform of h_k using the expressions from part (a) and the fact that $\mathcal{F}$ is linear.

$$\begin{aligned}
\mathcal{F}(h_0)(\xi) &= \mathcal{F}(g_0)(\xi) = e^{-\xi^2/2} = h_0(\xi) = i^0 h_0(\xi) \\
\mathcal{F}(h_1)(\xi) &= \mathcal{F}(2g_3 - 3g_1)(\xi) = 2i\xi^3 e^{-\xi^2/2} - 3i\xi e^{-\xi^2/2} = ih_1(\xi) = i^1 h_1(\xi) \\
\mathcal{F}(h_2)(\xi) &= \mathcal{F}(2g_2 - g_0)(\xi) = -2\xi^2 e^{-\xi^2/2} + e^{-\xi^2/2} = -h_2(\xi) = i^2 h_2(\xi) \\
\mathcal{F}(h_3)(\xi) &= \mathcal{F}(g_1)(\xi) = -i\xi e^{-\xi^2/2} = -ih_3(\xi) = i^3 h_3(\xi)
\end{aligned}$$

We conclude that the functions h_k, for $k = 0, 1, 2, 3$ satisfy the desired properties.

(c) Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$.

Solution Consider the following computations for $f \in \mathscr{S}(\mathbb{R})$ and $\xi \in \mathbb{R}$.

argue $\longrightarrow$

$\mathcal{F}(f) \in \mathscr{S}(\mathbb{R})$

$$\begin{aligned}
\mathcal{F}^2(f)(\xi) = \mathcal{F}(\mathcal{F}(f))(\xi) &= \int_{\mathbb{R}} \mathcal{F}(f)(x) e^{-iy\xi} dm(x) \\
&= \int_{\mathbb{R}} \mathcal{F}(f)(x) e^{iy(-\xi)} dm(x) \\
&= \mathcal{F}^*(\mathcal{F}(f))(-\xi) \\
&= f(-\xi)
\end{aligned}$$

The third equality follows from the definition of the inverse Fourier transform

$$\mathcal{F}^*(g)(\xi) = \int_{\mathbb{R}} g(x) e^{ix\xi} dm(x).$$

The fourth equality follows from the fact that $\mathcal{F}$ and $\mathcal{F}^*$ are inverse to each other on $\mathscr{S}(\mathbb{R})$ due to the Fourier inversion theorem, more specifically Corollary 12.12(iii).
Now for $f \in \mathscr{S}(\mathbb{R})$ and $\xi \in \mathbb{R}$, we get

$$\mathcal{F}^4(f)(\xi) = \mathcal{F}^2(\mathcal{F}^2(f))(\xi) = \mathcal{F}^2(f)(-\xi) = f(-(-\xi)) = f(\xi).$$

This shows that $\mathcal{F}^4(f) = f$ for all $f \in \mathscr{S}(\mathbb{R})$.

(d) Use (c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.

Solution Let $f \in \mathscr{S}(\mathbb{R})$ satisfy $\mathcal{F}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$. By part (c), we know that $\mathcal{F}^3(\mathcal{F}(f)) = \mathcal{F}(\mathcal{F}^3(f)) = f$, so $\mathcal{F}^3 = \mathcal{F}^*$ on $\mathscr{S}(\mathbb{R})$, that is, both $\mathcal{F}^3$ and $\mathcal{F}^*$ is the unique inverse function to $\mathcal{F}$ on $\mathscr{S}(\mathbb{R})$. Using the linearity of $\mathcal{F}$, we get

$$f = \mathcal{F}^*(\mathcal{F}(f)) = \mathcal{F}^*(\lambda f) = \mathcal{F}^3(\lambda f) = \lambda \mathcal{F}^3(f) = \lambda^4 f,$$

where we have used $\mathcal{F}(f) = \lambda f$ multiple times in the last inequality. Since f is non-zero and defined on $\mathbb{R}$, we conclude that $\lambda^4 = 1$. The solutions to this identity is exactly $\lambda \in \{1, i, -1, -i\}$.

To conclude, we have shown in part (b) that all the numbers $\{1, i, -1, -i\}$ are eigenvalues of $\mathcal{F}$. Furthermore, we have just shown that these are the only possible eigenvalues. Therefore, the eigenvalues of $\mathcal{F}$ are precisely $\{1, i, -1, -i\}$. ✓

Problem 5

Let $(x_n)_{n\geq 1}$ be a dense subset of $[0, 1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0, 1]$. Show that $\operatorname{supp}(\mu) = [0, 1]$.

non-empty

Solution Let us start by showing that all open subsets U of $[0, 1]$ have measure $\mu(U) > 0$. Let $U \subseteq [0, 1]$ be open. Since U is open and $(x_n)_{n\geq 1}$ is dense in $[0, 1]$, we have $x_N \in U$ for some $N \geq 1$ (if this was not the case, $(x_n)_{n\geq 1}$ would be contained in the closed set $[0, 1] \setminus U$, so it couldn't be dense in $[0, 1]$). We obtain

$$\mu(U) = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}(U) \geq 2^{-N}\delta_{x_N}(U) = 2^{-N} > 0.$$

non-empty!

So no open sets are null sets. Denote the union of all open null sets by N. We see that N is the empty set. The support of μ, denoted $\operatorname{supp}(\mu)$, is precisely defined as the complement of N, see Homework 8, Problem 3(a). Thus, $\operatorname{supp}(\mu) = [0, 1] \setminus N = [0, 1] \setminus \emptyset = [0, 1]$. This is what we wanted. ✓