

# The ground state energy of dilute 1d many-body quantum systems

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# Overview

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# Background

## The scattering length

### Theorem 1

For  $B_R = \{0 \leq |x| < R\} \subset \mathbb{R}^d$  with  $R > R_0 := \text{range}(v)$ , let  $\phi \in H^1(B_R)$  satisfy

$$-\Delta\phi + \frac{1}{2}v\phi = 0, \quad \text{on } B_R, \quad (1)$$

with boundary condition  $\phi(x) = 1$  for  $|x| = R$ . Then  $\phi(x) = f(|x|)$  for some  $f : (0, R] \rightarrow [0, \infty)$ , and for  $\text{range}(v) < r < R$ , we have

$$f(r) = \begin{cases} (r - a)/(R - a) & \text{for } d = 1 \\ \ln(r/a)/\ln(R/a) & \text{for } d = 2 \\ (1 - ar^{2-d})/(1 - aR^{2-d}) & \text{for } d \geq 3, \end{cases} \quad (2)$$

with some constant  $a$  called the **(s-wave) scattering length**.

# Model

We consider a many-body system of bosons that interacts via a repulsive pair potential  $v_{ij} = v(|x_i - x_j|)$ , with  $v = v_{\text{reg}} + v_{\text{h.c.}}$ .

$$\mathcal{E}(\psi) = \int_{\Lambda_L} \left( \sum_{i=1}^N |\nabla_i \psi|^2 + \sum_{i < j} v_{ij} |\psi|^2 \right) \quad \text{on } L^2(\Lambda_L)^{\otimes_{\text{sym}} N}. \quad (3)$$

The ground state energy is defined by

$$E(N, L) := \inf_{\psi \in \mathcal{D}(\mathcal{E}), \|\psi\|^2=1} \mathcal{E}(\psi).$$

## 2d and 3d

For  $\Lambda_L = [0, L]^d$ , let  $e(\rho) := \lim_{\substack{L \rightarrow \infty \\ N/L^d \rightarrow \rho}} E(N, L)/L^d$ .

Theorem 2 ( $d = 3$  result, Lee-Huang-Yang)

$$e(\rho) = 4\pi\rho^2 a \left( 1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3} + o(\sqrt{\rho a^3}) \right). \quad (4)$$

Theorem 3 ( $d = 2$  result, Fournais et al. 2022)

$$e(\rho) = 4\pi\rho^2 Y \left( 1 - Y |\log Y| + \left( 2\Gamma + \frac{1}{2} + \ln(\pi) \right) Y \right) + o(\rho^2 Y^2), \quad (5)$$

$$Y = |\ln(\rho a^2)|^{-1}.$$

# Main result

For the remainder of the talk,  $d = 1$ .

Theorem 4 (A., R. Reuvers, J. P. Solovej, 2022)

*Consider a Bose gas with repulsive interaction  $v = v_{\text{reg}} + v_{h.c.}$  as defined above. Define the density  $\rho = N/L$ . For  $\rho|a|$  and  $\rho R_0$  sufficiently small, the ground state energy can be expanded as*

$$E(N, L) = N \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a + \mathcal{O} \left( (\rho|a|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3} \right) \right), \quad (6)$$

*where  $a$  is the scattering length of  $v$ .*

# Examples

The hard core gas

Energy behaves like free Fermi energy in volume  $L - NR$ , i.e.

$$\begin{aligned} E_{\text{hard core}}(N, L) &= N \frac{\pi^2}{3} \rho^2 (1 - NR/L)^{-2} \\ &= E_0 (1 + 2\rho R + \mathcal{O}((\rho R)^2)) . \end{aligned} \quad (7)$$

Scattering length is  $a = R$ .

Lieb-Liniger model

Energy behaves asymptotically like

$$E_{LL}(N, L, c) = N \frac{\pi^2}{3} \rho^2 (1 - 4\rho/c + \mathcal{O}((\rho/c)^2)) , \quad (8)$$

with scattering length  $a = -\frac{2}{c}$ .

# Variational principle

To obtain an upper bound, we use the variational principle, *i.e.*

$$E(N, L) \leq \frac{\mathcal{E}(\Psi)}{\|\Psi\|^2}, \quad \text{for any } \Psi \in \mathcal{D}(\mathcal{E}).$$



# Trial state

Trial state has to encapture free Fermi energy, as well as corrections due to scattering processes. Hence we consider

$$\Psi(x) = \begin{cases} \omega(\mathcal{R}(x)) \frac{|\Psi_F(x)|}{\mathcal{R}(x)} & \text{if } \mathcal{R}(x) < b \\ |\Psi_F(x)| & \text{if } \mathcal{R}(x) \geq b, \end{cases}$$

where  $\omega$  is the suitably normalized solution to the two-body scattering equation,  $\Psi_F$  is the free Fermi ground state, and  $\mathcal{R}(x) := \min_{i < j} (|x_i - x_j|)$  is uniquely defined a.e.

# One-particle reduced density matrix

For the free Fermi gas we have

$$\begin{aligned}\gamma^{(1)}(x, y) &= \frac{2}{L} \sum_{j=1}^N \sin\left(\frac{\pi}{L} jx\right) \sin\left(\frac{\pi}{L} jy\right) \\ &= \frac{\pi}{L} \left( D_N\left(\pi \frac{x-y}{L}\right) + D_N\left(\pi \frac{x+y}{L}\right) \right),\end{aligned}\tag{9}$$

where  $D_N(x) = \frac{1}{2\pi} \sum_{k=-N}^N e^{ikx} = \frac{\sin((N+1/2)x)}{2\pi \sin(x/2)}$  is the Dirichlet kernel.

By Wick's theorem all derivatives of reduced density matrices are bounded by a constant times an appropriate power of  $\rho$ .

# Some useful bounds

## Lemma 1

$$\rho^{(2)}(x_1, x_2) \leq \left( \frac{\pi^2}{3} \rho^4 + f(x_2) \right) (x_1 - x_2)^2 + \mathcal{O}(\rho^6 (x_1 - x_2)^4),$$

with  $\int f(x_2) dx_2 \leq \text{const. } \rho^3 \log(N)$ .

## Lemma 2

*We have the following bounds*

$$\rho^{(3)}(x_1, x_2, x_3) \leq \text{const. } \rho^9 (x_1 - x_2)^2 (x_2 - x_3)^2 (x_1 - x_3)^2,$$

$$\rho^{(4)}(x_1, x_2, x_3, x_4) \leq \text{const. } \rho^8 (x_1 - x_2)^2 (x_3 - x_4)^2,$$

$$\left| \sum_{i=1}^2 \partial_{y_i}^2 \gamma^{(2)}(x_1, x_2; y_1, y_2) \Big|_{y=x} \right| \leq \text{const. } \rho^6 (x_1 - x_2)^2,$$
$$\vdots$$

# Collecting everything

## Upper bound

$$E \leq N \frac{\pi^2}{3} \rho^2 \frac{\left(1 + 2\rho a \frac{b}{b-a} + \text{const.} \left[ \frac{1}{N} + N(b\rho)^3 \left(1 + \rho b^2 \int v_{\text{reg}}\right)\right]\right)}{\|\Psi\|^2}, \quad (10)$$

where the finite measure  $v_{\text{reg}}$  is  $v$  with any hard core removed. By lemma 1 we know  $\|\Psi\|^2 \geq 1 - \text{const. } N(\rho b)^3$ .

## Localization

Divide into  $M$  smaller boxes with  $\tilde{N} = N/M$  particles in each, and make distance  $b$  between boxes (no interaction between boxes), and choose  $M$  such that  $\tilde{N} = (\rho b)^{-3/2} \gg 1$ .

# Upper Bound

After localization

$$E(N, L) \leq N \frac{\pi^2}{3} \rho^2 \frac{\left(1 + 2\rho a \frac{b}{b-a} + \text{const.} \frac{M}{N} + \text{const.} \tilde{N}(b\rho)^3 (1 + \rho b^2 \int v_{\text{reg}})\right)}{1 - \tilde{N}(\tilde{\rho}b)^3} \quad (11)$$

Choosing  $b = \max(\rho^{-1/5} |a|^{4/5}, R_0)$  we find

**Proposition 1 (Upper bound Theorem 4)**

*There exists a constant  $C_U > 0$  such that for  $\rho|a|$ ,  $\rho R_0 \leq C_U^{-1}$ , the ground state energy  $E^D(N, L)$  satisfies*

$$E^D(N, L) \leq N \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho a + C_U \left((\rho|a|)^{6/5} + (\rho R_0)^{3/2} + N^{-1}\right)\right). \quad (12)$$

# Lower bound

Proof of lower bound consists of the following steps:

- 1 Use Dyson's lemma to reduce to a nearest neighbor double delta-barrier potential.
- 2 Reduce to the Lieb Liniger model by discarding **a small part** of the wave function.
- 3 Use a known lower bound for the Lieb Liniger model.

# The Lieb-Liniger (LL) model

$$H_{LL} = - \sum_{i=1}^n \partial_i^2 + 2c \sum_{i < j} \delta(x_i - x_j). \quad (13)$$

Behavior in thermodynamic limit:  $\lim_{\substack{\ell \rightarrow \infty, \\ n/\ell \rightarrow \rho}} E_{LL}(n, \ell, c)/\ell = \rho^3 e(\gamma)$

with  $\gamma = c/\rho$ .

Lemma 3 (Lieb-Liniger lower bound)

Let  $\gamma > 0$ , then

$$e(\gamma) \geq \frac{\pi^2}{3} \left( \frac{\gamma}{\gamma + 2} \right)^2 \geq \frac{\pi^2}{3} \left( 1 - \frac{4}{\gamma} \right). \quad (14)$$

# Reducing to the LL model

## Lemma 4 (Dyson)

Let  $R > R_0 = \text{range}(v)$  and  $\varphi \in H^1(\mathbb{R})$ , then for any interval  $\mathcal{I} \ni 0$

$$\int_{\mathcal{I}} |\partial \varphi|^2 + \frac{1}{2} v |\varphi|^2 \geq \int_{\mathcal{I}} \frac{1}{R-a} (\delta_R + \delta_{-R}) |\varphi|^2, \quad (15)$$

where  $a$  is the  $s$ -wave scattering length.

Hence we have, denoting  $\mathfrak{r}_i(x) = \min_j (|x_i - x_j|)$

$$\begin{aligned} \int \sum_i |\partial_i \Psi|^2 + \sum_{i \neq j} \frac{1}{2} v_{ij} |\Psi|^2 \geq \\ \int \sum_i |\partial_i \Psi|^2 \chi_{\mathfrak{r}_i(x) > R} + \sum_i \frac{1}{R-a} \delta(\mathfrak{r}_i(x) - R) |\Psi|^2. \end{aligned} \quad (16)$$



# Reducing to the LL model

Define  $\psi \in L^2([0, \ell - (n-1)R]^n)$  by

$$\psi(x_1, x_2, \dots, x_n) = \Psi(x_1, R + x_2, \dots, (n-1)R + x_n),$$

for  $x_1 \leq x_2 \leq \dots \leq x_n$  and symmetrically extended.

Then

$$\begin{aligned} \mathcal{E}(\Psi) &\geq E_{LL}^N(n, \ell - (n-1)R, 2/(R-a)) \langle \psi | \psi \rangle \\ &\geq n \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho(a - R) + 2\rho R - \text{const.} \frac{1}{n^{2/3}} \right) \langle \psi | \psi \rangle. \end{aligned} \tag{17}$$

# Lower bound for mass of $\psi$

## Lemma 5

*Let  $\psi$  be defined as above, then*

$$1 - \langle \psi | \psi \rangle \leq 8 \left( R^2 \sum_{i < j} \int_{B_{ij}} |\partial_i \Psi|^2 + R(R - a) \sum_{i < j} \int v_{ij} |\Psi|^2 \right), \quad (18)$$

Combining lemmas 4 and 5 we have the following lemma:

## Lemma 6

*For  $n(\rho R)^2 \leq \frac{3}{16\pi^2} \frac{1}{8}$ ,  $\rho R \ll 1$  and  $R > 2|a|$  we have*

$$\langle \psi | \psi \rangle \geq 1 - \text{const.} \left( n(\rho R)^3 + n^{1/3}(\rho R)^2 \right). \quad (19)$$

# Lower bound

By the reduction to the LL model we find

## Proposition 2

*For assumptions as in lemma 6 we have*

$$E^N(n, \ell) \geq n \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a + \text{const.} \left( \frac{1}{n^{2/3}} + n(\rho R)^3 + n^{1/3}(\rho R)^2 \right) \right). \quad (20)$$

## Corollary 1

*For  $n = \text{const.}$   $(\rho R)^{-9/5}$  we have*

$$E^N(n, \ell) \geq n \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a - \text{const.} \left( (\rho R)^{6/5} + (\rho R)^{7/5} \right) \right). \quad (21)$$

# Lower bound localization

To prove the lower bound, we localize, as in the upper bound, to smaller boxes.

## Lemma 7

Let  $\Xi \geq 4$  be fixed and let  $n = m\Xi\rho\ell + n_0$  with  $n_0 \in [0, \Xi\rho\ell)$  for some  $m \in \mathbb{N}$  with  $n^* := \rho\ell = \mathcal{O}(\rho R)^{-9/5}$ . Furthermore, assume that  $\rho R \ll 1$  and let  $\mu = \pi^2\rho^2 \left(1 + \frac{8}{3}\rho a\right)$ , then

$$E^N(n, \ell) - \mu n \geq E^N(n_0, \ell) - \mu n_0. \quad (22)$$

## Proposition 3 (Lower bound Theorem 4)

There exists a constant  $C_L > 0$  such that the ground state energy  $E^N(N, L)$  satisfies

$$E^N(N, L) \geq N \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a - C_L \left( (\rho|a|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3} \right) \right). \quad (23)$$

# Spinless/spin-polarized fermions

Spinless Fermions are unitarily equivalent to Bosons with a zero b.c. at all planes of intersection, *i.e.* with an infinite delta potential. As a consequence we have the following corollary.

## Theorem 5 (spinless fermions)

*Consider a Fermi gas with repulsive interaction  $v = v_{\text{reg}} + v_{\text{h.c.}}$  as defined before. Let  $E_F(N, L)$  be its associated ground state energy. Write  $\rho = N/L$ . For  $\rho a_o$  and  $\rho R_0$  sufficiently small, the ground state energy can be expanded as*

$$E_F(N, L) = N \frac{\pi^2}{3} \rho^2 \left( 1 + 2\rho a_o + \mathcal{O} \left( (\rho R_0)^{6/5} + N^{-2/3} \right) \right), \quad (24)$$

*where  $a_o \geq 0$  is the odd wave scattering length of  $v$ .*

This is consistent with lower bound  $E_F(N, L) \geq E_0$ , since  $a_o \geq 0$ .

# A conjecture for spin-1/2 fermions

Two solvable model for spin-1/2 fermion:

The hard core gas

Ground state energy is independent of spin so

$$E_{\text{hard core}}(N, L) = N \frac{\pi^2}{3} \rho^2 (1 - NR/L)^{-2} \approx E_0(1 + 2\rho R). \quad (25)$$

Scattering length is  $a_e = a_o = R$ .

Yang-Gaudin model

Is the spin-1/2 version of the LL model, *i.e.*  $H_{YG} = H_{LL}$ . Behaves asymptotically like

$$E_{YG}(N, L, c) = N \frac{\pi^2}{3} \rho^2 \left( 1 - 4\rho \ln(2)/c + \mathcal{O}((\rho/c)^2) \right), \quad (26)$$

with scattering length  $a_e = -\frac{2}{c}$ ,  $a_o = 0$ .

# A conjecture for spin-1/2 fermions

Based on the two solvable cases, we expect

$$E(N, L) = N \frac{\pi^2}{3} \rho^2 \left( 1 + 2 \ln(2) \rho a_e + 2(1 - \ln(2)) \rho a_o \right. \\ \left. + \mathcal{O} \left( (\rho \max(|a_e|, a_o))^2 \right) \right) \quad (27)$$

Thanks for your attention!