# Functional Analysis, assignment 1

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## Problem 1a

I will show the three properties of being a norm.
The triangle inequality:

$$\begin{aligned} \|x+y\|_0 &= \|x+y\|_X + \|T(x+y)\|_Y \\ &\leq \|x\|_X + \|y\|_X + \|Tx+Ty\|_Y \\ &= \|x\|_X + \|Tx\|_Y + \|y\|_X + \|Ty\|_Y \\ &= \|x\|_0 + \|y\|_0 \forall x, y \in X \end{aligned}$$

First equality is by definition, second by T-linear.

Scalar-multiplication:

$$\begin{aligned} \|\alpha x\|_0 &= \|\alpha x|_X + \|T(\alpha x)\|_Y \\ &= \|\alpha x|_X + \|\alpha Tx\|_Y \\ &= |\alpha|\|x\|_X + |\alpha|\|Tx\|_Y \\ &= |\alpha|(\|x\|_X + \|Tx\|_Y) \\ &= |\alpha|\|x\|_0 \end{aligned}$$

$\alpha \in \mathbb{K}, x \in X$

Non-seminorm:

$$\|x\|_0 = 0 \Rightarrow \|x\|_X + \|Tx\|_Y = 0$$

So either $\|x\|_X = 0$ or $\|Tx\|_Y = 0$, but $\|x\|_X \geq 0$ and $\|Tx\|_Y \geq 0$.
So for $\|x\|_X = 0$ we may have $x = 0$ and if $x = 0$ then we will get that $\|0\|_0 = \|0\|_X + \|T0\|_Y = \|0\|_X + \|0\|_Y = 0$.
Hence $\|x\|_0 = 0 \Leftrightarrow x = 0$
It can now be concluded that $\|\cdot\|_0$ is a norm on $X$.

## Problem 1b

I want to show that any linear map $T : X \to Y$ is bounded, if $X$ is finite dimensional.
I Assume that $\dim V = n < \infty$ for a vector space $V$. I will now use theorem 1.6 which says that if $X$ finite then any two norms on $X$ is equivalent. I showed in a) that $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent norms on the linear map $T$, and gives us that $T$ will be bounded. Since $T$ is an arbitrary linear map, I get that all linear maps must be bounded where it applies that $\dim V = n < \infty$.

## Problem 1c

I will now assume that $X$ is infinite dimensional vector space and show that there exist a linear map $T : X \to Y$ which is not bounded.
Since it is given that $X$ is infinite I start by taking a Hamel-basis $B_X$, and I will define the basis as $B_X := \{b_i : i \in I\}$.
I will now let $b \in X$ such that the following applies:

$$T\left(\frac{b_i}{\|b_i\|}\right) = i \cdot y$$

where $y \in Y$ but $y \neq 0$ and $i \in \mathbb{N}$. If $i \notin \mathbb{N}$ then I will have

$$T\left(\frac{b_i}{\|b_i\|}\right) = 0$$

I let $N := \{b \in X : \|b\| \leq 1\}$ where $\left\{\frac{b_i}{\|b_i\|}\right\}_{i\in I} \subseteq \{b \in X : \|b\| \leq 1\}$

I.e $\left\{\frac{b_i}{\|b_i\|}\right\}_{i\in I} \subseteq N$

I also have that $\sup_{x\in N} \|Tb\| \geq i\|y\| > 0$ for $i \in I$. All this gives us that $T$ is not bounded.

## Problem 1d

As before $X$ is infinite. In c) I showed that $T$ is not bounded, i.e $X$ is not bounded. So if $X$ is not bounded then I have from a) that the two norms $\|\cdot\|_0$ and $\|\cdot\|_X$ cannot be equivalent on $X$. I will now look at $\|x\|_0 = \|x\|_X + \|Tx\|_Y$ to get $\|x\|_X \leq \|x\|_0$. If I just remove $\|Tx\|_Y$ I will end up with having what I wanted: $\|x\|_X \leq \|x\|_0$. I will now conclude $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space. So I got that the two norms are not equivalent on $X$ then by HW 3 P1 I have that $X$ cannot be complete. So I will now assume that $(X, \|\cdot\|_X)$ is a Banach space to see what happens. So if it applies it will be complete. But for $(X, \|\cdot\|_0)$ to be complete it should apply that the norms should be equivalent, but I just earlier said that they are not. Hence $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space.

## Problem 1e

To give an example of a vector space $X$ equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ that satisfies $\|x\|' \leq \|x\|$ for all $x \in X$ such that $(X, \|\cdot\|)$ is complete, while $(X, \|\cdot\|')$ is not, I will start by taking $\ell_1(\mathbb{N})$ with the norm $\|\cdot\|_1$ and $\|\cdot\|_\infty$
Then I have that $(\ell_1(\mathbb{N}), \|\cdot\|_1)$ is complete since $(\ell_1(\mathbb{N}), \|\cdot\|_p)$ is complete for $1 \leq p < \infty$ from lecture notes 1. I can now take a sequence $(x_1, ..., x_n)$ in $\ell_1(\mathbb{N})$ then I will get that

$$\|x\|_1 = \sum_{i=1}^{n} |x_n| \geq x_1 + x_2 + ... + x_n \geq \max_{i \in 1,...,n} \{|x_n|\} = \|x\|_\infty$$

i.e I now have that $\|x\|_\infty \leq \|x\|_1$.

I let $a_i = 1$ if $i \leq k$ and take a sequence $(a_n)_{n\in\mathbb{N}} = (a_1, a_2, ..., a_k, 0, 0, ...)$ then I will have that $\|a_n\|_\infty = 1$ but $\|a_n\|_1 = k$, so therefore can we say that there do not exist a $c$ such that $k \leq c \cdot 1$ since I always can choose a $k$ which is bigger. Hence The two norms are inequivalent.

I will now show $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete, by taking a sequence of sequences, i.e $((x_n)(c))_{n\in\mathbb{N}} = \frac{1}{c}$ for $1 \leq c \leq n$ and $(x_n)(c) = 0$ for $c > n$. Since all $x_n(c)$ have finite sum the one-norm I will get $x_n(c) \in \ell_1$. If I let $x(c) = \frac{1}{c}$ $\forall c \in \mathbb{N}$ then I see that:

$$\|x_n(c) - x(c)\|_\infty = \sup\{|x_n(c) - x(c)|\} = |\frac{1}{n+1}| \to 0$$

for $n \to \infty$.
I can now see that it is a Cauchy sequence with respect to the infinity-norm, which gives us that $\sum_{n=1}^{\infty} |\frac{1}{n+1}| \to \infty$ for $n \to \infty$ i.e $x(c) \notin \ell_1$
hence $(\ell_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete.

## Problem 2a

I want to show that $f$ is bounded, and to do this I let $\alpha, \beta \in \mathbb{C}$,
$\gamma = (a_1, b_1, 0, 0, 0, ...) \in M$ and $\delta = (a_2, b_2, 0, 0, 0, ...) \in M$, then I will have:

$$f(\alpha \cdot \gamma + \beta \cdot \delta) = f(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, 0, 0...)$$

$$= \alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2 = \alpha(a_1 + b_1) + \beta(a_2 + b_2)$$

$$= \alpha f(\gamma) + \beta f(\delta)$$

I do now have that $f$ is linear.

I will now show that $f$ is bounded. I will show that $\exists c > 0$ such that it apply $\|a+b\|_1 \leq C \cdot \|(a,b,0,0...)\|_p = C \cdot \|a,b\|_p$
From lecture notes I have that $\|a,b\|_p$ is a norm on $\mathbb{C}^2$. Furthermore I see that

$$\|a+b\|_1 = |a+b| \leq |a| + |b| = \|(a+b)\|_1$$

$$\leq C \cdot \|a,b\|_p = C\|(a,b,0,0...)\|_p$$

The first inequality is by triangular inequality, the second one is because we have that $\mathbb{C}^2$ is finite dimensional vector space. So then I will have that every norm in $\mathbb{C}^2$ is equivalent. This means that there exists a $c > 0$ such that the inequality will apply for all $(a,b) \in \mathbb{C}^2$.

I will now compute $\|f\|$:
I will claim that the following holds: $\|f\| = 2^{1-\frac{1}{p}}$
I will now show it by defining $d = \left(\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}}, 0, 0, ...\right) \in M$ where it apply that $\|d\|_p = 1$

I also have that:

$$\|f\| = \sup\{|a+b| : \|(a,b,0,0...)\|_p = 1\}$$

$$\geq \left|\frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}}\right| = \frac{2}{2^{\frac{1}{p}}} = 2^{1-1/p}$$

I.e I now have that $\|f\| \geq 2^{1-1/p}$.
The inequality applies because $\left|\frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}}\right| \in \{|a+b| : \|(a,b,0,0...)\|_p = 1\}$.

I will now show $\|f\| \leq 2^{1-1/p}$:
I have that

$$|a+b| \leq |a| + |b| = \|(a,b,0,0,...)\|_1$$

$$= \|(a \cdot 1, b \cdot 1, 0, 0, ...)\|_1 \leq \|(a,b,0,0,...)\|_p \cdot \|(1,1,0,0,...)\|_q$$

The inequality is by Hölder with $\frac{1}{p} + \frac{1}{q} = 1$, where I have that $p$ is fixed, so I just look at $q$. I choose $q = \frac{p}{p-1}$ such that $\frac{1}{p} + \frac{1}{q} = 1$ applies.

I will now let $\|(a,b,0,0,...)\|_p = 1$ and get

$$|a+b| \leq \|(1,1,0,0,...)\|q$$

$$= \left(\sum_{i=1}^{2} |1|^q\right)^{\frac{1}{q}} = 2^{\frac{1}{q}} = 2^{1-\frac{1}{p}}$$

The inequality will apply for all $\|(a, b, 0, 0, ...)\|_1$, then by this I have

$$\|f\| = \sup\{|a + b| : \|(a, b, 0, 0, ...)\|_p = 1\} \leq 2^{1-\frac{1}{p}}$$

Then for all $|a + b|$ in the supremum set I will get $\|a + b\| \leq 2^{1-\frac{1}{p}}$
Since I have shown that $\|f\| \leq 2^{1-\frac{1}{p}}$ and $\|f\| \geq 2^{1-\frac{1}{p}}$ I can conclude that $\|f\| = 2^{1-\frac{1}{p}}$.

## Problem 2b

I will start by showing that there exist a functional $F$. I have that $(\ell_p(\mathbb{N}), \|\cdot\|_p)$ is a normed vector space, where $M \subseteq (\ell_p(\mathbb{N}), \|\cdot\|_p)$. From a) I have that $f$ is both linear and bounded, i.e $f \in M^*$, hence I can use corollary 2.6 together with Hahn-Banach extension theorem to say that there exists $F \in \ell_p(\mathbb{N})^*$ such that $F|_M = f$ and that $\|F\| = \|f\|$.

I will now show uniqueness:
I start by recalling from HW 1 p 5 that $(\ell_p(\mathbb{N}))^* \cong \ell_q(\mathbb{N})$ where I notice that $1 = \frac{1}{p} + \frac{1}{q}$. I define a isometrically isomorphic function $T : \ell_q(\mathbb{N}) \to (\ell_p(\mathbb{N}))^*$, $T(x) = f_x$ and $f : \ell_p(\mathbb{N}) \to \mathbb{C}$ where I have that $f_x(y) = \sum_{n\in\mathbb{N}} x_n y_n$ for $x \in \ell_q(\mathbb{N})$ and $y \in \ell_p(\mathbb{N})$.

I will now let $F : \ell_p(\mathbb{N}) \to \mathbb{C}$ where I get $F(x_1, x_2, x_3, ...) = a + b$. I can now see that this is a Hahn-Banach extension of $f$.

I can now assume $F \neq \tilde{F}$ another Hahn-Banach extension, and then I will have that:
$\exists i \in \mathbb{N}, i > 2 : \tilde{F}(e_i) = c \neq 0$ where I notice that $e_i = (0, 0, ..., i, 0, ...)$.
Now I have that $F \in \ell_p(\mathbb{N}))^*$, hence there exists an unique $\xi \in \ell_q(\mathbb{N})$ such that $T(\xi) = F$
Hence $T(\xi) = (x_1, x_2, x_3, ...) = F(x_1, x_2, x_3, ...) = x_n + x_2$ Where I notice that $(x_1, x_2, x_3, ...) \in \ell_p(\mathbb{N})$
$T(\xi) = (x_1, x_2, x_3, ...) = \sum_{n\in\mathbb{N}} x_n \xi_n$ where $\xi = (\xi_1, \xi_2, ...))$ so I get that $\xi = (1, 1, 0, 0, ...)$.

Since I have that $\tilde{F} \in (\ell_p(\mathbb{N}))^*$ then I can find a $\phi \in \ell_q(\mathbb{N})$ such that $T(\phi) = \tilde{F}$ and since $T$ is bijective I have that $\phi \neq \xi$. So I will have $T(\xi)(x_1, x_2, x_3, 0, ...) = x_1 + x_2 = \sum_{n\in\mathbb{N}} x_n \phi_n$ because I have $\tilde{F}|_M = f$.
Hence

$$\phi = \xi + \sum_{i=4}^{\infty} \alpha_i e_i$$

for $x_i\mathbb{C}$ and $e_o = (0, 0, ..., i, 0, ...)$.
But then I will get that

$$\|\phi\|_q = \|\sum_{i=4}^{\infty} \alpha_i e_i\|_q = (1 + 1 + \sum_{i=4}^{\infty} |\alpha_i e_i|^q)^{\frac{1}{q}} > \|\phi\|_q = 2^{\frac{1}{q}}$$

Which is a contradiction and I now have that there is a unique linear functional $F$ on $\ell_p(\mathbb{N})$ extending $f$ and satisfying $\|F\| = \|f\|$.

# Problem 2c

I will show that if $p = 1$ then there are infinitely many linear functional $F$ on $\ell_1(\mathbb{N})$ extending $f$ and satisfying $\|F\| = \|f\|$.
So I start by letting $p = 1$ as given and will define $F_i : \ell_1(\mathbb{N}) \to \mathbb{C}$. The functions is given by $(x_1, x_2, ...) \mapsto x_1 + x_2 + x_i$ for $i > 2$.
$F_i$ is an extension and linear functional on $\ell_1(\mathbb{N})$, as a result of $F_i|_M(x) = x_1 + x_2 = f(x)$ where I have that $x \in M$. This means that $F_i$ extends $f$ and then $\|F_i\| \geq \|f\| = 2^{1-1/p} = 2^{1-1/1} = 1$

I will now look at the 1-norm on $F$:

$$\begin{aligned} \|F_i\|_1 &= \sup\{|F_i x| : \|x\|_1 = 1\} \\ &= \sup\{|x_1 + x_2 + x_i| : \|x\|_1 = 1\} \\ &= \sup\{|x_1| + |x_2| + |x_i| : \|x\|_1 = 1\} \\ &\leq 1 \end{aligned}$$

Before I got $\|F_i\| \geq 1$ and now I have $\|F_i\| \leq 1$, hence $\|F_i\| = 1$. This implies that $\|F_i\|$ is linear functional extending $f$. I can now conclude that there are infinitely many linear functional $F$ on $\ell_1(\mathbb{N})$ extending $f$ and satisfying $\|F\| = \|f\|$.

# Problem 3a

I have to show that no linear map $F : X \to \mathbb{K}^n$ is injective. I will do the exercise by contradiction.
I start by assuming that the map $F : X \to \mathbb{K}^n$ is injective.
Then I let $x_1, ..., x_{n+1}$ be linear independent in $X$ and $F(x_1), ..., F(x_{n+1})$ is linear dependent.
There exists $a_1, ..., a_{n+1} \in \mathbb{K}$ and not all are equal to zero where I have
$a_1 F(x_1)+, ..., +a_{n+1} F(x_{n+1}) = 0$ because we had $F(x_1), ..., F(x_{n+1})$ is linear dependent.

From linearity of $F$ I will now get
$0 = a_1 F(x_1)+, ..., +a_{n+1} F(x_{n+1}) = F(a_1 x_1+, ..., +a_{n+1} x_{n+1})$. This gives us

that $a_1x_1+,...,+a_{n+1}x_{n+1} = 0$ since we at first assumed that the map $F$ is injective. From this and by using that $x_1,...,x_{n+1}$ be linear independent we now have that $a_i = 0$. This implies a contradiction because we earlier said that not all $a's$ are equal to zero, but at least one is and now we get that all $a$ are equal to zero. So since I got to a contradiction I can now say that our map $F : X \to \mathbb{K}^n$ is not injective. From this I conclude that no linear map $F : X \to \mathbb{K}^n$ is injective.

## Problem 3b

I will show $\bigcap_{j=1}^n ker(f_j) \neq \{0\}$. I start by looking at the map $F : X \to \mathbb{K}^n$ which is given by $F(x) = (f_1(x), f_2(x), ..., f_n(x)), x \in X$.

We have from exercise a) that no linear map is injective, this implies that the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), ..., f_n(x)), x \in X$ is not injective. This gives us that $ker(F) \neq \{0\} \Rightarrow ker(f_1(x), f_2(x), ..., f_n(x)) \neq \{0\}$

So because I have $ker(F) \neq \{0\}$ there exist a $x \neq 0$ where it applies
$F(x) = (f_1(x), f_2(x), ..., f_n(x)) = 0$ and then I will have that each of them also will be equal to zero as: $f_1(x) = 0, f_2(x) = 0, ..., f_n(x) = 0$
Hence I can conclude $\{0\} \neq ker(F) = \bigcap_{j=1}^n ker(f_j)$

## Problem 3c

I will show that there exists a $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$.

From b) I have that $\bigcap_{j=1}^n ker(f_j) \neq \{0\}$, hence I will now pick $z$ non-zero in $\bigcap_{j=1}^n ker(f_j)$
From this I will now define $y = \frac{z}{\|z\|}$ and look at $f_j(y)$.

$f_j(y) = f_j\left(\frac{z}{\|z\|}\right) = \frac{1}{\|z\|} f_j(z)$. Since I chose a $z \in \bigcap_{j=1}^n ker(f_j)$ we will have that $f_j(z) = 0$ so $f_j(y) = 0$. This implies $y \in \bigcap_{j=1}^n ker(f_j)$ and hence $\|y\| = \|\frac{z}{\|z\|}\| = \frac{\|z\|}{\|z\|} = 1$. Hence I now have that there exists $y \in \bigcap_{j=1}^n ker(f_j) \subseteq X$ such that $\|y\| = 1$.

I will now show $\|y - x_j\| \geq \|x_j\|$. I know that $\|y - x_j\| = \|f_j\| \cdot \|y - x_j\|$ since $\|f_j\| = 1$, this applies because we have that $X$ is infinite dimensional normed vector space and by theorem 2.7(b) where $f_j \in X^*$.

$\|y - x_j\| = \|f_j\| \cdot \|y - x_j\| \geq \|f_j(y - x_j)\| = |f_j(y - x_j)| = |f_j(y) - f_j(x_j)| = |0 - \|x_j\|| = \|x_j\|$. The inequality applies by definition of the norm operator. The first equality after the inequality applies since the norm in $\mathbb{K}$ the absolute value. The second equality is by linearity, the third is from problem 3b.
Hence I can conclude $\|y - x_j\| \geq \|x_j\|$.

## Problem 3d

I will show that one cannot cover the unit sphere $S = \{x \in X : \|x\| = 1\}$ with a finite family of closed balls in $X$ such that none of the balls contains 0. I will call this closed balls $B_i$. I will show that $S$ cannot be covered, i.e I will show that $S \nsubseteq \bigcup_{i=1}^{n} B_i$. In other words I take a $x \in S$ and will now show that $x \notin \bigcup_{i=1}^{n} B_i$ this means that we can take a $x \in \bigcap_{j=1}^{n} ker(f_j) \bigcap S \subseteq S \nsubseteq \bigcup_{i=1}^{n} B_i$

I will now show that $B_i$ is convex to determine weather $x$ lies in $B_i$ or not
For every $x, y \in B_i$ and for $0 \leq \alpha \leq 1$ it applies that $\alpha x + (1-\alpha)y \in B_i$.

$$\|\alpha x + (1-\alpha)y - p\| = \|\alpha x - \alpha p + (1-\alpha)y - p + \alpha p\|$$

$$= \|\alpha(x-p) + (1-\alpha)y - p(1-\alpha)\| = \|\alpha(x-p) + (1-\alpha)(y-p)\|$$

$$\leq \|\alpha(x-p)\| + \|(1-\alpha)(y-p)\| = |\alpha| \cdot \|(x-p)\| + |(1-\alpha)| \cdot \|(y-p)\|$$

$$= \alpha\|(x-p)\| + (1-\alpha)\|(y-p)\| \leq \alpha r + (1-\alpha)r = \alpha r + r - \alpha r = r$$

I can now conclude that $B_i$ is convex since I showed $\|\alpha x + (1-\alpha)y - p\| \leq r$.

For $x$ to lie in $B_i$, where I have that $B_i$ is convex, it applies by Hahn-Banach that $Re(f_j)) \geq 1$.

So these $x$ which is in $\bigcap_{j=1}^{n} ker(f_j)$ do not lie in $B_i$, because if $x \in \bigcap_{j=1}^{n} ker(f_j)$ then I will have $f_j(x) = 0$ and if this applies then I will get that $Re(f_j(x)) = 0$ which is different from 1 as I said earlier. Hence I can conclude that $x \notin B_i$

From this I get $\bigcap_{j=1}^{n} ker(f_j) \bigcap B_i = \emptyset$ hence $\bigcap_{j=1}^{n} ker(f_j) \bigcap B_i \bigcap S = \emptyset$. This means that if I $x \in \bigcap_{j=1}^{n} ker(f_j) \bigcap B_i \bigcap S$ then $x \notin \bigcup_{i=1}^{n} B_i$

## Problem 3e

I start by showing that $S$ is not compact and will do it by contradiction.
I assume that $S$ is compact then for any $x \in S$ I will consider a open ball $B_x = \{v \in X : \|x - v\| < \frac{1}{2}\}$. So I take $x \in S$ then I will get $\|x - x\| = 0 < \frac{1}{2}$ so therefore by compactness $\{B_x\}_{x \in S}$ is an open covering of S. This implies $x \in B_x$ and $x \in \bigcup \{B_x\}_{x \in S}$, hence $S \subseteq \bigcup \{B_x\}_{x \in S}$ .

Compactness of $S$ applies that every open cover of this $S$ will have a finite subcover, the same applies with the open balls $\{B_x\}_{x \in S}$ will have a finite subcover $\{B_{x_1}, ..., B_{x_n}\}$ since $\{B_x\}_{x \in S}$ is an open cover of $S$.
I have that $\bigcup_{i=1}^{n} B_{x_i} \subseteq \bigcup_{i=1}^{n} \overline{B_{x_i}}$ since $B_{x_i} \subseteq \overline{B_{x_i}}$. So now I have that $B_{x_i}$ is a finite subcover and will get that $S \subseteq \bigcup_{i=1}^{n} B_{x_i}$, and $S \subseteq \bigcup_{i=1}^{n} \overline{B_{x_i}}$. I now have that $\{\overline{B_{x_1}}, ..., \overline{B_{x_n}}\}$ is a closed ball covering of $S$ and none of them will contain 0, because I have $\overline{B_x} = \{v \in X : \|x - v\| \leq \frac{1}{2}\}$

Since $x \in S$ I have $\|x - 0\| = \|x\| = 1$, but $1 > \frac{1}{2}$ then $0 \notin \overline{B_{x_i}}$. I have now shown that there exist a family with closed balls which is covering $S$ and contains 0. But this is a contradiction with problem 3d, where I showed that there do not exists a finite family of closed balls in $X$ such that none of the balls will contains 0. In this exercise I just found such a family:$\{\overline{B_{x_1}}, ..., \overline{B_{x_n}}\}$. So I can conclude that $S$ is non-compact.

I have that $S \subseteq B$ and $B$ is the closed unit ball. We have a property which says that if $B$ is compact then $S$ is compact since a closed subset of a compact space is again compact. But I will now use the contradiction of the statement. So since we showed earlier that $S$ is not compact then I will have that the closed $B$ in $X$ is neither compact.

## Problem 4a

To determine weather $E_n \subset L_1([0,1], m)$ is absorbing or not I first look at if it is convex or not. The definition of convex is that $\forall f, g \in E_n$ and $\forall 0 < \alpha < 1$ we have $\alpha f + (1-\alpha)g \in E_n$. I will start by showing

$$\left( \int_{[0,1]} |\alpha f + (1-\alpha)g|^3 dm \right) \leq n$$

To do that I will use Minkowski´s inequality:

$$\left( \int_{[0,1]} |\alpha f + (1-\alpha)g|^3 dm \right)^{\frac{1}{3}} \leq \left( \int_{[0,1]} |\alpha f|^3 dm \right)^{\frac{1}{3}} + \left( \int_{[0,1]} |(1-\alpha)g|^3 dm \right)^{\frac{1}{3}}$$

$$= \left( \int_{[0,1]} \alpha^3 |f|^3 dm \right)^{\frac{1}{3}} + \left( \int_{[0,1]} (1-\alpha)^3 |g|^3 dm \right)^{\frac{1}{3}}$$

$$= \alpha \left( \int_{[0,1]} |f|^3 dm \right)^{\frac{1}{3}} + (1-\alpha) \left( \int_{[0,1]} |g|^3 dm \right)^{\frac{1}{3}}$$

$$\leq \alpha n^{\frac{1}{3}} + (1-\alpha)^{\frac{1}{3}} = n^{\frac{1}{3}}$$

$f, g \in E_n$.
I can now say that the following applies

$$\left( \int_{[0,1]} |\alpha f + (1-\alpha)g|^3 dm \right) \leq n$$

and this gives us $\alpha f + (1-\alpha)g \in E_n$ hence $E_n$ is convex.

I will now see if $E_n$ is absorbing.
I have shown that $E_n$ is convex, so now I will show weather the following holds:

$\forall f \in L_1([0,1],m)\ \exists t > 0 : t^-1 f \in E_n$
I let $f(t) = t^{-\frac{1}{3}}$ and look at the following:

$$\|f\|_1 = \int_{[0,1]} f dm = \int_0^1 x^{-\frac{1}{3}} dx = \left[\frac{1}{-\frac{1}{3}+1} x^{-\frac{1}{3}}\right]_0^1 = \frac{3}{2} < \infty$$

Hence $f \in L_1([0,1],m)$ where we note that $f(t)$ is measurable.

I will now look at for any $t > 0$

$$\int_{[0,1]} |f|^3 dm = \int_0^1 |f|^3 dm = \int_0^1 \frac{1}{x} dx \to \infty$$

Hence $\int_0^1 \frac{1}{x} dx \approx \infty$. This means that $f \notin L_3([0,1],m)$ and this gives us that there do not exists $t > 0$ such that $t^-1 f \in E_n$

From $\int_{[0,1]} |f|^3 dm \approx \infty$ I get that $\int_{[0,1]} |t^-1 f|^3 dm \approx \infty$. Hence I now have that $\int_{[0,1]} |t^-1 f|^3 dm \not\leq n$, so now can I conclude that $E_n$ is not absorbing

## Problem 4b

I want to show that $E_n$ has empty interior in $L_1([0,1],m)$ for all $n \geq 1$. I will first look at the definition of a interior which says: The union of all open sets $U \subset E \subset X$, it is the largest open sets contained in $E$, and it is denoted as $E^\circ$.

I start by showing that $E_n^\circ = \emptyset$ and doing it by contradiction, i.e I will assume $E_n^\circ \neq \emptyset$. Hence I have $f \in E_n^\circ$ gives us the open ball

$$B(f, \epsilon := \{g \in L_1([0,1],m) : \|f - g\|_1 < \epsilon\} \subseteq E_n$$

for $\epsilon > 0$.

For $0 \neq g \in L_1([0,1],m)$ I get

$$\|f - (f + \frac{\epsilon}{2\|g\|_1} g)\|_1 = \|f - f - \frac{\epsilon}{2\|g\|_1} g\|_1 = -\|\frac{\epsilon}{2\|g\|_1} g\|_1 = |\frac{\epsilon}{2\|g\|_1}|\|g\|_1$$
$$= \frac{\epsilon}{2\|g\|_1}\|g\|_1 = \frac{\epsilon}{2} < \epsilon$$

I define h as $h := f + \frac{\epsilon}{2\|g\|_1} g \in B(f,\epsilon)$, from this I define g:
$g = (h - f)\frac{2\|g\|_1}{\epsilon}$

Since I have that $h \in B(f,\epsilon) \subseteq E_n$ I will have $h \in L_3([0,1],m)$ because we have that any function in $E_n$ also is in $L_3([0,1],m)$. So then because $f \in E_n$ I get $f \in L_3([0,1],m)$. From these informations I now have, I can say that $g \in L_3([0,1],m)$.
From all this I now get that $L_1([0,1],m) \subseteq L_3([0,1],m)$, but this is a contradiction by HW 2 and hence $E_n^\circ = \emptyset$ so $E_n$ have empty interior in $L_1([0,1],m)$.

## Problem 4c

I will now show that $E_n$ is closed in $L_1([0,1],m)$. I start by taking a sequence $(a_b)_{b\in\mathbb{N}}$ in $E_n$ and will show that the limit of $E_n$ will be in $E_n$. When I take a sequence $(a_b)_{b\in\mathbb{N}} \subseteq E_n$ I have $\|a_b - a\|_1 \to 0$ and I note by Bolzano-Weierstrass that the sequence $(a_{n_b})_{n_b\in\mathbb{N}}$ converges pointwise. I now have:

$$\|f\|_3^3 = \int_{[0,1]} |f|^3 dm \leq \lim_{n_b\to 0} \inf \int_{[0,1]} |f_{n_b}|^3 dm \leq \lim_{n_b\to 0} \inf n = n$$

The first inequality is by Fatou´s lemma, the second inequality is from $a_{n_b} \in E_n$. I can now see that $\|f\|_3^3 \leq n$, this means that $f \in E_n$ and I can conclude that $E_n$ is closed in $L_1([0,1],m)$.

## Problem 4d

I will now use b) and c) to say that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$. For something to be of first category I have by definition 3.12(ii) that here must exist a sequence of nowhere dense sets such that $L_3([0,1],m) = \bigcup_{n=1}^{\infty} E_n$. To show this I have to show that $Int(\overline{E_n}) = \emptyset$ and in exercise b) I showed that $Int(E_n) = \emptyset$ and in c) I showed that $E_n$ is closed. This implies $E_n = \overline{E_n}$ and this will now apply that $\emptyset = Int(E_n) = Int(\overline{E_n})$ hence I now get $Int(\overline{E_n}) = \emptyset$ as wanted. I can now say that $E_n$ is nowhere dense set and use it to show $L_3([0,1],m) = \bigcup_{n=1}^{\infty} E_n$.

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \{f \in L_1([0,1],m) : \int_{[0,1]} |f|^3 dm \leq n\}$$
$$= \{f \in L_1([0,1],m) : \int_{[0,1]} |f|^3 dm < \infty\}$$
$$= \{f \in L_1([0,1],m) : f \in L_3([0,1],m)\} = L_3([0,1],m)$$

since it was given that $L_3([0,1],m) \subsetneq L_1([0,1],m)$. I can now finish the exercise with saying that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

## Problem 5a

I suppose that $x_n \to x$ in norm and will determine weather $\|x_n\| \to \|x\|$.
I start by observing that

$$\|x\| = \|x - x_n + x_n\| \leq \|x - x_n\| + \|x_n\|$$

and we also observe that

$$\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\|$$

I will now use the reverse triangle inequality and combine the two expressions:

$$\big|\|x\| - \|x_n\|\big| \leq \|x - x_n\|$$

Since it is given that $x_n \to x$ for $\epsilon > 0$ there will exist $n_\epsilon \in \mathbb{N}$ such that $n \geq n_\epsilon$ gives us

$$\big|\|x\| - \|x_n\|\big| \leq \|x - x_n\| < \epsilon$$

Hence I can conclude that $\|x_n\| \to \|x\|$ as I wanted.

## Problem 5b

I suppose that $x_n \to x$ weakly and I will find out if $\|x_n\| \to \|x\|$.
I will show it by a counterexample.

I let $H = \ell_2(\mathbb{N})$ and $x_n = e_n$. I have that $H$ is separable so I look at $e_n$, where I will notice that $\langle e_n, e_m \rangle = \delta_{n,m} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$ I will now look at what $e_n$ is going towards weakly, and for that I assume $e_n \to 0$ weakly. I will start by taking a $x \in H$ then I have by Bessel's inequality that $\sum_n \big|\langle e_n, x \rangle\big|^2 \leq \|x\|^2 < \infty$ which converges.
From this I get $\big|\langle e_n, x \rangle\big|^2 \to \langle 0, x \rangle = 0$ i.e $\big|\langle e_n, x \rangle\big|^2 \to 0$ and hence $\langle e_n, x \rangle \to \langle 0, x \rangle$. since I have a Hilbert space and a sequence I recall from HW 4 $\langle e_n, x \rangle \to \langle 0, x \rangle \Leftrightarrow e_n \to 0$ weakly.
So I get that $e_n \to 0$ weakly since I showed $\langle e_n, x \rangle \to \langle 0, x \rangle$.

But can I from all this conclude that $\|e_n\| \to \|0\| = 0$
I know that $\|e_n\| = 1$ for every $n$, it applies that $\|e_n\| \not\to \|0\| = 0$. So I can now conclude that for $x_n \to x$ weakly it does not follow that $\|x_n\| \to \|x\|$.

## Problem 5c

I notice that $\|x_n\| \leq 1$ for all $n \geq 1$ and $x_n \to x$ weakly.
I will now find out if $\|x\| \leq 1$.
I start by looking at the property of a weak convergence, which says that the norm is sequentially weakly lower-semicontinous, i.e $\|x\| \leq \lim_{n\to\infty} \inf \|x_n\|$.

I know that $x_n \to x$ weakly then it applies that $\|x\| = \langle x, x \rangle = \lim_{n\to\infty} \langle x, x_n \rangle$ but I have that $\langle x, x_n \rangle \leq \|x_n\|$. So this implies
$\|x\| = \lim_{n\to\infty} \langle x, x_n \rangle \leq \lim_{n\to\infty} \inf \|x_n\|$. I can now finish with saying that $\|x\| \leq 1$ since $\|x_n\| \leq 1$.