Problem 1

a)

From Hw. 4 Pb. 2, we get that in order to show $f_N \to 0$ weakly, it suffices, to show that $T(f_N) \to T(0) = 0$ for alle $T \in H^*$. Using Riesz representation theorem (Hw. 2 Pb. 1) we get that for any $T \in H^*$ there exists $y \in H$ such that

$$T(f_n) = \langle f_N, y\rangle = \frac{1}{N}\langle\sum_{n=1}^{N^2} e_n, y\rangle = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, y\rangle.$$

Since $(e_n)_{n\in\mathbb{N}}$ is a basis, we know that any $y \in H$ can be written as $y = \sum_{m=1}^{\infty}\lambda_m e_m$ where $\{n \in \mathbb{N} : \lambda_m \neq 0\}$ is finite. Further since $(e_n)_{n\in\mathbb{N}}$ is orthonormal we know that these $\lambda_m = \langle y, e_m\rangle$, see for instance **Schilling Theorem 26.21**, hence $\{n \in \mathbb{N} : \langle y, e_m\rangle \neq 0\}$ is finite.
As a result of this we see that there exists $N' \in \mathbb{N}$ such that for all $N \geq N'$ we have that $\sum_{n=1}^{N^2}\langle e_n, y\rangle$ is constant. And therefore

$$T(f_N) = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, y\rangle \to 0 \qquad \text{for } N \to \infty$$

and we conclude $f_N \to 0$ weakly.
For $N \geq 1$ we have

$$||f_N||^2 = \langle f_N, f_N\rangle = \frac{1}{N^2}\langle\sum_{n=1}^{N^2} e_n, \sum_{m=1}^{N^2} e_m\rangle = \frac{1}{N^2}\sum_{n=1}^{N^2}\sum_{m=1}^{N^2}\langle e_n, e_m\rangle = \frac{1}{N^2}\sum_{n=1}^{N^2}\langle e_n, e_n\rangle = \frac{N^2}{N^2} = 1.$$

hence $||f_N|| = 1$.

b)

Note that since H is refleksive (**Proposition 2.10**), $\overline{B}^{||\cdot||}(0,1)$ is compact with respect to the weak topology (**Theorem 6.3**). Notice further $\mathrm{co}\{f_N : N \geq 1\} \subset \overline{B}^{||\cdot||}(0,1)$, since for any $g \in \mathrm{co}\{f_N : N \geq 1\}$ we have for some $n \in \mathbb{N}$, $(\alpha_i)_{i=1,\dots,n}$

$$||g|| = ||\sum_{i=1}^{n}\alpha_i f_i|| \leq \sum_{i=1}^{n}\alpha_i||f_i|| = \sum_{i=1}^{n}\alpha_i = 1$$

where we have used the triangle inequality and the definition of the convex hull.
Hence $K = \overline{\mathrm{co}\{f_N : N \geq 1\}}^{||\cdot||} \subset \overline{B}^{||\cdot||}(0,1)$. But from **Theorem 5.7** we get that $\overline{\mathrm{co}\{f_N : N \geq 1\}}^{\tau_w} = \overline{\mathrm{co}\{f_N : N \geq 1\}}^{||\cdot||} = K$, so actually K is a closed subset of a weakly compact set, and is therefore itself weakly compact **Folland Proporsition 4.22**.
Futher from Hw. 5 Pb.1 and part a) we get that there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathrm{co}\{f_N : N \geq 1\} \subset K$ such that $y_n \to 0$ in norm. Hence as K is closed, is follows that $0 \in K$.

c)

Lets first show that 0 is an extreme point. If it was not then there exists $0 \neq f, g \in \overline{\mathrm{co}\{f_N : N \in \mathbb{N}\}}^{||\cdot||}$ such that $0 = \alpha f + (1-\alpha)g$, but this would imply that for all e_n we have

$$0 = \langle\alpha g + (1-\alpha)f, e_n\rangle \Leftrightarrow \frac{\alpha - 1}{\alpha}\langle f, e_n\rangle = \langle g, e_n\rangle$$

This would further imply that either $\langle f, e_n\rangle$ or $\langle g, e_n\rangle$ must be negative as $\frac{\alpha-1}{\alpha}$ is negative. But this cannot be the case as $\langle f, e_n\rangle$ and $\langle g, e_n\rangle$, are just some scaling, and summation of innerproducts between elements of our ONB, which is alway greater or equal to 0. Hence there exists no proper convex combination adding to 0, and 0 must be an extreme point.

Lets now move on to show $\{f_N : N \in \mathbb{N}\}$ are extreme points. From **Theorem 7.8 - Krein-Milman** we know that

$$K = \overline{\text{co}\{f_N : N \in \mathbb{N}\}}^{||\cdot||} = \overline{\text{co}\{f_N : N \in \mathbb{N}\}}^{\tau_w} = \overline{\text{co}(\text{Ext}(\overline{\{f_N : N \in \mathbb{N}\}}^{\tau_w}))}^{\tau_w} = \overline{\text{co}(\text{Ext}(K))}^{\tau_w}$$

as K is a non-empty, weakly compact, convex set.

We now define $M = \{f_N : N \in \mathbb{N}\} \setminus f_M$ for some $f_M \in \{f_N : N \in \mathbb{N}\}$, and lets show that $\text{co}(\{f_N : N \in \mathbb{N}\}) \neq \text{co}(M)$.
Assume they were equal, the there would exist a convex combination of elements in M:
$\sum_{i=1}^{n} \alpha_i f_{N_i}, f_{N_i} \in M, \sum_{i=1}^{n} \alpha_i = 1$ such that this convex combination would be equal to f_M. We take this convex combination and compute

$$\begin{aligned}\langle \sum_{i=1}^{n} \alpha_i f_{N_i}, f_M\rangle &= \sum_{i=1}^{n} \alpha_i \frac{1}{N_i}\frac{1}{M}\sum_{p=1}^{N_i^2}\sum_{m=1}^{M^2}\langle e_p, e_m\rangle = \sum_{i=1}^{n} \alpha_i \frac{1}{N_i}\frac{1}{M}\sum_{p=1}^{\min(N_i^2, M^2)}\langle e_p, e_p\rangle \\ &= \sum_{i=1}^{n} \alpha_i \frac{1}{N_i}\frac{1}{M}\min(N_i^2, M^2) < \sum_{i=1}^{n} \alpha_i = 1\end{aligned}$$

which is a contradiction to a), where we showed $||f_N|| = 1$ for all $N \in \mathbb{N}$.
Hence $\text{co}(\{f_N : N \in \mathbb{N}\}) \neq \text{co}(M)$ and $\overline{\text{co}(\{f_N : N \in \mathbb{N}\})}^{\tau_w} \neq \overline{\text{co}(M)}^{\tau_w}$. Therefore we must have, by **Krein-Milman** that all f_N are extreme point, as if any of the $f_N\ 's$ were missing, the Krein-Milman equality would not hold.

d)

From **Theorem 7.9 - Milman** we know that

$$\text{Ext}(K) \subset \overline{\{f_N : N \in \mathbb{N}\}}^{\tau_w}$$

as we have earlier show $K = \overline{\text{co}(\{f_N : N \in \mathbb{N}\})}^{||\cdot||} = \overline{\text{co}(\{f_N : N \in \mathbb{N}\})}^{\tau_w}$ is weakly compact, convex and non-empty.
And since $\overline{\{f_N : N \in \mathbb{N}\}}^{\tau_w} = \{f_N : N \in \mathbb{N}\} \cup \{0\}$, K cannot have any other exstreme points than the ones we found in c).

Problem 2

a)

Let $S \in Y^*$, then $S \circ T \in X^*$ hence by Hw. 4 Pb. 2 we have $S(T(x_n)) = (S \circ T)(x_n) \to (S \circ T)(x) = S(T(x))$, and then again by Hw. 4 Pb. 2 and the fact that $S \in Y^*$ this implies $T(x_n) \to T(x)$ weakly.

b)

In order to show this result we will be using the double thinnning principle (See for instance **Theorem A.2 Stochastic Processes, Hansen**). Consider the sequence $(T(x_n))_{n\in\mathbb{N}}$. Then for any subsequence $(T(x_{n_k}))_{k\in\mathbb{N}}$ we know from Hw. 4 Pb. 2b) that $(x_{n_k})_{k\in\mathbb{N}}$ is bounded. So from **Proposition 8.2 (4)** we can find a further subsequence $(x_{n_{k_p}})_{p\in\mathbb{N}}$ such that $(T(x_{n_{k_p}}))_{p\in\mathbb{N}}$ converges to some $y \in Y$, hence also converge weakly to y. But from a) we know that $T(x_{n_{k_p}}) \to T(x)$ weakly, and hence by uniqueness of weak limits $y = T(x)$, and we can then conclude by double thinning that $T(x_n) \to T(x)$ in norm, as $n \to \infty$.

c)

Suppose T is not compact. Then by **Proposition 8.2** we have that $T(\overline{B_H(0,1)})$ is *not* totally bounded. Hence we have

$$\exists\delta > 0 \;\forall N \in \mathbb{N}\; \forall \text{ open balls } U_1, ..., U_N \subset Y \text{ with radius } \delta \text{ we have } T(\overline{B_H(0,1)}) \not\subset \bigcup_{i=1}^{N} U_i$$

as a result of this there exists infinetely many $x_n \in \overline{B_H(0,1)}$ such that $T(x_n) \in \bigcup_{i=1}^{N} U_i$ no matter how large N is. Now we are ready to construct a sequence $(x_n)_{n\in\mathbb{N}} \subset \overline{B_H(0,1)}$. Define

$$x_0 = 0 \qquad U_i = B(T(x_i), \delta) \subset Y \;\forall i \in \mathbb{N}_0$$
$$x_n \in T^{-1}\left(T(\overline{B_H(0,1)} \setminus \bigcup_{i=0}^{n-1} U_i\right) \bigcap \overline{B_H(0,1)}$$

where δ is chosed as above, and the existence of such x_n $'s$ also comes from the above considarations about totally boundedness.
This construction makes sure that for every $n \neq m$ $T(x_n)$ and $T(x_m)$ is the center tw0 balls with radius δ, where neither x_n nor x_m is contained in the other ball, hence we must have $||T(x_n) - T(x_m)|| \geq \delta$ for all $n \neq m$, hence *no subsequence of* $(T(x_n))_{n\in\mathbb{N}}$ *can be cauchy* and therefore no subsequence can be convergent.
Notice H is reflexive, so by **Theorem 5.9** we have that $\tau_w = \tau_{w^*}$. Further by **Theorem 5.13** $(\overline{B_H(0,1)}, \tau_{w^*})$ is metrizable, as H^* is seperable (can be seen by using **Riesz Representation Theorem, Hw. 2 Pb. 1**). At last as H is reflexive $\tau_{w^*} = \tau_{||\cdot||}$, and therefore $(\overline{B_H(0,1)})$ is compact by **Alaoglu's Theorem**, hence sequentially-compact, and therefore every sequence, must have a weakly convergent subsequence. In particular our constructed sequence $(x_n)_{n\in\mathbb{N}}$. By contraposition the desired result is shown.

d)

If we have a sequence $(x_n)_{n\in\mathbb{N}} \subset \ell_2$ where $x_n \to x$ weakly, then by a) we have that $T(x_n) \to T(x)$ weakly. By remark 5.3 this is equivalent to $||T(x_n) - T(x)|| \to 0$, as $(T(x_n))_{n\in\mathbb{N}} \subset \ell_1(\mathbb{N})$. So we can from c) conclude that $T \in \mathcal{K}(X, Y)$.

e)

Assume for contradiction that there exists $T \in \mathcal{K}(X, Y)$ which is onto. Then by Open Mapping Theorem we get that T is also open. When T is an open linear map we know that there exists $r > 0$ such that

$\underline{B_Y(0,r) \subset T(B_X(0,1))}$, and therefore also $\overline{B_Y(0,r)} \subset \overline{T(B_X(0,1))}$.
$\overline{T(B_X(0,1))}$ is compact from the definition of T, and then by **Folland Proposition 4.22** $\overline{B_Y(0,r)}$ is also compact. But this is a contradiction, as we in **Mandatory Assignment 1 Problem 3** showed that the closed unit ball in an infinite dimensional space is not compact, and the same argument scaled to a ball with radius r, will show that $\overline{B_Y(0,r)}$ cannot be compact. Therefore our initial assumption must have been false, and we conclude that *no* $T \in \mathcal{K}(X,Y)$ is onto.

f)

First we show that M is self adjoint, for any $f, g \in H$ we have

$$\langle M(f), g\rangle = \langle t \cdot f, g\rangle = \int_{[0,1]} t \cdot f(t) \cdot \overline{g(t)} dm(t) = \int_{[0,1]} f(t) \cdot \overline{t} \cdot \overline{g(t)} dm(t) = \rangle f, t \cdot g\rangle = \langle f, M(g)\rangle$$

where we used that $t \in [0,1]$ is real. Assume for contradiction that M was compact. Then by **Theorem 10.1** since $H = L_2([0,1], m)$ is infinite dimensional and seperable, and M is self adjoint, we have that there exist an ONB for H, consisting of eigenvector for M, with corresponding eigenvalues. But since we in Hw. 6 Pb. 3 showed that M has no eigenvalues, this is a contradiction. Hence M is *not* compact.

Problem 3

a)

Since $H = L_2([0,1], m)$ is a σ-finite measure space and $K(s,t) \in L_2([0,1] \times [0,1], m \otimes m)$:

$$||K(s,t)||_2^2 = \int_{[0,1]\times[0,1]} K(s,t) \ dm \otimes m \leq \int_{[0,1]\times[0,1]} 1 \ dm \otimes m = m([0,1]) \cdot m([0,1]) = 1 < \infty$$

we have by **Proposition 9.12** that T is compact, as it is the associated kernel operator.

b)

We want to show that T is self adjoint, for any $f, g \in H$ we have

$$\begin{aligned}\langle Tf, g\rangle &= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \ dm(t) \overline{g(s)} \ dm(s) = \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \overline{g(s)} \ dm(t) \ dm(s) \\ &= \int_{[0,1]} \int_{[0,1]} f(t) \overline{K(t,s) g(s)} \ dm(t) \ dm(s) = \int_{[0,1]} f(t) \int_{[0,1]} \overline{K(t,s) g(s)} \ dm(s) \ dm(t) \\ &= \langle f, Tg\rangle\end{aligned}$$

where we have used that $K(s,t) = K(t,s)$ is real, and Funibis theorem.

c)

We have that

$$\begin{aligned}(Tf)(s) &= \int_{[0,1]} K(s,t) f(t) dm(t) = \int_{[0,s]} K(s,t) f(t) dm(t) + \int_{[s,1]} K(s,t) f(t) dm(t) \\ &= (1-s) \int_{[0,s]} t f(t) dm(t) + s \int_{[s,1]} (1-t) f(t) dm(t)\end{aligned}$$

where we have used that when $t \in [0, s]$ then $K(s,t) = (1-s)t$ and when $t \in [s, 1]$ then $K(s,t) = (1-t)s$. This shows that $(Tf)(s)$ is continuous on $[0, 1]$, as it is composed of continuous functions. We futher see that

$$(Tf)(0) = 1 \cdot \int_{\{0\}} tf(t)dm(t) + 0 = 0$$
$$(Tf)(1) = 0 + 1 \cdot \int_{\{1\}} (1-t)f(t)dm(t) = 0$$

as any singleton is a nullset for the Lebesgue measure.

Problem 4

a)

We see that for any multi-indices α, β, we have that

$$\lim_{||x|| \to \infty} x^\beta \partial^\alpha e^{-x^2/2} = \lim_{||x|| \to \infty} \text{Pol}_{\beta+\alpha} e^{-x^2/2} = 0$$

hence $e^{-x^2/2} \in \mathscr{S}(\mathbb{R})$ and by Hw. 7 Pb. 1, $g_k(x) = x^k e^{-x^2/2} \in \mathscr{S}(\mathbb{R})$, for any $k \geq 0$.
Now for $k = 0, 1, 2, 3$ we can calculate the fourier transform of g_k

$$\mathcal{F}(g_0) = e^{-x^2/2} = g_0$$

by **Proposition 11.4**, where we used $||x|| = |x|$.
To calculate the rest of the Fourier transforms, we make use of **Proposition 11.13 d)**

$$\mathcal{F}(g_1) = \mathcal{F}(xg_0) = i\left(\frac{\partial}{\partial \xi}\mathcal{F}(g_0)\right) = i\left(\frac{\partial}{\partial \xi}g_0\right) = -e^{-xi^2/2}i\xi$$
$$\mathcal{F}(g_2) = \mathcal{F}(x^2 g_0) = i^2\left(\frac{\partial^2}{\partial \xi}\mathcal{F}(g_0)\right) = i^2\left(\frac{\partial^2}{\partial \xi}g_0\right) = e^{-x^2/2}(1-\xi^2)$$
$$\mathcal{F}(g_3) = \mathcal{F}(x^3 g_0) = i^3\left(\frac{\partial^3}{\partial \xi}\mathcal{F}(g_0)\right) = i^3\left(\frac{\partial^3}{\partial \xi}g_0\right) = e^{-\xi^2/2}i\xi^2(\xi^2-3)$$

b)

We see that

$$\mathcal{F}(g_0) = e^{-\xi^2/2} = g_0 = i^0 g_0$$
$$\mathcal{F}(g_3 - \frac{3}{2}g_1) = \mathcal{F}(g_3) - \frac{3}{2}\mathcal{F}(g_1) = e^{-\xi^2/2}i\xi(\xi^2-3) + \frac{3}{2}e^{-\xi^2/2}i\xi = e^{-\xi^2/2}i\xi(\xi^2 - \frac{3}{2})$$
$$= e^{-\xi^2/2}i\xi^3 - \frac{3}{2}e^{-\xi^2/2}i\xi = i^1(g_3 - \frac{3}{2}g_1)$$
$$\mathcal{F}(g_0 - 2g_2) = e^{-\xi^2/2} - 2e^{-\xi^2/2}(1-\xi^2) = 2e^{-xi^2/2}\xi^2 - e^{-\xi^2/2} = i^2(g_0 - 2g_2)$$
$$\mathcal{F}(g_1) = -e^{-\xi^2/2}i\xi = i^3 g_1$$

and can conclude

$$h_0 = g_0, \qquad h_1 = g_3 - \frac{3}{2}g_1, \qquad h_2 = g_0 - 2g_2, \qquad h_3 = g_1$$

c)

For any $f \in \mathscr{S}(\mathbb{R})$ we have

$$\begin{aligned}\mathcal{F}^2(f) &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)e^{-ixy}dm(x)e^{-iyz}dm(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(-x)e^{ixy}dm(x)e^{-iyz}dm(y) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} S_{-1}(x)e^{ixy}dm(x)e^{-iyz}dm(y) = \mathcal{F}(\mathcal{F}^*(S_{-1}(x))) = S_{-1}(x) = f(-x)\end{aligned}$$

$$\mathcal{F}^4(f(x)) = \mathcal{F}^2(\mathcal{F}^2(f(x))) = \mathcal{F}^2(f(-x)) = f(x)$$

d)

If $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$, then we have by linearity of the fourier transform that $\mathcal{F}^4(f) = \lambda^4 f$, but then also by c) that $f = \lambda^4 f$. The only complex numbers satisfying this equation are $\lambda \in \{1, i, -1, -i\}$. Futher we know from b) that for each of these possible values of λ, there actually exists $f \in \mathscr{S}(\mathbb{R})$ with the property that $\mathcal{F}(f) = \lambda f$. Hence the eigenvalues of $\mathcal{F}$ must be $\{1, i, -1, -i\}$.

Problem 5

If $\text{supp}(\mu)$ is the whole set it is defined on, then by Hw. 8 Pb. 3 a) we must have that $N = \emptyset$ or equivalently that $\mu(U) > 0$ whenever $U \subset [0,1]$ is open. So lets show that this holds. Let any open set $U \subset [0,1]$ be given. Then we can find $a, b \in [0,1]$ such that $K' = [a,b]$ is contained in U, and further it is compact, as it is closed and bounded. As $(x_n)_{n\in\mathbb{N}}$ is dense in $[0,1]$ we can find $N \in \mathbb{N}$ such that $x_N \in K'$. And now we use that μ is a Radon measure to conclude

$$\mu(U) = \sup\{\mu(K) : K \subset U \text{ is compact}\} \geq \mu(K') \geq \mu(x_N) = \frac{1}{2^N} > 0.$$

Hence $N = \emptyset$ and $\text{supp}(\mu) = [0,1]$.