FunkAn Assignment 2

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Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.

a)

Show that $f_N \to 0$ weakly, as $N \to \infty$, while $||f_N|| = 1$, for all $N \geq 1$.

$$||f_N||^2 = \langle N^{-1}\sum_{n=1}^{N^2} e_n, N^{-1}\sum_{n=1}^{N^2} e_n\rangle = N^{-2}\sum_{j,k=1}^{N^2}\langle e_j, e_k\rangle = N^{-2}\sum_{k=1}^{N^2}\langle e_k, e_k\rangle = N^{-2}\sum_{k=1}^{N^2}||e_n||^2 = \frac{N^2}{N^2} = 1$$

Where we used that $(e_n)_{n\geq 1}$ is an orthonormal basis so $\langle e_j, e_k\rangle = 0$ for $j \neq k$
Now i show that $f_N \to 0$ weakly.
By HMW 4 Pb 2 (or by definition in Folland, i will refer to this result as "definition" of weak convergence) we know that $f_N \to 0$ weakly $\Leftrightarrow F(f_N) \to F(0)$ for all $F \in H^*$. We also know that $F(0) = 0$ for all elements in the dual. By Theorem 5.25 Folland we can write $F(f_n) = \langle f_N, y\rangle$ where y is an unique element of H. Since (e_n) is an ONB we can write $y = \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i$ and as $||y|| < \infty$ for any ϵ there exists a K such that $||\sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i|| < \epsilon$.
Thus $|F(f_N)| = |\langle f_N, y\rangle| = |\langle f_N, \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i\rangle| = |\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle + \langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle|$. Which by the triangle inequality we get:

$$|\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i + \langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle| \leq |\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i| + |\langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle|$$

Firstly we bound the 2nd expression using Cauchy Schwartz as H is a Hilbert space.

$$|\langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle| \leq ||f_N|| \cdot ||\sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle|| < 1 \cdot \epsilon$$

Now to bound the 1st expression:

$$|\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i| = N^{-1}|\sum_{n=1}^{N^2}\langle e_n, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle| = N^{-1}\left|\sum_{n=1}^{N^2}\overline{\sum_{i=1}^{K}\langle y, e_i\rangle\langle e_n, e_i\rangle}\right| \leq N^{-1}\left|\sum_{n=1}^{K}\overline{\langle y, e_i\rangle}\right| < \epsilon \text{ for } N \to \infty$$

Where for the last inequalities we used e_n ONB and $\left|\sum_{n=1}^{K}\overline{\langle y, e_i\rangle}\right|$ being finite.
This shows that for all $F \in H^*$, $F(f_N) \to 0 = F(0)$ for $N \to \infty$ which shows that $f_N \to 0$ weakly.

b)

Let K be the norm closure of $co\{f_N : N \geq 1\}$. Argue that K is weakly compact, and that $0 \in K$.

Firstly we note that K, being the norm closure of a convex set, is convex so by Theorem 5.7 in the notes the norm and weak closures coincide. Thus we have (we omit the $N \geq 1$) $K = \overline{co\{f_n\}}^{||\cdot||} = \overline{co\{f_n\}}$. We know all Hilbert spaces are reflexive so by Theorem 6.3 in the notes $\overline{B_H(0,1)}$ is compact with respect to the weak topology. As the convex hull is the smallest set containing all convex combinations and all $||f_N|| = 1$ we have that $\overline{co\{f_n\}} \subset \overline{B_H(0,1)}$ as the closed unit ball is a convex set contaning all convex combinations of f_N. And as the closed unit ball is closed then K must be contained in it too. Thus K is a weakly closed subset of a weakly compact set and is thus weakly compact.

The sequence $(f_N)_{N\geq 1}$ lies in K as each f_N lies inside it. This sequence converges weakly to 0 thus it its in the weak closure of $co\{f_N : N \geq 1\}$ and hence in the norm closure, K.

c)

Show that 0, as well as each f_N, $N \geq 1$, are extreme points in K.

We will first show 0 is an extreme point.
Note that every element in $co\{f_N | N \geq 1\}$ will have a positive inner product with e_n as $\langle f_N, e_n\rangle$ is positive. Let $(x_n)_{n\geq 1}$ be a sequence in $co\{f_N | N \geq 1\}$ converging to x. Let $g_n \in H^*$ be given by $g_n(x) = \langle x, e_n\rangle$, these are continuous function so $\langle x_n, e_n\rangle \to \langle x, e_n\rangle$ for all n. Thus as all $\langle x_n, e_n\rangle \geq 0$ we must have that $\langle x, e_n\rangle \geq 0$. Therefore we have shown that each element in $\overline{co\{f_N | N \geq 1\}}$ will still have positive inner product with e_n.
Let 0 be given as a convex combination $0 = \alpha x + (1-\alpha)y$. Specifically we would also have $0 = \alpha\langle x, e_n\rangle + (1-\alpha)\langle y, e_n\rangle$ for all $n \geq 1$. But 0 is an extreme point of the positive real line thus for each n we have $\langle x, e_n\rangle = \langle y, e_n\rangle = 0$. But by Theorem 5.27(a) Folland we must have that $x = y = 0$. Hence we conclude that 0 is an extreme point of $\overline{co\{f_N | N \geq 1\}} = K$.

Now for the ugly part. Let $f_N = \alpha x + (1-\alpha)y$ be a convex combination in K. Where x is a limit point of $(x_n)_{n\geq 1}$ and y is a limit point of $(y_n)_{n\geq 1}$ ($(x_n), (y_n) \in co\{f_N | N \geq 1\}$). Thus we have that $\alpha(x_n)_{n\geq 1} + (1-\alpha)(y_n)_{n\geq 1} \to f_N$. As before note $g_{N^2}(x) = \langle x, e_{N^2}\rangle$. We can apply g_{N^2} (a continuous function) and get.

$$g_{N^2}(\alpha(x_n) + (1-\alpha)(y_n)) = \alpha g_{N^2}(x_n) + (1-\alpha)g_{N^2}(y_n) \to g_{N^2}(f_N) = \frac{1}{N}$$

We will now show that $g_{N^2}(x_n) \leq \frac{1}{N}$:
Note that if $j < N$ $g_{N^2}(f_j) = 0$ and if $j \geq N$ then $g_{N^2}(f_j) = \frac{1}{j} \leq \frac{1}{N}$ For simplicity we note the elements $x_n \in K$ as their convex combination $x_n = \sum_{k=1}^{\infty} \alpha_{n_k} f_k$ where we remember that the sum of the α_{n_k} is 1 and hence there is only a finite set of which they are non-zero thus can also be written as $x_n = \sum_{k=1}^{W_n} \alpha_{n_k} f_k$.

$$g_{N^2}(x_n) = \sum_{k=1}^{W_n} \alpha_{n_k} g_{N^2}(f_k) \leq \sum_{k=1}^{W_n} \alpha_{n_k} \frac{1}{N} = \frac{1}{N}$$

The exact same argument can be made for (y_n).
Therefore the only way for $\alpha g_{N^2}(x_n) + (1-\alpha)g_{N^2}(y_n) \to \frac{1}{N}$ to hold we must have that $g_{N^2}(x_n) \to \frac{1}{N}$ and $g_{N^2}(y_n) \to \frac{1}{N}$.
We know that $(x_n)_{n\geq 1}$ converges to a specific f_j if the sequence (α_{n_j}) converges to 1 ((α_{n_j}) is the sequence of j'th coefficent of the elements in the sequence $(x_n)_{n\geq 1}$).
We will show that if $g_{N^2}(x_n) \to \frac{1}{N}$ (and respectively for y_n) then $(x_n)_{n\geq 1}$ converges to f_N by showing that (α_{n_j}) converges to 1.
Assume that (α_{n_j}) does not converge to 1, therefore there must exist an $\epsilon > 0$ such that for every L there exist $n > L$ where $|1 - \alpha_{n_j}| > \epsilon$. As $\alpha_{n_j} \leq 1$ we have $r_n = 1 - \alpha_{n_j} > \epsilon$. Now we want to show the contradiction by showing $g_{N^2}(x_n) \not\to \frac{1}{N}$:

$$\left|\frac{1}{N} - g_{N^2}(\alpha(x_n) + (1-\alpha)(y_n))\right| = \frac{1}{N} - \alpha g_{N^2}(x_n) - (1-\alpha)g_{N^2}(y_n)$$

$$\geq \frac{1}{N} - \left(\alpha g_{N^2}(x_n) + (1-\alpha)\frac{1}{N}\right) \geq \alpha\frac{1}{N} - \left(\alpha\sum_{k=1}^{W_n} \alpha_{n_k} g_{N^2}(f_k)\right) = \alpha\frac{1}{N}(1-\alpha_{n_N}) - \left(\alpha \sum_{k=1, k\neq N}^{W_n} \alpha_{n_k} g_{N^2}(f_k)\right)$$

In the last equality we pulled out the N'th element of the sum. Now we use that $\sum_{i=1,i\neq N}^{W_n} \alpha_{n_k} = 1 - \alpha_{n_N} = r_n$ (by definition of convex combination coefficents) and that for $k \neq N$ we have $g_{N^2}(f_k) \leq \frac{1}{N+1}$

$$\geq \alpha \left(\frac{r_n}{N} - \frac{r_n}{N+1} \right) \geq \epsilon \cdot \alpha \left(\frac{1}{N} - \frac{1}{N+1} \right)$$

Which contradicts the assumption of $g_{N^2}(x_n) \to \frac{1}{N}$. The exact same argument can be made for $(y_n)_{n\geq 1}$.
Thus we know that (α_{n_j}) converges to 1 and as said before this implies that $(x_n)_{n\geq 1} \to x = f_N$ and (by the same argument) $(y_n) \to x = f_N$.
We finally conclude that for any convex combination in K such that $f_N = \alpha x + (1-\alpha)y$ we must have that $x = y = f_N$ making f_N an extreme point in K.

d)

Are there any other extreme points in K? Justify your answer. (An answer without justification will not be given any credit.)
We have that $K = \overline{co\{f_N\}}^{||\cdot||} = \overline{co\{f_N\}}^{w}$ and H with the weak topology is $LCTVS$ (top of page 27 lecture notes) thus by Milman (Theorem 7.9)

$$Ext(K) \subset \overline{\{f_N\}}^{w} =^{?} \{f_N, N \geq 1\} \cup \{0\}$$

Thus all the extreme points of K are contained in the set of f_N and 0, but we have shown that these points are extreme points. Therefore there are no more extreme points of K.

?: We have not shown the equality $\overline{\{f_N\}}^{w} = \{f_N, N \geq 1\} \cup \{0\}$, we will show it now.
As 0 is a weak limit point of f_n we have that $\overline{\{f_N\}}^{w} \supseteq \{f_N, N \geq 1\} \cup \{0\}$. To show the other way we will show that no sequence in $\{f_n\}$ has other weak limits. Assume that x is the weak limit of such a sequence then by "definition" of weak limit we must have that $\forall g \in H^*$ $g(x)$ is the limit of some sequence in $\{f_N\}$. Specifically we can use $H^* \ni g_1(x) := \langle x, e_1 \rangle$. We note that $g_1(\{f_N\}) = \{N^{-1} | \forall N \in \mathbb{N}\}$ which is a set whose only accumulation points are 0 and N^{-1}.
If N^{-1} is an accumulation point: By "definition" of weak convergence and $\{N^{-1}\}_{N\in\mathbb{N}}$ being discrete, any sequence $(f_{N_j})_{j\in\mathbb{N}} \in \{f_N\}$ where $g_1(f_{N_j}) = N_j^{-1} \to N^{-1}$ as $j \to \infty$ will weakly converge to f_N.
If 0 is an accumulation point $((f_{N_j})_{j\in\mathbb{N}} \in \{f_N\}$ and $g_1(f_{N_j}) = N_j^{-1} \to 0)$ then N_j goes to infinity as $j \to \infty$. Thus (f_{N_j}) must have a subsequence where each $N_{j_k} < N_{j_l}$ for $k < l$. This subsequence is also a subsequence of (f_N) so it must converge weakly to 0. Therefore (f_{N_j}) must also converge weakly to 0. Thus we have shown that any sequence in (f_N) must have weak limit points in the set $\{f_N, N \geq 1\} \cup \{0\}$

Problem 2

Let X and Y be infinite dimensional Banach spaces.

a)

Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

We again use HMW 4 Pb2 for the "definition" of weak convergence. And Theorem 7.13 for the existence of the Banach space adjoint which we denote T^*, where we note that all $T^*g(x_n)$ are elements in X^*

$x_n \to x$ weakly $\Leftrightarrow f(x_n) \to f(x), \forall f \in X^* \Rightarrow T^*g(x_n) \to T^*g(x) \Leftrightarrow g(Tx_n) \to g(Tx) \Leftrightarrow T(x_n) \to T(x)$ weakly

b)

Let $T \in \mathcal{K}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $||Tx_n - Tx|| \to 0$, as $n \to \infty$.

By Pb2 HMW 4 we know that $sup\{||x_n||\} < \infty$. So $\{x_1, x_2, ...\}$ is a bounded set, therefore T being compact implies $\overline{T(\{x_1, x_2, ..\})}$ is compact. I will state a result from Analysis 1 regarding norm convergence: If all subsequences of a sequence have convergent subsequence then the original sequence is convergent.
Let (Ty_l) be a subsequence of the sequence $(T(y_n))_{n\geq 1} = (T(x_n - x))_{n\geq 1}$, as T is compact one has that $\overline{T(\{y_1, y_2, ..\})}$ is compact (as (Ty_l) bounded) and thus there exists a converging subsequence (Ty_{l_j}) of (Ty_l) with $Ty_{l_j} \to \gamma$ for some γ.
Now to show that $\gamma = 0$: From (a) we know that $g(T(x_n)) \to g(T(x))$ for all $g \in Y^*$ thus specifically $g(T(y_{l_j})) = g(T(x_{l_j} - x)) \to g(T(x - x)) = 0$ showing $\gamma = 0$.
Thus we have that $Tx_{n_{l_j}}$ (subsequence of the subsequence Tx_{n_l}) converges to Tx therefore every subsequence of Tx_n has a convergent subsequence converging to Tx and thus the sequence itself must be convergent to Tx, showing $||Tx_n - Tx|| \to 0$.

c)

Let H be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H, Y)$ satisfies that $||Tx_n - Tx|| \to 0$ as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Assume that $||Tx_n - Tx|| \to 0$ as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$ and that T is not compact. T not being compact means that $T(\overline{B_H(0,1)}$ is not totally bounded (Def 8.1 and text below it). Thus there exists, by Proposition 8.2.(4), a sequence $(y_n)_{n\geq 1}$ in $T(\overline{B_H(0,1)})$ that has no convergent subsequences. But being in the image under T of the closed unit ball for each y_n we can pick a x_n such that $Tx_n = y_n$ for each n. Thus we have a sequence $(x_n)_{n\geq 1}$ inside the closed unit ball in H.
By theorem 6.3 in the notes $\overline{B_H(0,1)}$ is weakly compact thus $(x_n)_{n\geq 1}$ must have a weakly converging subsequence (x_{n_j}) and by b) we know that $T(x_{n_j})$ is a strongly converging sequence (i.e. $||Tx_{n_j} - Tx|| \to 0$) but this sequence is a converging subsequence of $(y_n)_{n\geq 1}$ which had no converging subsequences, thus we reach a contradiction and T must be a compact operator.

d)

Show that each $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ is compact.

Let $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$, we know that $l_2(\mathbb{N})$ is a separable Hilbert space. Thus we want to use c). So if we show its statement: "If $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ satisfies that $||Tx_n - Tx|| \to 0$ as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in $l_2(\mathbb{N})$ converging weakly to $x \in l_2(\mathbb{N})$, then $T \in \mathcal{K}(l_2(\mathbb{N}), l_1(\mathbb{N}))$".
Let $(x_n)_{n\geq 1}$ be a weakly converging sequence in $l_2(\mathbb{N})$ converging to x. By a) we know for any $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ that $Tx_n \to Tx$ weakly, by Remark 5.3 in the notes (the text below the remark) this implies that $||Tx_n - Tx|| \to 0$. Thus we have shown exactly the prerequisites for c) for any $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$. Thus by c) all $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ are compact.

e)

Show that no $T \in \mathcal{K}(X, Y)$ is surjective.

Assume per contradiction that K is a surjective compact map. By Theorem 5.10 Folland, T is an open map. And we know that T is open if and only if $T(B_X(0,1))$ contains a ball centered around 0_Y. Thus $B_Y(0,r) \subset T(B_X(0,1))$. Taking closure on both sides we get $\overline{B_Y(0,r)} \subset \overline{T(B_X(0,1))}$ as T is compact, the right hand side is compact and as $\overline{B_Y(0,r)}$ is a closed subset of a compact set it must be compact. But it is a contradiction with assignment 1 Pb3 e) where we showed that the unit ball in Y is not compact, thus any ball centered around 0 of some radius will not be compact. Hence our assumption that T was injective must be wrong.

f)

Let $H = L_2([0,1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0,1]$. Justify that M is self adjoint, but not compact

As H is a Hilbert space we check:

$$\langle Mf, g\rangle = \int_{[0,1]} Mf \cdot \overline{g} dm = \int_{[0,1]} t \cdot f \cdot \overline{g} dm = \int_{[0,1]} f \cdot \overline{t \cdot g} dm = \int_{[0,1]} f \cdot \overline{Mg} dm = \langle f, Mg\rangle$$

Thus M is self-adjoint.
Assume M is compact, then by the spectral theorem (Theorem 10.1 notes) H has an orthonormal basis of eigenvectors of M. But by problem 3a) in HMW 6 we know it has no eigenvalues thus we reach a contradiction and therefore M is not compact.

Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$ where m is the Lebesgue measure. Define K: $[0,1] \times [0,1] \to \mathbb{R}$ by (see the assignment text). And consider $T \in \mathcal{L}(H, H)$ defined by (see the assignment text).

a)

Justify that T is compact

$[0,1]$ is a compact Hausdorff space and m is a finite measure on it. K is piecewise continuous thus $K \in C([0,1] \times [0,1])$. Then by Theorem 9.6 in the notes, "the associated operator" T_k which is exactly T, is compact.

b)

Show that $T^* = T$.

We firstly note that if x is real:

$$\overline{\int_{\mathbb{R}} f(x)dx} = \overline{\int_{\mathbb{R}} \alpha(x)dx + i\int_{\mathbb{R}} \beta(x)dx} = \int_{\mathbb{R}} \alpha(x)dx - i\int_{\mathbb{R}} \beta(x)dx = \int_{\mathbb{R}} \alpha(x) - i\beta(x)dx = \int_{\mathbb{R}} \overline{\alpha(x) + i\beta(x)}dx = \int_{\mathbb{R}} \overline{f(x)}dx$$

Let $f, g \in H$. (That the integrals are finite is shown in the lecture notes in p.46 of lecture 9. So we can use Furbini)

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} \overline{g(s)} \int_{[0,1]} K(s,t)f(t)dm(t)dm(s) = \int_{[0,1]} \int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(s)dm(t) = \int_{[0,1]} f(t) \int_{[0,1]} K(s,t)\overline{g(s)}dm(s)dm(t) \\
&= \int_{[0,1]} f(t) \int_{[0,1]} \overline{K(s,t)g(s)}dm(s)dm(t) = \int_{[0,1]} f(t)\overline{\int_{[0,1]} K(s,t)g(s)dm(s)}dm(t) = \langle f, Tg\rangle
\end{aligned}$$

(where we used (s,t) are real)
Which shows that T is self-adjoint

c)

Show that (see the assignment text). Use this to show that Tf is continuous on $[0,1]$ and that $(Tf)(0) = (Tf)(1) = 0$.

As the point s is of measure 0, we know from MI we can split the integral in the following way:

$$\begin{aligned}
Tf(s) = \int_{[0,1]} K(s,t)f(t)dm(t) &= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)
\end{aligned}$$

Use this to show that Tf is continuous:
Firstly we put it back together

$$(1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t) = \int_{[0,1]} K(s,t)f(t)dm(t)$$

Then we use continuity lemma (Lemma 12.4 Schilling) where we note that exactly the same proof can be given for a closed set (like $[0,1]$) instead of an open one like $(0,1)$. Even further, we note that the lemma is only for functions into $\mathbb{R}$ but can be used for function into $\mathbb{C}$ when $(f(x) = a(x)+ib(x))$ a, b are real valued function:
$[0,1]$ is nondegenerate closed. $u : [0,1]\times[0,1]$ where $u(s,t) = K(s,t)f(t)$.
(a) $t \to u(s,t)$ is in $L_1([0,1], m)$ for every fixed $s \in [0,1]$ as its integrable (shown in the lecture notes in p.46 of lecture 9).
(b) $s \to u(s,t)$ is continuous for every fixed $t \in [0,1]$.
(c) $|u(s,t)| = |K(s,t)f(t)| \leq w(t) = |f(t)|$ for all $(s,t) \in [0,1]\times[0,1]$ (where $|f(t)| \in L_1$ by HMW2 Problem 2b).
Thus we conclude that $\int u(s,t)dm = \int k(s,t)f(t)dm$ is continuous on $[0,1]$.

$$Tf(0) = (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]}(1-t)f(t)dm(t) = 0+0 = 0$$

$$Tf(1) = (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]}(1-t)f(t)dm(t) = 0+0 = 0$$

Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$

a)

For each integer $k \geq 0$, set $g_k(x) = x^k e^{\frac{-x^2}{2}}$ for $x \in \mathbb{R}$. Justify that $g_k \in: \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$. Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

The function $x \to x^k e^{-\frac{x^2}{2}}$ is $C^\infty(\mathbb{R})$. Next notice that $\partial^\beta x^k e^{-\frac{x^2}{2}} = \frac{\partial^\beta}{\partial x^\beta} x^k e^{-\frac{x^2}{2}} = Pol(x)e^{-\frac{x^2}{2}}$ Where $Pol(x)$ is some polynomial in x. Therefore we get $x^\alpha \partial^\beta e^{-\frac{x^2}{2}} = Pol_2(x)e^{-\frac{x^2}{2}}$ where $Pol_2(x)$ is a gain some polynomial in x. But we know from MatIntro that $Pol_2(x)e^{-\frac{x^2}{2}} \to 0$ as $x \to \infty$ as the exponential goes faster to 0 than any polynomial. Thus we conclude that $g_k \in \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$.
Now to computing g_k for $k = 0, 1, 2, 3$. By proposition 11.12 (b) $g_k \in L_p(\mathbb{R})$ for all $1 \leq p < \infty$, specifically we must have that $g_k \in L_1(\mathbb{R})$. $\mathcal{F}(g_0)$ is calculated exactly on page 57 of the notes under Solution 1. As its a matter of just copy pasting what is written, i will omit all the justifications as its 100% exactly what is shown there. The conclusion is $\mathcal{F}(g_0)(\xi) = e^{-\frac{\xi^2}{2}}$.
For g_1 and so on we can use Proposition 11.13 c) and d). As all the partial derivatives of g_0 are in $L_1(\mathbb{R})$

$$\mathcal{F}(g_1) = \mathcal{F}(g_0(x)x) = i\frac{d\hat{g_0}(\xi)}{d\xi} = -i\xi e^{-\frac{\xi}{2}}$$

Which we use to calculate g_k for $k = 2, 3$:

$$\mathcal{F}(g_2) = \mathcal{F}(g_0(x)x^2) = i\frac{d^2\hat{g_0}(\xi)}{d\xi^2} = (1-\xi^2)e^{-\frac{\xi}{2}}$$

$$\mathcal{F}(g_3) = \mathcal{F}(g_0(x)x^3) = i\frac{d^3\hat{g_0}(\xi)}{d\xi^3} = i(\xi^3 - 3\xi)e^{-\frac{\xi}{2}}$$

b)

Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for k=0,1,2,3.

$$\mathcal{F}(h_0) = \mathcal{F}(g_0) = \mathcal{F}(e^{\frac{-x^2}{2}}) = e^{\frac{-\xi^2}{2}} = i^0 h_0$$

$$\mathcal{F}(h_1) = \mathcal{F}(g_3 - \frac{3}{2}g_1) = \mathcal{F}(e^{\frac{-x^2}{2}}(x^3 - \frac{3}{2}x)) = ie^{\frac{-\xi^2}{2}}(\xi^3 - \frac{3}{2}\xi) = i^1 h_1$$

$$\mathcal{F}(h_2) = \mathcal{F}(g_2 - \frac{1}{2}g_0) = \mathcal{F}(e^{\frac{-x^2}{2}}(x^2 - \frac{1}{2})) = e^{\frac{-\xi^2}{2}}((1-\xi^2) - \frac{1}{2}) = i^2 h_2$$

$$\mathcal{F}(h_3) = \mathcal{F}(g_1) = \mathcal{F}(xe^{\frac{-x^2}{2}}) = -i\xi e^{\frac{-\xi^2}{2}} = i^3 h_3$$

c)

Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$

By the definition of Fourier transform:

$$\hat{f}(\xi) = \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$$

$$\mathcal{F}^2(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{-i\xi x} dx$$

As $f \in \mathscr{S}(\mathbb{R})$ by definition 12.10 and corollary 12.12(iii) in the notes we know.

$$f(x) = \check{\hat{f}}(x) = \mathcal{F}^*(\hat{f}(\xi)) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} dx$$

By comparing the last two equations we see that $\mathcal{F}^2(f(x)) = f(-x)$. Thus $\mathcal{F}^4(f(x)) = \mathcal{F}^2(f(-x)) = f(x)$ for all $f \in \mathscr{S}(\mathbb{R})$

d)

Use (c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{\pm 1, \pm i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\lambda \in \{\pm 1, \pm i\}$.

From (c) we know that $\mathcal{F}^4(f) = f(x) = \lambda^4 f$ thus $\lambda^4 = 1$. As $\lambda \in \mathbb{C}$ the solutions are $\lambda = \{\pm 1, \pm i\}$. By the definition of eigenvalue ($\mathcal{F}f = \lambda f$) and by the fundamental theorem of algebra we know these 4 values are all the eigenvalues of $\mathcal{F}$.

Problem 5

Let $(x_n)_{n\geq 1}$ be a dense subset of $[0,1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0,1]$. Show that $\text{supp}(\mu) = [0,1]$.

Let N be the union of all open subsets U of $[0,1]$ such that $\mu(U) = 0$. By Problem 3 HMW 8 we know $supp(\mu) = N^c$. To show that $N^c = supp(\mu) = [0,1]$ we must show that $N = \emptyset$. To show that we must show that if an open set U has measure 0 then it must be the empty-set.
Assume U is an non-empty open set with $\mu(U) = 0$, by the definition of μ we must have that $x_n \notin U$ for any n. As U is non-empty and per the definition of open there must exist an open ball of radius ϵ around an element $x \in U$ of which all elements are contained in U. But by the defintion of dense in $[0,1]$, $B(x,\epsilon)$ must contain an element x_k of $(x_n)_{n\geq 1}$ which contradicts the assumption that $\mu(U) = 0$ as $\mu(U) \geq 2^{-k} > 0$. Thus the assumption of U being non-empty was wrong and we conclude that if we have an open set in $[0,1]$ such that $\mu(U) = 0$ it must be empty. Thus $N = \emptyset \Rightarrow N^c = supp(\mu) = [0,1]$.