Mandatory Assignment 1 FunkAn

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Problem 1

(a) Let us show that $||\cdot||_0$ is a norm. As it is the sum of two norms, it is positive, and if $||x||_0 = 0$, then $||x||_X = 0$, showing $x = 0$. By linearity of T, is holds for $x, y \in X$ and $\alpha \in \mathbb{K}$ that

$$||\alpha x||_0 = ||\alpha x||_X + ||T(\alpha x)||_Y = \alpha||x||_X + \alpha||Tx||_Y = \alpha||x||_0 \,.$$

Finally, the triangle inequality follows from the norms and linearity of T again:

$$\begin{aligned}||x+y||_0 &= ||x+y||_X + ||T(x+y)||_Y = ||x+y||_X + ||Tx+Ty||_Y \\ &\le ||x||_X + ||y||_X + ||Tx||_Y + ||Ty||_Y = ||x||_0 + ||y||_0 \,.\end{aligned}$$

Let us show that $||\cdot||_0$ and $||\cdot||_X$ are equivalent if and only if T is bounded.

Note that since $||\cdot||_Y$ is positive, we get that $||x||_X \le ||x||_0$ for free, for all $x \in X$. Now, if T is bounded, by definition there exists a $K \in \mathbb{R}_+$ such that for all $x \in X$, $||Tx||_Y \le K||x||_X$. Then we insert into the definition of $||\cdot||_0$:

$$||x||_0 = ||x||_X + ||Tx||_Y \le ||x||_X + K||x||_X = (K+1)||x||_X \,.$$

This shows equivalence.

Conversely, if the two norms are equivalent, we have the inequalities

$$c||x||_0 \le ||x||_X \le ||x||_0$$

for some $c \in \mathbb{R}_+$, for any $x \in X$. Then, using the definition of $||x||_0$ again, we see that

$$||Tx||_Y = ||x||_0 - ||x||_X \le \frac{1}{c}||x||_X - ||x||_X = \left(\frac{1}{c} - 1\right)||x||_X \,.$$

This shows that T is bounded, and we are done.

(b) Let us assume that X is finite dimensional, and show that any linear map $T : X \to Y$ is bounded.

Take some linear map $T : X \to Y$. Since X is finite dimensional, any two norms on X are equivalent, in particular $||\cdot||_X$ and $||\cdot||_0$, the latter defined as in (a). Then (a) tells us that T is bounded, which completes the proof.

(c) Let us show that if X is infinite dimensional, there exists an unbounded linear map $T : X \to Y$.

Since X is infinite dimensional, it has an infinite Hamel basis $(e_i)_{i\in I}$. We may assume the Hamel basis is normalized, i.e. $||e_i||_X = 1$. Since Y is non-zero, take some $y \in Y$, $y \neq 0$, and define $y_n := ny$ for $n \in \mathbb{N}$. As I is infinite, there exists an surjection $\phi : I \to \mathbb{N}$. Then, by the definition of a Hamel basis, there exists a linear map defined by $Te_i = y_{\phi(i)}$ for $i \in I$. Now, for every $n \in \mathbb{N}$, there exists some $i \in I$ such that $\phi(i) = n$, meaning

$$||Te_i||_Y = ||y_n||_Y = n||y||_Y \ .$$

As $||y||_Y$ is a constant and $||e_i||_X = 1$, this clearly shows that T is unbounded.

(d) Let us show that if X is infinite dimensional, there exists a norm $||\cdot||_0$ on X not equivalent to $||\cdot||_X$, satisfying $||x||_X \leq ||x||_0$ for all $x \in X$. In this situation, let us further show that $(X, ||\cdot||_0)$ is not complete if $(X, ||\cdot||_X)$ is a Banach space.

Simply take the unbounded linear map constructed in (c), and define $||x||_0 := ||x||_X + ||Tx||_Y$ for $x \in X$ as in (a). Then, since T is unbounded, (a) tells us that the norms are not equivalent. Clearly $||x||_X \leq ||x||_0$ is satisfied as well for all $x \in X$.

Now, $(X, ||\cdot||_0)$ cannot be complete, since otherwise Homework 3 Problem 1 would tell us that the norm would be equivalent to $||\cdot||_X$, a contradiction. Here we use both the inequality $||x||_X \leq ||x||_0$ and the fact that $(X, ||\cdot||_X)$ is a Banach space.

(e) Let us give an example of a vector space X equipped with two inequivalent norms $||\cdot||$ and $||\cdot||'$ satisfying $||x||' \leq ||x||$ for all $x \in X$, such that $(X, ||\cdot||)$ is complete, while $(X, ||\cdot||')$ is not.

Let $(X, ||\cdot||)$ be $(\ell_1(\mathbb{N}), ||\cdot||_1)$, and define $||\cdot||'$ by

$$||x||' = \sum_{n=1}^{\infty} \frac{|x_n|}{n^2} \ , \qquad x = (x_n)_{n\in\mathbb{N}} \in \ell_1(\mathbb{N}).$$

This is clearly a norm; all its properties are analogous to, and can be proven in the same fashion as, the norm properties of $||\cdot||_1$. Furthermore, $||x||' \leq ||x||_1$ for all $x \in \ell_1(\mathbb{N})$. Also, $(\ell_1(\mathbb{N}), ||\cdot||')$ is not complete: Take the sequence $(s_n)_{n\in\mathbb{N}} \subseteq \ell_1(\mathbb{N})$ defined by $s_n = (1, 1, \ldots, 1, 0, 0, \ldots)$ (n leading 1's). This is clearly Cauchy in $||\cdot||'$:

$$||s_m - s_k||' = ||(0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots)||' = \sum_{n=k+1}^{m} \frac{1}{n^2} \leq \sum_{n=k+1}^{\infty} \frac{1}{n^2} \to 0$$

for $k \to \infty$, $k \leq m$. But the sequence does not converge in $(\ell_1(\mathbb{N}), ||\cdot||')$ by definition of $\ell_1(\mathbb{N})$, since the norm $||s_n||_1$ diverges. This also shows that the norms are inequivalent, which completes the proof.

Problem 2

(a) Let $1 \leq p < \infty$. Let us show that $f : M \to \mathbb{C}$ given by $f(a, b, 0, 0, \dots) = a + b$ is bounded on $(M, || \cdot ||_p)$ and compute $||f||$.

Let $x = (a, b, 0, 0, \dots) \in M$. We note that $||x||_p = (|a|^p + |b|^p)^{\frac{1}{p}}$ and $|f(x)| = |a+b| \leq |a| + |b|$. We now see that

$$|f(x)|^p - ||x||_p^p \leq (|a| + |b|)^p - (|a|^p + |b|^p) = \sum_{r=1}^{p-1} \binom{p-1}{r} |a|^r |b|^{p-1-r|} .$$

Now assume $||x||_p = 1$. Then $|a|^p + |b|^p = 1$, showing $|a|, |b| \leq 1$. Assuming $|a| = |b| = 1$ to construct an upper bound, we insert in the above:

$$\begin{aligned} |f(x)|^p - ||x||_p^p \leq & \sum_{r=1}^{p-1} \binom{p-1}{r} = 2^{p-1} - 1 \\ & \Leftrightarrow |f(x)|^p \leq 2^{p-1} \\ & \Leftrightarrow |f(x)| \leq 2^{1-\frac{1}{p}} . \end{aligned}$$

This shows that f is bounded. I claim $||f|| = 2^{1-\frac{1}{p}}$. To show this, by the bound it is enough to find an element x with $||x||_p = 1$ attaining $|f(x)| = 2^{1-\frac{1}{p}}$. This is attained by $x = (2^{-\frac{1}{p}}, 2^{-\frac{1}{p}}, 0, 0, \dots)$. We calculate:

$$\begin{aligned} ||x||_p &= \left((2^{-\frac{1}{p}})^p + (2^{-\frac{1}{p}})^p\right)^{\frac{1}{p}} = \left(\frac{1}{2} + \frac{1}{2}\right)^{\frac{1}{p}} = 1 \\ |f(x)| &= |2^{-\frac{1}{p}} + 2^{-\frac{1}{p}}| = 2^{1-\frac{1}{p}} . \end{aligned}$$

This completes the proof.

(b) Let $1 < p < \infty$. Let us show that there exists a unique linear functional $F : \ell_p(\mathbb{N}) \to \mathbb{C}$ extending f such that $||F|| = ||f||$.

By Corollary 2.6, such an extension exists. For uniqueness, assume F and F' are both such extensions. Let $q > 1$ be the Hölder conjugate of p. By Homework 1 Problem 5, we have an isometric isomorphism $T : \ell_q(\mathbb{N}) \to \ell_p(\mathbb{N})^*$ defined by $T(x) = f_x$ for all $x = (x_n)_{n\in\mathbb{N}} \in \ell_q(\mathbb{N})$, where

$$f_x(y) = \sum_{n=1}^{\infty} x_n y_n , \quad \text{for all } \; y = (y_n)_{n\in\mathbb{N}} \in \ell_p(\mathbb{N}) .$$

Since T is a surjection, choose $x, x' \in \ell_q(\mathbb{N})$ such that $T(x) = F$, $T(x') = F'$. Since T is an isometry, we have that $||x||_q = ||x'||_q = ||f|| = 2^{1-\frac{1}{p}} = 2^{\frac{1}{q}}$. The last equality follows by p and q being conjugates. Furthermore, since F and F' extends f,

$$F((a, b, 0, \dots) = F'((a, b, 0, \dots)) = a + b ,$$

so must have that $x_1 = x_2 = x'_1 = x'_2 = 1$. But this completely determines x and x': Indeed,

$$||x||_q = \left(\sum_{n=1}^{\infty} |x_n|^q \right)^{\frac{1}{q}} = \left(2 + \sum_{n=3}^{\infty} |x_n|^q \right)^{\frac{1}{q}} \geq 2^{\frac{1}{q}} ,$$

but the norm attains this minimum if and only if $x_n = 0$ for all $n \geq 3$, and it must attain this minimum by assumption. Thus $x = (1, 1, 0, 0, \dots)$, and the same holds for x'. But then $x = x'$, showing that $F = F'$ by injectivity of T. This shows uniqueness.

(c) We consider the case $p = 1$. Let us show that there exist infinitely many linear functionals F on $\ell_1(\mathbb{N})$ extending f such that $||F|| = ||f||$.

First note that $||f||_1 = 2^{1-\frac{1}{1}} = 2^0 = 1$. Again by Homework 1 Problem 5, we know that $T : \ell_\infty(\mathbb{N}) \to \ell_1(\mathbb{N})^*$ defined similarly as above is an isometric isomorphism. Note that any $x = (x_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathbb{N})$ satisfying $||x||_\infty = 1$ with $x_1 = x_2 = 1$ is mapped to an extension F of f by T, with the correct norm as T is an isometry. But this means that all $s_i = (1, 1, 1, \dots, 1, 0, 0, \dots)$ ($i + 1$ leading 1's) for $i \in \mathbb{N}$ are sent to an extension of f with the same norm as f, since $||s_i||_\infty = 1$. As there are infinitely many s_i's and T is injective, this completes the proof.

Problem 3

(a) Let X be an infinite dimensional normed vector space, and let $n \in \mathbb{N}$. Let us show no map $F : X \to \mathbb{K}^n$ is injective.

Take a Hamel basis $(e_i)_{i\in I}$ of X. As I is infinite, we can take a subset of $(F(e_i))_{i\in I}$ of $n+1$ elements, which must be linearly dependent by the dimension of $\mathbb{K}^n$. If $\sum_{k=1}^{n+1} \alpha_k F(e_{n_k}) = 0$ is any non-trivial linear combination of 0 of such a subset, then $F\left(\sum_{k=1}^{n+1} \alpha_k e_{n_k}\right) = 0$, and $\sum_{k=1}^{n+1} \alpha_k e_{n_k} \neq 0$, since not all α_k are zero. This shows F not injective.

(b) let $n \in \mathbb{N}$ and let $f_1, \dots, f_n \in X^*$. Let us show that

$$\bigcap_{i=1}^{n} \ker f_i \neq \{0\} .$$

Define $F : X \to \mathbb{K}^n$ by $F(x) = (f_1(x), f_2(x), \dots, f_n(x))$. By (a), F is not injective, so $\ker F \neq \{0\}$. But $\ker F = \bigcap_{i=1}^{n} \ker f_i$, as $F(x)$ is zero if and only if all $f_i(x)$ are zero. This completes the proof.

(c) Let $x_1, \dots, x_n \in X$. Let us find a $y \in X$ with $||y|| = 1$ such that $||x_j|| \leq ||x_j - y||$ for all $1 \leq j \leq n$.

Using Theorem 2.7(b) for each x_j, we construct $f_1, f_2, \dots, f_n \in X^*$ such that $||f_j|| = 1$ and $f_j(x_j) = ||x_j||$ for all j. Using (b), we may take some non-zero $y \in \bigcap_{i=1}^{n} \ker f_i$, and as this intersection is a subspace, we may assume $||y|| = 1$. We claim this y satisfies our desired property. Indeed, for all j,

$$||x_j|| = |f_j(x_j)| = |f_j(x_j) - f_j(y)| = |f_j(x_j - y)| \leq ||f_j||\,||x_j - y|| = ||x_j - y|| .$$

This shows the desired property for y, and we are done.

(d) Let us show that one cannot cover the unit sphere $S = \{x \in X \mid ||x|| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

Let $B_1, B_2, \dots, B_n$ be a finite family of closed balls covering S, and let us show at least one ball contains 0. Let x_j and r_j be the center, respectively the radius, of B_j. Using (c), we can find a $y \in X$ with $||y|| = 1$ such that $||x_j|| \leq ||x_j - y||$ for all $1 \leq j \leq n$. Since $||y|| = 1$, $y \in S$, and since the balls cover S, there is some j_0 such that $y \in B_{j_0}$. But this means that

$$||x_{j_0} - 0|| = ||x_{j_0}|| \leq ||x_{j_0} - y|| \leq r_{j_0} .$$

This shows that $0 \in B_{j_0}$, and we are done.

(e) Let us Show that S is non-compact and argue that therefore the closed unit ball in X is non-compact.

First we note that the proof of (d) never really used that the balls are closed, and the analogous statement for open balls holds by a completely similar proof. Now, assume for contradiction that S is compact. Consider the open cover $\mathcal{B} = \{B(x, \frac{1}{2}) \mid x \in S\}$. By assumption, we may reduce this to a finite cover. But then by the analogous statement to (d) for open balls, 0 must be contained in one of the balls, but for any of these balls with center x,

$$||x - 0|| = ||x|| = 1 > \frac{1}{2}$$

which is a contradiction. Then S is non-compact. Because of this, the closed unit ball $\overline{B}$ in X cannot be compact. Indeed, if $\overline{B}$ was compact, then $\mathcal{B} \cup \{B(0, 1)\}$ would be an open cover of $\overline{B}$, and we could reduce it to a finite open cover. As $S \subseteq \overline{B}$, it would also be a finite open cover of S. In fact, if $B(0, 1)$ lies in this finite open cover of S, we could remove it and still have a cover, as $B(0, 1) \cap S = \emptyset$. But then we would have found a finite subset of $\mathcal{B}$ covering S, which we showed was impossible in the first part of this subproblem. Thus, we get a contradiction, showing that $\overline{B}$ is non-compact.

Problem 4

(a) Let $n \in \mathbb{N}$. Let us show that E_n is not absorbing.

By Homework 2 Problem 2(b), we may take $0 \neq f \in L_1([0,1],m) \setminus L_3([0,1],m)$. Then

$$||f||_3^3 = \int_{[0,1]} |f|^3 dm = \infty .$$

Then, for any $t > 0$, $tf \notin E_n$, as

$$\int_{[0,1]} |tf|^3 dm = t^3 \int_{[0,1]} |f|^3 dm = \infty .$$

This completes the proof.

(b) Let us show that E_n, for each $n \in \mathbb{N}$, has empty interior in $L_1([0,1],m)$.

Let $\varepsilon > 0$ be given. It is sufficient to show that for any $f \in E_n$, we have some g in $L_1([0,1],m) \setminus E_n$ such that $||f - g||_1 \leq \varepsilon$. Let $f \in E_n$ be given.

Define g by

$$g(x) = f(x) + x^{-\frac{1}{3}} \cdot \frac{3\varepsilon}{2} \quad \text{for } x \in X .$$

We make the following two calculations.

$$\begin{aligned}
||g||_1 &\leq ||f||_1 + \left|\left| x^{-\frac{1}{3}} \cdot \frac{3\varepsilon}{2} \right|\right|_1 = ||f||_1 + || + \frac{3\varepsilon}{2} \int_{[0,1]} x^{-\frac{1}{3}} dm \\
&= ||f||_1 + \frac{3\varepsilon}{2} \left[\frac{3}{2} x^{\frac{2}{3}} \right]_0^1 = ||f||_1 + \varepsilon < \infty \\
||g||_3 &\geq \frac{3\varepsilon}{2} \left|\left| x^{-\frac{1}{3}} \right|\right|_3 - ||f||_3 = \frac{3\varepsilon}{2} \int_{[0,1]} x^{-1} dm - ||f||_3 \\
&= \frac{3\varepsilon}{2} [\log(x)]_0^1 - ||f||_3 = \infty .
\end{aligned}$$

The first calculation shows that $g \in L_1([0,1],m)$, and the second shows that $g \notin E_n$. We used that f has finite 1-norm and 3-norm, as $f \in L_3([0,1],m)$. Finally, we see that, using a calculation made above,

$$||f - g||_1 = \left|\left| x^{-\frac{1}{3}} \cdot \frac{3\varepsilon}{2} \right|\right|_1 = \varepsilon .$$

This shows all our desired properties, and we are done.

(c) Let us show that E_n is closed in $L_1([0,1],m)$, for any $n \in \mathbb{N}$.

Take a sequence $(f_n)_{n\in\mathbb{N}} \subseteq E_n$ converging to f in $L_1([0,1],m)$, and let us show that $f \in E_n$. Since $(f_n)_{n\in\mathbb{N}}$ converges in $L_1([0,1],m)$ to f, we know by a result in An2 (corollary 12.8 in Shilling, first edition), that there is some subsequence f_{n_k} converging almost everywhere to f. This clearly gives us that $|f_{n_k}|^3$ converges almost everywhere to $|f|^3$. Then, by Fatou's lemma,

$$\int_{[0,1]} |f|^3 dm \leq \liminf \int_{[0,1]} |f_{n_k}|^3 dm \leq n \,.$$

The last inequality follows from the fact that $f_{n_k} \in E_n$. This shows that $f \in E_n$, and we are done.

(d) Let us show that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

First we see that $\mathrm{Int}(\overline{E_n}) = \mathrm{Int}(E_n) = \emptyset$ by respectively subproblems (c) and (b). Therefore, each E_n is nowhere dense. Next, note that $L_3([0,1],m) = \bigcup_{n\in\mathbb{N}} E_n$. Indeed, by definition, any $f \in L_1([0,1],m)$ has

$$||f||_3 < \infty \qquad \Longrightarrow \qquad \int_{[0,1]} ||f||^3 dm < \infty \,,$$

thus this integral is bounded by some natural number N, hence $f \in E_N$. The other inclusion is obvious. But this shows that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$, as wanted.

Problem 5

Let H be a separable infinite dimensional vectorspace with norm $||\cdot||$.

(a) Let us show that for $x, x_1, x_2, \dots$ in H, $x_n \to x$ in norm implies $||x_n|| \to ||x||$.

By the reverse triangle inequality,

$$|||x|| - ||x_n||| \leq ||x - x_n|| \,.$$

As $||x - x_n|| \to 0$ for $n \to \infty$ by assumption, we are done.

(b) Let us give an counterexample where $x_n \to x$ weakly for $n \to \infty$, but $||x_n||$ does not converge to $||x||$ for $n \to \infty$.

Let the Hilbert space be $H := \ell_2(\mathbb{N})$. Take the orthonormal basis $(e_n)_{n\in\mathbb{N}}$, where $e_n = (0, 0, \dots, 0, 1, 0, \dots)$ (1 on the n'th place), and define $x_n := e_n$. Our claim is that $x_n \to 0$ weakly. Clearly this would suffice, as $||x_n|| = 1$ for all $n \in \mathbb{N}$.

Let us show the weak convergence. By Homework 4 Problem 2(a), it is sufficient to show that $f(x_n) \to f(x)$ for all $f \in l_2(\mathbb{N})^*$. Recall that $T : l_2(\mathbb{N}) \to l_2(\mathbb{N})^*$ given by $T(x) = f_x$ is an isometric isomorphism (as defined in Problem 2), since 2 is its own Hölder conjugate. For any $s \in l_2(\mathbb{N})$, as $||s||_2 = \sum_{i=1}^{\infty} |s_i|^2 < \infty$, we know that $s_i \to 0$. Then we see that

$$f_s(x_n) = s_n \to 0 = f_s(0) \quad \text{for} n \to \infty \,.$$

As every $f \in l_2(\mathbb{N})^*$ has this form, by the isomorphism, we have shown that $f(x_n) \to f(x)$ for all $f \in l_2(\mathbb{N})^*$, which establishes weak convergence. Our counterexample is complete.

(c) Let us show that if $x_n \to x$ weakly and $||x_n|| \leq 1$ for all $n \in \mathbb{N}$, then $||x|| \leq 1$.

If $x = 0$, then we are done. Otherwise, by Theorem 2.7(b), there exists $f \in H^*$ such that $||f|| = 1$ and $f(x) = ||x||$. By Homework 4 Problem 2(a), $f(x_n)$ converges to $f(x)$ for $n \to \infty$. This means that

$$||x|| = |f(x)| = \lim_{n\to\infty} |f(x_n)| \,,$$

and it holds that $|f(x_n)| \leq ||f||\ ||x_n|| \leq 1$. Thus $||x|| \leq 1$, as the unit interval is closed.