spectral flow

June 2021

1 Introduction

2 Special case

For each $t \in \mathbb{R}$ let A_t be a self-adjoint operator on the Hilbert space H with domain $D(A_t) = W$ independent of t. We assume that W is compactly embedded and dense in H. Assume furthermore, that $t \mapsto A_t$ is continuously differentiable in the weak operator topology (WOT), *i.e.* a differentiable map, with derivative that is continuous in the WOT. In the following, we shall assume that A_t takes the form $A_t = \sum_j \lambda_j(t) \left|u_j\right\rangle \left\langle u_j\right|$. Define the operator $D_A = \frac{\mathrm{d}}{\mathrm{d}t} - A_t$, with domain

$$D(D_A) = \left\{ \sum_j \alpha_j(t) \left|u_j\right\rangle \,\middle|\, \int_{\mathbb{R}} \sum_j |\alpha_j(t)|^2 \,\mathrm{d}t < \infty,\ \int_{\mathbb{R}} \sum_j \left|\alpha_j'(t) - \lambda_j(t)\alpha_j(t)\right|^2 \mathrm{d}t < \infty \right\}.$$

Then we have the following result

Theorem 1. *Let D_A be as described above, and assume that A_t converges to invertible operators $A_\pm$ as $t \to \pm\infty$. Then D_A is Fredholm, and its index is given by the spectral flow of the family $(A_t)_{t\in\mathbb{R}}$.*

In order to show this theorem, we show first that to each element in the kernel of D_A or the kernel of D_A^*, we may associate an eigenvalue of A_t, $\lambda_j(t)$, crossing zero and odd number of times. More precisely we show that each element in $\ker(D_A)$ can be associated to eigenvalues $\lambda_j(t)$ with the properties $\lim_{t\to-\infty} \lambda_j(t) > 0$ and $\lim_{t\to\infty} \lambda_j(t) < 0$. Similarly each element in $\ker(D_A^*)$ can be associated to eigenvalues $\lambda_j(t)$ with the properties $\lim_{t\to-\infty} \lambda_j(t) < 0$ and $\lim_{t\to\infty} \lambda_j(t) > 0$.

Proposition 1. *Let D_A be as above, then $\ker(D_A)$ is spanned by $\{\beta_j(t) \left|u_j\right\rangle\}_{j\in M}$, where $M = \{j \,|\, \lim_{t\to-\infty} \lambda_j(t) > 0$ and $\lim_{t\to\infty} \lambda_j(t) < 0\}$ and $\beta_j(t) = e^{\int_0^t \lambda_j(s)\,\mathrm{d}s}$.*

Proof. It is clear that $\beta_j(t) \left|u_j\right\rangle$ is in the kernel under the assumptions. On the contrary if $\sum_j \alpha_j(t) \left|u_j\right\rangle \in \ker(D_A)$, then $\alpha_j'(t) - \lambda_j(t)\alpha_j(t) = 0$ for all α_j. Therefore $\alpha_j(t) = \alpha_j(0)\mathrm{e}^{\int_0^t \lambda_j(s)\,\mathrm{d}s}$, for all j. But then $\sum_j \alpha_j(t) \left|u_j\right\rangle \in D(D_A)$, only if $\lim_{t\to-\infty} \lambda_j(t) > 0$ and $\lim_{t\to\infty} \lambda_j(t) < 0$. Furthermore, we show below that only finitely many eigenvalues cross zero, and therefore, we may conclude that $\sum_j \alpha_j(t) \left|u_j\right\rangle$ is a finite sum, and hence $\sum_j \alpha_j(t) \left|u_j\right\rangle \in D(D_A)$ if $\lim_{t\to-\infty} \lambda_j(t) > 0$ and $\lim_{t\to\infty} \lambda_j(t) < 0$. □

Proposition 2. *Let D_A be as above, then $\ker(D_A^*)$ is spanned by $\{\beta_j(t) \left|u_j\right\rangle\}_{j\in M}$, where $M = \{j \,|\, \lim_{t\to-\infty} \lambda_j(t) < 0$ and $\lim_{t\to\infty} \lambda_j(t) > 0\}$ and $\beta_j(t) = e^{-\int_0^t \lambda_j(s)\,\mathrm{d}s}$.*

Proof. The proof is similar to the one for Proposition 1 □

We now show that the eigenvalues of A_t, can cross zero only finitely many times. We first need to establish that all eigenvalues must cross zero within some compact interval. This is a consequence of the following lemma

Lemma 1. *Let $(A_t)_{t\in\mathbb{R}}$ be a family of self-adjoint operators with t-independent domain W. Assume furthermore, $(A_t)_{t\in\mathbb{R}}$ converge to invertible operators $A^\pm$ in the norm-topology on $\mathcal{L}(W,H)$ and that $t\mapsto A_t$ is countinuously differentiable in the WOT. Then there exist $t_1,t_2\in\mathbb{R}$ and $c>0$ such that $|A_t|>c>0$ for $t<t_1$ or $t>t_2$.*

Proof. It is direct consequence of invertibility of $A^\pm$, that there exist $d>0$ such that $|A^\pm|>d$. Notice now that for invertible operators A and B we have $\frac{1}{A}-\frac{1}{B}=\frac{1}{A}(A-B)\frac{1}{B}$. Since (A_t+i) and $A\pm+i$ are invertible it follows that

$$\left\|\frac{1}{A_t+i}-\frac{1}{A^\pm+i}\right\|_{\mathcal{L}(H,H)}\leq\left\|\frac{1}{A_t+i}\right\|_{\mathcal{L}(H,H)}\left\|A_t-A^\pm\right\|_{\mathcal{L}(W,H)}\left\|\frac{1}{A^\pm+i}\right\|_{\mathcal{L}(H,W)}\leq\left\|A_t-A\right\|_{\mathcal{L}(W,H)}. \tag{1}$$

Thus we conclude that A_t converges to $A^\pm$ as $t\to\pm\infty$ in the norm resolvent sense. Thus for any $\epsilon>0$ we have that there exist t_1 and t_2 such that

$$\left\|\frac{1}{A_t+i}\right\|_{\mathcal{L}(H,H)}\leq\left\|A_t-A^\pm\right\|_{\mathcal{L}(W,H)}+\left\|\frac{1}{A^\pm+i}\right\|_{\mathcal{L}(H,H)}\leq\epsilon+\left\|\frac{1}{A^\pm+i}\right\|_{\mathcal{L}(H,H)} \tag{2}$$

for $t<t_1$ or $t>t_2$ from which it follows that

$$\sup_i\frac{1}{\left(|\lambda_i|^2+1\right)^{1/2}}\leq\epsilon+\sup_i\frac{1}{\left(|\lambda_i^\pm|^2+1\right)^{1/2}}. \tag{3}$$

Equivalently we have $\inf_i\left(|\lambda_i|^2+1\right)^{1/2}=\left(\inf_i|\lambda_i|^2+1\right)^{1/2}\geq\left(\epsilon+\frac{1}{\left(\inf_i|\lambda_i^\pm|^2+1\right)^{1/2}}\right)^{-1}$, and we see that

$$\left(\inf_i|\lambda_i|^2+1\right)^{1/2}\geq\left(\epsilon+\frac{1}{(d^2+1)^{1/2}}\right)^{-1} \tag{4}$$

$\inf_i|\lambda_i|\geq\left(\left(\epsilon+\frac{1}{(d^2+1)^{1/2}}\right)^{-2}-1\right)^{1/2}$, and the result follows by choosing $\epsilon<1-\frac{1}{(d^2+1)^{1/2}}$ □

We are now ready to show the result

Proposition 3. *Let $(A_t)_{t\in\mathbb{R}}$ be a family of self-adjoint operators with t-independent domain W. Assume furthermore, $(A_t)_{t\in\mathbb{R}}$ converge to invertible operators $A^\pm$ in the norm-topology on $\mathcal{L}(W,H)$ and that $t\mapsto A_t$ is countinuously differentiable in the WOT. Then only finitely many eigenvalues of A_t cross zero.*

Proof. It is known that A_t has discrete spectrum for all $t\in\mathbb{R}$. Now assume that infinitely many eigenvalues cross zero. By lemma 1, there exist t_1,t_2 such that all crossing happen in the interval $[t_1,t_2]$. Letting the crossing points define a sequence, it is clear by the Bolzano-Weierstrass theorem, that there exist a point, t^*, such that any interval I with $t^*\in I^o$ contains infinitely many crossings. It is then clear that $I_\epsilon=[t^*-\epsilon,t^*+\epsilon]$ contains infinitely many crossings for every $\epsilon>0$. Since $\dot{A}_t=\frac{\mathrm{d}A_t}{\mathrm{d}t}$ is continuous in the WOT, it holds that $f_x:\mathbb{R}\to H$ defines by $f_x(t)=\dot{A}_tx$ is continuous, when H is equipped with the weak topology. Therefore, $f_x([t^*-\epsilon,t^*+\epsilon])$ is weakly compact and hence norm bounded. We conclude that $\sup_{t\in I_\epsilon}\left\{\left\|\dot{A}_tx\right\|_H\right\}<\infty$ for all $x\in W$. By the uniform boundedness principle, it follows that $\sup_{t\in I_\epsilon}\left\{\left\|\dot{A}_t\right\|_{\mathcal{L}(W,H)}\right\}<\infty$. Thus we conclude that there exist $C > 0$ such that $\left|\lambda_j'(t)\right| \leq C\left(\sqrt{|\lambda_j(t)|^2 + 1}\right) \leq C(|\lambda_j(t)| + 1)$ for all $t \in I_\epsilon$. Letting $M_j = \max_{t \in I_\epsilon} |\lambda_j(t)|$, we see that by the mean value theorem, there exist a point $t' \in I_\epsilon$ where $\left|\lambda_j'(t)\right| \geq \frac{M}{2\epsilon}$, and therefore $\frac{M}{2\epsilon} \leq C(M+1)$ or equivalently $M \leq \frac{2\epsilon C}{1-2\epsilon C}$. This clearly contradicts the fact, that A_t^* has dicrete spectrum, as the eigenvalues of A_{t^*} accumulate at 0. Thereby, we conclude that the number of crossing eigenvalues must be finite. $\square$