## Assigment 1

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**Exercise 1 (24 points)** . Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be (non-zero) normed vector spaces over  $\mathbb{K}$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ 

> • [5p]. Let  $T: X \to Y$  be a linear map. Set  $||x||_0 = ||x||_X + ||Tx||_Y$ , for all  $x \in X$ . Show that  $\|\cdot\|_0$  is a norm on X. Show next that the two norms  $\|\cdot\|_X$  and  $\|\cdot\|_0$  are equivalent if and only if T is bounded.

Let's check the axioms of norm: for any  $\lambda \in \mathbb{K}$ ,  $x, y \in X$ 

1.  $\|\lambda x\|_0 = |\lambda| \|x\|_0$ .

$$\|\lambda x\|_0 = \|\lambda x\|_X + \|T(\lambda x)\|_Y = |\lambda| \|x\|_X + \|\lambda T(x)\|_Y =$$
$$= |\lambda| (\|x\|_X + \|T(x)\|_Y) = |\lambda| \|x\|_0.$$

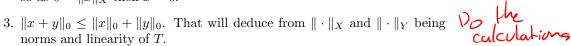
2.  $||x||_0 \ge 0$  for any  $x \in X$  and  $||x||_0 = 0 \iff x = 0$ . The first part is obvious, and for the second part:

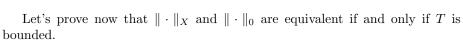
$$0 = ||x||_0;$$
  
$$0 = ||x||_X + ||T(x)||_Y;$$

as the two terms are non negative numbers:

$$0 = \|x\|_X = \|T(x)\|_Y,$$

so as  $0 = ||x||_X$  then x = 0.





•  $\Leftarrow$  If T is bounded then there exists  $||T|| = \sup\{||Tx||_Y, ||x||_X \le 1\}$  and  $||Tx||_Y \leq ||T|| ||x||_X$  for any  $x \in X$ . We have to prove there exist c, C > 0such that  $c||x||_X \le ||x||_0 \le C||x||_X$  for any  $x \in X$ . First, as  $||Tx||_Y \ge 0$ ,  $||x||_0 \ge ||x||_X$  for any  $x \in X$  (so we take c = 1). On the other hand:

$$||x||_0 = ||x||_X + ||T(x)||_Y \le ||x||_X + ||T|| ||x||_X = (1 + ||T||) ||x||_X.$$

So taking C = 1 + ||T||, the two norms are equivalent.

•  $\implies$  If there exists some C > 0 such that  $||x||_0 \le C||x||_X$ , then:

$$||x||_X + ||T(x)||_Y \le C||x||_X;$$
  
$$||T(x)||_Y \le (C-1)||x||_X$$

and that inequality implies T is bounded.



• [4p]. Show that any linear map  $T: X \to Y$  is bounded, if X is finite dimensional.

For the first part, we now T is bounded if and only if the two norms considered are equivalent on X, and for *Theorem 1.6* any 2 norms on X are equivalent when X is finite-dimensional.

• [5p]. Suppose that X is infinite dimensional. Show that there exists a linear map T: X → Y, which is not bounded (= not continuous). [Hint: Take a Hamel basis for X.] Let X be a vector space over K. An algebraic basis for X is a family (e<sub>i</sub>)<sub>i∈I</sub> of elements in X with the following property: For each vector space Y over K, and each family (y<sub>i</sub>)<sub>i∈I</sub> in Y there exists precisely one linear map T: X → Y satisfying T (e<sub>i</sub>) = y<sub>i</sub>, for all i ∈ I. One can show that this condition is equivalent to the more usual definition of an algebraic basis: for each x ∈ X, there is a unique family (λ<sub>i</sub>)<sub>i∈I</sub> in K for which the set {i ∈ I : λ<sub>i</sub> ≠ 0} is finite and x = ∑<sub>i∈I</sub> λ<sub>iei</sub>. When X is infinite dimensional, an algebraic basis is also called a Hamel basis. It is a consequence of Zorn's lemma that each infinite dimensional vector space admits a Hamel basis. You are free to use these facts without further justifications.

We take a Hamel basis  $(e_i) \subset X$  and define a norm on X such that this basis is orthogonal with respect to it, i.e. for  $x \in X$ , if  $x = \sum \lambda_i e_i$  then we define  $||x|| := \sqrt{\sum |\lambda_i|^2}$ .

Now we choose  $Y = \mathbb{K}$  and a countable subset of the Hamel basis  $(e_n)_{n \geq 0} \subset (e_i)$ . Also we consider the unique linear map such that  $T(e_n) = n$  and  $T(e_i) = 0$  otherwise.

Then T is linear and is not bounded, as:

$$|T| = \sup\{|Tx|, ||x|| = 1\} \ge \sup\{|T(e_i)|\} \ge \sup_{n \ge 0} n. = \infty$$

• [5p]. Suppose again that X is infinite dimensional. Argue that there exists a norm  $\|\cdot\|_0$  on X, which is not equivalent to the given norm  $\|\cdot\|_X$ , and which satisfies  $\|x\|_X \leq \|x\|_0$  for all  $x \in X$ . Conclude that  $(X, \|\cdot\|_0)$  is not complete if  $(X, \|\cdot\|_X)$  is a Banach space.

If we take the not bounded linear map T for the previous exercise, the first part show us that the norm  $\|x\|_0 := \|x\|_X + \|Tx\|_Y$  is not equivalent to  $\|x\|_X$ . And also, by constuction  $\|\cdot\|_0 \ge \|\cdot\|_X$ . Now, for HW3 Problem 1, if  $(X, \|\cdot\|_X)$  and  $(X, \|\cdot\|_0)$  were both complete, as  $\|\cdot\|_0 \ge \|\cdot\|_X$ , then the two norms would be equivalent and they are not, so  $(X, \|\cdot\|_0)$  cannot be complete.

This might change the norm structure / on I!

I is just some set, so MQI in general



• [5p]. Give an example of a vector space X equipped with two inequivalent norms  $\|\cdot\|$  and  $\|\cdot\|'$  satisfying  $\|x\|' \leq \|x\|$ , for all  $x \in X$ , such that  $(X, \|\cdot\|)$  is complete, while  $(X, \|\cdot\|')$  is not. [Hint: Take  $(X, \|\cdot\|) = (\ell_1(\mathbb{N}), \|\cdot\|_1)$  with a suitable choice of  $\|\cdot\|'$ ; or take  $(X, \|\cdot\|) = (L_2([0, 1], m), \|\cdot\|_2)$  with a suitable choice of  $\|\cdot\|'$ , where m is the Lebesgue measure.

We take  $X = \ell_1(\mathbb{N}) = \{(x_n) \subset \mathbb{K}, \sum_{n=1}^{\infty} |x_n| < \infty\}$ , and with norm  $\|\cdot\|_1$ . By HW1 Problem 5,  $(X, \|\cdot\|_1)$  is a Banach space. In addition, we can also consider the infinity norm  $||(x_n)||_{\infty} = \sup_{n>1} \{|x_n|\}$ , as it's well-defined on X.

It's clear that  $\|\cdot\|_{\infty} \leq \|\cdot\|_{1}$ , as for each element of the sequence  $|x_{n}| \leq$ Then  $(x_n) \in X$  and  $\|(x_n)\|_{\infty} = 1$ , but  $\|(x_n)\|_1 = [C] + 1 > C$ , so it's not true

that  $\|\cdot\|_1 \leq C\|\cdot\|_{\infty}$ . Then the two norms can't be equivalent.

 $(X, \|\cdot\|_{\infty})$  can't be a Banach space by the same argument of the previous part.

**Exercise 2 (20**points) . Let  $1 \le p < \infty$  be fixed, and consider the subspace M of the Banach space  $(\ell_p(\mathbb{N}), \|\cdot\|_p)$ , considered as a vector space over  $\mathbb{C}$ , given

$$M = \{(a, b, 0, 0, \ldots) : a, b \in \mathbb{C}\}\$$

Let  $f: M \to \mathbb{C}$  be given by  $f(a, b, 0, 0, 0, \ldots) = a + b$ , for all  $a, b \in \mathbb{C}$ 

• [8p]. Show that f is bounded on  $(M, \|\cdot\|_p)$  and compute  $\|f\|$ . (Answer depends on p.)

It's easy to show that f is bounded: for any  $x = (a, b, 0, ...) \in M$  we have:

$$|f(x)| = |a+b| \le |a| + |b| = |a|^{p^{1/p}} + |b|^{p^{1/p}} \le$$
  
  $\le ||x||_p + ||x||_p = 2||x||_p$ 

That shows that f is bounded. To compute the exact norm of f we will have to study

$$\sup_{\|x\|_p=1}\{|f(x)|\}=\sup_{|a|^p+|b|^p=1}\{|a+b|\}\leq \sup_{|a|^p+|b|^p=1}\{|a|+|b|\}$$

So we are going to study the maxima of the function  $h(t) = t + (1 - t^p)^{1/p}$ with  $t \in [0, 1]$ .

Computing its critical points and solving h'(t) = 0 we get to the equation

$$0 = 1 - (1 - t^p)^{\frac{1}{p} - 1} t^{p-1}$$

Is this immediate? It not, do calculations.

which has solution  $t_0 = (\frac{1}{2})^{1/p}$ . Analyzing the sign of h' we get to the conclusion that h reaches its maximum on  $t_0$ , which is the value  $h(t_0) = (\frac{1}{2})^{1/p} + (1 - \frac{1}{2}^{1/p}) = \frac{2}{21/p}$ .

So  $\sup_{\|x\|_p=1} \{|f(x)|\} \le \frac{2}{2^{1/p}}$ , and this value is actually attained for the vector  $x = ((\frac{1}{2})^{1/p}, (\frac{1}{2})^{1/p}, 0, \ldots)$ . So,  $\|f\| = \frac{2}{2^{1/p}}$ .

• [7p]. Show that if 1 , then there is a unique linear functional <math>F on  $\ell_p(\mathbb{N})$  extending f and satisfying ||F|| = ||f||.

We have a trivial extension of f given by  $F(a_1, a_2, a_3, ...) = a_1 + a_2$ . It's lineal and an extension of f, and verifies the norm condition, as we already have  $||F|| \ge ||f||$  and:

$$||F|| = \sup_{\sum_{n=1}^{\infty} |a_n|^p = 1} \{ |F(a_1, a_2, a_3, \ldots)| \} =$$

$$= \sup_{\sum_{n=1}^{\infty} |a_n|^p = 1} \{ |a_1 + a_2| \} \le \sup_{\sum_{n=1}^{\infty} |a_n|^p = 1, a_i = 0, i \ge 3} \{ |a_1| + |a_2| \} = ||f||$$

where in (\*) the inequality holds because we maximize  $|a_1 + a_2|$  if we maximize  $a_1^p$  and  $a_2^p$ , making  $a_i = 0, i \ge 3$ . So ||f|| = ||F||.

On the other hand let's prove uniqueness of F, i.e, that  $F(0,0,a_3,a_4,\ldots)=0$  for any  $(a_n)$  with  $a_1=a_2=0$ . Then let's suppose for some  $(a_n)$  that  $F(0,0,a_3,a_4,\ldots)\neq 0$ . Then, as  $F(0,0,a_3,a_4,\ldots)=\sum_i F(0,0,0\ldots,a_i,0,\ldots)$ , for some  $a_i$  we have  $F(0,0,0\ldots,a_i,0,\ldots)\neq 0$ . If we multiply  $a_i$  by  $\overline{F(0,0,0\ldots,a_i,0,\ldots)}$  and then normalize it, we can make  $|a_i|=1$  and  $\alpha:=F(0,0,0\ldots,a_i,0,\ldots)>0$ .

Now, for any 0 < r < 1 we define:

$$(x_n) = ((1-r^p)^{1/p} \frac{1}{2^{1/p}}, (1-r^p)^{1/p} \frac{1}{2^{1/p}}, 0, \dots, ra_i, 0, \dots).$$

It verifies  $\sum_{n=1}^{\infty} |x_n|^p = 1$ , so  $|F(x_n)| \leq \frac{2}{2^{1/p}}$ . On the other hand:

$$|F(x_n)| = F(x_n) = (1 - r^p)^{1/p} F(\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}}, 0, \ldots) + r\alpha =$$
  
=  $(1 - r^p)^{1/p} \frac{2}{2^{1/p}} + r\alpha$ 

Therefore, for any 0 < r < 1:

$$(1 - r^p)^{1/p} \frac{2}{2^{1/p}} + r\alpha \le \frac{2}{2^{1/p}},$$

$$r\alpha \le (1 - (1 - r^p)^{1/p}) \frac{2}{2^{1/p}},$$

$$\alpha \le \underbrace{\frac{(1 - (1 - r^p)^{1/p})}{r}}_{(\triangle)} \frac{2}{2^{1/p}}$$

If p > 1, we can compute the limit  $\lim_{r\to 0} (\triangle)$  (using L'Hôpital's rule) and conclude that it tends to 0. That makes  $\alpha = 0$  and we get our contradiction. We notice that for p=1 the argument doesn't work as  $\Delta=1$ 

> • [5p]. Show that if p = 1, then there are infinitely many linear functional F on  $\ell_1(\mathbb{N})$  extending f and satisfying ||F|| = ||f||.

We can extend the functional to  $F(a_1, a_2, a_3, ...) = a_1 + a_2$ . It's clearly linear, extends f and its norm (which verifies  $||F|| \ge ||f||$  due to being an extension of f):

$$||F|| = \sup_{\sum_{n=1}^{\infty} |a_n|=1} \{|F(a_1, a_2, a_3, \ldots)|\} =$$

$$= \sup_{\sum_{n=1}^{\infty} |a_n|=1} \{|a_1 + a_2|\} \le \sup_{\sum_{n=1}^{\infty} |a_n|=1} \{|a_1| + |a_2|\} \le 1 = ||f||$$

because  $\sum_{n=1}^{\infty} |a_n| = 1$  implies  $|a_1| + |a_2| \le 1$ .

On the other hand, we have another candidates for extending f:  $F_k(a_1, a_2, a_3, \ldots) = \sum_{i=1}^k a_i$  for any  $k \geq 3$ . Linearity and extension of  $F_k$  are clear, and the equality of the norms works by the same argument as before: verifies  $||F_k|| > ||f||$  due to being an extension of f and:

$$||F_k|| = \sup_{\sum_{n=1}^{\infty} |a_n|=1} \{ |F(a_1, a_2, a_3, \ldots)| \} =$$

$$= \sup_{\sum_{n=1}^{\infty} |a_n|=1} \{ \left| \sum_{i=1}^{k} a_i \right| \} \le \sup_{\sum_{n=1}^{\infty} |a_n|=1} \{ \sum_{i=1}^{k} |a_i| \} \le 1 = ||f||$$

because  $\sum_{n=1}^{\infty} |a_n| = 1$  implies  $\sum_{i=1}^{k} |a_i| \le 1$ . (We could also even consider an extension  $\bar{F}(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i$ , which will be well defined as that series converge absolutely and K is complete, and will be an extension by the same arguments as above).

Exercise 3 (25 points). Let X be an infinite dimensional normed vector space over  $\mathbb{K}$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ .

> • [5p]. Let  $n \geq 1$  be an integer. Show that no linear map  $F: X \to \mathbb{K}^n$ is injective.

If F were injective then X would be isomorphic to F(X) which would be a subspace of  $\mathbb{K}^n$ , so F(X) would be finite dimensional with X infinite-dimensional. Elaborate

• [5p]. Let  $n \geq 1$  be an integer and let  $f_1, f_2, \ldots, f_n \in X^*$ . Show that

$$\bigcap_{j=1}^{n} \ker (f_j) \neq \{0\}$$

[Hint: Consider the map  $F: X \to \mathbb{K}^n$  given by  $F(x) = (f_1(x), f_2(x), \dots, f_n(x)), x \in X$ .].

We consider  $F: X \to \mathbb{K}^n$  defined by  $F(x) = (f_1(x), f_2(x), \dots, f_n(x)), x \in X$ . By the first part, it can't be injective, then  $\ker F \neq \{0\}$ . On the other hand  $\ker F = \{x \in X, (f_1(x), f_2(x), \dots, f_n(x)) = 0\} = \{x \in X, f_i(x) = 0, i = 1, \dots, n\} = \bigcap_{i=1}^n \ker f_i$ .

• [5p]. Let  $x_1, x_2, \ldots, x_n \in X$ . Show that there exists  $y \in X$  such that ||y|| = 1 and  $||y - x_j|| \ge ||x_j||$  for all  $j = 1, 2, \ldots n$ . [Hint: Use Theorem 2.7 (b) from lectures to get started.]

Theorem 2.7b) In the lectures says that given X a normed space,  $0 \neq x \in X$  then there exists  $f \in X^*$  with ||f|| = 1 and f(x) = ||x||. First we notice that if some  $x_j = 0$  then there's nothing to prove for  $x_j$ , so we may assume all  $x_j$  are non-zero. Let's consider then  $f_i \in X^*$  for each  $x_i$  given by the theorem. By the second part in this exercise,  $\bigcap_{i=1}^n \ker f_i \neq \{0\}$ . We take  $0 \neq y \in \bigcap_{i=1}^n \ker f_i \neq \{0\}$ , which can be taken such that ||y|| = 1, by normalizing y. We have in consequence:

$$f(x_j - y) = f(x_j) - f(y) = f(x_j) = ||x_j||;$$
  
$$|f(x_j - y)| \le ||f|| ||x_j - y|| = ||x_j - y||$$

So we get  $||x_j|| \le ||x_j - y||$ .

• [5p]. Show that one cannot cover the unit sphere  $S = \{x \in X : \|x\| = 1\}$  with a finite family of closed balls in X such that none of the balls contains 0.

Assume by contradiction that there exists  $x_1, \ldots x_n \in X$ ,  $r_1, \ldots, r_n > 0$  such that  $S \subset \bigcup_{i=1}^n \bar{B}(x_i, r_i)$  such that  $0 \notin \bar{B}(x_i, r_i)$ , i.e.  $|x_i| > r_i$ . By the third part of this exercise there exists  $y \in S$  with  $||y - x_j|| \ge ||x_j|| > r_j$ . So  $y \notin \bar{B}(x_j, r_j)$  for all  $j = 1, \ldots, n$ .

• [5p]. Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

We'll be done when we give an open recover of S by open balls such that the closed balls don't contain 0. Then, if S were compact, there would be a finite recover  $S \subset \bigcup_{i=1}^n B(x_i,r_i)$ . But then applying closures (S is closed)  $S \subset \overline{\bigcup_{i=1}^n B(x_i,r_i)} = \bigcup_{i=1}^n \overline{B}(x_i,r_i)$ , and we will have a contradiction. We take the open cover by  $\{B(x,\frac{1}{2})\}_{x\in S}$ . Then  $S \subset \bigcup_{x\in S} B(x,\frac{1}{2})$  and  $0 \notin \overline{B}(x,\frac{1}{2})$ .

For the closed unit ball B, it's a closed space in X, same as S (as they are preimages of a continuous function in X, the norm), and then S is closed in B. So if B were compact, then S will be a closed subspace of a compact subspace, and therefore compact. So B can't be compact.

**Exercise 4 (20 points)** . Let  $L_1([0,1],m)$  and  $L_3([0,1],m)$  be the Lebesgue spaces on [0,1] Recall from HW2 that  $L_3([0,1],m) \subsetneq L_1([0,1],m)$ . For  $n \ge 1$ , define:

$$E_n := \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm \le n \right\}$$

• [6p]. Given  $n \ge 1$ , is the set  $E_n \subset L_1([0,1],m)$  absorbing? Justify.

It's not absorbing: if we consider  $h(x) = \frac{1}{x^{2/3}}$ , then  $h \in L_1([0,1], m)$ . However, for any  $0 < k \in \mathbb{K}$ :

$$\int_{[0,1]} \left| k \frac{1}{x^{2/3}} \right|^3 dx = k^3 \int_{[0,1]} \frac{1}{x^2} dx = \infty$$

So for any  $0 < k \in \mathbb{K}$ ,  $kh \notin E_n$ .

On the other hand  $E_n$  is actually convex: if  $f, g \in E_n$  and 0 < t < 1 then:

$$\int_{[0,1]} |tf + (1-t)g|^3 dm \le \int_{[0,1]} t |f|^3 + (1-t) |g|^3 dm \le tn + (1-t)n = n$$

So  $tf + (1-t)g \in E_n$ .

• [7p]. Show that  $E_n$  has empty interior in  $L_1([0,1],m)$ , for all  $n \geq 1$ .

We are going to show that given  $f \in E_n$ , for any r > 0, the open balls:  $B_1(f,r) = \{g \in L_1([0,1]) \int_{[0,1]} |f-g| \, dm < r\}$  verify that  $B_1(f,r) \not\subset E_n$ . In other words, for any  $E_n$ , any  $f \in E_n$  and any r > 0, we are going to give  $g_r \in B_1(f,r)$  with  $g_r \not\in E_n$ .

If f = 0, then we define  $h(x) := \frac{1}{x^{2/3}}$  and consider  $g_r(x) = \frac{r}{2}h(x)$ . As  $\int_{[0,1]} \frac{1}{x^{2/3}} dx = 1$ , then  $\int_{[0,1]} |g_r - 0| dm < r$ . However,  $g_r \notin E_n$  by our reasoning above on the part that the  $E_n$  are not absorbing.

In the general case, we consider  $g_r(x) := f(x) + \frac{r}{2}h(x)v(x)$ , with  $v : [0,1] \to \mathbb{K}$  such that |v(x)| = 1 and has the same angle as f(x) (i.e.  $v(x) = \frac{f(x)}{|f(x)|}$  if  $f(x) \neq 0$ , 1 otherwise), so that  $|g_r(x)| = |f(x)| + \frac{r}{2}|h(x)|$ .

Then  $g_r \in B_1(f,r)$  and:

$$\int_{[0,1]} |g_r| \, dm = \int_{[0,1]} \left| f + \frac{r}{2} h(x) v(x) \right|^3 dm \ge^* \int_{[0,1]} \frac{r^3}{8} h^3(x) dx = \infty.$$

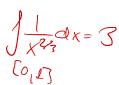
The inequality (\*) we'll come from expanding the binomial. Then  $g_r \notin E_n$ .

• [8p]. Show that  $E_n$  is closed in  $L_1([0,1], m)$ , for all  $n \ge 1$ .

We'll prove that for any sequence  $\{f_j\} \subset E_n$  such that there exists  $f \in L_1[0,1]$  with  $\lim_{j\to\infty} \|f_j - f\|_1 = 0$ , then  $f \in E_n$ .

We'll use the result that if  $f_j \to f$  in  $L_1$ , then there exists a subsequence  $f_{j_k}$  such that  $f_{j_k} \to f$  almost everywhere. That implies that  $|f_{j_k}|^3 \to |f|^3$  almost everywhere.

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Now we'll use the Fatou's lemma: given a sequence of non-negative measurable functions  $\{g_n\}$  then  $\liminf g_n$  is also measurable and:

$$\int \liminf g_n dm \le \liminf \int g_n dm.$$

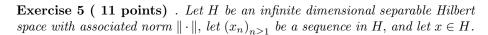
In our case with  $|f_{j_k}|^3$ , we have that  $\liminf |f_{j_k}|^3 = |f|^3$  so:

$$\int_{[0,1]} |f|^3 dm \le \liminf \int_{[0,1]} |f_{j_k}|^3 \le n$$

as  $f_{j_k} \in E_n$ . Then  $f \in E_n$ .

• [4p]. Conclude from (b) and (c) that  $L_3([0,1],m)$  is of first category in  $L_1([0,1],m)$ 

We want to show that  $L_3([0,1],m)$  is a countable union of nowhere dense sets. The sets  $E_n$  verify  $Int(\bar{E}_n) = Int(E_n) = \emptyset$ , so they are nowhere dense. Finally,  $L_3([0,1],m) = \bigcup_{n=1}^{\infty} E_n$  as if  $f \in L_3([0,1],m)$  then  $\int_{[0,1]} |f|^3 dm < \infty$ , so there exists some  $n \in \mathbb{N}$  such that  $f \in E_n$ .



• [2p]. Suppose that  $x_n \to x$  in norm, as  $n \to \infty$ . Does it follow that  $||x_n|| \to ||x||$ , as  $n \to \infty$ ? Give a proof or a counterexample.

 $x_n \to x$  in norm means  $||x_n - x|| \to 0$ . We want to show that  $||x_n|| \to ||x||$ , which is equivalent to  $|||x_n|| - ||x||| \to 0$ . We notice that:

$$||x_n|| = ||x_n - x + x|| \le ||x_n - x|| + ||x||,$$

therefore  $||x_n|| - ||x|| \le ||x_n - x||$ . Analogously, it can be shown that  $||x|| - ||x_n|| \le ||x_n - x||$ , so  $|||x_n|| - ||x||| \le ||x_n - x||$ . So:

$$0 \le |||x_n|| - ||x||| \le ||x_n - x||$$

then  $|||x_n|| - ||x||| \to 0$ , so  $||x_n|| \to ||x||$ .

• [5p]. Suppose that  $x_n \to x$  weakly, as  $n \to \infty$ . Does it follow that  $||x_n|| \to ||x||$ , as  $n \to \infty$ ? Give a proof or a counterexample. [Hint: Consider an orthonormal basis  $(e_n)_n \ge 1$  in H, and use HW4.]

As it has been said in the lectures, any separable Hilbert space admits countable orthonormal basis  $(e_n)_{n\geq 1}$  in H. By HW P2a)  $x_n\to x$  weakly if and only if  $f(x_n)\to f(x)$  for any  $f\in H^*$ . By the Riesz representation theorem we know



## It need not be.

that any  $f \in H^*$  is of the form  $f_y$  whith  $f_y(x) = \langle x, y \rangle$  for some unique  $y \in H$ . On the other hand, as  $(e_n)_{n \geq 1}$  is orthonormal basis, we get that for any  $y \in H$ ,  $y = \sum_n y_n e_n$  with  $\{y_n \neq 0\}$  finite and  $y_n = \langle y, e_n \rangle$ .

Then for any  $y \in H$ ,  $f_y(e_n) = \langle e_n, y \rangle = \overline{y_n}$ . As the set  $\{y_n \neq 0\}$  is finite,  $f_y(e_n) \to 0 = f_y(0)$ . So for any  $f \in H^*$  we have  $f(e_n) \to f(0)$ , so  $e_n \to 0$  weakly. However,  $||e_n|| = 1$  so  $||e_n|| \to ||0||$ .

(V)

• [4p]. Suppose that  $||x_n|| \le 1$ , for all  $n \ge 1$ , and that  $x_n \to x$  weakly, as  $n \to \infty$ . Is it true that  $||x|| \le 1$ ? Give a proof or a counterexample.

 $x_n \to x$  weakly is equivalent to  $f(x_n) \to f(x)$  for any  $f \in H^*$ , i.e,  $\langle x_n, y \rangle \to \langle x, y \rangle$  for any  $y \in H$ . In particular for x, we have  $\langle x_n, x \rangle \to \|x\|^2$ , so we have  $|\langle x_n, x \rangle| \to \|x\|^2$ .

On the other hand, we can apply Cauchy-Schwartz inequality:

$$|\langle x_n, x \rangle|^2 \le ||x_n|| ||x|| \le ||x||$$

so

$$|\langle x_n, x \rangle| \le ||x||^{1/2}$$

So  $\lim |\langle x_n, x \rangle| \le ||x||^{1/2}$ . Therefore  $||x||^2 \le ||x||^{1/2}$ , and that relation holds if and only if  $||x|| \le 1$ .

