Functional Analysis: Mandatory assignment 1

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Problem 1

a)

We start by showing $||\cdot||_0$ is a norm, so we check the 3 requirements:

$$\begin{aligned}(a)\ ||x+y||_0 = ||x+y||_X + ||T(x+y)||_Y &\leq ||x||_X + ||y||_Y + ||Tx+Ty||_Y \\ &\leq ||x||_X + ||y||_Y + ||Tx||_Y + ||Ty||_Y = ||x||_0 + ||y||_0\end{aligned}$$
$$(b)\ ||\alpha x||_0 = ||\alpha x||_X + ||T(\alpha x)||_Y = |\alpha| \cdot ||x|| + ||\alpha Tx||_Y = |\alpha| \cdot ||x|| + |\alpha| \cdot ||Tx||_Y$$

Because we know that T is linear and the know that $||\cdot||_X, ||\cdot||_Y$ are norms.

$$(c)\ ||x||_0 = ||x||_X + ||Tx||_Y = 0 \Leftrightarrow ||Tx||_Y = -||x||_X \Leftrightarrow ||Tx||_Y = ||x||_X = 0 \Leftrightarrow x = 0$$

Because norms are positive, so if the one norm is equal to the negative of the other, they can only be equal to 0. The rest comes again from T being linear and that $||\cdot||_X, ||\cdot||_Y$ are norms.
Next we show that the two norms are equivalent $\Leftrightarrow$ T is bounded:
"$\Leftarrow$"It is clear that $||x||_X \leq ||x||_0 = ||x||_X + ||Tx||_Y$ because norms are positive, so $||Tx||_Y \geq 0$. Now if T is Bounded, then $||Tx||_Y \leq ||T||||x||_X = C||x||_X$, which means that $||x||_0 = ||x||_X + ||Tx||_Y \leq ||x||_X + C||x||_X = (1-C)||x||_X = K||x||_X$, which means that the norms are equivalent.
"$\Rightarrow$"To show the other way we assume that the norms are equivalent, so we have that:

$$||x||_0 \leq C||x||_X \Rightarrow ||x||_X + ||Tx||_Y \leq C||x||_X \Rightarrow ||x||_Y \leq (C-1)||x||_X = K||x||_X$$

Where we subtract $||x||_X$ on both sides, so $||T|| \leq X < \infty$, so T is therefore bounded as we wanted.

b)

Assume X is finite dimensional, then by Theorem 1.6 (Lecture notes) we then have that every norms are equivalent and then by (a) we have that T is bounded, as we wanted.

c)

Take a Hamel basis $(e'_i)_{i\in I}$ as described in the assignment. Choose some $y' \in Y$, so that $y' \neq 0$ and take the family $(e_i)_{i\in I}$, where $e_i = \frac{e'_i}{||e'_i||_X}$, which is again a Hamel basis with norm 1. Now look at $y = \frac{y'}{||y'||_Y} \in Y$ and define the linear map T to take a e_i from the Hamel basis and send it to the index times this y, so $T(e_i) = i \cdot y$ for all $i \in \mathbb{N}$ and $T(e_i) = 0$ for other i. This linear map is not bounded for positive integers, because there is no constant, that can bound the norm of T. See that we have for e_i, where $i = k+1$ that $||T(e_i)||_Y = ||y \cdot i||_Y = k+1 \geq k = k||e_i||_X$

d)

We have that X is infinite dimensional, then we know from c) that there exists a linear map T, which is not bounded. But then we know that the norms $|| \cdot ||_X$ and $||\cdot||_0$ on the form $||x||_0 = ||x||_X + ||Tx||_Y$ is not equivalent (because if they were, then T would be bounded). We also know that $||x||_X \leq ||x||_0$ (just by definition of the zero norm). Now assume (X, V_X) is a Banach space (therefore complete) and then for proof by contradiction: Assume $(X, ||\cdot||_0)$ is also complete, but then by (Problem 1 HW3) we have that the norms is then equivalent, but this is a contradiction, so $(X, ||\cdot||_0)$ must not be complete.

e)

Let us look at $(L_2([0,1],m), ||\cdot||_2)$ and for $||\cdot||' = ||\cdot||_\infty$, we have that $||x||_\infty \leq ||x||_2$ (because we know $||\cdot||_\infty \leq ||\cdot||_2$ and $L_\infty([0,1],m) \subseteq L_2([0,1],m)$). We then know that $(L_2([0,1],m), ||\cdot||_2)$ is complete, and we have that $||\cdot||_2 \not\leq C||\cdot||_\infty$ for any C_1, because we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that $||(x_n)_{n\in\mathbb{N}}||_2 > C||(x_n)_{n\in\mathbb{N}}||_\infty$ (take for example the one we will construct later in this proof) so the norms are not equivalent.

Now let us look at if $(L_2([0,1],m), ||\cdot||_\infty)$ is complete:

Lets look at some sequence of sequences in $L_2([0,1],m)$ that is Cauchy with respect to $||\cdot||_\infty$ and converges to a sequence which is not in $L_2([0,1],m)$. Let each part of this sequence be defined, for $n \in \mathbb{N}$ and $i \in \mathbb{N}$ (two different index), by

$$x_{n_i} = \begin{cases} \frac{1}{n} & n \leq i \\ 0 & n > i \end{cases}$$

We have that each $(x_n)_{n_i}$ for each i is a sequence in $L_2([0,1],m)$, it is Cauchy for $k > i$ (because for $\epsilon > 0$ we have $||(x_{n_k}) - (x_{n_i})||_\infty = \frac{1}{i+1} < \epsilon$, because we can take $i > \frac{1-\epsilon}{\epsilon}$) and that the limit for $i \to \infty$ of the sequence, is equal to $(1/n)_{n\in\mathbb{N}}$, where $(\int_{[0,1]} (\frac{1}{n})^2 dm)^{\frac{1}{2}} = \infty$ and therefore is not in $L_2([0,1],m)$, so $(L_2([0,1],m), ||\cdot||_\infty)$ is not complete.

Problem 2

a)

To show f is bounded we have to show that $||f(x)||_p \leq C||x||_p$ and we have that, if we let $x = (a, b, 0, 0, 0, ...)$:

$$||f(x)||_p = ||a+b||_p \leq ||a||_p + ||b||_p = (|a|^p)^{\frac{1}{p}} + (|b|^p)^{\frac{1}{p}} = |a| + |b|$$

Because we know $||\cdot||_p$ is a norm (triangle inequality). And we know that $||x|| = (|a|^p + |b|^p)^{1/p}$...

Now we look at the norm of f: We use Jensen's inequality (Scilling Thm 13.13):

$$\frac{1}{2^p}|a+b|^p = |\frac{a+b}{2}|^p = |\frac{1}{2}a + \frac{1}{2}b|^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p = \frac{1}{2}(|a|^p + |b|^p)$$
$$\Rightarrow |a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$$

We use this to look at: $|f(a,b,0,0,...)| = |a+b| \leq 2^{\frac{p-1}{p}}(|a|^p + |b|^p) = 2^{\frac{p-1}{p}}||(a,b,0,...)||_p$, so f is bounded and $||f|| \leq 2^{\frac{p-1}{p}}$, now look at

$$|f(1,1,0,0,...)| = 2 = 2^{\frac{p-1}{p}}2^{\frac{1}{p}} = 2^{\frac{p-1}{p}}||(1,1,0,0,...)||_p$$

So we also have that $||f|| \geq 2^{\frac{p-1}{p}}$, therefore $||f|| = 2^{\frac{p-1}{p}}$

b)

We can use Hahn-Banach (Theorem 5.7 in Folland) to give us that there exists such a linear functional $F \in l_p^*$ extending f, such that $||F|| = ||f||$.

Now lets prove the uniqueness: So to prove uniqueness lets look at two different F, F' with the same properties related to f and see if they are equal. We know from Problem 5 HW1 that $(l_p(\mathbb{N})^*$ is isometrically isomorphic to $l_q(\mathbb{N})$ with $\frac{1}{p} + \frac{1}{q} = 1$ with the isometry $T : l_p^* \to l_q$, where $T(x) = f_x$ for $(x_n)_{n\geq 1} \in l_q$ and $f_x(y) = \sum_{n=1}^\infty x_n y_n$ for all $(y_n)_{n\geq 1} \in l_p$. Lets look at x and x' being the elements corresponding to F and F' respectively in l_q. Now we know:

$$||f|| = 2^{\frac{p-1}{p}} = ||F|| = ||x||_q = ||F'|| = ||x'||_q$$

Now lets look at an element in M $(a, b, 0, 0, ...) \in M$, where we know that f, F, F' are equal, so using the isometry on x, x' we have that

$$F(a,b,0,0,...) = a + b = (T(x))(a,b,0,0,...) = f_x(a,b,0,0,...) = x_1 a + x_2 b$$
$$F'(a,b,0,0,...) = a + b = (T(x))(a,b,0,0,...) = f_{x'}(a,b,0,0,...) = x_1' a + x_2' b$$

Therefore we must have, if we start by looking at F, that $||x||_q = (1^q + q^q + \sum_{n=3}^\infty |x_n|^q)^{\frac{1}{q}}$, because the first two terms of x must be 1. Then we have the inequality:

$$||x||_q = (1^q + q^q + \sum_{n=3}^\infty |x_n|^q)^{\frac{1}{q}} \geq (1^q + 1^q + 0)^{\frac{1}{q}} = 2^{\frac{1}{q}} = 2^{1-\frac{1}{p}} = 2^{\frac{p-1}{p}}$$

But we know that $||x|| = 2^{\frac{p-1}{p}}$, so this can only be possible if the rest of the terms in sum of x is equal to 0. The same can be done for F', which gives us that $x = x'$ and therefore $F = F'$, so this proves the uniqueness.

c)

Let us look at the linear functionals $F_n(x) = \sum_{i=1}^{n} |x_i|$, then we have that, for $n = 2$, we have that $F_n(x) = f(x)$ for $(x_n)_{n\in\mathbb{N}} \in l_1(\mathbb{N})$, and therefore it extends f on $l_1(\mathbb{N})$. We have that for $p = 1$ that $||f|| = 1$, so let's check if the norm of F is also equal to 1:

$$|F_n(x)| = |\sum_{i=1}^{n} |x_i|| \leq ||x||_1 = \sum_{i=1}^{n} |x_i| \Rightarrow ||F_n|| \leq 1$$

Now let's look at $x = (x_1, x_2, ..., x_n, 0, 0, ...) \in l_1(\mathbb{N})$, where all $a_i = 1$ for all $i \in \mathbb{N}$, then we have that

$$|F_n(x_1, x_2, ..., x_n, 0, 0, ...)| = |\sum_{i=1}^{n} 1| = \sum_{i=1}^{n} |x_i| + \sum_{i=n+1}^{\infty} |x_i| = ||x|| \Rightarrow ||F_n|| \geq 1$$

Which gives us that $||F_n|| = 1 = ||f||$, as we wanted.

Problem 3

a)

Let us assume for contradiction that there exist a linear map $F : X \to \mathbb{K}^n$ that is injective. This map would then map bijectively onto a basis (e_i) (with $0 < j \leq k$ elements) for $Im(T)$ and define the pre-image of the basis as (x_i), so $T(x_i) = e_i$. Then the span of these, will be mapped by T like this: $T(\sum_{i=1}^{j} \alpha_i x_i) = \sum_{i=1}^{j} \alpha_i T(x_i) = \sum_{i=1}^{j} \alpha_i e_i$, using T is linear. So the span maps bijectively to the image of T. But we have that X is infinte dimensional, so there exist and element, that is not in the span, so this element must be mapped to an element already mapped to by the span, but this is a contradiction to T being injective. This gives us that there is no linear map $T : X \to \mathbb{K}^n$ that is injective, as we wanted.

b)

As the hint suggests, let's look at the map $F : X \to \mathbb{K}^n$ given by $F(x) = (f_1(x), f_2(x), ..., f_n(x)), x \in X$. Now since this is a combination of linear maps, then we have that F is also linear and it is not injective by a). So we know that the kernel of F is not equal to $\{0\}$ and when we have that there is such an element in the kernel of F, then this element is also in the kernel of all f_i, so we get that:

$$\bigcap_{j=1}^{n} \ker(f_i) = \ker(F) \neq \{0\}$$

, as we wanted.

c)

We know there exists some $x_i \in X$, where $x_i \neq 0$, because we can choose the y in the $\bigcup_{j=1}^{n} \ker(f_j)$ because from b) we know there exists an element in the kernel, which is not zero. We then use Theorem 2.7 (b) (as the hint suggest) to get that there exist a $f_i \in X^*$ such that $||f_i|| = 1$ and $f_i(x_i) = x_i$Let's take an element y from the kernel which is not zero with norm $||y|| = 1$. We then pick $y = \frac{y'}{||y'||}$, then for all x_i we have that

$$||y - x_i|| \geq ||f_i(y - x_i)|| = ||f_i(y) - f_i(x_i)|| = || - f_i(x_i)|| = ||x_i||$$

Getting the inequality, that we wanted.

d)

Assume there is a family of closed balls in X such that none of them contains 0 and they cover S. Then the denote the balls $B_r(x_j) = \{||x_j|| \leq r\}$, where $x_1, x_2, ..., x_n$ is the center of the balls. Now lets look at the corresponding functionals f_i from c) and let y be an element in the kernel of these functionals not equal to zero. Then we have that $||y|| = 1$, so it is a point on S and then from c) we get that the distance $||y - x_j|| \geq ||x_j||$, which means that the distance from the center of the balls to the point on the unit sphere is greater that the distance from the balls center to 0, but this means that 0 is contained in the balls and that is a contradiction. So we cannot cover S with a finite family of closed balls in X, that does not contain 0.

e)

Assume for contradiction that A is compact, which means there exists open cover. Lets define this cover to be a family of open balls with a radius $r < 1$, such that 0 is not contained in any of the balls, with center in every point on S. Now because S is compact we know that we can take a finite subcover.
(Denote that the result from d) also holds for open balls, with the same proof, but with strict inequalities)
Then we have by d) that this is a contradiction, so S cannot be compact.

We know that S is closed in the closed unit ball in X. Now again assume for contradiction that the closed unit ball in X is compact, but this means that the closed subset S will also be compact, but this is a contradiction, so the closed unit ball in X must be non-compact.

Problem 4

a)

No, it is not absorbing, because if we take some element $f \in L_1([0,1], m)$, that we define $f(x) = \frac{1}{\sqrt[3]{x}}$, then we have that $\int_{[0,1]} |f|^3 dm = \int_{[0,1]} |\frac{1}{\sqrt[3]{x}}|^3 dm = \int_{[0,1]} \frac{1}{\sqrt{x}} dm$, which is equal to infinity. Therefore there is no constant $C \geq 0$, that will absorb the integral to E_n, because $\int_{[0,1]} |Cf|^3 dm = C \int_{[0,1]} |f|^3 dm$ and the integral for our specific f is equal to infinity.

b)

First we will prove that E_n is convex, because we will need it for the proof: If we take $f, g \in E_n$ and $0 \leq \alpha \leq 1$ and see that

$$\int_{[0,1]} |\alpha f + (1-\alpha)g| dm \leq \alpha \int_{[0,1]} |f| dm + (1-\alpha) \int_{[0,1]} |g| dm < \infty$$

$$||\alpha f + (1-\alpha)g||_3^3 \leq (|\alpha| ||f||_3 + |1-\alpha| ||g||_3)^3 \leq (|\alpha| \sqrt[3]{n} + |1-\alpha| \sqrt[3]{n}) = \sqrt[3]{n}^3 = n$$

So therefore $\alpha f + (1-\alpha)g \in E_n$, which means that E_n is convex.
Let us assume for contradiction that there is something in the interior (so it's not empty) . This means we can take a ball in E_n with some center f and a radius r, so $B_r(f)$. Now because E_n is convex we have that we can find an open ball around $-f$ and everything connecting these two balls, but this means that the ball $B_r(O)$ around origin is in E_n, but this means that E_n is absorbing and this is a contradiction to a), therefore E_n must have an empty interior in $L_1([0,1], m)$ for all $n \geq 1$

c)

To show that E_n is closed, we want to show that every sequence converges to at point in E_n, so take some sequence $\{f_k\}_{k \in \mathbb{N}} \subset E_n$, such that $||f_k - f||_1 \to 0$ (i.e f_k converges to f with respec to the one-norm). Now by Corollary 13.8 there exists a subsequence $|f_{k_i}|^3$ which converges pointwise almost everywhere to $|f|^3$. Now we can use Fatou's lemma 9.22 (Scilling) so see that

$$\int_{[0,1]} |f|^3 = \int_{[0,1]} \liminf_{n \to \infty} |f_{k_i}|^3 \leq \liminf_{n \to \infty} \int_{[0,1]} |f_{k_i}|^3 \leq n$$

So f is in E_n and E_n is therefore closed in $L_1([0,1], m)$ for $n \geq 1$.

d)

Now we have that E_n is closed and has empty interior, which gives us that it is nowhere dense. Now since $L_3([0,1], m) = \bigcup_{n \in \mathbb{N}} E_n$ and each of E_n is nowhere dense in $L_1([0,1], m)$, we have by Definition 3.12 (ii) (Lecture notes) that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$.

Problem 5

a)

We have that the norm of a Hilbert space is $||x|| = \sqrt{\langle x, x \rangle}$
It is true and let's give a proof: We have from Proposition 5.21 (Folland) that

$\langle x_n, x_n \rangle \to \langle x, x \rangle$, because $x_n \to x$, so by the definition of the norm, we then have that $||x_n|| = \sqrt{\langle x_n, x_n \rangle} \to \sqrt{\langle x, x \rangle} = ||x||$ as we wanted.

b)

This it not true and let's look at a counterexample: Pick an orthonormal countable basis for H, denoted $(e_n)_{n \geq 1}$ and we then use Problem 2 a) (HW4) to look at $f \in H^*$. Now use Problem 1 (HW2) to get there exists a unique $y \in H$, such that $f(e_n) = \langle y, e_n \rangle$ for all e_n in our basis. Now use 5.26 Bessle's inequality (Folland) to get that $\sum_{n=1}^{\infty} |\langle y, e_n \rangle|^2 \leq ||y||^2$, but this gives us that $|f(e_n)|^2 \to 0$ for $n \to \infty$, so e_n then converges weakly to 0. But this a contradiction to $||e_n|| = 1 \to 0||x||$ for $n \to \infty$.

c)

It is true and let's give a proof: If we have that $x_n \to x = 0$, the result will be trivial, so lets look at $x \neq 0$ and $x_n \neq 0$. So lets use Theorem 2.7 (b) to get that there exists an f, such that $||f|| = 1$ and $f(x) = ||x|| \leq 1$. Now because $x_n \to x$ weakly we have that $||x|| = ||f(x)|| = \lim_{n \to \infty} ||f(x_n)||$, but we also have that $||f(x_n)|| \leq ||f|| \cdot ||x_n|| \leq 1$ for every x_n. So this means that $||x|| \leq 1$ as we wanted.