Mandatory Assignment No. 2 for Functional Analysis 2020-2021
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In the following, the abbreviation "LN" stands for "Lecture Notes".

Problem 1

a)

By definition, the sequence, f_N, converges weakly to 0 if for all $y \in H^*$ we have $y(f_N) \to y(0)$ as $N \to \infty$. By the Riesz representation theorem any $y \in H^*$ is given by some $x \in H$ as $y(\cdot) = \langle \cdot, x\rangle$ hence f_N converges weakly to 0 if for all $x \in H$ we have $\langle f_N, x\rangle \to \langle 0, x\rangle = 0$ as $N \to \infty$. We see that we have to show that $\langle f_N, x\rangle = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, x\rangle \to 0$. We know that $\sum_{n=1}^{\infty}|\langle e_n, x\rangle|^2 < \infty$ i.e. for any $\epsilon > 0$ there exists $n' \in \mathbb{N}$ such that $\sum_{n=n'}^{\infty}|\langle e_n, x\rangle|^2 < \epsilon^2$. Notice that since eventually we let $N \to \infty$ we can in the following assume that $N^2 > n' + 1$. We have

$$\frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, x\rangle = \frac{1}{N}\sum_{n=1}^{n'-1}\langle e_n, x\rangle + \frac{1}{N}\sum_{n=n'}^{N^2}\langle e_n, x\rangle \tag{1}$$

where $\frac{1}{N}\sum_{n=1}^{n'-1}\langle e_n, x\rangle \to 0$ i.e. we need to show that $\frac{1}{N}\sum_{n=n'}^{N^2}\langle e_n, x\rangle \to 0$. Now, let b_n be a sequence with 1 in the mth to the N^2th entry. Then we can use the Cauchy Schwarz inequality to get

nth?

If you introduce b_n for this inequality then use it explicitly.

$$\left(\sum_{n=n'}^{N^2}|\langle e_n, x\rangle|\right)^2 = \left(\sum_{n=n'}^{N^2}|\langle e_n, x\rangle|\, b_n\right)^2 \leq (N^2 - n' + 1)\sum_{n=n'}^{N^2}|\langle e_n, x\rangle|^2 < (N^2 - n' + 1)\epsilon^2. \tag{2}$$

Or in other words

$$\frac{1}{N}\sum_{n=n'}^{N^2}|\langle e_n, x\rangle| < \sqrt{1 - \frac{n'+1}{N^2}}\,\epsilon \tag{3}$$

which when taking the limit $N \to \infty$ yields the desired. ✓ Furthermore, we have

$$\|f_N\|^2 = \frac{1}{N^2}\left\|\sum_{n=1}^{N^2} e_n\right\|^2 = \frac{1}{N^2}|\langle\sum_{n=1}^{N^2} e_n, \sum_{m=1}^{N^2} e_m\rangle| = \frac{1}{N^2}|\sum_{n,m=1}^{N^2}\delta_{nm}| = 1. \tag{4}$$ ✓

Don't need $|\cdot|$ here

b)

We let K be the norm closure of $\text{co}\{f_N | N \geq 1\}$. Since $\text{co}\{f_N | N \geq 1\}$ is a convex set we have, due to theorem 5.7 in the LN, that the weak closure and the norm closure coincide. As we have seen in lecture 2, every Hilbert space is reflexive and hence by theorem 6.3 in the LN we have that the weak closure of $B_H(0,1)$ is compact. Now from the fact that $\|f_N\| = 1$ it is easy to see that any convex combination of f_N's has norm less than or equal to 1: Let $\{\gamma_k\}_{k=1}^n \subset \mathbb{R}$ and $\{N_k\}_{k=1}^n \subset \mathbb{N}$ with $\sum_{k=1}^n \gamma_k = 1$, then $\left\|\sum_{k=1}^n \gamma_k f_{N_k}\right\| \leq \sum_{k=1}^n |\gamma_k|\left\|f_{N_k}\right\| = 1$. Since the unit ball is convex we also have that the weak and norm closures coincide and hence K is a weakly closed subset of of the weakly closed unit ball which is compact i.e. K is weakly compact. ✓

We know that 0 is in the weak closure of $\text{co}\{f_N | N \geq 1\}$ since for all $N \geq 1$ we have that $1f_N \in \text{co}\{f_N | N \geq 1\}$ and from **a)** f_N converges weakly to 0. But then, again, because of theorem 5.7 we have also that $0 \in K$. ✓

c) and d)

This is false.

Notice that

$$K = \{\sum_{N=1}^{n} \alpha_N f_N \mid \alpha_N \geq 0, \sum_{N=1}^{n} \alpha_N \leq 1, n \in \mathbb{N}\} \tag{5}$$

No. since every sequence in $\text{co}\{f_N \mid N \geq 1\}$ given by $\alpha_1 f_1 + \cdots + \alpha_{N'} f_{N'} + \alpha_N f_N$ for some $N' \in \mathbb{N}$ converges weakly to $\alpha_1 f_1 + \cdots + \alpha_{N'} f_{N'} \in K$ as $N \to \infty$ and $\alpha_1 + \cdots + \alpha_{N'} \leq 1$. Now, let $F = \{0\} \cup \{f_N \mid N \geq 1\}$, then we have that

$$\begin{aligned}\text{co}(F) &= \{\alpha_0 0 + \sum_{N=1}^{n} \alpha_N f_N \mid \alpha_0, \alpha_N \geq 0, \sum_{N=0}^{n} \alpha_N = 1, n \in \mathbb{N}\} \\ &= \{\sum_{N=1}^{n} \alpha_N f_N \mid \alpha_N \geq 0, \sum_{N=1}^{n} \alpha_N \leq 1, n \in \mathbb{N}\} = K. \end{aligned} \tag{6}$$

Since K is weakly closed we have that $\text{co}(F)$ is weakly closed and hence $K = \overline{\text{co}(F)}^{\tau}$. Since K is non-empty, compact and convex we can use Krein Milman to conclude that $\text{Ext}(K) = F$. This answers both c) and d). 0/0

Even if $K = \text{co}(F)$, you cannot conclude $\text{Ext}(K) = F$ immediately via Krein-Milman

Problem 2

a)

Notice that for any $y \in Y^*$ we have that $y \circ T \in X^*$ since a composition of continuous functions is continuous. Since $x_n \to x$ weakly we have by definition that for all $f \in X^*$, $f(x_n) \to f(x)$. Hence we know that $y \circ T(x_n) \to y \circ T(x)$ or equivalently $y(Tx_n) \to y(Tx)$ for all $y \in Y^*$ which shows the desired. ✓

b)

We first prove the following claim:

Claim: Let X be a topological space and x_n a sequence in X. If every subsequence of x_n has a subsequence converging to $x \in X$ then x_n converges to $x \in X$.

Proof. Suppose x_n does not converge to x. Then there exists a neighbourhood, U, of x such that for any $N \in \mathbb{N}$ there is $n' > N$ such that $x_{n'} \notin U$. Therefore there exists a subsequence of x_n which has no elements in U. Hence this subsequence has no sub-subsequence converging to x and we have the desired by counter-position. □

We have seen in HW4 problem 2b that if a sequence converges weakly then it is bounded. We also have from Proposition 8.2 in LN that since T is compact, every bounded sequence, x'_n, in X contains a subsequence, x'_{n_k} such that Tx'_{n_k} converges in Y. Now, if we take any subsequence of Tx_n, denoted Tx_{n_k} we know that x_{n_k} converges weakly to x in X (and is therefore bounded) and hence has a further subsequence, $x_{n_{k_j}}$ such that $Tx_{n_{k_j}}$ converges. We have then, from the claim, that Tx_n converges to some limit $y \in Y$. Now since every element of the dual $g \in Y^*$ is continuous we have that $g(Tx_n) \to g(y)$ i.e. Tx_n converges weakly to y. But since compact operators are a subset of bounded linear operators we have from a) that Tx_n converges weakly to Tx. This means that if we can prove that the limit of a weakly convergent sequence is unique, we are done. From Theorem 2.7(c) in LN we have that if $Tx \neq y$ then there exists $f \in Y^*$ such that $f(Tx) \neq f(y)$ and hence Tx_n would converge weakly to neither y nor Tx. Hence $y = Tx$ and we are done. (✓)

To what?

No, you haven't verified that every subseq. has a subseq. with the same limit

c)

Suppose that T is not compact. This implies by Proposition 8.2 that $T(\overline{B_H(0,1)})$ is not totally bounded. Hence there exists $\delta > 0$ such that any given collection of open balls, $\{B_Y(y_n, \delta)\}_{n=1}^N$ where $N \in \mathbb{N}$ does not cover $T(\overline{B_H(0,1)})$. Therefore we can build a sequence, y_n, in $T(\overline{B_H(0,1)})$ where the distance between y_n and y_m is at least δ for $n \neq m$. Now, there is sequence, x_n, in $\overline{B_H(0,1)}$ given such that $Tx_n = y_n$, or in other words $\|Tx_n - Tx_m\| \geq \delta$. The sequence, x_n, in H is certainly bounded.

Do it explicitly!

In general it is a subnet, not a subsequence

Notice that every Hilbert space is reflexive so by theorem 6.3 $\overline{B_H(0,1)}$ is weakly compact and hence x_n has a subsequence, x_{n_k}, that converges weakly to some element $x \in H$. By assumption this should imply that Tx_{n_k} converges strongly to Tx, but this contradicts what we found above.

How?

(✓)

d)

From remark 5.3 in the lecture notes we have that weak and strong convergence in $\ell_1(\mathbb{N})$ coincide. We know (From An2 and lecture 1) that $\ell_2(\mathbb{N})$ is a Hilbert space which is infinite dimensional and separable. Take a sequence, x_n, that converges weakly in $\ell_2(\mathbb{N})$ to x and let $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$. By a) we have that Tx_n converges weakly in $\ell_1(\mathbb{N})$ to Tx. But in $\ell_1(\mathbb{N})$ that means that Tx_n converges strongly to Tx. Hence by c) we have that $T \in \mathcal{K}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$. ✓

e)

Suppose for the sake of a contradiction that $T \in \mathcal{K}(X,Y)$ is onto. From the open mapping theorem we know that T is open. That means that there exists some $r > 0$ such that $B_Y(0,r) \subset T(B_X(0,1))$ which implies that $\overline{B_Y(0,r)} \subset \overline{T(B_X(0,1))}$. Since $\overline{T(B_X(0,1))}$ is compact by definition and $\overline{B_Y(0,r)}$ is a closed subset we know that $\overline{B_Y(0,r)}$ must be compact. Since Y is a metric space we know that compactness and sequential compactness are the same. But from Mandatory Assignment 1 we have that Y admits a Hamel basis, $(e_i)_{i \in I}$. Now take a countable, infinite subset, $\Lambda = \{\lambda_1, \lambda_2, \cdots\}$ of I. Assume WLOG that $\|e_i\| = r$ then $(e_{\lambda_i})_{i \geq 1}$ is a sequence that has no converging subsequence and hence $\overline{B_Y(0,r)}$ is not compact and we have reached a contradiction. How do you know this?

(✓)

f)

We see that for $f, g \in H$ we have

Do the calculations with the integrals!

$$\langle Mf(t), g(t) \rangle = \langle tf(t), g(t) \rangle = t \langle f(t), g(t) \rangle$$
$$= \langle f(t), t^* g(t) \rangle = \langle f(t), tg(t) \rangle = \langle f(t), Mg(t) \rangle, \quad (7)$$

??

0/0

so M is self-adjoint.

No, t is a variable not a scalar...

Now, note from HW4 prob 4a that H is separable and we know that H is infinite dimensional. Notice that the image of the unit ball $B_H(0,1)$ under M is $B_H(0,t)$ for $t \in [0,1]$ and by a similar argument as in e) we have that $\overline{B_H(0,t)}$ is non-compact and hence M is not compact.

Alternatively: Supposing that M is compact leads to a contradiction since we can use Theorem 10.1 to see that H has an ONB consisting of eigenvectors, $(e_i)_{i \geq 1}$ of M corresponding to eigenvalues, λ_i. But as we have seen in HW6 prob 3a, M has no eigenvalues, so we have a contradiction.

This is OK ✓

Note H separable and infinite-dim. as you did above.

Problem 3

a)

We want to use Prop 9.12 of the LN. Notice first that our measure space is certainly σ-finite since $m([0,1]) = 1$. We wish to show that $K \in L_2([0,1] \times [0,1], m \otimes m)$. We see that since $|K(s,t)|^2$ is a positive measurable function (it is continuous) we can use Tonelli's theorem

$$\int_{[0,1]\times[0,1]} |K(s,t)|^2 dm(s) \otimes m(t) \leq \int_{[0,1]} \left(\int_{[0,1]} dm(s) \right) dm(t) = 1. \tag{8}$$

Where we used the fact that $K(s,t) \leq 1$. Now, T is exactly the associated kernel operator (note, that $K(t,s) = K(s,t)$) and by Prop 9.12 it is Hilbert-Schmidt and hence compact.

b)

Let $f, g \in H$ and notice that we have

$$\langle Tf(t), g(t) \rangle = \int_{[0,1]} \left(\int_{[0,1]} K(s,t) f(s) \overline{g(t)} dm(s) \right) dm(t) \tag{9}$$

for any $t \in [0,1]$. Now, $K(s,t)f(s)\overline{g(t)}$ is measurable since it is a product of measurable functions. We see also that it is $m \otimes m$ integrable since

In principle it is not obvious that $f\overline{g}$ and or fg is $m\otimes m$-measurable but it is indeed true. (Not hard to show)

$$\begin{aligned}
\int_{[0,1]\times[0,1]} |K(t,s) f(s) \overline{g(t)}| dm(s) dm(t) &\leq \int_{[0,1]\times[0,1]} |f(s)\overline{g(t)}| dm(s) dm(t) \\
\leq \int_{[0,1]\times[0,1]} |f(s)||\overline{g(t)}| dm(s) dm(t) &\leq \int_{[0,1]} \left(\int_{[0,1]} |f(s)||\overline{g(t)}| dm(s) \right) dm(t) \\
&\leq \int_{[0,1]} |f(s)| dm(s) \int_{[0,1]} |\overline{g(t)}| dm(t) < \infty
\end{aligned} \tag{10}$$

where we used Tonelli's theorem in the third inequality. We used also that $L_2([0,1], m) \subset L_1([0,1], m)$ which we proved in An2. This means that we can use Fubini's theorem in (9) to get

$$\langle Tf(t), g(t) \rangle = \int_{[0,1]} \left(\int_{[0,1]} f(s) \overline{K(s,t) g(t)} dm(t) \right) dm(s) = \langle f(s), Tg(s) \rangle \tag{11}$$

for any $s \in [0,1]$. We used that $K(s,t) = \overline{K(s,t)}$. This shows the desired.

you also need $K(s,t) = K(t,s)$.

c)

Let $s \in [0,1]$ and $f \in H$. We see that

$$Tf(s) = \int_{[0,1]} K(s,t) f(t) dm(t) = \int_{[0,1]} \left((1-s)tf(t)\mathbf{1}_{t\leq s} + (1-t)sf(t)\mathbf{1}_{t>s} \right) dm(t) \tag{12}$$

by using the definition of K. Notice that $0 = s\int_{[0,1]}(1-t)f(t)\mathbf{1}_{t=s} dm(t)$ since the integrand is 0 a.e. Hence we get further that

$$\begin{aligned}
Tf(s) &= \int_{[0,1]} \left((1-s)tf(t)\mathbf{1}_{t\leq s} + (1-t)sf(t)(\mathbf{1}_{t>s} + \mathbf{1}_{t=s}) \right) dm(t) \\
&= (1-s)\int_{[0,s]} tf(t) dm(t) + s\int_{[s,1]} (1-t)f(t) dm(t).
\end{aligned} \tag{13}$$

Let $\Gamma(s) = \int_{[0,s]} tf(t)dm(t)$ and $\Gamma'(s) = \int_{[s,1]}(1-t)f(t)dm(t)$. We wish to show that these are continuous. Hence we want to show that given any sequence $(s_n)_{n\in\mathbb{N}} \subset [0,1]$ that converges to $s \in [0,1]$ we have that $\Gamma(s_n) \to \Gamma(s)$ and $\Gamma'(s_n) \to \Gamma'(s)$. We use the dominated convergence theorem:

Notice that both $tf(t)$ and $(1-t)f(t)$ are integrable since

$$\int_{[0,1]} |tf(t)|dm(t) \le \int_{[0,1]} |f(t)|dm(t) < \infty \tag{14}$$

and

$$\int_{[0,1]} |(1-t)f(t)|dm(t) \le \int_{[0,1]} |f(t)|dm(t) < \infty \tag{15}$$

This means (An2) that $|tf(t)|$ and $|(1-t)f(t)|$ are also integrable. Consider the sequences of functions given by $(tf(t)\mathbf{1}_{[0,s_n]}(t))_{n\in\mathbb{N}}$ and $((1-t)f(t)\mathbf{1}_{[s_n,1]}(t))_{n\in\mathbb{N}}$. These are subsets of the set of integrable functions since we have that

$$|tf(t)\mathbf{1}_{[0,s_n]}(t)| \le |tf(t)| \tag{16}$$

$$|(1-t)f(t)\mathbf{1}_{[s_n,1]}(t)| \le |(1-t)f(t)| \tag{17}$$

for all $n \in \mathbb{N}$. Now, take any $[0,1] \ni t' \neq s$ then there exists $N \in \mathbb{N}$ such that for all $n' > N$ we have $t'f(t')\mathbf{1}_{[0,s_{n'}]}(t') = t'f(t')\mathbf{1}_{[0,s]}(t')$ and since $m(\{t = s\}) = 0$ we know that $(tf(t)\mathbf{1}_{[0,s_n]}(t))_{n\in\mathbb{N}}$ converges pointwise to $tf(t)\mathbf{1}_{[0,s]}(t)$ a.e. Similarly $((1-t)f(t)\mathbf{1}_{[s_n,1]}(t))_{n\in\mathbb{N}}$ converges pointwise to $(1-t)f(t)\mathbf{1}_{[s_n,1]}(t)$ a.e. Hence we get by the dominated convergence theorem

$$\lim_{n\to\infty} \Gamma(s_n) = \lim_{n\to\infty} \int_{[0,1]} tf(t)\mathbf{1}_{[0,s_n]}(t)dm(t) = \int_{[0,1]} \lim_{n\to\infty} tf(t)\mathbf{1}_{[0,s_n]}(t)dm(t)$$
$$= \int_{[0,1]} tf(t)\mathbf{1}_{[0,s]}(t)dm(t) = \Gamma(s) \tag{18}$$

and similarly in the case $\Gamma'(s_n) \to \Gamma'(s)$. Since a product of continuous functions is continuous and a sum of continuous functions is continuous and $s \mapsto 1-s$ and $s \mapsto s$ are certainly continuous we can conclude that $Tf(s)$ is continuous.

Notice that $0 \le \int_{[0,s]} |tf(t)|dm(t) \le \sup\{|tf(t)|\}m([0,s])$. Now since when $s = 0$ we have $m(\{0\}) = 0$ and we have the convention that $\infty \cdot 0 = 0$ we see by the squeezing lemma that $\int_{\{0\}} |tf(t)|dm(t) = 0$ and by the triangle inequality we have $0 \le |\int_{\{0\}} tf(t)dm(t)| \le \int_{\{0\}} |tf(t)|dm(t) = 0$ so $|\int_{\{0\}} tf(t)dm(t)| = 0$ and hence $\int_{\{0\}} tf(t)dm(t) = 0$. The second term in $Tf(0)$ is trivially zero and hence $tf(0) = 0$. A completely similar line of argumentation shows that $Tf(1) = 0$.

Not needed

Problem 4

a)

Let $k, l \in \mathbb{N}_0$ and recall the product formula for derivatives: $\partial^l(fg) = \sum_{\alpha+\beta=l} \frac{l!}{\alpha!\beta!}(\partial^\alpha f)(\partial^\beta g)$. Notice that $\partial^\beta e^{-x^2/2} = \mathrm{Pol}_\beta(x)e^{-x^2/2}$. Using this on g_k gives us

$$\partial^l(x^k e^{-x^2/2}) = e^{-x^2/2} \sum_{\alpha+\beta=l} \frac{l!}{\alpha!\beta!}\partial^\alpha(x^k)\mathrm{Pol}_\beta(x) = e^{-x^2/2}\mathrm{Pol}_{k+l}(x), \tag{19}$$

so g_k is certainly in $C^\infty(\mathbb{R})$ since the derivative of any order is the product of two continuous functions. Also we see that for any $r \in \mathbb{N}_0$ we have

$$\lim_{x\to\infty} x^r \partial^l g_k = \lim_{x\to\infty} \mathrm{Pol}_{k+l+r}(x) e^{-x^2/2} = 0, \tag{20}$$

so $g_k \in \mathcal{S}(\mathbb{R})$ for any $k \in \mathbb{N}_0$.

Notice that

$$\mathcal{F}(g_0) = e^{-\xi^2/2}, \tag{21}$$

by proposition 11.4 in LN. It is easy to check that

$$\partial(g_0) = -g_1 \tag{22}$$
$$\partial^2(g_0) = g_2 - g_0 \tag{23}$$
$$\partial^3(g_0) = 3g_1 - g_3. \tag{24}$$

We have already seen that all the requirements for using prop. 11.13(b) of the lectures are fulfilled since Schwartz functions are a subset of L_1 functions. Hence we get

$$\mathcal{F}(g_1) = -\mathcal{F}(\partial g_0) = -i\xi e^{-\xi^2/2} \tag{25}$$

$$\mathcal{F}(g_2) = \mathcal{F}(\partial^2(g_0)) + \mathcal{F}(g_0) = -\xi^2 e^{\xi^2/2} + e^{-\xi^2/2} = (1-\xi^2)e^{\xi^2/2} \tag{26}$$

$$\mathcal{F}(g_3) = 3\mathcal{F}(g_1) - \mathcal{F}(\partial^3(g_0)) = -3i\xi e^{-\xi^2/2} + i\xi^3 e^{\xi^2/2} = (\xi^2-3)i\xi e^{-\xi^2/2}. \tag{27}$$

b)

We see that if we let $h_0 = g_0$, then

$$\mathcal{F}(h_0) = \mathcal{F}(g_0) = i^0 g_0 = i^0 h_0. \tag{28}$$

If we let $h_1 = g_3 - \frac{3}{2}g_1$, then

$$\mathcal{F}(h_1) = \mathcal{F}(g_3 - \tfrac{3}{2}g_1) = ih_1 \tag{29}$$

If we let $h_2 = g_2 - \frac{1}{2}g_0$, then

$$\mathcal{F}(h_2) = \mathcal{F}(g_2 - \tfrac{1}{2}g_0) = i^2 h_2 \tag{30}$$

and lastly if we let $h_3 = g_1$, then

$$\mathcal{F}(h_3) = \mathcal{F}(g_1) = i^3 h_3. \tag{31}$$

All of these are certainly Schwartz functions since they are linear combinations of Schwartz functions.

c)

We start by calculating $\mathcal{F}^2(f)$ which by definition is $\mathcal{F}(\mathcal{F}(f))$. We get

$$\mathcal{F}(\mathcal{F}(f))(\tau) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x) e^{-ix\xi} dm(x) \right] e^{-i\xi\tau} dm(\xi) \tag{32}$$

Now, since $|f(x)e^{-ix\xi}e^{-i\xi\tau}| \leq |f(x)| \in \mathcal{S}(\mathbb{R}) \subset L_1(\mathbb{R})$ we can use Fubini's theorem to change the order of integration to get

This does not show $f(x)e^{-i\xi x}e^{-i\xi\tau} \in L_1(\mathbb{R}\times\mathbb{R})$ which is in fact false.

$$\mathcal{F}(\mathcal{F}(f))(\tau) = \int_{\mathbb{R}} f(x) \Big[\underbrace{\int_{\mathbb{R}} e^{-i\xi\tau} e^{-i\xi x} dm(\xi)}_{\text{Not convergent.}} \Big] dm(x). \tag{33}$$

This is true when $e^{-i\xi\tau}$ is viewed as a temperate distribution and $\mathcal{F}$ is defined on the set of temperate distributions. However, we have not done so in this course.

Now, the term in the brackets is equal to $\mathcal{F}(e^{-i\xi\tau})(x)$ which is the Dirac-delta distribution $\delta(x+\tau)$ and hence we get

In this case this is abuse of notation. There is not an integral-formula for $\mathcal{F}(\omega)$ when ω is a temp. dist. as used in (33)

$$\mathcal{F}(\mathcal{F}(f))(\tau) = \mathcal{F}^2(f)(\tau) = f(-\tau). \tag{34}$$

In other words, we have $\mathcal{F}^2(f) = f \circ (-I)$ where I is the identity operator. Therefore we get

$$\mathcal{F}^4(f) = \mathcal{F}^2(\mathcal{F}^2(f)) = \mathcal{F}^2(f \circ (-I)) = (f \circ (-I)) \circ (-I) = f \quad (✓) \tag{35}$$

which was the desired.

d)

Let $f \in \mathcal{S}(\mathbb{R})$ be non-zero and $\lambda \in \mathbb{C}$ and suppose $\mathcal{F}(f) = \lambda f$. Applying $\mathcal{F}$ three times on both sides and using the result of c) gives by linearity of $\mathcal{F}$

$$f = \lambda^4 f \implies (1-\lambda^4) f = 0. \tag{36}$$

Since f is non-zero we have that $1 = \lambda^4$ which means exactly that $\lambda \in \{1, i, -1, -i\}$. The equation, $\mathcal{F}(f) = \lambda f$, is an eigenvalue equation hence any non-zero f satisfying this is an eigenfunction with eigenvalue λ. Hence the eigenvalues of $\mathcal{F}$ are exactly $\{1, i, -1, -i\}$. (✓)

It is not guaranteed that all of $\{1, i, -1, -i\}$ are attained. This follows from b.

Problem 5

a)

Let $x \in [0,1]$ and take any continuous $f : [0,1] \to [0,1]$ with compact support, K, such that $f(x) > 0$.

Notice first that

$$\int f d\mu = \int f d\Big(\sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}\Big) = \sum_{n=1}^{\infty} 2^{-n} f(x_n) \tag{37}$$

since f is positive and measurable (An2 or MI).

What is x? What is ε?

It is well known from the Heine-Cantor Theorem that f is uniformly continuous since it is a real continuous function defined on a closed and bounded (hence compact, by Heine-Borel) interval in $\mathbb{R}$. Hence if we pick $f(x) > \epsilon > 0$, then there exists $\delta > 0$ such that for any $x' \in [0,1]$ with $|x - x'| < \delta$ we have $|f(x) - f(x')| < f(x)$. Therefore for any $y \in (x-\delta, x+\delta) = A_\delta$ we have $f(y) > 0$ and hence A_δ is an open subset of K. What is K?

This is not uniform continuity, as ε depends on x.

?

Now, since $(x_n)_{n\geq 1}$ is dense in $[0,1]$ we know that there exists $m \in \mathbb{N}$ such that $x_m \in A_\delta$ and hence $f(x_m) > 0$. Therefore the sum in (37) is lower bounded by

$$2^{-m} f(x_m) > 0 \tag{38}$$

and it is now a consequence of HW8 prob 3b that $x \in \text{supp}(\mu)$. Hence $[0,1] \subset \text{supp}(\mu)$ and since we trivially have $\text{supp}(\mu) \subset [0,1]$ we conclude $\text{supp}(\mu) = [0,1]$. (✓)