Functional Analysis: Mandatory assignment 2

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Problem 1

a)

We use The Pythagorean Theorem (5.23 in Folland) to see that:

$$||f_N||^2 = ||N^{-1}\sum_{n=1}^{N^2} e_n||^2 = N^{-2}\sum_{n=1}^{N^2}||e_n||^2 = N^{-2}\sum_{n=1}^{N^2}1^2 = N^{-2}N^2 = 1$$

for all $N \geq 1$. Where we have that $(e_n)_{n\geq 1}$ is an orthonormal basis, so $\langle e_i, e_j\rangle = 0$ for $i \neq j$.
Now let us prove that $f_N \to 0$: By the definition of f_N converging weakly to 0, means that $F(f_N) \to F(0)$ for all $F \in H^*$ and we know that $F(0) = 0$ for all elements in the dual. We get from Theorem 5.25 in Folland that there exists a unique $y \in H$, such that $F(f_N) = \langle f_N, y\rangle$. We can write $y = \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i$ because y is an orthonormal basis and as $||y|| < \infty$ for any ε there exists a K such that $||\sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i|| < \varepsilon$. Therefore we have that

$$\begin{aligned}
|F(f_N)| = |\langle f_N, y\rangle| &= |\langle f_N, \sum_{i=1}^{\infty}\langle y, e_i\rangle e_i\rangle| \\
&= |\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle + \langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle| \\
&\leq |\langle f_N, \sum_{i=1}^{K}\langle y, e_i\rangle e_i\rangle| + |\langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle|
\end{aligned}$$

Where we use the triangle inequality. Now because we are in a Hilbert space we can use Cauchy Schwartz to bound the second term:

$$|\langle f_N, \sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i\rangle| \leq ||f_N|| \cdot ||\sum_{i=K+1}^{\infty}\langle y, e_i\rangle e_i|| < 1 \cdot \varepsilon$$

Now let's look at the other term:

$$
\begin{aligned}
|\langle f_N, \sum_{i=1}^{K} \langle y, e_i \rangle e_i \rangle| &= |\langle N^{-1} \sum_{n=1}^{N^2} e_n, \sum_{i=1}^{K} \langle y, e_i \rangle e_i \rangle| \\
&= N^{-1} |\sum_{n=1}^{N^2} \langle e_n, \sum_{i=1}^{K} \langle y, e_i \rangle e_i \rangle| \\
&= N^{-1} |\sum_{n=1}^{N^2} \overline{\sum_{i=1}^{K} \langle y, e_i \rangle \langle e_n, \rangle} | \\
&\leq N^{-1} |\sum_{i=1}^{K} \overline{\langle y, e_i \rangle}| < \varepsilon \text{ for } N \to \infty
\end{aligned}
$$

Where we use that e_n is an orthonormal basis and that $|\sum_{i=1}^{K} \overline{\langle y, e_i \rangle}|$ is finite. This gives us that for all $F \in H^*$, we get $F(f_N) \to F(0) = 0$ when $N \to \infty$, which is what we wanted. ✓

b)

Note w^* only exists on dual spaces.

(Following HW5 Problem 2 c)) We have that $co(A)$ is convex, where $A = \{f_N, N \geq 1\}$, so by Theorem 5.7 in the lecture notes we have that $K = \overline{co(A)} = \overline{co(A)}^w$ and we know that H is reflexive. This means we have that the weak topology is equal to the weak star topology, so $K = \overline{co(A)} = \overline{co(A)}^w = \overline{co(A)}^{w^*}$ and we have that $K = \overline{co(A)} \subset \overline{B(0,1)}$, because $A \subset \overline{B(0,1)}$.
Now this also holds for the weak star topology, and by Alaoglu's theorem (Theorem 6.1 in the lecture notes) the closed unit ball in the weak star topology is compact, and because $\overline{co(A)}^{w^*}$ is closed subset of a compact set, it is compact as well and this is equal to our $K = \overline{co(A)}$, so K is compact in the weak topology as we wanted.
We have that f_N for $N \geq 1$ is in K and from a) we have that $f_N \to 0$ weakly in K and therefore it is in the weak closure of the complex hull of $\{f_N : N \geq 1\}$ and therefore $0 \in K$. ✓

c)

Let us show that 0 is an extreme point, so lets look at 0 written as convex combination in K: $0 = \alpha x + (1-\alpha)y$ for $\alpha \in (0,1)$ and we have that $0 = \alpha \langle x, e_n \rangle + (1-\alpha)\langle y, e_n \rangle$ for all $n \geq 1$. We have that 0 is an extreme point on the real line therefore for each n we have that $\langle x, e_n \rangle = \langle y, e_n \rangle = 0$, but by completeness we have that this means $x = y = 0$, so 0 must be an extreme point in K. It is not 0/0

Now let us prove that each $f_N, N \geq 1$ are extreme points in K:
Let us look at f_N being a convex combination in K, so write $f_N = \alpha x + (1-\alpha)y$, where x and y are limit points of respectively $(x_n)_{n\geq1}, (y_n)_{n\geq1} \in co\{f_N : N \geq 1\}$. Then we have that $\alpha(x_n)_{n\geq1} + (1-\alpha)(y_n)_{n\geq1} \to f_N$. Let us now define $g_{N^2}(x) = \langle x, e_{N^2} \rangle$,

which is continuous and apply it to:

$$g_{N^2}(\alpha(x_n)_{n\geq 1} + (1-\alpha)(y_n)_{n\geq 1}) = \alpha g_{N^2}(x_n) + (1-\alpha) g_{N^2}(y_n) \to g_{N^2}(f_N) = \frac{1}{N}$$

We will now show that $g_{N^2}(x_n) \leq \frac{1}{N}$ (and the same argument to show $g_{N^2}(y_n) \leq \frac{1}{N}$):

Do you mean $x_n \in co\{f_n | n \in \mathbb{N}\}$?

First see that for $j < N$, then $g_{N^2}(f_j) = 0$ and for $j \geq N$ we get that $g_{N^2}(f_j) = \frac{1}{j} \leq \frac{1}{N}$. Now let us denote $x_n \in K$ by it's convex combination $x_n = \sum_{k=1}^{\infty} \alpha_{n_k} f_k$, where $\sum_{k=1}^{\infty} \alpha_{n_k} = 1$ and therefore there is only a finite set, ? where x_n is not-zero and we can therefore write $x_n = \sum_{k=1}^{W_n} \alpha_{n_k} f_k$. Therefore we have that

Why is this the case?

$$g_{N^2}(x_n) = \sum_{k=1}^{W_n} \alpha_{n_k} g_{N^2}(f_k) \leq \sum_{k=1}^{W_n} \alpha_{n_k} \frac{1}{N} = \frac{1}{N}$$

So if we have that $\alpha g_{N^2}(x_n) + (1-\alpha) g_{N^2}(y_n) \to \frac{1}{N}$ this must mean that $g_{N^2}(x_n) \to \frac{1}{N}$ and $g_{N^2}(y_n) \to \frac{1}{N}$.

We have that $(x_n)_{n\geq 1}$ converges to a specific f_j if the sequence (α_{n_j}) of the j'th coefficent of the elements in $(x_n)_{n\geq 1}$ convers to 1. So we will show that if $g_{N^2} \to \frac{1}{N}$, then $(x_n)_{n\geq 1}$ converges to f_N by showing that (α_{n_j}) to 1:

Let us assume for contradiction that it does not converge to 1, therefore there exists a $\varepsilon > 0$, so for every L there exists a $n > L$, so $|1 - \alpha_{n_j}| > \varepsilon$. Now because $\alpha_{n_j} \leq 1$ we have that $r_n = 1 - \alpha_{n_j} > \epsilon$. Then we see that

$$\begin{aligned}
&|\frac{1}{N} - g_{N^2}(\alpha(x_n) + (1-\alpha)(y_n))| = \frac{1}{N} - \alpha g_{N^2}(x_n) - (1-\alpha) g_{N^2}(y_n) \\
&\geq \frac{1}{N} - \alpha g_{N^2}(x_n) - (1-\alpha)\frac{1}{N} \\
&\geq \alpha \frac{1}{N} - (\alpha \sum_{i=1}^{W_n} \alpha_{n_i} g_{N^2}(f_i)) = \alpha \frac{1}{N}(1 - \alpha_{n_N}) - (\alpha \sum_{i=1, i\neq N}^{W_n} \alpha_{n_i} g_{N^2}(f_i))
\end{aligned}$$

Then we can use that $\sum_{i=1, i\neq N}^{W_n} a_{n_i} = 1 - a_{n_N} = r_n$ (because we know the sum of a_{n_i} is equal to 1) and that for $i \neq N$ then $g_{N^2}(f_i) \leq \frac{1}{N+1}$ to see that:

$$\alpha \frac{1}{N}(1 - \alpha_{n_N}) - (\alpha \sum_{i=1, i\neq N}^{W_n} \alpha_{n_i} g_{N^2}(f_i)) \geq \alpha(\frac{r_n}{N} - \frac{r_n}{N+1}) \geq \varepsilon\alpha(\frac{1}{N} - \frac{1}{N+1})$$

How? It is unclear that you reach a contradiction since $\frac{1}{N} - \frac{1}{N+1} \to 0$.

Which is a contradiction to $g_{N^2} \to \frac{1}{N}$ and therefore do (α_{n_j}) converge to 1 (and we can make the same argument for $(y_n)_{n\geq 1}$). This gives us then that $(x_n)_{n\geq 1}$ converge to $x = f_N$ and $(y_n)_{n\geq 1}$ converges to f_N.

Therefore we can conclude that any convex combination of $f_N = \alpha x + (1-\alpha) y$ implies that $x = y = f_N$ and therefore f_N is an extreme point for every $N \geq 1$.

d)

We use Milman theorem (Theorem 7.9 in the lecture notes) and look at $F = \{f_N : N \geq 1\}$, where $K = \overline{co(F)}$. Then we use the Milman theorem to say that $Ext(K) \subset \overline{F}^w = \{f_N :: \geq 1\} \cup \{0\}$, so there are not any other extreme points, than the ones mentioned in c). (✓)

Why is this true?

Problem 2

a)

We have by the definition of weak convergence of x_n, that for all $f \in X^*$, we have that $f(x_n) \to f(x)$. Let's take some $g \in Y^*$ and look at $g(Tx_n)$ which is also written $g(Tx_n) = T^\dagger g(x_n)$ by the definition 7.13 in the lecture notes, such that ? $g(Tx_n) \in X^*$, so let us define $f = T^\dagger g$, which gives us that $g(Tx_n) = T^\dagger g(x_n) = f(x_n) \to f(x) = T^\dagger g(x) = g(Tx)$, where we use the weak convergence of x_n. This is again the definition of weak convergence, so we have that $Tx_n \Rightarrow Tx$ weakly as $n \to \infty$ as we wanted. ✓

Wrong arrow

b)

So let's assume that for a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$ that $x_n \to x$ weakly as $n \to \infty$, then by Problem 2 b) of HW4 we have that: $\sup\{||x_n||, n \geq 1\} < \infty$. So $\{x_1, x_2, ...\}$ is bounded, which implies that because T is compact, then $\overline{T(\{x_1, x_2, ...\})}$ is compact.

with the same limit y

We have from a result from An1 that if all subsequences of a sequence have convergent a subsequence then the original sequence is convergent. with the same limit y

Now let us look at (Ty_n) being a subsequence of $(T(x_n - x))$. Now because T is compact we have that $\overline{T(\{y_1, y_2, ...\})}$ is compact as well and therefore there exists a converging subsequence (Ty_{n_i}) of (Ty_n) with $Ty_{n_i} \to \alpha$.

This is unclear.

Now let us prove that this α is equal to zero: We know from a) that Tx_n converges weakly to Tx, which means that for all $g \in Y^*$ that $g(T(x_n)) \to g(T(x))$, therefore $g(T(y_{n_j})) \to g(T(x - x)) = 0$, therefore $\alpha = 0$.

This gives us that Tx_{n_i} converges to Tx and therfore every subsequence of x_n has a convergent subsequence converging to Tx and therefore the sequence Tx_n converges to Tx, as we wanted to show. (✓)

c)

Let us assume that T is not compact (but the other assumptions hold), which means that $T(\overline{B_H(0,1)})$ is not totally bounded and that a sequence $(y_n)_{n\geq 1}$ in $T(\overline{B_H(0,1)})$ has no convergent subsequences, which we get by Proposition 8.2 in the lecture notes.

Now becuase we are in the image under T of the closed unit ball, we can for each y_n pick x_n so that $Tx_n = y_n$, which means we have a sequence $(x_n)_{n\geq 1}$ inside the closed unit ball in H.

Generally, this will only be a subnet.

Now because H is reflexive we have by Theorem 6.3 in the notes that $\overline{B_H(0,1)}$ is compact wrt the weak topology, which means that $(x_n)_{n\geq 1}$ has a weak converging subsequence (x_{n_i}) and then by b) we have that $T(x_{n_i})$ converges strongly, but we know from Proposition 8.2 that $(y_n)_{n\geq 1}$ has no converging subsequences, but $T(x_{n_i})$ is a subsequence of $(y_n)_{n\geq 1}$, so this is a contradiction and therefore T must be compact. (✓)

d)

The assumption says that $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ and by a) we have that for a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$ that $x_n \to x$ weakly, when $n \to \infty$ implies $Tx_n \to Tx$ weakly. Now by Remark 5.3 we have that when a sequence Tx_n in $l_1(\mathbb{N})$ converges weakly, this implies that Tx_n converges in norm, i.e $||Tx_n - Tx|| \to 0$ as $n \to \infty$. With this we can use c), because now we have strong convergence of $Tx_n \to Tx$, which says that this implies that T is compact, as we wanted to show. ✓

e)

Assume that $T \in \mathcal{K}(X,Y)$ and that T is onto, then we have that T is open and then by Folland page 162 we have that $B_Y(0,r) = r \cdot B_Y(0,1) \subseteq T(B_X(0,1)$. We have that $T(B_X(0,1)$ has compact closure by the definition of T being compact and then by compactness this implies that $B_Y(0,r)$ has compact closure (because a closed subset of a compact set is compact). But by Riesz lemma, the closure of $B_Y(0,r) = r \cdot B_Y(0,1)$ being compact implies that Y is finite dimensional , which is a contradiction to the assumptions of Y being an infinite dimensional Banach space, so there is no $T \in \mathcal{K}(X,Y)$ that is onto. ✓

f)

M?

Let us that by showing that T is self-adjoint: To show that it is self-adjoint we have to show that $\langle Tf, g\rangle = \langle f, Tg\rangle$. So we see that ← What are f,g?

$$\langle Tf, g\rangle = \int_{[0,1]} (Mf)(t) \cdot \overline{g}(t) dm(t) = \int_{[0,1]} f(t)t \cdot \overline{g(t)} dm(t)$$

$$= \int_{[0,1]} f(t)\overline{t \cdot g(t)} dm = \int_{[0,1]} f(t)\overline{(Mg)(t)} dm(t) = \langle f, Tg\rangle$$ ✓

Remember to note that $L^2([0,1], m)$ is separable and infinite-dimensional.

Now if we assume that M is compact, then by The Spectral Theorem (Theorem 10.1 in the lecture notes) we have a contradiction, because it has been shown in the lectures that M has no eigenvalues, but Theorem 10.1 says that if $T = M$ is self-adjoint and compact then H has an orthonormal basis consisting of the eigenvectors for M, but since M has no eigenvalues this is a contradiction.

Problem 3

a)

We want to use Theorem 9.6 from the lecture notes. We start by checking the assumptions: We have that the lebesgue measure is finite and Borel on $[0,1]$ and we have that $K(s,t)$ is piecewise continuous, so therefore continuous and we have that the associated operator by definition is exactly our $(Tf)(s)$ and then we can use the theorem to say that T is compact, as we wanted to show.

Piecewise cont $\not\Rightarrow$ cont.

$T = T_{\tilde{K}}$, $\tilde{K}(s,t) = K(t,s)$!

b)

To show that it is self-adjoint we have to show that $\langle Tf, g\rangle = \langle f, Tg\rangle$. So we see that:

$$\langle Tf, g\rangle = \int_{[0,1]} Tf\overline{g}dm = \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s)$$
$$= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s)$$

Now we can use Tonelli-Fubini because we know that the integrals are finite (shown at page 46 of Lecture 9), so we get that That is not enough. only if you show $K \in L_2([0,1]^2)$

$$\int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(t)dm(s) = \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\overline{g(s)}dm(s)dm(t)$$
$$= \int_{[0,1]} f(t)\int_{[0,1]} K(s,t)\overline{g(s)}dm(s)dm(t)$$
$$= \int_{[0,1]} f(t)\int_{[0,1]} \overline{K(s,t)g(s)}dm(s)dm(t)$$

Where we use that $\overline{K(s,t} = K(s,t)$ and then we have that this is equal to: $\int_{[0,1]} f(t)\overline{\int_{[0,1]} K(s,t)g(s)dm(s)}dm(t) = \langle f, Tg\rangle$ because we are in $L_2([0,1], m)$, so we have shown that T is self-adjoint.

only if $K(s,t)=K(t,s)$

c)

We know from measure-theory that the integral of a piecewise function can be split up, like this:

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t) = \int_{[0,s]}(1-s)tf(t)dm(t) + \int_{[s,1]}(1-t)sf(t)dm(t)$$
$$= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t), \quad s\in[0,1], \quad f \in H$$

To show that Tf is continuous we will use the continuity lemma from Measure Theory (Lemma 12.4 in Schilling) for $u(s,t) = K(s,t)f(t)$.
Note that the same proof can be given for a closed set $[0,1]$ instead of an open interval $(0,1)$ and denote that we can use this lemma for functions into $\mathbb{C}$ when $f(x) = a(x) + ib(x)$, where a, b are real valued functions. show or justify.
To use this lemma we have to ensure the three statements hold:

1. $t \mapsto u(s,t)$ is in $\mathcal{L}^1(m)$ for every fixed $s \in [0,1]$
2. $s \mapsto u(s,t)$ is continuous for every fixed $t \in [0,1]$
3. $|u(s,t)| \le |w(t)|$ for all $(s,t) \in [0,1]\times[0,1]$ and some $w \in \mathcal{L}^1(m)$

1) holds because we have that $L_2 \subseteq L_1$, This implies $f \in L_1([0,1])$ what about $k(s,\cdot)f$
2) holds because when we fix t in $u(s,t) = K(s,t)f(t)$, then $f(t)$ is a constant and $K(s,t)$ is piecewise continuous function and therefore continuous and lastly Same error as above
3) holds because we can take $w = |f(t)| \in L_2 \subseteq L_1$ and we have that $|K(s,t)f(t)| \leq |f(t)| = w(t)$ because $0 \leq K(s,t) \leq 1$.
Then the continuity lemma says that $U(s) = \int u(s,t)dm(t) = \int_{[0,1]} K(s,t)f(t)dm(t) = (Tf)(s)$ is continuous, as we wanted.
Lastly we look at $(Tf)(0)$ and $(Tf)(1)$:

$$(Tf)(0) = (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]} (1-t)f(t)dm(t) = \int_{[0,0]} tf(t)dm(t) = 0$$

$$(Tf)(1) = (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]} (1-t)f(t)dm(t) = \int_{[1,1]} (1-t)f(t)dm(t) = 0$$

Problem 4

a)

To show that $g_k \in \mathscr{S}(\mathbb{R})$, we have to show that $\lim_{||x||\to\infty} x^\beta \partial^\alpha g_k(x) = 0$ from Definition 11.10 in the notes. This upholds because we have that $e^{-x^2/2}$ goes faster to zero than x^k, x^β for any k, β or any polynomial we get from differentiating our function g_k, so $g_k \in \mathscr{S}(\mathbb{R})$.

To compute $\mathcal{F}(g_k)$ we use Proposition 11.13 d) from the notes, so we calculate $i^{|k|}(\partial^k \hat{f})(\xi)$ for $k = 0,1,2,3$, where $f = e^{-x^2/2}$ and we know that $\hat{f}(\xi) = e^{-\xi^2/2}$ from the solution to Proposition 11.4 in the notes. So we calculate:

$$k = 0 : i^0(\partial^0 \hat{f})(\xi) = \hat{f}(\xi) = e^{-\xi^2/2}$$
$$k = 1 : i^1(\partial^1 \hat{f})(\xi) = i(\partial e^{-x^2/2})(\xi) = i\xi e^{-\xi^2/2}$$
$$k = 2 : i^2(\partial^2 \hat{f})(\xi) = -1 \cdot (\partial^2 e^{-x^2/2})(\xi) = -1(\xi^2 - 1)e^{-\xi^2/2} = (1-\xi^2)e^{-\xi^2/2}$$
$$k = 3 : i^3(\partial^3 \hat{f})(\xi) = -i(\partial^3 e^{-x^2/2})(\xi) = -i\xi(\xi^2 - 3)e^{-\xi^2/2} = i(3\xi e^{-\xi^2/2} - \xi^3 e^{-\xi^2/2})$$

b)

k=0: We can just look at $h_0 = g_0$ and then we get $\mathcal{F}(h_0) = \mathcal{F}(g_0) = \mathcal{F}(e^{-x^2/2}) = e^{-\xi^2/2} = i^0 h_0$, as we wanted.
k=3: We can again simply take $h_3 = g_1$ and then we get $\mathcal{F}(h_3) = \mathcal{F}(g_1) = \mathcal{F}(xe^{-x^2/2}) = -ie^{-\xi^2/2} = i^3 h_0$.
For h_1 and h_2 we can look at the following linear combinations of g_0, g_1, g_2, g_3 and

get:

$$\begin{aligned}
\mathbf{k=1}: \mathcal{F}(h_1) &= \mathcal{F}(g_3 - \frac{3}{2}g_1) = \mathcal{F}(x^3 e^{-x^2/2} - \frac{3}{2}xe^{-x^2/2}) \\
&= i3\xi e^{-\xi^2/2} - i\xi^3 e^{-\xi^2/2} - \frac{3}{2}i\xi e^{-\xi^2/2} = i(\frac{3}{2}\xi e^{-\xi^2/2} - \xi^3 e^{-\xi^2/2}) = i^1 h_1 \\
\mathbf{k=2}: \mathcal{F}(h_2) &= \mathcal{F}(g_2 - \frac{1}{2}g_0) = \mathcal{F}(x^2 e^{-x^2/2} - \frac{1}{2}e^{-x^2/2}) \\
&= (1-\xi^2)e^{-\xi^2/2} - \frac{1}{2}e^{-\xi^2/2} = -1(\xi^2 e^{-\xi^2/2} - \frac{1}{2}e^{-\xi^2/2}) = i^2 h_2
\end{aligned}$$

c)

We start by computing $\mathcal{F}^2(f) = \mathcal{F}(\mathcal{F}(f))$. We have by the definition that the Fourier transformation is defined as:

$$\mathcal{F}(f(x)) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}dx \tag{0.1}$$

$$\mathcal{F}^2(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(\xi)e^{-ix\xi}dx \tag{0.2}$$

Now because we have $f \in \mathscr{S}(\mathbb{R})$ we get from Corollary 12.12 iii) and the definition of inverse Fourier transformation that:

$$f(x) = \mathcal{F}^*(\hat{f}(\xi)) = \int_{\hat{R}} \hat{f}(\xi)e^{ix\xi}dx \tag{0.3}$$

Now by comparing (0.2) and (0.3) we see that $\mathcal{F}^2(f(x)) = f(-x)$. This then gives us that $\mathcal{F}^4(f(x)) = \mathcal{F}^2(\mathcal{F}^2(f(x))) = \mathcal{F}^2(f(-x)) = f(x)$ for all $f \in \mathscr{S}(\mathbb{R})$, which is what we wanted to prove.

d)

We know from c), that if $\mathcal{F}(f) = \lambda f$, then $\mathcal{F}^4(f) = \lambda^4 f = f$, which means that $\lambda^4 = 1$, which is only possible for $\lambda = \{1, -1, i, -i\}$.
We have by definition of eigenvalues, $\mathcal{F}(f) = \lambda f$ and the fundamental theorem of algebra, that the values of λ is exactly $\mathcal{F}$'s eigenvalues, therefore the eigenvalues of $\mathcal{F}$ is $\{\pm 1, \pm i\}$.

$\frac{1}{2}/1$

NO. 4.b gives you that $\{1, i, -1, -i\}$ are eigenvalues

Problem 5

We have from Problem 3 HW8 we have that $\text{supp}(\mu) = N^c$, where N is the union of all open subsets U of $[0,1]$, such that $\mu(U) = 0$. Because we want to show that $\text{supp}(\mu) = [0,1]$ we have to prove that $N = \emptyset$, therefore we have to prove that an open set U, which has measure 0 must be the empty set.
Let us assume for contradiction that U is a non-empty set and has measure $\mu(U) = 0$, which by the definition of μ means that $x_n \notin U$ for any $n \geq 1$. Then, because U is open, we have that there must exist some $x \in U$ which is the center of a ball with

radius ε, where all the elements are contained in U. But because $(x_n)_{n\geq 1}$ is a dense subset of $[0,1]$, this means that $B(x,\varepsilon)$ contains an element of $(x_n)_{n\geq 1}$, which is a contradiction to $\mu(U) = 0$. Therefore if U is an open ball in $[0,1]$ with $\mu(U) = 0$ it must be empty and then $\operatorname{supp}(U) = [0,1]$, as we wanted.