Advanced Mathematical Physics, Assignment 1

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1 Stability in two dimensions

We define the energy functional for a particle in $\mathbb{R}^2$ as $\mathcal{E}(\psi) = T_\psi + V_\psi$, with

$$T_\psi = \int_{\mathbb{R}^2} |\nabla\psi(x)|^2 \,\mathrm{d}x, \quad \text{and} \quad V_\psi = \int V(x)|\psi(x)|^2 \,\mathrm{d}x. \tag{1.1}$$

The ground state energy is defined by

$$E_0 = \inf\{\mathcal{E}(\psi),\ \psi \in H^1(\mathbb{R}^2),\ \|\psi\|_2 = 1,\ V_\psi \text{ well defined.}\}. \tag{1.2}$$

Now assuming that $V \in L^{1+\epsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ we prove that $E_0 > -\infty$.

Proof. Let $V = v + w$ with $v \in L^{1+\epsilon}(\mathbb{R}^2)$ and $w \in L^\infty(\mathbb{R}^2)$. Notice first, that by Sobolev's inequality we have

$$\|\nabla\psi\|_2^2 \geq S_{2,p}\|\psi\|_2^{\frac{-4}{p-2}}\|\psi\|_p^{\frac{2p}{p-2}}, \qquad 2 < p < \infty. \tag{1.3}$$

It follows that $\psi \in L^p(\mathbb{R}^2)$ for $2 < p < \infty$, whenever $\psi \in H^1(\mathbb{R}^2)$. Assuming that V_ψ is well defined we know from Hölder's inequality that

$$\begin{aligned} V_\psi = \int V(x)|\psi(x)|^2 \,\mathrm{d}x &\geq \int v(x)|\psi(x)|^2 \,\mathrm{d}x - \|w\|_\infty\|\psi\|_2^2 \\ &\geq -\|v\|_q \||\psi|^2\|_{\frac{q}{q-1}} - \|w\|_\infty\|\psi\|_2^2 \\ &= -\|v\|_q\|\psi\|_{\frac{2q}{q-1}}^2 - \|w\|_\infty\|\psi\|_2^2. \end{aligned} \tag{1.4}$$

Thus setting $p = \frac{2q}{q-1} = 2 + \frac{2}{\epsilon}$, with $\epsilon > 0$, we find that

$$V_\psi \geq -\|v\|_{1+\epsilon}\|\psi\|_p^2 - \|w\|_\infty\|\psi\|_2^2. \tag{1.5}$$

Now using Sobolev's inequality we find that

$$T_\psi \geq S_{2,p}\|\psi\|_2^{\frac{-4}{p-2}}\|\psi\|_p^{\frac{2p}{p-2}} = S_{2,p}\|\psi\|_2^{\frac{-4}{p-2}}\|\psi\|_p^{2(1+\epsilon)}. \tag{1.6}$$

Thus we conclude that $\mathcal{E}(\psi) \geq S_{2,p}\|\psi\|_2^{\frac{-4}{p-2}}\|\psi\|_p^{2(1+\epsilon)} - \|v\|_{1+\epsilon}\|\psi\|_p^2 - \|w\|_\infty\|\psi\|_2^2$. Consider now the case in which $\psi \in H^1(\mathbb{R}^2)$, $\|\psi\|_2 = 1$ and V_ψ is well defined. It then follows that

$$\mathcal{E}(\psi) \geq S_{2,p}\|\psi\|_p^{2(1+\epsilon)} - \|v\|_{1+\epsilon}\|\psi\|_p^2 - \|w\|_\infty. \tag{1.7}$$

Therefore, we may conclude that

$$\begin{aligned} E_0 &= \inf\{\mathcal{E}(\psi) : \psi \in H^1(\mathbb{R}^2),\ \|\psi\|_2 = 1,\ V_\psi \text{ well defined}\} \\ &\geq \inf\{S_{2,p}\|\psi\|_p^{2(1+\epsilon)} - \|v\|_{1+\epsilon}\|\psi\|_p^2 - \|w\|_\infty : \psi \in H^1(\mathbb{R}^2),\ \|\psi\|_2 = 1,\ V_\psi \text{ well defined}\} \\ &\geq \inf\{S_{2,p}x^{(1+\epsilon)} - \|v\|_{1+\epsilon}x - \|w\|_\infty : x \in \mathbb{R},\ x \geq 0\} > -\infty, \end{aligned} \tag{1.8}$$

where we have used that fact that
$\{\|\psi\|_p^2 : \psi \in H^1(\mathbb{R}^2),\ \|\psi\|_2 = 1,\ V_\psi \text{ well defined}\} \subseteq \{x \in \mathbb{R} : x \geq 0\}$ □

2 Stability of hydrogen through ground state positivity

(a)

Let $\Omega \in \mathbb{R}^3$ be an open set and $V \in \mathcal{C}(\Omega)$. Assume that $\psi \in \mathcal{C}^2(\Omega)$ satisfies $(-\Delta + V)\psi = E\psi$ for some $E \in \mathbb{R}$ and furthermore $\psi > 0$. Then it holds that

$$\int_\Omega |(\nabla\varphi)(x)|^2 \,\mathrm{d}x + \int_\Omega V(x)|\varphi(x)|^2 \,\mathrm{d}x \geq E \int_\Omega |\varphi(x)|^2 \,\mathrm{d}x, \tag{2.1}$$

for all $\varphi \in \mathcal{C}_0^1(\Omega)$.

Proof. Let $\varphi \in \mathcal{C}_0^1(\Omega)$, and write $\varphi = g\psi$. Since $\psi > 0$ we clearly have $g = \varphi/\psi \in \mathcal{C}_0^1(\Omega)$. Notice that $\nabla\varphi = (\nabla g)\psi + g(\nabla\psi)$ and therefore

$$|\nabla\varphi|^2 = |\psi|^2|\nabla g|^2 + |g|^2|\nabla\psi|^2 + (\nabla g)(\nabla\psi)\bar{g}\psi + (\nabla\psi)(\nabla\bar{g})\psi g \tag{2.2}$$

Using that $(\nabla g)(\nabla\psi)\bar{g}\psi = \nabla \cdot (g(\nabla\psi)\bar{g}\psi) - |g|^2(\Delta\psi)\psi - g(\nabla\psi)(\nabla\bar{g})\psi - |g|^2|\nabla\psi|^2$, we find

$$|\nabla\varphi|^2 = |\psi|^2|\nabla g|^2 + \nabla \cdot (g(\nabla\psi)\bar{g}\psi) - |g|^2(\Delta\psi)\psi. \tag{2.3}$$

Applying Stokes' (or Gauss') theorem, as well as using the fact that g has compact support[1] we conclude

$$\int_\Omega |(\nabla\varphi)(x)|^2 \,\mathrm{d}x = \int_\Omega |\psi(x)|^2|\nabla g(x)|^2 - |g(x)|^2(\Delta\psi(x))\psi(x) \,\mathrm{d}x \geq \int_\Omega |g(x)|^2\psi(x)(-\Delta\psi(x)). \tag{2.4}$$

[1]Notice that since g is continuous, the support of g, $\mathrm{supp}(g) = \{x \in \mathbb{R}^3 : f(x) \neq 0\}$, is necessarily open. However, $S = \overline{\mathrm{supp}(g)}$ is compact by assumption. Furthermore, by continuity of g, we must have $g|_{\partial S} = 0$. Thus we may split the integral

$$\int_\Omega \nabla \cdot (g(\nabla\psi)\bar{g}\psi) \,\mathrm{d}x = \int_S \nabla \cdot (g(\nabla\psi)\bar{g}\psi) \,\mathrm{d}x + \int_{\Omega\setminus S} \nabla \cdot (g(\nabla\psi)\bar{g}\psi) \,\mathrm{d}x = \int_{\partial S} (g(\nabla\psi)\bar{g}\psi) \cdot \hat{n} \,\mathrm{d}a = 0.$$

Therefore we conclude

$$\begin{aligned}\int_\Omega |(\nabla\varphi)(x)|^2\,\mathrm{d}x + \int_\Omega V(x)|\varphi(x)|^2\,\mathrm{d}x &\geq \int_\Omega |g(x)|^2\psi(x)(-\Delta\psi(x)) + |g(x)|^2\psi(x)(V(x)\psi(x))\,\mathrm{d}x \\ &= \int_\Omega |g(x)|^2\psi(x)\left[(-\Delta + V(x))\psi(x)\right]\mathrm{d}x \\ &= E\int_\Omega |g(x)|^2|\psi(x)|^2\,\mathrm{d}x \\ &= E\int_\Omega |\varphi(x)|^2\,\mathrm{d}x\end{aligned} \tag{2.5}$$

this conclude the proof. □

(b)

Consider now the function $\psi(x) = \exp(-\alpha\,|x|)$. We show that this function indeed satisfies $\psi \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\})$ and that there exist an α such that $(-\Delta - Z/\,|x|)\psi = E_0\psi$ for some E_0. First we notice that ψ is a composition of $\mathcal{C}^\infty(\mathbb{R}^3 \setminus \{0\})$, thus $\psi \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\}) \subset \mathcal{C}^\infty(\mathbb{R}^3 \setminus \{0\})$. Furthermore, by going to spherical coordinates (r, θ, φ), with θ the azimuthal angle and φ the polar angle, we can express $\tilde{\psi}(r, \theta, \phi) := \psi(x(r, \theta, \varphi)) = \exp(-\alpha r)$. It is well known that the Laplacian on $\mathcal{C}^2(\mathbb{R} \setminus \{0\})$, Δ, can be exressed in polar coordinates as

$$\Delta\phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi\frac{\partial\phi}{\partial\varphi}) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\phi}{\partial^2\theta}, \quad r > 0,\ 0 \leq \theta < 2\pi,\ 0 \leq \varphi \leq \pi. \tag{2.6}$$

Thereby we see that

$$\begin{aligned}(-\Delta - Z/\,|x|)\psi(x)|_{x=x(r,\theta,\varphi)} = (-\Delta - Z/r)\tilde{\psi}(r, \theta, \varphi) &= -\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\exp(-\alpha r)) - Z/r\exp(-\alpha r) \\ &= (-\alpha^2 + 2\alpha/r - Z/r)\exp(\alpha r).\end{aligned}$$

Thus choosing $\alpha = Z/2$ we find that $(-\Delta - Z/r)\psi = E_0\psi$, with $E_0 = -Z^2/4$. From problem 2.(a) with $\Omega = \mathbb{R}^3 \setminus \{0\}$, which is clearly open, we then conclude that for all $\varphi \in \mathcal{C}_0^1(\mathbb{R}^3 \setminus \{0\})$ we have

$$\int_{\mathbb{R}^3\setminus\{0\}} |(\nabla\varphi)(x)|^2\,\mathrm{d}x - \int_{\mathbb{R}^3\setminus\{0\}} \frac{Z}{|x|}|\varphi(x)|^2\,\mathrm{d}x \geq E\int_{\mathbb{R}^3\setminus\{0\}} |\varphi(x)|^2\,\mathrm{d}x. \tag{2.7}$$

3 Lieb-Thirring inequalities in one dimension

We show that in one dimension a Lieb-Thirring inequality of the form

$$\sum_{j\geq 0} |E_j|^\gamma \leq L_\gamma \int_{\mathbb{R}} V_-(x)^{\gamma+1/2}\,\mathrm{d}x, \tag{3.1}$$

cannot hold for $0 \leq \gamma < 1/2$. We show this by contradiction.