Assignment 2, Functional Analysis

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Problem 1. *Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$ for all $N \geq 1$.*

(a) We show that the f_N converge weakly to 0 as $N \to \infty$ using the characterization of weak convergence of HW4 Problem 2. That is, we prove that for any functional $g\colon H \longrightarrow \mathbb{K}$, which by the Riesz representation theorem is of the form $g = \langle \cdot, x\rangle$ for some $x \in H$, we have that $\langle f_N, x\rangle$ converges weakly to 0 as $N \to \infty$. Look at

$$\lim_{N\to\infty} \langle f_N, x\rangle = \lim_{N\to\infty} N^{-1} \sum_{n=1}^{N^2} \langle e_n, x\rangle,$$

which is 0, because by Bessel's inequality $\sum_{n=1}^{\infty}\langle e_n, x\rangle^2 \leq \|x\|^2$ bounded, hence convergent since the terms are positive, and the product of two convergent sequences converges to the product of the limits. On the other hand, $\|f_N\| = 1$ because $\|\cdot\| \geq 0$ and

$$\|f_N\|^2 = \langle N^{-1}\sum_{n=1}^{N^2} e_n, N^{-1}\sum_{n=1}^{N^2} e_n\rangle = \frac{1}{N^2}\sum_{n,m=1}^{N^2} \delta_n^m = \frac{1}{N^2}N^2 = 1.$$

(b) *Notation*: $F = \{f_N \ : \ N \geq 1\}$ and $B = \{x \in H \ : \ \|x\| \leq 1\}$.

First of all, notice that since $\mathrm{co}(F)$ is convex, its norm closure equals its weak closure (Theorem 5.7), so K is closed in the weak topology. Also, $F \subseteq B$ by (a), where the later is convex, thus $\mathrm{co}(F) \subseteq B$; taking norm closure we get $K \subseteq B$. Thus, if we show that B is compact in the weak topology, we will obtain that K is also weakly compact, because closed subsets of compact ones are also compact. The norm closed unit ball B is weakly compact by Theorem 6.3, because H is reflexive, since it is a Hilbert space (Proposition 2.10).

The fact that $0 \in K$ follows because $(f_N)_{N\geq 1} \subseteq K$ converges weakly to 0, and we have just proven that K is weakly closed.

(c) Let $(e_i)_{i\geq 1}$ be an orthonormal basis for H. First, we show that each f_N is an extreme point of K. Suppose that we have $f_N = \alpha x_1 + (1-\alpha)x_2$ for some $0 < \alpha < 1$ and $x_1, x_2 \in K$. We must show that $x_1 = x_2 = f_N$. In part (a) we saw that $\|f_N\| = 1$, and it followed that $K \subseteq \overline{B}_H(0,1)$. Thus, we have

$$1 = \|f_N\| = \|\alpha x_1 + (1-\alpha)x_2\| \leq \alpha\|x_1\| + (1-\alpha)\|x_2\| \leq 1,$$

so both inequalities must be equalities. From the first of them we deduce that $\alpha x_1 = \lambda(1-\alpha)x_2$ for some $\lambda \in \mathbb{R}$, since it is only then that the triangular inequality is actually an equality; from the second we deduce that $\|x_1\| = \|x_2\| = 1$, because these norms are not greater than 1, as we have pointed out before. Putting both together, we get $\lambda = \pm\alpha/(1-\alpha) \in \mathbb{R}$, therefore $x_1 = \pm x_2$. Now we prove something that will allow us to conclude that $x_1 = x_2$.

Claim: If $x \in K$, then $\langle x, e_i \rangle \geq 0$ for every $i \geq 1$.

Proof. If $x \in K = \overline{\mathrm{co}(F)}$, then it is a limit of a convergent sequence in norm of points in the convex hull of F, i.e., $x = \lim_{k\to\infty} x^k$ with

$$x^k = \sum_{j=1}^{n} a_j f_{N_i}, \qquad \text{with } a_j > 0, \text{ and } \sum_{j=1}^{n} a_j = 1,$$

where, in fact, all indices and coefficients depend on k. Fix e_i arbitrary in the orthonormal basis, and notice that the limit comes out in what follows because $\langle \cdot, e_i \rangle \colon (H, \|\cdot\|) \longrightarrow \mathbb{K}$ is continuous:

$$\langle x, e_i \rangle = \lim_{k\to\infty} \langle x^k, e_i \rangle.$$

This quantity is non-negative since $\langle x^k, e_i \rangle$ is either $a_i > 0$ as above, or zero. $\square$

To finish, it remains to prove that 0 is also an extreme point of K; this makes sense because $0 \in K$ by part (b). Suppose that we have $0 = \alpha x_1 + (1-\alpha)x_2$ with $0 < \alpha < 1$. By the claim, we have $\langle x_1, e_i \rangle, \langle x_2, e_i \rangle \geq 0$ for every $i \geq 1$. Also, $\alpha, (1-\alpha) > 0$, so we deduce that $\langle x_1, e_i \rangle = \langle x_2, e_i \rangle = 0$ for all $i \geq 1$. Thus $x_1 = x_2 = 0$.

(d) Write $F = \{f_N\}_{N\geq 1}$, and notice that, since the norm closure agrees with the weak closure on convex sets such as $\mathrm{co}(F)$, we have that $K = \overline{\mathrm{co}(F)}^{w}$. The other hypothesis of the Milman theorem are also satisfied: (H, τ_w) is a LCTVS, K is non-empty, compact (in τ_w by (b)) and convex subset of H. By the mentioned theorem, we conclude that $\mathrm{Ext}(K) \subseteq \overline{F}^{w} = F \cup \{0\}$, so there are no other extreme points in K. The last equality holds because $F \cup \{0\}$ is compact and (H, τ_w) is Hausdorff (a convergent sequence whith its limit point, which is unique in a Hausdorff space, is compact. This is a standard proof with an oper cover argument), hence it is closed, while F is not because $0 \notin F$; recall that the closure is the smallest closed set which contains F.

Problem 2. *Let X and Y be infinite dimensional Banach spaces.*

(a) Let $T \in \mathcal{L}(X,Y)$, and let $(x_n)_{n\geq 1}$ be a sequence in X converging weakly to $x \in X$. We show that $(Tx_n)_{n\geq 1}$ converges weakly to Tx in Y, by using the characterization of HW4 Problem 2 (a). Thus, let $f \colon Y \longrightarrow \mathbb{K}$ be any functional, and now consider the functional $fT \colon X \longrightarrow \mathbb{K}$, obtained by precomposing with the bounded linear functional T. Then, using the characterization of weak convergence with $(x_n)_{n\geq 1}$, we obtain that $f(Tx_n)$ converges to $f(Tx)$ in $\mathbb{K}$.

(b) Let $T \in \mathcal{L}(X,Y)$ compact, and let $(x_n)_{n\geq 1}$ be a sequence in X converging weakly to $x \in X$. We show that $\|Tx_n - Tx\| \longrightarrow 0$, by proving that every subsequence of $(Tx_n)_{n\geq 1}$ admits a further subsequence which converges in norm to Tx.

Let $(Tx_{n_k})_{n\geq 1}$ be a subsequence; note that $(x_{n_k})_{k\geq 1}$ also converges weakly to x. By HW4 Problem 2(b), the sequence $(x_{n_k})_{k\geq 1}$ is bounded, so there exists a further subsequence $(x_{n_{k(i)}})_{i\geq 1}$ such that $(Tx_{n_{k(i)}})_{i\geq 1}$ converges in norm (in particular weakly) to some $y \in Y$, by the characterizations of compactness of T in Proposition 8.2. Finally, the limit point y of this sequence must be Tx, because $(Tx_n)_{n\geq 1}$ converges weakly to Tx by part (a), so also does $(Tx_{n_{k(i)}})_{i\geq 1}$; since (H, τ_w) is a Hausdorff topological space, the limit is unique, and we get $y = Tx$.

(c) We prove that T is compact using the characterization of Proposition 8.2. Let $(x_n)_{n\geq 1}$ be a bounded sequence in X, which by scaling if necessary, we may assume that it is contained in the unit ball $B_H(0,1)$. Since H is a Hilbert space, it is reflexive (Proposition 2.10), hence $\overline{B}_H(0,1)$ is compact with respect to the weak topology, by Theorem 6.3.

Claim: $\overline{B}_H(0,1)$ with the weak topology is metrizable.

Proof. Since H is separable, it follows that the Hilbert space H^* is separable. Indeed, the map $F\colon H \longrightarrow H^*$ defined by $y \mapsto \langle \cdot, y\rangle$ is onto, by the Riesz Representation Theorem, and continuous; indeed, it is actually an isometry, in particular continuous, because $\|F(y)\| = \|\langle \cdot, y\rangle\| \leq \|y\|$, by the Cauchy-Schwartz inequality after writing out the definition of the norm of the functional, and $\|y\|^2 = \langle y, y\rangle \leq \|F(y)\|\|y\|$, by boundedness of $F(y)$, implies $\|y\| \leq \|F(y)\|$. Thus, if A is a countable dense subset of H, then $F(A)$ is a countable dense subset of H^*, by continuity and surjectivity; indeed, $H^* = F(H) = F(\overline{A}) \subseteq \overline{F(A)} \subseteq H^*$.

Now, by Theorem 5.13, $\overline{B}_{(H^*)^*}(0,1)$ with the weak* topology is metrizable. Notice that the weak and weak* topology agree on $(H^*)^*$ because H^* is reflexive, by Theorem 5.9. The conclusion follows from reflexivity: $\Lambda\colon (H^*)^* \cong H$. □

Consequently, the compactness of $\overline{B}_H(0,1)$ is equivalent to its sequential compactness, so there exists a weakly convergent subsequence $(x_{n_k})_{k\geq 1}$. By the hypothesis, we get that $(Tx_{n_k})_{k\geq 1}$ converges in norm in Y. This proves that T is compact.

(d) Let $(x_n)_{n\geq 1}$ be a sequence converging to x weakly in X, and $T \in \mathcal{L}(l_2(\mathbb{N}), l_1(\mathbb{N}))$ be arbitrary. Then, by part (a), $(Tx_n)_{n\geq 1}$ converges weakly to Tx, or equivalently, it converges in norm to Tx, by Remark 5.3. Notice that $l_2(\mathbb{N})$ is a separable (HW4 P4) infinite Hilbert space, and that we have just shown that the conditions under which part (c) ensures that T is compact hold.

(e) Suppose that $T \in \mathcal{L}(X, Y)$ is onto. Then T is open by the Open Mapping Theorem. In particular, $TB_X(0,1)$ is open, so there exists an open ball $B_Y(0,r)$, for small enough $r > 0$, centered at $T(0) = 0$, and which is contained in $TB_X(0,1)$. Thus, we have

$$\overline{B_Y(0,r)} \subseteq \overline{TB_X(0,1)}.$$

Now, notice that $\overline{B_Y(0,r)}$ is not compact by Problem 3(e) in the first assignment, because Y is infinite dimensional and the mentioned set is homeomorphic to the closed unit ball by scaling.

If T were compact, i.e., $\overline{TB_X(0,1)}$ compact, then $\overline{B_Y(0,r)}$ would also be compact, because it is a closed subset. This is a contradiction, hence T is not compact.

(f) M is self-adjoint:

$$\begin{aligned}\langle Mf, g\rangle &= \int_{[0,1]} (tf(t))\overline{g(t)}dm(t) \\ &= \int_{[0,1]} f(t)(\overline{tg(t)})dm(t) \\ &= \langle f, Mg\rangle.\end{aligned}$$

Suppose that M is compact. Since we have just shown that M is self-adjoint, and $L_2([0,1],m)$ is a separable (HW4 P4) infinite dimensional Hilbert space, the Spectral Theorem ensures the existence of infinitely many eigenvalues for M. However, we showed in HW6 Problem 3 that M has no eigenvalues. We conclude that M is not compact.

Problem 3. *Consider the Hilbert space $H = L_2([0,1],m)$, where m is the Lebesgue measure.*

$T = T_{\tilde{K}}$ for $\tilde{K}(s,t) = K(t,s) = K(s,t)$

(a) The operator T is the kernel operator associated to K, and it is compact by Theorem 9.6. The hypothesis are satisfied because K is continuous on $[0,1]\times[0,1]$ (it is clearly piece-wise continuous and these agree on the intersection), the topological space $[0,1]$ is compact and Hausdorff, and the Lebesgue measure is a finite Borel measure on $[0,1]$.

(b) The operator T is self-adjoint:

$$\begin{aligned}\langle Tf, g\rangle &= \int_{[0,1]} \left(\int_{[0,1]} K(s,t)f(t)dm(t)\right)\overline{g(s)}dm(s) \\ &= \int_{[0,1]}\int_{[0,1]} f(t)\overline{K(s,t)g(s)}dm(s)dm(t) \\ &= \int_{[0,1]} f(t)\overline{\int_{[0,1]} K(s,t)g(s)dm(s)}dm(t) \\ &= \langle f, Tg\rangle,\end{aligned}$$

where in the second equality we first pulled $\overline{g(s)}$ inside, then we applied Fubini's Theorem, and finally we have used that $K(s,t) = \overline{K(s,t)}$ because it is real valued. The hypothesis of Fubini's Theorem are satisfied because the Lebesgue measure is σ-finite on $[0,1]$, and the product $K(s,t)f(t)\overline{g(s)}$ is integrable on $[0,1]\times[0,1]$; this holds because $K(s,t)\leq M$ is bounded on $[0,1]\times[0,1]$, because K is continuous and the domain is compact, so we have

$$\begin{aligned}\int_{[0,1]\times[0,1]} |K(s,t)f(t)\overline{g(s)}|dm(s,t) &\leq M\int_{[0,1]\times[0,1]} |f(t)||g(s)|dm(s,t) \\ &= \|f\|_1\|g\|_1 < \infty.\end{aligned}$$

In the last equality we have used that $|f(t)|$ and $|g(s)|$ are non-negative, and measurable functions because $f,g\in L_2([0,1],m)$, so applying Tonelli and using that each is independent of the other variable, we get what we wrote. The $\|\cdot\|_1$-norms are finite because $L_2([0,1],m)\subseteq L_1([0,1],m)$, by HW2 Problem 2. ✓

(c) The stated formula for $(Tf)(s)$ is obtained directly from the definition of K: for a fixed s, the function $K(s,t)$ is given by $(1-s)t$ for t in the interval than you need to show $f \in L_2([0,1]) \Rightarrow tf \in L_2([0,1])$

$[0, s]$, and $s(1-t)$ for t in the interval $[s, 1]$; to get Tf we integrate over t, so $(1-s)$ and s come out of the integral. write this out.

With $(Tf)(s)$ expressed in this way, we see that it is continuous because it is a product and sum of continuous functions. Indeed, the integrals are continuous in s. The argument for this, which works in more generality, is as follows. Let $h \in L_2([0,1], m)$, and we want to see that $H(s) := \int_{[0,s]} h dm$ is continuous in $s \in [0, 1]$. Consider any convergent sequence $(s_n)_{n \geq 1}$ to a fixed s_0, and we show that the $H(s_n)$ converge to $H(s_0)$. First, we consider every $H(s_n) = \int_{[0,1]} h \cdot \chi_{[0,s_n]} dm$ over the same measure space. Then, Lebesgue Dominated Convergence Theorem gives the result, since pointwise convergence is clear, and each function is dominated by $|h|$, which is integrable. why? $h \in L_2([0,1])$

It is clear that $(Tf)(0) = (Tf)(1) = 0$ because in both cases one of the integrals gets multiplied by zero and the other integrals is over a point, a set of measure zero. write this out!

Problem 4. *Consider the Schwartz space $\mathcal{S}(\mathbb{R})$.*

(a) In HW7 Problem 1 we first saw that $e^{-x^2} \in \mathcal{S}(\mathbb{R})$, and then, by part (d) in that problem, $S_{\sqrt{2}}(e^{-x^2}) = e^{-x^2/2} \in \mathcal{S}(\mathbb{R})$, and finally, by part (a) in the same, we get $x^k e^{-x^2/2} \in \mathcal{S}(\mathbb{R})$ for all $k \geq 0$.

Write $\varphi(x) := e^{-x^2/2}$ and $g_k := x^k \varphi$. Let us compute $\mathcal{F}(g_k)$. Notice that each g_k is in $L_1(\mathbb{R})$, by problem $1(c)$ in HW7. Hence, by Proposition 11.13, we have

$$\mathcal{F}(g_k) = i^k(\partial^k \hat{\varphi}).$$

This gives the following, using that $\mathcal{F}(\varphi) = \varphi$ by Proposition 11.4:

$$\begin{aligned}
\mathcal{F}(g_0) &= \varphi, \\
\mathcal{F}(g_1) &= -i\xi\varphi, \\
\mathcal{F}(g_2) &= i^2(-\varphi + \xi^2\varphi) = \varphi - \xi^2\varphi, \\
\mathcal{F}(g_3) &= i^3(\xi\varphi + 2\xi\varphi - \xi^3\varphi) = i(\xi^3\varphi - 3\xi\varphi).
\end{aligned}$$

✓

(b) We write the computations of part (a) as follows:

$$\begin{aligned}
\mathcal{F}(g_0) &= g_0, \\
\mathcal{F}(g_1) &= -ig_1, \\
\mathcal{F}(g_2) &= g_0 - g_2, \\
\mathcal{F}(g_3) &= ig_3 - 3ig_1.
\end{aligned}$$

We can already see that we can put $h_0 = g_0$ and $h_3 = g_1$. For $k = 1, 2$, we set up a system of linear equations by imposing

$$\mathcal{F}(ag_1 + bg_2 + cg_3 + dg_4) = i^k(ag_1 + bg_2 + cg_3 + dg_4),$$

(use linearity of $\mathcal{F}$ to develop the left hand side expression). All in all, we get the following:

$$\begin{aligned} h_0 &= g_0, \\ h_1 &= -\frac{3}{2}g_1 + g_3, \\ h_2 &= -\frac{1}{2}g_0 + g_2, \\ h_3 &= g_1. \end{aligned}$$

(c) Let $f \in \mathcal{S}(\mathbb{R})$. We prove that $\mathcal{F}^4(f) = f$. First of all, notice that $\mathcal{F}(f)$ is also a Schartz function by Proposition 11.13 (e), so it makes sense to iterate $\mathcal{F}$. Our claim follows from the fact that $\mathcal{F}^2(f)(x) = f(-x)$ for all $x \in \mathbb{R}$, because then applying $\mathcal{F}^2$ again gives back f. Proof of the fact:

$$\begin{aligned} \mathcal{F}^2(f)(x) &= \int_{\mathbb{R}} \hat{f}(\xi)e^{-i\langle\xi,x\rangle}dm(\xi) \\ &= \int_{\mathbb{R}} \hat{f}(\xi)e^{i\langle\xi,-x\rangle}dm(\xi) \\ &= \mathcal{F}^*\mathcal{F}(f)(-x) \\ &= f(-x), \end{aligned}$$

where in the last step we have used that $\mathcal{F}$ and $\mathcal{F}^*$ are mutual inverses on $\mathcal{S}(\mathbb{R})$ (Corollary 12.12).

(d) Suppose that $f \in \mathcal{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for $\lambda \in \mathbb{C}$. Then, $\mathcal{F}^4(f) = \lambda^4 f$ by linearity, and by part (c) we get $\lambda^4 = 1$. The solutions to the later equation are $\lambda = 1, i, -1, -i$. This does not show that they are in fact eigenvalues.

Problem 5. *Let (x_n) be a dense subset of $[0,1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0,1]$.*

By definition (HW3 Problem 3), $\mathrm{supp}(\mu)$ is the complement of the union of all open sets of $[0,1]$ on which μ vanishes. Since $(x_n)_{n\geq 1}$ is dense in $[0,1]$, any non-empty open set U in $[0,1]$ contains some x_i, thus $2^{-i} = \delta_{x_i}(U) \leq \mu(U)$, since all the summands are positive. So μ only vanishes on the empty set, thus $\mathrm{supp}(\mu) = [0,1]$.