

FUNCTIONAL ANALYSIS

Mandatory Assignment 2

Jonas Uglebjerg (krc974)

Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n \geq 1}$. Set $f_N = N^{-1} \sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.

(a) To show that $f_N \rightarrow 0$ weakly, as $N \rightarrow \infty$, while $\|f_N\| = 1$, for all $N \geq 1$, we want to show that

$$\langle f_N, y \rangle \rightarrow \langle 0, y \rangle = 0, \quad \forall y \in H$$

We can write y as a linear combination of the basis

$$y = \sum_i^{\infty} \alpha_i e_i$$

and now we can calculate $\langle f_N, y \rangle$ by linearity of the first coordinate in the inner product:

$$\begin{aligned} \langle f_N, y \rangle &= \left\langle \frac{1}{N} (e_1 + e_2 + \dots + e_{N^2}), \alpha_1 e_1 + \alpha_2 e_2 + \dots \right\rangle \\ &= \frac{1}{N} \left(\langle e_1, \alpha_1 e_1 + \alpha_2 e_2 + \dots \rangle + \langle e_2, \alpha_1 e_1 + \alpha_2 e_2 + \dots \rangle + \dots + \langle e_{N^2}, \alpha_1 e_1 + \alpha_2 e_2 + \dots \rangle \right) \\ &= \frac{1}{N} \left(\overline{\alpha_1} \langle e_1, e_1 \rangle + \overline{\alpha_1} \langle e_2, e_1 \rangle + \dots + \overline{\alpha_1} \langle e_{N^2}, e_1 \rangle + \overline{\alpha_2} \langle e_1, e_2 \rangle + \dots + \overline{\alpha_2} \langle e_{N^2}, e_2 \rangle + \dots \right) \end{aligned}$$

We know that $\langle e_i, e_j \rangle = 0$ when $i \neq j$ and $\langle e_i, e_j \rangle = 1$ when $i = j$, since $(e_n)_{n \geq 1}$ is a orthonormal basis, which means:

$$\begin{aligned} \langle f_N, y \rangle &= \frac{1}{N} \left(\overline{\alpha_1} \langle e_1, e_1 \rangle + \overline{\alpha_1} \langle e_2, e_1 \rangle + \dots + \overline{\alpha_1} \langle e_{N^2}, e_1 \rangle + \overline{\alpha_2} \langle e_1, e_2 \rangle + \dots + \overline{\alpha_2} \langle e_{N^2}, e_2 \rangle + \dots \right) \\ &= \frac{1}{N} \left(\overline{\alpha_1} \langle e_1, e_1 \rangle + \overline{\alpha_2} \langle e_2, e_2 \rangle + \dots + \overline{\alpha_{N^2}} \langle e_{N^2}, e_{N^2} \rangle \right) = \sum_{i=1}^{N^2} \frac{\overline{\alpha_i}}{N} \end{aligned}$$

It is clear that $\sum_{i=1}^{N^2} \frac{\overline{\alpha_i}}{N} \rightarrow 0$ as $N \rightarrow \infty$, since $\frac{\overline{\alpha_i}}{N} \rightarrow 0$ as $N \rightarrow \infty$ for all $i \in \mathbb{N}$. Therefore, it is shown that $f_N \rightarrow 0$ weakly, as $N \rightarrow \infty$.

To show $\|f_N\| = 1$ for all $N \geq 1$ we once again use that the inner product of the bases is 1 or 0.

$$\begin{aligned} \|f_N\| &= \sqrt{\langle f_N, f_N \rangle} = \sqrt{\left\langle \frac{1}{N} (e_1 + \dots + e_{N^2}), \frac{1}{N} (e_1 + \dots + e_{N^2}) \right\rangle} \\ &= \sqrt{\frac{1}{N^2} \langle e_1 + \dots + e_{N^2}, e_1 + \dots + e_{N^2} \rangle} \\ &= \sqrt{\frac{1}{N^2} (\langle e_1, e_1 \rangle + \dots + \langle e_{N^2}, e_{N^2} \rangle)} = \sqrt{\frac{1}{N^2} \cdot N^2} = \sqrt{1} = 1 \end{aligned}$$

(b) Let K be the norm closure of $\text{co}\{f_N : N \geq 1\}$. To argue that K is weakly compact, and that $0 \in K$, let $M = \text{co}\{f_N : N \geq 1\}$, which means $K = \overline{M}^{\|\cdot\|}$. Since M is convex, according to definition 7.7 (Musats notes), we use theorem 5.7 to conclude $K = \overline{M}^{\|\cdot\|} = \overline{M}^{\mathcal{T}_w}$. Let $x \in M$, which means that $x = \sum_{i=1}^n \alpha_i f_{N_i}$, $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$. The following shows that K is bounded, remember from part (a) that $\|f_N\| = 1$:

$$\left\| \sum_{i=1}^n \alpha_i f_{N_i} \right\| = \left\| \alpha_1 f_{N_1} + \dots + \alpha_n f_{N_n} \right\| \leq \alpha_1 \|f_{N_1}\| + \dots + \alpha_n \|f_{N_n}\| = \sum_{i=1}^n \alpha_i = 1$$

which means that if $x \in M$, then $\|x\| \leq 1$, and then by closure $x \in K$ makes $\|x\| \leq 1$, and hence K is bounded. Since K is a bounded, convex set of H , which is a reflexive Banach space (because H is a Hilbert space), K is weakly compact.

Furthermore, we want to show that $0 \in K$. This follows by part (a). Since $f_N \rightarrow 0$ weakly, as $N \rightarrow \infty$, and all $f_N \in M$ and $K = \overline{M}^{\mathcal{T}_w}$ it must follow that $0 \in K$.

(c) To show that 0, as well as each f_N , $N \geq 1$, are extreme points in K , note that $f_N = N^{-1} \sum_{n=1}^{N^2} e_i$ only have non-negative coordinates. If we look at the convex hull

$$\text{co}\{f_N : N \geq 1\} = \left\{ \sum_{i=1}^n \alpha_i f_{N_i} : \alpha_i > 0, \sum_{i=1}^n \alpha_i = 1, n \in \mathbb{N} \right\}$$

it is clear that the elements of $\text{co}\{f_N : N \geq 1\}$ only have non-negative coordinates, too. Although we look at the closure of the convex hull, there are still only non-negative coordinates. This means if $x \in K \subset H$, then x only have non-negative coordinates. Let $x = 0 \in K$, and let $0 < \alpha < 1$ such that $\alpha x_1 + (1 - \alpha)x_2 = x$. Look at the i 'th coordinate

$$\alpha x_{1,i} + (1 - \alpha)x_{2,i} = x_i = 0$$

Since $\alpha > 0$ and $(1 - \alpha) > 0$ one of $x_{1,i}$ or $x_{2,i}$ must be negative or $x_{1,i} = x_{2,i} = 0$. Since $x_{1,i}$ and $x_{2,i}$ can not be negative, it must hold that $x_{1,i} = x_{2,i} = 0$, which according to definition 7.1 (Musats notes) means that 0 is an extreme point.

To show that f_N are extreme points remember from part (b) that if $x \in K$, then $\|x\| \leq 1$. Look at $x = f_N \in K$, where $N \geq 1$, that means $\|x\| = \|f_N\| = 1$. Let $0 < \alpha < 1$ and $x_1, x_2 \in K$, which means $\|x_1\| \leq 1$ and $\|x_2\| \leq 1$. Let $x = \alpha x_1 + (1 - \alpha)x_2$, then the following must hold

$$\|x\| = \|\alpha x_1 + (1 - \alpha)x_2\| \leq \alpha \|x_1\| + \|x_2\| - \alpha \|x_2\| = \alpha(\|x_1\| - \|x_2\|) + \|x_2\|$$

Since $\|x\| = 1$ and $\alpha > 0$ it is easily seen that $\|x_1\| \geq \|x_2\|$, because otherwise we will get that $1 = \|x\| < 1$. Since x_1 and x_2 can switch places, we can conclude that $\|x_1\| = \|x_2\|$. This means that

$$1 = \|x\| \leq \|x_2\| \leq 1$$

so $\|x\| = \|x_1\| = \|x_2\| = 1$. This means $x, x_1, x_2 \in \partial B(0, 1)$. This means that $x = \alpha x_1 + (1 - \alpha)x_2$ only is possible if $x_1 = x_2 = x$.

Problem 2

Let X and Y be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X, Y)$. To show that $x : n \rightarrow x$ weakly, as $n \rightarrow \infty$, implies that $Tx_n \rightarrow Tx$ weakly, as $n \rightarrow \infty$, for a sequence $(x_n)_{n \geq 1}$ in X and $x \in X$, we use that $Tx_n \rightarrow Tx$ weakly if and only if $g(Tx_n) \rightarrow g(Tx)$ for all $g \in Y^*$. Since $g \in Y^*$ we know that g is a bounded linear function, just like T . Hence

$$\begin{aligned} |g(Tx_n) - g(Tx)| &= |g(Tx_n - Tx)| \\ &\leq \|g\|_{Y^*} \|Tx_n - Tx\|_Y \\ &= \|g\|_{Y^*} \|T(x_n - x)\|_Y \\ &\leq \|g\|_{Y^*} \|T\| \|x_n - x\|_X \end{aligned}$$

Since $x_n \rightarrow x$ weakly, we know that $|f(x_n) - f(x)| \rightarrow 0$, $\forall f \in X^*$. Therefore, we can conclude that

$$\|x_n - x\|_X = \sup_{f \in X^* \setminus \{0\}} \left(\frac{|f(x_n - x)|}{\|f\|_{X^*}} \right)$$

Choose $f \in X^*$ such that $\|f\|_{X^*} = 1$. For a given $\varepsilon > 0$, the following must hold

$$\|x_n - x\|_X < \frac{|f(x_n - x)|}{\|f\|_{X^*}} + \frac{\varepsilon}{2\|T\|} = |f(x_n - x)| + \frac{\varepsilon}{2\|T\|}$$

Since $x_n \rightarrow x$ weakly, there exists $N \in \mathbb{N}$ such that

$$|f(x_n) - f(x)| = |f(x_n - x)| < \frac{\varepsilon}{2\|T\|} \quad \text{for } n > N$$

All this gives us:

$$\begin{aligned} |g(Tx_n) - g(Tx)| &\leq \|g\|_{Y^*} \|T\| \|x_n - x\|_X \\ &< \|g\|_{Y^*} \|T\| \left(|f(x_n - x)| + \frac{\varepsilon}{2\|T\|} \right) \\ &< \|g\|_{Y^*} \|T\| \left(\frac{\varepsilon}{2\|T\|} + \frac{\varepsilon}{2\|T\|} \right) \\ &= \|g\|_{Y^*} \varepsilon \end{aligned}$$

Since g is bounded one could choose $\varepsilon' = \frac{\varepsilon}{\|g\|_{Y^*}}$ and it is clear that $Tx_n \rightarrow Tx$ weakly

(b) Let $T \in \mathcal{K}(X, Y)$, which means that T is compact. To show that $x_n \rightarrow x$ weakly, as $n \rightarrow \infty$, implies that $\|Tx_n - Tx\| \rightarrow 0$, as $n \rightarrow \infty$, for a sequence $(x_n)_{n \geq 1}$ in X and $x \in X$, let $(x_n)_{n \geq 1}$ be a sequence in X and suppose $x_n \rightarrow x$ weakly, as $n \rightarrow \infty$. For contradiction suppose that $\|Tx_n - Tx\| \not\rightarrow 0$ as $n \rightarrow \infty$. This means that for $\varepsilon > 0$ there exists a subsequence $(x_{n_k})_{k \geq 1}$ such that

$$\|Tx_{n_k} - Tx\| > \varepsilon \quad \text{for all } k \geq 1$$

Because $x_n \rightarrow x$ weakly as $n \rightarrow \infty$, we know that $x_{n_k} \rightarrow x$ weakly as $k \rightarrow \infty$. Using proposition 8.2 (Musat's notes) and the compactness of T and the fact that $(x_{n_k})_{k \geq 1}$ is bounded, we know that there exists a subsequence such that $\|Tx_{n_{k_i}} - Tx'\| \rightarrow 0$ as $i \rightarrow \infty$ for some $x' \in X$. Because $x_{n_k} \rightarrow x$ weakly as $k \rightarrow \infty$, we know from part (a) that $Tx_{n_k} \rightarrow Tx$ weakly as $k \rightarrow \infty$, and subsequently $Tx_{n_{k_i}} \rightarrow Tx$ weakly as $i \rightarrow \infty$.

If a sequence converges by norm to something, it must converge weakly to the same. This is true since if $(y_n)_{n \geq 1}$ is a sequence in Y and $\|y_n - y\| \rightarrow 0$ as $n \rightarrow \infty$, then let $g \in Y^*$ then

$$|g(y_n) - g(y)| = |g(y_n - y)| \leq \|g\| \|y_n - y\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

because $\|y_n - y\| \rightarrow 0$. So $y_n \rightarrow y$ weakly as $n \rightarrow \infty$.

So when $\|Tx_{n_{k_i}} - Tx'\| \rightarrow 0$ as $i \rightarrow \infty$ and $Tx_{n_{k_i}} \rightarrow Tx$ weakly as $i \rightarrow \infty$ we can conclude that $Tx = Tx'$. This means that $\|Tx_{n_{k_i}} - Tx\| \rightarrow 0$ as $i \rightarrow \infty$ but this contradicts that fact that $\|Tx_{n_{k_i}} - Tx\| > \varepsilon$ for all $k \geq 1$ and therefore

$$\|Tx_n - Tx\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

(c) Let H be a separable infinite dimensional Hilbert space. To show that if $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \rightarrow 0$, as $n \rightarrow \infty$, whenever $(x_n)_{n \geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$, we have to prove that T is compact. Suppose T is not compact for contradiction. If T is not compact then $T(\overline{B_H(0, 1)})$ is not totally bounded, according to proposition 8.2 (Musat's notes). This means that there exists $\delta > 0$ such that every finite union of open balls with radius δ does not cover $T(\overline{B_H(0, 1)})$.

Now we define a sequence $(x_n)_{n \geq 1}$. Let's start by chosen $x_1 \in \overline{B_H(0, 1)}$ at random. Then $B_Y(Tx_1, \delta)$ will not cover $T(\overline{B_H(0, 1)})$ because it is not totally bounded. Now choose $x_2 \in \overline{B_H(0, 1)}$ such that $Tx_2 \in (B_Y(Tx_1, \delta))^c$. Again $\bigcup_{i=1}^2 (B_Y(Tx_i, \delta))$ will not cover $T(\overline{B_H(0, 1)})$ and so forth. Let $x_{n+1} \in \overline{B_H(0, 1)}$ such that $Tx_{n+1} \in (\bigcup_{i=1}^n B_Y(Tx_i, \delta))^c$. For this constructed sequence $(x_n)_{n \geq 1}$ we know that $\|Tx_n - Tx_m\| \geq \delta$ for $n \neq m$.

Since H is a separable Hilbert space, then so is the dual space H^* . By theorem 6.1 (Musat's notes) the closed unit ball $\overline{B_{H^{**}}(0, 1)}$ is compact in the w^* -topology. By theorem 5.13 (Musat's notes) $(\overline{B_{H^{**}}(0, 1)}, \tau_{w^*})$ is metrizable. This means $\overline{B_{H^{**}}(0, 1)}$ is compact in the w^* -topology, and sequences in $\overline{B_{H^{**}}(0, 1)}$ will have a converging subsequence. If we consider a sequence $(z_n)_{n \geq 1}$ in $\overline{B_H(0, 1)}$, then $(\hat{z})_{n \geq 1}$ will be a corresponding sequence in $\overline{B_{H^{**}}(0, 1)}$. $(\hat{z})_{n \geq 1}$ will have a converging subsequence $(\hat{z}_{n_k})_{n \geq 1}$ in $\overline{B_{H^{**}}(0, 1)}$ as $k \rightarrow \infty$ in the w^* -topology.

Let $f \in H^*$ then $f(z_{n_k}) = \hat{z}_{n_k} f \rightarrow \hat{z} f = f(z)$ as $k \rightarrow \infty$. This means that all sequences in $\overline{B_H(0, 1)}$ must converge weakly, including the subsequence of the previously constructed sequence, that is $(x_{n_k})_{k \geq 1}$ as $k \rightarrow \infty$. Since $x_{n_k} \rightarrow x$ weakly as $k \rightarrow \infty$ then $\|Tx_{n_k} - Tx\| \rightarrow 0$ by assumption. But since $\|Tx_m - Tx_n\| \geq \delta$ for $m \neq n$ we also know that $\|Tx_{n_k} - Tx\| \not\rightarrow 0$ as $k \rightarrow \infty$. This contradiction shows that T is compact.

(d) To show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact note that $\ell_2(\mathbb{N})$ and $\ell_1(\mathbb{N})$ are Banach spaces, so the requirements in part (a) are fulfilled. So if we have a sequence $(x_n)_{n \geq 1}$ in $\ell_2(\mathbb{N})$ and $x_n \rightarrow x \in \ell_2(\mathbb{N})$ weakly as $n \rightarrow \infty$, then $Tx_n \rightarrow Tx$ weakly as $n \rightarrow \infty$ in $\ell_1(\mathbb{N})$. According to remark 5.3 (Musat's notes) a sequence converges weakly in $\ell_1(\mathbb{N})$ if and only if it converges in norm. This means that

$$\|Tx_n - Tx\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Thus by part (c) T is compact, as $\ell_2(\mathbb{N})$ is a (separable) Hilbert space according to page 3 in Musat's notes).

(e) To show that no $T \in \mathcal{K}(X, Y)$ is onto, suppose for contradiction that T is onto. According to theorem 3.15 (Musat's notes) then T is open. As X and Y are normed vector spaces and T is open then there exists $r > 0$, such that $B_Y(0, r) \subset T(B_X(0, 1))$, according to page 18 in Musat's notes. Since the closure preserves inclusion $\overline{B_Y(0, r)} \subset \overline{T(B_X(0, 1))}$. We know that $\overline{B_Y(0, r)} = r\overline{B_Y(0, 1)}$. From problem 3e in mandatory assignment 1 we know that a closed unit-ball in an infinite dimensional vector space is not compact. This means that $\overline{B_Y(0, 1)}$ is not compact, and then $r\overline{B_Y(0, 1)}$ is not compact. But at the same time $r\overline{B_Y(0, 1)}$ is a closed subset of $\overline{T(B_X(0, 1))}$ and $\overline{T(B_X(0, 1))}$ is compact because T is a compact operator. This implies that $r\overline{B_Y(0, 1)}$ is compact. This is a contradiction and hence no $T \in \mathcal{K}(X, Y)$ can be onto.

(f) Let $H = L_2([0, 1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0, 1]$. To justify that M is self-adjoint, we need to show that $\langle Mf, g \rangle = \langle f, Mg \rangle$ for all $f, g \in H$. Let $t \in [0, 1]$ which means that $t = \overline{t}$, then

$$\begin{aligned} \langle Mf, g \rangle &= \int_{[0, 1]} Mf(t) \overline{g(t)} dm(t) \\ &= \int_{[0, 1]} tf(t) \overline{g(t)} dm(t) \\ &= \int_{[0, 1]} f(t) \overline{tg(t)} dm(t) \\ &= \int_{[0, 1]} f(t) \overline{tg(t)} dm(t) \\ &= \int_{[0, 1]} f(t) \overline{Mg(t)} dm(t) \\ &= \langle f, Mg \rangle \end{aligned}$$

So $M = M^*$ and M is self-adjoint.

To justify that M is not compact, we suppose M is compact for contradiction. From HW4 (problem 4) we know that H is separable. Since H is separable, infinite-dimensional Hilbert space and M is self-adjoint and assumed compact, then by theorem 10.1 (Musat's notes) H has an orthonormal basis consisting of eigenvalues $\lambda_n \in \mathbb{R}$. But in HW6 (problem 3) we showed that M has no eigenvalues. Here is the contradiction, that makes M non compact.

Problem 3

Consider the Hilbert space $H = L_2([0, 1], m)$, where m is the Lebesgue measure. Define $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ by

$$K(s, t) = \begin{cases} (1-s)t, & \text{if } 0 \leq t \leq s \leq 1, \\ (1-t)s, & \text{if } 0 \leq s \leq t \leq 1, \end{cases}$$

and consider $T \in \mathcal{L}(H, H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s, t)f(t)dm(t), \quad s \in [0, 1], \quad f \in H.$$

(a) To justify that T is compact, we just have to use proposition 9.12 (Musat's notes), where $X = Y = [0, 1]$ and $\mu = \nu = m$.

This needs a lot of elaboration!

(b) To show that $T = T^*$ we have to show that $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all $f, g \in H$. Notice that $K(s, t) = K(t, s)$

$$\begin{aligned} \langle Tf, g \rangle &= \int_{[0,1]} Tf(s)\overline{g(s)}dm(s) \\ &= \int_{[0,1]} \int_{[0,1]} K(s, t)f(t)dm(t)\overline{g(s)}dm(s) \\ &= \int_{[0,1]} \int_{[0,1]} f(t)K(t, s)dm(t)\overline{g(s)}dm(t)dm(s) \\ &= \int_{[0,1]} \int_{[0,1]} f(t)\overline{K(t, s)g(s)}dm(t)dm(s) \\ &\stackrel{\text{(switch the integrals)}}{=} \int_{[0,1]} \int_{[0,1]} f(t)\overline{K(t, s)g(s)}dm(s)dm(t) \\ &= \int_{[0,1]} f(t) \int_{[0,1]} \overline{K(t, s)g(s)}dm(s)dm(t) \\ &= \int_{[0,1]} f(t)\overline{Tg(t)}dm(t) \\ &= \langle f, Tg \rangle \end{aligned}$$

← real

*By what thm?
and why justified*

Hence $T = T^*$.

(c) To show that

$$(Tf)(s) = (1-s) \int_{[0,s]} tf(t)dm(t) + s \int_{[s,1]} (1-t)f(t)dm(t), \quad s \in [0,1], \quad f \in H$$

split up the integral

$$\begin{aligned} Tf(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\ &= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\ &= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\ &= (1-s) \int_{[0,s]} tf(t)dm(t) + s \int_{[s,1]} (1-t)f(t)dm(t) \end{aligned}$$

To show that Tf is continuous on $[0,1]$ remember that $f \in H$ is continuous. It is clear that $(1-s)$ and s are continuous and so are the integrals. *why?*

To show that $(Tf)(0) = (Tf)(1) = 0$ look at the following calculation:

$$\begin{aligned} (Tf)(0) &= (1-0) \int_{[0,0]} tf(t)dm(t) + 0 \int_{[0,1]} (1-t)f(t)dm(t) \\ &= \int_{[0,0]} tf(t)dm(t) = 0 \end{aligned}$$

$$\begin{aligned} (Tf)(1) &= (1-1) \int_{[0,1]} tf(t)dm(t) + 1 \int_{[1,1]} (1-t)f(t)dm(t) \\ &= \int_{[1,1]} (1-t)f(t)dm(t) = 0 \end{aligned}$$

This holds since we integrate over singletons $[0,0] = \{0\}$ and $[1,1] = \{1\}$.

Problem 4

Consider the Schwartz space $\mathcal{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$.

(a) For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$.

To justify that $g_k \in \mathcal{S}(\mathbb{R})$, for all integers $k \geq 0$, we use definition 11.10 (Musat's notes). First notice that $g_k \in C^\infty(\mathbb{R})$ for all $k \geq 0$. Secondly we have to show that

$$\lim_{\|x\| \rightarrow \infty} x^\beta \partial^\alpha g_k(x) = 0$$

for all multi-indices α, β . Look at the following calculations, first where $\alpha = 1$:

$$\frac{\partial}{\partial x} g_k(x) = kx^{k-1}e^{-\frac{x^2}{2}} + x^k \cdot (-x)e^{-\frac{x^2}{2}} = (kx^{k-1} - x^{k+1})e^{-\frac{x^2}{2}}$$

It is clear that, if we continue with the differentiations, we will get a polynomial for every $\alpha \in \mathbb{N}$ like this

$$\frac{\partial^\alpha}{\partial x^\alpha} g_k(x) = \text{Pol}(x) \cdot e^{-\frac{x^2}{2}}$$

which means that

$$x^\beta \frac{\partial^\alpha}{\partial x^\alpha} g_k(x) = \text{Pol}(x) \cdot e^{-\frac{x^2}{2}}$$

We know from previous courses that

$$\lim_{\|x\| \rightarrow \infty} (\text{Pol}(x) \cdot e^{-\frac{x^2}{2}}) = 0$$

and now it is shown that $g_k \in \mathcal{S}(\mathbb{R})$.

To compute the Fourier transform of g_0, g_1, g_2 and g_3 we start with g_0 . Note that $g_0(x) = e^{-\frac{x^2}{2}}$. By proposition 11.4 (Musat's notes) we get $\mathcal{F}(g_0) = \hat{g}_0(\xi) = e^{-\frac{\xi^2}{2}}$. Looking at $g_1(x) = xe^{-\frac{x^2}{2}}$, and using that $e^{-\frac{x^2}{2}}$ and $\sin(x\xi)$ are even, and $\cos(x\xi)$ and x are odd, we know that

$$\begin{aligned} \mathcal{F}(g_1) = \hat{g}_1(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} xe^{-\frac{x^2}{2}} \cos(x\xi) dx - \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} xe^{-\frac{x^2}{2}} \sin(x\xi) dx \\ &= -\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} xe^{-\frac{x^2}{2}} \sin(x\xi) dx \\ &= -\frac{2i}{\sqrt{2\pi}} \int_0^\infty xe^{-\frac{x^2}{2}} \sin(x\xi) dx \\ &= -\frac{2i}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} \xi e^{-\frac{\xi^2}{2}} \\ &= -i\xi e^{-\frac{\xi^2}{2}} \end{aligned}$$

Now we do the same form of calculation for the Fourier transform for g_2 and g_3 . Remember that x^2 is even and x^3 is odd:

$$\begin{aligned} \mathcal{F}(g_2) = \hat{g}_2(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} \cos(x\xi) dx - \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} \sin(x\xi) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} \cos(x\xi) dx \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-\frac{x^2}{2}} \cos(x\xi) dx \\ &= \frac{2}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} (1 - \xi^2) e^{-\frac{\xi^2}{2}} \\ &= (1 - \xi^2) e^{-\frac{\xi^2}{2}} \end{aligned}$$

$$\begin{aligned} \mathcal{F}(g_3) = \hat{g}_3(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^3 e^{-\frac{x^2}{2}} \cos(x\xi) dx - \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x^3 e^{-\frac{x^2}{2}} \sin(x\xi) dx \\ &= -\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x^3 e^{-\frac{x^2}{2}} \sin(x\xi) dx \\ &= -\frac{2i}{\sqrt{2\pi}} \int_0^\infty x^3 e^{-\frac{x^2}{2}} \sin(x\xi) dx \\ &= -\frac{2i}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} (3\xi - \xi^3) e^{-\frac{\xi^2}{2}} \\ &= -i(3\xi - \xi^3) e^{-\frac{\xi^2}{2}} \end{aligned}$$

(b) To find non-zero functions $h_k \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$ we first have to find a h_0 , such that $\mathcal{F}(h_0) = h_0$. If we let $h_0 = g_0$ then

$$\mathcal{F}(h_0) = \mathcal{F}(g_0) = e^{-\frac{x^2}{2}} = g_0 = h_0$$

and it is clear this h_0 works.

Now we have to find a h_1 such that $\mathcal{F}(h_1) = i \cdot h_1$. Remember \mathcal{F} is linear. If we let $h_1 = 2g_3 - 3g_1$ then

$$\begin{aligned} \mathcal{F}(h_1) &= \mathcal{F}(2g_3 - 3g_1) = 2\mathcal{F}(g_3) - 3\mathcal{F}(g_1) = 2(-i(3x - x^3)e^{-\frac{x^2}{2}}) - 3(-i)xe^{-\frac{x^2}{2}} \\ &= i(-6xe^{-\frac{x^2}{2}} + 2x^3e^{-\frac{x^2}{2}} + 3xe^{-\frac{x^2}{2}}) = i(2x^3e^{-\frac{x^2}{2}} + (3 - 6)xe^{-\frac{x^2}{2}}) \\ &= i(2x^3e^{-\frac{x^2}{2}} - 3xe^{-\frac{x^2}{2}}) = i(2g_3 - 3g_1) = ih_1 \end{aligned}$$

and it is clear this h_1 works.

Now we have to find a h_2 such that $\mathcal{F}(h_2) = -h_2$. If we let $h_2 = g_0 - 2g_2$ then

$$\begin{aligned} \mathcal{F}(h_2) &= \mathcal{F}(g_0 - 2g_2) = \mathcal{F}(g_0) - 2\mathcal{F}(g_2) = e^{-\frac{x^2}{2}} - 2(1 - x^2)e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} - 2e^{-\frac{x^2}{2}} + 2x^2e^{-\frac{x^2}{2}} \\ &= -e^{-\frac{x^2}{2}} + 2x^2e^{-\frac{x^2}{2}} = -(e^{-\frac{x^2}{2}} - 2x^2e^{-\frac{x^2}{2}}) = -(g_0 - 2g_2) = -h_2 \end{aligned}$$

and it is clear this h_2 works.

Now we have to find a h_3 such that $\mathcal{F}(h_3) = -i \cdot h_3$. If we let $h_3 = g_1$ then

$$\mathcal{F}(h_3) = \mathcal{F}(g_1) = -ixe^{-\frac{x^2}{2}} = -ig_1 = -i \cdot h_3$$

and it is clear this h_3 works.



(c) To show that $\mathcal{F}^4(f) = f$, for all $f \in \mathcal{S}(\mathbb{R})$, we use that

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ix\xi} dx$$

According to corollary 12.14 (Musat's notes) everything below is well-defined as all functions are Schwartz functions.

Futhermore, we know from definition 12.10 (Musat's notes) that

$$\mathcal{F}^*(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{ix\xi} dx$$

which means that

$$\mathcal{F}(f)(\xi) = \mathcal{F}^*(f)(-\xi) \quad \forall \xi \in \mathbb{R}$$

From this it is easily seen that

$$\mathcal{F}(\mathcal{F}(f)(\xi)) = \mathcal{F}(\mathcal{F}^*(f)(-\xi)) \quad \forall \xi \in \mathbb{R}$$

$$\mathcal{F}^2(f)(\xi) = f(-\xi) \quad \forall \xi \in \mathbb{R}$$

and then it is clear that

$$\mathcal{F}^4(f)(\xi) = \mathcal{F}^2(\mathcal{F}^2(f)(\xi)) = \mathcal{F}^2(f)(-\xi) = f(-(-\xi)) = f(\xi) \quad \forall \xi \in \mathbb{R}$$

and it is shown that $\mathcal{F}^4(f) = f$.

(d) To show that if $f \in \mathcal{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$, we use the fact that $\mathcal{F}^4(f) = f$ known from part (c). Combined with $\mathcal{F}(f) = \lambda f$ we get (remember \mathcal{F} is linear)

$$f = \mathcal{F}^4(f) = \mathcal{F}^3(\lambda f) = \mathcal{F}^2(\lambda^2 f) = \mathcal{F}(\lambda^3 f) = \lambda^4 f$$

Now we just have to solve $\lambda^4 = 1$ and in the complex numbers we have exactly that $\lambda \in \{1, i, -1, -i\}$.

To conclude that the eigenvalues of \mathcal{F} precisely are $\{1, i, -1, -i\}$ we assume for contradiction that $\mu \notin \{1, i, -1, -i\}$ is an eigenvalue, but then $\mathcal{F}(f) = \mu f$ and we have just shown that $\mu \in \{1, i, -1, -i\}$ which is a contradiction.

But you also have to show that there exist f_n $n=0,1,2,3$ with $\mathcal{F}f_n = i^n f_n$, so all in fact are eigenvalues.
(4.6)

Problem 5

Let $(x_n)_{n \geq 1}$ be a dense subset of $[0, 1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}$ on $[0, 1]$. To show that $\text{supp}(\mu) = [0, 1]$ we use problem 3 from HW8. Let us look at the open sets of $[0, 1]$ with measure 0. But an open set $U \subset [0, 1]$ ($U \neq \emptyset$) will contain an open interval (a, b) , where $0 \leq a < b \leq 1$. Because $(x_n)_{n \geq 1}$ is dense in $[0, 1]$ it will have elements in the interval (a, b) and therefore $0 < \mu((a, b)) \leq \mu(U)$. Hence the only open μ -null set is \emptyset . Let N be the union of all open μ -null sets, which is the largest open μ -null set of $[0, 1]$ according to problem 3 from HW8. Then $N = \emptyset$ and $\text{supp}(\mu) = N^c = [0, 1]$.