FunkAn Mandatory 1

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Problem 1

Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a)

Let $T : X \to Y$ be a linear map. Set $||x||_0 = ||x||_X + ||Tx||_Y$, for all $x \in X$. Show that $||\cdot||_0$ is a norm on X. Show next that the two norms $||\cdot||_X$ and $||\cdot||_0$ are equivalent if and only if T is bounded.

First note that since $||\cdot||_X$ and $||\cdot||_Y$ are norms we have that $||\cdot||_X : X \to [0,\infty)$ and $||\cdot||_Y : Y \to [0,\infty)$, so clearly $||\cdot||_0 : X \to [0,\infty)$.

Now let $x, y \in X$. Since T is linear and $||\cdot||_X$ and $||\cdot||_Y$ are norms we have that

$$\begin{aligned}||x+y||_0 &= ||x+y||_X + ||T(x+y)||_Y \\ &= ||x+y||_X + ||Tx+Ty||_Y \\ &\leq ||x||_X + ||y||_X + ||Tx||_Y + ||Ty||_Y \\ &= ||x||_0 + ||y||_0\end{aligned}$$

Further for $\alpha \in \mathbb{K}$ and $x \in X$, then

$$\begin{aligned}||\alpha x||_0 &= ||\alpha x||_X + ||T(\alpha x)||_Y \\ &= ||\alpha x||_X + ||\alpha T(x)||_Y \\ &= |\alpha|||x||_X + |\alpha|||T(x)||_Y \\ &= |\alpha|||x||_0\end{aligned}$$

Again we have used that T is linear and $||\cdot||_X$ and $||\cdot||_Y$ are norms.

We will show that $||x||_0 \Leftrightarrow x = 0$, $\forall x \in X$.
So suppose $||x||_0 = 0$ for some $x \in X$, then $||x||_X + ||Tx||_Y = 0$, so $||x||_X = -||Tx||_Y$. Now since both of $||\cdot||_X$ and $||\cdot||_Y$ maps into $[0,\infty)$ this is true if and only if $||x||_X = -||Tx||_Y = 0$. Now since $||\cdot||_X, ||\cdot||_Y$ are norms and T is linear this holds if and only if $x = 0$. So $||x||_0 \Leftrightarrow x = 0$, $\forall x \in X$.

In conclusion $||\cdot||_X$ is a norm on X.

Now we will show that $||\cdot||_X$ and $||\cdot||_0$ are equivalent if and only if T is bounded.

" $\Rightarrow$ " Suppose $||\cdot||_X$ and $||\cdot||_0$ are equivalent. By definition 1.4 this means there exists constants $0 < C_1 \leq C_2 < \infty$ such that for all $x \in X$,

$$C_1||x||_X \leq ||x||_0 \leq C_2||x||_X,$$

which is equivalent to

$$C_1||x||_X \leq ||x||_X + ||Tx||_Y \leq C_2||x||_X$$

Notice that $||Tx||_Y \leq ||x||_X + ||Tx||_Y$ for all $x \in X$, which implies that $||Tx||_Y \leq C_2||x||_X$. So by proposition 1.10 then T is bounded.

" $\Leftarrow$ " Assume T is bounded. Then by proposition 1.10 there exists a constant $C > 0$ such that

$$||Tx||_Y \leq C||x||_X, \quad \forall x \in X$$

Now we see that for all $x \in X$ we have

$$\begin{aligned} ||x||_X &\leq ||x||_0 \\ &= ||x||_X + ||Tx||_Y \\ &\leq ||x||_X + C||x||_X \\ &= (1 + C)||x||_X \end{aligned}$$

Thus $||\cdot||_X$ and $||\cdot||_0$ are equivalent.

(b)

Show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.

Suppose X is finite dimensional. Given a linear map $T : X \to Y$, define the function $||\cdot||_0 : X \to [0, \infty)$ by

$$||x||_0 = ||x||_X + ||Tx||_Y, \quad \forall x \in X.$$

This is a norm by (a). Now by Theroem 1.6, since X is finite dimensional, then any two norms on X are equivalent. So in particular $||\cdot||_X$ and $||\cdot||_0$ are equivalent. Hence T must be bounded by (a).

(c)

Suppose that X is infinite dimensional. Show that there exists a linear map $T : X \to Y$, which is not bounded.

We consider a Hamel basis $\{e_i\}_{i\in I}$ where $||e_i|| = 1$ for each $i \in I$. Since X is infinite dimensional there exists an infinite countable subset $\{e_n\}_{n\in\mathbb{N}} \subseteq \{e_i\}_{i\in I}$. Careful with the notation.
Now we define the linear map $T : X \to Y$ for all $e_i \in \{e_i\}_{i\in I}$ by

$$T(e_i) = \begin{cases} iy' & \text{for } \ e_i \in \{e_n\}_{n\in\mathbb{N}} \\ 0 & \text{for } \ e_i \notin \{e_n\}_{n\in\mathbb{N}} \end{cases}$$

for some element $y' \in Y$ with unit norm, which exists since $(Y, ||\cdot||_Y)$ is non-zero. Hence

$$||Te_i|| = ||iy'|| = i||y'|| = i, \quad \forall e_i \in \{e_n\}_{n\in\mathbb{N}}.$$

Now for every constant $C > 0$, there exists some $i > C$ satisfying

$$||Te_i|| = i > C = C||e_i||.$$

So by proposition 1.10 the linear map T is not bounded.

(d)

Suppose again that X is infinite dimensional. Argue that there exists a norm $||\cdot||_0$ on X, which is not equivalent to the given norm $||\cdot||_X$, and which satisfies $||x||_X \le ||x||_0$, for all $x \in X$. Conclude that $(X, ||\cdot||_0)$ is not complete if $(X, ||\cdot||_X)$ is a Banach space.

Assume X is infinite dimensional. Then let $T : X \to Y$ be a linear map which is not bounded. We know that such a map exists by (c). Now define the norm $||\cdot||_0 : X \to [0, \infty)$, by

$$||x||_0 = ||x||_X + ||Tx||_Y, \quad \forall x \in X$$

which is a norm by (a). Then we have from (a) that $||\cdot||_0$ and $||\cdot||_X$ are not equivalent, since T is not bounded. Notice that for all $x \in X$ these satisfies

$$||x||_X \le ||x||_X + ||Tx||_Y = ||x||_0.$$

Now if $(X, ||\cdot||_X)$ is a Banach space, it is complete with respect to $||\cdot||_X$. Then assume to reach a contradiction that $(X, ||\cdot||_0)$ is complete. Then since X is complete with respect to both norms and $||x||_X \le ||x||_0$ for all $x \in X$, we get by Homework 3, problem 1 that $||\cdot||_0$ and $||\cdot||_X$ are equivalent. But this is a contradiction, so $(X, ||\cdot||_0)$ is not complete. ✓

(e)

Give an example of a vector space X equipped with two inequivalent norms $||\cdot||$ and $||\cdot||'$ satisfying $||x||' \le ||x||$, for all $x \in X$, such that $(X, ||\cdot||)$ is complete, while $(X, ||\cdot||')$ is not.

Consider the vector space $\ell_1(\mathbb{N})$ equipped with the 1-norm. We know that $(\ell_1(\mathbb{N}), ||\cdot||_1)$ is indeed a Banach space. Now we wish to show that $(\ell_1(\mathbb{N}), ||\cdot||_\infty)$ is not complete, where $||x||_\infty = \sup\{|x_n| : n \ge 1\}$. First notice that these norms satisfies $||x||_\infty \le ||x||_1$, since for all $x \in \ell_1(\mathbb{N})$ we have that

$$||x||_\infty = \sup\{|x_n| : n \ge 1\} \le \sum_{n=1}^{\infty} |x_n|. = ||x||_1$$

Now we wish to find a sequence $(x_n)_{n\in N}$ that is cauchy with respect to $||\cdot||_\infty$, which does not converge in $\ell_1(\mathbb{N})$. Let $(x_n)_{n\ge1}$ be the sequence of sequences in $\ell_1(\mathbb{N})$ defined by $x_n^{(k)} = \frac{1}{k}$ if $k \le n$ and $x_n^{(k)} = 0$ otherwise, i.e.

$$\begin{aligned} x_1 &= (1, 0, 0, \dots) \\ x_2 &= (1, 1/2, 0, 0, \dots) \\ x_3 &= (1, 1/2, 1/3, 0, 0 \dots) \\ &\vdots \\ x_n &= (1, 1/2, 1/3, \dots, 1/n, 0, 0, \dots) \end{aligned}$$

Now to show this is cauchy, we first note that for $m > n$, then

$$||x_n - x_m||_\infty = \sup\{(0, \dots, 0, \frac{1}{n+1}, \frac{1}{n+2}, \dots, \frac{1}{m}, 0, \dots)\} = \frac{1}{n+1}$$

Now let $N := \frac{1}{\varepsilon} - 1$, then for all $n, m > N$, we have

$$||x_n - x_m||_\infty = \frac{1}{\min\{n, m\} + 1} \le \frac{1}{N+1} = \varepsilon.$$

So $(x_n)_{n\geq 1}$ is cauchy. Now this sequence of sequences converges to $x = \left(\frac{1}{n}\right)_{n\geq 1}$, but

$$||x||_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right| = \infty,$$

so $x \notin \ell_1(\mathbb{N})$, which is exactly what we wanted.
The fact that $(x_n)_{n\geq 1}$ converges to $x = \left(\frac{1}{n}\right)_{n\geq 1}$ is easily seen by choosing N as before, since

$$||x_n - x||_\infty = \frac{1}{n+1} \leq \frac{1}{N+1} = \varepsilon, \quad \forall n \geq N.$$

Hence $x_n \to x$, when $n \to \infty$ with respect to $||\cdot||_\infty$. In conlusion $(\ell_1(\mathbb{N}), ||\cdot||_\infty)$ is not complete.
From the above and by Homework 3, Problem 1 we also note that $||\cdot||_1$ and $||\cdot||_\infty$ are not equivalent.

Problem 2

Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(\ell_p(\mathbb{N}), ||\cdot||_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, 0, \dots) : a, b \in \mathbb{C}\}.$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, \dots) = a + b$, for all $a, b \in \mathbb{C}$.

(a)

Show that f is bounded on $(M, ||\cdot||_p)$ and compute $||f||$.

Let $(x_n)_{n\geq 1} = (x_1, x_2, 0, 0, \dots)$ and $(y_n)_{n\geq 1} = (1, 1, 0, 0, \dots)$ be sequences in M.
Suppose $1 < p < \infty$ and notice that $y \in \left(\ell_{\frac{p}{p-1}}, ||\cdot||_{\frac{p}{p-1}}\right)$. As $\frac{1}{p} + \frac{1}{\frac{p}{p-1}} = 1$, we obtain by using Hölders inequality the following

$$\begin{aligned}
|fx| &\leq \sum_{n=1}^{\infty} |x_n y_n| \\
&\leq ||x||_p ||y||_{\frac{p}{p-1}} \\
&= ||x||_p \left(|1|^{\frac{p}{p-1}} + |1|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&= 2^{\frac{p-1}{p}} ||x||_p
\end{aligned}$$

Assume $p = 1$, then $y \in (\ell_\infty, ||\cdot||_\infty)$. So by Hölders inequality

$$|fx| \leq \sum_{n=1}^{\infty} |x_n y_n| \leq ||x||_1 ||y||_\infty = ||x||_1$$

Thus for $1 \leq p < \infty$ we have that $|fx| \leq 2^{\frac{p-1}{p}} ||x||_p$, for all $x \in M$. Hence f is bounded on $(M, ||\cdot||_p)$.

Now we will compute $||f||$.
Since $|fx| \leq 2^{\frac{p-1}{p}} ||x||_p$, for all $x \in M$ we have that

$$||f|| = \inf\{C > 0 : |fx| \leq C||x||_p\} \leq 2^{\frac{p-1}{p}}.$$

Now let z be the sequence

$$z = \left(\frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{1}{p}}}, 0, 0, \dots \right) \in M.$$

Then we have

$$||z||_p = \left(\left| \frac{1}{2^{\frac{1}{p}}} \right|^p + \left| \frac{1}{2^{\frac{1}{p}}} \right|^p \right)^{\frac{1}{p}} = \left(\frac{1}{2} + \frac{1}{2} \right)^{\frac{1}{p}} = 1^{\frac{1}{p}} = 1.$$

Thus since

$$|fz| = \left| \frac{1}{2^{\frac{1}{p}}} + \frac{1}{2^{\frac{1}{p}}} \right| = \frac{2}{2^{\frac{1}{p}}} = 2^{1-\frac{1}{p}} = 2^{\frac{p-1}{p}},$$

we get that

$$||f|| = \sup\{|fx| : ||x||_p = 1\} \geq 2^{\frac{p-1}{p}}.$$

Hence $||f|| = 2^{\frac{p-1}{p}}$ for $1 \leq p < \infty$.

(b)

Show that if $1 < p < \infty$, then there is a unique linear functional F on $\ell_p(\mathbb{N})$ extending f and satisfying $||F|| = ||f||$.

Since $f \in M^*$ corollary 2.6 implies the existence of such a linear functional $F \in (\ell_p(\mathbb{N}))^*$ extending f and satisfying $||F|| = ||f||$. Now define F as $F(x_1, x_2, x_3, \dots) = x_1 + x_2$, then it is clear that $F|_M = f$ and that $||F|| = ||f||$. Note by Homework 1, problem 5 we know that $(\ell_p(\mathbb{N}))^*$ is isometrically isomorphic to $\ell_q(\mathbb{N})$ for $1 < p < \infty$, when $\frac{1}{p} + \frac{1}{q} = 1$. To satisfy this property we let $q = \frac{p}{p-1}$.

Then we can write $F(x) = \sum_{n=1}^{\infty} x_n y_n$, where $y = (y_n)_{n\geq1} \in \ell_q(\mathbb{N})$ and $x = (x_n)_{n\geq1} \in \ell_p(\mathbb{N})$.
Now since $F(x)$ must satisfy $F|_M = f$, we get that the sequence $(y_n)_{n\geq1}$ is on the form

$$(y_n)_{n\geq1} = (1, 1, x_3, x_4, \dots).$$

Actually since $||F|| = ||f||$ it must be on the form $(y_n)_{n\geq1} = (1, 1, 0, 0, \dots)$, which we will see is the only possibility for $(y_n)_{n\geq1}$ and thus that F is determined uniquely.
So assume to reach a contradiction that there exists another linear functional $F' \in (\ell_p(\mathbb{N}))^*$ such that $F'|_M = f$ and $||F'|| = ||f||$, defined by $F'(x) = \sum_{n=1}^{\infty} x_n y_n$ with $y = (y_n)_{n\geq1}$, $x = (x_n)_{n\geq1}$, where $|y_n| \neq 0$ for some $n \neq 1, 2$, meaning $y \neq (1, 1, 0, \dots)$.
Since we know there is an isometric isomorphism $F' \mapsto y \in \ell_q(\mathbb{N})$ we see that

Should be ≥?

$$||F'|| = ||y||_q = \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{\frac{1}{q}} = (|1|^q + |1|^q + |y_n|^q)^{\frac{1}{q}} = \left(2 + |y_n|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} > ||f||$$

Hence we reach a contradiction since $||F'|| = ||f||$, so such an F' doesn't exist, thus F is unique.

(c)

Show that if $p = 1$, then there are infinitely many linear functionals F on $\ell_1(N)$ extending f and satisfying $||F|| = ||f||$.

Suppose that $p = 1$. Then we know that there exists a linear functional F on $\ell_1(\mathbb{N})$ satisfying $F|_M = f$ and $||F|| = ||f||$ by corollary 2.6.

Maybe be more explicit in saying that F is of this form via a concrete identification $\ell^1(\mathbb{N})^* \cong \ell^\infty(\mathbb{N})$.

Define $F \in \ell_1(\mathbb{N})$ as in (b) such that $F(x) = \sum_{n=1}^{\infty} x_n y_n$ and F satisfies $F|_M = f$ and $||F|| = ||f||$ for some $(y_n)_{n\geq 1} \in \ell_\infty(\mathbb{N})$ and $(x_n)_{n\geq 1} \in \ell_1(\mathbb{N})$.
Note that $||f|| = 2^{\frac{1-1}{1}} = 2^0 = 1$.
Then, since $(\ell_1(\mathbb{N}))^* \cong \ell_\infty(\mathbb{N})$ (isometrically isomorphism) we have that

$$||F|| = ||y||_\infty = \sup\{|y_n| : n \geq 1\}.$$

Now take any sequence

$$y' = (y'_n)_{n\geq 1} = (1, 1, y_3, y_4, \dots) \in \ell_\infty(\mathbb{N}) \text{ where } |y'_n| \leq 1 \text{ for all } n \in \mathbb{N},$$

then we have

$$||y'||_\infty = \max\{|1|, |1|, |x_3|, |x_4|, \dots\} = 1 = ||f||$$

Remember to verify $F|_M = f$.

Hence every $F(x) = \sum_{n=1}^{\infty} x_n y'_n$ satisfies the wanted conditions. Now since there are infinitely many sequences y'_n, we get infinitely many linear functionals F which extends f and satisfies $||F|| = ||f||$. ✓

Problem 3

Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a)

Let $n \geq 1$ be an integer. Show that no linear map $F : X \to \mathbb{K}^n$ is injective.

Assume to reach a contradiction that some linear map $F : X \to \mathbb{K}^n$ is injective. Now we know that $F : X \to \mathrm{Im}(F)$ is surjective, hence there is a bijection between X and the image of F. Since $\mathrm{Im}(F)$ is a subspace of $\mathbb{K}^n$ there is in fact an isomorphism of vectorspaces. But $\mathrm{Im}(F) \subseteq \mathbb{K}^n$ so we must have $\dim(\mathrm{Im}(F)) \leq n$. But since X is infinite dimensional, there cannot be an isomorphism between these vectorspaces. So we reach a contradiction, hence F cannot be injective. ✓

Show this.

(b)

Let $n \geq 1$ be an integer and let $f_1, f_2, ..., f_n \in X^*$. Show that $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$.

Let $F : X \to \mathbb{K}^n$ be given by $F(x) = (f_1(x), f_2(x), \dots, f_n(x))$, $x \in X$. Linearity of F follows from linearity of $f_i \in X^*$, $i \in \{1, \dots, n\}$. By (a) we know that F is not injective, i.e. $\ker F \neq 0$, meaning there exists some $x_1 \neq 0$ such that $F(x_1) = 0$. Notice that

$$F(x_1) = F(f_1(x_1), f_2(x_1), \dots, f_n(x_1)) = (0, 0, \dots, 0),$$

implies that $f_1(x_1) = 0, f_2(x_1) = 0, \dots, f_n(x_1) = 0$, so $x_1 \in \ker(f_j)$ for $j \in \{1, \dots, n\}$.
Hence $\bigcap_{j=1}^{n} \ker(f_j) \neq \{0\}$ since $x_1 \neq 0$. ✓

(c)

Let $x_1, x_2, ..., x_n \in X$. Show that there exists $y \in X$ such that $||y|| = 1$ and $||y - x_j|| \geq ||x_j||$ for all $j = 1, 2, \dots, n$

Note that if $x_1, \dots, x_n = 0$ we can easily choose any y with $||y|| = 1$, such that $||y - x_j|| = ||y|| \geq ||x_j|| = 0$.

So assume $x_1, \dots, x_n \neq 0$. Then for every x_i, $i \in \{1, \dots, n\}$ we can use Theorem 2.7(b), so there exists $f_i \in X^*$ such that $||f_i|| = 1$ and $f_i(x_i) = ||x_i||$.

Now define $f : X \to \mathbb{K}^n$ by $F(x) = (f_1(x), f_2(x), \dots)$ for $x \in X$, which is linear since $f_i \in X^*$. Hence by (a) and (b) this map is injective. So $\ker F \neq \{0\}$, i.e. there exists some $0 \neq \xi \in X$ such that

Not

$$F(\xi) = (f_1(\xi), f_2(\xi), \dots) = (0, 0, \dots).$$

Then choose $y = \frac{\xi}{||\xi||}$, such that $||y|| = 1$. Notice that since $f_j(y) = \frac{1}{||\xi||} f_j(\xi) = 0$ and $||x_j|| = f_j(x_j)$ for each $j = 1, \dots, n$ we get that

$$||x_j|| = f_j(x_j) = f_j(x_j) - f_j(y) = f_j(x_j - y),$$

since f_j is linear. Then we have that

$$\frac{||x_j||}{||x_j - y||} = \frac{f_j(x_j - y)}{||x_j - y||}, \quad \forall j = 1, 2, \dots, n.$$

Now recall that the operator norm is defined as

$$||x_j - y|| = \sup\{|f_j(x_j - y)| : ||x_j - y|| \leq 1\},$$

hence

$$\frac{||x_j||}{||x_j - y||} = \frac{f_j(x_j - y)}{||x_j - y||} \leq 1, \quad \forall j = 1, 2, \dots n.$$

so we obtain the inequality

$$||x_j|| \leq ||x_j - y|| = ||y - x_j||, \quad \forall j = 1, 2, \dots n.$$

(d)

Show that one cannot cover the unit sphere $S = \{x \in X : ||x|| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

Assume there is an arbitrary finite family of closed balls $B_1, \dots, B_n$ in X which covers S, where B_i have centrum c_i and radius r_i. We will show that at least one of the balls must have radius large enough, such that 0 is contained in the ball.

Consider the centrum of the balls $c_1, \dots, c_n \in X$.
By (c) we get that there exists $y \in X$ such that $||y|| = 1$ and

$$||y - c_i|| \geq ||c_i||, \forall i = 1, \dots, n.$$

Now since y has unit norm it must be contained in the unit sphere, so $y \in S$. But then y also lies in one of the closed balls covering S. So $y \in B_k$ for some $k \in \{1, \dots, n\}$.
Notice that both y and c_k lies in B_k, and that $||y - c_k|| \geq ||c_k||$.

Since $||c_k||$ is the distance from 0 to the centrum c_k, we have that the radius r_k of B_k must satisfy

$$r_k \geq ||y - c_k|| \geq ||c_k|| = ||c_k - 0||$$

Hence 0 must be contained in B_k. So there is no finite family of closed balls in X which covers S and where none of the balls contains 0.

(e)

Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

Assume to reach a contradiction that S is compact. Then each of its open covers has a finite subcover. Now consider the open cover $\cup_{i\in\mathbb{N}} B_i$, defined by taking an open ball with radius $\frac{1}{2}$ around each $x_i \in S$, such that for every $x_i \in S$, we have a ball $B_i(x_i, 1/2) = \{x \in X : ||x - x_i|| < 1/2)$. Now since S is compact we know that there is a finite subcover, so there exists some finite set $K \subseteq \mathbb{N}$ such that $\{B_i\}_{i\in K} \subseteq \{B_i\}_{i\in\mathbb{N}}$ and

$$S \subseteq \bigcup_{i\in K} B_i$$

But then S must also be contained in the union of the closed balls, so we have that

$$S \subseteq \bigcup_{i\in K} \bar{B}_i,$$

where $\bar{B}_i(x_i, 1/2) = \{x \in X : ||x - x_i|| \leq 1/2)$. Hence $\cup_{i\in K} \bar{B}_k$ is a finite closed covering of S. Now note that since S is the unit sphere and the closed balls $\bar{B}_i$, $i \in K$ have radius $1/2$, then 0 is clearly not contained in any of the balls. So we have a finite cover of closed balls in X which does not contain 0, but this contradicts (d). So S is not compact.

For the second part, assume that the closed unit ball is compact. We consider the closed unit ball $U = \{x \in X : ||x|| = 1\}$ equipped with the subspace topology. Then since the complement of S in U is the open unit ball $\{x \in X : ||x|| < 1\}$, which is clearly open, then S is closed in U. Now every closed subset of a compact space is again compact, so S is compact. Hence we reach a contradiction, so the closed unit ball is non-compact.

Problem 4

Let $L_1([0,1], m)$ and $L_3([0,1], m)$ be the Lebesgue spaces on $[0,1]$.
Recall from HW2 that $L_3([0,1], m) \subsetneq (L_1([0,1], m)$. For $n \geq 1$, define

$$E_n := \left\{ f \in L_1([0,1], m) : \int_{[0,1]} |f|^3 dm \leq n. \right\}$$

(a)

Given $n \geq 1$, is the set $E_n \subset L_1([0,1], m)$ absorbing?

We know from HW2 that $L_3([0,1], m)$ is a proper subspace of $L_1([0,1], m)$, so there exists some measurable function $0 \neq f \in L_1([0,1], m) \setminus L_3([0,1], m)$. Since $f \notin L_3([0,1], m)$ then

$$\left(\int_{[0,1]} |f|^3 dm \right)^{\frac{1}{3}} = \infty, \quad \text{hence} \int_{[0,1]} |f|^3 dm = \infty$$

Thus for all $t > 0$ we have that

$$\int_{[0,1]} |t^{-1}f|^3 dm = \int_{[0,1]} (t^{-1})^3 |f|^3 dm$$
$$= (t^{-1})^3 \int_{[0,1]} |f|^3 dm = \infty$$

So $t^{-1}f \notin E_n$. This proofs E_n is not absorbing.

(b)

Show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.

First note that if E_n has empty interior in $L_1([0,1],m)$, then E_n contains no open sets of $L_1([0,1],m)$, other than the empty set. Assume to reach a contradiction that there is an open ball around some $f \in E_n$. We define the open ball for some $r > 0$ by

$$B(f,r) = \{g \in L_1([0,1],m) : ||f-g||_3 < r\} \subseteq E_n$$

Now take an arbitrary $0 \neq y \in L_1([0,1],m)$, then we wish to reach a contradiction by showing that y also belongs to $L_3([0,1],m)$.
First we construct g as

$$g = f + \frac{r}{2} \cdot \frac{y}{||y||} \in L_1([0,1],m)$$

replace $||\cdot||_3$ with $||\cdot||_1$ then it is correct.

$||y||_3$

such that $g \in B(f,r)$ since

$$||f-g||_3 = \Big|\Big|\frac{r}{2} \cdot \frac{y}{||y||}\Big|\Big|_3 = \frac{r}{2} \cdot \frac{||y||}{||y||} = \frac{r}{2}.$$

1/2

$||y||_1$

Now we see that y must be on the form

$$y = (g-f)\frac{2}{r}||y||.$$

Then since $E_n \subseteq L_3([0,1],m)$ we have that $f, g \in L_3([0,1],m)$. Hence we must have that $y \in L_3([0,1],m)$. So $L_1([0,1],m) \subseteq L_3([0,1],m)$, but this contradicts the fact that $L_3([0,1],m)$ is a proper subspace of $L_1([0,1],m)$. Thus for every $f \in E_n$ and every $r > 0$, there exist no open ball $B(f,r)$ which is non-empty. In conclusion, E_n has an empty interior.

(c)

Show that E_n is closed in $L_1([0,1],m)$, for all $n \geq 1$.

Let $(f_n)_{n\geq 1}$ be a sequence in E_n with limit f, i.e.

$$||f_n - f||_1 \to 0, \quad \text{when } n \to \infty.$$

In order for E_n to be closed we will show that $f \in E_n$.
First notice that by Corollary 12.8 in Schilling there exists a subsequence $(f_{n(k)})_{k\geq 1}$ that converges pointwise almost everywhere to f, i.e.

$$\lim_{n\to\infty} ||f_{n(k)} - f||_1 = 0 \text{ almost everywhere}$$

Now by Fatous Lemma we obtain

$$\begin{aligned}\int_{[0,1]} |f(x)|^3 dm &= \int_{[0,1]} \liminf_{n\to\infty} |f_{n(k)}(x)|^3 dm \\ &\leq \liminf \int_{[0,1]} |f_{n(k)}(x)|^3 dm \\ &\leq n\end{aligned}$$

Thus $f \in E_n$, so E_n is closed.

(d)

Conclude from (b) and (c) that $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

Since E_n is closed and has empty interior by (a) and (b), the sequence $(E_n)_{n\geq 1}$ has nowhere dense subsets by definition 3.12(i). Further we see that $L_3([0,1],m) = \cup_{n=1}^{\infty} E_n$.

Consists of ?

$L_3([0,1],m) \subseteq \cup_{n=1}^{\infty} E_n$:
Let $f \in L_3([0,1],m)$, then $\left(\int_{[0,1]} |f|^3 dm\right)^{1/3} < \infty$, hence $\left(\int_{[0,1]} |f|^3 dm\right)^{1/3} = r$, for some $r \in \mathbb{R}$. Choose n as the first integer greater than r^3, this means there exists an $n \geq 1$ such that $\int_{[0,1]} |f|^3 dm \leq n$, thus $f \in E_n$.

$\cup_{n=1}^{\infty} E_n \subseteq L_3([0,1],m)$:
Let $f \in \cup_{n=1}^{\infty} E_n$, then $f \in E_n$, for some $n \geq 1$. So $\int_{[0,1]} |f|^3 dm \leq n$, hence $\left(\int_{[0,1]} |f|^3 dm\right)^{1/3} < \infty$, thus $f \in L_3([0,1],m)$.

By definition 3.12(ii) then $L_3([0,1],m)$ is of first category in $L_1([0,1],m)$.

Problem 5

Let H be an infinite dimensional separable Hilbert space with associated norm $||\cdot||$, let $(x_n)_{n\geq 1}$ be a sequence in H, and let $x \in H$.

(a)

Suppose that $x_n \to x$ in norm, as $n \to \infty$. Does it follow that $||x_n|| \to ||x||$, as $n \to \infty$?

We wish to proof the above statement. Assume that x_n converges to x in norm, i.e.

$$\lim_{n\to\infty} ||x_n - x|| = 0.$$

Then we get from the reversed triangle inequality that

$$\big| ||x_n|| - ||x|| \big| \leq ||x_n - x|| \to 0, \quad \text{when } n \to \infty.$$

So $\lim_{n\to\infty} \big| ||x_n|| - ||x|| \big| = 0$, hence we have $||x_n|| \to ||x||$ as $n \to \infty$.

(b)

Suppose that $x_n \to x$ weakly, as $n \to \infty$. Does it follow that $||x_n|| \to ||x||$, as $n \to \infty$?

We will give a counterexample for the above statement. Since H is an infinite dimensional separable Hilbert space we can consider the countable orthonormal basis $(e_n)_{n\geq 1}$ (Lecture 8, p.44). We wish to show that the sequence $(e_n)_{n\geq 1}$ converges weakly to 0. From Homework 4, problem 2 we know that the sequence $(e_n)_{n\geq 1}$ in H converges to 0 in the weak topology τ_ω on X if and only if the net $(f(e_n))_{n\geq 1}$ converges to $f(0)$ when $n \to \infty$, for every $f \in H^*$.
Now since H is a Hilbert space, then by Riesz representation theorem every $f \in H^*$ is on the form $f(y) = \langle y, x\rangle$, for every $x \in H$ and some $y \in H$. So we want to show that the inner product $\langle e_n, x\rangle$ converges to $\langle 0, x\rangle$ for some $x \in H$. all.
It follows from Bessels inequality that for an orthonormal basis $(e_n)_{n\geq 1}$ then

$$\sum_{k=1}^{\infty} < e_n, x >^2 \leq ||x||^2, \qquad \text{for any } x \in H.$$

should be $\sum_{k=1}^{\infty} |\langle e_k, x\rangle|^2 \leq \|x\|^2$.

So since $||x||^2 < \infty$ the series above converges. Now a series converges if the terms goes to zero, so we must have that $\langle e_n, x\rangle^2 \to 0$, when $n \to \infty$. This implies that

$$\lim_{n\to\infty} < e_n, x >= 0 =< 0, x > .$$

Hence $(e_n)_{n\geq 1} \xrightarrow{\omega} 0$ by HW4.

Now notice that $||e_n|| = 1$ for every $n \geq 1$. But the norm of 0 is always zero, so $||e_n||$ does not converge to $||0||$. Hence it does not follow that $||x_n|| \to ||x||$, as $n \to \infty$, when $x_n \to x$, $n \to \infty$. ✓

(c)

Suppose that $||x_n|| \leq 1$, for all $n \geq 1$, and that $x_n \to x$ weakly, as $n \to \infty$. Is it true that $||x|| \leq 1$?

We will prove this statement. Assume $(x_n)_{n\geq 1}$ in H converges weakly to $x \in H$ and that $||x_n|| \leq 1$.

If $x = 0$, it is clear that $||x|| \leq 1$.

Suppose $x \neq 0$, then since H is a normed vector space, we get from Theorem 2.7(b) that there exists $f \in X^*$ such that $||f|| = 1$ and $f(x) = ||x||$. Note that from Homework 4, problem 2 we have that

$$f(x) = \lim_{n\to\infty} f(x_n)$$

Hence we get that

$$||x|| = f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} |f(x_n)| \leq \sup_{n\to\infty} ||x_n|| \leq 1$$ ✓

$\sup_{n\in\mathbb{N}}$