One-dimensional Dilute Quantum Gases and Their Ground State Energies

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PhD defense
June 6, 2023

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Overview

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Motivation (Bosons)

- 2D and 3D dilute Bose gases are well-studied in the mathematical physics literature.
- 2D and 3D results are related to Bose-Einstein condensation (BEC).
- No BEC is expected in 1D.
- In 1D the hard core and Lieb-Liniger models are solvable.
- Our result is consistent with the absence of BEC in 1D.
- On the contrary it suggests that the 1D dilute bose gas shares features with the Fermi gas.

Motivation (Fermions)

- 2D and 3D dilute Fermi gases are well-studied in the mathematical physics literature.
- In 1D the hard core and Yang-Gaudin models are solvable.
- In 1962 E. H. Lieb and D. C. Mattis showed that one-dimensional Fermi gases are antiferromagnetic (contradicting standard perturbative tight-binding methods).
- Hence the standard justification of the Heisenberg model of magnetism is too simple.
- Our result will break ground in rigorously justifying the Heisenberg antiferromagnet as an effective model in 1D.

Many-Body Quantum Mechanics

For a system of N bosons/fermions in region $\Omega \in \mathbb{R}^d$, we define for $\sigma \in [0, \infty]$ **the energy quadratic forms**

$$\mathcal{E}_{(v,\sigma)}(\Psi) = \int_{\Omega^N} \sum_{i=1}^N |\nabla_i \Psi|^2 + \sum_{i<j} v(x_i - x_j) |\Psi|^2 + \sigma \int_{\partial(\Omega^N)} |\Psi|^2, \quad (1)$$

with domain $\mathcal{D}\left(\mathcal{E}_{(v,\sigma)}\right) = \{\Psi \in (C^\infty(\Omega^N))_{\text{b/f}} | \mathcal{E}_{(v,\sigma)}(\Psi) < \infty\}$.

Definition 1

We say a potential $v \geq 0$ is **allowed** in dimension d, if $\mathcal{E}_{(v,\sigma)}$ is closable on $\mathcal{H}_{(v,\sigma)} := \overline{\mathcal{D}\left(\mathcal{E}_{(v,\sigma)}\right)}^{\|\cdot\|_2} \subset L^2_{s/a}(\Omega^N)$ for any $\sigma \in [0, \infty]$.

Proposition 1

Let $d = 1$, then any potential of the form $v = v_{\sigma-\text{finite}} + v_{\text{meas.}} + c\delta_0$, with $c \in \{0, \infty\}$, is allowed.

The Scattering Length

Theorem 2

For $B_R = \{0 \leq |x| < R\} \subset \mathbb{R}^d$ with $R > R_0 := \text{range}(v)$, let $\phi \in H^1(B_R)$ satisfy

$$-\Delta\phi + \frac{1}{2}v\phi = 0, \qquad \text{on } B_R, \tag{2}$$

with boundary condition $\phi(x) = 1$ for $|x| = R$. Then $\phi(x) = f(|x|)$ for some $f : (0, R] \to [0, \infty)$, and for $\text{range}(v) < r < R$, we have

$$f(r) = \begin{cases} (r-a)/(R-a) & \text{for } d = 1 \\ \ln(r/a)/\ln(R/a) & \text{for } d = 2 \\ (1 - ar^{2-d})/(1 - aR^{2-d}) & \text{for } d \geq 3, \end{cases} \tag{3}$$

*with some constant a called the (s-wave) **scattering length**.*

2D and 3D

For $\Lambda_L = [0, L]^d$, let $e(\rho) := \lim_{\substack{L \to \infty \\ N/L^d \to \rho}} E(N, L)/L^d$.

Theorem 3 $\left(d = 3 \text{ result, Lee-Huang-Yang 1957}[1]\right)$

$$e(\rho) = 4\pi\rho^2 a \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho a^3})\right). \tag{4}$$

Theorem 4 $\left(d = 2 \text{ result}[2]\right)$

$$e(\rho) = 4\pi\rho^2 Y \left(1 - Y|\log Y| + \left(2\Gamma + \frac{1}{2} + \ln(\pi)\right)Y\right) + o\left(\rho^2 Y^2\right), \tag{5}$$

$Y = \left|\ln(\rho a^2)\right|^{-1}$.

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[a] Upper bound: Yau-Yin 2009, Basti-Cenatiempo-Schlein 2021. Lower bound: Fournais-Solovej 2021

[b] Fournais-Girardot-Junge-Morin-Olivieri 2022

Bosons Main Result

For the remainder of the presentation, $d = 1$.

Theorem 5 (A., R. Reuvers, J. P. Solovej, 2022)

Consider a Bose gas with repulsive interaction $v = v_{reg} + v_{h.c.}$. Define the density $\rho = N/L$. For $\rho|a|$ and ρR_0 sufficiently small, the ground state energy can be expanded as

$$E(N, L) = N\frac{\pi^2}{3}\rho^2 \left(1 + 2\rho a + \mathcal{O}\left((\rho|a|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3}\right)\right),$$
(6)

where a is the scattering length of v.

Examples

The hard core gas

Energy behaves like free Fermi energy in volume $L - NR$, *i.e.*

$$\begin{aligned}
E_{\text{hard core}}(N, L) &= N\frac{\pi^2}{3}\rho^2(1 - NR/L)^{-2} \\
&= E_0\left(1 + 2\rho R + \mathcal{O}\left((\rho R)^2\right)\right).
\end{aligned} \tag{7}$$

Scattering length is $a = R$.

Lieb-Liniger model

Energy behaves asymptotically like

$$E_{LL}(N, L, c) = N\frac{\pi^2}{3}\rho^2\left(1 - 4\rho/c + \mathcal{O}\left((\rho/c)^2\right)\right), \tag{8}$$

with scattering length $a = -\frac{2}{c}$.

Variational Principle

To obtain an upper bound, we use the variational principle, *i.e.*

$$E(N, L) \leq \frac{\mathcal{E}(\Psi)}{\|\Psi\|^2}, \quad \text{for any } \Psi \in \mathcal{D}(\mathcal{E}).$$

Trial State

Trial state has to capture free Fermi energy, as well as corrections due to scattering processes. Hence we consider

$$\Psi(x) = \begin{cases} \omega(\mathcal{R}(x)) \frac{|\Psi_F(x)|}{\mathcal{R}(x)} & \text{if } \mathcal{R}(x) < b \\ |\Psi_F(x)| & \text{if } \mathcal{R}(x) \geq b, \end{cases}$$

where ω is the suitably normalized solution to the two-body scattering equation, Ψ_F is the free Fermi ground state, and $\mathcal{R}(x) := \min_{i<j}(|x_i - x_j|)$ is uniquely defined a.e.

One-particle Reduced Density Matrix

For the free Fermi gas we have

$$\gamma^{(1)}(x,y) = \frac{2}{L} \sum_{j=1}^{N} \sin\left(\frac{\pi}{L}jx\right) \sin\left(\frac{\pi}{L}jy\right)$$

$$= \frac{\pi}{L}\left(D_N\left(\pi\frac{x-y}{L}\right) + D_N\left(\pi\frac{x+y}{L}\right)\right), \qquad (9)$$

where $D_N(x) = \frac{1}{2\pi}\sum_{k=-N}^{N} e^{ikx} = \frac{\sin((N+1/2)x)}{2\pi\sin(x/2)}$ is the Dirichlet kernel.

By Wick's theorem all derivatives of reduced density matrices are bounded by a constant times an appropriate power of ρ.

Some Useful Bounds

Lemma 1

$$\rho^{(2)}(x_1, x_2) \leq \left(\frac{\pi^2}{3}\rho^4 + f(x_2)\right)(x_1 - x_2)^2 + \mathcal{O}(\rho^6(x_1 - x_2)^4),$$

with $\int f(x_2) \, dx_2 \leq$ const. $\rho^3 \log(N)$.

Lemma 2

We have the following bounds

$$\rho^{(3)}(x_1, x_2, x_3) \leq \text{const. } \rho^9(x_1 - x_2)^2(x_2 - x_3)^2(x_1 - x_3)^2,$$

$$\rho^{(4)}(x_1, x_2, x_3, x_4) \leq \text{const. } \rho^8(x_1 - x_2)^2(x_3 - x_4)^2,$$

$$\left|\sum_{i=1}^{2} \partial_{y_i}^2 \gamma^{(2)}(x_1, x_2; y_1, y_2)\Big|_{y=x}\right| \leq \text{const. } \rho^6(x_1 - x_2)^2,$$
$$\vdots$$

Collecting Everything

Upper bound

$$E \leq N\frac{\pi^2}{3}\rho^2\frac{\left(1 + 2\rho a\frac{b}{b-a} + \text{const. } \left[\frac{1}{N} + N(b\rho)^3\left(1 + \rho b^2\int v_{\text{reg}}\right)\right]\right)}{\|\Psi\|^2},$$

(10)

where the finite measure v_{reg} is v with any hard core removed. By lemma 1 we know $\|\Psi\|^2 \geq 1 - \text{const. } N(\rho b)^3$.

Localization

Divide into M smaller boxes with $\tilde{N} = N/M$ particles in each, and make distance b between boxes (no interaction between boxes), and choose M such that $\tilde{N} = (\rho b)^{-3/2} \gg 1$.

Upper Bound

After localization

$$E(N, L) \leq N\frac{\pi^2}{3}\rho^2 \frac{\left(1 + 2\rho a\frac{b}{b-a} + \text{const. } \frac{M}{N} + \text{const. } \tilde{N}(b\rho)^3\left(1 + \rho b^2 \int v_{\text{reg}}\right)\right)}{1 - \tilde{N}(\tilde{\rho}b)^3}$$

$$(11)$$

Choosing $b = \max(\rho^{-1/5}|a|^{4/5}, R_0)$ we find

Proposition 2 (Upper bound Theorem 5)

There exists a constant $C_U > 0$ such that for $\rho|a|$, $\rho R_0 \leq C_U^{-1}$, the ground state energy $E^D(N, L)$ satisfies

$$E^D(N, L) \leq N\frac{\pi^2}{3}\rho^2\left(1 + 2\rho a + C_U\left((\rho|a|)^{6/5} + (\rho R_0)^{3/2} + N^{-1}\right)\right).$$

$$(12)$$

Lower Bound

Proof of lower bound consists of the following steps:

1. Use Dyson's lemma to reduce to a nearest neighbor double delta-barrier potential.
2. Reduce to the Lieb Liniger model by discarding **a small part** of the wave function.
3. Use a known lower bound for the Lieb Liniger model.

The Lieb-Liniger (LL) model

$$H_{LL} = -\sum_{i=1}^{n} \partial_i^2 + 2c \sum_{i<j} \delta(x_i - x_j). \tag{13}$$

Behavior in thermodynamic limit: $\lim_{\substack{\ell \to \infty, \\ n/\ell \to \rho}} E_{LL}(n, \ell, c)/\ell = \rho^3 e(\gamma)$

with $\gamma = c/\rho$.

Lemma 3 (Lieb-Liniger lower bound)

Let $\gamma > 0$, then

$$e(\gamma) \geq \frac{\pi^2}{3} \left(\frac{\gamma}{\gamma + 2} \right)^2 \geq \frac{\pi^2}{3} \left(1 - \frac{4}{\gamma} \right). \tag{14}$$

Reducing to the LL Model

Lemma 4 (Dyson)

Let $R > R_0 = \text{range}(v)$ and $\varphi \in H^1(\mathbb{R})$, then for any interval $\mathcal{I} \ni 0$

$$\int_{\mathcal{I}} |\partial\varphi|^2 + \frac{1}{2} v \, |\varphi|^2 \geq \int_{\mathcal{I}} \frac{1}{R-a} \left(\delta_R + \delta_{-R} \right) |\varphi|^2, \qquad (15)$$

where a is the s-wave scattering length.

Hence we have, denoting $\mathfrak{r}_i(x) = \min_j(|x_i - x_j|)$

$$\int \sum_i |\partial_i \Psi|^2 + \sum_{i \neq j} \frac{1}{2} v_{ij} \, |\Psi|^2 \geq$$

$$\int \sum_i |\partial_i \Psi|^2 \, \chi_{\mathfrak{r}_i(x) > R} + \sum_i \frac{1}{R-a} \delta(\mathfrak{r}_i(x) - R) \, |\Psi|^2. \qquad (16)$$

Reducing to the LL Model

Define $\psi \in L^2([0, \ell - (n-1)R]^n)$ by

$$\psi(x_1, x_2, ..., x_n) = \Psi(x_1, R + x_2, ..., (n-1)R + x_n),$$

for $x_1 \leq x_2 \leq ... \leq x_n$ and symmetrically extended.

Then

$$\mathcal{E}(\Psi) \geq E_{LL}^N(n, \ell - (n-1)R, 2/(R-a)) \langle \psi | \psi \rangle$$

$$\geq n \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho(a - \cancel{R}) + \cancel{2\rho R} - \text{const.} \ \frac{1}{n^{2/3}} \right) \langle \psi | \psi \rangle .$$

$$(17)$$

Lower Bound for Mass of ψ

Lemma 5

Let ψ be defined as above, then

$$1 - \langle \psi | \psi \rangle \leq 8 \left(R^2 \sum_{i<j} \int_{B_{ij}} |\partial_i \Psi|^2 + R(R-a) \sum_{i<j} \int v_{ij} |\Psi|^2 \right), \quad (18)$$

Combining lemmas 4 and 5 we have the following lemma:

Lemma 6

For $n(\rho R)^2 \leq \frac{3}{16\pi^2} \frac{1}{8}$, $\rho R \ll 1$ and $R > 2|a|$ we have

$$\langle \psi | \psi \rangle \geq 1 - \text{const.} \left(n(\rho R)^3 + n^{1/3}(\rho R)^2 \right). \quad (19)$$

Lower Bound

By the reduction to the LL model we find

Proposition 3

For assumptions as in lemma 6 we have

$$E^N(n, \ell) \geq n\frac{\pi^2}{3}\rho^2 \left(1 + 2\rho a + \text{const.} \ \left(\frac{1}{n^{2/3}} + n(\rho R)^3 + n^{1/3}(\rho R)^2 \right) \right).$$
(20)

Corollary 1

For $n = \text{const.} \ (\rho R)^{-9/5}$ we have

$$E^N(n, \ell) \geq n\frac{\pi^2}{3}\rho^2 \left(1 + 2\rho a - \text{const.} \ \left((\rho R)^{6/5} + (\rho R)^{7/5} \right) \right).$$
(21)

Lower Bound Localization

To prove the lower bound, we localize, as in the upper bound, to smaller boxes.

Lemma 7

Let $\Xi \geq 4$ be fixed and let $n = m\Xi\rho\ell + n_0$ with $n_0 \in [0, \Xi\rho\ell)$ for some $m \in \mathbb{N}$ with $n^* := \rho\ell = \mathcal{O}(\rho R)^{-9/5}$. Furthermore, assume that $\rho R \ll 1$ and let $\mu = \pi^2 \rho^2 \left(1 + \frac{8}{3}\rho a\right)$, then

$$E^N(n, \ell) - \mu n \geq E^N(n_0, \ell) - \mu n_0. \tag{22}$$

Proposition 4 (Lower bound Theorem 5)

There exists a constant $C_L > 0$ such that the ground state energy $E^N(N, L)$ satisfies

$$E^N(N, L) \geq N\frac{\pi^2}{3}\rho^2 \left(1 + 2\rho a - C_L \left((\rho |a|)^{6/5} + (\rho R_0)^{6/5} + N^{-2/3}\right)\right). \tag{23}$$

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Spinless/Spin-Polarized Fermions

Spinless fermions are unitarily equivalent to bosons with a zero b.c. at all planes of intersection, *i.e.* with an infinite delta potential. As a consequence we have the following corollary.

Theorem 6 (Spin-polarized fermions)

Consider a spin-polarized Fermi gas with repulsive interaction $v = v_{reg} + v_{h.c.}$ as defined before. Let $E_F(N, L)$ be its associated ground state energy. Write $\rho = N/L$. For ρa_o and ρR_0 sufficiently small, the ground state energy can be expanded as

$$E_F(N, L) = N\frac{\pi^2}{3}\rho^2 \left(1 + 2\rho a_o + \mathcal{O}\left((\rho R_0)^{6/5} + N^{-2/3}\right)\right), \quad (24)$$

where $a_o \geq 0$ is the odd wave scattering length of v.

This is consistent with lower bound $E_F(N, L) \geq E_0$, since $a_o \geq 0$.

Two solvable model for spin-1/2 fermions

The hard core gas

Ground state energy is independent of spin so

$$E_{\text{hard core}}(N, L) = N\frac{\pi^2}{3}\rho^2(1 - NR/L)^{-2} \approx E_0(1 + 2\rho R). \quad (25)$$

Scattering length is $a_e = a_o = R$.

Yang-Gaudin model

Is the spin-1/2 version of the LL model, *i.e.* $H_{YG} = H_{LL}$. Behaves asymptotically like

$$E_{YG}(N, L, c) = N\frac{\pi^2}{3}\rho^2\left(1 - 4\rho\ln(2)/c + \mathcal{O}\left((\rho/c)^2\right)\right), \quad (26)$$

with scattering length $a_e = -\frac{2}{c}$, $a_o = 0$.

A Conjecture for Spin-1/2 Fermions

Based on the two solvable cases, we expect

Conjecture 1

Let $v \geq 0$ satisfy the assumption from above, then the ground state energy of the dilute spin–1/2 Fermi gas satisfies

$$E = N\frac{\pi^2}{3}\rho^2 \left(1 + 2\rho \left(\ln(2)a_e + (1 - \ln(2))a_o\right) + \mathcal{O}(\rho^2 \max(|a_e|, a_o)^2)\right). \tag{27}$$

$$E(N, L) = N\frac{\pi^2}{3}\rho^2 \Big(1 + 2\ln(2)\rho a_e + 2(1 - \ln(2))\rho a_o \\ + \mathcal{O}\left((\rho \max(|a_e|, a_o))^2\right)\Big) \tag{28}$$

Spin–$1/2$ Fermions Main Result (Upper Bound)

Theorem 7

Let $v \geq 0$ satisfy the assumption from above, then the ground state energy of the dilute spin–$1/2$ Fermi gas satisfies

$$E \leq N \frac{\pi^2}{3} \rho^2 \left(1 + 2\rho \left(\ln(2) a_e + (1 - \ln(2)) a_o \right) + \mathcal{O} \left((\rho R)^{6/5} + N^{-1} \right) \right), \tag{29}$$

with $R = \max(|a_e|, a_o, R_0)$.

Trial State

One the sector

$$\{1, 2, ..., N\} = \{0 < x_1 < x_2 < \ldots < x_N < L\}$$

we define the trial state by

$$\Psi_\chi = \begin{cases} \frac{\Psi_F}{\mathcal{R}} \left(\left(\eta \omega_e^{\mathcal{R}} + (1-\eta)\omega_o^{\mathcal{R}}\right) \mathrm{P}_s^{\mathcal{R}} + \omega_o^{\mathcal{R}} \mathrm{P}_t^{\mathcal{R}} \right) \chi, & \mathcal{R}(x) < b \\ \Psi_F \chi, & \mathcal{R}(x) \geq b \end{cases}, \quad (30)$$

where χ is some spin state, $b > R_0$, $\mathcal{R}(x) = \min_{i<j} |x_i - x_j|$,
$\omega_{e/o}^{\mathcal{R}}(x) := \omega_{e/o}(\mathcal{R}(x)) = b f_{e/o}(\mathcal{R}(x))$ and

$$\eta(x) := \begin{cases} 0, & \text{if } \mathcal{R}_2(x) \leq b \\ \left(\frac{\mathcal{R}_2(x)}{b} - 1\right), & \text{if } b < \mathcal{R}_2(x) < 2b \\ 1, & \text{if } \mathcal{R}_2(x) \geq 2b. \end{cases} \quad (31)$$

with $\mathcal{R}_2(x) = \min_{(i,j)\neq(k,l)} \max(|x_i - x_j|, |x_k - x_l|)$.

Trial state energy is the free Fermi energy with a correction of the form

$$2\rho \left((a_o - a_e) \left\langle \chi \left| \frac{1}{N} \sum_i S_i \cdot S_{i+1} \right| \chi \right\rangle + \frac{1}{4} a_e + \frac{3}{4} a_o \right) E_F.$$

Proof:

Antiferromagnetic Heisenberg Chain

The (periodic) antiferromagnetic Heisenberg chain

$$H = \sum_{i=1}^{N} S_i \cdot S_{i+1}, \text{ with } S_{N+1} := S_1$$

Ground state energy per site of the infinite chain is know due to Hulthen

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Lemma 8

Let $|GS_{HAF}\rangle$ denote the ground state of the periodic antiferromagnetic Heisenberg chain. Then

$$\lim_{N\to\infty} \left\langle GS_{HAF} \left| \frac{1}{N} \sum_{k=1}^{N} S_k \cdot S_{k+1} \right| GS_{HAF} \right\rangle = \frac{1}{4} - \ln(2) \qquad (32)$$

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Control of the error for a finite chain

Lemma 9

Let $|\mathrm{GS_{HAF}}\rangle$ denote the ground state of the periodic antiferromagnetic Heisenberg chain. Then

$$\left\langle \mathrm{GS_{HAF}} \left| \frac{1}{N} \sum_{k=1}^{N} S_k \cdot S_{k+1} \right| \mathrm{GS_{HAF}} \right\rangle = \frac{1}{4} - \ln(2) + \mathcal{O}(N^{-1}) \quad (33)$$

Proof.

Upper bound: Truncate longer of length $M > N$ chain at length N. Lower bound: Construct trial state for longer chain of length mN by m copies of length N chain. Use translation invariance and uniqueness of the ground state:
$\frac{1}{mN}(E_{mN} - m) \le \frac{1}{N} E_N \le \frac{1}{M} E_M + 1$. $\qquad \square$

Lower Bound in Terms of LLH Model

> **Lemma 10 (Dyson's lemma spin–$1/2$ fermions)**
>
> Let $R > R_0 = \text{range}(v)$ and
> $\varphi \in \left(H^1_{\text{even}}(\mathbb{R}) \otimes P_s \left(\left(\mathbb{C}^2 \right)^2 \right) \right) \oplus \left(H^1_{\text{odd}}(\mathbb{R}) \otimes P_t \left(\left(\mathbb{C}^2 \right)^2 \right) \right)$, then for
> any interval $\mathcal{I} \ni 0$
>
> $$\int_{\mathcal{I}} |\partial \varphi|^2 + \frac{1}{2} v \, |\varphi|^2 \geq \int_{\mathcal{I}} \overline{\varphi} \left(\frac{1}{R - a_e} P_s + \frac{1}{R - a_o} P_t \right) (\delta_R + \delta_{-R}) \, \varphi, \tag{34}$$
>
> where $a_{e/o}$ is the even/odd-wave scattering length.

The Lieb-Liniger-Heisenberg model:

$$H_{LLH} = -\sum_i \partial_i^2 + 2 \sum_{i<j} \left(c' \, \tilde{P}_s^{i,j} + c \, \tilde{P}_t^{i,j} \right) \delta(x_i - x_j), \tag{35}$$

where the spin projectors, $\tilde{P}_{s/t}$ are defined on the sector $\{\sigma\}$ to be

$$\tilde{P}_{s/t}^{ij} = P_{s/t}^{\sigma^{-1}(i)\sigma^{-1}(j)}.$$

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Proposition 5

For $n(\rho R)^2 \leq \frac{3}{16\pi^2}\frac{1}{8}$, $\rho R \leq \frac{1}{2}$ and $R > 2\max(|a_e|, a_o, R_0)$ we have

$$
E^N(N, L) \geq E^N_{LLH}\left(N, \tilde{L}, \frac{2}{R - a_e}, \frac{2}{R - a_o}\right) \\
\times \left(1 - \text{const.}\left(n(\rho R)^3 + n^{1/3}(\rho R)^2\right)\right). \tag{36}
$$

Remark 1

The Lieb-Liniger-Heisenberg model is not exactly solvable. Thus no available good lower bound.

Conclusion and Outlook

We have shown that:

- Interaction pair-potential of the form $v = v_{\sigma\text{-finite}} + v_{\text{meas.}} + c\delta_0$ give rise to a unique Hamiltonian.

- The ground state energy of the dilute Bose (spin polarized Fermi or anyon) gas in one dimension can be expanded in terms of the diluteness parameter to next-to-leading order (universality).

- The solvable models of spin–$1/2$ fermionic systems are consistent with a similar expansion.

- The ground state energy of the dilute spin–$1/2$ Fermi gas can be upper bounded by this expansion.

- The solvable models of spin–$1/2$ fermionic systems are consistent with a similar expansion.

- The ground state energy of the dilute spin–$1/2$ Fermi gas can be lower bounded by the ground state energy of the Lieb-Liniger-Heisenberg model.

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Interesting future problems:

- Showing a ground state energy expansion to order $(\rho a)^2$ for the dilute Bose gas.

- Proving that the given upper bound for the Fermi gas is tight, *i.e.* matching lower bound. For example by showing lower bound for LLH model.

- Fully understanding the connection to antiferromagnetism.

- Understanding the momentum distribution of the dilute Bose gas (similar universality).

- Generalize spin–$1/2$ Fermi results to higher spin.

Thanks for your attention!