Mandatory assignment 1, FunkAn 2020

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If not specified otherwise, references are to the Lecture Notes.
When refering to 'Schilling', we refer to the book 'Measures, Integrals and Martingales' by René L. Schilling, Second Edition, 2017.

Problem 1

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be (non-zero) normed vector spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $T_X \to Y$ be a linear map. Set $\|x\|_0 = \|x\|_X + \|Tx\|_Y$, for all $x \in X$. Show that $\|x\|_0$ is a norm on X. Show next that the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ are equivalent if and only if T is bounded.

Solution To show $\|\cdot\|_0$ is a norm, let $x, y \in X$ and $\alpha \in \mathbb{K}$. Using properties of norms and linearity, we obtain

1. $$\begin{aligned}\|x+y\|_0 &= \|x+y\|_X + \big\|T(x+y)\big\|_Y \\ &= \|x+y\|_X + \|Tx+Ty\|_Y \\ &\le \|x\|_X + \|y\|_X + \|Tx\|_Y + \|Ty\|_Y \\ &= \|x\|_0 + \|y\|_0\end{aligned}$$

2. $$\|\alpha x\|_0 = \|\alpha x\|_X + \big\|T(\alpha x)\big\|_Y = \|\alpha x\|_X + \|\alpha Tx\|_Y = |\alpha| \left(\|x\|_X + \|Tx\|_Y\right) = |\alpha|\|x\|_0$$

3. Clearly, we have $\|0\|_0 = \|0\|_X + \big\|T(0)\big\|_Y = 0$ since $T(0) = 0$. For the converse, assume $\|x\|_0 = 0$, that is $\|x\|_X + \|Tx\|_Y = 0$. Since norms are non-negative, we deduce $\|x\|_X = 0$ and therefore $x = 0$. □

It follows from the above that $\|\cdot\|_0$ is a norm on X.

For the second part, assume T is bounded. For any $x \in X$, we have the following inequalities.

$$\|x\|_X \le \|x\|_X + \|Tx\|_Y \le \|x\|_X + \|T\|\|x\|_X = \big(\|T\| + 1\big)\|x\|_X$$

By the definition $\|x\|_0 = \|x\|_X + \|Tx\|_Y$, the two norms $\|\cdot\|_X$ and $\|\cdot\|_0$ on X are equivalent.
Conversely, assume the two norms are equivalent, that is, there exists $0 < C_1 \le C_2 < \infty$ such that $C_1\|x\|_X \le \|x\|_0 \le C_2\|x\|_X$ for all $x \in X$. Note that C_2 must be greater than 1 since $\|x\|_X + \|Tx\|_Y \le C_2\|x\|_X$ for all $x \in X$ and T is arbitrary. For $x \in X$, we have

$$\|Tx\|_Y = \|x\|_0 - \|x\|_X \le (C_2 - 1)\|x\|_X .$$

Since $C_2 - 1 > 0$, T is bounded.

(b) Show that any linear map $T : X \to Y$ is bounded, if X is finite dimensional.

Solution Let a linear map $T : X \to Y$ be given. Since X is finite dimensional, say of dimension $n \in \mathbb{N}$, let $(u_1, ..., u_n)$ be a basis of X. Denote $C_{max} = \max_{i=1,...,n} \|Tu_i\|_Y$.
For any $y \in X$ there exist unique scalars $\alpha_1, ..., \alpha_n$ such that $y = \sum_{i=1}^n \alpha_i u_i$. Define a norm $\|y\|_\infty := \max\{|\alpha_1|, ..., |\alpha_n|\}$ on X. This is indeed a norm by the proof of Theorem 1.6, Lecture Notes 1. Since X is finite dimensional, all norms are equivalent by Theorem 1.6. Hence there exists $C > 0$ such that $\|y\|_\infty \leq C\|y\|_X$.

Let $x \in X$ and pick unique scalars $a_1, ..., a_n \in \mathbb{K}$ such that $x = \sum_{i=1}^n a_i u_i$. We obtain

$$\begin{aligned}
\|Tx\|_Y &= \left\| \sum_{i=1}^n a_i T u_i \right\|_Y \\
&\leq \sum_{i=1}^n |a_i| \|Tu_i\|_Y \\
&\leq C_{max} \sum_{i=1}^n |a_i| \\
&\leq C_{max} \cdot n \cdot \|x\|_\infty \\
&\leq C_{max} \cdot n \cdot C \cdot \|x\|_X
\end{aligned} \qquad \square$$

This shows that T is bounded.

(c) Suppose X is infinite dimensional. Show that there exists a linear map $T : X \to Y$, which is not bounded (= not continuous).

Solution Let $(e_i)_{i \in I}$ be a Hamel basis for X. Since X is infinite dimensional, I contains a subset with cardinality equal to the natural numbers. Identify this subset with the natural numbers such that $\mathbb{N} \subset I$. Fix an element $y \in Y$ with $\|y\|_Y = 1$. Consider the family $(y_i)_{i \in I}$ of Y defined by $y_i = 2^i y \|e_i\|_X$ if $i \in \mathbb{N}$ and $y_i = 0$ if $i \in I \setminus \mathbb{N}$.

By the definition of a Hamel basis (as given in the assignment text) there exists (exactly) one linear map $T : X \to Y$ satisfying $T(e_i) = y_i$ for all $i \in I$. Now, for $i \in \mathbb{N}$, we get

$$\left\| T(e_i) \right\|_Y = \left\| 2^i y \|e_i\|_X \right\|_Y = 2^i \|e_i\|_X .$$

Letting i tends towards infinity, we see that T is not bounded.

(d) Suppose again that X is infinite dimensional. Argue that there exists a norm $\|\cdot\|_0$ on X, which is not equivalent to the given norm $\|\cdot\|_X$, and which satisfies $\|x\|_X \leq \|x\|_0$ for all $x \in X$. Conclude that $(X, \|\cdot\|_0)$ is not complete if $(X, \|\cdot\|_X)$ is a Banach space.

Solution Since X is infinite dimensional, let $T : X \to Y$ be a linear, unbounded map. This is possible due to (c). Consider the norm from (a) given by $\|x\|_0 := \|x\|_X + \|Tx\|_Y$ for $x \in X$. Since T is not bounded, the two norms $\|\cdot\|_0$ and $\|\cdot\|_X$ are not equivalent (due to (a)). And clearly, $\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_0$ for all $x \in X$.

Finally from Homework 3, Problem 1, we conclude that since the norms are not equivalent, X cannot be complete with respect to both of them. So since $(X, \|\cdot\|_X)$ is a Banach space, $(X, \|\cdot\|_0)$ is not complete.

(e) Give an example of a vector space X equipped with two inequivalent norms $\|\cdot\|$ and $\|\cdot\|'$ satisfying $\|x\|' \leq \|x\|$ for all $x \in X$, such that $(X, \|\cdot\|)$ is complete, while $(X, \|\cdot\|')$ is not.

Solution Consider the space $l_1(\mathbb{N})$ equipped with the two norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$. As remarked in Lecture Notes 1, page 3, $l_1(\mathbb{N})$ equipped with $\|\cdot\|_1$ is a Banach space.
Also, the two norms are not equivalent: Consider sequences $x_n = (1, 1, ..., 1, 0, 0, ...) \in l_1(\mathbb{N})$ starting with n 1's and trailing zeroes. We have $\|x_n\|_\infty = 1$ for all $n \in \mathbb{N}$, but $\|x_n\|_1 = n$. Therefore, there is no $C > 0$ such that $\|x\|_1 \leq C\|x\|_\infty$ for all $x \in l_1(\mathbb{N})$. Hence, the norms are not equivalent.

We also have

$$\|x\|_\infty = \sup_{n\in\mathbb{N}}\{|x_n|\} \leq \sum_{n=1}^{\infty} |x_n| = \|x\|_1$$

for all $x \in l_1(\mathbb{N})$.

Finally, we must show that $(l_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete. So consider the sequence $(x_n)_{n\in\mathbb{N}}$ of sequences $x_n = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, 0, 0, ...)$. To show this is Cauchy, let $\varepsilon > 0$ be given. Let $N > \frac{1}{\varepsilon}$. Pick $n, m \geq N$, assume for simplicity that $n < m$. We get

$$\|x_m - x_n\|_\infty = \left\|(0, ..., 0, \frac{1}{n+1}, ..., \frac{1}{m}, 0, ...)\right\|_\infty = \frac{1}{n+1} < \varepsilon.$$

So $(x_n)_{n\in\mathbb{N}}$ is Cauchy, but the limit of the sequence in $l_\infty(\mathbb{N})$ is $x = (1, 1/2, 1/3, ...)$. This sequence x is not in $l_1(\mathbb{N})$, since the harmonic series diverges, so $(l_1(\mathbb{N}), \|\cdot\|_\infty)$ is not complete. $\square$

Problem 2

Let $1 \leq p < \infty$ be fixed, and consider the subspace M of the Banach space $(l_p(\mathbb{N}), \|\cdot\|_p)$, considered as a vector space over $\mathbb{C}$, given by

$$M = \{(a, b, 0, 0, ...) \mid a, b \in \mathbb{C}\}.$$

Let $f : M \to \mathbb{C}$ be given by $f(a, b, 0, 0, 0, ...) = a + b$, for all $a, b \in \mathbb{C}$.

(a) Show that f is bounded on $(M, \|\cdot\|_p)$ and compute $\|f\|$.

Solution For $x = (a, b, 0, 0, ...) \in M$, we have $\|x\|_p = |a|^p + |b|^p$. We want to determine the norm of f,

$$\|f\| = \sup_{x \in M,\ \|x\|_p = 1} |f(a, b, 0, 0, ...)| = \sup_{|a|^p + |b|^p = 1} |a + b|.$$

Note that p and $\frac{p}{p-1}$ are conjugate numbers. Let $x = (a, b, 0, ...) \in M$ with $\|x\|_p^p = |a|^p + |b|^p = 1$. Using Hölders inequality, we get

$$|a + b| \leq |a| + |b| = \|(a, b, 0, ...)\|_1 \leq \|(a, b, 0, ...)\|_p \|(1, 1, 0, ...)\|_{\frac{p}{p-1}} = 2^{\frac{p-1}{p}} \qquad \square$$

So we have found an upper bound on $\|f\|$. Now, take $a = b = \frac{1}{2^{1/p}}$ and put $x = (a, b, 0, ...)$. Then $\|x\|_p = \left(|a|^p + |b|^p\right)^{1/p} = 1^{1/p} = 1$. Furthermore, $|a+b| = 2 \cdot \frac{1}{2^{1/p}} = 2^{1-1/p} = 2^{\frac{p-1}{p}}$. So we indeed get $\|f\| = 2^{1-1/p}$. This of course also shows that f is bounded, since for any $x \in M$, $|f(x)| \leq \|f\| \|x\|_p$.

(b) Show that if $1 < p < \infty$, then there is a unique linear functional F on $l_p(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Solution We will show that the linear functional $F : l_p(\mathbb{N}) \to \mathbb{C}$ by $F(x) = F(x_1, x_2, x_3, ...) = x_1 + x_2$ is the unique extension of f satisfying $\|F\| = \|f\|$.
Let $G : l_p(\mathbb{N}) \to \mathbb{C}$ be a linear functional which is an extension of f and satisfies $\|G\| = \|f\|$. This also means that G is bounded, so $G \in l_p(\mathbb{N})^*$. From Homework 1, Problem 5, we know that the dual space is isometrically isomorphic to $l_q(\mathbb{N})$. The isomorphism is given by $T : l_q(\mathbb{N}) \to l_p(\mathbb{N})^*$ which is given by $T(x)(y) = \sum_{n=1}^{\infty} x_n y_n$ for all $y = (y_n)_{n \geq 1} \in l_p(\mathbb{N})$ (also due to Homework 1, Problem 5).

Denote $x = (x_1, x_2, ...) := T^{-1}(G) \in l_q(\mathbb{N})$. Hence we have

$$G(y) = T(x)(y) = \sum_{n=1}^{\infty} x_n y_n$$

for all $y \in l_p(\mathbb{N})$. The goal is now to determine x. Since G is an extension of f,

$$1 = f(1, 0, 0, ...) = G(1, 0, 0, ...) = x_1$$

Similarly, $1 = f(0,1,0,0,...) = G(0,1,0,0,...) = x_2$. We remember that p and q are conjugates, which ensures that $q = p/(p-1)$. Furthermore, T and T^{-1} are isometries, so $\|x\|_q = \|G\|$. We now have

$$\|x\|_q = \|x\|_{\frac{p}{p-1}} = \left(\sum_{n=1}^{\infty} |x_n|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \overset{\dagger}{\geq} (1+1)^{\frac{p-1}{p}} = 2^{1-1/p} = \|f\| = \|G\| = \|x\|_q .$$

Hence, we require equality at †. This amount to equality in

$$|1|^{\frac{p}{p-1}} + |1|^{\frac{p}{p-1}} + \sum_{n=3}^{\infty} |x_n|^{\frac{p}{p-1}} = 1 + 1.$$

So we conclude that $x_3 = x_4 = ... = 0$ and $x = (1,1,0,0,...)$. Now, we conclude that

$$G(y) = G(y_1, y_2, y_3, ...) = \sum_{n=1} x_n y_n = y_1 + y_2.$$

This determines G completely. We see now that $G \in l_p(\mathbb{N})^*$ is the unique linear functional extending f and satisfying $\|G\| = \|f\|$.

(c) Show that if $p = 1$, then there are infinitely many linear functionals F of $l_1(\mathbb{N})$ extending f and satisfying $\|F\| = \|f\|$.

Solution For all $n \in \mathbb{N}$, $n \geq 2$, define $F_n : l_1(\mathbb{N}) \to \mathbb{C}$ by $F_n(x) = F_n(x_1, x_2, ...) = \sum_{i=1}^{n} x_i$. Just like f, F_n is also linear. It follows from the facts that sums respect summation and scaling. Clearly, for $x = (a, b, 0, ...) \in M$, we have $F_n(x) = a + b = f(x)$, so F_n is an extension of f for all $n \in \mathbb{N}$. Furthermore, for $x \in l_1(\mathbb{N})$ with $\|x\|_1 \leq 1$, we have

$$|F_n(x)| = \left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{\infty} |x_i| = \|x\|_1 \leq 1.$$

Note that for $p = 1$, $\|f\| = 1$. So we have

$$\|F_n\| = \sup_{\|x\|_1 \leq 1} |F_n(x)| \leq 1 = \|f\| .$$

Since F_n is an extension of f, we of course have $\|F_n\| \geq \|f\|$. Hence we have $\|F_n\| = \|f\|$. So for all $n \in \mathbb{N}$, $n \geq 2$ we have specified a normpreserving extension of f. This means there are infinitely many. □

Problem 3

Let X be an infinite dimensional normed vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(a) Let $n \geq 1$. Show that no linear map $F : X \to \mathbb{K}^n$ is injective.

Solution Let $F : X \to \mathbb{K}^n$ be an arbitraty linear map. Since X is infinite dimensional, it is possible to find $n+1$ elements that are linearly independent (if this was not the case, the dimension of X would be maximally n). Denote them $x_1, ..., x_{n+1}$. Consider the span $Y := \text{Span}\{x_1, ..., x_{n+1}\}$. This is an $(n+1)$-dimensional subspace $Y \subset X$. Consider the linear identity embedding $1 : Y \to X$ and the composite map $1 \circ F : Y \to \mathbb{K}^n$. Since $1 \circ F$ is linear and Y is of a higher finite dimension than $\mathbb{K}^n$, the map $1 \circ F$ is not injective (follows from basic linear algebra, e.g. rank-nullity theorem). Since 1 is injective, F cannot be injective. So no linear map $F : X \to \mathbb{K}^n$ is injective.

(b) Let $n \geq 1$ be an integer and let $f_1, ..., f_n \in X^*$. Show that

$$\bigcap_{j=1}^{n} \ker f_j \neq \{0\}.$$

Solution Consider the map $F : X \to \mathbb{K}^n$ given by

$$F(x) = (f_1(x), ..., f_n(x))$$

for $x \in X$. Since $f_1, ..., f_n$ are linear, F is also linear. By (a), it is not injective which means that the kernel is non-zero. We get

$$\bigcap_{j=1}^{n} \ker f_j = \ker F \neq \{0\}$$

where the first equality follows from the equivalences:

$$\begin{aligned} x \in \bigcap_{j=1}^{n} \ker f_j &\Leftrightarrow \forall j = 1, ..., n : f_j(x) = 0 \\ &\Leftrightarrow F(x) = (f_1(x), ..., f_n(x)) = 0 \in \mathbb{K}^n \\ &\Leftrightarrow x \in \ker F \end{aligned}$$

$\square$

(c) Let $x_1, ..., x_n \in X$. Show that there exists $y \in X$ such that $\|y\| = 1$ and $\|y - x_j\| \geq \|x_j\|$ for all $j = 1, ..., n$.

Solution If $x_j = 0$ for some $j = 1, ..., n$, the condition $\|y - x_j\| \geq \|x_j\|$ is trivially fulfilled, and we will ignore this x_j when choosing y in the following. Therefore we may assume that all x_j's are non-zero.
Consider the following for $j = 1, ..., n$: By Theorem 2.7 of Lecture Notes 2, since $x_j \neq 0$ there exists $f_j \in X^*$ such that $\|f\|_j = 1$ and $f_j(x_j) = \|x\|_j$. By (b), we have $\bigcap_{i=j}^n \ker f_j \neq \{0\}$. So pick $y \in \bigcap_{i=j}^n \ker f_j$ with $\|y\| = 1$. Now, for all $j = 1, ..., n$, we obtain

$$\|x_j\| = f_j(x_j) = |-f_j(x_j)| = |f_j(y) - f_j(x_j)| = |f_j(y - x_j)| \leq \|f_j\| \|y - x_j\| = \|y - x_j\|.$$

This is what we wanted.

(d) Show that one cannot cover the unit sphere $S = \{x \in X \mid \|x\| = 1\}$ with a finite family of closed balls in X such that none of the balls contains 0.

Solution Suppose we have $x_1, ..., x_n \in X$ and $r_1, ..., r_n > 0$ and balls $\overline{B}(x_i, r_i)$ for $i = 1, ..., n$ that do not contain 0. Having the balls not contain 0 is exactly imposing the restriction $\|x_i\| > r_i$ for all $i = 1, ..., n$.
By (c), pick $y \in X$ with $\|y\| = 1$ and $\|y - x_i\| \geq \|x_i\|$ for all $i = 1, ..., n$. Note that $y \in S$ and

$$\|y - x_i\| \geq \|x_i\| > r_i,$$

so y is not contained in any of the balls $\overline{B}(x_i, r_i)$, $i = 1, ..., n$. This is what we wanted.

(e) Show that S is non-compact and deduce further that the closed unit ball in X is non-compact.

Solution Note that the statement in (d) also holds for open balls. If the balls in the statement were open, we would impose the restriction $\|x_i\| \geq r_i$ for all $i = 1, ..., n$ to make sure 0 was not contained in any of the balls. And then we would obtain $\|y - x_i\| \geq \|x_i\| \geq r_i$, which would indeed show that y is not contained in any of the balls $B(x_i, r_i)$, $i = 1, ..., n$.

To show S is non-compact, consider the family of open balls $B(x, \frac{1}{2})$ for all $x \in S$. None of these balls contain zero, since $\|x\| = 1 > \frac{1}{2}$ for all $x \in S$. So this is a family of open sets covering S. By (d) applied to open balls (as discussed above), there is no finite subfamily of these open balls that cover all of S. Hence, S is non-compact.

If the closed unit ball was compact, S would also be compact as it is a closed subset of the closed unit ball (S is closed since S is defined as the norm preimage of a singleton). This is not the case, and therefore the closed unit ball is also not compact. □

Problem 4

Let $L_1([0,1],m)$ and $L_3([0,1],m)$ (for short, we write L_1 and L_3 respectively) be the Lebesgue spaces on $[0,1]$. Recall from HW2 that $L_3 \subsetneq L_1$. For $n \geq 1$, define

$$E_n := \{f \in L_1 \mid \int_{[0,1]} |f|^3 dm \leq n\}.$$

(a) Given $n \geq 1$, is the set $E_n \subset L_1$ absorbing? Justify.

Solution The set E_n is not absorbing for any $n \geq 1$.
Since L_3 is properly contained in L_1, pick $f \in L_1 \setminus L_3$. Note that $f \neq 0$ since $0 \in L_3$. We have $\int_{[0,1]} |f| < \infty$, and $\int_{[0,1]} |f|^3 = \infty$. Let $t > 0$. Then we also get

$$\int_{[0,1]} |t^{-1}f|^3 = \frac{1}{t^3} \int_{[0,1]} |f|^3 = \infty$$

So $t^{-1}f$ does not lie in E_n. This holds for all $t > 0$, so E_n is not absorbing.

(b) Show that E_n has empty interior in $L_1([0,1],m)$, for all $n \geq 1$.

Solution Let $n \geq 1$ be fixed, and let $f \in E_n$ be arbitrary. To show that E_n has empty interior, let $\varepsilon > 0$ be given. Our goal is to construct $g \in L_1 \setminus E_n$ with $\|f-g\|_1 < \varepsilon$.
We may assume $\varepsilon < 1$. Pick $M := \frac{4n}{\sqrt{\varepsilon}}$. In particular, we have $M > 2n$. Denote

$$A = \{x \in [0,1] \mid |f(x)| < M\} \subseteq [0,1].$$

We must have $m(A) \geq 1/2$ by the following argument: For $x \in [0,1] \setminus A$, we have $|f| \geq M > 2n$ and hence $|f|^3 > 2n$. This gives us

$$n \geq \int_{[0,1]} |f|^3 dm \geq \int_{[0,1]\setminus A} 2n dm = 2n \cdot m([0,1] \setminus A).$$

From here it follows that $m([0,1] \setminus A) \leq 1/2$ and hence $m(A) \geq 1/2$.

Now, let B be a subset of A with $m(B) = \frac{\varepsilon}{4M}$. This is possible since $m(B) = \frac{\varepsilon}{4M} \leq \frac{1}{8n} < \frac{1}{2} \leq m(A)$.

Let us define $g \in L_1([0,1],m)$ by

$$g(x) = \begin{cases} f(x) + 2M, & x \in B \\ f(x), & x \notin B \end{cases}$$

First of all, since $f \in L_1$ we also get $g \in L_1$. We further get

$$\|g-f\|_1 = \int_{[0,1]} |g-f| dm = \int_B 2M dm = 2M \cdot m(B) = 2M \cdot \frac{\varepsilon}{4M} = \frac{1}{2}\varepsilon < \varepsilon.$$

Remember that on $B \subset A$, we have $|f(x)| < M$. In the following, we will use the following inequality which is derived using the reverse triangle inequality:

$$|2M + f(x)| \geq \big|2M - |-f(x)|\big| = 2M - |f(x)| > M$$

Finally, we obtain

$$\int_{[0,1]} |g|^3 dm \geq \int_B |f(x) + 2M|^3 dm \geq \int_B M^3 dm = M^3 \cdot m(B)$$

$$= M^3 \frac{\varepsilon}{4M} = \frac{1}{4} M^2 \varepsilon = \frac{1}{4} \left(\frac{4n}{\sqrt{\varepsilon}} \right)^2 \varepsilon = 4n^2 > n$$

So $g \in L_1([0,1], m) \setminus E_n$ and $\|f - g\|_1 < \varepsilon$. Since $f \in E_n$ was arbitrary and ε was arbitrary (less than 1), we conclude that E_n has empty interior.

(c) Show that E_n is closed in $L_1([0,1], m)$, for all $n \geq 1$.

Solution Let $n \geq 1$ be fixed. Let $(f_k)_{k \in \mathbb{N}} \subseteq E_n$ be a sequence in E_n which converges in $\|\cdot\|_1$ to $f \in L_1([0,1], m)$. To show E_n is closed, we must show that $f \in E_n$.

By Corollary 13.8 of Schilling, there exists a subsequence $(f_{k(l)})_{l \geq 1}$ such that $f_{k(l)}(x) \to f(x)$ as $l \to \infty$ for a.e. $x \in [0,1]$. We deduce that $|f_{k(l)}|^3 \to |f|^3$ as $l \to \infty$ for a.e. $x \in [0,1]$. Note also that since f and f_k are L_1-functions for all $k \geq 1$, they are measurable, and hence $|f_k|^3$ and $|f|^3$ are positive and measurable for all $k \geq 1$.

Now, we will use Fatou's lemma (Theorem 9.11 of Schilling) on $(f_{k(l)})_{l \geq 1}$. As stated, the theorem requires that $f_{k(l)}(x) \to f(x)$ for all $x \in [0,1]$. But since the theorem gives an inequality on integrals, pointwise convergence for a.e. $x \in [0,1]$ is sufficient. We get

$$\int_{[0,1]} |f|^3 dm \leq \lim_{l \to \infty} \int_{[0,1]} |f_{k(l)}|^3 dm \leq n$$

The last inequality follows from the fact that $f_{k(l)} \in E_n$, so we have $\int_{[0,1]} |f_{k(l)}|^3 dm \leq n$ for all $l \in \mathbb{N}$. Since the inequality is weak, it also holds in the limit. We conclude that $f \in E_n$, and hence E_n is closed.

(d) Conclude from (b) and (c) that $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$.

Solution We know from (c) fra E_n is closed for all $n \in \mathbb{N}$ and hence we know from (b) that $\text{Int}(\overline{E}_n = \text{Int}(E_n) = \emptyset$. This shows that E_n is nowhere dense for all $n \in \mathbb{N}$. Also, it is clear by definition that $\bigcup_{n=1}^{\infty} E_n = L_3([0,1], m)$. We conclude that $(E_n)_{n \in \mathbb{N}} \subset L_3$ is a sequence of nowhere dense sets such that $L_3([0,1], m) = \bigcup_{n=1}^{\infty} E_n$. So $L_3([0,1], m)$ is of first category in $L_1([0,1], m)$. $\square$

Problem 5

Let H be an infinite dimensional separable Hilbert space with associated norm $\|\cdot\|$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H, and let $x \in H$.

(a) Suppose that $x_n \to x$ in norm as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$ as $n \to \infty$? Give a proof or a counterexample.

Solution We give a proof. The assumption $x_n \to x$ as $n \to \infty$ can be spelled out as $\|x_n - x\| \to 0$ as $n \to \infty$. By the reverse triangle inequality, we have

$$\big|\|x_n\| - \|x\|\big| \leq \|x_n - x\| \to 0$$

as $n \to \infty$. We conclude that $\|x_n\| \to \|x\|$ as $n \to \infty$.

(b) Suppose that $x_n \to x$ weakly as $n \to \infty$. Does it follow that $\|x_n\| \to \|x\|$ as $n \to \infty$? Give a proof or a counterexample.

Solution We give a counterexample, i.e. we specify a sequence $(x_n)_{n\geq 1}$ in H such that $x_n \to x$ weakly, but $\|x_n\| \nrightarrow \|x\|$.

Since H is separable, there exists an orthonormal basis $(e_n)_{n\geq 1}$ in H. Let us show that this sequence converges weakly to 0. We will use Homework 4, Problem 2(a), to show that $(e_n)_{n\geq 1}$ converges weakly to 0. Since H is a Hilbert space, any functional $f \in H^*$ is of the form $f(x) = \langle x, y\rangle$ for some $y \in H$. So let us show that $\langle e_n, y\rangle$ converges to $\langle 0, y\rangle = 0$. Since H has an orthonormal basis, Theorem 16.21 of Schilling gives us that

$$\sum_{n=1}^{\infty} |\langle e_n, y\rangle|^2 = \sum_{n=1}^{\infty} |\langle y, e_n\rangle|^2 = \|y\|^2 < \infty.$$

So in particular, the sequence $|\langle e_n, y\rangle|^2$ converges to 0 as $n \to \infty$. We get that $|\langle e_n, y\rangle| \to 0$ as $n \to \infty$. And since $|\cdot|$ is a norm on $\mathbb{K}$, we deduce that $\langle e_n, y\rangle \to 0 = \langle 0, y\rangle$ as $n \to \infty$. By HW4, Problem 2(a), we conclude that $(e_n)_{n\geq 1} \to 0$ weakly as $n \to \infty$.

Since $(e_n)_{n\geq 1}$ is an orthonormal basis in H, we have $\|e_n\| = 1$ for all $n \geq 1$. So we see that $\|e_n\| = 1 \to 1 \neq 0$ as $n \to \infty$. In particular, $\|e_n\|$ does not converge to $\|0\| = 0$.
In conclusion, we have found a sequence with $e_n \to 0$ weakly, but $\|e_n\| \nrightarrow \|0\| = 0$ as $n \to \infty$.

(c) Suppose that $\|x_n\| \leq 1$, for all $n \geq 1$, and that $x_n \to x$ weakly as $n \to \infty$. Is it true that $\|x\| \leq 1$? Give a proof or a counterexample.

Solution We give a proof. Consider the set

$$A = \{y \in H \mid \|y\| \leq 1\}.$$

This set is closed in norm, $\overline{A}^{\|\cdot\|} = A$. Let us show it is convex. Let $x, y \in A$, i.e. $\|x\|, \|y\| \leq 1$ and $0 \leq \alpha \leq 1$. We get

$$\left\|\alpha x + (1-\alpha)y\right\| \leq |\alpha|\|x\| + |1-\alpha|\|y\| \leq \alpha + (1-\alpha) = 1.$$

So indeed, A is convex. By Theorem 5.7 of Lecture Notes 5, we get $A = \overline{A}^{\|\cdot\|} = \overline{A}^{\tau_w}$, that is, the norm closure and weak closure of A are equal, and since A is closed in norm, it is also closed weakly. Therefore since $\|x_n\| \in A$ by assumption, and $x_n \to x$ weakly as $n \to \infty$, we have $x \in A$, i.e. $\|x\| \leq 1$.