

FunkAn Mandatory 2

Line Lindholm Jensen, zmj832

January 2021

Problem 1

Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n \geq 1}$.
Set $f_N = N^{-1} \sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.

(a)

Show that $f_N \rightarrow 0$ weakly, as $N \rightarrow \infty$, while $\|f_N\| = 1$, for all $N \geq 1$.

First notice that $\|f_N\| = 1$, since by Pythagoras identity we have

$$\|f_N\| = \left\| N^{-1} \sum_{n=1}^{N^2} e_n \right\| = N^{-1} \left\| \sum_{n=1}^{N^2} e_n \right\| = N^{-1} \left(\sum_{n=1}^{N^2} \|e_n\|^2 \right)^{1/2} = \frac{1}{N} \sqrt{N^2} = 1. \quad \checkmark$$

We wish to show that the sequence $(f_N)_{N \geq 1}$ converges weakly to 0, as $n \rightarrow \infty$. From Homework 4, problem 2 we know that the sequence $(f_N)_{N \geq 1}$ in H converges to 0 in the weak topology τ_w on H if and only if the net $(g(f_N))_{N \geq 1}$ converges to $g(0)$ when $N \rightarrow \infty$, for every $g \in H^*$.

for some $x \in H$, for every $y \in H$.

Now since H is a Hilbert space, then by Riesz representation theorem every $g \in H^*$ is on the form $g(y) = \langle y, x \rangle$, for every $x, y \in H$. So we want to show that the inner product $\langle f_N, x \rangle$ converges to $\langle 0, x \rangle = 0$ for all $x \in H$.

First note we can write $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$, for all $x \in H$. Then by Parsevals identity we have

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \|x\|^2 < \infty,$$

meaning there exists $M_1 \geq 1$ such that for $\varepsilon > 0$, then

$$\sum_{i=M_1+1}^{\infty} |\langle x, e_i \rangle|^2 < \left(\frac{\varepsilon}{2}\right)^2$$

Now we see that

$$\begin{aligned}
|\langle f_N, x \rangle| &= \left| \langle f_N, \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \rangle \right| \\
&\leq \left| \langle f_N, \sum_{i=1}^M \langle x, e_i \rangle e_i \rangle \right| + \left| \langle f_N, \sum_{i=M+1}^{\infty} \langle x, e_i \rangle e_i \rangle \right| \\
&= \left| \langle N^{-1} \sum_{n=1}^{N^2} e_n, \sum_{i=1}^M \langle x, e_i \rangle e_i \rangle \right| + \left| \langle f_N, \sum_{i=M+1}^{\infty} \langle x, e_i \rangle e_i \rangle \right| \\
&= \left| N^{-1} \sum_{n=1}^{N^2} \sum_{i=1}^M \overline{\langle x, e_i \rangle} \langle e_n, e_i \rangle \right| + \left| \langle f_N, \sum_{i=M+1}^{\infty} \langle x, e_i \rangle e_i \rangle \right| \\
&\leq N^{-1} \sum_{i=1}^M |\overline{\langle x, e_i \rangle}| \sum_{n=1}^{N^2} |\langle e_n, e_i \rangle| + \|f_N\| \left\| \sum_{i=M+1}^{\infty} \langle x, e_i \rangle e_i \right\| \\
&\leq N^{-1} \sum_{i=1}^M |\overline{\langle x, e_i \rangle}| + \left(\sum_{i=M+1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{1/2}
\end{aligned}$$

where we have used Cauchy-Schwarz, Pythagoras identity and that $\|f_N\| = 1$. Now notice that $N^{-1} \rightarrow 0$ when $N \rightarrow \infty$. Thus for every $\varepsilon > 0$ we can choose $N_1 \geq 1$ large enough such that

$$N^{-1} \sum_{i=1}^M |\overline{\langle x, e_i \rangle}| < \frac{\varepsilon}{2}.$$

← This N_1 will depend on M .

Then for all $N, M \geq N_1, M_1$ we have that

$$|\langle f_N, x \rangle| \leq N^{-1} \sum_{i=1}^M |\overline{\langle x, e_i \rangle}| + \left(\sum_{i=M+1}^{\infty} |\langle x, e_i \rangle|^2 \right)^{1/2} < \frac{\varepsilon}{2} + \sqrt{\left(\frac{\varepsilon}{2}\right)^2} = \varepsilon.$$

So in conclusion when $N \rightarrow \infty$, then $|\langle f_N, x \rangle| \rightarrow 0$, for all $x \in H$. Hence $f_N \rightarrow 0$ weakly, as $N \rightarrow \infty$. ✓

(b)

Let K be the norm closure of $\text{co}\{f_N : N \geq 1\}$. Argue that K is weakly compact, and that $0 \in K$.

Since the convex hull by definition is a convex subset of H , we get by theorem 5.7 that

$$K = \overline{\text{co}\{f_N : N \geq 1\}}^{||\cdot||} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w},$$

hull

so K is the weak closure of $\text{co}\{f_N : N \geq 1\}$. Note that H is a Hilbert space, so it is reflexive by proposition 2.10. Now from Theorem 6.3 we get that it is reflexive if and only if $\overline{B}_H(0, 1)$ is compact with respect to the weak topology τ_w on H . So we want to show that K is in fact a subset of the closed unit ball, i.e. $K \subseteq \overline{B}_H(0, 1)$, since closed subsets of compact sets are again compact. Recall that we showed in (a) that f_N has unit length for all $N \geq 1$, i.e. $\|f_N\| = 1$, hence $f_N \in \overline{B}_H(0, 1)$, $\forall N \geq 1$. Now take an arbitrary element of the convex hull $\sum_{i=1}^n \alpha_i x_i$, where $x_i \in \{f_N : N \geq 1\}$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$, $n \in \mathbb{N}$. Then

$$\left\| \sum_{i=1}^n \alpha_i x_i \right\| \leq \sum_{i=1}^n |\alpha_i| \|x_i\| = \sum_{i=1}^n |\alpha_i| = 1,$$

so $\text{co}\{f_N : N \geq 1\} \subseteq \overline{B_H}(0, 1)$. Hence $K = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w} \subseteq \overline{\overline{B_H}(0, 1)} = \overline{B_H}(0, 1)$. Thus since $\overline{B_H}(0, 1)$ is weakly compact and K is a closed subset in the weak topology, then K is also weakly compact. ✓

From (a) $f_N \rightarrow 0$ weakly as $n \rightarrow \infty$, so 0 is contained in the weakly closure, i.e. $0 \in \overline{\{f_N : N \geq 1\}}^{\tau_w}$. By definition the convex hull is the smallest convex set containing $\{f_N : N \geq 1\}$, so clearly we have that $\{f_N : N \geq 1\} \subseteq \text{co}\{f_N : N \geq 1\}$. Hence $0 \in \overline{\{f_N : N \geq 1\}}^{\tau_w} \subseteq \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w} = K$. ✓

(c)

Show that 0, as well as each f_N , $N \geq 1$, are extreme points in K .

First we wish to show that $0 \in \text{Ext}(K)$. Suppose that 0 can be written as a convex combination, $0 = \alpha x + (1 - \alpha)y$ for $0 < \alpha < 1$ and some $x, y \in K$, where K is the norm closure of $\text{co}\{f_N : N \geq 1\}$. Then we know there exists sequences $(x_n)_{n \geq 1}, (y_n)_{n \geq 1} \subseteq \text{co}\{f_N : N \geq 1\}$, such that $(x_n)_{n \geq 1}, (y_n)_{n \geq 1}$ converges, respectively, to x and y in norm, when $n \rightarrow \infty$. Now let e_m be an arbitrary element from the orthonormal basis $(e_n)_{n \geq 1}$, then we notice that

$$0 = \langle 0, e_m \rangle = \langle \alpha x + (1 - \alpha)y, e_m \rangle = \alpha \langle x, e_m \rangle + (1 - \alpha) \langle y, e_m \rangle. \quad (1)$$

We wish to show that $\langle x, e_m \rangle, \langle y, e_m \rangle \geq 0$, since this implies $\langle x, e_m \rangle = \langle y, e_m \rangle = 0$, and we would be done. So consider $\langle x, e_m \rangle$, then

$$\langle x, e_m \rangle = \langle \lim_{n \rightarrow \infty} x_n, e_m \rangle = \lim_{n \rightarrow \infty} \langle x_n, e_m \rangle,$$

by continuity of the inner product. Now note that every element x_n in $\text{co}\{f_N : N \geq 1\}$ is of the form $x_n = \sum_{i=1}^k \alpha_i f_{N_i}$, where $\alpha_i > 0$, and $\sum_{i=1}^k \alpha_i = 1$. Consider $\langle x_n, e_m \rangle$, then

miss elements like $\alpha_i = 20$, otherwise you $\frac{1}{2}f_1 + \frac{1}{2}f_3$.

$$\langle x_n, e_m \rangle = \langle \sum_{i=1}^k \alpha_i f_{N_i}, e_m \rangle = \sum_{i=1}^k \alpha_i \langle f_{N_i}, e_m \rangle.$$

Now notice that

$$\langle f_{N_i}, e_m \rangle = \langle N_i^{-1} \sum_{n=1}^{N_i^2} e_n, e_m \rangle = N_i^{-1} \sum_{n=1}^{N_i^2} \langle e_n, e_m \rangle = \begin{cases} 0 & m > N_i^2 \\ N_i^{-1} & m \leq N_i^2 \end{cases}$$

So in particular $\langle f_{N_i}, e_m \rangle \geq 0$, which implies $\langle x_n, e_m \rangle \geq 0$, hence $\langle x, e_m \rangle = \lim_{n \rightarrow \infty} \langle x_n, e_m \rangle \geq 0$. So $\langle x, e_m \rangle \geq 0$ and in a similar way we obtain that $\langle y, e_m \rangle \geq 0$. Hence from (1) we have that

$$0 = \alpha \langle x, e_m \rangle + (1 - \alpha) \langle y, e_m \rangle$$

and since $\alpha, (1 - \alpha) > 0$ and both $\langle x, e_m \rangle, \langle y, e_m \rangle \geq 0$, we must have $\langle x, e_m \rangle = \langle y, e_m \rangle = 0$, hence $x = y = 0$ by completeness (5.27 Folland). So 0 is not a proper convex combination of two other points in K , i.e. 0 is an extreme point in K . ✓

Now we will show that $f_N \in \text{Ext}(K)$ for every $N \geq 1$. Fix N and suppose that f_N can be written as a convex combination, $f_N = \alpha x + (1 - \alpha)y$ for $0 < \alpha < 1$ and some $x, y \in K$. Note that $\|x\| \leq 1$ since $x \in K \subseteq \overline{B_H}(0, 1)$, as we showed in (b). Hence $|\langle x, f_N \rangle| \leq \|x\| \leq 1$, by Cauchy-Schwarz. In the same way we obtain $\|y\| \leq 1$ and $|\langle y, f_N \rangle| \leq 1$.

Then by the triangle inequality and linearity of the inner product,

$$\begin{aligned} 1 &= |\langle f_N, f_N \rangle| = |\langle \alpha x + (1 - \alpha)y, f_N \rangle| \\ &\leq \alpha |\langle x, f_N \rangle| + (1 - \alpha) |\langle y, f_N \rangle| \end{aligned}$$

implies $|\langle x, f_N \rangle| = |\langle y, f_N \rangle| = 1$. Now note since $\|f_N\| = 1$ and $\|x\| \leq 1$, then

$$\|x\| \|f_N\| = \|x\| \leq |\langle x, f_N \rangle|.$$

Hence we have equality in Cauchy-Schwarz, thus

$$\|x\| \|f_N\| = |\langle x, f_N \rangle| = 1,$$

so $1 = \|x\| = \|f_N\|$.

Now by Lemma 26.3 in Schilling, equality in Cauchy-Schwarz implies there exists some $\lambda > 0$ such that $f_N = \lambda x$. But then $\|f_N\| = |\lambda| \|x\|$, so $\lambda = 1$, hence in particular $f_N = x$. In a similar way we get that $f_N = y$. Thus $f_N = x = y$, so we conclude that f_N for every $N \geq 1$ are extreme points in K .

(d)

Are there any other extreme points in K ?

Denote $F = \{f_N : N \geq 1\} \subseteq K$, such that $K = \overline{\text{co}(F)}^{\tau_w}$.

Notice that from (c) we know $F \cup \{0\} \subseteq \text{Ext}(K)$, so we will show that $\text{Ext}(K) \subseteq F \cup \{0\}$.

We claim that the closure of a convex set X is convex. Suppose X is convex, then for all $x, y \in \overline{X}$, there exists sequences $(x_n)_{n \geq 1}, (y_n)_{n \geq 1} \subseteq X$, such that $x_n \rightarrow x$ and $y_n \rightarrow y$, when $n \rightarrow \infty$. Then since X is convex, $\alpha x_n + (1 - \alpha)y_n \in X$, for all $n \geq 1$ and every $0 < \alpha < 1$. Now by linearity of limits

$$\alpha x_n + (1 - \alpha)y_n \rightarrow \alpha x + (1 - \alpha)y,$$

when $n \rightarrow \infty$, which implies $\alpha x + (1 - \alpha)y \in \overline{X}$. So \overline{X} is convex, which proofs the claim.

Notice from the above and from (b) that K is a non-empty, convex, weakly compact subset of H and $K = \overline{\text{co}(F)}^{\tau_w}$. Further (H, τ_w) is a Hilbert space with the weak topology, so it is $LCTVS$.

Hence Theorem 7.9 yields $\text{Ext}(K) \subseteq \overline{F}^{\tau_w}$. Now we claim that \overline{F}^{τ_w} is equal to $F \cup \{0\}$. Notice that \overline{F}^{τ_w} is the union of F and all the weak limit points in F . By (a) $f_N \rightarrow 0$ weakly as $N \rightarrow \infty$, so since the weak topology is hausdorff and since there are infinitely many elements in F that lies close to 0, then 0 must be the only limit of $(f_N)_{N \geq 1}$. So in particular 0 is the only accumulation point of $(f_N)_{N \geq 1}$, hence every weakly convergent sequence in F must either converge to 0 or an element in F . Hence $\text{Ext}(K) \subseteq \overline{F}^{\tau_w} = F \cup \{0\}$.

In conclusion there are no other extreme points in K , besides 0 and $f_N, \forall N \geq 1$.

Problem 2

Let X and Y be infinite dimensional Banach spaces.

(a)

Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n \geq 1}$ in X and $x \in X$, show that $x_n \rightarrow x$ weakly, as $n \rightarrow \infty$, implies that $Tx_n \rightarrow Tx$ weakly, as $n \rightarrow \infty$.

Suppose $T \in \mathcal{L}(X, Y)$ and $(x_n)_{n \geq 1}$ is a sequence in X , where $x \in X$, such that $x_n \rightarrow x$ weakly, as $n \rightarrow \infty$.

From Homework 4, problem 2 we know that $x_n \rightarrow x$ in the weak topology τ_w if and only if $f(x_n)$ converges weakly to $f(x)$ when $n \rightarrow \infty$, for every $f \in X^*$. Notice that $y \circ T \in X^*$, for every $y \in Y^*$. Then we see that

$$y(Tx_n) = y \circ T(x_n) \rightarrow y \circ T(x) = y(Tx)$$

weakly, as $n \rightarrow \infty$. Hence $Tx_n \rightarrow Tx$ weakly when $n \rightarrow \infty$ by Homework 4, problem 2.

(b)

Let $T \in \mathcal{K}(X, Y)$. For a sequence $(x_n)_{n \geq 1}$ in X and $x \in X$, show that $x_n \rightarrow x$ weakly, as $n \rightarrow \infty$, implies that $\|Tx_n - Tx\| \rightarrow 0$, as $n \rightarrow \infty$.

Suppose $T \in \mathcal{K}(X, Y)$ and $(x_n)_{n \geq 1}$ is a sequence in X for $x \in X$ such that $x_n \rightarrow x$ weakly, as $n \rightarrow \infty$. By Homework 4, problem 2 then $\sup\{\|x_n\| : n \geq 1\} < \infty$, so $(x_n)_{n \geq 1}$ is bounded, and specially every subsequence $(x_{n_k})_{k \geq 1}$ is bounded. By proposition 8.2 since T is compact and $(x_{n_k})_{k \geq 1}$ is bounded, then there exists a subsequence $(x_{n_{k_j}})_{j \geq 1}$ such that $\|Tx_{n_{k_j}} - y\| \rightarrow 0$ as $j \rightarrow \infty$, for some $y \in Y$. Since norm convergence implies weak convergence, we have that $Tx_{n_{k_j}} \rightarrow y$ weakly when $j \rightarrow \infty$.

Notice by (a) since $T \in \mathcal{K}(X, Y)$ implies $T \in \mathcal{L}(X, Y)$, then $Tx_n \rightarrow Tx$ weakly as $n \rightarrow \infty$. But then we must also have that $Tx_{n_{k_j}} \rightarrow Tx$ weakly when $j \rightarrow \infty$, for every (sub)subsequence $(x_{n_{k_j}})_{j \geq 1}$.

Now since the weak topology τ_w is Hausdorff, then by uniqueness of limits $y = Tx$, so $\|Tx_{n_{k_j}} - Tx\| \rightarrow 0$ as $j \rightarrow \infty$. Hence every subsequence $(Tx_{n_{k_j}})_{j \geq 1}$ of $(Tx_n)_{n \geq 1}$ has a convergent subsequence $(Tx_{n_{k_j}})_{j \geq 1}$ such that $\|Tx_{n_{k_j}} - Tx\| \rightarrow 0$ when $j \rightarrow \infty$.

Now assume for contradiction that $\|Tx_n - Tx\| \not\rightarrow 0$, when $n \rightarrow \infty$. Then there exists $\varepsilon > 0$ such that for every $K \in \mathbb{N}$ we can choose some $n_K > K$ such that $\|Tx_{n_K} - Tx\| \geq \varepsilon$. Now for every K choose the smallest n_K such that this is satisfied, then we can construct a sequence $(x_{n_K})_{K \geq 1}$, such that $\|Tx_{n_K} - Tx\| \geq \varepsilon$ for all $K \geq 1$. Note that $(x_{n_K})_{K \geq 1}$ is a subsequence of $(x_n)_{n \geq 1}$, so $(Tx_{n_K})_{K \geq 1}$ is in particular a subsequence of $(Tx_n)_{n \geq 1}$, hence $(Tx_{n_K})_{K \geq 1}$ contains a convergent subsequence, but this is a contradiction since $\|Tx_{n_K} - Tx\| \geq \varepsilon$ for all $K \geq 1$. Hence, $\|Tx_n - Tx\| \rightarrow 0$, as $n \rightarrow \infty$.

(c)

Let H be a separable infinite dimensional Hilbert space. If $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \rightarrow 0$, as $n \rightarrow \infty$, whenever $(x_n)_{n \geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Suppose for contraposition that T is not compact. By proposition 8.2 then $T(\overline{B_H(0, 1)})$ is not totally bounded, meaning there exists $\delta > 0$ such that $\overline{B_H(0, 1)}$ is not contained in a finite union of open balls with radius δ . We wish to construct a sequence $(x_n)_{n \geq 1}$ in $\overline{B_H(0, 1)}$ recursively. First let $x_1 \in \overline{B_H(0, 1)}$. Then we may assume for some $n \in \mathbb{N}$, that we have found x_2, \dots, x_n such that $\|Tx_j - Tx_k\| \geq \delta$, for all $j \neq k \leq n$. Then define the set $S = T(\overline{B_H(0, 1)}) \setminus (\cup_{i=1}^n B_Y(Tx_i, \delta))$, for $x_i \in \overline{B_H(0, 1)}$. Notice that S is non-empty, since otherwise $T(\overline{B_H(0, 1)}) \subseteq \cup_{i=1}^n B_Y(Tx_i, \delta)$, but this contradicts T not being totally bounded. Now choose $x_{n+1} \in \overline{B_H(0, 1)}$ such that $Tx_{n+1} \in S$. Then $Tx_{n+1} \in (\cup_{i=1}^n B_Y(Tx_i, \delta))^c = \cap_{i=1}^n (B_Y(Tx_i, \delta))^c$, so $Tx_{n+1} \notin B_Y(Tx_i, \delta)$ for any $i \leq n$. Thus x_{n+1} satisfies that $\|Tx_{n+1} - Tx_i\| \geq \delta$ for every $i \leq n$. In this way we can obtain a sequence $(x_n)_{n \geq 1}$ in $\overline{B_H(0, 1)}$ satisfying $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$.

Note that H is a Hilbert space, then H is reflexive by proposition 2.10 and by Theorem 6.3 it follows that $\overline{B_H(0, 1)}$ is compact with respect to the weak topology. Note that since isometries preserves metric properties and $y \mapsto F_y$ is an isometry for $y \in H, F_y \in H^*$, then H and H^* has similar metric properties. Hence H being separable implies that H^* is separable. Now by theorem 5.13 then $(\overline{B_H(0, 1)}, \tau_w)$ is metrizable, since there is an isometric isomorphism between H and its double dual H^{**} . Notice that

$(\overline{B_H(0,1)}, \tau_w)$ is compact if and only if it is sequentially compact, so $(\overline{B_H(0,1)}, \tau_w)$ is weakly sequentially compact, meaning every sequence in $\overline{B_H(0,1)}$ has a subsequence that converges in $\overline{B_H(0,1)}$. Thus we have a subsequence $(x_{n_k})_{k \geq 1} \subseteq (x_n)_{n \geq 1} \subseteq \overline{B_H(0,1)}$ such that $\|Tx_{n_k} - Tx\| \rightarrow 0$ as $k \rightarrow \infty$. But $\|Tx_n - Tx_m\| \geq \delta$ for every $n \neq m$, so $\|Tx_{n_k} - Tx\| \not\rightarrow 0$ as $k \rightarrow \infty$. Hence we reach a contradiction, so if $x_n \rightarrow x$ as $n \rightarrow \infty$ implies $\|Tx_n - Tx\| \rightarrow 0$ as $n \rightarrow \infty$, then T is compact.

(d)

Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

Let $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ and let $(x_n)_{n \geq 1}$ be a sequence in $\ell_2(\mathbb{N})$ such that $x_n \rightarrow x$ weakly when $n \rightarrow \infty$. Then by (a) we have $Tx_n \rightarrow Tx$ weakly, as $n \rightarrow \infty$. Now by remark 5.3 weakly convergence coincides with norm convergence in $\ell_1(\mathbb{N})$, hence $\|Tx_n - Tx\| \rightarrow 0$, as $n \rightarrow \infty$. So we obtain that $x_n \rightarrow x$ weakly as $n \rightarrow \infty$ implies $Tx_n \rightarrow Tx$ in norm, as $n \rightarrow \infty$. Hence by (c) T is compact.

(e)

Show that no $T \in \mathcal{K}(X, Y)$ is onto.

Suppose to reach a contradiction that T is onto, i.e. surjective. By the open mapping theorem (3.15), then T is open. Now by the note on page 18 we know this means there exists some $r > 0$, such that $B_Y(0, r) \subset T(B_X(0, 1))$. Thus in particular $\overline{B_Y(0, r)} \subset \overline{T(B_X(0, 1))}$. Now we know that $\overline{T(B_X(0, 1))}$ is compact, by the definition of T being compact. So since closed subset of compact sets are again compact, then in particular $\overline{B_Y(0, r)}$ is compact.

We claim that it is possible to scale the unit ball by some constant $r > 0$, such that $r\overline{B(0, 1)} = \overline{B(0, r)}$, for all $r > 0$. Let $x \in r\overline{B(0, 1)}$, then there exists $x' \in \overline{B(0, 1)}$ such that $x = rx'$. Thus $\|x\| = r\|x'\| \leq r$, so $x \in \overline{B(0, r)}$. For the other way, take $x \in \overline{B(0, r)}$, then $x = \frac{x}{r}r$, so $\|\frac{x}{r}\| = \frac{\|x\|}{r} \leq 1$, since $\|x\| \leq r$. Hence $\frac{x}{r} \in \overline{B(0, 1)}$, so $x \in r\overline{B(0, 1)}$. This proves the claim.

Now for all $r > 0$ we consider the function $f : Y \rightarrow Y$, given by $x \mapsto \frac{1}{r}x$, which is definitely continuous. Since $B_Y(0, r)$ is compact we get that $f(\overline{B_Y(0, r)}) = \frac{1}{r}\overline{B_Y(0, r)} = \overline{B_Y(0, 1)}$ is compact, since the image of a compact set under a continuous function is compact. Now from Mandatory assignment 1, problem 3(e) we know that $B_Y(0, 1)$ is non-compact, hence we reach a contradiction.

In conclusion no $T \in \mathcal{K}(X, Y)$ is onto.

(f)

Let $H = L_2([0, 1], m)$, and consider the operator $M \in L(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0, 1]$. Justify that M is self-adjoint, but not compact.

First note that $t = \bar{t}$, since $t \in [0, 1]$. Hence for $f, g \in H$ we have

$$\langle Mf(t), g(t) \rangle = \langle tf(t), g(t) \rangle = \langle f(t), \bar{t}g(t) \rangle = \langle f(t), tg(t) \rangle = \langle f(t), Mg(t) \rangle$$

Thus M is self-adjoint, i.e. $M = M^*$.

Now assume to reach a contradiction that T is compact. Note that $H = L_2([0, 1], m)$ is an infinite dimensional Hilbert space, which is indeed separable by Homework 4, problem 4. Since M is a self-adjoint,

compact operator on H , then the spectral theorem for self-adjoint compact operators yields H has an orthonormal basis consisting of eigenvectors of M with corresponding eigenvalues. But by Homework 6, problem 3 then M has no eigenvectors, so there's a contradiction. Hence M is not compact. ✓

Problem 3

Consider the Hilbert space $H = L_2([0, 1], m)$, where m is the Lebesgue measure. Define $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ by

$$K(s, t) = \begin{cases} (1-s)t, & \text{if } 0 \leq t \leq s \leq 1 \\ (1-t)s, & \text{if } 0 \leq s < t \leq 1 \end{cases}$$

and consider $T \in \mathcal{L}(H, H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s, t)f(t)dm(t), \quad s \in [0, 1], \quad f \in H.$$

(a)

Justify that T is compact.

Notice that $[0, 1]$ is a compact Hausdorff topological space, since it is a closed and bounded interval in \mathbb{R} . Furthermore since $m([0, 1]) = 1 < \infty$, then m is a finite Borel measure on $[0, 1]$. K is clearly a continuous function, since $(1-s)t$ is continuous for $0 \leq t \leq s \leq 1$, and $(1-t)s$ is continuous for $0 \leq s < t \leq 1$. Notice for $(s, t) \in [0, 1] \times [0, 1]$, then $|K(s, t)| \leq 1$ and we have

$$\int_{[0,1] \times [0,1]} |K(s, t)| d(m \otimes m)(s, t) \leq \int_{[0,1]} \int_{[0,1]} 1 \cdot dm(t)dm(s) = 1 < \infty,$$

where we have used Tonelli's Theorem. Hence $K \in L_2([0, 1] \times [0, 1], m \otimes m)$. Then T is the associated operator T_K as we defined on page 46. Hence T is compact by Theorem 9.6. ✓

(b)

Show that $T = T^*$.

First we will argue that we can use Fubini's theorem on the integral:

$$\int_{[0,1] \times [0,1]} K(s, t)f(t)\overline{g(s)}dm(t)dm(s), \quad f, g \in H.$$


Note that $|K(s, t)| \leq 1$. Then by using Tonelli's theorem and the fact that $f, g \in L_2([0, 1], m)$ implies $f, g \in L_1([0, 1], m)$ as we showed in Homework 2, we get that

$$\begin{aligned} \int_{[0,1] \times [0,1]} |K(s, t)f(t)\overline{g(s)}| dm(t)dm(s) &= \int_{[0,1]} \int_{[0,1]} |K(s, t)| |f(t)| |\overline{g(s)}| dm(t)dm(s) \\ &\leq \int_{[0,1]} \int_{[0,1]} |f(t)| |\overline{g(s)}| dm(t)dm(s) \\ &= \int_{[0,1]} |f(t)| dm(t) \int_{[0,1]} |\overline{g(s)}| dm(s) < \infty, \end{aligned}$$

since $f, g \in L_1([0, 1], m)$.

Now let $f, g \in H$. We wish to show that $\langle Tf, g \rangle = \langle f, Tg \rangle$. We see that

$$\begin{aligned}
\langle Tf, g \rangle &= \int_{[0,1]} Tf(s) \overline{g(s)} dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s, t) f(t) dm(t) \overline{g(s)} dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s, t) f(t) \overline{g(s)} dm(t) dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s, t) f(t) \overline{g(s)} dm(s) dm(t) \\
&= \int_{[0,1]} f(t) \int_{[0,1]} \overline{K(s, t)} \overline{g(s)} dm(s) dm(t) \\
&= \int_{[0,1]} f(t) \int_{[0,1]} \overline{K(t, s)} \overline{g(s)} dm(s) dm(t) \\
&= \int_{[0,1]} f(t) \overline{Tg(t)} dm(t) \\
&= \langle f, Tg \rangle,
\end{aligned}$$

where we have used that since K is real we have $K(s, t) = \overline{K(t, s)}$ and from the definition of K we see that $K(s, t) = K(t, s)$. 

(c)

Show that

$$(Tf)(s) = (1-s) \int_{[0,s]} tf(t) dm(t) + s \int_{[s,1]} (1-t)f(t) dm(t), \quad s \in [0, 1], \quad f \in H.$$

Use this to show that Tf is continuous on $[0, 1]$, and that $(Tf)(0) = (Tf)(1) = 0$.

Notice that by linearity of integrals we obtain the wanted equation,

$$\begin{aligned}
(Tf)(s) &= \int_{[0,1]} K(s, t) f(t) dm(t) \\
&= \int_{[0,s]} K(s, t) f(t) dm(t) + \int_{[s,1]} K(s, t) f(t) dm(t) \\
&= \int_{[0,s]} (1-s)tf(t) dm(t) + \int_{[s,1]} (1-t)sf(t) dm(t) \\
&= (1-s) \int_{[0,s]} tf(t) dm(t) + s \int_{[s,1]} (1-t)f(t) dm(t).
\end{aligned}$$

Now to show that Tf is continuous we will show by proposition 1.10 that there exists a constant $c > 0$ such that $\|Tf\| \leq c\|f\|$, for all $f \in H$. Note that $|t| \leq 1$ and $|s| \leq 1$, so we can bound these. In particular

$|1-s| \leq 1$ and $|1-t| \leq 1$, thus we have

$$\begin{aligned} \|Tf\| &= \|Tf(s)\| = \left| (1-s) \int_{[0,s]} tf(t)dm(t) + s \int_{[s,1]} (1-t)f(t)dm(t) \right| \\ &\leq |1-s| \int_{[0,s]} |t||f(t)|dm(t) + |s| \int_{[s,1]} |(1-t)||f(t)|dm(t) \\ &\leq \int_{[0,s]} |f(t)|dm(t) + \int_{[s,1]} |f(t)|dm(t) \\ &= \int_{[0,1]} |f(t)|dm(t) < \infty, \end{aligned}$$

by linearity of integrals and since $f \in L_2([0,1], m) \subset L_1([0,1], m)$ (HW.2). Thus there exists some constant $c > 0$ such that $\|Tf\| \leq c$, so specially $\|Tf\| \leq c\|f\|$.

Note that integrals over the areas $[0,0]$ and $[1,1]$ gives 0, hence

$$\begin{aligned} (Tf)(0) &= \int_{[0,0]} tf(t)dm(t) + 0 \cdot \int_{[0,1]} (1-t)f(t)dm(t) = 0 \\ (Tf)(1) &= 0 \cdot \int_{[0,1]} tf(t)dm(t) + \int_{[1,1]} (1-t)f(t)dm(t) = 0 \end{aligned}$$

Thus $(Tf)(0) = (Tf)(1) = 0$.

no $c = \|T\|_2$.
This shows T cont.
Not Tf. \checkmark

Problem 4

Consider the Schwartz space $\mathcal{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$.

(a)

For each integer $k \geq 0$, set $g_k(x) = x^k e^{-\frac{1}{2}x^2}$, for $x \in \mathbb{R}$. Justify that $g_k \in \mathcal{S}(\mathbb{R})$, for all integers $k \geq 0$. Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

Notice that $g_k(x) \in C^\infty(\mathbb{R}), \forall x \in \mathbb{R}$. First we will show that $e^{-\frac{1}{2}x^2} \in \mathcal{S}(\mathbb{R})$.

We claim that $\partial^\alpha e^{-\frac{1}{2}x^2} = \text{Pol}_\alpha(x) \cdot e^{-\frac{1}{2}x^2}$, for $x \in \mathbb{R}$, where $\text{Pol}_\alpha(x)$ is some polynomial depending on $\alpha \in \mathbb{N}$. We proof the claim by induction.

When $\alpha = 0$, then


$$\partial^0 e^{-\frac{1}{2}x^2} = e^{-\frac{1}{2}x^2},$$

so the base case holds. Assume the claim holds up to $\alpha = n-1$, then we have that


$$\partial^n e^{-\frac{1}{2}x^2} = \partial \text{Pol}_{n-1}(x) e^{-\frac{1}{2}x^2} = \text{Pol}_n(x) e^{-\frac{1}{2}x^2} + \text{Pol}_1(x) \text{Pol}_{n-1}(x) e^{-\frac{1}{2}x^2} = \text{Pol}_n(x) e^{-\frac{1}{2}x^2}.$$

So the claim holds for all $\alpha \in \mathbb{N}$.

Now notice that $\partial^\alpha e^{-\frac{1}{2}x^2} = \text{Pol}_\alpha(x) e^{-\frac{1}{2}x^2} \rightarrow 0$ when $|x| \rightarrow \infty$, since $e^{-\frac{1}{2}x^2}$ goes to zero faster than any polynomial grows. Further $x^\beta \partial^\alpha e^{-\frac{1}{2}x^2} \rightarrow 0$ when $|x| \rightarrow \infty$, for all non-negative integers α, β , since we

simply multiply with a polynomial. Hence $e^{-x^2/2} \in \mathcal{S}(\mathbb{R})$. Now by Homework 7, exercise 1 we get that $x^k e^{-x^2/2} \in \mathcal{S}(\mathbb{R})$, for all $k \geq 0$. 

Next we will compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$. From proposition 11.4 we get that $\mathcal{F}(g_0) = \hat{g}_0(\xi) = e^{-\frac{1}{2}\xi^2}$, for $\xi \in \mathbb{R}$. Now notice that $e^{-\frac{1}{2}x^2} \in L_1(\mathbb{R})$ and $g_k(x) = x^k e^{-\frac{1}{2}x^2} \in L_1(\mathbb{R})$, since $\mathcal{S}(\mathbb{R}) \subset L_1(\mathbb{R})$. Then by proposition 11.13 we have that $\partial^k \hat{g}_0$ exists and we can compute $\mathcal{F}(g_k)$, for $k = 1, 2, 3$:

$$\begin{aligned}\mathcal{F}(g_1) &= \hat{g}_1(\xi) = i(\partial \hat{g}_0)(\xi) = -i\xi e^{-\frac{1}{2}\xi^2} \\ \mathcal{F}(g_2) &= \hat{g}_2(\xi) = i^2(\partial^2 \hat{g}_0)(\xi) = i(\partial \hat{g}_1)(\xi) = (1 - \xi^2)e^{-\frac{1}{2}\xi^2} \\ \mathcal{F}(g_3) &= \hat{g}_3(\xi) = i^3(\partial^3 \hat{g}_0)(\xi) = i(\partial \hat{g}_2)(\xi) = (i\xi^3 - 3i\xi)e^{-\frac{1}{2}\xi^2}\end{aligned}$$


(b)

Find non-zero functions $h_k \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

Let h_k be of the form $h_k = \alpha_0 g_0 + \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3$. Note that $h_k \in \mathcal{S}(\mathbb{R})$, since linear combinations of Schwartz functions are again Schwartz functions, by the continuity of limits.

Then since the Fourier transformation is linear we have that

$$\begin{aligned}\mathcal{F}(h_k) &= \alpha_0 \hat{g}_0 + \alpha_1 \hat{g}_1 + \alpha_2 \hat{g}_2 + \alpha_3 \hat{g}_3 \\ &= e^{-\frac{1}{2}x^2} (\alpha_0 - ix\alpha_1 + (1 - x^2)\alpha_2 + (ix^3 - 3ix)\alpha_3).\end{aligned}$$

Further we see that

$$i^k h_k = e^{-\frac{1}{2}x^2} (i^k \alpha_0 + i^k \alpha_1 x + i^k \alpha_2 x^2 + i^k \alpha_3 x^3).$$

Now solving $\mathcal{F}(h_k) = i^k h_k$ for $k = 1$ we have

$$\alpha_0 - ix\alpha_1 + (1 - x^2)\alpha_2 + (ix^3 - 3ix)\alpha_3 = i^k \alpha_0 + i^k \alpha_1 x + i^k \alpha_2 x^2 + i^k \alpha_3 x^3,$$

hence

$$(1 - i)\alpha_0 - 2ix\alpha_1 + (1 - x^2 - ix^2)\alpha_2 - 3ix\alpha_3 = 0.$$

Now choose $\alpha_0 = \alpha_2 = 0$, $\alpha_1 = 3$ and $\alpha_3 = -2$ such that the above is satisfied.

For $k = 2$ we solve

$$2\alpha_0 + (x - ix)\alpha_1 + \alpha_2 + (ix^3 - 3ix + x^3)\alpha_3 = 0$$


Then choose $\alpha_1 = \alpha_3 = 0$, $\alpha_0 = 1$ and $\alpha_2 = -2$.

For $k = 3$ we have

$$(1 + i)\alpha_0 + (1 - x^2 + ix^2)\alpha_2 + (-x^3 - 3ix)\alpha_3 = 0$$

Then choose $\alpha_0 = \alpha_2 = \alpha_3 = 0$ and $\alpha_1 = 1$.

In conclusion we have found the following non-zero functions $h_k \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$

$$\begin{aligned}h_0 &= g_0 \\ h_1 &= 3g_1 - 2g_3 \\ h_2 &= g_0 - 2g_2 \\ h_3 &= g_1\end{aligned}$$


(c)

Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathcal{S}(\mathbb{R})$

Consider the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}) \subseteq L_1(\mathbb{R})$,


$$\mathcal{F}(f) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dm(x).$$

Now since $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$, then $\hat{f} \in L_1(\mathbb{R})$, so we can find the Fourier transformation of \hat{f} , thus

$$\mathcal{F}(\mathcal{F}(f)) = \hat{\hat{f}}(\xi) = \int_{\mathbb{R}} \hat{f}(x) e^{-i\xi x} dm(x)$$

Again since $\hat{\hat{f}}(\xi) \in L_1(\mathbb{R})$, we may note that $\mathcal{F}(\mathcal{F}(f)) = \hat{\hat{f}}(\xi) = \check{\hat{f}}(-\xi)$. Now since $f \in \mathcal{S}(\mathbb{R})$, then f is in particular C^∞ , so in fact continuous. Then by Theorem 12.11 we have that $\check{\hat{f}}(-\xi) = f(-\xi)$. Thus $\mathcal{F}^2(f(\xi)) = f(-\xi)$. So we get that

$$\mathcal{F}^4(f(\xi)) = \mathcal{F}^2(\mathcal{F}^2(f(\xi))) = \mathcal{F}^2(f(-\xi)) = f(\xi),$$

hence we obtain that $\mathcal{F}^4(f) = f$. 


(d)

Use (c) to show that if $f \in \mathcal{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of \mathcal{F} precisely are $\{1, i, -1, -i\}$.

Assume $\mathcal{F}(f) = \lambda f$, for $f \in \mathcal{S}(\mathbb{R})$ non-zero and some $\lambda \in \mathbb{C}$. Notice since the Fourier transformation is linear we can always pull constants out. Hence using (c) we get that

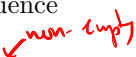
$$f = \mathcal{F}^4(f) = \mathcal{F}^3(\mathcal{F}(f)) = \mathcal{F}^3(\lambda f) = \lambda \mathcal{F}^3(f) = \lambda^2 \mathcal{F}^2(f) = \lambda^3 \mathcal{F}(f) = \lambda^4 f$$


Hence $\lambda^4 = 1$, and $\lambda - 1$ has 4 roots for $\lambda \in \mathbb{C}$, so there are the following possibilities $\lambda \in \{1, -1, i, -i\}$.

Since the eigenvalues λ of \mathcal{F} must satisfy $\mathcal{F}(f) = \lambda f$, it is clear from the above that $\{1, -1, i, -i\}$ are the only possible eigenvalues. Now note from (b) that $\{1, -1, i, -i\}$ are in fact eigenvalues of \mathcal{F} , since $\{i^k : k = 1, 2, 3, 4\} = \{1, -1, i, -i\}$. Hence the eigenvalues of \mathcal{F} are precisely $\{1, i, -1, -i\}$. 

Problem 5

Let $(x_n)_{n \geq 1}$ be a dense subset of $[0, 1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}$ on $[0, 1]$. Show that $\text{supp}(\mu) = [0, 1]$.

Note that $(x_n)_{n \geq 1}$ being dense in $[0, 1]$ means $\overline{\{x_n\}_{n \geq 1}} = [0, 1]$, or equivalent that all non-empty open sets in $[0, 1]$ will intersect the set $\{x_n\}_{n \geq 1}$, where $\{x_n\}_{n \geq 1}$ is the set consisting of elements from the sequence $(x_n)_{n \geq 1}$. 

Let N be the union of all open subsets $U \subseteq [0, 1]$ that satisfies $\mu(U) = 0$. Note that since all open sets $U \subseteq [0, 1]$ contains at least one element from $(x_n)_{n \geq 1}$, then as $\delta_{x_n}(U) = 1$ when $x_n \in U$, we must have $\mu(U) > 0$ for every $U \subseteq [0, 1]$. Hence N must be the empty set. 

Now since $[0, 1]$ is indeed a locally compact Hausdorff topological space we get from Homework 8, problem 3(a) that the support of μ is the complement to N . So since $N = \emptyset$ we get $\text{supp}(\mu) = [0, 1]$, which is what we wanted. 