Functional Analysis Assignment 2

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Problem 1 *Let H be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Set $f_N := N^{-1}\sum_{n=1}^{N^2} e_n$, for all $N \geq 1$.*

a) *Show that $f_N \rightharpoonup 0$ as $N \to \infty$, while $\|f_N\| = 1$ for all $N \geq 1$.*

Let $N \in \mathbb{N}$, $x \in H$ arbitrary. We are to prove

$$\langle f_N, x\rangle_H = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, x\rangle \overset{N\to\infty}{\to} 0.$$

From Bessel's Inequality (for example as presented in T26.19iii) in Analysis2's book by Schilling), we necessarily have that $e_n \rightharpoonup 0$ in H, and as such, with the absolute-value function being continuous we therefore have

$$|\langle e_n, x\rangle| \overset{n\to\infty}{\to} |0| = 0,$$

and by the same argument as all n'th roots are continuous we have

$$(|\langle e_n, x\rangle|)^{\frac{1}{n}} \overset{n\to\infty}{\to} = 0.$$

As such we therefore also have $\limsup_{n\to\infty}(|\langle e_n, x\rangle|)^{\frac{1}{n}} = 0 < 1$, such that we by the root test for infinite series might conclude that $\exists q \in \mathbb{R}$ such that

$$\sum_{n=1}^{\infty}|\langle e_n, x\rangle| = q.$$

By completeness of H we might by T1.7 in LN therefore conclude, that the "non-absolute" sum $\sum_{n=1}^{\infty}\langle e_n, x\rangle = q_1$ will also converge to something inevitably smaller than q, such that we altogether have

$$\langle f_N, x\rangle_H = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, x\rangle \overset{N\to\infty}{\to} 0 \cdot q_1 = 0.$$

Now for showing $\|f_N\| = 1$, observe $\forall N \in \mathbb{N}$ $\|f_N\| = 1$ as picking an arbitrary $N \in \mathbb{N}$ yields

$$\begin{aligned}
\|f_N\|^2 \equiv \langle f_N, f_N\rangle \equiv & \left\langle \frac{1}{N}\sum_{i=1}^{N^2} e_i, \frac{1}{N}\sum_{j=1}^{N^2} e_j \right\rangle \\
=& \frac{1}{N^2}\left\langle \sum_{i=1}^{N^2} e_i, \sum_{j=1}^{N^2} e_j \right\rangle \\
=& \frac{1}{N^2}\sum_{i=1}^{N^2}\sum_{j=1}^{N^2} \langle e_i, e_j\rangle \\
\overset{\text{ortho.}}{=}& \frac{1}{N^2}\sum_{i=1}^{N^2} \langle e_i, e_i\rangle \\
\equiv& \frac{1}{N^2}\sum_{i=1}^{N^2} \|e_i\|^2 \\
\overset{\text{normal}}{=}& \frac{1}{N^2}\sum_{i=1}^{N^2} 1 \\
=& \frac{1}{N^2}N^2 = 1,
\end{aligned} \tag{1}$$

such that $\|f_N\| = 1, \forall N \in \mathbb{N}$.

b) ***Let K be the norm-closure of $co\{f_N : N \geq 1\}$. Argue that K is weakly compact, and that $0 \in K$.***

Observe that as $\|f_N\| \overset{(1)}{=} 1 \, \forall N \in \mathbb{N}$, we have for any $f \in K$ that f can be written as $f = \sum_{i=1}^n \alpha_i f_{N_i}$, such that $\|f\| = \|\sum_{i=1}^n \alpha_i f_{N_i}\| \leq \sum_{i=1}^n \alpha_i \|f_{N_i}\| = \sum_{i=1}^n \alpha_i = 1$, ie. we have $K \subseteq \overline{B}_H(0,1)$.

As any Hilbert space is a reflexive Banach space (P2.10 in LN) we may by P6.3 in LN (through Alaoglu's Theorem) conclude that $\overline{B}_H(0,1)$ is weakly compact.

As the convex hull of a set in particular will be convex, we have that with K being defined as the norm-closure of this hull, K will not only be norm-closed, but also weakly closed by T5.7 in LN, as the two forms of closure coincide on convex sets.

As closed subsets of compact sets are compact for some topology (ex. see P4.22 in Folland as a reference), we in particular have that as K is weakly closed and contained in the weakly compact $\overline{B}_H(0,1)$, K will be weakly compact.

By a) note also that 0 will reside in the weak closure of $\{f_N : N \geq 1\}$, and as such also within the weak closure of its convex hull. By T5.7 in LN once again, we note therefore that $0 \in K$.

c) ***Show that 0 as well as each f_N, $N \geq 1$, are extreme points in K.***

First, we will approach the problem of 0 being an extreme point of K. We may therefore towards a contradiction assume that $K \ni 0 \notin Ext(K)$, such that we may write $0 = \alpha k_1 + \beta k_2$, for $k_1, k_2 \in K$, $k_1 \neq k_2$ and $\alpha + \beta = 1$, $\alpha, \beta > 0$.
With K being the norm-closure of the convex hull of the f_N's we note that the inner product of any $p \in K$ with any of the basis vectors will be positive ie. $\langle p, e_i\rangle \geq 0 \, \forall i \in \mathbb{N}$, as this is the case for our f_N's. We will

therefore for each $i \in \mathbb{N}$ be able to write $0 = \alpha \langle e_i, k_1 \rangle + \beta \langle e_i, k_2 \rangle$, having taken inner products with e_i on both sides.
But as $\alpha, \beta > 0$, and $\langle e_i, k_1 \rangle, \langle e_i, k_2 \rangle \geq 0$, we must therefore again for each $i \in \mathbb{N}$ have $\langle e_i, k_1 \rangle = \langle e_i, k_2 \rangle = 0$, in order for the sum with its coefficients to be 0. But with $(e_n)_{n\in\mathbb{N}}$ constituting an ONB for H, we will necessarily have that $\langle e_i, k_1 \rangle = \langle e_i, k_2 \rangle = 0 \ \forall i \in \mathbb{N} \Rightarrow k_1 = k_2 = 0$. (for example by definition of an ONB in Schillings Definition 26.20) $k_1 = k_2 = 0$ will be a contradiction with us being able to write 0 as a non-degenerate convex combination of elements in K, so we conclude therefore that we must have $0 \in Ext(K)$.

We will show that each f_N is an extreme point of K. Fix $N \in \mathbb{N}$, and assume once again that the object of our affection, in this case f_N, can be written as a combination of elements of K on the form $f_N = \alpha k_1 + (1-\alpha) k_2,\ k_1, k_2 \in K,\ 0 \leq \alpha \leq 1$. With $k_1, k_2 \in K$ we may, using the results of the previous subproblems, write these themselves as limits of convex combinations on the form $k_1 = \sum_{n=1}^{\infty} \beta_n f_{N_n},\ k_2 = \sum_{m=1}^{\infty} \gamma_m f_{N_m}$, with $1 = \sum_{n=1}^{\infty} \beta_n = \sum_{m=1}^{\infty} \gamma_m,\ \beta_n, \gamma_m \geq 0,\ \forall n, m \in \mathbb{N}$. Also recall the definition of f_N as

$$f_N := \frac{1}{N} \sum_{i=1}^{N^2} e_i \tag{2}$$

We will therefore be able to rewrite f_N as

$$\begin{aligned} f_N &= \alpha \sum_{n=1}^{\infty} \beta_n f_{N_n} + (1-\alpha) \sum_{m=1}^{\infty} \gamma_m f_{N_m} \\ &= \alpha \sum_{n=1}^{\infty} \beta_n \frac{1}{N_n} \sum_{i=1}^{N_n^2} e_i + (1-\alpha) \sum_{m=1}^{\infty} \gamma_m \frac{1}{N_m} \sum_{j=1}^{N_m^2} e_j. \end{aligned} \tag{3}$$

With f_N on the form of (2) note that by taking the inner product of f_N with e_{N^2} gets us

$$\begin{aligned} \langle f_N, e_{N^2} \rangle &\equiv \left\langle \frac{1}{N} \sum_{i=1}^{N^2} e_i, e_{N^2} \right\rangle \\ &= \frac{1}{N} \sum_{i=1}^{N^2} \langle e_i, e_{N^2} \rangle \\ &= \frac{1}{N} \sum_{i=1}^{N^2-1} \langle e_i, e_{N^2} \rangle + \frac{1}{N} \langle e_{N^2}, e_{N^2} \rangle \\ &\overset{orthonormality}{=} \frac{1}{N}. \end{aligned} \tag{4}$$

Taking inner product with e_{N^2} on the right hand side of (3) we may combine this with (4) such that as

$$\langle e_q, e_p \rangle = \begin{cases} 0, & q \neq p \\ 1, & q = p, \end{cases} \tag{5}$$

we get

$$\begin{aligned} \frac{1}{N} &\overset{(4)}{=} \left\langle \alpha \sum_{n=1}^{\infty} \beta_n \frac{1}{N_n} \sum_{i=1}^{N_n^2} e_i + (1-\alpha) \sum_{m=1}^{\infty} \gamma_m \frac{1}{N_m} \sum_{j=1}^{N_m^2} e_j, e_{N^2} \right\rangle \\ &= \left\langle \alpha \sum_{n=1}^{\infty} \beta_n \frac{1}{N_n} \sum_{i=1}^{N_n^2} e_i, e_{N^2} \right\rangle + \left\langle (1-\alpha) \sum_{m=1}^{\infty} \gamma_m \frac{1}{N_m} \sum_{j=1}^{N_m^2} e_j, e_{N^2} \right\rangle \\ &\overset{(5)}{=} \alpha \sum_{n \in \mathcal{G}_n} \frac{1}{N_n} \beta_n + (1-\alpha) \sum_{m \in \mathcal{G}_m} \frac{1}{N_m} \gamma_m, \end{aligned} \tag{6}$$

with $\mathcal{G}_n, \mathcal{G}_m$ respectively denoting the set of all indices n, m for which the N_n, N_m's with $N_n^2, N_m^2 < N^2$ are "discarded".

This however is where our conclusion kicks in, as we note that the subsequences $N_n, N_m \geq 1$, $\beta_n, \gamma_m \geq 0$, such that as (6) is constantly equal to $\frac{1}{N}$, while the sums are also monotone yields that we necessarily must have $N = N_n = N_m$, in which have we by definition of k_1, k_2 must have $f_{Nn} = f_{Nm} = f_N$, such that $f_N \in Ext(K)$.

d) *Are there any other extreme points in K? Justify your answer.*

The result that the only extreme points of K are to be found amongst the f_N's and 0 will follow from the Milman Theorem, and an application of the Open Covering Theorem.
Choose $F := \{f_N \,|\, N \geq \mathbb{N}\} \cup \{0\}$. With conditions for use of Milman's theorem proven above, we have that $Ext(K) \subseteq \overline{F}^{\tau}$. By showing F to be weakly closed itself, we will get the desired result, as there will be no more extreme points to be found.
To this end, we will through the use of the Open Covering Theorem prove F to be weakly compact which itself implies F weakly closed.

For $\mathcal{A}$ being some open cover of F, note that the (unique) weak convergence of $f_N \rightharpoonup 0$ in particular entails that the f_N's will eventually be contained within some open neighbourhood of 0, that we might denote $\mathcal{A}_0$ which again in particular will be of $\mathcal{A}$. Also by eventuality, existence of a $j \in \mathbb{N}$ such that $\forall n \in \mathbb{N} : n \geq j : f_n \in \mathcal{A}_0$, such that for each $N < j$ we might also choose some open set $\mathcal{A}_N$ from $\mathcal{A}$, that contains f_N.

In conclusion we may therefore point to the finite open covering $\mathcal{A}_f := \mathcal{A}_0 \cup \{\mathcal{A}_N : N = 1, \ldots, j-1\}$, of F, which has cardinality $\#\mathcal{A}_f = 1 + j - 1 = j < \infty$. By the Open Covering Theorem, F will therefore be (weakly) compact, and thus in particular (weakly) closed, such that Milman Theorem guarantees $F \equiv \{f_N \,|\, N \geq \mathbb{N}\} \cup \{0\}$ to be an exhaustive set of extreme points for K.

Problem 2 *Let X and Y be infinite dimensional Banach spaces.*

a) *Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \rightharpoonup x$, as $n \to \infty$, implies that $Tx_n \rightharpoonup Tx$, as $n \to \infty$.*

Using HW4 Problem 2, we might reduce the problem of showing $Tx_n \overset{n\to\infty}{\rightharpoonup} Tx$ to showing that $f(Tx_n) \to f(Tx)$ for all $f \in X^*$. Note in this regard that $f(T) \in X^*$. As $x_n \rightharpoonup x$ we will by P5.4 in LN used on $f \circ T$ be able to conclude that

$$f(Tx_n) \equiv (f \circ T)(x_n) \underset{P5.4}{\overset{n\to\infty}{\to}} (f \circ T)(x) \equiv f(Tx),$$

which by HW4 P2 is sufficient to conclude that $Tx_n \rightharpoonup Tx$.

b) *Let $T \in \mathcal{K}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \rightharpoonup x$, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$.*

We will be using subproblem 2a) as we note that for $T \in \mathcal{K}(X, Y) \subseteq \mathcal{L}(X, Y)$.
Still assuming $(x_n)_{n\in\mathbb{N}} \subseteq X$, converges weakly to $x \in X$, by HW4 Problem 2 we might conclude that the sequence be bounded in X.
From P8.2(1)$\Leftrightarrow$(4) in LN, we therefore have in particular that $(x_n)_{n\in\mathbb{N}}$ contains a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ converges in Y.

Now suppose towards a contradiction that for some $\varepsilon > 0$, $\|Tx_{n_k} - Tx\| \geq \varepsilon$, $\forall k \in \mathbb{N}$. This will ultimately be at odds with $(Tx_{n_k})_{k\in\mathbb{N}}$ converging necessarily weakly to Tx in Y, as we then again might use compactness of T to then choose a further sub-subsequence of (x_n), call it $\left(x_{n_{k_t}}\right)$ under which $Tx_{n_{k_t}}$ would converge strongly to Tx, a contradiction.

c) *Let H be a separable infinite dimensional Hilbert space. Show that if $T \in \mathcal{L}(H, Y)$ satisfies $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.*

By the hint we show the contrapositive statement to the one above. Suppose therefore T not to be compact. By the TFAE of P8.2 in LN we may therefore conclude that $T(\overline{B_H(0,1)})$ is not totally bounded, such that we by definition of being totally bounded might point to some $\delta > 0$ such that $T(\overline{B_H(0,1)})$ is not covered by a union of finitely many open balls in Y ($T : H \to Y$, some Banach Space) of radius $\delta > 0$.

By the hint, we will want to construct a sequence $(x_n)_{n\in\mathbb{N}}$ in the closed unit ball of H such that $\|Tx_n - Tx_m\| \geq \delta$, $n \neq m$. Pick therefore some $x_1 \in \overline{B_H(0,1)}$, such that $Tx_1 \in T(\overline{B_H(0,1)})$. By non-total boundedness, as $B_Y(T(x_1), \delta)$ does not cover $T(\overline{B_H(0,1)})$, we may then by non-totally boundedness find some other $x_2 \in \overline{B_H(0,1)}$ such that $Tx_2 \notin B_Y(Tx_1, \delta)$, also satisfying that $B_Y(Tx_1, \delta) \cup B_Y(Tx_2, \delta) \not\supseteq T(\overline{B_H(0,1)})$.

Repeating this process we might pick our $x_{n\geq 3} \in \overline{B_H(0,1)}$'s, successively such that $Tx_{n+1} \in T(\overline{B_H(0,1)}) \setminus \bigcup_{i=1}^{n} B_Y(Tx_i, \delta), n \in \mathbb{N}$. This sequence in particular comes with the extrinsic property that $\|Tx_n - Tx_m\|_Y \geq \delta$, $n \neq m$, that we by the hint searched for, with $(x_n)_{n\in\mathbb{N}} \subseteq \overline{B_H(0,1)}$ as well.
In order to find a weakly convergent subsequence of $(x_n)_{n\in\mathbb{N}}$, note that as H is assumed to be a seperable infinite dimensional Hilbert space, we have by T5.13 in LN that $\overline{B_{H^*}(0,1)}$ is metrizable in ω^*-topology, and as Hilbert spaces are in particular reflexive the ω and ω^*-topologies will coincide on H^* by T5.9 in LN. By T6.3, reflexivity grants us weak compactness of $\overline{B_H(0,1)}$, so that we can deem that $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence, $(x_{n_k})_{k\in\mathbb{N}}$. Weak convergence or not however, we will still have $\left\|Tx_{n_k} - Tx_{n_j}\right\| \geq 0$, thus completing the proof of the contrapositive.

d) *Show that each $T \in \mathcal{L}\left(\ell_2\left(\mathbb{N}\right), \ell_1\left(\mathbb{N}\right)\right)$ is compact.*

By the hint, note that for $\ell_2(\mathbb{N}) \supseteq (x_n)_{n\in\mathbb{N}} \overset{n\to\infty}{\rightharpoonup} x \in \ell_2(\mathbb{N})$, we by 2a) have that for arbitrary $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$, that $Tx_n \overset{n\to\infty}{\rightharpoonup} Tx$. By Remark 5.3 a property of $\ell_1(\mathbb{N})$ is the fact that a sequence weakly if and only if it converges in norm, so we therefore get

$$\begin{aligned}
&\ell_2(\mathbb{N}) \ni x_n \overset{n\to\infty}{\rightharpoonup} x \in \ell_2(\mathbb{N}) \\
&\qquad \overset{2a)}{\Rightarrow} \forall T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N})) : \ell_1(\mathbb{N}) \ni Tx_n \overset{n\to\infty}{\rightharpoonup} Tx \in \ell_1(\mathbb{N}) \\
&\qquad \overset{\text{Remark 5.3}}{\Leftrightarrow} \|Tx_n - Tx\|_{\ell_1(\mathbb{N})} \overset{n\to\infty}{\to} 0 \\
&\qquad \overset{2c)}{\Rightarrow} T \in \mathcal{K}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N})),
\end{aligned}$$

as desired.

e) *Show that no $T \in \mathcal{K}\left(X, Y\right)$ is onto.*

In this subproblem we will be making use of the conclusions drawn from the third problem of the first mandatory assignment, conjured forward by contradictory supposition. Suppose therefore that $T \in \mathcal{K}\left(X, Y\right)$ is onto. Note that as X, Y assumed complete normed (Banach) vector spaces, the T3.15 in LN Open Mapping Theorem (which utilises the onto assumption) grants us that T is open, ie. (iff) $\exists r > 0 : B_Y(0, r) \subseteq T(B_X(0,1))$. We might then choose some $r_1 \leq r$ such that $\overline{B_Y(0, r_1)}^{\|\cdot\|_Y} \subseteq T(B_X(0,1))$.

Note also that by definition D8.1 in LN, as T is compact $\overline{T(B_X(0,1))}^{\|\cdot\|_Y}$ will be compact in Y, such that as $\overline{B_Y(0,r_1)}^{\|\cdot\|_Y} \subseteq T(B_X(0,1)) \subseteq \overline{T(B_X(0,1))}^{\|\cdot\|_Y}$, $\overline{B_Y(0,r_1)}^{\|\cdot\|_Y}$ will be a norm-closed subset of a norm compact set, and therefore itself compact - something at odds with the compactness conclusions of Problem 3 in the first FunkAn mandatory assignment. We may therefore conclude that for X, Y infinite dimensional Banach spaces, no $T \in \mathcal{K}(X,Y)$ will be onto.

f) Let $H = L_2([0,1], m)$, and consider the operator $M \in \mathcal{L}(H,H)$ given by $M(f(t)) := tf(t)$, for $f \in H$ and $t \in [0,1]$. Justify that M is self-adjoined, but not compact.

In accordance with the definition of a Hilbert space-adjoined, given on page 41 in LN, let $f, y \in H$, and note that

$$\begin{aligned}
\langle Mf, y\rangle_H &\equiv \langle tf(t), y\rangle \\
&= \int_{[0,1]} tf(t)\overline{y(t)}dm(t) \\
&\overset{t\in[0,1]\subseteq\mathbb{R}}{=} \int_{[0,1]} f(t)\overline{ty(t)}dm(t) \\
&\equiv \langle f, My\rangle,
\end{aligned}$$

so M is self-adjoined.

M will however not be compact by HW6 Problem 3 which tells us that M has no eigenvalues, and the contrapositive of The Spectral Theorem for self-adjoined compact operators, as M would have eigenvectors with corresponding eigenvalues, if it were compact.

Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$, ***with*** m ***denoting the Lebesgue measure. Define*** $K : [0,1] \times [0,1] \to \mathbb{R}$ ***by***

$$K(s,t) := \begin{cases} (1-s)\,t, & 0 \le t \le s \le 1, \\ (1-t)\,s, & 0 \le s < t \le 1, \end{cases}$$

and consider $T \in \mathcal{L}(H,H)$ ***defined by***

$$(Tf)(s) := \int_{[0,1]} K(s,t)f(t)dm(t), \quad s \in [0,1], \; f \in H.$$

a) *Justify that* T *is compact.*

Notice first that for m being the Lebesgue-measure (appropriately restricted), $\left([0,1], \mathbb{B}_{[0,1]}, m\right)$ is known (from such courses as An2 or MI) to be σ-finite. We will use P9.12 in LN to argue that T is compact, as we show it to be a Hilbert-Schmidt operator. It will therefore suffice to show T to be the associated kernel operator of K which requires showing $K \in L_2\left([0,1]^2, m_{\otimes 2}\right)$. Observe that $K \in C^0\left([0,1]^2\right)$, and that $K(s,t) \in [0,1]$ such that we might estimate

$$\int_{[0,1]^2} |K(s,t)|^2 \, dm_2(s,t) \le \int_{[0,1]^2} 1 dm_2(s,t) = 1 < \infty,$$

such that $K \in L_2\left([0,1]^2, m_{\otimes 2}\right)$. By P9.12 and the to K associated Kernel operator, T, will therefore be Hilbert-Schmidt in particular compact by P9.11 in LN.

b) ***Show that*** $T = T^*$.

Note that as $T \in \mathcal{L}(H, H)$, H a Hilbert space, that we by page 41 in LN need to show that for all $f, g \in H$, we will have

$$\langle Tf, g \rangle_H = \langle f, Tg \rangle_H \,,$$

in order to show $T = T^*$.
So, let $f, g \in H \equiv L_2\left([0,1], m\right)$ arbitrary. Heed

$$\begin{aligned}
\langle Tf, g \rangle &= \int_{[0,1]} (Tf)(s)\overline{g(s)}\, dm(s) \\
&\equiv \int_{[0,1]} \int_{[0,1]} K(s,t) f(t)\, dm(t) \overline{g(s)}\, dm(s) \\
&= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \overline{g(s)}\, dm(t)\, dm(s).
\end{aligned} \tag{7}$$

We would like to use Fubinis theorem to collect the double integral of (7) into a product, only to unfold them again in the opposite order of integration. Towards this, observe

$$\left| K(s,t) f(t) \overline{g(s)} \right| = |K(s,t)| \cdot \left| f(t)\overline{g(s)} \right| \leq 1 \cdot \left| f(t)\overline{g(s)} \right|. \tag{8}$$

Looking at (8) note as $f, g \in H \equiv L_2\left([0,1], m\right)$, (and therefore also $\overline{g} \in L_2\left([0,1], m\right)$) we might by the Cauchy-Schwartz inequality conclude $f\overline{g} \in L_1\left([0,1], m\right)$. Through this observation coupled with σ-finiteness of $\left([0,1], \mathbb{B}_{[0,1]}, m\right)$, we may use Fubini to further (7);

$$\begin{aligned}
\langle Tf, g \rangle &\overset{(7)}{=} \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \overline{g(s)}\, dm(t)\, dm(s) \\
&\overset{\dagger_1}{=} \int_{[0,1]} \int_{[0,1]} K(s,t) f(t) \overline{g(s)}\, dm(s)\, dm(t) \\
&= \int_{[0,1]} f(t) \int_{[0,1]} K(s,t) \overline{g(s)}\, dm(s)\, dm(t) \\
&\overset{\dagger_2}{=} \int_{[0,1]} f(t) \int_{[0,1]} \overline{K(s,t) g(s)}\, dm(s)\, dm(t) \\
&\overset{\dagger_3}{=} \int_{[0,1]} f(t) \overline{\int_{[0,1]} K(s,t) g(s)\, dm(s)}\, dm(t) \\
&= \langle f, Tg \rangle,
\end{aligned}$$

as requested with $\dagger_1$ attributed to the use of Fubinis Theorem twice, $\dagger_2$ to complex conjugation of a product being the product of the complex conjugations, and $\overline{K} = K$ as $K \in [0,1]$, and $\dagger_3$ being a complex analysis result.

c) ***Show that***

$$(Tf)(s) = (1-s) \int_{[0,s]} t f(t) dm(t) + s \int_{[s,1]} (1-t) f(t) dm(t), \quad s \in [0,1], \; f \in H. \tag{9}$$

Use this to show that Tf ***is continuous on*** $[0,1]$, ***and that*** $(Tf)(0) = (Tf)(1) = 0$. Choose $f \in H \equiv L_2\left([0,1], m\right)$, $s \in [0,1]$. Through repeated use of different types of linearity of the integral, note that

$$(Tf)(s) \equiv \int_{[0,1]} K(s,t) f(t) dm(t)$$

$$
\begin{aligned}
&= \int_{[0,s]} K(s,t) f(t) dm(t) + \int_{(s,1]} K(s,t) f(t) dm(t) \\
&= \int_{[0,s]} (1-s)\, t f(t) dm(t) + \int_{(s,1]} (1-t)\, s f(t) dm(t) \\
&= (1-s) \int_{[0,s]} t f(t) dm(t) + s \int_{(s,1]} (1-t)\, f(t) dm(t) \\
&= (1-s) \int_{[0,s]} t f(t) dm(t) + 0 + s \int_{(s,1]} (1-t)\, f(t) dm(t) \\
&\overset{(*)}{=} (1-s) \int_{[0,s]} t f(t) dm(t) + s \int_{\{s\}} (1-t)\, f(t) dm(t) + s \int_{(s,1]} (1-t)\, f(t) dm(t) \\
&= (1-s) \int_{[0,s]} t f(t) dm(t) + s \int_{[s,1]} (1-t)\, f(t) dm(t), \qquad (10)
\end{aligned}
$$

with (*) attributed to $\{s\}$ being a m-nullset, such that integrating over the set also yields zero. Note that (10) is on the form (9), as requested.

We have $(Tf)(0) = (Tf)(1) = 0$ as

$$
(Tf)(0) \overset{(10)}{=} (1-0) \int_{\{0\}} t f(t) dm(t) + 0 \int_{[0,1]} (1-t)\, f(t) dm(t) = 0 + 0 = 0,
$$

and

$$
(Tf)(1) \overset{(10)}{=} (1-1) \int_{[0,1]} t f(t) dm(t) + 1 \int_{\{1\}} (1-t)\, f(t) dm(t) = 0 + 0 = 0.
$$

In proving Tf to be continuous, it would suffice to prove the integrals of (10) to be continuous. Note that for $x, y \in [0,1]$, $x \leq y$ WLOG we have

$$
\begin{aligned}
\left| \int_{[0,x]} t f(t) dm(t) - \int_{[0,y]} t f(t) dm(t) \right| &= \left| - \int_{(x,y]} t f(t) dm(t) \right| \\
&= \left| \int_{(x,y]} t f(t) dm(t) \right| \\
&\leq \int_{(x,y]} |t f(t) dm(t)| \qquad (11) \\
&= \int \left| 1_{(x,y]}(t) t f(t) dm(t) \right| \\
&\leq \left\| 1_{(x,y]} \right\|_2 \| t f(t) \|_2 \\
&= m\left((x,y] \right)^{\frac{1}{2}} \| t f(t) \|_2 \\
&= (y-x)^{\frac{1}{2}} \| t f(t) \|_2 ,
\end{aligned}
$$

with the bounding done by the triangle inequality and Cauchy-Schwarz respectively. We may by (11) note that the integrals of the form $\int_{[0,x]} t f(t) dm(t)$ satisfy a Hölder condition, and will therefore in particular be continuous. And as we might ascribe the second integral of (10) the desired form through the rewrite

$$
\begin{aligned}
\int_{[s,1]} (1-t)\, f(t) dm(t) &= \int_{[0,1]} (1-t)\, f(t) dm(t) - \int_{[0,s)} (1-t)\, f(t) dm(t) \\
&= \int_{[0,1]} (1-t)\, f(t) dm(t) - \int_{[0,s]} (1-t)\, f(t) dm(t).
\end{aligned}
$$

And as composition of continuous functions are continuous, we necessarily have by (10) that (Tf) is continuous on $[0,1]$.

Problem 4 ***Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F}: \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.***

a) ***For each integer $k \geq 0$, set $g_k(x) := x^k e^{\frac{-x^2}{2}}, x \in \mathbb{R}$. Justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all $k \geq 0$. Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.***

Making use of HW7P1 thrice, we will prove $g_k \in \mathscr{S}(\mathbb{R})\, \forall k \geq 0$.
Note that for $x \in \mathbb{R}$,

$$\mathscr{S}(\mathbb{R}) \overset{HW7P1}{\ni} e^{-\|x\|^2} = e^{-|x|^2} = e^{-x^2}.$$

By HW7P1d) we may thus with the choice of $\mathbb{R} \setminus \{0\} \ni a = \sqrt{\frac{1}{2}}$ conclude that

$$S_a\left(e^{-x^2}\right) \equiv e^{-\left(\frac{x}{a}\right)^2} \doteq e^{-\frac{x^2}{2}} \in \mathscr{S}(\mathbb{R}).$$

Finally choosing $x = x, \alpha = k \geq 0$, (a multi-index on $\mathbb{R}$) we deduce $g_k \in \mathscr{S}(\mathbb{R})$ as

$$e^{-\frac{x^2}{2}} \in \mathscr{S}(\mathbb{R}) \overset{HW7P1a)}{\Rightarrow} x^k e^{-\frac{x^2}{2}} \equiv g_k(x) \in \mathscr{S}(\mathbb{R}),\ \forall k \geq 0$$

for all $x \in \mathbb{R}$.
As the Fourier transform is well defined on $\mathscr{S}(\mathbb{R})$, we may thus for $\xi \in \hat{\mathbb{R}}$ start off by noting

$$\mathcal{F}(g_0)(\xi) \equiv \widehat{g_0}(\xi) \overset{P11.4}{=} g_0(\xi) \equiv e^{-\frac{\xi^2}{2}}. \tag{12}$$

As $g_k(\xi) = \xi^k g_0(\xi)$, $k \geq 0$, and $\mathscr{S}(\mathbb{R}) \overset{HW7P1c)}{\subseteq} L_1(\mathbb{R})$, we may by P11.13d) in LN conclude that

$$\begin{aligned} g_1(\xi) \equiv \widehat{g_1}(\xi) \equiv \widehat{\xi^1 g_0}(\xi) &\overset{P11.13}{=} i^{|1|}\left(\frac{d^1}{d\xi^1}\widehat{g_0}(\xi)\right) \\ &\overset{(12)}{=} i\left(\frac{d^1}{d\xi^1}g_0(\xi)\right) \\ &= -i\xi g_0(\xi) \\ &\equiv -i\xi e^{-\frac{\xi^2}{2}} \equiv -ig_1(\xi), \end{aligned} \tag{13}$$

$$\begin{aligned} \mathcal{F}(g_2)(\xi) \equiv \widehat{g_2}(\xi) \equiv \widehat{\xi^2 g_0}(\xi) &\overset{P11.13}{=} i^2\left(\frac{d^2}{d\xi^2}g_0(\xi)\right) \\ &= -\left(-e^{-\frac{\xi^2}{2}} + \xi^2 e^{-\frac{\xi^2}{2}}\right) \\ &= e^{-\frac{\xi^2}{2}}\left(1 - \xi^2\right) = -g_2(\xi) + g_0(\xi), \end{aligned} \tag{14}$$

$$\begin{aligned} \mathcal{F}(g_3)(\xi) \equiv \widehat{g_3}(\xi) \equiv \widehat{\xi^3 g_0}(\xi) &\overset{P11.13}{=} i^3\left(\frac{d^3}{d\xi^3}g_0(\xi)\right) \\ &= -i\left(3\xi e^{-\frac{\xi^2}{2}} - \xi^3 e^{-\frac{\xi^2}{2}}\right) \\ &= i\xi e^{-\frac{\xi^2}{2}}\left(\xi^2 - 3\right) = ig_3(\xi) - 3ig_1(\xi). \end{aligned} \tag{15}$$

b) ***Find non-zero functions*** $h_k \in \mathscr{S}(\mathbb{R})$ ***such that*** $\mathcal{F}(h_k) = i^k h_k$, ***for*** $k = 0, 1, 2, 3$.

Espy that as $i^0 = 1$ we may by (12) fairly immediately pick $h_0(\xi) := g_0(\xi)$.
As $i^3 = -i$, we may also deduce $h_3(\xi) := g_1(\xi)$ makes a sufficient pick by (13). Notice also that for $h_2(\xi) := g_2(\xi) - \frac{1}{2}g_0(\xi)$, we have

$$\begin{aligned}\mathcal{F}(h_2(\xi)) \equiv \mathcal{F}(g_2(\xi) - \frac{1}{2}g_0(\xi)) &\overset{\mathcal{F}lin.}{\underset{P11.8}{=}} \mathcal{F}(g_2)(\xi) - \frac{1}{2}\mathcal{F}(g_0)(\xi) \\ &= -g_2(\xi) + g_0(\xi) - \frac{1}{2}g_0(\xi) \\ &\overset{(14),(12)}{=} -\left(g_2(\xi) - \frac{1}{2}g_0(\xi)\right) = i^2 h_2.\end{aligned}$$

And that we may finally choose $h_1(\xi) := g_3(\xi) - \frac{3}{2}g_1(\xi)$, as have

$$\begin{aligned}\mathcal{F}(h_1(\xi)) \equiv \mathcal{F}(g_3(\xi) - \frac{3}{2}g_1(\xi)) &\overset{\mathcal{F}lin.}{\underset{P11.8}{=}} \mathcal{F}(g_3)(\xi) - \frac{3}{2}\mathcal{F}(g_1)(\xi) \\ &\overset{(15),(13)}{=} ig_3(\xi) - 3ig_1(\xi) - \left(-\frac{3}{2}ig_1(\xi)\right) \\ &= i\left(g_3(\xi) - \frac{3}{2}g_1(\xi)\right) \equiv i^1 h_1.\end{aligned}$$

Note in particular, that all of the above h_i's are non-zero.

c) ***Show that*** $\mathcal{F}^4(f) = f$, ***for all*** $f \in \mathscr{S}(\mathbb{R})$.

Choose arbitrary $f \in \mathscr{S}(\mathbb{R})$, and define $F := \mathcal{F}(f)$. Note by C12.12 LN that F is unitary on $\mathscr{S}(\mathbb{R})$ such that we may in particular conclude $\mathcal{F}^*(F) = f$, for $\mathcal{F}^*$ being the inverse Fourier transform on $\mathbb{R}$ (or atleast our designated copy of $\mathbb{R}$, $\hat{\mathbb{R}}$) which itself by D12.10 may be written on the form $\mathcal{F}^*(f)(x) \equiv \check{f}(x) := \int_{\hat{\mathbb{R}}} f(\xi)e^{i\langle x,\xi\rangle}dm(\xi)$, such that

$$f(x) = \mathcal{F}^*(F)(x) \equiv \int_{\hat{\mathbb{R}}} F(\xi)e^{i\langle x,\xi\rangle}dm(\xi), \tag{16}$$

$x \in \mathbb{R}$. Consequently observe that for $x \in \mathbb{R}$

$$\begin{aligned}\mathcal{F}^2(f)(x) \equiv \mathcal{F}(\mathcal{F}(f))(x) \equiv &\mathcal{F}(F)(x) \\ &= \int_{\mathbb{R}} F(\xi)e^{-i\langle x,\xi\rangle}dm(\xi) \\ &= \int_{\mathbb{R}} F(\xi)e^{-i\langle x,\xi\rangle}dm(\xi) \\ &= \int_{\mathbb{R}} F(\xi)e^{i\langle -x,\xi\rangle}dm(\xi) \\ &\overset{(16)}{=} f(-x).\end{aligned} \tag{17}$$

Hence we may deduce that

$$\mathcal{F}^4(f)(x) \equiv \mathcal{F}^2(\mathcal{F}^2(f))(x) \overset{(17)}{=} \mathcal{F}^2(f(-x)) \overset{(17)}{=} f(x), \tag{18}$$

as requested.

d) *Use c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$ then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.*

Assuming $\mathcal{F}(f) = \lambda f$ for $f \not\equiv 0$, $\lambda \in \mathbb{C}$, heed, using repeated linearity

$$\begin{aligned} \mathcal{F}^4(f) = \mathcal{F}^3(\lambda f) \equiv \mathcal{F}(\mathcal{F}(\mathcal{F}(\lambda f))) \\ =& \mathcal{F}(\mathcal{F}(\lambda \mathcal{F}(f))) \\ =& \mathcal{F}(\lambda \mathcal{F}(\mathcal{F}(f))) \\ =& \lambda \mathcal{F}(\mathcal{F}(\mathcal{F}(f))) \equiv \lambda \mathcal{F}^3(f). \end{aligned} \tag{19}$$

Repeating the process of (19) three more times, we get

$$\begin{aligned} f &\overset{(18)}{=} \mathcal{F}^4(f) \\ &\overset{(19)}{=} \lambda \mathcal{F}^3(f) \\ &\overset{(19)}{=} \lambda^2 \mathcal{F}^2(f) \\ &\overset{(19)}{=} \lambda^3 \mathcal{F}(f) \\ &\overset{(19)}{=} \lambda^4 f, \end{aligned}$$

consequently yielding the equation

$$f - \lambda^4 f = f\left(1 - \lambda^4\right) = 0. \tag{20}$$

By assumption $f \not\equiv 0$, such that we by (20) therefore must have $1 - \lambda^4 = 0 \Leftrightarrow \lambda^4 = 1 \Leftrightarrow \lambda \in \{1, i, -1, -i\}$, by the "zero-rule of products". The eigenvalues of $\mathcal{F}$ will therefore be amongst the four forth-roots of unity, by the fundamental theorem of algebra.

Problem 5 *Let $(x_n)_{n\in\mathbb{N}}$ be a dense subset of $[0,1]$ and consider the Radon measure $\mu := \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$. Show that $supp(\mu) = [0,1]$.*

Note that a subset B of some topological space (X, τ) is dense if and only if all neighbourhoods of some $x \in X$ contain some point of B, such that B will be dense if and only if all open, non-empty subsets of X will have non-empty intersection with B.

Thus, assuming $(x_n)_{n\in\mathbb{N}}$ dense in $[0,1]$, yields that for any non-empty open subset $Q \subset [0,1]$ there will exist some $t \in [0,1]$ such that $t \in Q \cap (x_n)_{n\in\mathbb{N}}$, and hence there will exist some $N \in \mathbb{N}$ such that $t = x_N \in Q$. By the definition of μ we will therefore have

$$\begin{aligned} \mu(Q) \equiv \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}(Q) &\overset{(*)}{=} \sum_{n=1}^{N-1} 2^{-n}\delta_{x_n}(Q) + 2^{-N}\delta_{x_N}(Q) + \sum_{n=N+1}^{\infty} 2^{-n}\delta_{x_n}(Q) \\ &\geq 2^{-N}\delta_{x_N}(Q) = 2^{-N} > 0, \end{aligned} \tag{21}$$

yielding $Q \subseteq supp(\mu)$, observing that the split (*) is well defined as $\mu \leq \sum_{n=1}^{\infty} 2^{-n}$. Developing (21) further, note for $Q_n := \left(0 + \frac{1}{n}, 1 - \frac{1}{n}\right)$, that $\mu(Q_n) \overset{(21)}{>} 0, \forall n \in \mathbb{N}$ while $\bigcup_{n=1}^{\infty} Q_n = [0,1]$, such that $supp(\mu) \supseteq \bigcup_{n=1}^{\infty} Q_n = [0,1]$, and as μ is defined within $[0,1]$ and therefore $supp(\mu) \subseteq [0,1]$, we therefore have $supp(\mu) = [0,1]$.