Advanced Mathematical Physics, Assignment 1

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1 Stability in two dimensions

We define the energy functional for a particle in $\mathbb{R}^2$ as $\mathcal{E}(\psi) = T_\psi + V_\psi$, with

$$T_\psi = \int_{\mathbb{R}^2} |\nabla\psi(x)|^2 \, \mathrm{d}x, \quad \text{and} \quad V_\psi = \int V(x)|\psi(x)|^2 \, \mathrm{d}x. \tag{1.1}$$

The ground state energy is defined by

$$E_0 = \inf\{\mathcal{E}(\psi), \ \psi \in H^1(\mathbb{R}^2), \ \|\psi\|_2 = 1, \ V_\psi \text{ well defined.}\}. \tag{1.2}$$

Now assuming that $V \in L^{1+\epsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ we prove that $E_0 > -\infty$.

Proof. Let $V = v + w$ with $v \in L^{1+\epsilon}(\mathbb{R}^2)$ and $w \in L^\infty(\mathbb{R}^2)$. Notice first, that by Sobolev's inequality we have

$$\|\nabla\psi\|_2^2 \geq S_{2,p} \|\psi\|_2^{\frac{-4}{p-2}} \|\psi\|_p^{\frac{2p}{p-2}}, \qquad 2 < p < \infty. \tag{1.3}$$

It follows that $\psi \in L^p(\mathbb{R}^2)$ for $2 < p < \infty$, whenever $\psi \in H^1(\mathbb{R}^2)$. Assuming that V_ψ is well defined we know from Hölder's inequality that

$$\begin{aligned} V_\psi = \int V(x)|\psi(x)|^2 \, \mathrm{d}x &\geq \int v(x)|\psi(x)|^2 \, \mathrm{d}x - \|w\|_\infty \|\psi\|_2^2 \\ &\geq -\|v\|_q \||\psi|^2\|_{\frac{q}{q-1}} - \|w\|_\infty \|\psi\|_2^2 \\ &= -\|v\|_q \|\psi\|_{\frac{2q}{q-1}}^2 - \|w\|_\infty \|\psi\|_2^2. \end{aligned} \tag{1.4}$$

Thus setting $p = \frac{2q}{q-1} = 2 + \frac{2}{\epsilon}$, with $\epsilon > 0$, we find that

$$V_\psi \geq -\|v\|_{1+\epsilon} \|\psi\|_p^2 - \|w\|_\infty \|\psi\|_2^2. \tag{1.5}$$

Now using Sobolev's inequality we find that

$$T_\psi \geq S_{2,p} \|\psi\|_2^{\frac{-4}{p-2}} \|\psi\|_p^{\frac{2p}{p-2}} = S_{2,p} \|\psi\|_2^{\frac{-4}{p-2}} \|\psi\|_p^{2(1+\epsilon)}. \tag{1.6}$$

Thus we conclude that $\mathcal{E}(\psi) \geq S_{2,p} \|\psi\|_2^{\frac{-4}{p-2}} \|\psi\|_p^{2(1+\epsilon)} - \|v\|_{1+\epsilon} \|\psi\|_p^2 - \|w\|_\infty \|\psi\|_2^2$. Consider now the case in which $\psi \in H^1(\mathbb{R}^2)$, $\|\psi\|_2 = 1$ and V_ψ is well defined. It then follows that

$$\mathcal{E}(\psi) \geq S_{2,p}\|\psi\|_p^{2(1+\epsilon)} - \|v\|_{1+\epsilon}\|\psi\|_p^2 - \|w\|_\infty. \tag{1.7}$$

Therefore, we may conclude that

$$\begin{aligned}
E_0 &= \inf\{\mathcal{E}(\psi) : \psi \in H^1(\mathbb{R}^2),\ \|\psi\|_2 = 1,\ V_\psi \text{ well defined}\} \\
&\geq \inf\{S_{2,p}\|\psi\|_p^{2(1+\epsilon)} - \|v\|_{1+\epsilon}\|\psi\|_p^2 - \|w\|_\infty : \psi \in H^1(\mathbb{R}^2),\ \|\psi\|_2 = 1,\ V_\psi \text{ well defined}\} \\
&\geq \inf\{S_{2,p}x^{(1+\epsilon)} - \|v\|_{1+\epsilon}x - \|w\|_\infty : x \in \mathbb{R},\ x \geq 0\} > -\infty,
\end{aligned} \tag{1.8}$$

where we have used that fact that
$\{\|\psi\|_p^2 : \psi \in H^1(\mathbb{R}^2),\ \|\psi\|_2 = 1,\ V_\psi \text{ well defined}\} \subseteq \{x \in \mathbb{R} : x \geq 0\}$ □

2 Stability of hydrogen through ground state positivity

(a)

Let $\Omega \in \mathbb{R}^3$ be an open set and $V \in \mathcal{C}(\Omega)$. Assume that $\psi \in \mathcal{C}^2(\Omega)$ satisfies $(-\Delta + V)\psi = E\psi$ for some $E \in \mathbb{R}$ and furthermore $\psi > 0$. Then it holds that

$$\int_\Omega |(\nabla\varphi)(x)|^2 \,\mathrm{d}x + \int_\Omega V(x)|\varphi(x)|^2 \,\mathrm{d}x \geq E\int_\Omega |\varphi(x)|^2 \,\mathrm{d}x, \tag{2.1}$$

for all $\varphi \in \mathcal{C}_0^1(\Omega)$.

Proof. Let $\varphi \in \mathcal{C}_0^1(\Omega)$, and write $\varphi = g\psi$. Since $\psi > 0$ we clearly have $g = \varphi/\psi \in \mathcal{C}_0^1(\Omega)$. Notice that $\nabla\varphi = (\nabla g)\psi + g(\nabla\psi)$ and therefore

$$|\nabla\varphi|^2 = |\psi|^2|\nabla g|^2 + |g|^2|\nabla\psi|^2 + (\nabla g)(\nabla\psi)\bar{g}\psi + (\nabla\psi)(\nabla\bar{g})\psi g. \tag{2.2}$$

Using that $(\nabla g)(\nabla\psi)\bar{g}\psi = \nabla\cdot(g(\nabla\psi)\bar{g}\psi) - |g|^2(\Delta\psi)\psi - g(\nabla\psi)(\nabla\bar{g})\psi - |g|^2|\nabla\psi|^2$, we find

$$|\nabla\varphi|^2 = |\psi|^2|\nabla g|^2 + \nabla\cdot(g(\nabla\psi)\bar{g}\psi) - |g|^2(\Delta\psi)\psi. \tag{2.3}$$

Applying Stokes' (or Gauss') theorem, as well as using the fact that g has compact support[1] we conclude

$$\int_\Omega |(\nabla\varphi)(x)|^2 \,\mathrm{d}x = \int_\Omega |\psi(x)|^2|\nabla g(x)|^2 - |g(x)|^2(\Delta\psi(x))\psi(x) \,\mathrm{d}x \geq \int_\Omega |g(x)|^2\psi(x)(-\Delta\psi(x)). \tag{2.4}$$

[1]Notice that since g is continuous, the support of g, $\mathrm{supp}(g) = \{x \in \mathbb{R}^3 : f(x) \neq 0\}$, is necessarily open. However, $S = \overline{\mathrm{supp}(g)}$ is compact by assumption. Furthermore, by continuity of g, we must have $g|_{\partial S} = 0$. Thus we may split the integral

$$\int_\Omega \nabla\cdot(g(\nabla\psi)\bar{g}\psi) \,\mathrm{d}x = \int_S \nabla\cdot(g(\nabla\psi)\bar{g}\psi) \,\mathrm{d}x + \int_{\Omega\setminus S} \nabla\cdot(g(\nabla\psi)\bar{g}\psi) \,\mathrm{d}x = \int_{\partial S} (g(\nabla\psi)\bar{g}\psi)\cdot\hat{n} \,\mathrm{d}a = 0.$$

Therefore we conclude

$$\begin{aligned}\int_\Omega |(\nabla\varphi)(x)|^2\,\mathrm{d}x + \int_\Omega V(x)|\varphi(x)|^2\,\mathrm{d}x &\geq \int_\Omega |g(x)|^2\psi(x)(-\Delta\psi(x)) + |g(x)|^2\psi(x)(V(x)\psi(x))\,\mathrm{d}x \\ &= \int_\Omega |g(x)|^2\psi(x)\left[(-\Delta + V(x))\psi(x)\right]\mathrm{d}x \\ &= E\int_\Omega |g(x)|^2|\psi(x)|^2\,\mathrm{d}x \\ &= E\int_\Omega |\varphi(x)|^2\,\mathrm{d}x\end{aligned} \tag{2.5}$$

this concludes the proof. □

(b)

Consider now the function $\psi(x) = \exp(-\alpha\,|x|)$. We show that this function indeed satisfies $\psi \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\})$ and that there exist an α such that $(-\Delta - Z/\,|x|)\psi = E_0\psi$ for some E_0. First we notice that ψ is a composition of $\mathcal{C}^\infty(\mathbb{R}^3 \setminus \{0\})$, thus $\psi \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\}) \subset \mathcal{C}^\infty(\mathbb{R}^3 \setminus \{0\})$. Furthermore, by going to spherical coordinates (r, θ, φ), with θ the azimuthal angle and φ the polar angle, we can express $\tilde{\psi}(r, \theta, \phi) := \psi(x(r, \theta, \varphi)) = \exp(-\alpha r)$. It is well known that the Laplacian on $\mathcal{C}^2(\mathbb{R} \setminus \{0\})$, Δ, can be exressed in polar coordinates as[2]

$$\Delta\phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi\frac{\partial\phi}{\partial\varphi}) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\phi}{\partial^2\theta}, \quad r > 0,\ 0 \leq \theta < 2\pi,\ 0 \leq \varphi \leq \pi. \tag{2.6}$$

Thereby we see that

$$\begin{aligned}(-\Delta - Z/\,|x|)\psi(x)|_{x=x(r,\theta,\varphi)} = (-\Delta - Z/r)\tilde{\psi}(r, \theta, \varphi) &= -\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\exp(-\alpha r)) - Z/r\exp(-\alpha r) \\ &= (-\alpha^2 + 2\alpha/r - Z/r)\exp(\alpha r).\end{aligned}$$

Thus choosing $\alpha = Z/2$ we find that $(-\Delta - Z/r)\psi = E_0\psi$, with $E_0 = -Z^2/4$. From problem 2.(a) with $\Omega = \mathbb{R}^3 \setminus \{0\}$, which is clearly open, we then conclude that for all $\varphi \in \mathcal{C}_0^1(\mathbb{R}^3 \setminus \{0\})$ we have

$$\int_{\mathbb{R}^3\setminus\{0\}} |(\nabla\varphi)(x)|^2\,\mathrm{d}x - \int_{\mathbb{R}^3\setminus\{0\}} \frac{Z}{|x|}|\varphi(x)|^2\,\mathrm{d}x \geq E\int_{\mathbb{R}^3\setminus\{0\}} |\varphi(x)|^2\,\mathrm{d}x. \tag{2.7}$$

3 Lieb-Thirring inequalities in one dimension

We show that in one dimension a Lieb-Thirring inequality of the form

$$\sum_{j\geq 0} |E_j|^\gamma \leq L_\gamma \int_{\mathbb{R}} V_-(x)^{\gamma+1/2}\,\mathrm{d}x, \tag{3.1}$$

[2]Notice that we use Δ to denote the Laplacian in both spherical and Cartesian coordinates.

cannot hold for $0 \leq \gamma < 1/2$. We show this by contradiction. Consider the Hamiltonian $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha(\alpha+1)(\tanh(x)^2 - 1)$ with eigenfunction $\psi(x) = \frac{1}{\cosh(x)^\alpha}$, $\alpha > 0$

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \alpha(\alpha+1)(\tanh(x)^2 - 1)\psi(x) = -\alpha^2\psi(x). \tag{3.2}$$

This can be seen by the following calculations:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cosh(x)^\alpha}\right) = -\alpha\frac{\sinh(x)}{\cosh(x)^{\alpha+1}}, \tag{3.3}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{\cosh(x)^\alpha}\right) = -\alpha\frac{1}{\cosh(x)^\alpha} + \alpha(\alpha+1)\frac{1}{\cosh(x)^\alpha}\tanh(x)^2. \tag{3.4}$$

From which it clearly follows that

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \alpha(\alpha+1)(\tanh(x)^2 - 1)\psi(x) = -\alpha^2\psi(x). \tag{3.5}$$

The potential of H, is clearly given by $V(x) = \alpha(\alpha+1)(\tanh(x)^2 - 1)$. Since $\tanh(x) < 1$ for $x \in \mathbb{R}$. we have that $V_-(x) = \alpha(\alpha+1)(1 - \tanh(x)^2)$. Assume now that a Lieb-Thirring inequality of the form (3.1) with $0 \leq \gamma < 1/2$ holds. Let us then compute the right-hand side of the inequality with the potential $V_-(x) = \alpha(\alpha+1)(1 - \tanh(x)^2)$

$$\begin{aligned} L_\gamma\alpha(\alpha+1)\int_{\mathbb{R}}(1-\tanh(x)^2)^{\gamma+1/2}\,\mathrm{d}x &= L_\gamma\alpha(\alpha+1)\int_{(-1,1)}(1-u^2)^{\gamma-1/2}\,\mathrm{d}u \\ &= 2L_\gamma\alpha(\alpha+1)\int_{(0,1)}(1+u)^{\gamma-1/2}(1-u)^{\gamma-1/2}\,\mathrm{d}u, \end{aligned} \tag{3.6}$$

where we have made the change of variables $u = \tanh(x)$ in the first line, Notice that then $\frac{\mathrm{d}u}{\mathrm{d}x} = 1 - \tanh(x)^2$. In the second line we simply exploited the fact that the integrand is even in u and factorized the integrand. Since the integrand is positive we can by monotone convergence theorem express it as a limit of integrals over the intervals $(1/n, 1)$ with $n \to \infty$. Then we can rewrite all these integrals to Riemann integrals. By a simple comparison to integrals of the type $\int_0^1 \frac{1}{x^p}dx$, it is clear that the integral of (3.6) is convergent if and only if $\gamma < 1/2$. In this case we simply define $C_\gamma = 2L_\gamma\int_{(-1,1)}(1-u^2)^{\gamma-1/2}du$, and we see that the Lieb-Thirring inequality is of the form

$$\sum_j |E_j|^\gamma \leq \alpha(\alpha+1)C_\gamma. \tag{3.7}$$

On the other hand we know that $\alpha^{2\gamma} \leq \sum_j |E_j|^\gamma$, since we have shown $-\alpha^2$ to be one of the energies. Thus we conclude that

$$\alpha^{2\gamma} \leq \alpha(\alpha+1)C_\gamma, \quad 0 \leq \gamma < 1/2 \tag{3.8}$$

However, this is clear a contradiction since $\alpha > 0$ was chosen arbitrarily. To see this, simply choose $0 < \alpha < 1$ such that $\alpha^{2\gamma-1} > 2C_\gamma$. This conlcudes that in one dimension, there can be no Lieb-Thirring inequality of the form (3.1) with $0 \leq \gamma < 1/2$.

4 Thomas-Fermi theory

Notation: We say in the following that an integral $\int f(x)\,\mathrm{d}x$ act as a bounded linear functional on some L^p-space if the linear functional $F : L^p \ni g \mapsto \int f(x)g(x)\,\mathrm{d}x \in \mathbb{C}$ is bounded on L^p

Let $\rho \in L^1(\mathbb{R}^3) \cap L^{5/3}(\mathbb{R}^3)$, $\rho > 0$. The direct Couloumb energy is defined as

$$D(\rho) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.1}$$

Consider then the *Thomas-Fermi* energy function

$$\mathcal{E}^{TF}(\rho) = \int_{\mathbb{R}^3} \rho(x)^{5/3}\,\mathrm{d}x - \int_{\mathbb{R}^3} \frac{\rho(x)}{|x|} + D(\rho), \tag{4.2}$$

and for a fixed $N > 0$ the minimization problem

$$\begin{aligned} &, E_0 := \inf\left\{\mathcal{E}^{TF}(\rho) : \rho \in \mathcal{D}_N\right\} \\ &\mathcal{D}_N := \left\{\rho \in L^1(\mathbb{R}^3) \cap L^{5/3}(\mathbb{R}^3) : \rho \geq 0,\ \|\rho\|_1 \leq N\right\}. \end{aligned} \tag{4.3}$$

(a)

We prove that for $\rho \in \mathcal{D}_N$, we have

$$\int_{\mathbb{R}^3} \frac{1}{|x|}\rho(x)\,\mathrm{d}x \leq cN + d\,\|\rho\|_{5/3}\,, \tag{4.4}$$

with some constants $c, d > 0$ independent of ρ. To see this, let $a > 0$, we then split the integral

$$\int_{\mathbb{R}^3} \frac{1}{|x|}\rho(x)\,\mathrm{d}x = \int_{|x|\leq a} \frac{1}{|x|}\rho(x)\,\mathrm{d}x + \int_{|x|>a} \frac{1}{|x|}\rho(x)\,\mathrm{d}x \leq \int_{|x|\leq a} \frac{1}{|x|}\rho(x)\,\mathrm{d}x + a^{-1}\,\|\rho\|_1\,. \tag{4.5}$$

For the remaining integral we use Hölder's inequality with $q = 5/2$ and $p = 5/3$. Then we get

$$\int_{|x|\leq a} \frac{1}{|x|}\rho(x)\,\mathrm{d}x \leq \left|\int_{|x|\leq a} \frac{1}{|x|^{5/2}}\,\mathrm{d}x\right|^{\frac{2}{5}} \|\rho\|_{5/3} = \left|4\pi \int_{(0,a)} \frac{1}{r^{1/2}}\,\mathrm{d}r\right|^{\frac{2}{5}} \|\rho\|_{5/3} = (8\pi\sqrt{a})^{2/5}\,\|\rho\|_{5/3} \tag{4.6}$$

where we in the second equality changed to spherical coordinates with Jacobian $r^2\sin(\varphi)$ and computed the angular integrals directly. Thus we have for $\rho \in \mathcal{D}_N$

$$\int_{\mathbb{R}^3} \frac{1}{|x|}\rho(x)\,\mathrm{d}x \leq a^{-1}N + (8\pi\sqrt{a})^{2/5}\,\|\rho\|_{5/3}\,, \tag{4.7}$$

where we used that $\|\rho\|_1 \leq N$. Knowing that $0 \leq D(\rho) < \infty$ and choosing $0 < a \leq 1/(8\pi)^2$ we may conclude that

$$E_0 \geq \inf\left\{(1-(8\pi\sqrt{a})^{2/5})\,\|\rho\|_{5/3} - a^{-1}N + D(\rho) : \rho \in \mathcal{D}_N\right\} \geq -a^{-1}N > -\infty. \tag{4.8}$$

(b)

Let $(\rho^j)_{j\geq 1} \subset \mathcal{D}_N$ be a sequence such that $\mathcal{E}^{TF}(\rho^j) \to E_0$. Then $\left\|\rho^j\right\|_{5/3}$ is bounded. From Banach-Alaoglu's theorem as well as the fact that the predual of $L^{5/3}(\mathbb{R}^3)$, namely $L^{5/2}(\mathbb{R}^3)$, is reflexive[3], we may conclude after restricting to a subsequence that we have $\rho^j \rightharpoonup \rho_0$ for some $\rho_0 \in L^{5/3}(\mathbb{R}^3)$. We prove now that

$$\|\rho_0\|_{5/3} \leq \liminf_{j\geq 1} \left\|\rho^j\right\|_{5/3}. \tag{4.9}$$

In fact we can prove the more general statement: Let X be a Banach space and $(x^j)_{j\geq 1} \subset X$ a sequence converging weakly to $x \in X$, then $\|x\| \leq \liminf_{j\geq 1} \left\|x^j\right\|$.

Proof. By the Hahn-Banach theorem, there exist a linear functional $f : X \to \mathbb{C}$ such that $f(x) = \|x\|$ and such that $\|f\| = 1$. By weak convergence of x^j we then have

$$\|x\| = |f(x)| = \liminf_{j\geq 1} \left|f(x^j)\right| \leq \liminf_{j\geq 1} \left\|x^j\right\|, \tag{4.10}$$

which proves the claim. □

Now since $L^{5/3}(\mathbb{R}^3)$ is a Banach space by the Riez-Fischer theorem, we have desired result.

(c)

We prove now that $\rho_0 > 0$ almost everywhere. First we notice that ρ_0 is measurable. Consider therefore the set $M_R = \{x \in \mathbb{R}^3 : \rho_0(x) < 0\} \cap B_R(0)$, where $B_R(0)$ denotes the ball of radius R centred at 0. Assume for contradiction that this set has measure greater than zero, $\lambda(M_R) > 0$ for some $R > 0$. Then $\int_{M_R} \rho_0(x)\,\mathrm{d}x < 0$. However, this is a contradiction, since $\int_{M_R} \rho^j(x)\,\mathrm{d}x \geq 0$ and $\int_{M_R} \mathrm{d}x$ acts as a bounded linear functional on $L^{5/3}(\mathbb{R}^3)$. As we have already established weak convergence of ρ^j in $L^{5/3}(\mathbb{R}^3)$, we may conclude that $\int_{M_R} \rho_0(x)\,\mathrm{d}x = \lim_{j\to\infty} \int_{M_R} \rho^j(x)\,\mathrm{d}x \geq 0$. Thus we have established that $\rho_0 \geq 0$ on $B_R(0)$ a.e. for all $R > 0$, from which it follows that $\rho_0 \geq 0$ a.e.

Now that we have established that $\rho_0 > 0$ a.e., we show that $\int_{\mathbb{R}^3} \rho_0\,\mathrm{d}x \leq N$. To see this consider the sequence $(\chi_{B_n(0)}\rho_0)_{n\geq 1}$. By the monotone convergence theorem we know that

$$\int_{\mathbb{R}^3} \rho_0(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^3} \chi_{B_n(0)}(x)\rho_0(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{B_n(0)} \rho_0(x)\,\mathrm{d}x \tag{4.11}$$

[3] Clearly $L^p(\mathbb{R}^3)$ is reflexive for all $1 < p < \infty$ as $(L^p(\mathbb{R}^3))^* = L^q(\mathbb{R}^3)$, with $1/p + 1/q = 1$.

Now since $\int_{B_n(0)} \mathrm{d}x$ acts as a bounded linear functional on $L^{5/3}(\mathbb{R}^3)$ we may conclude from weak convergence of ρ^j that

$$\int_{B_n(0)} \rho_0(x)\,\mathrm{d}x = \lim_{j\to\infty} \int_{B_n(0)} \rho^j(x)\,\mathrm{d}x. \tag{4.12}$$

From the fact that $\int_{B_n(0)} \rho^j\,\mathrm{d}x \leq \int_{\mathbb{R}^3} \rho^j(x)\,\mathrm{d}x \leq N$ we may conclude that

$$\int_{B_n(0)} \rho_0(x)\,\mathrm{d}x \leq N, \quad \text{for all } n \geq 1. \tag{4.13}$$

Thus it follows from the MCT that $\int_{\mathbb{R}^3} \rho_0(x)\,\mathrm{d}x \leq N$, so $\rho_0 \in \mathcal{D}_N$.

(d)

It can be shown that $\rho^j \rightharpoonup \rho_0$ in $L^q(\mathbb{R}^3)$ for some $1 < q < 3/2$. Using this we can show that

$$\int_{\mathbb{R}^3} \frac{1}{|x|}\rho^j(x)\,\mathrm{d}x \to \int_{\mathbb{R}^3} \frac{1}{|x|}\rho_0(x)\,\mathrm{d}x. \tag{4.14}$$

To see this, we split the integral in two

$$\int_{\mathbb{R}^3} \frac{1}{|x|}\rho^j(x)\,\mathrm{d}x = \int_{|x|\leq 1} \frac{1}{|x|}\rho^j(x)\,\mathrm{d}x + \int_{|x|>1} \frac{1}{|x|}\rho^j(x)\,\mathrm{d}x. \tag{4.15}$$

We then notice that the integral $\int_{|x|\leq 1} \frac{1}{|x|} dx$ acts as a bounded linear function on $L^{5/3}(\mathbb{R}^3)$. This can be seen by the fact that $\chi_{|x|\leq 1}\frac{1}{|x|} \in L^{5/2}(\mathbb{R}^3)$. Thus by Hölder's inequality we have for $f \in L^{5/3}(\mathbb{R}^3)$ that

$$\left| \int_{|x|\leq 1} \frac{1}{|x|} f(x)\,\mathrm{d}x \right| \leq \left\| \chi_{|x|\leq 1}\frac{1}{|x|} \right\|_{5/2} \|f\|_{5/3}. \tag{4.16}$$

Therefore, we may conclude the convergence of the first integral by weak convergence of ρ^j in $L^{5/3}(\mathbb{R}^3)$. For the second integral, we instead use that $\chi_{|x|>1}\frac{1}{|x|} \in L_p(\mathbb{R}^3)$ for all $p > 3$. Again by Hölder's inequality $\int_{|x|>1} \frac{1}{|x|}\,\mathrm{d}x$ acts as a bounded linear functional on $L^q(\mathbb{R}^3)$ for all $1 < q < 3/2$. Thus, by the fact that ρ^j converges weakly in $L_q(\mathbb{R}^3)$ for some $1 < q < 3/2$ we may conclude the convergence of the second integral. Thereby we have

$$\underbrace{\int_{|x|\leq 1} \frac{1}{|x|}\rho^j(x) \to \int_{|x|\leq 1} \frac{1}{|x|}\rho_0(x),}_{\text{weak convergence in } L^{5/3}(\mathbb{R}^3)} \qquad \underbrace{\int_{|x|>1} \frac{1}{|x|}\rho^j(x) \to \int_{|x|>1} \frac{1}{|x|}\rho_0(x)}_{\text{weak convergence in } L^q(\mathbb{R}^3) \text{ for some } 1<q<3/2}. \tag{4.17}$$

From which we obtain the desired result

$$\int_{\mathbb{R}^3} \frac{1}{|x|}\rho^j(x) \to \int_{\mathbb{R}^3} \frac{1}{|x|}\rho_0(x). \tag{4.18}$$

(e)

Collecting all the result from problem $4.(a)$ to $4.(b)$ and assuming that $D(\rho_0) \leq \liminf_{j\to\infty} D(\rho^j)$, we can now show that

$$\mathcal{E}^{TF}(\rho_0) = E_0, \tag{4.19}$$

i.e. ρ_0 is a minimizer of $\mathcal{E}^{TF}$.