Mandatory Assignment 2, Functional Analysis

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Problem 1

Part (a)

It follows from **Homework week 4, problem 2(a)** that f_N converges weakly to 0 as $N \to \infty$ if and only if $T(f_n)$ converges to $T(0) = 0$ for all $T \in H^*$. We also know from Riesz representation theorem (**Homework week 2, problem 1**) that for every bounded linear functional $T \in H^*$ there ~~exits~~ exists a unique $y \in H$ such that $T(F_N) = \langle F_n, y\rangle$. Now for a given $y \in H$ we have from linearity (of the first coordinate) in inner product that

$$\langle F_n, y\rangle = \langle \frac{1}{N}\sum_{n=1}^{N^2} e_n, y\rangle = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, y\rangle$$

Since $(e_n)_{n\in\mathbb{N}}$ is an ONB we have that $\overline{y} = \overline{\sum_{n=1}^{\infty}\langle y, e_n\rangle} = \sum_{n=1}^{\infty}\overline{\langle y, e_n\rangle} = \sum_{n=1}^{\infty}\langle e_n, y\rangle$, so $\sum_{n=1}^{N^2}\langle e_n, y\rangle$ converges to $\overline{y}$ for $N \to \infty$. Therefore we have for all $y \in H$ (and hence all $T \in H^*$ that $T(F_n) = \langle F_n, y\rangle = \frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, y\rangle$ converges to $0 = \langle 0, y\rangle = T(0)$ for $N \to \infty$ and hence F_N converges weakly to 0. ✓
Could be more explicit here.
Using Parseval's Identity (**Folland, Theorem 5.27 (b)**)We have for all $N \in \mathbb{N}$ that

$$\|F_N\|^2 = \sum_{i=1}^{\infty}|\langle F_N, e_i\rangle|^2 = \sum_{i=1}^{\infty}|\frac{1}{N}\sum_{n=1}^{N^2}\langle e_n, e_i\rangle|^2 = \frac{1}{N^2}\sum_{i=1}^{N^2}|\langle e_i, e_i\rangle|^2 = \frac{1}{N^2}N^2 = 1$$

Where we have used that $(e_n) \subset H$ is an ONB so $\langle e_n, e_i\rangle$ is equal to 0 if $n \neq i$ and equal to 1 if $n = i$. ✓

Part (b)

For a given $g \in \text{co}\{F_N : N \in \mathbb{N}\}$ there exits a $n \in \mathbb{N}$ such that $g = \sum_{i=1}^{n}\alpha_i f_i$ where $\alpha_i \geq 0$ and $\sum_{i=1}^{n}\alpha_i = 1$. Then we have that

$$\|g\| = \left\|\sum_{i=1}^{n}\alpha_i F_i\right\| \leq \sum_{i=1}^{n}\alpha_i\|F_i\| = \sum_{i=1}^{n}\alpha_i = 1$$

Therefore we can conclude that $\text{co}\{F_N : N \in \mathbb{N}\} \subset \overline{B}_H(0,1)$. Since $\overline{B}_H(0,1)$ is closed with respect to the norm topology, we have that $\overline{\text{co}\{F_N : N \in \mathbb{N}\}}^{\|\cdot\|} \subset \overline{B}_H(0,1)$. Since $\text{co}\{F_N : N \in \mathbb{N}\}$ is a convex subset of H it follows from **Theorem 5.7** that $\overline{\text{co}\{F_N : N \in \mathbb{N}\}}^{\|\cdot\|} = \overline{\text{co}\{F_N : N \in \mathbb{N}\}}^{\tau_w}$. Now as H is a Hilbet space and therefore reflexive we can use **Theorem 6.3** to say that $\overline{B}_H(0,1)$ is compact with respect to τ_w. We now have that

$$\overline{\text{co}\{F_N : N \in \mathbb{N}\}}^{\|\cdot\|} = \overline{\text{co}\{F_N : N \in \mathbb{N}\}}^{\tau_w} \subset \overline{\overline{B}_H(0,1)}^{\tau_w}$$

Using that a closed subset of a compact set is compact (**Folland, Proposition 4.22**) we can therefore conclude that $K = \overline{\text{co}\{F_N : N \in \mathbb{N}\}}^{\tau_w}$ is weakly compact. ✓ Since $F_N \to 0$ weakly from **Problem 1(a)** we can now use **Homework 5, Problem 1** to say that there exits a sequence $(y_n)_{n\in\mathbb{N}} \subset \text{co}\{F_N : N \in \mathbb{N}\}$ such that $(y_n)_{n\in\mathbb{N}}$ converges in norm to 0. Since $K = \overline{\text{co}\{F_N : N \in \mathbb{N}\}}^{\|\cdot\|}$ we know that K contains all the limits points of norm convergence and hence $0 \in K$. ✓

Part (c)

For a given $g \in \text{co}\{F_N : N \in \mathbb{N}\}$ we have that for all $j \in \mathbb{N}$ that

$$\langle g, e_i\rangle = \langle \sum_{i=1}^{n} \alpha_i f_i, e_j\rangle = \sum_{i=1}^{n} \alpha_i \langle \frac{1}{i} \sum_{k=1}^{i^2} e_k, e_j\rangle = \sum_{i=1}^{n} \frac{\alpha_i}{i} \sum_{k=1}^{i^2} \langle e_k, e_j\rangle = \sum_{i=1}^{n} \frac{\alpha_i}{i} 1_{(i^2 \geq j)}$$

Where $1_{(i^2\geq j)}$ is indicator function. We then note that $\langle g, e_i\rangle \geq 0$ and is real valued since α_i has to be real valued. Now for a given $h \in K$ we have that since $K = \overline{\text{co}\{F_N : N \in \mathbb{N}\}}^{\|\cdot\|}$ then there exits a sequence $(h_n)_{n\in\mathbb{N}} \subset \text{co}\{F_N : N \in \mathbb{N}\}$ such that $h_n \to h$ for $n \to \infty$. It now follows Hilbert space theory (**Folland**, Proposition 5.21) that $\langle h_n, e_i\rangle \to \langle h, e_i\rangle$ for all $n \to \infty$ for all $i \in \mathbb{N}$. Therefore we have that $\langle h, e_i\rangle$ is real valued and non-negative for all $h \in K$. Now assume that 0 is not a extreme point, then there exits $\alpha \in (0, 1)$ and $h_1, h_2 \in K$ different from 0 such that $0 = \alpha h_1 + (1 - \alpha)h_2$. Since $h_1 \neq 0$ we know that there exits a $n \in \mathbb{N}$ such that $\langle h_1, e_n\rangle \neq 0$. Using linearity of the inner product we get that

$$\begin{aligned}\langle 0, e_n\rangle &= \langle \alpha h_1 + (1-\alpha)h_2, e_n\rangle \Rightarrow \\ 0 &= \alpha\langle h_1, e_n\rangle(1-\alpha) + \langle h_2, e_n\rangle \Rightarrow \\ \frac{\alpha}{\alpha-1}\langle h_1, e_n\rangle &= \langle h_2, e_n\rangle\end{aligned}$$

Since $\langle h_1, e_n\rangle \neq 0$ and we know that $\langle h_1, e_n\rangle \in \mathbb{R}_+$ then it follows that $\langle h_2, e_n\rangle$ has to be negative, since $\frac{\alpha}{\alpha-1} < 0$, which is a contradiction. Hence 0 cannot be written as a convex combination so $0 \in \text{Ext}(K)$. ✓

Since K is a non-empty, compact and convex subset of H with respect to the weak topology, it now follows from Krein-Milman (**Theorem 7.8**) that $K = \overline{\text{co}(\text{Ext}(K))}^{\tau_w}$. We now assume that $\text{Ext}(K) \subset F \cup \{0\}$ which will be proved in **Part (d)**. Assume that $F_{n_0} \notin \text{Ext}(K)$, let $F' = \{F_N : N \in \mathbb{N} \setminus \{n_0\}\}$ and let $K' = \overline{\text{co}(F' \cup \{0\})}^{\tau_w}$. Using Krein-Milman we must then have that $K' = K$ and hence $f_{n_0} \in K'$. We can reuse the arguments of **Problem 1(b)** to say that since $\text{co}(F' \cup \{0\})$ is Convex compact then $K' = \overline{\text{co}(F' \cup \{0\})}^{\tau_w} = \overline{\text{co}(F' \cup \{0\})}^{\|\cdot\|}$ per **Theorem 5.7**. So $f_{n_0} \in K'$ if and only if there exits a sequence $g_n \in \text{co}(F' \cup \{0\})$ such that $\|g_n - f_{n_0}\| \to 0$. It follows from **Homework week 5, Problem 2(b)** that $\text{co}(F' \cup \{0\})$ is given by

How so? Be more explicit!

Why is this applicable here?

$$\text{co}(F' \cup \{0\}) = \left\{ \sum_{\substack{i=1\\ i\neq n_0}}^{n} \alpha_i f_i, n \in \mathbb{N}, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i \leq 1 \right\}$$

For a $g \in \text{co}(F' \cup \{0\})$ we have that

$$\langle g, f_{n_0}\rangle = \langle \sum_{\substack{i=1\\ i\neq n_0}}^{k} \alpha_i f_i, f_{n_0}\rangle = \sum_{\substack{i=1\\ i\neq n_0}}^{n} \alpha_i \langle f_i, f_{n_0}\rangle = \sum_{\substack{i=1\\ i\neq n_0}}^{n} \alpha_i \frac{1}{i}\frac{1}{n_0} \langle \sum_{n=1}^{i^2} e_n, \sum_{m=1}^{n_0^2} e_m\rangle = \sum_{\substack{i=1\\ i\neq n_0}}^{n} \alpha_i \frac{1}{i}\frac{1}{n_0} \min\{i^2, n_0^2\}$$

Where we have that $\frac{1}{i}\frac{1}{n_0}\min\{i^2, n_0^2\} = \min\left\{\frac{i}{n_0}, \frac{n_0}{i}\right\}$ and we see that for all $i \in \mathbb{N} \setminus n_0$ we have that $\min\left\{\frac{i}{n_0}, \frac{n_0}{i}\right\} \leq \frac{n_0}{n_0+1}$. This holds since for $i < n_0$ we have that $\frac{i}{n_0} < \frac{n_0}{n_0+1}$ and for $i > n_0$ we have that $\frac{n_0}{i} \leq \frac{n_0}{n_0+1}$. Therefore we can conclude that for all $g \in \text{co}(F' \cup \{0\})$ we have that $\langle g, f_{n_0}\rangle \leq \frac{n_0}{n_0+1} < 1$. Hence for any sequence $g_n \to f_{n_0}$ we would have from **Folland, Proposition 5.21** that $\langle g_n, f_{n_0}\rangle \to \langle f_{n_0}, f_{n_0}\rangle = \|f_{n_0}\|^2 = 1$. But this contradicts our conclusion about $\langle g, f_{n_0}\rangle \leq \frac{n_0}{n_0+1} < 1$ and therefore no sequence in $\text{co}(F' \cup \{0\})$ converges to f_{n_0} and therefore $f_{n_0} \notin K'$. Hence we can conclude that $K' \neq K$ and f_{n_0} then has to be an Extreme point. ✓

Calculate.

Part (d)

We know that the weak topology is a LCTVS and that K is weakly compact and convex set. Since $\{F_N : N \in \mathbb{N}\}$ is a subset such that $K = \overline{\text{co}(F)}^{\tau_w}$ we have from Milman (**Theorem 7.9**) that $\text{Ext}(K) \subset \overline{F}^{\tau_w}$. Since $\overline{F}^{\tau_w} \subset K$ and K is weakly compact it follows from **Folland, Proposition 4.22** that $\overline{F}^{\tau_w}$ is weakly compact. It follows from the definition of weak convergence that since $F_N \to 0$ weakly for $N \to \infty$ then $0 \in \overline{F}^{\tau_w}$. Now since F is countable set in the Hilbert space we know that the closure of F contain exactly all of its limitpoints. But a sequence in $(h_n)_{n\in\mathbb{N}} \subset F$ will either have a $n_0 \in \mathbb{N}$ such that $F_{n_0} = h_n$ for infinitely many n or each F_N will only appear finitely many times. If this happens then we have that $h_n \to 0$ weakly. So if $F_{n_0} = h_n$ for infinitely many n we must have that if $(h_n)_{n\in\mathbb{N}}$ converges then the convergence point has to be f_{n_0} since weak convergence is unique. Therefore we have that $\overline{F}^{\tau_w} = \{f_N : N \in \mathbb{N}\} \cup \{0\}$ and then it follows that there are no other extreme points in K since $\text{Ext}(K) = \overline{F}^{\tau_w}$

Could be writte a bit neater.

✓

We note that weak converges is unique, since it is equivalent with convergence of $f(x_n) \to f(x)$ for every $f \in X^*$, and Hahn-Banach Extension Theorem (**Theorem 2.7**) tells us that the bounded linear functionals separate points.

Problem 2

Part (a)

Wrong notation. Should be $T^\dagger(y^*) \in X^*$.

We note that from **Theorem 7.13** we know that there for all $T \in \mathcal{L}(X,Y)$ exits a unique adjoint operator $T^\dagger$ such that $(T^\dagger y^*)(x) = y^*(Tx)$ for all $x \in X$ for all $y^* \in Y^*$ and we note that $T^\dagger \circ y^* \in X^*$. From **Homework week 4, Problem 2(a)** we know that if x_n converges weakly to x then $f(x_n)$ converges to $f(x)$ for all $f \in X^*$ so it especially holds for $T^\dagger \circ y^*$. Hence for a given $T \in \mathcal{L}(X,Y)$ we know that for all $y^* \in Y^*$ it holds that if x_n converges weakly to x then $T^\dagger \circ y^*(x_n) = y^*(T(x_n))$ converges to $T^\dagger \circ y^*(x) = y^*(T(x))$ and we can conclude, using **Homework week 4, Problem 2(a)**, that $T(x_n)$ converges weakly to $T(x)$, as $n \to \infty$. ✓

Part (b)

We first proof that in a Banach space, a sequence converges in norm, then it also converges weakly. So for $x_n \to x$ in norm we have that for all $f \in X^*$ it holds that

$$\|f(x_n) - f(x)\| = \|f(x_n - x)\| \leq \|f\| \, \|x_n - x\|$$

Where we used linearity of f and that it is bounded. Since $x_n \to x$ we know that $\|x_n - x\| \to 0$ for $n \to \infty$ and hence $f(x_n) \to f(x)$ and it now follows form **Homework week 4, Problem 2(a)** that x_n converges weakly to x.
For a given $T \in \mathcal{K}(X,Y)$ and a weakly convergent sequence, $(x_n)_{n\in\mathbb{N}} \subset X$, it follows from **Homework 4, Problem 2(b)** that $\sup_{n\mathbb{N}} \|x_n\| < \infty$ and it therefore also holds for any subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Using compactness of T we have from **Proposition 8.2** that the bounded sequence $(x_{n_k})_{k\in\mathbb{N}}$ contains a subsequence (so a subsubsequence of $(x_n)_{n\to\infty}$) $\left(x_{n_{k_p}}\right)_{p\in\mathbb{N}}$ such that $T\left(x_{n_{k_p}}\right)$ converges to a $y \in Y$. Then it follows from the small results above that $T\left(x_{n_{k_p}}\right) \to y$ weakly. We know that every compact operator is linear, so **Problem 2(a)** tells us that $T(x_n) \to T(x)$ weakly and using uniqueness of weak convergence we get that $T\left(x_{n_{k_p}}\right)$ has to converge to weakly $T(x)$, which only happens it it also converges in norm to $T(x)$. Since we have proved that every subsequence of $T(x_n)_{n\in\mathbb{N}}$ has a convergent subsubsequence with the same limit, it now follows that $T(x_n)$ converges to $T(x)$. ✓

Part (c)

Assume that for a given $T \in \mathcal{L}(H, Y)$ and that T is not a compact operator. Then we have from **Proposition 8.2** that $T(\overline{B}_H(0,1))$ is not totally bounded. This means that there exits a $\delta > 0$ such that such that for all $N \in \mathbb{N}$ and all sets of open balls $U_1, \ldots, U_N$ with radius δ we have that $T(\overline{B}_H(0,1)) \not\subset \cup_{i=1}^{N} U_i$. Then we can let $x_1 = 0$ and let $U_1 = B_Y(T(x_1), \delta)$ then since $T(\overline{B}_H(0,1)) \setminus U_1 \neq \emptyset$ there exits a $x_2 \in \overline{B}_H(0,1)$ such that $T(x_2) \in T(\overline{B}_H(0,1)) \setminus U_1$ and we let $U_2 = B_Y(T(x_2), \delta)$. So in general we define x_n by observing that $T(\overline{B}_H(0,1)) \setminus \cup_{i=1}^{n-1} U_i \neq \emptyset$ so there exits a $x_n \in \overline{B}_H(0,1)$ such that $T(x_n) \in T(\overline{B}_H(0,1)) \setminus \cup_{i=1}^{n-1} U_i$ and we define $U_n = B_Y(T(x_n), \delta)$. This defines a sequence in $\overline{B}_H(0,1)$ where it holds that $(x_n)_{n\in\mathbb{N}} \subset \overline{B}_H(0,1)$ and that $T(x_n) \in U_m$ if and only if $n = m$ hence we have that $\left\|T(x_n) - T_(x_m)\right\| \geq \delta$ for all $n \neq m$.

For a given sequence $(x_n)_{n\in\mathbb{N}} \subset \overline{B}_h(0,1)$ that lies in the closed unit ball we have that we can define the sequence in $f_n(y) = \langle y, x_n \rangle$ for all $y \in H$ and it follows form Riesz Representation Theorem that $(f_n)_{n\in\mathbb{N}} \subset H^*$ and we also have since $\|f_n\| = \|x_n\| \leq 1$ it lies in the closed unit ball of H^*. Since H is a separable Hilber space we know from **Theorem 5.13** that $\overline{B}_{H^*}(0,1)$ is metrizable with respect to the weak* topology and from Alaoglu's Theorem (**Theorem 6.1**) that the closed unit ball in H^* is compact with respect to the weak* topology. This means that $\overline{B}_{H^*}(0,1)$ is sequential compact with respect to τ_w^* and the sequence $(f_n)_{n\in\mathbb{N}} \subset \overline{B}_{H^*}(0,1)$ contains a subsequence $(f_{n_k})_{n\in\mathbb{N}}$ that converges to a element, $f_0 \in \overline{B}_{H^*}(0,1)$ in the weak* topology. Let $x_o \in H$ be the element such that $f_0(y) = \langle y, x_0 \rangle$. It now follows from **Homework week 4, Problem 2(c)** that since $f_{n_k} \to f_0$ in the weak* topology we have that $f_{n_k}(y) \to f_0(y)$ for all $y \in H$, and . This is equivalent to $\langle y, x_{n_k} \rangle \to \langle y, x_0 \rangle$ which again is equivalent to convergence of the conjugated so $\langle x_{n_k}, y \rangle \to \langle x_0, y \rangle$. This in turn shows that per **Homework week 4, Problem 2(a)** we have x_{n_k} converges weakly to x_0.

Hence take a given $T \in \mathcal{L}(H, Y)$ where it holds that $\|T(x_n) - T(x)\| \to 0$ for $n \to \infty$ for all weakly convergent sequences x_n. Now assume that $T \notin \mathcal{K}(H, Y)$ and look at the sequence
$\|Tx_n - Tx_m\| \geq \delta$
defined so that $(x_n)_{n\in\mathbb{N}} \subset \overline{B}_H(0,1)$ and that $\|x_n - x_m\| \geq \delta$. Then we know that there exits a subsequence such that $x_{n_k} \to x$ weakly and it still holds that $\left\|T(x_{n_k}) - T(x_{n_p})\right\| \geq \delta$ for all $k \neq p$. But then the subsequence is not Cauchy and hence not convergent to any point in Y, especially $T(x)$. This is in contradiction with the assumptions about T and hence it has to be compact.

Part (d)

For a given $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{B}))$ and a given sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x weakly, it follows from **Problem 2(a)** that $T(x_n)$ converges weakly to $T(x)$ and then we know from **Remark 5.3** that $T(x_n)$ converges in norm to $T(x)$, since it is in $\ell_1(\mathbb{N})$. Knowing that $\ell_2(\mathbb{N})$ is a separable infinite dimensional Hilbert space we can now invoke the result of **Problem 2(c)** to say that $T \in \mathcal{K}(X, Y)$.

Part (e)

For a given $T \in \mathcal{K}(X, Y)$ assume that T is surjective. It now follows from the Open mapping theorem (**Theorem 3.15** that T is open. We therefore know that there exits a $r > 0$ such that $B_Y(0, r) \subset T(B_X(0,1))$. Since T is a compact operator it follows from **Definition 8.1** that $\overline{T(B_X(0,1))}$ is compact in Y and using **Folland, Proposition 4.22** we know that then $\overline{B_Y(0,r)}$ also has to be compact in Y. But this is a contradiction since we in **Mandatory Assignment 1, problem 3(e)** proved that the closed unit ball is non-compact in an infinite dimensional normed space. We now have a contradiction since we for a normed space have that

$\overline{B_Y(0,r)}$ is compact if and only if $\overline{B}_Y(0,1)$ is compact. Why is this true? ✓

Part (f)

We have for all $f, g \in L_2([0,1], m)$ that

$$\langle M(f), g\rangle = \int_{[0,1]} tf(t)\overline{g(t)}dm(t) = \int_{[0,1]} tf(t)\overline{g(t)}dm(t) = \int_{[0,1]} f(t)\overline{tg(t)}dm(t) = \langle f, M(g)\rangle$$

using that $t \in [0,1]$ so $\bar{t} = t$, and therefore M is a self-adjoint operator. We also know from **Homework week 6, problem 3(a)** that M has no eigenvalues. Hence if M was compact we could use The Spectral Theorem (**Theorem 10.1**), since $L_2([0,1], m)$ is a separable per **Homework week 4, problem 4(a)**, and conclude that $L_2([0,1], m)$ has an ONB consiting of eigenvectors. But this is a contradiction since M has no eigenvalues and therefore also has no eigenvectors. ✓

Problem 3

Part (a)

We have that $([0,1], \mathbb{B}([0,1]), m)$ is a finite, and then also σ-finite measure space with Hilbert space $H = L_2([0,1], m)$. For the product measure space $([0,1]\times[0,1], \mathbb{B}([0,1])\otimes\mathbb{B}([0,1]), m\otimes m)$ we have that $K \in L_2([0,1]\times[0,1], m\otimes m)$ since using Tonelli's theorem (**Folland, Theorem 2.37**) we get that

$$\|K\|^2 = \int_{[0,1]\times[0,1]} |K(s,t)|^2 dm\otimes m(s,t) = \int_{[0,1]}\int_{[0,1]} |K(s,t)|^2 dm(s)dm(t) \leq 1$$

Where we have used that $|K(s,t)| \leq 1$ for all $s,t \in [0,1]$. It now follows that if we define $T \in \mathcal{L}(H,H)$ by

$$(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t) \quad s \in [0,1],\ f \in H$$

it follows the construction of ($\square$) in Lecture 9. It now follows form **Proposition 9.12** that T is Hilbert-Schmidt and hence per **Proposition 9.11** also compact. ✓

Part (b)

We first note that $K(s,t) = K(t,s)$, which trivially holds if $s = t$. Otherwise we can , without loss of generality, assume that $s > t$ and then $K(s,t) = (1-s)t = K(t,s)$. It now follows that for all $f, g \in H$ we have that

$$\begin{aligned}\langle T(f), g\rangle &= \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)dm(t)\overline{g(s)}dm(s)\\ &\overset{(*)}{=} \int_{[0,1]} f(t)\int_{[0,1]} K(s,t)\overline{g(s)}dm(s)dm(t)\\ &\overset{(**)}{=} \int_{[0,1]} f(t)\overline{\int_{[0,1]} K(t,s)g(s)dm(s)}dm(t)\\ &= \int_{0,1]} f(t)\overline{Tg(t)}dm(t) = \langle f, T(g)\rangle\end{aligned}$$

(triangle for integrals

why? +(Cauchy-Schwartz twice)?)

Where we at $(*)$ used that $\int_{[0,1]}\left|\int_{[0,1]} K(s,t)f(t)dm(t)g(s)\right| dm(s) \leq \|K\|_2 \|f\|_2 \|g\|_2 < \infty$ so we can use Fubini's theorem (**Folland, Theorem 2.37**). At $(**)$ we used that $K(s,t) = K(t,s)$ and that it is real valued so $\overline{K(s,t)} = K(s,t)$. It thereby follows that $T = T^*$ so the operator is self-adjoint. write out

Part (c)

We have that for all $s \in [0,1]$, for all $f \in H$ that

$$\begin{aligned} Tf(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) = \int_{[0,s]} K(s,t)f(t)dm(t) +_{(s,1]} K(s,t)f(t)dm(t) \\ &= \int_{[0,s]} (1-s)tf(t)dm(t) + (1-s)\int_{(s,1]} s(1-t)f(t)dm(t) \\ &= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t) \end{aligned}$$

Where we used that $[0,1]$ is the disjoint union of $[0,s]$ and $(s,1]$ and that we know the value of $K(s,t)$ in the two intervals since either $t \leq s$ og $s < t$. Lastly we used linearity of integrals and that the set $\{s\}$ can be added to the right integrals since it as a lebesgue null-set. It is therefore clear that

$$\begin{aligned} Tf(0) &= (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]} tf(t)dm(t) = 1\int_{\{0\}} tf(t)dm(t) = 0 \\ Tf(1) &= (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]} tf(t)dm(t) = 1\int_{\{1\}} tf(t)dm(t) = 0 \end{aligned}$$

For the two integrals we have, using from **Homework Week 2, Problem 2(b)** $L_2([0,1],m) \subset L_1([0,1],m)$ so $\int_{[0,1]} |f(t)|dm(t) \leq \|f\|_2$ and we get that for all $h > 0$

The implication is the other way.

$\frac{1}{2}$ ← Cauchy-Schwartz ?

$$\left|\int_{[0,s+h]} tf(t)dm(t) - \int_{[0,s]} tf(t)dm(t)\right| = \left|\int_{(s,s+h]} tf(t)dm(t)\right| \leq \left(\int_{(s,s+h]} dm(t)\right)^{\frac{1}{2}} \|f\|_2 = h^{\frac{1}{2}}\|f\|_2$$

$$\left|\int_{[0,s+h]} (1-t)f(t)dm(t) - \int_{[0,s]} (1-t)f(t)dm(t)\right| = \left|\int_{(s,s+h]} (1-t)f(t)dm(t)\right| \leq h^{\frac{1}{2}}\|f\|_2$$

Where we used that $t, (1-t) \in [0,1]$. Therefore since the difference goes til zero as $h \to 0$ we have that the two integrals are continuous. Since $s \mapsto s$ and $s \mapsto 1-s$ also are continuous it now follows that $Tf(s)$ is build out of continuous functions, so it is continuous.

(✓)

Problem 4

part (a)

We have from **Homework week 7, Problem 1** that $g(x) = e^{-x^2}$ belongs to $\mathscr{S}(\mathbb{R})$. Using **Homework week 7, problem 1(d)** we get that $S_{\sqrt{2}}(f)(x) = e^{-\left(\frac{x}{\sqrt{2}}\right)^2} = e^{-\frac{x^2}{2}} = g_0(x)$ also belongs to $\mathscr{S}(\mathbb{R})$.For $k \in \{1,2,3\}$ we have that $g_k(x) = x^k g_0(x)$ and it now follows from **Homework week 7, Problem 1(a)** that $g_k \in \mathscr{S}(\mathbb{R})$.

We have from **Proposition 11.4** that for $g_0(x) = e^{-\frac{x^2}{2}}$ it holds that $\mathcal{F}(g_0)(\xi) = \hat{g}_0(\xi) = g_o(\xi) = e^{-\frac{\xi^2}{2}}$. For $k \in \{1,2,3\}$ we have that since $g_0(x), g_k(x) \in \mathscr{S}(\mathbb{N}) \subset L_1(\mathbb{N})$ we can use **Proposition 11.13(d)** to say that

$$\mathcal{F}(g_k)(\xi) = \widehat{g_k}(\xi) = \left(x^k g_0\right)^\wedge(\xi) = i^k\left(\partial^k \widehat{g_0}\right)(\xi) = i^k g_0^{(k)}(\xi)$$

Since $g_0'(x) = -xe^{\frac{x^2}{2}}$, $g_0''(x) = -e^{-\frac{x^2}{2}} + x^2e^{-\frac{x^2}{2}}$ and $g_0^{(3)}(x) = 3xe^{-\frac{x^2}{2}} - x^3e^{-\frac{x^2}{2}}$ we get that

$$\begin{aligned}
\mathcal{F}(g_0)(\xi) &= \widehat{g_0}(\xi) = e^{-\frac{\xi^2}{2}} = g_0(\xi) \\
\mathcal{F}(g_1)(\xi) &= \widehat{g_1}(\xi) = i\left(-\xi e^{-\frac{\xi^2}{2}}\right) = -ig_1(\xi) \\
\mathcal{F}(g_2)(\xi) &= \widehat{g_2}(\xi) = i^2\left(-e^{-\frac{\xi^2}{2}} + \xi^2 e^{-\frac{\xi^2}{2}}\right) = g_0(\xi) - g_2(\xi) \\
\mathcal{F}(g_3)(\xi) &= \widehat{g_3}(\xi) = i^3\left(3\xi e^{-\frac{\xi^2}{2}} - \xi^3 e^{-\frac{\xi^2}{2}}\right) = -3ig_1(\xi) + ig_3(\xi)
\end{aligned}$$

part b

Let $h_0(x) = g_0(x)$, $h_1(x) = 3g_1(x) - 2g_3(x)$ $h_2(x) = g_0(x) - 2g_2(x)$ and $h_3(x) = g_1(x)$, where we know that if $f_1, f_2 \in \mathscr{S}(\mathbb{R})$ then $\alpha f_1 + \beta f_2 \in \mathscr{S}(\mathbb{R})$ so $h_1, h_2, h_3 \in \mathscr{S}(\mathbb{R})$. We then get by linearity of the Fourier transformation that

$$\begin{aligned}
\widehat{h_0}(\xi) &= \widehat{g_0}(\xi) = g_0(\xi) = h_0(\xi) \\
\widehat{h_1}(\xi) &= 3\widehat{g_1}(\xi) - 2\widehat{g_3}(\xi) = 3\left(-ig_1(\xi)\right) - 2\left(-3ig_1(\xi) + ig_3(\xi)\right) = 3ig_1(\xi) - 2ig_3(\xi) = i(h_1(\xi)) \\
\widehat{h_2}(\xi) &= \widehat{g_0}(\xi) - 2\widehat{g_2}(\xi) = g_0(\xi) - 2\left(g_o(\xi) - g_2(\xi)\right) = -\left(g_0(\xi) - g_2(\xi)\right) = -h_2(\xi) \\
\widehat{h_3}(\xi) &= \widehat{g_1}(\xi) = -ig_1(\xi) = -ih_3(\xi)
\end{aligned}$$

Part (c)

For a given $f \in \mathscr{S}(\mathbb{R})$ we note that from **Homework week 7, Problem 1(d)** we proved that $S_a(f) \in \mathscr{S}(\mathbb{R})$, for all $a \in \mathbb{R} \setminus \{0\}$. It now follows that for all $f \in \mathscr{S}(\mathbb{R})$ we have that

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dm(x) \stackrel{*}{=} \int_{\mathbb{R}} f(-y)e^{-i(-y)\xi}dm(y) = \int_{\mathbb{R}} \left(S_{-1}f\right)(y)e^{iy\xi}dm(y) = \mathcal{F}^*\left(S_{-1}f\right)(\xi)$$

Where we at $*$ used change of variable $x = -y$ and that the Lebesgue measure is invariant under reflections (**Schilling, Theorem 4.7**). As $S_{-1}f \in \mathscr{S}(\mathbb{R})$ it now follows from Fourier inversion theorem (**Theorem 12.11**) that

$$\mathcal{F}^2(f)(\xi) = \mathcal{F}\left(\mathcal{F}(f)\right)(\xi) = \mathcal{F}\left(\mathcal{F}^*\left(S_{-1}f\right)\right)(\xi) = \left(S_{-1}f\right)(\xi)$$

We therefore get that

$$\mathcal{F}^4(f)(\xi) = \mathcal{F}^2\left(\mathcal{F}^2(f)\right)(\xi) = \mathcal{F}^2\left(S_{-1}f\right)(\xi) = \left(S_{-1}S_{-1}f\right)(\xi) = f(\xi)$$

Part (d)

$\neq 0$

For $f \in \mathscr{S}(\mathbb{R})$ where $\mathcal{F} = \lambda f$ for $\lambda \in \mathbb{C}$ we have, using linearity of Fourier transform, that $\mathcal{F}^4(f) = \lambda^4 f$. From **Problem 4(c)** we also have that $\mathcal{F}(f) = f$, hence we get that $\lambda^4 = 1$ which happens if and only if $\lambda \in \{1, i, -1, -i\}$. We can therefore conclude that any eigenvalue for $\mathcal{F}$ has to be in the set $\{1, i, -1, -i\}$. We have in **Problem 4(b)** found four functions in $\mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$ for $k \in \{0, 1, 2, 3\}$ and we can therefore conclude that the eigenvalues for $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.

Problem 5

For a given $x_0 \in [0,1]$ and for a given continuous function $f : [0,1] \to [0,1]$ with compact support where $f(x_0) > 0$ it follows from continuity of f that there for $\epsilon = \frac{f(x_0)}{2} > 0$ exits a $\delta > 0$ such that $f(x) > \frac{f(x_0)}{x}$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Since $(x_n)_{n\in\mathbb{N}}$ is dense in $[0,1]$ we know that there for the non-empty open set $(x_0 - \delta, x_0 + \delta)$ exits a $n_0 \in \mathbb{N}$ such that $x_{n_0} \in (x_0 - \delta, x_0 + \delta)$ and therefore $f(x_{n_0}) > \frac{f(x_0)}{2} > 0$. We therefore have that

$$\int_{[0,1]} f d\mu \geq \int_{(x_0-\delta,x_0+\delta)} f d\mu \geq \sum_{n=1}^{\infty} 2^{-n} f(x_n)\delta_{x_n}(x_0 - \delta, x_0 + \delta) \geq 2^{-n_0} f(x_{n_0}) > 0$$

Be carefull here.

It now follows from **Homework week 8, Problem 3(b)** that $x_0 \in \text{supp}(\mu)$. This arguments holds for all $x_0 \in [0,1]$ and hence $\text{supp}(\mu) = [0,1]$ ✓