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RANK ONE PERTURBATIONS OF NOT SEMIBOUNDED OPERATORS

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Rank one perturbations of selfadjoint operators which are not necessarily semibounded are studied in the present paper. It is proven that such perturbations are uniquely defined, if they are bounded in the sense of forms. We also show that form unbounded rank one perturbations can be uniquely defined if the original operator and the perturbation are homogeneous with respect to a certain one parameter semigroup. The perturbed operator is defined using the extension theory for symmetric operators. The resolvent of the perturbed operator is calculated using Krein's formula. It is proven that every rank one perturbation can be approximated in the operator norm. We prove that some form unbounded perturbations can be approximated in the strong resolvent sense without renormalization of the coupling constant only if the original operator is not semibounded. The present approach is applied to study first derivative and Dirac operators with point interaction, in one dimension.

1 INTRODUCTION.

In the present paper we study rank one perturbations of selfadjoint operators which are not semibounded. Let A be a selfadjoint operator acting in the Hilbert space $\mathcal{H}$. Let φ be a linear bounded functional on the domain of the operator. A rank one perturbation of the operator A is the operator formally given by

$$A_\alpha = A + \alpha(\varphi, *)\varphi, \tag{1}$$

where α is a real parameter and $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{H}$. If the operator A is semibounded and φ is an element from the domain of the quadratic form associated with the resolvent of the operator, then the operator A_α can be defined uniquely even for infinite values of the parameter α [18,8]. If φ does not belong to the domain of the quadratic form associated with the resolvent of the operator A, then the operator A_α does not posses a unique definition and the family of selfadjoint operators corresponding to the formal expression (1) is defined by an extra real parameter c [3]. If this parameter can be fixed using some additional information, then the selfadjoint operator corresponding to the formal expression (1) is uniquely defined.

The theory of rank one perturbations can be developed even in the case where the perturbation is not form bounded. An approach based on the analytical properties of the Q-function associated with a symmetric operator and its selfadjoint extensions has been developed recently by S. Hassi and H. de Snoo [10]. This allowed them to study rank one perturbations of selfadjoint operators which are not necessarily semibounded. Their approach is based on the fact that the original operator A and any rank one perturbation of it are two different selfadjoint extensions of a certain symmetric operator with deficiency indices $(1,1)$. The relations between two selfadjoint operators whose resolvents differ by a rank one operator have also been studied in [10] and [11]. However the answer to the question about which selfadjoint operator corresponds to the given formal expression (1) does not follow directly from the approach developed in [10] and [11]. This question for generalized rank one perturbations has been discussed in [16].

It will be shown in the present paper that the theory of rank one perturbations of nonsemibounded operators can be developed in a way similar to the form bounded case. We introduce two scales of Hilbert spaces: one associated with the operator $|A|$ and the second associated with the operator A and the vector φ. Two important cases should be treated separately. If φ is an element from the domain of the quadratic form associated with the resolvent of A then the selfadjoint operator corresponding to the formal expression (1) is defined uniquely (in analogy with H_{-1} perturbations of semibounded operators). If φ does not belong to this domain then there exists a one parameter family of selfadjoint operators corresponding to the formal expression (1) (in analogy with H_{-2} perturbations of semibounded operators). These perturbations will be named *form unbounded*. We show that if the original operator A and the vector φ are homogeneous with respect to a certain unitary group then the selfadjoint operator corresponding to the formal expression (1) can be defined uniquely even in the case of form unbounded perturbations. Approximations of rank one perturbations in the strong resolvent sense and in the sense of linear operators in Banach spaces are discussed. It is shown that every rank one perturbation of a selfadjoint operator can be approximated by operators with bounded perturbations in the operator norm. It is also proven that form unbounded perturbations can be approximated in the strong resolvent sense if certain additional conditions are satisfied. This shows where the main difference between the approximations of semibounded and not semibounded operators lies.

In the last section the methods we developed are applied to study singular perturbations of the first derivative operator in dimension one and of the one dimensional Dirac operator. It is shown that using the scaling properties of the Dirac operator even unbounded perturbations can be defined uniquely.

2 PRELIMINARIES.

2.1 Rank one perturbations and extension theory.

In this section we are going to study rank one perturbations of a selfadjoint operator A acting in the separable Hilbert space $\mathcal{H}$. A is allowed to be a not semibounded operator. Rank one perturbations of such an operator are formally defined by the standard formula

$$A_\alpha = A + \alpha(\varphi, \cdot)\varphi. \tag{2}$$

Our goal is to determine a selfadjoint operator corresponding to this formal expression. If

φ is an element from the Hilbert space then the operator A_α is selfadjoint on the domain $D(A)$ of the selfadjoint operator A and its action is determined by the formula (2). If $\varphi \notin \mathcal{H}$ then the operator A_α can be defined using the quadratic form formalism. This approach can be developed only if φ is a certain bounded linear functional on the domain of the operator A. The quadratic form approach can be used for semibounded operators due to the one to one correspondence between the semibounded selfadjoint operators and semibounded closed quadratic forms. If the operator A is not semibounded then the extension theory for symmetric operators can be used to define the selfadjoint operator corresponding to the formal rank one perturbation. If φ is a bounded linear functional on the domain of the operator A then the selfadjoint operator corresponding to the formal expression (2) coincides with one of the selfadjoint extensions of the symmetric operator $A^0 = A|_{D_\varphi}$, where $D_\varphi = \{\psi \in D(A) | (\varphi, \psi) = 0\}$. The condition $(\varphi, \psi) = 0$ is well defined for $\psi \in D(A)$ since φ is a linear bounded functional on the domain of the operator A.

LEMMA 2.1 *Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$ with the domain $D(A)$ equipped with the norm equal to the graph norm of the operator and let φ be a bounded linear functional on $D(A)$ i.e. $\varphi \in (D(A))^*$ which can not be extended as a bounded linear functional to $\mathcal{H}$. Then the restriction A^0 of the operator A to the domain of functions $D(A^0) = D_\varphi = \{\psi \in D(A) : (\varphi, \psi) = 0\}$ is a densely defined symmetric operator in $\mathcal{H}$ with deficiency indices $(1, 1)$.*

PROOF. We prove first that the restricted operator is densely defined. The operator A is densely defined and thus for every $f \in \mathcal{H}$ there exists a sequence $f_n \in D(A)$ converging to f in the Hilbert space norm:

$$\lim_{n\to\infty} \| f - f_n \|_{\mathcal{H}} = 0.$$

The functional φ is not a bounded functional on the Hilbert space. It follows that there exists a sequence $\psi_n \in D(A)$ with unit norm $\| \psi_n \|_{\mathcal{H}} = 1$ such that the corresponding sequence (φ, ψ_n) diverges to infinity. This sequence can be chosen in such a way that $\lim_{n\to\infty} \frac{(\varphi, f_n)}{(\varphi, \psi_n)} = 0$. Then the sequence $f_n - \frac{(\varphi, f_n)}{(\varphi, \psi_n)}\psi_n$ belongs to the domain of the restricted operator

$$(\varphi, f_n - \frac{(\varphi, f_n)}{(\varphi, \psi_n)}\psi_n) = 0$$

and converges to the element f in the Hilbert space norm

$$\| f_n - \frac{(\varphi, f_n)}{(\varphi, \psi_n)}\psi_n - f \|_{\mathcal{H}} \leq \| f_n - f \|_{\mathcal{H}} + \left| \frac{(\varphi, f_n)}{(\varphi, \psi_n)} \right| \to_{n\to\infty} 0.$$

Thus the operator A^0 is densely defined.

The deficiency elements of the operator A_0 for every λ with $\Im\lambda \neq 0$ ($\Im$ denoting the imaginary part) are equal to $g_\lambda = (A - \lambda)^{-1}\varphi$. The latter equality has to be understood in the generalized sense i.e. g_λ is the bounded linear functional which acts on every $\psi \in \mathcal{H}$ according to the following formula

$$(g_\lambda, \psi) = ((A - \lambda)^{-1}\varphi, \psi) = (\varphi, (A - \bar{\lambda})^{-1}\psi) \leq$$

$$\leq \| \varphi \|_{D(A)^*} \left(\| \frac{A}{A-\bar{\lambda}} \psi \|_{\mathcal{H}} + \| \frac{1}{A-\bar{\lambda}} \psi \|_{\mathcal{H}} \right) \leq \| \varphi \|_{D(A)^*} C_\lambda (1 + |\Im \lambda|^{-1}) \| \psi \|_{\mathcal{H}},$$

where C_λ is a certain constant and $^-$ denotes complex conjugation. Let $\psi \in D_\varphi$ then the following equalities hold

$$(\psi, A^* g_\lambda) = (A\psi, g_\lambda) = (\frac{A}{A-\bar{\lambda}} \psi, \varphi) =$$

$$= (\frac{\bar{\lambda}}{A-\bar{\lambda}} \psi, \varphi) = \lambda (\psi, g_\lambda).$$

It follows that g_λ is a deficiency element corresponding to the restricted operator and the complex number λ. The deficiency element is unique. Hence the lemma is proven. □

We are going to normalize the vector φ in such a way that $\| \frac{1}{A-i} \varphi \|_{\mathcal{H}} = 1$.

The domain of the adjoint operator is equal to the following direct linear sum $D(A^{0*}) = D_\varphi \dot{+} \mathcal{L}\{\frac{1}{A-i}\varphi, \frac{1}{A+i}\varphi\}$. Every element ψ from the domain of the adjoint operator possesses the following representation

$$\psi = \hat{\psi} + a_+(\psi) \frac{1}{A-i} \varphi + a_-(\psi) \frac{1}{A+i} \varphi, \tag{3}$$

where $\hat{\psi} \in D_\varphi, a_\pm(\psi) \in \mathbf{C}$. The adjoint operator A^{0*} acts as follows on every $\psi \in D(A^{0*})$

$$A^{0*} (\hat{\psi} + a_+(\psi) \frac{1}{A-i} \varphi + a_-(\psi) \frac{1}{A+i} \varphi) =$$

$$= A\hat{\psi} + a_+(\psi) \frac{i}{A-i} \varphi + a_-(\psi) \frac{-i}{A+i} \varphi.$$

Let $\psi, \eta \in D(A^{0*})$ then the boundary form of the adjoint operator is equal to

$$(A^{0*}\psi, \eta) - (\psi, A^{0*}\eta) =$$

$$= 2i (\frac{1}{A-i} \varphi, \frac{1}{A-i} \varphi) \left(\overline{a_-(\psi)} a_-(\eta) - \overline{a_+(\psi)} a_+(\eta) \right) =$$

$$= 2i \left(\overline{a_-(\psi)} a_-(\eta) - \overline{a_+(\psi)} a_+(\eta) \right).$$

All selfadjoint extensions of the operator A^0 can be parametrized by one unimodular parameter $v \in \mathbf{C}$; $|v| = 1$. Every selfadjoint extension A_v coincides with the restriction of the operator A^{0*} to the domain $D_v = \{\psi \in D(A^{0*}) : a_-(\psi) = v a_+(\psi)\}$. The extension corresponding to $v = -1$ coincides with the original operator $A_0 = A$.

2.2 Scales of Hilbert spaces.

We are going to consider two different scales of Hilbert spaces. The first set of spaces

$$\mathcal{H}_2(|A|) \subset \mathcal{H}_1(|A|) \subset \mathcal{H} \subset \mathcal{H}_{-1}(|A|) \subset \mathcal{H}_{-2}(|A|)$$

is the standard scale of Hilbert spaces associated with the nonnegative operator $|A|$. The second scale of Hilbert spaces

$$\mathcal{H}_2(|A|) = D(A) \subset B_\varphi(A) \subset \mathcal{H} \subset B_\varphi(A)^* \subset D(A)^* = \mathcal{H}_{-2}(|A|) \tag{4}$$

is associated with the operator A and the functional φ. $B_\varphi(A)$ denotes here the domain of the adjoint operator A^{0*}. We are going to study the second scale of Hilbert spaces in more detail without making the requirement that the operator A be semibounded. Let us consider the domain $D(A) = \mathcal{H}_2(|A|)$ of the operator A. It is a Hilbert space with the norm equal to the graph norm of the operator

$$\parallel \psi \parallel_{D(A)} = \parallel A\psi \parallel_{\mathcal{H}} + \parallel \psi \parallel_{\mathcal{H}} = \parallel (A-i)\psi \parallel_{\mathcal{H}} .$$

Every element $\psi \in B_\varphi(A) = D(A^{0*})$ possesses the following unique representation

$$\psi = \tilde{\psi} + \frac{b(\psi)}{2}\left(\frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi\right), \tag{5}$$

where $\tilde{\psi} \in D(A)$. This representation is related to the representation (3) via the formulas

$$\tilde{\psi} = \hat{\psi} + (a_+(\psi) - a_-(\psi))\frac{i}{A^2+1}\varphi;$$

$$b(\psi) = a_+(\psi) + a_-(\psi).$$

The norm in the space $B_\varphi(A)$ can be defined by the following formula

$$\parallel \psi \parallel_{B_\varphi(A)} = \parallel \tilde{\psi} \parallel_{D(A)} + |b(\psi)|.$$

If $\psi \in D(A)$, then the following equality holds

$$\parallel \psi \parallel_{B_\varphi(A)} = \parallel \psi \parallel_{D(A)} .$$

The space $B_\varphi(A)$ is a one dimensional extension of the space $D(A) = \mathcal{H}_2(|A|)$ in the sense that $B_\varphi(A)$ is isomorphic with $D(A)\dot{+}\mathbf{C}$. The natural embedding ρ is defined using the formula (5):

$$\rho : D(A)\dot{+}\mathbf{C} \to B_\varphi(A)$$

$$\psi = (\tilde{\psi}, b(\psi)) \mapsto \tilde{\psi} + \frac{b(\psi)}{2}\left(\frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi\right).$$

The map ρ is invertible and preserves the norm. The spaces $B_\varphi(A)$ and $D(A)\dot{+}\mathbf{C}$ will henceforth be identified. Let $\psi \in B_\varphi(A)$ then the following inequality holds

$$\parallel \psi \parallel_{\mathcal{H}} \leq \parallel \psi \parallel_{B_\varphi(A)} .$$

It follows that $B_\varphi(A)$ is a subspace of $\mathcal{H}$. Consider now the spaces of bounded linear functionals on $D(A)$ resp. $B_\varphi(A)$ denoted correspondingly by $D(A)^* = \mathcal{H}_{-2}(|A|)$ and $B_\varphi(A)^*$. The norms in these spaces are defined as usual in the theory of dual spaces. The inclusion (4) is now obvious.

Consider the resolvent $\frac{1}{A-i}$ of the operator A acting in the generalized sense. Let $\varphi \in D(A)^*$ then $\frac{1}{A-i}\varphi$ is the linear functional which acts on every $\psi \in \mathcal{H}$ in accordance to the following formula

$$\left(\frac{1}{A-i}\varphi, \psi\right) = \left(\varphi, \frac{1}{A+i}\psi\right) \leq \parallel \varphi \parallel_{D(A)^*} \parallel \psi \parallel_{\mathcal{H}} .$$

It follows that $\frac{1}{A-i}\varphi$ is a bounded linear functional on $\mathcal{H}$ and thus it can be identified with an element from the Hilbert space $\mathcal{H}$. The norm in the space $D(A)^*$ is equal to

$$\| \psi \|_{D(A)^*} = \| \frac{1}{A-i}\psi \|_{\mathcal{H}} .$$

The space $B_\varphi(A)^*$ is equal to the sum of two spaces $B_\varphi(A)^* = D(A)^* \times \mathbf{C}$. Every element $\varphi \in B_\varphi(A)^*$ is equal to the sum of two functionals $\varphi = \tilde{\varphi} + c(\varphi)$, where $\tilde{\varphi} \in D(A)^*, c(\varphi) \in \mathbf{C}$. Let $\psi = (\tilde{\psi}, b(\psi)) \in B_\varphi(A)$ then

$$(\varphi, \psi) = (\tilde{\varphi}, \tilde{\psi}) + \bar{c}(\varphi)b(\psi).$$

The norm in this space is equal to

$$\| \varphi \|_{B_\varphi(A)^*} = \max_{0 \leq \chi \leq 1} \left((1-\chi) \| \tilde{\varphi} \|_{D(A)^*} + \chi |c(\varphi)| \right).$$

REMARK. *The second scale of Hilbert spaces is constructed using the functional φ. This determines the main difference between the two scales of Hilbert spaces.*

The quadratic form corresponding to the selfadjoint operator A is not positive definite. We are going to use the quadratic form of the positive operator $|A|$ to define form bounded and form unbounded perturbations of not semibounded operators. A vector φ defines a form bounded rank one perturbation of the operator A if it belongs to the Hilbert space $\mathcal{H}_{-1}(|A|)$. Vectors φ in $\mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(|A|)$ define form unbounded perturbations.

3 FORM BOUNDED PERTURBATIONS.

We study here form bounded perturbations of selfadjoint operators defined by vectors $\varphi \notin \mathcal{H}$.

THEOREM 3.1 *Let $\varphi \in \mathcal{H}_{-1}(|A|) \setminus \mathcal{H}$ then the domain of the selfadjoint operator $A_\alpha = A + \alpha(\varphi, \cdot)\varphi$ coincides with the following set*

$$D(A_\alpha) = \{\psi \in B_\varphi(A) | a_+(\psi) = -\frac{1 + \alpha(\varphi, \frac{1}{A+i}\varphi)}{1 + \alpha(\varphi, \frac{1}{A-i}\varphi)} a_-(\psi)\}. \tag{6}$$

A_α is a selfadjoint extension of A^0. For $\alpha = 0$ we have $A_0 = A$.

PROOF. The operator A_α is defined as a linear operator acting in the Hilbert spaces $B_\varphi(A) = D(A^{0*}) \to D(A)^*$. The linear operator A is defined in the generalized sense

$$(A\psi, \phi) = (\psi, A\phi)$$

for any $\psi \in \mathcal{H}, \phi \in D(A)$. The quadratic form $Q[\varphi, \varphi] = (\varphi, (\frac{1}{A-i} + \frac{1}{A+i})\varphi)$ is finite

$$|Q[\varphi, \varphi]| = \left| (\varphi, \frac{2A}{A^2+1}\varphi) \right| \leq 4 \| \frac{1}{(|A|+1)^{1/2}}\varphi \|^2 .$$

Let $\psi \in B_\varphi(A)$ then the representation (3) is valid and the linear operator is acting as follows

$$A_\alpha \psi = (A + \alpha(\varphi, \cdot)\varphi) \left(\hat{\psi} + a_+(\psi)\frac{1}{A-i}\varphi + a_-(\psi)\frac{1}{A+i}\varphi \right) =$$

$$= A\hat{\psi} + \alpha(\varphi,\hat{\psi})\varphi + a_+(\psi)A\frac{1}{A-i}\varphi +$$

$$+\alpha a_+(\psi)(\varphi,\frac{1}{A-i}\varphi)\varphi + a_-(\psi)A\frac{1}{A+i}\varphi + \alpha a_-(\psi)(\varphi,\frac{1}{A+i}\varphi)\varphi =$$

$$= A\hat{\psi} + a_+(\psi)\frac{i}{A-i}\varphi + a_-(\psi)\frac{-i}{A+i}\varphi +$$

$$+\left\{a_+(\psi)\left(1+\alpha(\varphi,\frac{1}{A-i}\varphi)\right) + a_-(\psi)\left(1+\alpha(\varphi,\frac{1}{A+i}\varphi)\right)\right\}\varphi. \tag{7}$$

Here we used the fact that $(\varphi,\hat{\psi}) = 0$. The domain of the selfadjoint operator A_α coincides with the following set $\{\psi \in D(A^{0*})|A_\alpha\psi \in \mathcal{H}\}$. The element $A_\alpha\psi$ belongs to the Hilbert space if and only if the following equation is satisfied

$$a_+(\psi) = -\frac{1+\alpha(\varphi,\frac{1}{A+i}\varphi)}{1+\alpha(\varphi,\frac{1}{A-i}\varphi)}a_-(\psi). \tag{8}$$

The parameter

$$v = -\frac{1+\alpha(\varphi,\frac{1}{A+i}\varphi)}{1+\alpha(\varphi,\frac{1}{A-i}\varphi)}$$

has modulus 1 and the adjoint operator A^{0*} restricted to the domain of functions from $B_\varphi(A)$ satisfying the boundary conditions (8) is selfadjoint. The restrictions of the operators A_α and A^{0*} to the described domain are identical since the expression in the figure brackets in the formula (7) vanishes for the elements satisfying the boundary conditions (8).

If $\alpha = 0$ then the parameter v has the value $v = -1$ and the corresponding operator coincides with the original operator A. Thus the theorem is proven. □

Considering different $\alpha \in \mathbf{R}\cup\{\infty\}$ all selfadjoint extensions of the symmetric operator A^0 can be obtained. The formula (8) establishes the one to one correspondence between the parameters α and v.

4 FORM UNBOUNDED PERTURBATIONS.

Consider form unbounded rank one perturbations defined by the same heuristic expression (2). Any selfadjoint operator corresponding to this heuristic expression is an extension of the symmetric operator A^0. The heuristic expression (2) is defined on the domain of the adjoint operator A^{0*} if the bounded functional φ is defined on this domain. The distribution φ is first defined on the domain of the operator A, but the element $\frac{1}{A-i}\varphi$ does not belong to this domain in the case of form unbounded perturbations. The distribution φ has to be extended as a bounded linear functional to the domain $B_\varphi(A) = D(A^{0*})$.

LEMMA 4.1 *Every extension of the functional φ to the domain $D(A^{0*})$ is defined by one parameter $c \in \mathbf{C}$. Let $\psi = \tilde{\psi} + \frac{b(\psi)}{2}\left(\frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi\right) \in B_\varphi(A)$, then the extended functional φ_c acts as follows*

$$(\varphi_c,\psi) = (\varphi,\tilde{\psi}) + \bar{c}b(\psi). \tag{9}$$

This extension defines a real quadratic form $Q[\psi,\psi] = (\psi,\left(\frac{1}{A-i}+\frac{1}{A+i}\right)\psi)$ with domain $D(Q) = \mathcal{H}\dot{+}\mathcal{L}\{\varphi\}$ if and only if the parameter c is real.

PROOF. The linear functional φ_c defined by the formula (9) is bounded and defined on any element ψ from the domain of the adjoint operator. The quadratic form corresponding to this extension is real if the parameter c is real.

Consider now an arbitrary bounded linear extension $\hat{\varphi}$ of the functional φ to the domain of the adjoint operator. Let $\psi = \tilde{\psi} + \frac{b(\psi)}{2}\left(\frac{1}{A-i} + \frac{1}{A+i}\right)\varphi$ be an element from the domain $D(A^{0*})$ of the adjoint operator. The following formula follows from the fact that $\hat{\varphi}$ is a linear functional

$$(\hat{\varphi}, \psi) = (\varphi, \tilde{\psi}) + \frac{b(\psi)}{2}(\hat{\varphi}, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi).$$

Thus every bounded linear extension of the functional φ is defined by the associated parameter

$$\bar{c} = \frac{(\hat{\varphi}, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi)}{2}.$$

Let us consider an arbitrary element $\psi = \tilde{\psi} + q(\psi)\varphi \in D(Q)$, where $\tilde{\psi} \in \mathcal{H}, q(\psi) \in \mathbf{C}$. Then the quadratic form can be calculated as follows

$$Q[\psi, \psi] = (\tilde{\psi} + q(\psi)\hat{\varphi}, \left(\frac{1}{A-i} + \frac{1}{A+i}\right)(\tilde{\psi} + q(\psi)\hat{\varphi})) =$$

$$= (\tilde{\psi}, \left(\frac{1}{A-i} + \frac{1}{A+i}\right)\tilde{\psi}) + (q(\psi)\varphi, \left(\frac{1}{A-i} + \frac{1}{A+i}\right)\tilde{\psi}) +$$

$$+(\tilde{\psi}, \left(\frac{1}{A-i} + \frac{1}{A+i}\right)q(\psi)\varphi) + |q(\psi)|^2(\hat{\varphi}, \left(\frac{1}{A-i} + \frac{1}{A+i}\right)\varphi) =$$

$$= 2\Re\left((\tilde{\psi}, \frac{1}{A-i}\tilde{\psi}) + (q(\psi)\varphi, \left(\frac{1}{A-i} + \frac{1}{A+i}\right)\tilde{\psi})\right) + |q(\psi)|^2 2\bar{c}.$$

The latter formula shows that the quadratic form is real if and only if the parameter c is real. □

The following definition will be used henceforth.

DEFINITION 4.1 *Let $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(|A|)$, then the functional φ_c is the linear bounded extension of the functional φ to the domain $B_\varphi(A)$ defined by the condition*

$$(\varphi_c, \left(\frac{1}{A-i} + \frac{1}{A+i}\right)\varphi) = 2c, \tag{10}$$

where $c \in \mathbf{R}$.

The following theorem describes the domain of the selfadjoint operator corresponding to the heuristic expression (2).

THEOREM 4.1 *Let φ_c be a linear bounded extension of the form unbounded functional φ and let $\| \frac{1}{A-i}\varphi \|_{\mathcal{H}} = 1$. The domain of the selfadjoint operator*

$$A_\alpha = A + \alpha(\varphi_c, \cdot)\varphi$$

coincides with the following set

$$D(A_\alpha) = \{\psi \in D(A^{0*}) : a_-(\psi) = -\frac{1+\alpha(c+i)}{1+\alpha(c-i)}a_+(\psi)\}.$$

A_α is a selfadjoint extension of A^0. For $\alpha = 0$ we have $A_0 = A$.

PROOF. The proof is similar to the one of theorem 3.1. The linear operator A_α acts as follows on the domain $D(A^{0*}) \ni \psi$

$$A_\alpha \psi = (A + \alpha(\varphi_c, \cdot)\varphi)(\hat{\psi} + a_+(\psi)\frac{1}{A-i}\varphi + a_-(\psi)\frac{1}{A+i}\varphi) =$$

$$= A\hat{\psi} + a_+(\psi)\left[\varphi + \frac{i}{A-i}\varphi + \alpha(c + i \parallel \frac{1}{A-i}\varphi \parallel^2)\varphi\right] +$$

$$+a_-(\psi)\left[\varphi + \frac{-i}{A+i}\varphi + \alpha(c - i \parallel \frac{1}{A-i}\varphi \parallel^2)\varphi\right] =$$

$$= A\hat{\psi} + a_+(\psi)\frac{i}{A-i}\varphi + a_-\frac{-i}{A+i}\varphi +$$

$$+ \{a_+(\psi)\,(1 + \alpha(c+i)) + a_-(\psi)\,(1 + \alpha(c-i))\}\,\varphi.$$

Here we used the fact that the element φ has unit norm in the space $D(A)^*$. The range of the linear operator A_α does not belong to the Hilbert space. The domain of the selfadjoint operator A_α is equal to the following set

$$D(A_\alpha) = \{\psi \in B_\varphi(A) : A_\alpha\psi \in \mathcal{H}\}.$$

The element $A_\alpha\psi$ belongs to $\mathcal{H}$ if and only if the following conditions are satisfied

$$a_-(\psi) = -\frac{1 + \alpha(c+i)}{1 + \alpha(c-i)}a_+(\psi).$$

The absolute value of $v = -\frac{1+\alpha(c+i)}{1+\alpha(c-i)}$ is equal to unity. The operator A^{0*} restricted to the domain of functions satisfying the latter condition is selfadjoint and coincides with the operator A_α restricted to the same domain. □

5 FORM UNBOUNDED PERTURBATIONS OF THE HOMOGENEOUS OPERATORS.

This section is devoted to the investigation of the form unbounded rank one perturbations in the case where the original operator and the element φ are homogeneous with respect to a certain group of unitary transformations of the Hilbert space $\mathcal{H}$. The extension of the functional φ can be defined uniquely using the homogeneity properties of the operator and the perturbation.

LEMMA 5.1 *Let the selfadjoint operator A and vector $\varphi \in (D(A))^*$ be homogeneous with respect to a certain unitary (multiplicative) group $G(t)$, i.e. there exist real constants β, γ such that*

$$G(t)A = t^{-\beta}AG(t); \tag{11}$$

$$(G(t)\varphi, \psi) = (\varphi, G(1/t)\psi) = t^\gamma(\varphi, \psi) \tag{12}$$

for every $\psi \in D(A)$. Then φ can be extended as a homogeneous linear bounded functional to the domain $B_\varphi(A)$ if and only if

$$f(t) = i\frac{1-t^\beta}{1-t^{-\beta-2\gamma}}(\varphi, \frac{1}{(A-i)(A-t^\beta i)}\varphi) \tag{13}$$

does not depend on $t \neq 1$.

PROOF. Consider an arbitrary linear bounded extension φ_c of the functional φ which is defined by the parameter

$$2c = (\varphi_c, \left(\frac{1}{A-i} + \frac{1}{A+i}\right)\varphi)$$

(see Lemma 4.1). Suppose that this extension is homogeneous and thus satisfies the equation (12). Then the function $f(t)$ can be calculated according to

$$\begin{aligned} f(t) &= i\frac{1-t^\beta}{1-t^{-\beta-2\gamma}}(\varphi, \frac{1}{(A-i)(A-t^\beta i)}\varphi) = \\ &= \frac{1}{1-t^{-\beta-2\gamma}}(\varphi_c, \left(\frac{1}{A-i} - \frac{1}{A-t^\beta i}\right)\varphi) = \\ &= \frac{1}{1-t^{-\beta-2\gamma}}\left\{(\varphi_c, \frac{1}{A-i}\varphi) - (\varphi_c, \frac{1}{A-t^\beta i}\varphi)\right\} = \\ &= \frac{1}{1-t^{-\beta-2\gamma}}\left\{(\varphi_c, \frac{1}{A-i}\varphi) - t^{-\gamma}(\varphi_c, \frac{1}{A-t^\beta i}G(t)\varphi)\right\} = \\ &= \frac{1}{1-t^{-\beta-2\gamma}}\left\{(\varphi_c, \frac{1}{A-i}\varphi) - t^{-\beta-\gamma}(\varphi_c, G(t)\frac{1}{A-i}\varphi)\right\} = \\ &= (\varphi_c, \frac{1}{A-i}\varphi). \end{aligned}$$

It follows that for any extension φ_c the function $f(t)$ is equal to a certain constant which is determined by the extension.

Suppose conversely that the function $f(t)$ is equal to a certain given constant c. Let us define the extension of the functional φ by the following condition

$$\frac{1}{2}(\varphi_c, \left(\frac{1}{A-i} + \frac{1}{A+i}\right)\varphi) = c = f(t) - i(\varphi, \frac{1}{A^2+1}\varphi).$$

It is necessary to show that the extension of the functional is homogeneous. For this it is enough to prove this property for the elements $\frac{1}{A-i}\varphi$ and $\frac{1}{A+i}\varphi$. But we have

$$\begin{aligned} (G(1/t)\varphi_c, \frac{1}{A-i}\varphi) &= (\varphi_c, G(t)\frac{1}{A-i}\varphi) = t^{\gamma+\beta}(\varphi_c, \frac{1}{A-t^\beta i}\varphi) = \\ &= t^{\gamma+\beta}\left((\varphi_c, \frac{1}{A-i}\varphi) + (t^\beta i - i)(\varphi, \frac{1}{(A-i)(A-t^\beta i)}\varphi)\right) = \\ &= t^{\gamma+\beta}\left((\varphi_c, \frac{1}{A-i}\varphi) - (1-t^{-\beta-2\gamma})(\varphi_c, \frac{1}{A-i}\varphi)\right) = \end{aligned}$$

$$= t^{-\gamma}(\varphi_c, \varphi).$$

Similarly one can prove that $(G(1/t)\varphi_c, \frac{1}{A+i}\varphi) = t^{-\gamma}(\varphi_c, \frac{1}{A+i}\varphi)$, and the lemma is proven. □

REMARK. *If the unitary group G consists of only two elements then the homogeneous extension can always be constructed and it is unique. This condition is true for example for the first derivative operator and Dirac operators in one dimension perturbed by a delta potential. The group of the unitary transformations coincides with the symmetry group with respect to the origin. This group consists of only two elements.*

6 RESOLVENT FORMULAS.

The resolvent of the perturbed operator can be calculated explicitly using the M.Krein's formula. To calculate the resolvent one has to solve the following equation

$$(A_\alpha - z)^{-1} f = \psi$$

for every $f \in \mathcal{H}$ and ψ being an element from $D(A_\alpha)$ (z is in the resolvent set of A_α). We apply first the operator $A_\alpha - z$ to the latter equality

$$f = (A_\alpha - z)\psi = (A + \alpha(\varphi, \cdot)\varphi - z)(\hat{\psi} + a_+(\psi)\frac{1}{A-i}\varphi + a_-(\psi)\frac{1}{A+i}\varphi) =$$

$$= (A-z)\hat{\psi} + a_+(\psi)(i-z)\frac{1}{A-i}\varphi + a_-(\psi)(-i-z)\frac{1}{A+i}\varphi +$$

$$\left\{a_+(\psi)(1+\alpha(\varphi, \frac{1}{A-i}\varphi)) + a_-(\psi)(1+\alpha(\varphi, \frac{1}{A+i}\varphi))\right\}\varphi =$$

$$= (A-z)\hat{\psi} + a_+(\psi)(i-z)\frac{1}{A-i}\varphi + a_-(\psi)(-i-z)\frac{1}{A+i}\varphi.$$

Here we used that $(\varphi, \hat{\psi}) = 0$. The expression in the figure brackets $\{\ \}$ is equal to zero due to the boundary conditions which define the domain of the operator A_α. By applying the resolvent $(A-z)^{-1}$ of the original operator to the latter equality we obtain

$$\frac{1}{A-z} f = \hat{\psi} + a_+(\psi)(i-z)\frac{1}{A-z}\frac{1}{A-i}\varphi + a_-(\psi)(-i-z)\frac{1}{A-z}\frac{1}{A+i}\varphi.$$

Projection on φ gives the following equation

$$(\varphi, \frac{1}{A-z}f) = 0 + a_+(\psi)(i-z)(\varphi, \frac{1}{A-z}\frac{1}{A-i}\varphi) + a_-(\psi)(-i-z)(\varphi, \frac{1}{A-z}\frac{1}{A+i}\varphi).$$

The coefficients $a_\pm(\psi)$ also satisfy the boundary conditions and we thus get a 2×2 linear system to determine the coefficients. This system has the following solution

$$a_+(\psi) = -i\frac{1+\alpha(\varphi, \frac{1}{A+i}\varphi)}{2(\varphi, \frac{1}{A^2+1}\varphi)}\frac{(\varphi, \frac{1}{A-z}f)}{1+\alpha(\varphi, \frac{1}{A-z}\varphi)};$$

$$a_-(\psi) = i\frac{1+\alpha(\varphi, \frac{1}{A-i}\varphi)}{2(\varphi, \frac{1}{A^2+1}\varphi)}\frac{(\varphi, \frac{1}{A-z}f)}{1+\alpha(\varphi, \frac{1}{A-z}\varphi)}.$$

The element $\hat{\psi}$ can also be calculated

$$\hat{\psi} = \frac{1}{A-z}f-$$

$$+i\frac{1+\alpha(\varphi_c, \frac{1}{A+i}\varphi)}{2(\varphi, \frac{1}{A^2+1}\varphi)}\frac{(\varphi_c, \frac{1}{A-z}f)}{1+\alpha(\varphi_c, \frac{1}{A-z}\varphi)}(i-z)\frac{1}{A-z}\frac{1}{A-i}\varphi+$$

$$-i\frac{1+\alpha(\varphi_c, \frac{1}{A-i}\varphi)}{2(\varphi, \frac{1}{A^2+1}\varphi)}\frac{(\varphi_c, \frac{1}{A-z}f)}{1+\alpha(\varphi_c, \frac{1}{A-z}\varphi)}(-i-z)\frac{1}{A-z}\frac{1}{A+i}\varphi.$$

Krein's formula can be written as follows

$$\frac{1}{A_\alpha - z}f = \frac{1}{A-z}f - \frac{\alpha}{1+\alpha(\varphi_c, \frac{1}{A-z}\varphi)}(\frac{1}{A-\bar{z}}\varphi, f)\frac{1}{A-z}\varphi. \tag{14}$$

Define (for $\Im z \neq 0$)

$$F_\alpha(z) = (\varphi_c, \frac{1}{A_\alpha - z}\varphi).$$

The function $F(z) = F_0(z)$ can be calculated in terms of the original distribution φ and the extension parameter c

$$F(z) = \frac{1}{2}\left[(\varphi_c, \frac{1}{A-i}\varphi) + (\varphi_c, \frac{1}{A+i}\varphi)\right] +$$

$$+\frac{i}{2}\left[(\varphi, \frac{1}{A-z}\frac{1}{A-i}\varphi) - (\varphi, \frac{1}{A-z}\frac{1}{A-i}\varphi)\right] +$$

$$+\frac{z}{2}\left[(\varphi, \frac{1}{A-z}\frac{1}{A-i}\varphi) + (\varphi, \frac{1}{A-z}\frac{1}{A-i}\varphi)\right] =$$

$$= c + (\varphi, \frac{1+zA}{A-z}\frac{1}{A^2+1}\varphi) = c + (\frac{1}{A-i}\varphi, \frac{1+zA}{A-z}\frac{1}{A-i}\varphi).$$

We remark that the element $\frac{1}{A-i}\varphi$ belongs to the Hilbert space and it follows that $F(z)$ is a Nevanlinna function (holomorphic function in the upper halfplane with positive imaginary part there). Thus there exists a certain measure $d\mu(\lambda)$ on $\mathbf{R}$ such that

$$F(z) = c + \int_{-\infty}^{\infty} \frac{1+z\lambda}{\lambda - z}\frac{1}{\lambda^2+1}d\mu(\lambda)$$

and the following condition is satisfied

$$\int_{-\infty}^{\infty} \frac{d\mu(\lambda)}{\lambda^2+1} < \infty.$$

Considering different selfadjoint operators A and functionals φ one can get any Nevanlinna function without linear term. We recall that any Nevanlinna function $F(z)$, $z \in \mathbf{C}$, possesses the following representation

$$F(z) = a + bz + \int_{\mathbf{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2+1}\right) d\sigma(\lambda),$$

where $a \in \mathbf{R}, b \geq 0$ and where the function $\sigma(\lambda)$ is nondecreasing and satisfies $\int_{\mathbf{R}} \frac{d\sigma(\lambda)}{\lambda^2+1} < \infty$. It has been proven in [9,17] that the linear term in the representation of the Q-function of a rank one perturbation is not equal to zero only if this perturbation is not an operator but a selfadjoint relation. We note that the linear term can be present for rank two perturbations of selfadjoint operators.

All five critical formulas for the rank one perturbation [18] can be written in the same form as for the case of a semibounded operator A:

$$F_\alpha(z) = \frac{F(z)}{1+\alpha F(z)}; \tag{15}$$

$$\frac{1}{A_\alpha - z}\varphi = \frac{1}{1-\alpha F(z)} \frac{1}{A-z}\varphi; \tag{16}$$

$$\frac{1}{A_\alpha - z} = \frac{1}{A-z} - \frac{\alpha}{1+\alpha F(z)} (\frac{1}{A-\bar{z}}\varphi, \cdot) \frac{1}{A-z}\varphi; \tag{17}$$

$$Tr\left[\frac{1}{A-z} - \frac{1}{A_\alpha - z}\right] = \frac{d}{dz}\ln(1+\alpha F(z)). \tag{18}$$

$$\int_{-\infty}^{\infty} [d\mu_\alpha(E)] d\alpha = dE, \tag{19}$$

where μ_α is the spectral measure corresponding to the operator A_α. We shall prove the latter formula in more details

LEMMA 6.1 *Let $f \in L_1(\mathbf{R})$ then $f \in L_1(\mathbf{R}, d\mu_\alpha)$ for almost every α,*

$$\int f(E) d\mu_\alpha(E) \in L_1(\mathbf{R}, d\alpha)$$

and

$$\int \left(\int f(E) d\mu_\alpha(E)\right) d\alpha = \int f(E) dE.$$

PROOF. The proof is similar to the proof of the Theorem 1.8 from [18]. It is sufficient to prove the result for the functions $f_z(E) = \frac{1}{E-z} - \frac{1}{E+i}$ for any $z \in \mathbf{C} \setminus \mathbf{R}$. Considering the closed contour in the upper half plane one can calculate

$$\int_{\mathbf{R}} f_z(E) dE = \begin{cases} 0 & \Im z < 0 \\ 2\pi i & \Im z > 0. \end{cases}$$

On the other hand

$$\int_{\mathbf{R}} f_z(E) d\mu_\alpha(E) = \int_{\mathbf{R}} \left(\frac{1}{E-z} - \frac{1}{E+i}\right) d\mu_\alpha(E) =$$

$$= \int_{\mathbf{R}} \left(\left[\frac{1}{E-z} - \frac{E}{E^2+1}\right] - \left[\frac{1}{E+i} - \frac{E}{E^2+1}\right]\right) d\mu_\alpha(E) =$$

$$= F_\alpha(z) - F_\alpha(-i) = \frac{1}{\alpha + F(z)^{-1}} - \frac{1}{\alpha + F(-i)^{-1}} \equiv h_z(\alpha).$$

The function $h_z(\alpha)$ has either two poles in the lower half plane if $\Im z < 0$ or one in each half plane if $\Im z > 0$. It follows that

$$\int_{\mathbf{R}} h_z(\alpha) d\alpha = \begin{cases} 0 & \Im z < 0 \\ 2\pi i & \Im z > 0. \end{cases}$$

The lemma is proven. □

It is important that the result does not depend on the parameter c.

7 APPROXIMATIONS OF THE RANK ONE PERTURBATIONS.

7.1 Convergence of linear operators.

We are going to study the convergence of the operators in the operator norm. We concentrate our attention to form unbounded perturbations. The operator $A_\alpha = A + \alpha(\varphi_c, \cdot)\varphi$ is a linear operator which acts from $B_\varphi(A)$ to $D(A)^*$. Let $\varphi_n \in \mathcal{H}$ then the operator $A_\alpha^n = A + \alpha(\varphi_n, \cdot)\varphi_n$ can be defined on the same domain.

LEMMA 7.1 *Let f be an element from $\mathcal{H} \setminus D(A)$ and φ be an element from $D(A)^*$; then for any c there exists a sequence φ_n of elements from $\mathcal{H}$ converging to φ in the $D(A)^*$ norm such that (f, φ_n) converges to c.*

PROOF. The Hilbert space $\mathcal{H}$ is a dense subspace of the Hilbert space $D(A)^*$. This follows from the fact that the domain of the operator is dense in the Hilbert space. It follows that there exists a sequence $\tilde{\varphi}_n \in \mathcal{H}$ converging to φ in the $D(A)^*$ norm. If the sequence $(f, \tilde{\varphi}_n) = a_n$ does not converge to c consider a sequence $\psi_n \in \mathcal{H}$ with unit $D(A)^*$ norm $\| \psi_n \|_{D(A)^*} = 1$ such that (f, ψ_n) diverges to ∞. Such a sequence exists because $f \notin D(A)$. The subsequence can be chosen in such a way that $\frac{c - a_n}{(f, \psi_n)} \to 0$. We keep the same notation for the chosen subsequence. Consider the sequence

$$\varphi_n = \tilde{\varphi}_n + \frac{c - a_n}{(f, \psi_n)} \psi_n.$$

The following estimates are valid because $\| \psi_n \|_{D(A)^*} = 1$:

$$\| \varphi_n - \varphi \|_{D(A)^*} \leq \| \tilde{\varphi}_n - \varphi \|_{D(A)^*} + \left| \frac{c - a_n}{(f, \psi_n)} \right|.$$

It follows that φ_n converges to φ in $D(A)^*$ norm. The sequence $(f, \varphi_n) = a_n + c - a_n = c$ obviously converges to c, and the lemma is proven. □

THEOREM 7.1 *Let the sequence $\varphi_n \in \mathcal{H}$ converge to φ in $D(A)^*$ and assume $(\varphi_n, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi)$ converges to $2c$, then the sequence of linear operators*

$$A_\alpha^n = A + \alpha(\varphi_n, \cdot)\varphi_n$$

defined on the domain $B_\varphi(A)$ converges in the operator norm to the operator A_α, as $n \to \infty$.

PROOF. Consider an arbitrary element $g = \tilde{g} + b(g)\left(\frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi\right) \in B_\varphi(A)$, $\tilde{g} \in D(A), b(g) \in \mathbf{C}$. Then the following estimates are valid

$$\| (A_\alpha^n - A_\alpha) g \|_{D(A)^*} = |\alpha| \, \| (\varphi_n, g)\varphi_n - (\varphi_c, g)\varphi \|_{D(A)^*} =$$

$$
\begin{aligned}
&= |\alpha| \parallel (\varphi_n, \tilde{g})\varphi_n + b(g)(\varphi_n, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi)\varphi_n - \\
&\quad -(\varphi, \tilde{g})\varphi - b(g)(\varphi_c, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi)\varphi \parallel_{D(A)^*} \leq \\
&\leq |\alpha| \Big(|(\varphi_n, \tilde{g}) - (\varphi, \tilde{g})| \parallel \varphi_n \parallel_{D(A)^*} + |(\varphi, \tilde{g})| \parallel \varphi_n - \varphi \parallel_{D(A)^*} + \\
&+ |b(g)| |(\varphi_n, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi) - 2c| \parallel \varphi_n \parallel_{D(A)^*} + 2|a(g)|\, |c| \parallel \varphi_n - \varphi \parallel_{D(A)^*} \Big) \leq \\
&\leq |\alpha| \Big(\parallel \varphi_n \parallel_{D(A)^*} \parallel \varphi_n - \varphi \parallel_{D(A)^*} \parallel \tilde{g} \parallel_{D(A)} + \parallel \varphi_n - \varphi \parallel_{D(A)^*} \parallel \varphi \parallel_{D(A)^*} \parallel \tilde{g} \parallel_{D(A)} + \\
&+ |(\varphi_n, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi) - 2c| \parallel \varphi_n \parallel_{D(A)^*} |b(g)| + 2|c| \parallel \varphi_n - \varphi \parallel_{D(A)^*} |b(g)| \Big) \leq \\
&\leq |\alpha| \Big\{ \Big(\parallel \varphi_n \parallel_{D(A)^*} + \parallel \varphi \parallel_{D(A)^*} + 2|c| \Big) \parallel \varphi_n - \varphi \parallel_{D(A)^*} + \\
&+ \parallel \varphi_n \parallel_{D(A)^*} |(\varphi_n, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi) - 2c| \Big\} \parallel g \parallel_{B_\varphi(A)} .
\end{aligned}
$$

The following estimate is valid for the norm of the linear operators

$$
\begin{aligned}
&\parallel A_\alpha^n - A_\alpha \parallel_{B_\varphi(A) \to D(A)^*} \leq \\
&\leq |\alpha| \Big\{ \Big(\parallel \varphi_n \parallel_{D(A)^*} + \parallel \varphi \parallel_{D(A)^*} + 2|c| \Big) \parallel \varphi_n - \varphi \parallel_{D(A)^*} + \\
&+ \parallel \varphi_n \parallel_{D(A)^*} |(\varphi_n, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi) - 2c| \Big\} .
\end{aligned}
$$

The sequence φ_n converges to φ in the $D(A)^*$ norm, the sequence $\parallel \varphi_n \parallel_{D(A)^*}$ is bounded and the sequence $(\varphi_n, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi)$ converges to $2c$. It follows that the linear operators A_α^n converge in the operator norm to A_α, as $n \to \infty$. □.

THEOREM 7.2 *Let $\varphi \in D(A)^* \setminus \mathcal{H}$ then there exists a sequence $\varphi_n \in \mathcal{H}$ converging to φ in $D(A)^*$ norm such that the sequence of linear operators*

$$A_\alpha^n = A + \alpha(\varphi_n, \cdot)\varphi_n$$

defined on the domain $B_\varphi(A)$ converges in the operator norm to the operator $A_\alpha = A + \alpha(\varphi_c, \cdot)\varphi$.

PROOF. The element $\frac{1}{A-i}\varphi$ belongs to the Hilbert space $\mathcal{H}$ but does not belong to the domain of the operator. The same is true for the element

$$\frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi = 2\frac{1}{A-i}\varphi - 2i\frac{1}{A^2+1}\varphi$$

since $\frac{1}{A^2+1}\varphi \in D(A)$. It follows from Lemma 7.1 that there exists a sequence φ_n converging to φ in $D(A)^*$ norm and such that $(\varphi_n, \frac{1}{A-i}\varphi + \frac{1}{A+i}\varphi)$ converges to $2c$. It follows from Theorem 7.1 that the operators A_α^n converge to A_α in the operator norm. □

The approximating sequence φ_n can be constructed using the spectral representation of the original operator A. If the element φ belongs to $\mathcal{H}_{-2}(A)$ then there exists a certain Borel measure $d\mu(\lambda)$ on $\mathbf{R}$ such that

$$(\frac{1}{A-i}\varphi, \frac{1+zA}{A-z}\frac{1}{A-i}\varphi) = \int_{-\infty}^{+\infty} \frac{1+z\lambda}{\lambda - z}\frac{1}{\lambda^2+1} d\mu(\lambda)$$

and $\int_{-\infty}^{+\infty} \frac{d\mu(\lambda)}{\lambda^2+1} < \infty$. Consider the spaces $\mathcal{H}_{-1,-2}(A)$ and $\mathcal{H}_{-2,-1}(A)$ formed by the elements from $\mathcal{H}_{-2}(A)$ satisfying the following additional conditions $\int_{-\infty}^{0} \frac{|\lambda| d\mu(\lambda)}{\lambda^2+1} < \infty$ and $\int_{0}^{\infty} \frac{|\lambda| d\mu(\lambda)}{\lambda^2+1} < \infty$ respectively. The following lemma can be proven

LEMMA 7.2 *Let $\varphi \in \mathcal{H}_{-2}(A) \setminus (\mathcal{H}_{-1,-2}(A) \cup \mathcal{H}_{-2,-1}(A))$ then there exist two sequences $c_n, d_n \to \infty$ such that $\lim_{n\to\infty} \int_{-c_n}^{d_n} \frac{\lambda}{\lambda^2+1} d\mu(\lambda) = c$.*

PROOF. The convergence of the integral $\int_{-\infty}^{+\infty} \frac{\lambda}{\lambda^2+1} d\mu(\lambda)$ implies that two sequences

$$n \int_{n}^{n+1} \frac{\lambda}{\lambda^2+1} d\mu(\lambda), \quad n = 1, 2, \ldots$$

and

$$n \int_{-n-1}^{-n} \frac{\lambda}{\lambda^2+1} d\mu(\lambda), \quad n = 1, 2, \ldots$$

have zero limits when $n \to \infty$ and have different signs. The sums of both sequences are diverging. It follows that the sequence $\int_{-n}^{m} \frac{\lambda}{\lambda^2+1} d\mu(\lambda)$ can converge to any real number when $n, m \to \infty$. Consider the approximating sequence of the elements from the Hilbert space $\varphi_n = E_{(-c_n, d_n)}(A)\varphi$, where $E(A)$ denotes the family of spectral projectors for the operator A. The following limit holds

$$\lim_{n\to\infty} (\varphi_n, \frac{A}{A^2+1} \varphi_n) = c.$$

□

The sequence φ_n will be used in what follows to construct the approximations of rank one perturbations in the strong resolvent sense.

7.2 Strong resolvent convergence.

We study in this subsection the strong resolvent convergence of the operators. It has been shown that the difference of the resolvents of the original and the perturbed operators has rank one.

THEOREM 7.3 *Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$ and φ be an element from $\mathcal{H}_{-1}(|A|)$. Let the sequence $\varphi_n \in \mathcal{H}$ converge to φ in the norm $\mathcal{H}_{-1}(|A|)$. Then the sequence of operators $A_\alpha^n = A + \alpha(\varphi_n, \cdot)\varphi_n$ converge to the operator $A_\alpha = A + \alpha(\varphi, \cdot)\varphi$ in the strong resolvent sense for all z, $\Im z \neq 0$.*

PROOF. Since the $\{\frac{1}{A_\alpha^n - z}\}$ are uniformly bounded it is enough to prove the weak convergence of the resolvents. Consider two arbitrary vectors ψ_1, ψ_2 from the Hilbert space. The convergence in the space $\mathcal{H}_{-1}(|A|)$ implies

$$\lim_{n\to\infty} (\frac{1}{A-\bar{z}}(\varphi - \varphi_n), \psi_1) = 0; \tag{20}$$

$$\lim_{n\to\infty} (\psi_2, \frac{1}{A-z}(\varphi_n - \varphi)) = 0. \tag{21}$$

Moreover the quadratic form of the resolvent converges at the point $z = i$ and similarly at all other points

$$|(\varphi_n, \frac{1}{A-z}\varphi_n) - (\varphi, \frac{1}{A-z}\varphi)| \leq$$

$$\leq 2 \parallel \varphi_n - \varphi \parallel_{\mathcal{H}_{-1}(|A|)} \left(\parallel \varphi_n \parallel_{\mathcal{H}_{-1}(|A|)} + \parallel \varphi \parallel_{\mathcal{H}_{-1}(|A|)} \right) \to 0.$$

We have for the difference of the resolvents

$$(\psi_2, \frac{1}{A_\alpha^n - z}\psi_1) - (\psi_2, \frac{1}{A_\alpha - z}\psi_1) =$$

$$= \frac{\alpha}{1+\alpha(\varphi, \frac{1}{A-z}\varphi)}(\frac{1}{A-\bar{z}}\varphi, \psi_1)(\psi_2, \frac{1}{A-z}\varphi) -$$

$$- \frac{\alpha}{1+\alpha(\varphi_n, \frac{1}{A-z}\varphi_n)}(\frac{1}{A-\bar{z}}\varphi_n, \psi_1)(\psi_2, \frac{1}{A-z}\varphi_n).$$

The weak resolvent convergence follows from the formulas (20,21) and the convergence of the quadratic form of the resolvent. The denominator in the first quotient does not vanish because $\Im z \neq 0$. The theorem is thus proven. □

THEOREM 7.4 *Let A be a selfadjoint operator and φ be a functional from $D(A)^*$. Let φ_n be any sequence from the Hilbert space converging to φ in $D(A)^*$ and let $\lim_{n\to\infty}(\varphi_n, \frac{1}{A-i}\varphi_n) = c$. Then the sequence of selfadjoint operators*

$$A_\alpha^n = A + \alpha(\varphi_n, \cdot)\varphi_n$$

converges to A_α in the strong resolvent sense. If $\lim_{n\to\infty}|(\varphi_n, \frac{1}{A-i}\varphi_n)| = \infty$, then the operators A_α^n converge to the original operator in the strong resolvent sense.

PROOF. The first part of the theorem can be proven using the fact that the convergence in $\mathcal{H}_{-2}$ implies weak convergence of the resolvents and formulas (20,21) hold for every $\psi_1, \psi_2 \in \mathcal{H}$. Calculations similar to the ones carried out during the proof of the theorem 7.3 lead to the result which has to be proven. One has only to take into account that $\lim_{n\to\infty}(\varphi_n, \frac{1}{A-z}\varphi_n) = (\varphi_c, \frac{1}{A-z}\varphi)$.

Consider now the case where $\lim_{n\to\infty}|(\varphi_n, \frac{1}{A-i}\varphi_n)| = \infty$. The difference of the resolvents is the rank one operator

$$\frac{1}{A_\alpha^n - z} - \frac{1}{A-z} = -\frac{\alpha}{1+\alpha(\varphi_n, \frac{1}{A-z}\varphi_n)}\left(\frac{1}{A-\bar{z}}\varphi_n, \cdot\right)\frac{1}{A-z}\varphi_n.$$

The first term on the right hand side of the latter equality converges to zero. It follows that the difference of the resolvents converges weakly to zero since $\frac{1}{A-z}\varphi_n$ and $\frac{1}{A-\bar{z}}\varphi_n$ converge to $\frac{1}{A-z}\varphi$ and $\frac{1}{A-\bar{z}}\varphi$ respectively. □

Let $\varphi \in \mathcal{H}_{-2}(A) \setminus (\mathcal{H}_{-1,-2}(A) \cup \mathcal{H}_{-2,-1}(A))$ then it is possible to construct a sequence of operators converging to A_α in the strong resolvent sense according to the following:

THEOREM 7.5 *Let $\varphi \in \mathcal{H}_{-2}(A) \setminus (\mathcal{H}_{-1,-2}(A) \cup \mathcal{H}_{-2,-1}(A))$ then there exist two sequences $c_n, d_n \to \infty$ such that $\varphi_n = E_{(-c_n,d_n)}(A)\varphi$ determines the sequence of selfadjoint operators $A_\alpha^n = A + \alpha(\varphi_n, \cdot)\varphi_n$ with bounded perturbations converging to the perturbed operator $A_\alpha = A + \alpha(\varphi, \cdot)\varphi$ in the strong resolvent sense.*

PROOF. All statements follow easily from lemma 7.2 and theorem 7.4. □

The latter theorem shows how to construct the approximating sequence φ_n leading to the approximations of the operator A_α in the strong resolvent sense.

7.3 Conclusions.

If $\varphi \in \mathcal{H}_{-1}(|A|)$ then the sequence $\varphi_n \in \mathcal{H}$ converging to φ in the $\mathcal{H}_{-1}$ norm defines a sequence of selfadjoint operators converging to the perturbed operator A_α in the strong resolvent sense. If $\varphi \in \mathcal{H}_{-2}(A) \setminus (\mathcal{H}_{-1,-2}(A) \cup \mathcal{H}_{-2,-1}(A))$ then there exist a sequence φ_n converging to φ in the $\mathcal{H}_{-2}$ norm such that the sequence of corresponding perturbed operators is converging to A_α in the strong resolvent sense. If $\varphi \in \mathcal{H}_{-1,-2}(A) \setminus \mathcal{H}_{-1}(|A|)$ or $\varphi \in \mathcal{H}_{-2,-1}(A) \setminus \mathcal{H}_{-1}(|A|)$ then every sequence φ_n converging to φ in $\mathcal{H}_{-2}$ norm defines a sequence of selfadjoint operators converging to the original operator in the strong resolvent sense. It follows that not every form unbounded rank one perturbation can be approximated in the strong resolvent sense by operators with bounded perturbations.

Approximations in the sense of linear operators can be constructed for every rank one perturbation. If the perturbation is form bounded then every sequence φ_n converging to φ in the $\mathcal{H}_{-1}$ norm determines a sequence of operators converging to the perturbed operator in the norm of linear operators. If $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(|A|)$ then one can prove only the existence of the approximating sequence.

8 EXAMPLES.

8.1 Perturbations of the first derivative operator.

Consider rank one perturbations defined by the heuristic expression

$$A_\alpha = \frac{1}{i}\frac{d}{dx} + \alpha\delta = \frac{1}{i}\frac{d}{dx} + \alpha(\delta, \cdot)\delta. \tag{22}$$

The operator A_α can be considered as a rank one perturbation of the selfadjoint not semi-bounded operator $A = \frac{1}{i}\frac{d}{dx}$ with the domain $D(A) = W_2^1(\mathbf{R})$. The element δ defines a bounded linear functional on $W_2^1(\mathbf{R})$ due to the embedding theorem. But the element $\frac{1}{A-i}\delta = ie^{-x}\Theta(x)$ does not belong to the domain of the operator A. ($\Theta(x)$ denotes here the Heaviside step function.) The restriction A^0 of the operator A to the domain of functions $D(A^0) = \{\psi \in W_2^1(\mathbf{R}) : \psi(0) = 0\}$ has deficiency indices $(1,1)$. Every function ψ from the domain of the adjoint operator possesses the standard representation

$$\psi(x) = \hat{\psi} + ia_+(\psi)e^{-x}\Theta(x) - ia_-(\psi)e^{x}\Theta(-x),$$

where $\hat{\psi}(0) = 0$. Thus the coefficients $a_\pm(\psi)$ can be calculated from the boundary values of the function ψ at the origin

$$\psi(\pm 0) = \pm ia_\pm(\psi).$$

Consider the group of the central symmetries of the real line:

$$G(1) = I, G(-1) = J,$$

$$G(-1)^2 = G(1),$$

where I and J are the identity and inversion operators defined by the following formulas in the generalized sense

$$(If)(x) = f(x);$$

$$(Jf)(x) = f(-x).$$

The original operator and the functional δ are homogeneous with respect to this group

$$AG(t) = tG(t)A,$$

$$G(t)\delta = \delta.$$

The parameters β and γ for this problem are equal to -1 and 0 correspondingly. The group consists of only two elements and the extension of the functional δ can be defined using the parameter $f(-1)$

$$f(-1) = i(\delta, \frac{1}{(A-i)(A+i)}\delta) = i(\frac{1}{A-i}\delta, \frac{1}{A-i}\delta) = i\int_0^\infty e^{-2x}dx = \frac{i}{2}.$$

It follows that the parameter c which defines the unique homogeneous extension is equal to

$$c = f(-1) - i(\delta, \frac{1}{A^2+1}\delta) = 0.$$

We are going to keep the same notation δ for the extended functional.

It follows from theorem 4.1 that the selfadjoint operator A_α corresponding to the heuristic expression (22) is defined on the domain of functions satisfying the following conditions

$$a_-(\psi) = -\frac{1+i\frac{\alpha}{2}}{1-i\frac{\alpha}{2}}a_+(\psi).$$

Thus the operator A_α is the selfadjoint operator $\frac{1}{i}\frac{d}{dx}$ defined on the following domain

$$D(A_\alpha) = \left\{\psi \in W_2^1(\mathbf{R}\setminus\{0\}) : \psi(-0) = \frac{1+i\frac{\alpha}{2}}{1-i\frac{\alpha}{2}}\psi(+0)\right\}.$$

The spectral analysis of the operator A_α can easily be carried out. The spectrum of each operator A_α is absolutely continuous and covers the whole real line $(-\infty, +\infty)$.

8.2 Dirac operator perturbed by a pseudopotential.

A similar analysis can be carried out for the one dimensional Dirac operator with the delta potential described heuristically by

$$H_\alpha = \begin{pmatrix} m & -i\frac{d}{dx} \\ -i\frac{d}{dx} & -m \end{pmatrix} + V\vec{\delta} \tag{23}$$

$$V = V^* = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix},$$

where $v_{11}, v_{22} \in \mathbf{R}, v_{12} = \overline{v_{21}} \in \mathbf{C}$. This family of Dirac operators with pseudopotentials is described by four real parameters. The original operator

$$H = \begin{pmatrix} m & -i\frac{d}{dx} \\ -i\frac{d}{dx} & -m \end{pmatrix}$$

is defined on the two component functions $f = (f_1, f_2) \in L_2(\mathbf{R}) \oplus L_2(\mathbf{R})$ from the domain $D(H) = W_2^1(\mathbf{R}) \oplus W_2^1(\mathbf{R})$. Two delta functions δ_1, δ_2 defined as follows $(\delta_i, f) = f_i(0), i = 1, 2$ are bounded linear functionals on the domain of the original operator. The delta function $\vec{\delta}$ is the linear map

$$\vec{\delta} : W_2^1(\mathbf{R}) \oplus W_2^1(\mathbf{R}) \to \mathbf{C}^2,$$
$$(\vec{\delta}, f) = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix}.$$

The product of the delta function and an arbitrary continuous function f is equal to

$$(f\vec{\delta}, \psi) = (\vec{\delta}, \begin{pmatrix} f_1\psi_1 \\ f_2\psi_2 \end{pmatrix}) = \begin{pmatrix} f_1(0)\psi(0) \\ f_2(0)\psi(0) \end{pmatrix} =$$
$$= \begin{pmatrix} f_1(0) & 0 \\ 0 & f_2(0) \end{pmatrix} (\vec{\delta}, \varphi).$$

The heuristic expression (23) can be written as follows

$$H_\alpha = \begin{pmatrix} m & -i\frac{d}{dx} \\ -i\frac{d}{dx} & -m \end{pmatrix} + V\ diag\{(\delta_i, \cdot)\}\vec{\delta}. \tag{24}$$

This operator can be considered as a rank two perturbation of the selfadjoint not semibounded original operator H. In accordance with the developed approach we restrict the original operator H to the domain of functions $D_\delta(H) = \{\psi \in D(H) : \vec{\delta}\psi = 0\}$. The restricted operator H^0 has deficiency indices $(2, 2)$. The adjoint operator H^{0*} is defined on the domain $W_2^1(\mathbf{R} \setminus \{0\}) \oplus W_2^1(\mathbf{R} \setminus \{0\})$. To determine the perturbed operator, the bounded linear functionals δ_i have to be extended to a set of functions which are discontinuous at the origin. The delta functions are homogeneous with respect to the group of central symmetries of the real line and the extension we are discussing is uniquely determined by

$$(\delta_i, f) = \frac{f_i(+0) + f_i(-0)}{2}, \ i = 1, 2.$$

This extension allows one to define the perturbed linear operator on the domain $W_2^1(\mathbf{R} \setminus \{0\}) \oplus W_2^1(\mathbf{R} \setminus \{0\})$ since the boundary values at the origin of the functions from this domain are well defined. The domain of the perturbed selfadjoint operator coincides with the set of all function $\psi \in L_2(\mathbf{R}) \oplus L_2(\mathbf{R})$, such that

$$\begin{pmatrix} m & -i\frac{d}{dx} \\ -i\frac{d}{dx} & -m \end{pmatrix} \psi + V\vec{\delta}(x)\psi \in L_2(\mathbf{R}) \oplus L_2(\mathbf{R}).$$

Let us calculate the distribution

$$f = \begin{pmatrix} m & -i\frac{d}{dx} \\ -i\frac{d}{dx} & -m \end{pmatrix} \psi + V\vec{\delta}(x)\psi$$

for any function ψ from the domain of the adjoint operator H_0^*. Every such distribution can be presented in the following form

$$f = \tilde{f} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ diag\{\psi_1(+0) - \psi_1(-0), \psi_2(+0) - \psi_2(-0)\}\vec{\delta} +$$

$$+\frac{1}{2}V\, diag\{\psi_1(+0)+\psi_1(-0),\psi_2(+0)+\psi_2(-0)\}\vec{\delta},$$

where $\tilde{f} \in L_2(\mathbf{R}) \oplus L_2(\mathbf{R})$. The vector f belongs to the Hilbert space if and only if the coefficient in front of the delta function $\vec{\delta}$ is equal to zero. We get the following boundary conditions for the function ψ at the origin:

$$\left(\frac{1}{2}V - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\psi(+0) = -\left(\frac{1}{2}V + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\psi(-0).$$

These boundary conditions can be written in the form:

$$\psi(+0) = \Lambda\psi(-0); \tag{25}$$

$$\Lambda = -\left(\frac{1}{2}V - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)^{-1}\left(\frac{1}{2}V + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

One can show, that

$$\Lambda\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Lambda^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and it follows that the operator H^{0*} restricted to the domain of functions satisfying the boundary conditions (25) is selfadjoint ([5]).

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