# Funk.An. Mandatory 2

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## Problem 1

### a)

We know that since $(e_n)_{n\geq 1}$ is a orthonormal basis for a separable Hilbert space $H$, and $(f_N)_{N\geq 1}$ is a sequence of vectors in $H$, $f_N$ converges weakly to $x \in H$ if and only if it is bounded and converges to $x$ pointwise, i.e. there exists $M > 0$ such that $|| f_N || \leq M$ for all $N \in \mathbb{N}$ and $\langle f_N, e_n\rangle \to \langle f, e_n\rangle$ for all $n \in \mathbb{N}$.
By definition of weak convergence and the Riesz representation theorem we have that if $f_N \to f$, then $(f_N)_{N\geq 1}$ is bounded, and $\langle f_N, e_n\rangle \to \langle f, e_n\rangle$ for $N \to \infty$ for all $n \in \mathbb{N}$. So lets assume that there exists $M > 0$ such that $|| f_N || \leq M$ for all $N \in \mathbb{N}$ and $\langle f_N, e_n\rangle \to \langle f, e_n\rangle$ for $n \to \infty$, for all $n \in \mathbb{N}$. Then by linearity of the inner product we have that $\langle f_N, y\rangle \to \langle f, y\rangle$ for $n \to \infty$, where $y = \sum_{i=1}^{p}\langle y, e_i\rangle e_i$ is a finite sum of the basis vectors. So for all $v \in H$ we can let $\epsilon > 0$ and $p \in \mathbb{N}$ such that $\sum_{k=p+1}^{\infty} \mid \langle v, e_k\rangle \mid^2 < \left(\frac{\epsilon}{2(M+||f||)}\right)^2$.
Then since $\langle f_N, \sum_{k=1}^{p}\langle v, e_k\rangle e_k\rangle \to \langle f, \sum_{k=1}^{p}\langle v, e_k\rangle e_k\rangle$ for $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $\mid \langle f_N, \sum_{k=1}^{p}\langle v, e_k\rangle e_k\rangle - \langle f_N, \sum_{k=1}^{p}\langle v, e_k\rangle e_k\rangle \mid < \frac{\epsilon}{2}$.
Then for every $n \geq n_0$ we have that $\mid \langle f_N, v\rangle - \langle f, v\rangle \mid = \mid \langle f_N, \sum_{k=1}^{p}\langle v, e_k\rangle e_k\rangle - \langle f, \sum_{k=1}^{p}\langle v, e_k\rangle e_k\rangle + \langle f_N, \sum_{k=p+1}^{\infty}\langle v, e_k\rangle e_k\rangle - \langle f, \sum_{k=p+1}^{\infty}\langle v, e_k\rangle e_k\rangle \mid = \mid \langle f_N, \sum_{k=1}^{p}\langle v, e_k\rangle e_k\rangle - \langle f, \sum_{k=1}^{p}\langle v, e_k\rangle e_k\rangle \mid + \mid \langle f_N - f, \sum_{k=p+1}^{\infty}\langle v, e_k\rangle e_k\rangle \mid < \frac{\epsilon}{2} + || f_N - f || \sqrt{\sum_{k=p+1}^{\infty} \mid \langle v, e_k\rangle \mid^2} < \epsilon$, since $|| f_N - f || < M + || f ||$. So we have that $f_N \to f$, where in our case $f_N \to 0$ weakly as $N \to \infty$.
We have that $|| f_N ||^2 = \frac{1}{N^2} || \sum_{n=1}^{N^2} e_n ||^2 = \frac{1}{N^2} \cdot N^2 = 1$, which means that $|| f_N || = 1$.

### b)

Weakly compact (compact w.r.t the weak topology), (def. 7.7 convex hull), (thm. 5.7 convex subset)

For $K$ to be weakly compact it need to be compact with respect to the weak topology. By theorem 5.7 we have that the norm closure is the same as the weak closure. So we have that $K$ is weakly closed and bounded, which means that it is weakly compact. And we also have that $0 \in \overline{\{f_N : N \geq 1\}}^{\tau_w}$ and hence also $0 \in \overline{co\{f_N : N \geq 1\}}^{\tau_w}$, so by this we get that $0 \in \overline{co\{f_N : N \geq 1\}}^{||\cdot||_\infty}$, so $0 \in K$.

### c)

By definition 7.8 in the lecture notes, we have that $K$ is the closed convex hull of its extreme points, in particular we have that $K = \overline{\text{co}(\text{Ext}(K))}^{\tau}$, this means that $\text{Ext}(K) = \{f_N : N \geq 1\}$, and we also have that $0 \in \text{Ext(K)}$, since $f_N \to 0$ as $N \to \infty$ by (a), so $0 \in K$ and by definition 7.1 we have that $0 = \alpha x_1 + (1-\alpha)x_2 \Rightarrow 0 = x_1 = x_2$, for all $x_1, x_2 \in K$ and $0 < \alpha < 1$.

### d)

Any other extreme points in $K$? $K$ does not have any other extreme points since $x_0 \in \text{Ext}(K) \Leftrightarrow$ if $x_0 \in K$, then $\{x_0\}$ is a face of $K$ by remark 7.4, since $K$ is non-empty and convex by construction, and also by remark 7.4 we have that if $(F_\alpha)_{\alpha\in I}$ is a collection of faces of $K$, then $\cap_{\alpha\in I} F_\alpha$ is a face of $K$, and by (c) and theorem

7.8 we have $\mathrm{Ext}(K) = \{f_N : N \geq 1\}$ so $f_N$ are faces, and the intersection of them are 0 by (a), and by theorem 7.9 we have that $\mathrm{Ext}(K) \subset \overline{\{f_N : N \geq 1\}}^{\tau}$, so there is no other extreme points in $K$.

# Problem 2

## a)

Assume $x_n \to x$ weakly $\Leftrightarrow$ $f(x_n) \to f(x)$ weakly for every $f \in X^*$, then we know that $Tx_n \to Tx$ weakly $\Leftrightarrow$ $g(Tx_n) \to g(Tx)$ weakly for every $g \in Y^*$, and for every $g \in Y^*$ we have that $g \circ T \in X^*$. So we have that $x_n \to x$ weakly $\Leftrightarrow$ $(g \circ T)(x_n) \to (g \circ T)(x) \Leftrightarrow g(Tx_n) \to g(Tx) \Leftrightarrow Tx_n \to Tx$ weakly as $n \to \infty$.

## b)

Since $T$ is compact, we know that every sequence is sent to a sequence that has a subsequence. We can prove this by cantradiction, lets assume that $Tx_n$ does not converge to $Tx$ in norm i.e. $|| Tx_n - Tx || \nrightarrow 0$, then there must exists a subsequence $(Tx_{n_k})_{k \in I}$ with $| (Tx_{n_k})_{k \in I} - Tx | > \epsilon > 0$, then we can take a norm convergent subsequence of this subsequence, which will not converge weakly to $Tx$, since it converges in norm to another point, which contradicts (a). This means that $|| Tx_n - Tx || \to 0$ as $n \to \infty$.

## c)

Assume that $T$ is not compact, then we have that $T(B_H)$ is not totally bounded, i.e. there exists a $\delta > 0$ such that there is no finite number of balls with radius $\delta$ that cover $\overline{T(B_H)}$. So we can choose some $Tx_1 \in T(B_H)$, and for each n, we have that $Tx_{n+1} \in T(B_H) \setminus \bigcup_{k=1}^{n} \overline{B(x_k, \delta)}$, which is non-empty by definition of $\delta$.
So we have a sequence $(x_n) \in B_H$ such that $|| Tx_n - Tx_m || \geq \delta$ for all $m \neq n$. Then by the negation of proposition 8.2 (4), we get that the sequence $(x_n)$ must have a weakly convergent subsequence. Then this weakly convergent subsequence must satisfy that $| (x_{n_k}) - (x_{n_l}) | \geq \delta$ for all $k \neq l$. So it cannot be mapped to a sequence which converges in norm, since it would have to be cauchy. This leads to a contradiction, since we have that if a sequence doesn't converge in a Banach space, then it isn't cauchy, so $T$ must be compact i.e. $T \in \mathcal{K}(H, Y)$.

## d)

Compactness Firstly we take a sequence $(x_n)_{n \geq 1}$ in $X$ and $x \in X$ such that $x_n \to x$ weakly, as $n \to \infty$. Then by (a) we have that $Tx_n \to Tx$ weakly as $n \to \infty$, since $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$. But since we have that weak convergence is the same as norm convergens in $\ell_1(\mathbb{N})$, we get that $|| Tx_n - Tx || \to 0$, as $n \to \infty$. Then by (c) we have that $T \in \mathcal{K}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ as we wanted.

## e)

Assume that $T$ is compact and open, then $T(B(0,1)) \subset T(\overline{B(0,1)})$, where $T(B(0,1))$ is open, since $T$ is an open map, by the open mapping theorem. So $T(\overline{B(0,1)})$ contains some ball $\overline{B(y,r)}$ for $y \in Y$ and $0 < r < 1$. And since $T$ is compact, we have that the closure of $T(\overline{B(0,1)}) = \overline{T(\overline{B(0,1)})}$ is compact, where $\overline{B(0,1)}$ is the closed unit-ball, since there always is a closed unitball in a Banach space. This means that the closed subset $\overline{B(y,r)}$ is compact as well, by the first assignment problem 3(e). Then we can conclude that $Y$ has a compact unit-ball, and hence $Y$ is finite-dimensional, which means that no $T \in \mathcal{K}(X, Y)$ can be onto.

## f)

$M$ is self-adjoint by the definition of a Hilbert space adjoint, since $M^* = W^{-1} M^{\dagger} V = M$, where $V : H \to H^*$ an $W : H \to H^*$ by remark 7.15 in the lecture notes. And it is onto by definition, so it cannot be compact by (e).

## Problem 3

### a)

$T = T_{\tilde{k}}$ for $\tilde{k}(s,t) = k(t,s)$

We notice that $(Tf)(s)$ is the same function used in proposition 9.12, which gives us that $T$ is Hilber-Schmidt and hence compact, for $K \in L_2([0,1] \times [0,1], m)$

← you need to show this then.

### b)

By theorem 8.8 we have that $T^*$ is compact, since $T$ is compcat by (a). Then by the note in HW. 6 problem 4 (c), we are in the case where $\mathcal{K}(H,H)$ is a closed two sided ideal in $\mathcal{L}(H,H)$. Then by theorem 7.12 we get that $TSf = STf$, where there exists $f \in H$ such that $Tf = f$ for all $T \in \mathcal{L}(H,H)$, where $S \in \mathcal{L}(H,H)$, then we can take the adjoint $T^*$ on both sides and get that $T^*TSf = T^*STf = T^*Sf \Leftrightarrow Sf = ST^*f \Leftrightarrow T^*f = f = Tf$ for all $T \in \mathcal{L}(H,H)$, this means that $T = T^*$ as we wanted.

what is $S$?

true anyway?

NO!

Does not make sense to me.

### c)

Since $(Tf)(s) = \int_{[0,1]} K(s,t)f(t)\,dm(t)$, $s \in [0,1]$, $f \in H$, we can split the integral up in the two cases where $t \in [0,s]$ and $t \in [s,1]$, which gives that $(Tf)(s) = \int_{[0,s]}(1-s)tf(t)\,dm(t) + \int_{[s,1]}(1-t)sf(t)\,dm(t)$, $s \in [0,1]$, $f \in H$, and since we integrate over $t$ and not $s$ or $(1-s)$, we can move them out of the integrals so we get that $(Tf)(s) = (1-s)\int_{[0,s]} tf(t)\,dm(t) + s\int_{[s,1]}(1-t)f(t)\,dm(t)$, $s \in [0,1]$, $f \in H$. ✓

For $Tf$ to be continuous on $[0,1]$ we need that $\lim_{s\to p^\pm}(Tf)(s) = (Tf)(p)$ for every $p \in (0,1)$, which is obviesly true by the construction of (Tf)(s), since $\lim_{s\to p^\pm}(Tf)(s) = (1-p^\pm)\int_{[0,p^\pm]} tf(t)\,dm(t) + p^\pm\int_{[p^\pm,1]}(1-t)f(t)\,dm(t)$, where $s$ would be in $[p^+,1]$ for $\lim_{s\to p^+}(Tf)(s)$, and $s$ would be in $[0,p^-]$ for $\lim_{s\to p^-}(Tf)(s)$, and that $\lim_{s\to o^+}(Tf)(s) = (Tf)(0)$, where $\lim_{s\to o^+}(Tf)(s) = (1-0^+)\int_{[0,0^+]} tf(t)\,dm(t) + 0^+\int_{[0^+,1]}(1-t)f(t)\,dm(t) = (Tf)(0) = 0$, since the first integral would be an integral over one single point $\{0\}$ which would be equal to 0,

This is not obvious.

and that $\lim_{s\to 1^-}(Tf)(s) = (1-1^-)\int_{[0,1^-]} tf(t)\,dm(t) + 1^-\int_{[1^-,1]}(1-t)f(t)\,dm(t) = (Tf)(1) = 0$, since the second integral would become the integral over a single point $\{1\}$ which would then be 0, and since $(1-1^-)\int_{[0,1^-]} tf(t)\,dm(t) = 0$

(✓)

# Problem 4

$S(\mathbb{R})$ is the Schwarz space.

## a)

Schwarts space and Fourier transform. Lecture 11 (combination of prop. 11.4 and 11.13 (e)) (remark 11.12 (a))

By proposition 11.4 in the lecture notes, we have that for $g_0(x) = \varphi(x) = e^{-\frac{1}{2}x^2}, x \in \mathbb{R}$, then $\hat{g}_0(\xi) = \hat{\varphi}(\xi) = e^{-\frac{1}{2}\xi^2}, \xi \in \hat{\mathbb{R}}$, since we are in the case where $n = 1$ for $\mathbb{R}^n$ in the formula. Then by proposition 11.13 (e) in the lecture notes we have that $\varphi(x) \in S(\mathbb{R})$, where $\hat{\varphi}(x) \subset S(\mathbb{R})$. And then by remark 11.12 (a) in the lecture notes we have that $\hat{\varphi}(x) \in S(\mathbb{R}) = e^{-\frac{1}{2}x^2}$ implies that $x^\alpha \partial^\beta \hat{\varphi}(x) \in S(\mathbb{R})$, for all multi-indices $\alpha, \beta$. And this means that $x^k e^{-\frac{1}{2}x^2} \in S(\mathbb{R})$ for all integers $k \geq 0$, when we let $\alpha = k$ and $\beta = 0$, since $\hat{\varphi} = \varphi$ in this case by eksample 11.3 in the lecture notes. So we have that $g_k(x) = x^k \varphi(x)$ which is Schwarts for all integers $k \geq 0$ by HW7 problem 1.

you don't show $\varphi(x) \in S(\mathbb{R})$

Then we have that $\mathcal{F}(g_k(x)) = \mathcal{F}(x^k e^{-\frac{1}{2}x^2}) = (x^k\varphi)^\wedge(\xi) = i^k(\partial^k\hat{\varphi})(\xi), \xi \in \mathbb{R}$, by proposition 11.13 (d) in the lecture notes. So for $k = 0$ we have that $\mathcal{F}(g_0) = \mathcal{F}(\varphi) = \mathcal{F}(e^{-\frac{1}{2}x^2}) = (x^0\varphi)^\wedge(\xi) = i^0(\partial^0\hat{\varphi})(\xi) = \hat{\varphi}(\xi) = e^{-\frac{1}{2}\xi^2}$, by the proof in propositione 11.4 in the lecture notes, and by theorem 11.13 (d) so $\mathcal{F}(g_0) = \hat{\varphi}$,
and for $k = 1$: $\mathcal{F}(g_1) = (x^1\varphi)^\wedge(\xi) = i^1(\partial^1\hat{\varphi})(\xi) = -i\xi e^{-\frac{1}{2}\xi^2}$,
and for $k = 2$: $\mathcal{F}(g_2) = (x^2\varphi)^\wedge(\xi) = i^2(\partial^2\hat{\varphi})(\xi) = (-1)(-e^{-\frac{1}{2}\xi^2} + \xi^2 e^{-\frac{1}{2}\xi^2}) = e^{-\frac{1}{2}\xi^2} - \xi^2 e^{-\frac{1}{2}\xi^2}$,
and for $k = 3$ $\mathcal{F}(g_3) = (x^3\varphi)^\wedge(\xi) = i^3(\partial^3\hat{\varphi})(\xi) = -i(3\xi e^{-\frac{1}{2}\xi^2} - \xi^3 e^{-\frac{1}{2}\xi^2}) = i\xi e^{-\frac{1}{2}\xi^2}(\xi^2 - 3)$.

✓

## b)

By (a) we have that $\mathcal{F}(g_0) = e^{-\frac{1}{2}\xi^2} = i^0 g_0 \Rightarrow h_0 = g_0$, which is non-zero, ✓

$\mathcal{F}(g_1) = -i\xi e^{-\frac{1}{2}\xi^2} = -ig_1 = -i^1 g_1 \Rightarrow h_1 = -g_1$, $h_3 = g_1$ ✓ This will not work.

$\mathcal{F}(g_2) = e^{-\frac{1}{2}\xi^2} - \xi^2 e^{-\frac{1}{2}\xi^2} = g_0 - g_2 = -i^2(g_0 - g_2) \Rightarrow h_2 = -(g_0 - g_2) = g_2 - g_0$, and that
$\mathcal{F}(g_3) = -i(3\xi e^{-\frac{1}{2}\xi^2} - \xi^3 e^{-\frac{1}{2}\xi^2}) = -i(3g_1 - g_3) = i^3(3g_1 - g_3) \Rightarrow h_3 = 3g_1 - g_3$.
Which is all non-zero since $g_k(x) \neq 0$ for $k = 0, 1, 2, 3$.

## c)

We have that $f \in S(\mathbb{R})$, so $f \in C^\infty(\mathbb{R})$ by definition 11.10 in the lecture notes. In general we have that $\mathcal{F}^2(f) = \mathcal{F}(\mathcal{F}(f)) = \mathcal{F}(\hat{f}) = \int_\mathbb{R} \hat{f}(\xi)e^{-i\xi\tau} \, \mathrm{dm}(\xi) = \int_\mathbb{R}\int_\mathbb{R} f(x)e^{-ix\xi} \, \mathrm{dm}(x) \; e^{-i\xi\tau} \, \mathrm{dm}(\xi) =$
c/c $\int_\mathbb{R} f(x) \, \mathrm{dm}(x) \; \int_\mathbb{R} e^{-ix\xi}e^{-i\xi\tau} \, \mathrm{dm}(\xi)$ for $x \in \mathbb{R}, \xi \in \hat{\mathbb{R}}, \tau \in \hat{\hat{\mathbb{R}}}$. This map takes $\hat{f} \in L_1(\hat{\mathbb{R}})$ and sends it into $C_0(\mathbb{R})$ by definition of the fourier transform $\mathcal{F}$, since $f \in S(\mathbb{R}) \subset L_1(\mathbb{R})$ and since $\mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R})$. Then by definition 12.10 in the lecture notes we have that $\int_\mathbb{R} \hat{f}(\xi)e^{-i\xi\tau} \, \mathrm{dm}(\xi) = \check{f}(\tau) = \mathcal{F}^*(\hat{f}), \tau \in \mathbb{R}$. c/e

Furthermore we have that for $f$ being a Schwartz-function, then by corollary. 12.14 we can find a Schwartz-function $g$ such that $\mathcal{F}(f) = \hat{f} = g$, which is an isomorphism?, then we can calculate $\mathcal{F}^2(f)$ by using that $\mathcal{F}^*(g) = f$, so $\mathcal{F}^2(f)(\xi) = \mathcal{F}(\mathcal{F}(f))(\xi) = \mathcal{F}(\mathcal{F}(\mathcal{F}^*(g)))(\xi) = \mathcal{F}(g)(\xi) = \int_\mathbb{R} g(x)e^{-ix\xi} \, \mathrm{dm}(x) = \mathcal{F}^*(g)(-\xi) = f(-\xi)$, this means that $\mathcal{F}^4(f)(\xi) = (\mathcal{F}^2(\mathcal{F}^2(f)))(\xi) = \mathcal{F}^2(f)(-\xi) = f(\xi)$, by corollary 12.12 (iii) in the lecture notes, this means that $\mathcal{F}^4(f) = f$ for all $f \in S(\mathbb{R})$. ✓

## d)

Take $f \in S(\mathbb{R})$ non-zero, with $\mathcal{F}(f) = \lambda f$ for some $\lambda \in \mathbb{C}$. This means that $\mathcal{F}^2(f) = \mathcal{F}(\lambda f) = \lambda^2 f$, and that $\mathcal{F}^4(f) = \lambda^4 f = f$, which implies that $\lambda^4 = 1 \Leftrightarrow \lambda \in \{1, i, -1, -i\}$, and by (b) we have that $\mathcal{F}(h_k) = i^k h_k = \lambda h_k$, which for $h = 0$ gives that: $\mathcal{F}(h_0) = i^0 h_0 = h_0$ and for ✓
$h = 1$: $\mathcal{F}(h_1) = i^1 h_1 = ih_1$,
$h = 2$: $\mathcal{F}(h_2) = i^2 h_2 = -h_2$ and lastly for

✓

$h = 3$: $\mathcal{F}(h_3) = i^3 h_3 = -ih_3$.
So since $\mathcal{F}^4(f) = f$ for all $f \in S(\mathbb{R})$ we have that $\mathcal{F}^5(f) = \mathcal{F}(f)$, so $\{1, i, -1, -i\}$ are precisely the the eigenvalues of $\mathcal{F}$.

what does this have to do with the rest?

## Problem 5

We have the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0,1]$, where $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n} = \frac{1}{2}\delta_{x_1} + \frac{1}{2^2}\delta_{x_2} + \cdots + \frac{1}{2^n}\delta_{x_n} + \cdots$ on $[0,1]$. Then by problem 3 HW.8 we have that $\text{supp}\mu = \{x\}$ for some $x \in [0,1] \Leftrightarrow \mu = c\delta_x$ for some $c > 0$, where $\delta_x$ is the Dirak mass at $x$. This combined with the fact that $(x_n)_{n\geq 1}$ is dense in $[0,1]$, gives us that $\text{supp}(\mu) = \text{supp}(\sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}) = \text{supp}(\sum_{n=1}^{\infty} \mu_n) = \sum_{n=1}^{\infty} \text{supp}(\mu_n) = \sum_{n=1}^{\infty}\{x_n\} = [0,1]$, for $\mu_n = c\delta_{x_n}$.