# Mandatory Assignment 2

# Functional Analysis


Katrine Fredensborg Dedenroth

Student ID: cph273


**Problem 1** Let $H$ be an infinite dimensional separable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$, for all $n \geq 1$. why?

a) We wish to show, that $f_N \to 0$ weakly, as $N \to \infty$. Let $x = \sum_{n=1}^{\infty} a_n e_n$ be any element of $H$. Then $\langle x, f_N\rangle = N^{-1}\sum_{n=1}^{N^2} a_n$. Given that $a_n \in \ell_2$ we must show, that $N^{-1}\sum_{n=1}^{N^2} a_n \to 0$. Let $\epsilon > 0$. Choose $m$ such that $\sum_m^{\infty}|a_n|^2 < \epsilon$. Since $N^{-1}\sum_1^{m-1} a_n \to 0$ if suffices to show that $N^{-1}\sum_m^{N^2} a_n \to 0$.. 3?

Let $K$ be the norm closure of $\text{co}\{f_N : N \geq 1\}$.

b) We wish to argue that $K$ is weakly compact, and that $0 \in K$. Hilbert spaces are reflexive and by Alaoglu's Theorem any weakly bounded sets in them are weakly compact. Moreover, the weak closure of a convex set is the same as its norm closure. It follows that $K$ is weakly closed and bounded, hence weakly compact. Since $0$ is in the weak closure of $\{f_N : N \geq 1\}$, it is also in the weak closure of its convex hull, hence $0$ is in the norm closure of $\text{co}\{f_N : N \geq 1\}$. Thus, we have $0 \in K$. ✓

bounded and weakly closed.

c) We wish to show that $0$, as well as each $f_N$, $N \geq 1$, are an extreme point in $K$.

First, we wish to show, that $0 \in \text{Ext}(K)$. Assume for contradiction, that $0 \notin \text{Ext}(K)$. Then we can write $0$ as a non-trivial convex combination of distinct elements of $K$, that is, $0 = pu + qv$ for $u, v \in K$ for which $u \neq v$, and $p, q > 0$ for which $p + q = 1$. But every element of $K$ is contained in the intersection of the closed convex half spaces $H_k = \{u \in H : \langle e_k, u\rangle \geq 0\}$. Why? Thus, for each $k$ we have $u, v \in H_k$, hence $0 = p\langle e_k, u\rangle + q\langle e_k, v\rangle$. Since $0$ is an extreme point of the set of non-negative reals, this imply that $u = v = 0$, which is a contradiction to the supposition. Thus, $0 \in \text{Ext}(K)$. ✓

d) We wish to find out whether there are any other extreme points in $K$ or not.

Let $F = \{f_N\} \cup \{0\}$. By c) we have that $F \subseteq \mathrm{Ext}(K)$. Since we know $0 \in K$, we see that the closed convex hull of $F$ is equal to $K$. If there was any extreme point $e \in \mathrm{Ext}(K)$ not in $F$, then we could strictly separate $e$ from the closed convex hull of $F$ with a hyperplane. But $K$ is the closed convex hull of $F$ and by the Krein-Milman Theorem also of $\mathrm{Ext}(K)$, so this cannot be. Hence, there would be a contradiction. Why? Need details!

**Problem 2** Let $X$ and $Y$ be infinite dimensional Banach spaces.

a) Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in $X$ and $x \in X$, we wish to show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

We know by problem 2 HW4 that $x_n \to x$ weakly, as $n \to \infty$, holds if and only if $Fx_n \to Fx$ for all $F \in X^*$. Now, take $G \in Y^*$. Then the composition $G \circ T \in X^*$ meaning $(G \circ T)(x_n) \to (G \circ T)(x)$ as $n \to \infty$ for all $G \in Y^*$. This means exactly that $Tx_n \to Tx$ weakly, as $n \to \infty$. ✓

b) Let $T \in \mathcal{K}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in $X$ and $x \in X$, we wish to show that $x_n \to x$ weakly, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$.

Suppose for $x \in X$ that $x_n \to x$ weakly, as $n \to \infty$, and that $\|Tx_n - Tx\| \nrightarrow 0$, as $n \to \infty$. Then there exists a subsequence $\left(x_{n_k}\right)_{k\geq 1}$ and $\epsilon > 0$ such that $\left\|Tx_{n_k} - Tx\right\| > \epsilon$ for all $k \geq 1$. Since $x_n \to x$ weakly, as $n \to \infty$, then $x_{n_k} \to x$ weakly, as $k \to \infty$, whereas $\left(x_{n_k}\right)_{k\geq 1}$ is bounded. This means that it has a subsequence $\left(x_{n_{k_i}}\right)_{i\geq 1}$ such that $\left\|Tx_{n_{k_i}} - Tx'\right\| \to 0$, as $i \to \infty$, for some $x' \in X$. Why is $(Tx_{n_{k_i}})$ convergent? And why is the limit inside $T(X)$?
Now, since $x_{n_k} \to x$ weakly, as $k \to \infty$, then by 2(a), $Tx_{n_k} \to Tx$ weakly, as $k \to \infty$, and so especially $Tx_{n_{k_i}} \to Tx$ weakly, as $i \to \infty$. However, if something converges by norm to something, then it must weakly converge to the same thing. This follows from the fact, that if $(y_n)_{n\geq 1}$ is in some Banach space $Y$, then for all $G \in Y^*$ we have $|Gy_n - Gy| \leq C\|y_n - y\|$ for some constant $C > 0$, so if $\|y_n - y\| \to 0$ then $|Gy_n - Gy| \to 0$, meaning $y_n \to y$ weakly, as $n \to \infty$. Hence, we can conclude, that $Tx' = Tx$, which means that $\left\|Tx_{n_{k_i}} - Tx\right\| \to 0$ as $i \to \infty$, but this contradicts the fact, that $\left\|Tx_{n_k} - Tx\right\| > \epsilon$ for all $k \geq 1$, and so we must have that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$. (✓)

c) Let $H$ be a separable infinite dimensional Hilbert space. We wish to show that if $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in $H$ converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Take $T \in \mathcal{L}(H, Y)$ such that whenever $(x_n)_{n\geq 1} \in X$ satisfies $x_n \to x$ weakly as $n \to \infty$, then $\|Tx_n - Tx\| \to 0$. Furthermore, suppose that $T$ is not compact. This holds if and only if

$T(B_X(0,1))$ is not totally bounded, i.e. There exists $\delta > 0$ such that every finite union of open balls with radius $\delta$ does not cover $T(B_X(0,1))$.
Define a sequence $(x_n)_{n\geq 1}$ recursively. Now, we take $x_1 \in B_X(0,1)$. Suppose we found $x_2, x_3, \dots, x_n$ such that $\|Tx_q - Tx_r\| \geq \delta$ for all $q, r \leq n$, $q \neq r$. Now, consider the set

$$T(B_X(0,1)) \cap \left( \bigcup_{i=1}^{n} B_Y(Tx_i, \delta) \right)^c .$$

$T(B_X(0,1))$

This is non-empty, or else $T(B_X(0,1)) \subset \bigcup_{i=1}^{n} B_Y(Tx_i, \delta)$, but this is not true, since $T$ is not totally bounded. Thus, we may pick $x_{n+1} \in B_X(0,1)$ such that $Tx_{n+1} \in T(B_X(0,1)) \cap (\bigcup_{i=1}^{n} B_Y(Tx_i, \delta))^c$. So $Tx_{n+1} \in (\bigcup_{i=1}^{n} B_Y(Tx_i, \delta))^c = \bigcap_{i=1}^{n} (B_Y(Tx_i, \delta))^c$, which means that $Tx_{n+1} \notin B_Y(Tx_i, \delta)$ for all $i \geq n$, meaning $\|Tx_{n+1} - Tx_i\| \geq \delta$ for all $i \geq n$. Continuing this way, we obtain a sequence $(x_n)_{n\geq 1}$ such that $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$.

H

As $X$ is a Banach space, then so is $X^*$, and by Alaoglu's Theorem we may conclude, that the closed unit ball $\bar{B}_{X^{**}}(0,1)$ is compact in the $w^*$-topology. As $X$ is reflexive then $X^{**}$ is separable, thus, $X^*$ is separable. By Theorem 5.13 in the lecture notes, we get that $(\bar{B}_{X^{**}}(0,1), \tau_{w^*})$ is metriziable. So as $\bar{B}_{X^{**}}(0,1)$ is compact in the $w^*$-topology, then it is also sequentially compact. in the $w^*$-topology.
Now, consider $(z_n)_{n\geq 1} \in \bar{B}_X(0,1)$ then $(\hat{z}_n)_{n\geq 1} \in \bar{B}_{X^{**}}(0,1)$. As $\bar{B}_{X^{**}}(0,1)$ is sequentially compact in the $w^*$-topology, then $(\hat{z}_n)_{n\geq 1}$ has a convergent subsequence $(\hat{z}_{n_k})_{k\geq 1}$, i.e. $\hat{z}_{n_k} \to \hat{z}$ as $k \to \infty$ is the $w^*$-topology. This holds if and only if $f(\hat{z}_{n_k}) = \hat{z}_{n_k}(f) \to \hat{z}(f) = f(z)$ for all $f \in X^*$ as $k \to \infty$, meaning $\bar{B}_X(0,1)$ is weakly sequentially compact.
As $\bar{B}_X(0,1)$ is weakly sequentially compact, we let $(\hat{z}_{n_k})_{k\geq 1}$ be the weakly convergent subsequence of $(x_n)_{n\geq 1}$. However as $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$, then $\|Tx_{n_k} - Tx\| \nrightarrow 0$ as $k \to \infty$. But this is a contradiction and hence, $T$ must be compact.

a

d) We wish to show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact. weakly, I assume?

Let $(x_n)_{n\geq 1} \in X$ and suppose further that $x_n \to x$ as $n \to \infty$. Then by a) we know that $Tx_n \to Tx$ weakly in $\ell_1(\mathbb{N})$, but by remark 5.3 we know that this holds if and only if $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. This means by c) that $T$ is compact.

e) We wish to show that no $T \in \mathcal{K}(X, Y)$ is onto. Suppose that $T \in \mathcal{L}(X, Y)$ is compact and onto. By the Open mapping Theorem $T$ is open. As $X, Y$ are normed vector spaces and $T$ is open then there exists $r > 0$ such that $B_Y(0, r) \subset T(B_X(0,1))$. As closure preserves inclusion, we get $\overline{B_Y(0, r)} \subset \overline{T(B_X(0,1))}$. Since $T$ is a compact operator, then $\overline{T(B_X(0,1))}$ is compact, thus, $\overline{B_Y(0, r)}$ is compact. Now, lets consider different values of $r$, and see what happens.
$r = 1$: Then we have $\overline{B_Y(0, r)} = \overline{B_Y(0,1)}$ is compact, which is a contradiction by Mandatory Assignment 1.
$r > 1$: Then $\overline{B_Y(0,1)}$ is a closed set of the compact set $\overline{B_Y(0, r)}$, meaning that $\overline{B_Y(0,1)}$ , which is a contradiction by Mandatory assignment 1.

$r < 1$: consider $f: Y \to Y$ by $x \to \frac{1}{r}x$, which is clearly continuous. Since the image under a continuous function of a compact set is compact, then we get that $f\left(\overline{B_Y(0,r)}\right) = \overline{B_Y(0,1)}$ is compact, which is a contradiction by Mandatory Assignment 1.

Thus, no $T \in \mathcal{K}(\mathrm{X},\mathrm{Y})$ is open, and hence, no $T \in \mathcal{K}(\mathrm{X},\mathrm{Y})$ is onto, by the Open mapping Theorem. ✓

**Problem 3** Consider the Hilbert space $H = L_2([0,1], m)$, where $m$ is the Lebesgue measure. Define $K : [0,1] \times [0,1] \to \mathbb{R}$ by

$$K(s,t) = \begin{cases} (1-s)t, & \text{if } 0 \le t \le s \le 1 \\ (1-t)s, & \text{if } 0 \le s < t \le 1, \end{cases}$$

and consider $T \in \mathcal{L}(H,H)$ defined by

$$(Tf)(s) = \int_{[0,1]} K(s,t) f(t) dm(t), \qquad s \in [0,1], \qquad f \in H.$$

a) We wish to justify, that $T$ is compact. First, we show that $K$ is continuous, and this is equivalent to showing that the function $s \mapsto K(s,t)$ is continuous for all $t \in [0,1]$, and that the function $t \mapsto K(s,t)$ is continuous for all $s \in [0,1]$. For a given $t \in [0,1]$ consider the function $K_t : [0,1] \to \mathbb{R}$ given by $K_t(s) = K(s,t)$. This is easily seen to be continuous when restricted to either $[0,t)$ or $(t,1]$. Furthermore, we notice that $K_t(s) \to (1-t)t$, when $s$ approaches $t$ from both the left and the right. Thus, $K_t$ is continuous, and due to the definition of $K$, continuity of $t \mapsto K(s,t)$ is shown in a similar way. Hence $K$ is contiuous, and we can conclude (by the lectures), that $T$ is compact.

By what result

you should argue that $[0,1]$ is locally compact Hausdorff + finite measure space. + $T = T_K$

b) We wish to show, that $T = T^*$. Since $H$ is a Hilbert space, we know that $T = T^*$ is equivalent to $\langle Tf, g\rangle = \langle f, Tg\rangle$, for any $f, g \in H$. Let $f, g \in H$ be given and consider

$$\langle Tf, g\rangle = \int_{[0,1]} \left( \int_{[0,1]} K(s,t) f(t) dm(t) \right) \overline{g(s)} dm(s)$$
$$= \int_{[0,1]} \int_{[0,1]} K(s,t) f(t)\, \overline{g(s)} dm(t) dm(s).$$

No, integrable

In order of using Fubini's Theorem on changing the order of integration we need to show that $K(s,t)f(t)\overline{g(s)}$ is measurable on $[0,1] \times [0,1]$ with the corresponding Lebesgue measure. Since $K$ is continuous it is also measurable, and thus we only need look at $f(t)\overline{g(s)}$. We know that the function $h : [0,1] \times [0,1] \to \mathbb{R}$ defined by $h(s,t) = f(t)\overline{g(s)}$ is measurable if the integral

This is true in general.

$$\int_{[0,1]} \int_{[0,1]} f(t)\, \overline{g(s)} dm(t) dm(s) = \int_{[0,1]} \left( \int_{[0,1]} f(t) dm(t) \right) \overline{g(s)} dm(s)$$

Don't need this

exists. Since $f \in H = L_2([0,1], m)$, we know from HW that we also have $f \in L_1([0,1], m)$. Thus,

$$\left|\int_{[0,1]}\left(\int_{[0,1]} f(t)dm(t)\right)\overline{g(s)}dm(s)\right| \leq \int_{[0,1]}\left(\int_{[0,1]} f(t)dm(t)\right)\overline{g(s)}dm(s)$$
$$= \|f\|_1 \int_{[0,1]} g(s)dm(s) = \|f\|_1\|g\|_1 < \infty.$$

This does not show $k(s,t)f(t)\overline{g(s)} \in L_1([0,1]^2)$

Hence $h$ is measurable and $K(s,t)h(s,t) = K(s,t)f(t)\overline{g(s)}$ is measurable on $[0,1]\times[0,1]$. Using Fubini's Theorem we see

$$\langle Tf,g\rangle = \int_{[0,1]}\int_{[0,1]} K(s,t)f(t)\,\overline{g(s)}dm(s)dm(t)$$
$$= \int_{[0,1]} f(t)\overline{\int_{[0,1]} K(s,t)g(s)dm(s)}\,dm(t).$$

Now, it suffices to show, that for any $s_0, t_0 \in [0,1]$, $K(s_0,t_0) = K(t_0,s_0)$. But due to the way $K$ is defined, this is clear. Thus, we see

$$\langle Tf,g\rangle = \int_{[0,1]} f(t)\overline{\int_{[0,1]} K(s,t)g(s)dm(s)}\,dm(t)$$
$$= \int_{[0,1]} f(t)\overline{\int_{[0,1]} K(t,s)g(s)dm(s)}\,dm(t) = \langle f,Tg\rangle.$$

Hence, we have $T = T^*$.

c) We wish to show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t), \qquad s\in[0,1], \qquad f\in H.$$

Let $s\in[0,1]$ be given. Then we know

$$\int_{[0,1]} K(s,t)f(t)dm(t) = \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t).$$

By the definition of $K$ we then get that

$$\int_{[0,1]} K(s,t)f(t)dm(t) = \int_{[0,s]}(1-s)tf(t)dm(t) + \int_{[s,1]}(1-t)sf(t)dm(t)$$
$$= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]}(1-t)f(t)dm(t),$$

which is what we wanted.

We now wish to use the above to show that $Tf$ is continuous on $[0,1]$. For $Tf$ to be continuous it now suffices to show, that the following functions are continuous

$$s\mapsto \int_{[0,s]} tf(t)dm(t) \text{ and } s\mapsto\int_{[s,1]}(1-t)f(t)dm(t).$$

Given $s, s_0\in[0,1]$ we have

$$\left|\int_{[0,s]} tf(t)dm(t) - \int_{[0,s_0]} tf(t)dm(t)\right| \leq \int_{[s_0,s]}|tf(t)|dm(t) \leq \int_{[s_0,s]}|f(t)|dm(t)$$
$$\leq \|f\|_1 < \infty.$$

Thus,

$$\int_{[s_0,s]}|f(t)|dm(t) \to 0 \text{ as } s\to s_0,$$

why?

This has not been shown

hence, the function $s \longmapsto \int_{[0,s]} tf(t)dm(t)$ is continuous.
We can show that $s \longmapsto \int_{[s,1]}(1-t)f(t)dm(t)$ is continuous in a similar way. Thus, $Tf$ is composed of continuous functions, and hence, $Tf$ is continuous itself.

Now, we wish to show that $(Tf)(0) = (Tf)(1) = 0$. If we have $s = 0$, then the first integral of $(Tf)(s)$ will be an integral of an $\mathcal{L}_2$-function on a null-set which is 0, and the other integral will be multiplied by a zero, and hence we have a sum of two zeros, and $(Tf)(0) = 0$. Choosing $s = 1$ we would get a similar result and thus, $Tf(1) = 0$.