Homework for week 2, FunkAn, Fall 2018

Problem 1: Recall the Riesz representation theorem for bounded linear functionals on Hilbert spaces, namely, that if H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $F\colon H \to \mathbb{C}$ is a bounded linear functional on H, then there exists $y \in H$ such that

$$F(x) = \langle x, y \rangle, \quad x \in H\,.$$

Prove this statement, and use it further to show the following uniqueness result for the Hahn-Banach extension theorem in the case of closed subspaces of Hilbert spaces and seminorm p given by the norm:

Let H be a Hilbert space and M a closed subspace of H. If f is a bounded linear functional on M, then there is a unique extension F of f to H such that $\|F\| = \|f\|$.

Problem 2: [For this exercise you should first recall Hölder's inequality (see, e.g., Section 6.1 in Folland).] Let $1 < q < r < \infty$ and use Hölder's inequality to show that the following inclusions hold. Give examples proving that these inclusions are strict.

(a) $l_1(\mathbb{N}) \subsetneq l_q(\mathbb{N}) \subsetneq l_r(\mathbb{N}) \subsetneq l_\infty(\mathbb{N})\,.$

(b) $L_\infty([0,1],m) \subsetneq L_r([0,1],m) \subsetneq L_q([0,1],m) \subsetneq L_1([0,1],m)$, where m stands for Lebesgue measure.

Do the inclusions in (b) remain valid if we replace the measure space $([0,1],m)$ by $(\mathbb{R},m)$?

Next, suppose that $f \in L_p(\mathbb{R},m) \cap L_\infty(\mathbb{R},m)$, for some $1 \le p < \infty$. Show that $f \in L_q(\mathbb{R},m)$, for all $q > p$.

If time permits, then also show that the inclusion maps in (a) are continuous and compute their norms.

Problem 3: Define $T\colon c(\mathbb{N}) \to c_0(\mathbb{N})$ by setting $Tx = y$, for every $x = (x_n)_{n\ge1} \in c(\mathbb{N})$, where $y = (y_n)_{n\ge1}$ is given by

$$y_1 = \lim_{n\to\infty} x_n, \quad y_{n+1} = x_n - y_1, \quad n \ge 1\,.$$

(a) Prove that T is a one-to-one and onto linear operator.

(b) Show that T is an isomorphism (of Banach spaces) and compute $\|T\|$ and $\|T^{-1}\|$.

(c) Let $F\colon c(\mathbb{N}) \to c_0(\mathbb{N})$ be any linear isomorphism. Show that there exist $u, v \in c(\mathbb{N})$ such that $\|u\| = \|v\| = 1$ and $\|u+v\| = 2$, and moreover, with the property that, if for $n \in \mathbb{N}$ one has $|F(u)(n)| \ge 1/2$, then $F(v)(n) = 0$. Use this to conclude that F cannot be an isometry.
(Hint: Try with $u = (1,1,1,\cdots)$ and a suitably chosen $v \in c_0(\mathbb{N}) \subseteq c(\mathbb{N})$.)

Remark: The purpose Problem 3 is to justify the assertion that $c(\mathbb{N})$ and $c_0(\mathbb{N})$ are isomorphic, but not *isometrically* isomorphic. Yet, as we saw already, they have the same dual space, namely $l_1(\mathbb{N})$ (cf. Remark 1.15, Lecture 1).

Problem 4: Let $\iota\colon l_1(\mathbb{N}) \to (l_\infty(\mathbb{N}))^*$ be the canonical map given by

$$\iota(x)(y)\colon = \sum_{n=1}^{\infty} x_n y_n\,, \quad x = (x_n)_{n\geq 1} \in l_1(\mathbb{N}),\; y = (y_n)_{n\geq 1} \in l_\infty(\mathbb{N})\,.$$

Prove that ι is an isometry which is not onto.

Hint: In order to show that ι is not onto, consider the linear functional $f : c(\mathbb{N}) \to \mathbb{C}$ given by $f((x_n)_{n\geq 2}) = \lim_{n\to\infty} x_n$, for all $(x_n)_{n\geq 1} \in c(\mathbb{N})$. Apply the Complex Hahn-Banach extension Theorem (or rather, its corollary from Lecture 2) to obtain a linear extension F of f to $l_\infty(\mathbb{N})$ with $\|F\| = \|f\|$. Show that $F \in (l_\infty(\mathbb{N}))^* \setminus \iota(l_1(\mathbb{N}))$.

The problem below contains further examples of Banach spaces. It will be covered in the exercise classes if time permits.

Problem 5: Let X be a locally compact Hausdorff topological space.

(a) Show that $(C_b(X), \|\cdot\|_\infty)$ is a Banach space.

(b) Prove that $(C_0(X), \|\cdot\|_\infty)$ is a closed subspace of $C_b(X)$, and hence, a Banach space.

[The relevant spaces above were introduced in Lecture 1.]