QUADRATIC FORMS FOR THE FERMIONIC UNITARY GAS MODEL

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We consider a quantum system in dimension three composed by a group of N identical fermions, with mass $1/2$, interacting via zero-range interaction with a group of M identical fermions of a different type, with mass $m/2$. Exploiting a renormalization procedure, we construct the corresponding quadratic form and define the so-called Skornyakov–Ter-Martirosyan extension H_α, which is the natural candidate as a possible Hamiltonian of the system. It is shown that if the form is unbounded from below then H_α is not a self-adjoint and bounded from below operator, and this in particular suggests that the so-called Thomas effect could occur. In the special case $N = 2$, $M = 1$ we prove that this is in fact the case when a suitable condition on the parameter m is satisfied.

Keywords: zero-range interactions, unitary gas, Skornyakov–Ter-Martirosyan extension.

1. Introduction

In many models in condensed matter physics and statistical mechanics, a gas of n quantum particles in $\mathbb{R}^3$ is described through the formal Hamiltonian

$$H = -\sum_{i=1}^{n} \frac{1}{2m_i}\Delta_{x_i} + \mu \sum_{\substack{i,j=1 \\ i<j}}^{n} \delta(x_i - x_j) \tag{1.1}$$

where $x_i \in \mathbb{R}^3$, m_i is the mass of the i-th particle, Δ_{x_i} is the free Laplacian relative to the coordinate x_i and $\mu \in \mathbb{R}$ is the strength of the δ, or zero-range, interaction acting between each pair of particles of the gas. To simplify the notation we fix $\hbar = 1$.

One reason of interest for the Hamiltonian (1.1) is that it is a simple but nontrivial modification of the free Hamiltonian and then it can be used for concrete computations of relevant physical properties of the quantum gas. It is worth mentioning that in recent years these models have been widely used in the physical literature. They are studied for systems of bosons or fermions, possibly with harmonic confining potential, in the unitary limit, i.e. for infinite two-body scattering length (see the review [3] and also [5, 19, 20]).

From the mathematical point of view a Hamiltonian of the type (1.1) in the appropriate Hilbert space is defined as a self-adjoint extension of the free Hamiltonian restricted to a domain of smooth functions vanishing on each hyperplane $x_i = x_j$. The most used techniques for the concrete construction of such extensions are Krein's theory of self-adjoint extensions and limiting procedure of smooth approximating Hamiltonians (in the sense of the resolvent or the quadratic form).

The problem is completely understood in the case $n = 2$, where one is reduced to study a fixed δ interaction in the relative coordinate and all the self-adjoint extensions can be explicitly constructed (we refer to [1] for a complete mathematical analysis of this case).

A direct generalization of the same construction to the case $n > 2$ naturally leads to the definition of the so-called Skornyakov–Ter-Martirosyan extension H_α. Roughly speaking, such extension is a symmetric operator acting on a set of functions ψ which are smooth outside the hyperplanes $x_i = x_j$, $i, j = 1, \dots, n$, while on each hyperplane they exhibit the following singular behaviour

$$\psi \simeq \frac{\mathfrak{f}_{ij}}{|x_i - x_j|} + \alpha\, \mathfrak{f}_{ij} + o(1) \quad \text{for } |x_i - x_j| \to 0, \tag{1.2}$$

where $\mathfrak{f}_{ij}$ is a function defined on the hyperplane $x_i = x_j$ and α is a real parameter. One can see that α^{-1} is proportional to the two-body scattering length and therefore the unitary limit is obtained for $\alpha = 0$.

As a matter of fact the operator H_α is not self-adjoint and all its self-adjoint extensions are unbounded from below due to the presence of an infinite sequence of energy levels E_k going to $-\infty$ for $k \to \infty$. This result was first rigorously proved for a system of (at least) three identical bosons in [16] (see also [13] for the case of three different particles) using the theory of self-adjoint extensions. This effect, known as Thomas effect, prevents H_α from being a good physical Hamiltonian. Notice that the effect is absent in dimension two [7].

It is expected that the Thomas effect could be absent if the Hilbert space of states is appropriately restricted, e.g. introducing suitable symmetry constraints on the wave function. A typical example is the antisymmetry requirement. If the wave function is antisymmetric under the exchange of the coordinates of two particles of the system then such two particles cannot "see" the zero-range interaction and therefore the whole interaction term in the Hamiltonian is less singular.

In this paper we shall consider the case of a system in dimension three made of two subsystems $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ consists of N identical fermions of one kind and $\mathcal{B}$ of M identical fermions of another kind. We assume that no interaction is present between particles of the same species while each particle of $\mathcal{A}$ interacts with each particle of $\mathcal{B}$ through a zero-range potential. Without loss of generality, we fix the mass of a particle in $\mathcal{A}$ equal to $1/2$ and the mass of a particle in $\mathcal{B}$ equal to $m/2$.

The first mathematical problem is to construct the corresponding Skornyakov–Ter-Martirosyan extension and to show that it is self-adjoint and bounded from below or, on the contrary, it is only symmetric and, possibly, there is Thomas effect. In this generality the problem is open and one can only stress that the answer seems to be strongly dependent on the physical parameters m, N, M.

Some partial results are available in the simpler case $M = 1$. In [15] it is proved that for $M = 1$, $N \leq 4$ and m sufficiently large the Skornyakov–Ter-Martirosyan extension is self-adjoint and bounded from below. The case $M = 1$, $N = 3$ has been approached in [5] where, exploiting analytical and numerical arguments, it is shown that there is Thomas effect if $m^{-1} > 13.384$. For $M = 1$, $N = 2$ it is known in the physical literature (see e.g. [3] and references therein) that for $m^{-1} > 13.607$ the Thomas effect is present while for $m^{-1} < 13.607$ the Hamiltonian is expected to be bounded from below. As rigorous results in this case, we mention [14,17] for $m = 1$ and [18], where it is proved that for $m^{-1} > 13.607$ the Skornyakov–Ter-Martirosyan extension is not self-adjoint and any self-adjoint extension is unbounded from below.

These results are not suprprising. For small m the dynamics can be described through the Born–Oppenheimer approximation, at least on a variational level. Therefore the light particle undergoes the influence of two attractive delta potentials whose centers parametrically depend on the heavy particle position. It is known, see [1], that the ground state energy of such reduced system diverges to $-\infty$ when the distance between the two centers goes to 0. This provide a simple heuristic explanation for the physical behaviour of the three body problem.

Our aim here is to propose a quadratic form approach for the study of the Hamiltonian of the system composed by the subsystems $\mathcal{A}$ and $\mathcal{B}$. More precisely, following the line of [7], we construct a renormalized quadratic form F_α which is naturally associated to the formal Hamiltonian of the system for any N and M (Section 2).

We also introduce an independent definition of the Skornyakov–Ter-Martirosyan extension H_α for the same system and we show that $(u, H_\alpha u) = F_\alpha(u)$ for any $u \in D(H_\alpha)$ (Section 3).

The relation between F_α and H_α is briefly discussed exploiting the Birman–Krein extension theory of a positive symmetric operator [2,4,12]. It is proved that if F_α is unbounded from below then H_α is not a self-adjoint and bounded from below operator (Section 4).

We apply the above general result to the simplest case $M = 1$, $N = 2$, studying in particular the restriction of F_α to the subspaces of fixed angular momentum l. For any fixed l, we find that if an explicit condition on m (see (5.12)) is satisfied then the quadratic form is unbounded from below. It is conjectured that for l even the condition is never satisfied while for l odd it generalizes the result obtained for $l = 1$ in [18] (Section 5).

In the appendix the renormalization procedure for the derivation of the quadratic form F_α is briefly described and some results from potential theory are recalled.

We finally collect here some notation that will be used in the paper.

- With an abuse of notation, the scalar product and the norm of various L^2-spaces introduced throughout the paper will be all denoted by the same symbols $(\cdot,\cdot)$, $\|\cdot\|$.
- $L^2_a(\mathbb{R}^{3(N+M)})$ is the space of L^2-functions in $\mathbb{R}^{3(N+M)}$ which are antisymmetric in the first N coordinates and in the second M coordinates.
- $H^s(\mathbb{R}^d)$ is the standard Sobolev space of order $s \in \mathbb{R}$ in dimension $d \in \mathbb{N}$.
- $H^s_a(\mathbb{R}^{3(N+M)}) = H^s(\mathbb{R}^{3(N+M)}) \cap L^2_a(\mathbb{R}^{3(N+M)})$.
- $\hat{f}$ is the Fourier transform of f.
- $\boldsymbol{x}_N = (x_1, \dots, x_N) \in \mathbb{R}^{3N}$, $x_i \in \mathbb{R}^3$, $\boldsymbol{y}_M = (y_1, \dots, y_M) \in \mathbb{R}^{3M}$, $y_j \in \mathbb{R}^3$.
- $\hat{\boldsymbol{p}}_i = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_N) \in \mathbb{R}^{3(N-1)}$.
- $f\left(\hat{\boldsymbol{p}}_i, r, \hat{\boldsymbol{k}}_j, s\right) = f\left(p_1, \dots, p_{i-1}, r, p_{i+1}, \dots, p_N, k_1, \dots, k_{j-1}, s, k_{j+1}, \dots, k_M\right)$.
- $\displaystyle\sum_{(i,j)} = \sum_{i=1}^N \sum_{j=1}^M, \quad \sum_{(i,j)\neq(l,h)} = \sum_{\substack{i=1\\ i\neq l}}^N \sum_{\substack{j=1\\ j\neq h}}^M, \quad \sum_{(i,j)\neq(l,h)}' = \sum_{\substack{i,l=1\\ (i,j)\neq(l,h)}}^N \sum_{j,h=1}^M .$

2. The quadratic form and its wellposedness

In this section we introduce and study the quadratic form naturally associated to the Hamiltonian of a system composed by two species $\mathcal{A}$ and $\mathcal{B}$ of identical fermions described in the introduction. In the Hilbert space of the system $L^2_a(\mathbb{R}^{3(N+M)})$ the formal Hamiltonian describing the dynamics is

$$(Hu)(\boldsymbol{x}_N, \boldsymbol{y}_M) = (H_0 u)\,(\boldsymbol{x}_N, \boldsymbol{y}_M) - \mu \sum_{(i,j)} \delta(y_j - x_i) u(\boldsymbol{x}_N, \boldsymbol{y}_M), \tag{2.1}$$

$$H_0 = -\Delta_{\boldsymbol{x}_N} - \frac{1}{m}\Delta_{\boldsymbol{y}_M}. \tag{2.2}$$

In order to give a precise meaning as an operator in $L^2_a(\mathbb{R}^{3(N+M)})$ to the formal expression (2.1) we exploit a renormalization procedure for the corresponding quadratic form, following the line of [7]. The first step is to introduce an ultra-violet cut-off and to define a regularized Hamiltonian H_R, $R > 0$; then we consider the corresponding quadratic form which is re-written in a suitable form; finally we renormalize the coupling constant μ and we explicitly compute the limit for $R \to \infty$. The procedure is described with some details in the appendix. We underline that our aim is only to identify the limit quadratic form and therefore no attempt is made to give a rigorous proof of the convergence. The limit quadratic form is denoted by F_α, $D(F_\alpha)$, $\alpha \in \mathbb{R}$, and it is explicitly given by

$$D(F_\alpha) = \Big\{u \in L^2_a(\mathbb{R}^{3(N+M)}) \mid u = w^\lambda + G^\lambda \xi,\ w^\lambda \in H^1_a(\mathbb{R}^{3(N+M)}),\ \xi \in H^{1/2}(\mathbb{R}^{3(N+M-1)})\Big\} \tag{2.3}$$

$$F_\alpha(u) = \mathcal{F}^\lambda(u) + \Phi^\lambda_\alpha(\xi) \tag{2.4}$$

where

$$\mathcal{F}^\lambda(u) = \int d\boldsymbol{p}_N d\boldsymbol{k}_M \Big[(h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda)|(\hat{u} - \widehat{G^\lambda \xi})(\boldsymbol{p}_N, \boldsymbol{k}_M)|^2 - \lambda |\hat{u}(\boldsymbol{p}_N, \boldsymbol{k}_M)|^2\Big], \tag{2.5}$$

$$\Phi^\lambda_\alpha(\xi) = NM\Big[\alpha \|\xi\|^2 + \Phi^\lambda_0(\xi) + \Phi^\lambda_1(\xi) + \Phi^\lambda_2(\xi) + \Phi^\lambda_3(\xi)\Big], \tag{2.6}$$

and

$$\Phi^\lambda_0(\xi) = 2\pi^2 \left(\frac{2m}{m+1}\right)^{3/2} \int dq\, d\hat{\boldsymbol{p}}_1\, d\hat{\boldsymbol{k}}_1 \sqrt{h_1(q, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1) + \lambda}\; |\hat{\xi}(q, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)|^2, \tag{2.7}$$

$$\Phi^\lambda_1(\xi) = (N-1) \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M \frac{\overline{\hat{\xi}}\left(\frac{p_1+k_1}{\sqrt{2}}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1\right) \hat{\xi}\left(\frac{p_2+k_1}{\sqrt{2}}, \hat{\boldsymbol{p}}_2, \hat{\boldsymbol{k}}_1\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda}, \tag{2.8}$$

$$\Phi^\lambda_2(\xi) = (M-1) \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M \frac{\overline{\hat{\xi}}\left(\frac{p_1+k_1}{\sqrt{2}}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1\right) \hat{\xi}\left(\frac{p_1+k_2}{\sqrt{2}}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_2\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda}, \tag{2.9}$$

$$\Phi^\lambda_3(\xi) = -(N-1)(M-1) \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M \frac{\overline{\hat{\xi}}\left(\frac{p_1+k_1}{\sqrt{2}}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1\right) \hat{\xi}\left(\frac{p_2+k_2}{\sqrt{2}}, \hat{\boldsymbol{p}}_2, \hat{\boldsymbol{k}}_2\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda}. \tag{2.10}$$

In the above definition we have denoted

$$h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) = \boldsymbol{p}_N^2 + \frac{\boldsymbol{k}_M^2}{m}, \tag{2.11}$$

$$h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \frac{2}{m+1} q^2 + \hat{\boldsymbol{p}}_i^2 + \frac{\hat{\boldsymbol{k}}_j^2}{m}, \tag{2.12}$$

$$\widehat{G^\lambda \xi}(\boldsymbol{p}_N, \boldsymbol{k}_M) = \sum_{(i,j)} \frac{(-1)^{i+j} \hat{\xi}\left(\frac{p_i+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda}. \tag{2.13}$$

In the following we shall often refer to $G^\lambda \xi$ as the potential while ξ will be called the charge. The condition $G^\lambda \xi \in L^2_a(\mathbb{R}^{3(N+M)})$ implies that $\hat{\xi}$ must be antisymmetric in the variables $\hat{\boldsymbol{p}}_i$ and in the variables $\hat{\boldsymbol{k}}_j$. We also notice that if $\mathcal{A}$ and $\mathcal{B}$ would consist of distinguishable particles the definition of the quadratic form would be the same with

$$\widehat{G^\lambda \xi}(\boldsymbol{p}_N, \boldsymbol{k}_M) = \sum_{(i,j)} \widehat{G^\lambda \xi_{ij}}(\boldsymbol{p}_N, \boldsymbol{k}_M) \equiv \sum_{(i,j)} \frac{\hat{\xi}_{ij}\left(\frac{p_i+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda}. \tag{2.14}$$

Since we want to consider fermions, the final step of the construction is to take into account the requirement of antisymmetry by particle index exchange. This requires (see more details in Appendix) that

$$\hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = (-1)^{i+j}\, \hat{\xi}_{11}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \equiv (-1)^{i+j}\, \hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j). \tag{2.15}$$

Let us show well-posedness of our definition of the quadratic form. It is easy to see that for $\xi \in H^{1/2}(\mathbb{R}^{3(N+M-1)})$ the potential $G^\lambda \xi$ belongs to $L^2(\mathbb{R}^{3(N+M)})$ but it does not belong to $H^1(\mathbb{R}^{3(N+M)})$ and therefore the decomposition $u = w^\lambda + G^\lambda \xi$ for the generic element of the form domain (2.3) is unique. It remains to show that the quadratic form on the charges (2.6) is well defined for $\xi \in H^{1/2}(\mathbb{R}^{3(N+M-1)})$. For the proof we shall exploit the following lemma.

LEMMA 2.1. *Let us consider the following integral operator in* $L^2(\mathbb{R}^3)$

$$(Qu)(x) = \int dx' \frac{u(x')}{\sqrt{|x|}(x^2 + x'^2)\sqrt{|x'|}}. \tag{2.16}$$

Then Q is a bounded, positive operator with

$$\|Q\| = 2\pi^2. \tag{2.17}$$

Proof: Introducing spherical coordinates $x = (r, z)$ and denoting the standard measure on S^2 by dz, we have

$$(u, Qv) = \int dz\, dz' \int_0^\infty dr \int_0^\infty dr'\, \frac{(r\, r')^{3/2}}{r^2 + r'^2}\, \bar{u}(r, z) v(r', z'). \tag{2.18}$$

The operator $\tilde{Q}$ in $L^2(\mathbb{R}^+, r^2 dr)$ with integral kernel

$$\tilde{Q}(r, r') = \frac{(r\, r')^{3/2}}{r^2 + r'^2} \tag{2.19}$$

can be explicitly diagonalized. It is sufficient to introduce the unitary operator

$$\mathcal{D} \,:\, L^2(\mathbb{R}^+, r^2 dr) \to L^2(\mathbb{R}), \qquad (\mathcal{D} f)(y) = e^{3y/2} f(e^y), \tag{2.20}$$

and to observe that

$$(\mathcal{D}\tilde{Q}\mathcal{D}^{-1} g)(y) = \frac{1}{2} \int dy'\, \frac{g(y')}{\mathrm{ch}(y - y')}. \tag{2.21}$$

Taking the Fourier transform, the above operator is reduced to the multiplication operator (see e.g. [8])

$$(\widehat{\mathcal{D}\tilde{Q}\mathcal{D}^{-1}} g)(k) = \frac{\pi}{2\, \mathrm{ch}\, \frac{\pi}{2} k}\, \hat{g}(k) \tag{2.22}$$

and therefore $\tilde{Q}$ is positive and its norm is $\frac{\pi}{2}$. Using this fact in (2.18) we have

$$(u, Qv) = \int dk \, \frac{\pi}{2 \operatorname{ch} \frac{\pi}{2} k} \int dz \, \widehat{\mathcal{D}\bar{u}}(k, z) \int dz' \, \widehat{\mathcal{D}v}(k, z'). \tag{2.23}$$

From (2.23) it easily follows that Q is positive and its norm is $2\pi^2$. □

In the next proposition we show that $\Phi_\alpha^\lambda(\xi)$ is well defined for $\xi \in H^{1/2}(\mathbb{R}^{3(N+M-1)})$.

PROPOSITION 2.2. *There exist positive constants $C_i = C_i(N, M, m, \lambda)$ such that*

$$|\Phi_i^\lambda(\xi)| \le C_i \Phi_0^\lambda(\xi), \qquad i = 1, 2, 3, \tag{2.24}$$

for any $\xi \in H^{1/2}(\mathbb{R}^{3(N+M-1)})$.

Proof: Let us first consider Φ_1^λ defined in (2.8). We introduce the change of the integration variables $(p_1, p_2, k_1) \to (x, y, z)$ given by

$$\begin{cases} x = \left(\dfrac{m+1}{m(m+2)^2}\right)^{1/2} \big(k_1 + p_1 - (m+1)p_2\big), \\ y = \left(\dfrac{m+1}{m(m+2)^2}\right)^{1/2} \big(k_1 + p_2 - (m+1)p_1\big), \\ z = \left(\dfrac{1}{m(m+2)}\right)^{1/2} (k_1 + p_1 + p_2) \end{cases} \tag{2.25}$$

with inverse given by

$$\begin{cases} p_1 = \left(\dfrac{m}{m+2}\right)^{1/2} z - \left(\dfrac{m}{m+1}\right)^{1/2} y, \\ p_2 = \left(\dfrac{m}{m+2}\right)^{1/2} z - \left(\dfrac{m}{m+1}\right)^{1/2} x, \\ k_1 = \left(\dfrac{m}{m+1}\right)^{1/2} (x + y) + \left(\dfrac{m^3}{m+2}\right)^{1/2} z. \end{cases} \tag{2.26}$$

Moreover we define

$$\eta(x, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1) = \hat{\xi}\left(\sqrt{\frac{m(m+1)^2}{2(m+2)}}\, p_2 + \sqrt{\frac{m}{2(m+1)}}\, x, \sqrt{\frac{m}{m+2}}\, p_2 - \sqrt{\frac{m}{m+1}}\, x, p_3, \dots, p_N, \hat{\boldsymbol{k}}_1\right). \tag{2.27}$$

Then we have

$$\Phi_1^\lambda(\xi) = (N-1)|J_1| \int d\hat{\boldsymbol{p}}_1 \, d\hat{\boldsymbol{k}}_1 \int dx \, dy \, \frac{\bar{\eta}(x, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)\eta(y, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)}{x^2 + y^2 + \frac{2}{m+1} x \cdot y + \hat{\boldsymbol{p}}_1^2 + \frac{1}{m} \hat{\boldsymbol{k}}_1^2 + \lambda}, \tag{2.28}$$

where

$$|J_1| = \left|\frac{\partial(p_1, p_2, k_1)}{\partial(x, y, z)}\right| = \left(\frac{m^{3/2}\sqrt{m+2}}{m+1}\right)^3 \tag{2.29}$$

is the Jacobian of the transformation of coordinates (2.25). Taking into account

$$x^2 + y^2 + \frac{2}{m+1} x \cdot y + \hat{\boldsymbol{p}}_1^2 + \frac{\hat{\boldsymbol{k}}_1^2}{m} \geq \frac{m}{m+1}(x^2 + y^2) \tag{2.30}$$

we have the following estimate:

$$|\Phi_1^\lambda(\xi)| \leq (N-1)|J_1| \frac{m+1}{m} \int d\hat{\boldsymbol{p}}_1 d\hat{\boldsymbol{k}}_1 \int dx\, dy\, \frac{\sqrt{|x|}|\bar{\eta}(x, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)|\sqrt{|y|}|\eta(y, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)|}{\sqrt{|x|}(x^2 + y^2)\sqrt{|y|}}. \tag{2.31}$$

Using the estimate (2.17) we obtain

$$|\Phi_1^\lambda(\xi)| \leq 2\pi^2 (N-1)|J_1| \frac{m+1}{m} \int d\hat{\boldsymbol{p}}_1 d\hat{\boldsymbol{k}}_1 \int dx\, |x||\eta(x, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)|^2. \tag{2.32}$$

Let us rewrite also $\Phi_0^\lambda(\xi)$ in terms of η defined by (2.27). It is convenient to introduce a further change of coordinates $(q, q_2) \to (x, p_2)$ given by

$$\begin{cases} x = \sqrt{\dfrac{m+1}{m(m+2)^2}}\left(\sqrt{2}\, q - (m+1) q_2\right), \\ p_2 = \dfrac{1}{\sqrt{m(m+2)}}\left(\sqrt{2}\, q + q_2\right), \end{cases} \tag{2.33}$$

with inverse given by

$$\begin{cases} q = \sqrt{\dfrac{m}{2(m+1)}} x + \sqrt{\dfrac{m(m+1)^2}{2(m+2)}} p_2, \\ q_2 = -\sqrt{\dfrac{m}{m+1}} x + \sqrt{\dfrac{m}{m+2}} p_2. \end{cases} \tag{2.34}$$

The diagonal term now reads

$$\Phi_0^\lambda(\xi) = 2\pi^2 \left(\frac{m}{m+1}\right)|J_2| \cdot \int d\hat{\boldsymbol{p}}_1\, d\hat{\boldsymbol{k}}_1\, dx \sqrt{\frac{m(m+2)}{(m+1)^2} x^2 + (m-1) p_2^2 + \hat{\boldsymbol{p}}_1^2 + \frac{\hat{\boldsymbol{k}}_1^2}{m} + \lambda}\, |\eta(x, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)|^2, \tag{2.35}$$

where

$$|J_2| = m^3 \left(\frac{m+2}{2(m+1)}\right)^{3/2} \tag{2.36}$$

is the Jacobian of the transformation of coordinates (2.33). By the trivial estimate

$$\sqrt{\frac{m(m+2)}{(m+1)^2}x^2+(m-1)p_2^2+\hat{\boldsymbol{p}}_1^2+\frac{\hat{\boldsymbol{k}}_1^2}{m}+\lambda} \geq \sqrt{\frac{m(m+2)}{(m+1)^2}}\,|x| \tag{2.37}$$

and (2.32), (2.35) we conclude

$$|\Phi_1^\lambda(\xi)| \leq C_1\,\Phi_0^\lambda(\xi), \qquad C_1 = \max\left\{1, (N-1)\frac{2^{3/2}(m+1)^2}{m^{3/2}\sqrt{m_2}}\right\}. \tag{2.38}$$

Proceeding exactly in the same way we also have

$$|\Phi_2^\lambda(\xi)| \leq C_2\,\Phi_0^\lambda(\xi), \qquad C_2 = \max\left\{1, (N-1)\frac{2^{3/2}(m+1)^2}{m^{3/2}\sqrt{m_2}}\right\}. \tag{2.39}$$

Let us consider Φ_3^λ defined in (2.10). Introducing the coordinates

$$v=\frac{p_1+k_1}{\sqrt{2}}, \qquad x=\frac{p_1-k_1}{\sqrt{2}}, \qquad z=\frac{p_2+k_2}{\sqrt{2}}, \qquad y=\frac{p_2-k_2}{\sqrt{2}}, \tag{2.40}$$

and exploiting the fact that $h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) \geq m^{-1}(x^2+y^2)$, we have

$$\begin{aligned}|\Phi_3^\lambda(\xi)| &\leq (N-1)(M-1)m\int dp_3\dots dp_N dk_3\dots dk_M dv dz\\ &\cdot\int dx\,dy\frac{\left|\hat{\xi}\left(v, \frac{z+y}{\sqrt{2}}, p_3,\dots,p_N,\frac{z-y}{\sqrt{2}},k_3,\dots,k_M\right)\right|\left|\hat{\xi}\left(z, \frac{v+x}{\sqrt{2}}, p_3,\dots,p_N,\frac{v-x}{\sqrt{2}},k_3,\dots,k_M\right)\right|}{x^2+y^2}.\end{aligned} \tag{2.41}$$

Using the estimate (2.17) we also obtain

$$\begin{aligned}|\Phi_3^\lambda(\xi)| &\leq (N-1)(M-1)\,2\pi^2 m\int dp_3\dots dp_N dk_3\dots dk_M dv dz\\ &\qquad\cdot\int dx\sqrt{x^2+v^2}\left|\hat{\xi}\left(z, \frac{v+x}{\sqrt{2}}, p_3,\dots,p_N,\frac{v-x}{\sqrt{2}},k_3,\dots,k_M\right)\right|^2\\ &= (N-1)(M-1)\,2\pi^2 m\int dz\,d\hat{\boldsymbol{p}}_1\,d\hat{\boldsymbol{k}}_1\sqrt{p_2^2+k_2^2}\,|\hat{\xi}(z,\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1)|^2\\ &\leq (N-1)(M-1)\frac{(m+1)^{3/2}}{2^{3/2}m^{1/2}}\,\Phi_0^\lambda(\xi)\;\equiv\;C_3\Phi_0^\lambda(\xi)\end{aligned} \tag{2.42}$$

and the proof of (2.24) is concluded. □

Since the norm induced by Φ_0^λ is clearly equivalent to the $H^{1/2}$ norm, it immediately follows from Proposition 2.2 that $\Phi_\alpha^\lambda(\xi) < +\infty$ if $\xi \in H^{1/2}(\mathbb{R}^{3(N+M-1)})$ and therefore the well-posedness of the definition of F_α is proved.

3. The Skornyakov–Ter-Martirosyan extension

In this section we introduce the Skornyakov–Ter-Martirosyan extension H_α, i.e. the symmetric operator which is usually considered as a possible candidate for the description of the dynamics of our system. Then we show that the energy form associated with it coincides with the quadratic form defined in the previous sections. We define the operator H_α as follows. Let us introduce the $3(N+M-1)$-dimensional hyperplanes in $\mathbb{R}^{3(N+M)}$,

$$\Gamma_{ij} = \left\{ (\boldsymbol{x}_N, \boldsymbol{y}_M) \in \mathbb{R}^{3(N+M)} \mid x_i = y_j \right\} \tag{3.1}$$

and the open domain

$$\Omega = \mathbb{R}^{3(N+M)} \setminus \bigcup_{(i,j)} \Gamma_{ij}. \tag{3.2}$$

Then

$$\begin{aligned} D(H_\alpha) = \Big\{ & u \in L^2_a(\mathbb{R}^{3(N+M)}) \mid u = w^\lambda + G^\lambda \xi,\ \xi \in H^{3/2}(\mathbb{R}^{3(N+M-1)}), \\ & w^\lambda \in H^2_a(\mathbb{R}^{3(N+M)}), \\ & 8\pi^{3/2} \widehat{w^\lambda|_{\Gamma_{ij}}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \Big(\alpha\, \hat{\xi}_{ij} + \sum_{(l,h)} \mathcal{T}^\lambda_{ij,lh} \hat{\xi}_{lh} \Big)(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \Big\}, \end{aligned} \tag{3.3}$$

$$(H_\alpha + \lambda)u = (H_0 + \lambda) w^\lambda, \tag{3.4}$$

where the operator $\mathcal{T}^\lambda_{ij,lh}$ acting on the surface charges $\hat{\xi}_{lh}$ is defined in the following way

$$\begin{aligned} & \mathcal{T}^\lambda_{ij,lh} \hat{\xi}_{lh}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \\ & = \begin{cases} 2\pi^2 \left(\dfrac{2m}{m+1} \right)^{3/2} \sqrt{h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}\ \hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) & (i,j) = (l,h), \\ -8\pi^{3/2}\ \widehat{G^\lambda \xi_{lh}|_{\Gamma_{ij}}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) & (i,j) \neq (l,h). \end{cases} \end{aligned} \tag{3.5}$$

It is useful to give explicit expressions for the nondiagonal terms in (3.5). We distinguish the three possible cases: $l \neq i$ and $h \neq j$, $l = i$ and $h \neq j$, $l \neq i$ and $h = j$.

(1) $l \neq i$ and $h \neq j$:

$$\begin{aligned} & G^\lambda \xi_{lh}|_{\Gamma_{ij}}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \\ & = \frac{1}{(2\pi)^{\frac{3}{2}(N+M)}} \int d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j\, e^{i\left(\hat{\boldsymbol{p}}_i \cdot \hat{\boldsymbol{x}}_i + \hat{\boldsymbol{k}}_j \cdot \hat{\boldsymbol{y}}_j\right)} \int dp_i\, dk_j\, e^{i(p_i + k_j) x_i} \frac{\hat{\xi}_{lh}\left(\frac{p_l + k_h}{\sqrt{2}}, \hat{\boldsymbol{p}}_l, \hat{\boldsymbol{k}}_h \right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda} \\ & = \frac{(-1)^{l+h}}{(2\pi)^{\frac{3}{2}(N+M)}} \int dq\, d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\, e^{i\left(\sqrt{2}q \cdot x_i + \hat{\boldsymbol{p}}_i \cdot \hat{\boldsymbol{x}}_i + \hat{\boldsymbol{k}}_j \cdot \hat{\boldsymbol{y}}_j\right)} \\ & \qquad \cdot \int ds\, \frac{\hat{\xi}\left(\frac{p_l + k_h}{\sqrt{2}}, \hat{\boldsymbol{p}}_l|_{p_i = \frac{q+s}{\sqrt{2}}}, \hat{\boldsymbol{k}}_h|_{k_j = \frac{q-s}{\sqrt{2}}} \right)}{\tilde{h}_0(q, s, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}, \end{aligned} \tag{3.6}$$

where

$$\tilde{h}_0(q, s, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \frac{m+1}{2m} q^2 + \frac{m+1}{2m} s^2 + \frac{m-1}{m} q \cdot s + \hat{\boldsymbol{p}}_i^2 + \frac{1}{m} \hat{\boldsymbol{k}}_j^2. \tag{3.7}$$

Then the Fourier transform reads

$$\widehat{G^\lambda \xi_{lh}|_{\Gamma_{ij}}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \frac{(-1)^{l+h}}{8\pi^{3/2}} \int ds \, \frac{\hat{\xi}\left(\frac{p_l + k_h}{\sqrt{2}}, \hat{\boldsymbol{p}}_l|_{p_i = \frac{q+s}{\sqrt{2}}}, \hat{\boldsymbol{k}}_h|_{k_j = \frac{q-s}{\sqrt{2}}}\right)}{\tilde{h}_0(q, s, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}. \tag{3.8}$$

A similar computation can be done for the other two cases.
(2) $l = i$ and $h \neq j$;

$$\widehat{G^\lambda \xi_{ih}|_{\Gamma_{ij}}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \frac{(-1)^{i+h}}{8\pi^{3/2}} \int ds \, \frac{\hat{\xi}\left(\frac{q+s}{2} + \frac{k_h}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_h|_{k_j = \frac{q-s}{\sqrt{2}}}\right)}{\tilde{h}_0(q, s, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}. \tag{3.9}$$

(3) $l \neq i$ and $h = j$;

$$\widehat{G^\lambda \xi_{lj}|_{\Gamma_{ij}}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \frac{(-1)^{l+j}}{8\pi^{3/2}} \int ds \, \frac{\hat{\xi}\left(\frac{p_l}{\sqrt{2}} + \frac{q-s}{2}, \hat{\boldsymbol{p}}_l|_{p_i = \frac{q+s}{\sqrt{2}}}, \hat{\boldsymbol{k}}_j\right)}{\tilde{h}_0(q, s, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}. \tag{3.10}$$

The last equality in (3.3) should be considered as the boundary condition satisfied by u on Γ_{ij} and it connects the regular and the singular part of an element of $D(H_\alpha)$. It is easy to verify that the operator H_α,$D(H_\alpha)$ is independent of the choice of $\lambda > 0$ and it is symmetric.

In the next proposition we show that our definition of H_α,$D(H_\alpha)$ coincides with the standard definition usually found in the literature, except for an irrelevant modification of the coupling constant α.

PROPOSITION 3.1. *Let $u \in D(H_\alpha)$. Then*

$$H_\alpha u|_\Omega = H_0 u|_\Omega, \tag{3.11}$$

$$\lim_{|x_i - y_j| \to 0} |x_i - y_j| u(\boldsymbol{x}_N, \boldsymbol{y}_M) = \mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j), \tag{3.12}$$

$$\lim_{|x_i - y_j| \to 0} \left(u(\boldsymbol{x}_N, \boldsymbol{y}_M) - \frac{\mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j)}{|x_i - y_j|} \right) = \alpha_0 \, \mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j), \tag{3.13}$$

where

$$\mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) = (-1)^{i+j} \, \frac{2\sqrt{\pi}\, m}{m+1} \, \xi(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j), \tag{3.14}$$

$$\alpha_0 = \frac{\sqrt{2}(m+1)}{8\pi^2 m} \, \alpha. \tag{3.15}$$

Proof: Taking into account (3.4) and the fact that $(H_0 + \lambda) G^\lambda \xi_{ij}|_\Omega = 0$ (see (6.23)), we have

$$H_\alpha u|_\Omega = (H_\alpha+\lambda)u|_\Omega - \lambda u|_\Omega = (H_0+\lambda)w^\lambda|_\Omega - \lambda u|_\Omega$$
$$= (H_0+\lambda)\Big(u - \sum_{(i,j)} G^\lambda \xi_{ij}\Big)|_\Omega - \lambda u|_\Omega = H_0 u|_\Omega. \tag{3.16}$$

Let us characterize the singularity of an element of (3.3) at the hyperplane Γ_{ij}. Exploiting (6.24), for $|x_i - y_j| \to 0$ we have

$$u(\boldsymbol{x}_N, \boldsymbol{y}_M) = w^\lambda(\boldsymbol{x}_N, \boldsymbol{y}_M) + \sum_{(l,h)\neq(i,j)} G^\lambda \xi_{lh}(\boldsymbol{x}_N, \boldsymbol{y}_M) + G^\lambda \xi_{ij}(\boldsymbol{x}_N, \boldsymbol{y}_M)$$
$$= w^\lambda(\boldsymbol{x}_N, \boldsymbol{y}_M) + \sum_{(l,h)\neq(i,j)} G^\lambda \xi_{lh}(\boldsymbol{x}_N, \boldsymbol{y}_M) + \frac{1}{|x_i - y_j|}\frac{2\sqrt{\pi}\, m}{m+1}\, \xi_{ij}(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j)$$
$$- \frac{2\pi^2 (2m)^{3/2}}{(2\pi)^{\frac{3}{2}(N+M)}(m+1)^{3/2}}$$
$$\cdot \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j\, e^{i\left(\sqrt{2}x_i q + \hat{\boldsymbol{x}}_i\cdot\hat{\boldsymbol{p}}_i + \hat{\boldsymbol{y}}_j\cdot\hat{\boldsymbol{k}}_j\right)} \sqrt{h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}\; \hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + o(1). \tag{3.17}$$

We notice that

$$\lim_{|x_i - y_j|\to 0} |x_i - y_j| \Big(w^\lambda(\boldsymbol{x}_N, \boldsymbol{y}_M) + \sum_{(l,h)\neq(i,j)} G^\lambda \xi_{lh}(\boldsymbol{x}_N, \boldsymbol{y}_M)\Big) = 0. \tag{3.18}$$

Therefore from (3.17) we obtain (3.12). Moreover

$$\lim_{|x_i - y_j|\to 0}\left(u(\boldsymbol{x}_N, \boldsymbol{y}_M) - \frac{\mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j)}{|x_i - y_j|}\right)$$
$$= \lim_{|x_i - y_j|\to 0}\left(u(\boldsymbol{x}_N, \boldsymbol{y}_M) - \frac{2\sqrt{\pi}\, m\, \xi_{ij}(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j)}{|x_i - y_j|(m+1)}\right)$$
$$= w^\lambda(\boldsymbol{x}_N, \boldsymbol{y}_M)|_{\Gamma_{ij}} + \sum_{(l,h)\neq(i,j)} G^\lambda \xi_{lh}(\boldsymbol{x}_N, \boldsymbol{y}_M)|_{\Gamma_{ij}}$$
$$- \frac{2\pi^2}{(2\pi)^{\frac{3}{2}(N+M)}} \left(\frac{2m}{m+1}\right)^{3/2}$$
$$\cdot \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j\, e^{i\left(\sqrt{2}x_i q + \hat{\boldsymbol{x}}_i\cdot\hat{\boldsymbol{p}}_i + \hat{\boldsymbol{y}}_j\cdot\hat{\boldsymbol{k}}_j\right)} \sqrt{h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}\; \hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)$$
$$\equiv f(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j). \tag{3.19}$$

The computation of f is more easily done in the Fourier space. Exploiting (3.5) we have

$$\widehat{f}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)$$
$$= \widehat{w|_{\Gamma_{ij}}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \sum_{(l,h)\neq(i,j)} \widehat{G^\lambda \xi_{lh}|_{\Gamma_{ij}}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) - \frac{1}{8\pi^{3/2}}(\mathcal{T}^\lambda_{ij,ij}\hat{\xi}_{ij})(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)$$

$$= \widehat{w|_{\Gamma_{ij}}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) - \frac{1}{8\pi^{3/2}}\Big(\sum_{(l,h)} \mathcal{T}^{\lambda}_{ij,lh}\hat{\xi}_{lh}\Big)(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)$$

$$= \frac{\alpha}{8\pi^{3/2}}(-1)^{i+j}\hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j), \tag{3.20}$$

where, in the last line, we have used the boundary condition in (3.3). Taking the inverse Fourier transform and using (3.14), (3.15), we find

$$f(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) = \alpha_0\, \mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \tag{3.21}$$

concluding the proof of the proposition. □

The next step is to verify that the mean value of the operator H_α coincides with our quadratic form F_α restricted to $D(H_\alpha)$.

PROPOSITION 3.2. *If $u \in D(H_\alpha)$ then $(u, H_\alpha u) = F_\alpha(u)$.*

Proof: Let us introduce the tubular neighbourhood $\Gamma^{\varepsilon}_{ij}$, for $\varepsilon > 0$, of the hyperplane Γ_{ij},

$$\Gamma^{\varepsilon}_{ij} = \Big\{(\boldsymbol{x}_N, \boldsymbol{y}_M) \in \mathbb{R}^{3(N+M)} \,|\, |x_i - y_j| \le \varepsilon\Big\} \tag{3.22}$$

and the open domain

$$\Omega^{\varepsilon} = \mathbb{R}^{3(N+M)} \setminus \bigcup_{(i,j)} \Gamma^{\varepsilon}_{ij}. \tag{3.23}$$

Taking into account (3.11), for any $u \in D(H_\alpha)$ we can write

$$(u, H_\alpha u) = \lim_{\varepsilon\to 0}\int_{\Omega^{\varepsilon}} d\boldsymbol{x}_N d\boldsymbol{y}_M\, \bar{u}\, H_0 u. \tag{3.24}$$

The r.h.s. of (3.24) can be computed using Definition (3.3) and Eq. (6.23) proved in Proposition 6.1. In fact we have

$$\begin{aligned}
&(u, H_\alpha u)\\
&= \lim_{\varepsilon\to 0}\int_{z^{\varepsilon}} d\boldsymbol{x}_N\, d\boldsymbol{y}_M \Big(\overline{w^{\lambda}} + \sum_{(i,j)} G^{\lambda}\bar{\xi}_{ij}\Big)(H_0+\lambda)\Big(w^{\lambda} + \sum_{(i,j)} G^{\lambda}\xi_{ij}\Big) - \lambda\int d\boldsymbol{x}_N\, d\boldsymbol{y}_M\, |u|^2\\
&= \int d\boldsymbol{x}_N\, d\boldsymbol{y}_M\, \overline{w^{\lambda}}(H_0+\lambda)w^{\lambda} - \lambda\int d\boldsymbol{x}_N\, d\boldsymbol{y}_M\, |u|^2 + \sum_{(i,j)}\int d\boldsymbol{x}_N\, d\boldsymbol{y}_M\, G^{\lambda}\bar{\xi}_{ij}\,(H_0+\lambda)w^{\lambda}\\
&= \mathcal{F}^{\lambda}(u) + 8\pi^{3/2}\sum_{(i,j)}\int dq\, d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\, \bar{\hat{\xi}}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)\widehat{w^{\lambda}|_{\Gamma_{ij}}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)\\
&= \mathcal{F}^{\lambda}(u) + \sum_{(i,j)}\int dq\, d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\, \bar{\hat{\xi}}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)\Bigg(\alpha\,\hat{\xi}_{ij} + \sum_{(l,h)}\mathcal{T}^{\lambda}_{ij,lh}\hat{\xi}_{lh}\Bigg)(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)
\end{aligned} \tag{3.25}$$

where in the last line we have used the boundary condition satisfied by u on Γ_{ij} (see (3.3)). Now we closely look at the last term appearing in r.h.s. of (3.25) and show that they reconstruct $\Phi^\lambda_\alpha(\xi)$. First we have

$$\alpha \sum_{(i,j)} \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j\, \bar{\hat{\xi}}_{ij}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)\widehat{\xi_{ij}}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j) = \alpha NM\|\xi\|^2. \tag{3.26}$$

Using (3.5), the diagonal terms can be written as

$$\sum_{(i,j)} \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j\, \bar{\hat{\xi}}_{ij}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)\mathcal{T}^\lambda_{ij,ij}\hat{\xi}_{ij}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)$$
$$= 2\pi^2\left(\frac{2m}{m+1}\right)^{3/2} \sum_{(i,j)} \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j \sqrt{h_1(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\lambda}\,|\xi(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)|^2 = NM\Phi^\lambda_0(\xi). \tag{3.27}$$

Concerning the nondiagonal terms, we use the explicit expression of $\widehat{G^\lambda\xi_{lh}|_{\Gamma_{ij}}}$ and we find

$$\sum_{(i,j)\neq(l,h)}{}' \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j\, \bar{\hat{\xi}}_{ij}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)\mathcal{T}^\lambda_{ij,lh}\hat{\xi}_{lh}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j) = NM(\Phi^\lambda_1(\xi)+\Phi^\lambda_2(\xi)+\Phi^\lambda_3(\xi)). \tag{3.28}$$

The proof of the proposition is concluded. □

4. On the relation between the form F_α and the operator H_α

For any physical application the crucial point is to show that the symmetric operator H_α is a good Hamiltonian for our system, i.e. to prove that it is self-adjoint and bounded from below (stability condition). As we already remarked, in general the problem is open and we expect that the answer can be positive only under appropriate conditions on the physical parameters of the system N, M, m. Exploiting the representation theorem of self-adjoint operators (see e.g. [11]), the result could be obtained proving that the associated quadratic form F_α is closed and bounded from below.

On the other hand one can also prove a “negative” result. More precisely, we show that if the form F_α is unbounded from below then H_α is not a self-adjoint and bounded from below operator in $L^2_a(\mathbb{R}^{3(N+M)})$.

First we recall some facts from Birman–Krein theory of positive extensions of a given symmetric and positive operator on a Hilbert space [4, 9, 12]. Proofs can also be found in [2] where a detailed discussion of the original Russian literature is given.

Let S_0 be a symmetric and positive operator on a Hilbert space $\mathcal{H}$ and let $\mathcal{N}$ be the kernel of S_0^*. We shall denote the Friedrichs extension of S_0 by S_F. Notice that since S_0 is positive then S_F is positive and has a bounded inverse.

The main result of the Birman–Krein theory is that the positive self-adjoint extensions of S_0 are in a one-to-one correspondence with positive operators on $\mathcal{N}$.

More precisely, if S is a positive self-adjoint extension of S_0 then there exists a positive operator $B : D(B) \subseteq \mathcal{N} \to \mathcal{N}$ such that

$$D(S) = \{u \in \mathcal{H} \,|\, u = \phi + S_F^{-1}(Bf + g) + f,\ \phi \in D(S_0),\ f \in D(B),\ g \in \mathcal{N} \cap D(B)^{\perp}\} \tag{4.1}$$

and

$$Su = S_0^*|_{D(S)}u = S_0\phi + Bf + g. \tag{4.2}$$

Notice that the closure of $D(B)$ may be a proper subspace of $\mathcal{N}$.

Let us specialize this general result to our concrete case. We have $\mathcal{H} = L^2_a(\mathbb{R}^{3(N+M)})$ and $S_0 = \tilde{H}_0 + \lambda$, $\lambda > 0$, where $\tilde{H}_0$ is the free Hamiltonian restricted to

$$D(\tilde{H}_0) = \big\{u \in L^2_a(\mathbb{R}^{3(N+M)}) \,|\, u \in H^2(\mathbb{R}^{3(N+M)}),\ u|_{\Gamma_{ij}} = 0,\ \ i = 1,\dots,N,\ j = 1,\dots,M\big\}. \tag{4.3}$$

Moreover (see e.g. [9], [15])

$$\mathcal{N} = \big\{u \in L^2_a(\mathbb{R}^{3(N+M)}) \,|\, u = G^{\lambda}\mu,\ \mu \in H^{-1/2}(\mathbb{R}^{3(N+M-1)})\big\}. \tag{4.4}$$

The Friedrichs extension of $\tilde{H}_0 + \lambda$ is $H_F + \lambda$, where H_F is the free Laplacian with domain

$$D(H_F) = \big\{u \in L^2_a(\mathbb{R}^{3(N+M)}) \,|\, u \in H^2(\mathbb{R}^{3(N+M)})\big\}. \tag{4.5}$$

We shall denote $\mathcal{G}^{\lambda} = (H_F + \lambda)^{-1}$. Notice that $\mathcal{G}^{\lambda} : L^2(\mathbb{R}^{3(N+M)}) \to H^2(\mathbb{R}^{3(N+M)})$ while $G^{\lambda} : H^{-1/2}(\mathbb{R}^{3(N+M-1)}) \to L^2(\mathbb{R}^{3(N+M)})$ even if they act in the same way as multiplication operators in Fourier space. By the Birman–Krein theory we have that any self-adjoint positive extension of $\tilde{H}_0 + \lambda$ is given by $H_B + \lambda$, where

$$D(H_B) = \big\{u \in L^2_a(\mathbb{R}^{3(N+M)}) \,|\, u = \varphi^{\lambda} + \mathcal{G}^{\lambda}(BG^{\lambda}\mu + G^{\lambda}\nu) + G^{\lambda}\mu,\ \varphi^{\lambda} \in D(\tilde{H}_0),\ \mu, \nu \in H^{-1/2}(\mathbb{R}^{3(N+M-1)}),\ G^{\lambda}\mu \in D(B),\ G^{\lambda}\nu \in \mathcal{N} \cap D(B)^{\perp}\big\}, \tag{4.6}$$

$$(H_B + \lambda)u = (H_0 + \lambda)\varphi^{\lambda} + BG^{\lambda}\mu + G^{\lambda}\nu, \tag{4.7}$$

where $B : D(B) \subseteq \mathcal{N} \to \mathcal{N}$ is a positive operator. Exploiting this fact we can prove the following result.

PROPOSITION 4.1. *If the form F_α is unbounded from below then the Skornyakov–Ter-Martirosyan extension H_α, defined by* (3.3) *and* (3.4)*, is not a self-adjoint and bounded from below operator in $L^2_a(\mathbb{R}^{3(N+M)})$.*

Proof: We shall prove the proposition by contradiction. Let us assume that $H_\alpha + \lambda$ is a positive, self-adjoint extension of $\tilde{H}_0 + \lambda$ for a sufficiently large $\lambda > 0$. Then there exists a positive operator B in $\mathcal{N}$ such that $D(H_\alpha) = D(H_B)$. In particular, for any $u = w^{\lambda} + G^{\lambda}\xi \in D(H_\alpha)$ there exist $\varphi^{\lambda} \in D(\tilde{H}_0)$ and $\mu, \nu \in H^{-1/2}(\mathbb{R}^{3(N+M-1)})$, with $G^{\lambda}\mu \in D(B)$ and $G^{\lambda}\nu \in \mathcal{N} \cap D(B)^{\perp}$, such that the following identity holds

$$u \equiv w^{\lambda} + G^{\lambda}\xi = \varphi^{\lambda} + \mathcal{G}^{\lambda}(BG^{\lambda}\mu + G^{\lambda}\nu) + G^{\lambda}\mu. \tag{4.8}$$

From (4.8) we conclude that $G^\lambda \xi = G^\lambda \mu$ and therefore, by (6.25), it follows $\xi = \mu$ and $\mu \in H^{3/2}(\mathbb{R}^{3(N+M-1)})$. Moreover, from Propositions 3.1 and 6.1 we obtain

$$\begin{aligned}
&\lim_{|x_i - y_j| \to 0} \left(u(\boldsymbol{x}_N, \boldsymbol{y}_M) - \frac{2\sqrt{\pi}\, m}{(m+1)|x_i - y_j|} \xi_{ij}(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \right) \\
&= \frac{\alpha}{(2\pi)^{3/2}} \xi_{ij}(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \\
&= \frac{1}{(2\pi)^{3/2}} \Big[8\pi^{3/2} \big(\mathcal{G}^\lambda (B G^\lambda \xi + G^\lambda \nu) \big)(\boldsymbol{x}_N, \boldsymbol{y}_M)|_{\Gamma_{ij}} - \sum_{(l,h)} \big(\mathscr{F}^{-1} \mathcal{T}^\lambda_{ij,lh} \hat{\xi}_{lh} \big)(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \Big],
\end{aligned} \tag{4.9}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. Formula (4.9) holds in particular for $\xi = 0$ and this means that $\nu = 0$. Then in the Fourier space formula (4.9) reads

$$\Big(\alpha\, \hat{\xi}_{ij} + \sum_{(l,h)} \mathcal{T}^\lambda_{ij,lh}\, \hat{\xi}_{lh} \Big)(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = 8\pi^{3/2} \big(\mathcal{G}^\lambda \widehat{B G^\lambda \xi}|_{\Gamma_{ij}} \big)(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j). \tag{4.10}$$

From (4.10) we obtain

$$\Phi^\lambda_\alpha(\xi) \equiv \sum_{(i,j)} \Big(\hat{\xi}_{ij},\, \alpha\, \hat{\xi}_{ij} + \sum_{(l,h)} \mathcal{T}^\lambda_{ij,lh}\, \hat{\xi}_{lh} \Big) = 8\pi^{3/2} \sum_{(i,j)} \big(\hat{\xi}_{ij},\, \mathcal{G}^\lambda \widehat{B G^\lambda \xi}|_{\Gamma_{ij}} \big). \tag{4.11}$$

By a direct computation one sees that the r.h.s. of (4.11) equals $8\pi^{3/2}(G^\lambda \xi, B\, G^\lambda \xi)$ and therefore we conclude

$$\Phi^\lambda_\alpha(\xi) = 8\pi^{3/2}(G^\lambda \xi, B\, G^\lambda \xi) \geq 0. \tag{4.12}$$

On the other hand if (4.12) holds for any $\lambda > 0$ sufficiently large then the form F_α is bounded from below and we obtain a contradiction. □

5. The case $M = 1$ and $N = 2$

From now on we shall limit ourselves to the special case $M = 1$ and $N = 2$. The quadratic form Φ^λ_α (see (2.6)) reads

$$\Phi^\lambda_\alpha(\xi) = 2\alpha \|\xi\|^2 + 2\Phi^\lambda_0(\xi) + 2\Phi^\lambda_1(\xi), \tag{5.1}$$

$$\Phi^\lambda_0(\xi) = 2\pi^2 \left(\frac{2m}{m+1} \right)^{3/2} \int dq\, dp\, \sqrt{h_1(q,p) + \lambda}\; |\hat{\xi}(q,p)|^2,$$

$$h_1(q,p) = \frac{2}{m+1} q^2 + p^2, \tag{5.2}$$

$$\Phi^\lambda_1(\xi) = \int dp_1\, dp_2\, dk\, \frac{\overline{\hat{\xi}}\left(\frac{p_1 + k}{\sqrt{2}}, p_2 \right) \hat{\xi}\left(\frac{p_2 + k}{\sqrt{2}}, p_1 \right)}{h_0(p_1, p_2, k) + \lambda}, \quad h_0(p_1, p_2, k) = p_1^2 + p_2^2 + \frac{1}{m} k^2. \tag{5.3}$$

The regular part $\mathcal{F}$ of the quadratic form is written as in (2.5) where the potential $G^\lambda\xi$ is now given by

$$\widehat{G^\lambda\xi}(p_1, p_2, k) = \frac{\hat{\xi}\left(\frac{p_1+k}{\sqrt{2}}, p_2\right) - \hat{\xi}\left(\frac{p_2+k}{\sqrt{2}}, p_1\right)}{h_0(p_1, p_2, k) + \lambda}. \tag{5.4}$$

Notice that no symmetry condition is required on ξ to ensure that $G^\lambda\xi \in L^2_a(\mathbb{R}^9)$. With the above notation we have

$$F_\alpha(u) = \mathcal{F}^\lambda(u) + \Phi^\lambda_\alpha(\xi). \tag{5.5}$$

In the next proposition we give a detailed analysis of the subspaces where the form (5.5) is unbounded from below. It is well known, see [18], that H_α can be unbounded from below in the subspace corresponding to p-wave for the charges. Here we shall show that, depending on the value of m, this is not the only case. For this pourpose it is more convenient to write Φ^λ_α using the η function introduced in (2.27), and use the coordinates (2.25) also on the total Hilbert space $L^2_a(\mathbb{R}^9)$. With an abuse of notation, we write

$$\Phi^\lambda_\alpha(\eta) = 2\alpha m^3 \left(\frac{m+2}{2(m+1)}\right)^{3/2} \|\eta\|^2 + 2\Phi^\lambda_0(\eta) + 2\Phi^\lambda_1(\eta), \tag{5.6}$$

$$\Phi^\lambda_0(\eta) = \frac{2\pi^2 m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dx\, dp \sqrt{\frac{m(m+2)}{(m+1)^2}x^2 + mp^2 + \lambda}\; |\eta(x, p)|^2, \tag{5.7}$$

$$\Phi^\lambda_1(\eta) = \frac{m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dp\, dx\, dy \frac{\overline{\eta}(x, p)\eta(y, p)}{x^2 + y^2 + \frac{2}{m+1}x \cdot y + p^2 + \lambda}, \tag{5.8}$$

$$G^\lambda\eta(x, y, p) = \frac{\eta(x, p) - \eta(y, p)}{x^2 + y^2 + \frac{2}{m+1}x \cdot y + p^2 + \lambda}. \tag{5.9}$$

First we notice that in these coordinates the quadratic form F_α has some simple decomposition properties. In fact, there is an orthogonal decomposition

$$L^2_a(\mathbb{R}^9) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l \tag{5.10}$$

such that if $u \in \mathcal{H}_l$ and $u' \in \mathcal{H}_{l'}$ then

$$F_\alpha(u, u') = 0 \qquad \text{if } l \neq l'. \tag{5.11}$$

The charge space is $L^2(\mathbb{R}^6) = L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$. We shall decompose the first $L^2(\mathbb{R}^3)$, corresponding to the x variable, into subspaces of fixed angular momentum $\widetilde{\mathcal{H}}_l$. With this choice we have

$$L^2(\mathbb{R}^6) = \bigoplus_{l=0}^{\infty} \widetilde{\mathcal{H}}_l \otimes L^2(\mathbb{R}^3).$$

The subspaces $\mathcal{H}_l$ is defined as the subspace generated by the charges in $\widetilde{\mathcal{H}}_l$, that is

$$\mathcal{H}_l = \left\{ G^\lambda \eta \;\; \text{s.t.} \;\; \eta \in \widetilde{\mathcal{H}}_l \otimes L^2(\mathbb{R}^3) \right\}.$$

If we take $u = w^\lambda + G^\lambda \eta \in \mathcal{H}_l$ then, due to the uniqueness of the decomposition, we have that both w^λ and $G^\lambda \eta$ belong to $\mathcal{H}_l$. Morover it is straightforward to verify that the Laplacian is invariant by rotations in these coordinates and that (5.11) holds true.

PROPOSITION 5.1. *Let m and l be such that*

$$1 + \frac{(m+1)}{\pi\sqrt{m(m+2)}} \int_{-1}^{1} dy\, P_l(y) \frac{\arccos\left(y/(m+1)\right)}{\sin \arccos\left(y/(m+1)\right)} < 0, \tag{5.12}$$

where P_l is the Legendre polynomial of order l. Then

$$\inf_{u \in \mathcal{H}_l} F_\alpha(u) = -\infty. \tag{5.13}$$

Proof: We fix $\lambda > 0$ and consider a trial function of the form

$$u = G^\lambda \eta, \qquad \eta \in \widetilde{\mathcal{H}}_l \otimes L^2(\mathbb{R}^3). \tag{5.14}$$

Therefore we have $F_\alpha(u) = \Phi_\alpha^\lambda(\eta) - \lambda \|u\|^2$. Let us take $\eta(x, p) = f(x)g(p)$, with $f \in \widetilde{\mathcal{H}}_l$ and $g \in \mathcal{S}(\mathbb{R}^3)$, and define the sequence u_n by the following rescaling of η,

$$\eta_n(x, p) = \frac{1}{n} f\left(\frac{x}{n}\right) g(p), \qquad u_n = G^\lambda \eta_n. \tag{5.15}$$

Exploiting the estimate (6.25), one can easily check that the sequence u_n satisfies the condition $\inf_n \|u_n\| > 0$. Using (5.6) and (5.8) we have

$$\Phi_0^\lambda(\eta_n) = n^2 \frac{2\pi^2 m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dx\, d\hat{p} \sqrt{\frac{m(m+2)}{(m+1)^2} x^2 + \frac{m\, p^2 + \lambda}{n^2}}\, |f(x)|^2 |g(p)|^2, \tag{5.16}$$

$$\Phi_1^\lambda(\eta_n) = n^2 \frac{m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dp\, dx\, dy \frac{\overline{f}(x) f(y) |g(p)|^2}{x^2 + y^2 + 2x/(m+1) x \cdot y + (p^2 + \lambda)/n^2}. \tag{5.17}$$

Let us compute the leading terms of $\Phi_0^\lambda(\eta_n)$ and $\Phi_1^\lambda(\eta_n)$ for $n \to \infty$. Using the inequality

$$\sqrt{\frac{m(m+2)}{(m+1)^2} x^2 + \frac{mp^2 + \lambda}{n^2}} - \frac{\sqrt{m(m+2)}}{m+1} |x| \le \frac{1}{n} \sqrt{mp^2 + \lambda}$$

it follows

$$\Phi_0^\lambda(\eta_n) = n^2 \frac{2\pi^2 m^5 (m+2)^2}{(m+1)^4} \int dx\, |x|\, |f(x)|^2 + \mathcal{O}(n). \tag{5.18}$$

Moreover we prove that

$$\Phi_1^\lambda(\eta_n) = n^2 \frac{m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dx\, dy\, \frac{\overline{f}(x) f(y)}{x^2 + y^2 + 2x \cdot y/(m+1)} + \mathcal{O}(n). \tag{5.19}$$

In fact we have

$$\begin{aligned} \Phi_1^\lambda(\eta_n) &- n^2 \frac{m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dx\, dy\, \frac{\overline{f}(x) f(y)}{x^2 + y^2 + 2x \cdot y/(m+1)} \\ &= n^2 \frac{m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dp |g(p)|^2 \int dx\, dy\, \overline{f}(x) f(y)\, T_n(x, y; p^2 + \lambda), \end{aligned} \tag{5.20}$$

where we have defined

$$\begin{aligned} &T_n(x, y; p^2 + \lambda) \\ &= -\frac{p^2 + \lambda}{n^2} \frac{1}{\big(x^2 + y^2 + 2x \cdot y/(m+1)\big)\big(x^2 + y^2 + 2x \cdot y/(m+1) + (p^2 + \lambda)/n^2\big)} \end{aligned} \tag{5.21}$$

For any n, p, λ, the integral kernel (5.21) defines a Hilbert–Schmidt operator and therefore its norm can be estimated as

$$\begin{aligned} \|T_n(p^2 + \lambda)\|^2 &\leqslant \int dx\, dy\, |T_n(x, y; p^2 + \lambda)|^2 \\ &\leqslant \frac{(p^2 + \lambda)^2 (m+1)^4}{n^4 m^4} \int dx\, dy\, \frac{1}{(x^2 + y^2)^2 \big(x^2 + y^2 + (m+1)(p^2 + \lambda)/n^2 m\big)^2} \\ &= \frac{c}{n^2} \frac{(p^2 + \lambda)(m+1)^3}{m^3}, \end{aligned} \tag{5.22}$$

where c is a numerical constant. Using this estimate in (5.20), we obtain (5.19). Moreover we notice that

$$\alpha \|\eta_n\|^2 = \mathcal{O}(n). \tag{5.23}$$

Therefore, from (5.18), (5.19), (5.23), we have

$$F_\alpha(u_n) = \Phi_\alpha^\lambda(\eta_n) - \lambda \|u_n\|^2 = n^2 \frac{4\pi^2 m^5 (m+2)^2}{(m+1)^4}\, \widetilde{\Phi}(f) + \mathcal{O}(n), \tag{5.24}$$

where

$$\widetilde{\Phi}(f) = \int dx |x| |f(x)|^2 + \frac{1}{2\pi^2} \frac{m+1}{\sqrt{m(m+2)}} \int dx\, dy \frac{\overline{f}(x) f(y)}{x^2 + y^2 + 2x \cdot y/(m+1)}. \tag{5.25}$$

Thus the problem is reduced to find f such that $\widetilde{\Phi}(f) < 0$. We introduce polar coordinates (ρ, θ, φ) in $\mathbb{R}^3$ and denote the standard measure on S^2 by dz. We further specialize our choice of the trial function by

$$f(\rho, \theta, \varphi) = \frac{1}{\|P_l\|_2} a(\rho)\, P_l(\cos\theta), \tag{5.26}$$

where the radial part a will be specified later. Then we have

$$\begin{aligned}\widetilde{\Phi}(f) = 2\pi \int_0^{+\infty} d\rho\, \rho^3 |a(\rho)|^2 + \frac{1}{2\pi^2} \frac{m+1}{\sqrt{m(m+2)}} \frac{1}{\|P_l\|_2^2} \int_0^{+\infty} d\rho_1\, d\rho_2\, \rho_1^2 \rho_2^2\, \overline{a}(\rho_1) a(\rho_2) \\ \times \int_{S^2} dz_1\, dz_2 \frac{P_l(\cos\theta_1)\, P_l(\cos\theta_2)}{\rho_1^2 + \rho_2^2 + \frac{2}{m+1}\rho_1\rho_2 \cos\theta_{12}}\end{aligned} \tag{5.27}$$

where θ_{12} is the angle between x and y. With the change of variable $e^x = \rho$ we arrive at

$$\begin{aligned}\widetilde{\Phi}(f) = 2\pi \int_{\mathbb{R}} dx\, |e^{2x} a(e^x)|^2 + \frac{1}{4\pi^2} \frac{m+1}{\sqrt{m(m+2)}} \frac{1}{\|P_l\|_2^2} \int_{\mathbb{R}} dx_1\, dx_2\, e^{2x_1} \overline{a}(e^{x_1}) e^{2x_2} a(e^{x_2}) \\ \times \int_{S^2} dz_1 dz_2 \frac{P_l(\cos\theta_1)\, P_l(\cos\theta_2)}{\mathrm{ch}(x_1 - x_2) + \frac{1}{m+1}\cos\theta_{12}},\end{aligned} \tag{5.28}$$

Both terms appearing in (5.28) can be diagonalized by Fourier transform, see e.g. [8], and we get

$$\widetilde{\Phi}(f) = \int dk\, |d(k)|^2\, S_l(k), \qquad d(k) = \frac{1}{\sqrt{2\pi}} \int dx\, e^{-ixk} e^{2x} a(e^x), \tag{5.29}$$

where $S_l(k)$ is the smooth and even function given by

$$S_l(k) = 2\pi + \frac{1}{2\pi} \frac{m+1}{\sqrt{m(m+2)}} \frac{1}{\|P_l\|_2^2} \int_{S^2} dz_1\, dz_2\, P_l(\cos\theta_1)\, P_l(\cos\theta_2) \frac{\mathrm{sh}\, \gamma k}{\sin\gamma\ \mathrm{sh}\, \pi k} \tag{5.30}$$

and $\gamma = \arccos\left(\frac{\cos\theta_{12}}{m+1}\right)$. Now we study the sign of $S_l(0)$ and, in particular, we show that if (5.12) holds then $S(0) < 0$. We have

$$S_l(0) = 2\pi + \frac{1}{2\pi^2} \frac{m+1}{\sqrt{m(m+2)}} \frac{1}{\|P_l\|_2^2} \int_{S^2} dz_1\, dz_2\, P_l(\cos\theta_1)\, P_l(\cos\theta_2) \frac{\gamma}{\sin\gamma}. \tag{5.31}$$

Let us briefly focus on the angular integral appearing in (5.31). Introducing $\varphi_{12} = \varphi_1 - \varphi_2$ and performing the change of variables $(\varphi_1, \theta_1, \varphi_2, \theta_2) \to (\varphi_1, \theta_1, \varphi_{12}, \theta_{12})$, the integral takes the form

$$\int_{S^2} dz_{12} \frac{\gamma(\theta_{12})}{\sin\gamma(\theta_{12})} \int_{S^2} dz_1 P_l(\cos\theta_1) P_l(\cos\theta_1 \cos\theta_{12} + \sin\theta_1 \sin\theta_{12} \cos(\varphi_1 - \varphi_{12})). \tag{5.32}$$

Now we exploit the additional properties of Legendre polynomials, see [10],

$$
\begin{aligned}
&P_l(\cos\theta_1\cos\theta_{12}+\sin\theta_1\sin\theta_{12}\cos(\varphi_{12}-\varphi_1))\\
&=P_l(\cos\theta_1)P_l(\cos\theta_{12})+2\sum_{k=1}^{\infty}\frac{\Gamma(l-k+1)}{\Gamma(l+k+1)}P_l^k(\cos\theta_1)P_l^k(\cos\theta_{12})\cos(k(\varphi_{12}-\varphi_1)).
\end{aligned}
\tag{5.33}
$$

The integral over φ_{12} kills all the terms in the series in (5.33) and we obtain

$$
\int_{S^2} dz_1\,dz_2\;P_l(\cos\theta_1)\;P_l(\cos\theta_2)\frac{\gamma}{\sin\gamma}=4\pi^2\|P_l\|_2^2\int_{-1}^{1}dy\;P_l(y)\frac{\arccos\,(y/(m+1))}{\sin\arccos\,(y/(m+1))}.
\tag{5.34}
$$

Subsituting in (5.31), we arrive at

$$
S_l(0)=2\pi\left[1+\frac{m+1}{\pi\sqrt{m(m+2)}}\int_{-1}^{1}dy\;P_l(y)\frac{\arccos\,(y/(m+1))}{\sin\arccos\,(y/(m+1))}\right],
\tag{5.35}
$$

and $S_l(0)<0$ by (5.12). Then it is sufficient to fix the radial function a such that $d(k)$ is, roughly speaking, supported around $k=0$. We choose

$$
a(\rho)=c\;\frac{\sqrt{\beta}}{\rho^2\,\mathrm{ch}\left(\frac{1}{2}(\rho^{\beta}+\rho^{-\beta})\right)},\qquad \beta>0.
\tag{5.36}
$$

A straightforward calculation gives

$$
d(k)=\frac{c}{\sqrt{\beta}}\;\widehat{h}\left(\frac{k}{\beta}\right),\qquad h(x)=\frac{1}{\mathrm{ch}(\mathrm{ch}\,x)}.
\tag{5.37}
$$

We fix c such that $\|d\|=\|h\|=1$. For β sufficiently small, possibly depending on l, $\widetilde{\Phi}(f)<0$ and the proof is complete. $\square$

We conclude this section with a discussion of more detailed discussion condition (5.12) for $l=0,1,2,3$. We shall make use of the identities $\arccos z=\pi/2-\arcsin z$, $\cos\arcsin z=\sin\arccos z$ and of the explicit expression of the first Legendre polynomials.

For $l=0$, a straightforward computation shows that the l.h.s. of (5.12) reads

$$
1+\frac{m+1}{\sqrt{m(m+2)}}\int_0^1 dy\;\frac{1}{\cos\arcsin\,(y/(m+1))}
$$

and this means that condition (5.12) is never satisfied for any $m>0$.

For $l=1$, condition (5.12) reads

$$
1-\frac{2(m+1)^3}{\pi\sqrt{m(m+2)}}\int_0^{\arcsin\frac{1}{m+1}}x\;\sin x\,dx<0
$$

and therefore it reduces to the unboundedness conditions appearead in [18] which is satisfied for $m^{-1}>13.607$.

For $l=2$, condition (5.12) cannot be satisfied. It is convenient to decompose the integral into a positive and a negative part according to the sign of the Legendre polynomial and estimate each term separately taking into account the monotonicity of the other the other part of the integrand. In facts, we have

$$
\begin{aligned}
1+\frac{m+1}{\pi\sqrt{m(m+2)}}\int_0^1 dy\,(3y^2-1)\frac{1}{\cos\arcsin\left(y/(m+1)\right)} \\
&= 1+\frac{m+1}{\pi\sqrt{m(m+2)}}\int_0^{\frac{1}{\sqrt{3}}} dy\,(3y^2-1)\frac{1}{\cos\arcsin\left(y/(m+1)\right)} \\
&\quad+\frac{m+1}{\pi\sqrt{m(m+2)}}\int_{\frac{1}{\sqrt{3}}}^1 dy\,(3y^2-1)\frac{1}{\cos\arcsin\left(y/(m+1)\right)} \\
&\geq 1+\frac{m+1}{\pi\sqrt{m(m+2)}}\int_0^{\frac{1}{\sqrt{3}}} dy\,(3y^2-1)\frac{1}{\cos\arcsin\left(\frac{1}{\sqrt{3}}(m+1)\right)} \\
&\quad+\frac{m+1}{\pi\sqrt{m(m+2)}}\int_{\frac{1}{\sqrt{3}}}^1 dy\,(3y^2-1)\frac{1}{\cos\arcsin\left(\frac{1}{\sqrt{3}}(m+1)\right)}=1.
\end{aligned}
$$

For $l=3$, condition (5.12) is satisfied at least for sufficiently small m. This can be easily seen estimating the integral as in the previous case.

$$
\begin{aligned}
1-\frac{2(m+1)}{\pi\sqrt{m(m+2)}}\int_0^1 dy\,(5y^3-3y)\frac{\arcsin\left(y/(m+1)\right)}{\cos\arcsin\left(y/(m+1)\right)} \\
&\leq 1-\frac{2(m+1)}{\pi\sqrt{m(m+2)}}\int_0^{\sqrt{\frac{3}{5}}} dy\,(5y^3-3y)\frac{\arcsin\left(\sqrt{3/5}/(m+1)\right)}{\cos\arcsin\left(\frac{1}{m+1}\sqrt{3/5}\right)} \\
&\quad+-\frac{2(m+1)}{\pi\sqrt{m(m+2)}}\int_{\sqrt{\frac{3}{5}}}^1 dy\,(5y^3-3y)\frac{\arcsin\left(\sqrt{3/5}/(m+1)\right)}{\cos\arcsin\left(\frac{1}{m+1}\sqrt{3/5}\right)} \\
&= 1-\frac{m+1}{2\pi\sqrt{m(m+2)}}\frac{\arcsin\left(\sqrt{3/5}/(m+1)\right)}{\cos\arcsin\left(\frac{1}{m+1}\sqrt{3/5}\right)}.
\end{aligned}
$$

A numerical analysis of condition (5.12) shows that for $l=3$ it is satisfied for $m^{-1}>10.659$.

The structure exhibited by these examples can be extended to the general case. It can be proved, see [6], that condition (5.12) is never satisfied for even l and it is satisfied for m less than a critical value $m^*(l)$ for odd l.

6. Appendix: Renormalization procedure and properties of the potential

In order to obtain a well-defined operator corresponding to the formal Hamiltonian (2.1), the first step is to introduce a regularization and this is more conveniently done in the Fourier space. Using the representation

$$\delta(y_j - x_i) = \frac{1}{(2\pi)^3} \int dw\, e^{iw(y_j - x_i)}, \tag{6.1}$$

a direct computation yields

$$\begin{aligned}(\hat{H}\hat{u})(\boldsymbol{p}_N, \boldsymbol{k}_M) &= h_0(\boldsymbol{p}_N, \boldsymbol{k}_M)\hat{u}(\boldsymbol{p}_N, \boldsymbol{k}_M) - \frac{\mu}{(2\pi)^3} \sum_{(i,j)} \int dz\, \hat{u}(\hat{\boldsymbol{p}}_i, p_i + z, \hat{\boldsymbol{k}}_j, k_j - z) \\ &= h_0(\boldsymbol{p}_N, \boldsymbol{k}_M)\hat{u}(\boldsymbol{p}_N, \boldsymbol{k}_M) \\ &\quad - \frac{2^{3/2}\mu}{(2\pi)^3} \sum_{(i,j)} \int dz\, \hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{p_i + k_j}{2} + \frac{z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{k_j - z}{2} - \frac{z}{\sqrt{2}}\Big).\end{aligned} \tag{6.2}$$

A natural regularization of (6.2) is the following Hamiltonian depending on the cut-off $R > 0$

$$\begin{aligned}(\hat{H}_R\hat{u})(\boldsymbol{p}_N, \boldsymbol{k}_M) &= h_0(\boldsymbol{p}_N, \boldsymbol{k}_M)\hat{u}(\boldsymbol{p}_N, \boldsymbol{k}_M) \\ &- \mu_R \sum_{(i,j)} 1_R\Big(\frac{p_i - k_j}{\sqrt{2}}\Big) \int dz\, 1_R(z)\hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{p_i + k_j}{2} + \frac{z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{k_j - z}{2} - \frac{z}{\sqrt{2}}\Big),\end{aligned} \tag{6.3}$$

where μ_R is a new coupling constant explicitly dependent on the cut-off and 1_R is the characteristic function of the ball in $\mathbb{R}^3$ of radius R and center in the origin. It is obviously true that (6.3) defines a lower bounded self-adjoint operator for any $R > 0$ with the same domain of the free Hamiltonian. The next step is to compute the quadratic form associated to (6.3) and then to take the limit $R \to \infty$ for a suitably chosen μ_R. The identification of the limit is easier if one introduces the following "volume charges" for $i = 1, \dots, N$, $j = 1, \dots, M$,

$$\begin{aligned}&\hat{\rho}_{ij}^R(\boldsymbol{p}_N, \boldsymbol{k}_M) \\ &\quad = \mu_R 1_R\Big(\frac{p_i - k_j}{\sqrt{2}}\Big) \int dz\, 1_R(z)\hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{p_i + k_j}{2} + \frac{z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{k_j - z}{2} - \frac{z}{\sqrt{2}}\Big),\end{aligned} \tag{6.4}$$

and the corresponding "potentials" produced by $\hat{\rho}_{ij}^R$,

$$\widehat{G^\lambda \rho^R}(\boldsymbol{p}_N, \boldsymbol{k}_M) = \sum_{(i,j)} \widehat{G^\lambda \rho_{ij}^R}(\boldsymbol{p}_N, \boldsymbol{k}_M) = \sum_{(i,j)} \frac{\hat{\rho}_{ij}^R(\boldsymbol{p}_N, \boldsymbol{k}_M)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda}, \tag{6.5}$$

where $\lambda > 0$. Hence a direct computation yields

$$\begin{aligned}(\hat{u}, \hat{H}_R\hat{u}) &= \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, h_0|\hat{u}|^2 - \sum_{(i,j)} \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, \overline{\hat{u}}\, \hat{\rho}_{ij}^R \\ &= \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M \Big[(h_0 + \lambda)|\hat{u} - \widehat{G^\lambda \rho^R}|^2 - \lambda|\hat{u}|^2\Big] - \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M (h_0 + \lambda)|\widehat{G^\lambda \rho^R}|^2 \\ &\quad + 2\,\mathrm{Re} \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, \overline{\hat{u}}\, (h_0 + \lambda)\widehat{G^\lambda \rho^R} - \sum_{(i,j)} \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, \overline{\hat{u}}\, \hat{\rho}_{ij}^R\end{aligned}$$

$$
\begin{aligned}
&= \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M \Big[(h_0+\lambda)|\hat{u} - \widehat{G^\lambda \rho^R}|^2 - \lambda |\hat{u}|^2 \Big] \\
&\quad - \sum_{(i,j)\neq(l,h)}{}' \int d\boldsymbol{p}_N \boldsymbol{k}_M \frac{\overline{\hat{\rho}_{ij}^R}\, \hat{\rho}_{lh}^R}{h_0+\lambda} - \sum_{(i,j)} \int d\boldsymbol{p}_N \boldsymbol{k}_M \frac{|\hat{\rho}_{ij}^R|^2}{h_0+\lambda} \\
&\quad + \sum_{(i,j)} \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, \overline{\hat{u}}\, \hat{\rho}_{ij}^R, \qquad (6.6)
\end{aligned}
$$

where we have used (6.3), (6.4), (6.5) and the fact that

$$
\operatorname{Im} \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, \overline{\hat{u}}\, \hat{\rho}_{ij}^R = 0. \qquad (6.7)
$$

Let us define the following "surface charges" for $i = 1, \ldots, N,\ j = 1, \ldots, M,$

$$
\hat{\xi}_{ij}^R(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \mu_R \int dz\, 1_R(z) \hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{q+z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{q-z}{\sqrt{2}} \Big). \qquad (6.8)
$$

We notice that

$$
\hat{\xi}_{ij}^R\Big(\frac{p_i+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j \Big) = \mu_R \int dz\, 1_R(z) \hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{p_i+k_j}{2} + \frac{z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{p_i+k_j}{2} - \frac{z}{\sqrt{2}} \Big), \qquad (6.9)
$$

and therefore

$$
\hat{\rho}_{ij}^R(\boldsymbol{p}_N, \boldsymbol{k}_M) = 1_R\Big(\frac{p_i-k_j}{\sqrt{2}} \Big) \hat{\xi}_{ij}^R\Big(\frac{p_i+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j \Big). \qquad (6.10)
$$

Let us rewrite the last two integrals in (6.6). We have

$$
\int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, \overline{\hat{u}}\, \hat{\rho}_{ij}^R = \mu_R^{-1} \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j\, |\hat{\xi}_{ij}^R(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 \qquad (6.11)
$$

and

$$
\begin{aligned}
\int d\boldsymbol{p}_N\, d\boldsymbol{k}_M \frac{|\hat{\rho}_{ij}^R|^2}{h_0+\lambda} &= \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, 1_R\Big(\frac{p_i-k_j}{\sqrt{2}} \Big) \frac{|\hat{\xi}_{ij}^R\Big((p_i+k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j \Big)|^2}{h_0+\lambda} \\
&= \int dq\, d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\, |\hat{\xi}_{ij}^R(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2\, \mathcal{I}_R(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \qquad (6.12)
\end{aligned}
$$

where we have introduced the integration variables $q = (p_i+k_j)/\sqrt{2}$, $z = (p_i-k_j)/\sqrt{2}$ and we have defined

$$
\begin{aligned}
&\mathcal{I}_R(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \int dz\, \frac{1_R(z)}{(m+1)z^2/2m + \frac{m-1}{m}\, q\cdot z + \gamma}, \\
&\gamma = \frac{m+1}{2m} q^2 + \hat{\boldsymbol{p}}_i^2 + \frac{1}{m} \hat{\boldsymbol{k}}_j^2 + \lambda.
\end{aligned} \qquad (6.13)
$$

Using (6.11), (6.12) in (6.6), we have

$$
\begin{aligned}
(\hat{u}, \hat{H}_R\hat{u}) = &\int d\boldsymbol{p}_N\, d\boldsymbol{k}_M \Big[(h_0+\lambda)|\hat{u} - \widehat{G^\lambda \rho^R}|^2 - \lambda |\hat{u}|^2 \Big] \\
&+ \sum_{(i,j)} \int dq\, d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\, |\hat{\xi}^R_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 \left(\mu_R^{-1} - \mathcal{I}_R(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)\right) \\
&- \sum_{(i,j)\neq(l,h)}' \int d\boldsymbol{p}_N \boldsymbol{k}_M \frac{\overline{\hat{\rho}^R_{ij}}\, \hat{\rho}^R_{lh}}{h_0+\lambda}.
\end{aligned} \tag{6.14}
$$

For $R \to \infty$ one has

$$
\begin{aligned}
\mathcal{I}_R(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) &= \frac{2m}{m+1} \int dz\, \frac{1_R(z)}{z^2} \\
&\quad - \frac{2m}{m+1} \int dz\, \frac{\frac{m-1}{m} q\cdot z + \gamma}{z^2 \left((m+1)z^2/2m + \frac{m-1}{m} q\cdot z + \gamma \right)} + o(1) \\
&= \frac{8\pi\, m}{m+1} R - 2\pi^2 \left(\frac{2m}{m+1}\right)^{3/2} \sqrt{\frac{2}{m+1} q^2 + \hat{\boldsymbol{p}}_i^2 + \frac{1}{m} \hat{\boldsymbol{k}}_j^2 + \lambda} + o(1),
\end{aligned} \tag{6.15}
$$

where in the last line we have used the explicit integration

$$
\int dz\, \frac{\delta\cdot z + \gamma}{z^2 (z^2 + \delta\cdot z + \gamma)} = \pi^2 \sqrt{4\gamma - \delta^2}, \qquad \delta^2 < 4\gamma. \tag{6.16}
$$

Therefore, in order to obtain a nontrivial limit for $R \to \infty$, we fix

$$
\mu_R^{-1} = \frac{8\pi\, m}{m+1} R + \alpha \tag{6.17}
$$

where $\alpha \in \mathbb{R}$ is a new coupling constant. At least formally, with this choice we can remove the cut-off and define the renormalized quadratic form as the limit of (6.14) for $R \to \infty$. More precisely, we are lead to the following definition of quadratic form

$$
\begin{aligned}
G_\alpha(u) = &\int d\boldsymbol{p}_N\, d\boldsymbol{k}_M \Big[(h_0(\boldsymbol{p}_N, \boldsymbol{k}_M)+\lambda) \Big| \Big(\hat{u} - \sum_{(i,j)} \widehat{G^\lambda \xi_{ij}}\Big)(\boldsymbol{p}_N, \boldsymbol{k}_M)\Big|^2 - \lambda |\hat{u}(\boldsymbol{p}_N, \boldsymbol{k}_M)|^2 \Big] \\
&+ \sum_{(i,j)} \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j \left(\alpha + 2\pi^2 \left(\frac{2m}{m+1}\right)^{3/2} \sqrt{h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda} \right) |\hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 \\
&- \sum_{(i,j)\neq(l,h)}' \int d\boldsymbol{p}_N d\boldsymbol{k}_M \frac{\overline{\hat{\xi}_{ij}}\left((p_i+k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right) \hat{\xi}_{lh}\left((p_l+k_h)/\sqrt{2}, \hat{\boldsymbol{p}}_l, \hat{\boldsymbol{k}}_h\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M)+\lambda},
\end{aligned} \tag{6.18}
$$

where

$$\hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \lim_{R\to\infty} \hat{\xi}^R_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \tag{6.19}$$

and the potential produced by the surface charges $\hat{\xi}_{ij}$ was introduced in (2.13).

In the quadratic form (6.18) the particles in the two groups $\mathcal{A}$ and $\mathcal{B}$ are still considered distinguishable. Since we want to describe fermions, the final step of the construction is to take into account the requirement of antisymmetry. From (6.19), (6.8) it is easily seen that

$$\hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = (-1)^{i+j}\, \hat{\xi}_{11}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \equiv (-1)^{i+j}\, \hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \tag{6.20}$$

which in particular means that the interaction is completely described by the unique surface charge $\hat{\xi}_{11} \equiv \hat{\xi}$. We also denote

$$\widehat{G^\lambda \xi}(\boldsymbol{p}_N, \boldsymbol{k}_M) = \sum_{(i,j)} \frac{(-1)^{i+j}\hat{\xi}\left((p_i + k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda} \tag{6.21}$$

Moreover for the products of surface charges in (6.18) we have

$$\begin{aligned}
&\overline{\hat{\xi}_{ij}}\left((p_i + k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right) \hat{\xi}_{lh}\left((p_l + k_h)/\sqrt{2}, \hat{\boldsymbol{p}}_l, \hat{\boldsymbol{k}}_h\right)\\
&\qquad = \overline{\hat{\xi}}\left((p_1 + k_1)/\sqrt{2}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1\right) \hat{\xi}\left((p_2 + k_2)/\sqrt{2}, \hat{\boldsymbol{p}}_2, \hat{\boldsymbol{k}}_2\right) \quad \text{if} \quad i \neq l, \quad j \neq h.\\
&\overline{\hat{\xi}_{ij}}\left((p_i + k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right) \hat{\xi}_{ih}\left((p_i + k_h)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_h\right)\\
&\qquad = -\overline{\hat{\xi}}\left((p_1 + k_1)/\sqrt{2}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1\right) \hat{\xi}\left((p_1 + k_2)/\sqrt{2}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_2\right) \quad \text{if} \quad i = l, \quad j \neq h.\\
&\overline{\hat{\xi}_{ij}}\left((p_i + k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right) \hat{\xi}_{lj}\left((p_l + k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_l, \hat{\boldsymbol{k}}_j\right)\\
&\qquad = -\overline{\hat{\xi}}\left((p_1 + k_1)/\sqrt{2}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1\right) \hat{\xi}\left((p_2 + k_1)/\sqrt{2}, \hat{\boldsymbol{p}}_2, \hat{\boldsymbol{k}}_1\right) \quad \text{if} \quad i \neq l, \quad j = h.
\end{aligned} \tag{6.22}$$

Taking into account the above symmetry constraints in (6.18), we finally arrive at the quadratic form (2.4), (2.5) and (2.6).

In the next proposition we collect some useful properties of the potential produced by the surface charges ξ_{ij}.

PROPOSITION 6.1. *For $\xi_{ij} \in L^2(\mathbb{R}^{3(N+M-1)})$ the corresponding potential $G^\lambda \xi_{ij}(\boldsymbol{x}_N, \boldsymbol{y}_M)$ satisfies*

$$\left[(H_0 + \lambda)\, G^\lambda \xi_{ij}\right](\boldsymbol{x}_N, \boldsymbol{y}_M) = 8\pi^{3/2} \xi_{ij}(\sqrt{2}\, x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j)\, \delta(x_i - y_j) \tag{6.23}$$

in the distributional sense. For $\xi_{ij} \in H^1(\mathbb{R}^{3(N+M-1)})$ the singularity for $|x_i - y_j| \to 0$ is characterized as follows

$$(G^\lambda \xi_{ij})(\boldsymbol{x}_N, \boldsymbol{y}_M) = \frac{1}{|x_i - y_j|} \frac{2\sqrt{\pi}\, m}{m+1} \xi_{ij}(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) - \frac{2\pi^2 (2m)^{3/2}}{(2\pi)^{\frac{3}{2}(N+M)}(m+1)^{3/2}} \cdot \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j\, e^{i\left(\sqrt{2}x_i q + \hat{\boldsymbol{x}}_i\cdot\hat{\boldsymbol{p}}_i + \hat{\boldsymbol{y}}_j\cdot\hat{\boldsymbol{k}}_j\right)} \sqrt{h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}\; \hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + o(1). \tag{6.24}$$

Moreover for $\xi_{ij} \in H^{-1/2}(\mathbb{R}^{3(N+M-1)})$ *one has*

$$c_1 \int dq\, d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \frac{|\hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2}{\sqrt{q^2 + \hat{\boldsymbol{p}}_i^2 + \hat{\boldsymbol{k}}_j^2/m + \lambda}} \le \|G^\lambda \xi_{ij}\|^2 \le c_2 \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j \frac{|\hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2}{\sqrt{q^2 + \hat{\boldsymbol{p}}_i^2 + \hat{\boldsymbol{k}}_j^2/m + \lambda}}, \tag{6.25}$$

where $c_1 = \pi^2 \min\{m, 1\}$, $c_2 = \pi^2 \max\{m, 1\}$.

Proof: Let us fix a test function ϕ and let us consider the definition (2.14) of $\widehat{G^\lambda \xi_{ij}}$. Then, using Fourier transform, we have

$$\begin{aligned} &\int d\boldsymbol{x}_N\, d\boldsymbol{y}_M\, \overline{\phi}(\boldsymbol{x}_N, \boldsymbol{y}_M) \left[(H_0 + \lambda)\, G^\lambda \xi_{ij}\right](\boldsymbol{x}_N, \boldsymbol{y}_M) \\ &= \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, \overline{\hat{\phi}}(\boldsymbol{p}_N, \boldsymbol{k}_M) \hat{\xi}_{ij}\left((p_1 + k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right) \\ &= \frac{8\pi^{3/2}}{(2\pi)^{\frac{3}{2}(N+M)}} \int dx_i\, d\hat{\boldsymbol{x}}_i\, d\hat{\boldsymbol{y}}_j\, \xi_{ij}(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \cdot \int d\boldsymbol{p}_N\, d\boldsymbol{k}_M\, \overline{\hat{\phi}}(\boldsymbol{p}_N, \boldsymbol{k}_M)\, e^{-i[(p_i + k_j)\cdot x_i + \hat{\boldsymbol{p}}_i \cdot \hat{\boldsymbol{x}}_i + \hat{\boldsymbol{k}}_j \cdot \hat{\boldsymbol{y}}_j)]} \\ &= 8\pi^{3/2} \int dx_i\, d\hat{\boldsymbol{x}}_i\, d\hat{\boldsymbol{y}}_j\, \xi_{ij}(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \overline{\phi}(\hat{\boldsymbol{x}}_i, x_i, \hat{\boldsymbol{y}}_j, x_i) \end{aligned} \tag{6.26}$$

and this proves (6.23). From (2.14) we also have

$$\begin{aligned} G^\lambda \xi_{ij}(\boldsymbol{x}_N, \boldsymbol{y}_M) &= \frac{1}{(2\pi)^{\frac{3}{2}(N+M)}} \int d\boldsymbol{p}_N d\boldsymbol{k}_M\, e^{i(\boldsymbol{x}_N\cdot \boldsymbol{p}_N + \boldsymbol{y}_M \cdot \boldsymbol{k}_M)} \frac{\hat{\xi}_{ij}\left((p_1 + k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda} \\ &= \frac{1}{(2\pi)^{\frac{3}{2}(N+M)}} \int dq\, d\hat{\boldsymbol{p}}_i\, d\hat{\boldsymbol{k}}_j\, e^{i\left[\frac{x_i + y_j}{\sqrt{2}}\cdot q + \hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{p}}_i + \hat{\boldsymbol{y}}_j \cdot \hat{\boldsymbol{k}}_j\right]} \hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)\, \mathcal{L}(x_i - y_j, q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j), \end{aligned} \tag{6.27}$$

where

$$\mathcal{L}(x_i - y_j, q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \int dz \, \frac{e^{i(x_i - y_j)z/\sqrt{2}}}{\frac{m+1}{2m} z^2 + \frac{m-1}{m} q \cdot z + \gamma},$$
$$\gamma = \frac{m+1}{2m} q^2 + \hat{\boldsymbol{p}}_i^2 + \frac{1}{m} \hat{\boldsymbol{k}}_j^2 + \lambda. \tag{6.28}$$

For $|x_i - y_j| \to 0$ the last integral is given by

$$\begin{aligned}
&\mathcal{L}(x_i - y_j, q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)\\
&= \frac{2m}{m+1} \int dz \, \frac{e^{i(x_i - y_j)/\sqrt{2}\cdot z}}{z^2} - \frac{2m}{m+1} \int dz \, \frac{\frac{m-1}{m} q \cdot z + \gamma}{z^2 \left(\frac{m+1}{2m} z^2 + \frac{m-1}{m} q \cdot z + \gamma \right)} + o(1)\\
&= \frac{4\sqrt{2}\pi^2 m}{m+1} \frac{1}{|x_i - y_j|} - 2\pi^2 \left(\frac{2m}{m+1} \right)^{3/2} \sqrt{h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda} + o(1)
\end{aligned} \tag{6.29}$$

where we have used the explicit integration (6.16). Using (6.29) in (6.27) we obtain (6.24).

Finally for the proof of (6.25) we observe that

$$\|G^\lambda \xi_{ij}\|^2 = \int d\boldsymbol{p}_N d\boldsymbol{k}_M \, \frac{\left|\hat{\xi}\left((p_i + k_j)/\sqrt{2}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right)\right|^2}{\left(h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda\right)^2}. \tag{6.30}$$

Introducing the coordinates $q = (p_i + k_j)/\sqrt{2}$, $v = (p_i - k_j)/\sqrt{2}$ and using the elementary inequality $-\frac{1}{2}(v^2 + q^2) \le v \cdot q \le \frac{1}{2}(v^2 + q^2)$ we have

$$c_1 \int dq \, d\hat{\boldsymbol{p}}_i \, d\hat{\boldsymbol{k}}_j \, |\hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 \mathcal{M} \le \|G^\lambda \xi_{ij}\|^2 \le c_2 \int dq \, d\hat{\boldsymbol{p}}_i \, d\hat{\boldsymbol{k}}_j \, |\hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 \mathcal{M}, \tag{6.31}$$

where

$$\mathcal{M} = \frac{1}{\pi^2} \int dv \, \frac{1}{\left(v^2 + q^2 + \hat{\boldsymbol{p}}_i^2 + \hat{k}_j^2/m + \lambda\right)^2} \tag{6.32}$$

By an explicit computation of the above integral we obtain (6.25). □

REFERENCES

[1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden: *Solvable Models in Quantum Mechanics*, Springer, New York 1988.

[2] A. Alonso and B. Simon: The Birman–Krein–Vishik theory of self-adjoint extensions of semibounded operators, *J. Operator Theory* **4** (1980), 251–270.

[3] E. Braaten and H. W. Hammer: Universality in few-body systems with large scattering length, *Phys. Rep.* **428** (2006), 259–390.

[4] M. S. Birman: On the self-adjoint extensions of positive definite operators (in Russian), *Math. Sb.* **38** (1956), 431–450.

[5] Y. Castin, C. Mora and L. Pricoupenko: Four-body Efimov effect, arXiv:1006.4720v 1.

[6] M. Correggi, G. Dell'Antonio, D. Finco A. Michelangeli and A. Teta: Zero-range interactions for n identical fermions plus a different particle, in preparation.

[7] G. Dell'Antonio, R. Figari and A. Teta: Hamiltonians for systems of N particles interacting through point interactions, *Ann. Inst. Henri Poincare, Phys. Theo.* **60** (1994), 253–290.

[8] A. Erdely et al,: *Tables of Integrals Transforms*, Mc Graw-Hill, New York 1954.

[9] G. Flamand: Mathematical theory of non-relativistic two- and three-particle systems with point interactions, in *Cargese Lectures in Theoretical Physics*, Lurcat F. ed., 1967, 247–287.

[10] I. S. Gradshteyn and I. M. Ryzhik: *Table of Integrals, Series, and Products*, Academiv Press 2007.

[11] T. Kato: *Perturbation Theory for Linear Operators*, Springer, Berlin 1980.

[12] M. Krein: The theory of self-adjoint extensions of semibounded hermitian transformations and its applications, *Rec. Math. (Math. Sb.)* **20** (1947), 431–495.

[13] A. M. Melnikov and R. A. Minlos: On the pointlike interaction of three different particles, *Adv. Soviet Math.* **5** (1991), 99–112.

[14] R. A. Minlos: On the point interaction of three particles, *Lect. Notes in Physics* **324**, Springer, 1989.

[15] R. A. Minlos: On point-like interaction between n fermions and another particle, *Moscow Math. Journal* **11** (2011), 113–127.

[16] R. A. Minlos and L. Faddeev: On the point interaction for a three-particle system in Quantum Mechanics, *Soviet Phys. Dokl.* **6** (1962), 1072–1074.

[17] R. A. Minlos and M. S. Shermatov: On pointlike interaction of three quantum particles, *Vestnik Mosk, Univ. Ser. I Math. Mekh.* **6** (1989), 7–14.

[18] M. K. Shermatov: Point Interaction between two fermions and a particle of a different nature, *Theo. Math. Phys.* **136** (2003), 1119–1130.

[19] F. Werner and Y. Castin: Unitary gas in an isotropic harmonic trap: symmetry properties and applications, *Phys. Rev. A* **74** (2006), 053–604.

[20] F. Werner and Y. Castin: Unitary quantum three-body problem in a harmonic trap, *Phys. Rev. Lett.* **97** (2006), 150–401.