Notes on stability of the Fermi gas with point interactions

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Dated: August 4, 2021

Contents

We study in these notes the Fermi gas, *i.e.* a many-body system of spin-$\frac{1}{2}$ fermions or more generally just two species of fermions. The specific gas we study is interacting *via.* point interactions *or* zero-range interactions. We will restrict to the case where the two species can have different mass, but all fermions in one species have equal mass. The relevant quantity in this case is the relative mass of the two. Thus by setting the mass of one species to 1 and the mass of the other to m we have en mass-ratio of m. Formally the system we are studying can thus be described by the Hamiltonian

$$H=-\frac{1}{2m}\sum_{j=1}^{M}\Delta_{y_j}-\frac{1}{2}\sum_{i=1}^{N}\Delta_{x_i}+\gamma\sum_{i=1}^{N}\sum_{j=1}^{M}\delta(x_i-y_j), \tag{0.1}$$

where $x_i\in\mathbb{R}^3$ for all $i\in\{1,...,N\}$ and $y_j\in\mathbb{R}^3$ for all $j\in\{1,...,M\}$. Notice that we also restrict to the case of equal coupling between all particles. These formally defined Hamiltonians are clearly ill-defined as the δ-function is a temperate distribution and thus is only defined on the Schwartz functions. However, restricting the domain to the Schwartz functions will not make the Laplacians self-adjoint. Furthermore, the codomain of δ is not in $L^2(\mathbb{R}^{3(N+M)})$. Thus no self-adjoint operator on $L^2(\mathbb{R}^{3(N+M)})$ of this form exists. As a quadratic form $\langle\psi|H\psi\rangle$ might make sense. Since the δ-function can be made to makes sense only on continuous functions, there exist no sensible domain of this quadratic form such that it is closed. If such a domain existed the Laplacians would be closed on it, however, this is only true for $H^1(\mathbb{R}^3)$ which contains non-continuous functions (defined a.e.), making the δ-functions ill-defined.

One way of rigorously studying such formal Hamiltonians is to consider self-adjoint extensions of more well defined Hamiltonians. While this approach is very successful in the $N=M=1$ case, it becomes increasingly difficult as the number of self-adjoint extension become infinite already at the $N=2$ and $M=1$ case. In [3] quadratic forms where developed in order to describe systems of the form (0.1). These quadratic forms are generally more well defined, however their origin and connection to the formal Hamiltonian might be obscured as they need regularization and renormalization procedures to make sense of the point interactions. We will in these notes aim to construct the quadratic form corresponding to the formal expression in (0.1), and show that they can be reaches by considering a sequence of rank one perturbations. We aim at showing that operators corresponding to these rank one perturbations actually converge, in the strong resolvent sense, to the operator of the quadratic form given by [3]. Furthermore, it is our hope that this will shed light on the stability of these systems which has only been shown the cases of $(N,M)=(N,1)$ and $(N,M)=(2,2)$. We start out by considering the simpler case which is $(N,M)=(N,1)$ also denoted the $N+1$ case.

1 Formal Hamiltonian for the $N+1$ case

The formal Hamiltonian of (0.1) can be rewritten in the $N+1$ case by separating the centre of mass. Notice that this indeed already restricts the set of possible self-adjoint Hamiltonians mimicking (0.1) as this asserts the translational invariance of the Hamiltonian. Thus this separation of the centre of mass restricts to couplings that are independent of the centre of mass coordinate. Defining the centre of mass and the relative coordinates by

$$X = \frac{my + \sum_{i=1}^{N} x_i}{m+N}, \quad \tilde{x}_i = x_i - y, \tag{1.1}$$

we obtain that

$$\begin{aligned}\Delta_{x_i} = \sum_{j=1}^{3} \partial_{x_i^j}\partial_{x_i^j} = \sum_{j=1}^{3} \left(\frac{\partial X^j}{\partial x_i^j}\partial_{X^j} + \frac{\partial \tilde{x}_i{}^j}{\partial x_i^j}\partial_{\tilde{x}_i^j} \right) \left(\frac{\partial X^j}{\partial x_i^j}\partial_{X^j} + \frac{\partial \tilde{x}_i{}^j}{\partial x_i^j}\partial_{\tilde{x}_i^j} \right) \\ = \frac{1}{(m+N)^2}\Delta_X + \Delta_{\tilde{x}_i} + \frac{2}{m+N}\nabla_X \cdot \nabla_{\tilde{x}_i},\end{aligned} \tag{1.2}$$

$$\begin{aligned}\Delta_y &= \sum_{j=1}^{3} \partial_{y^j}\partial_{y^j} = \sum_{j=1}^{3} \left(\frac{\partial X^j}{\partial y^j}\partial_{X^j} + \sum_{i=1}^{N} \frac{\partial \tilde{x}_i{}^j}{\partial y^j}\partial_{\tilde{x}_i^j} \right) \left(\frac{\partial X^j}{\partial y^j}\partial_{X^j} + \sum_{i=1}^{N} \frac{\partial \tilde{x}_i{}^j}{\partial y^j}\partial_{\tilde{x}_i^j} \right) \\ &= \frac{m^2}{(m+N)^2}\Delta_X + \sum_{i=1}^{N}\Delta_{\tilde{x}_i} + 2\sum_{\substack{(i,j)=(1,1)\\ i<j}}^{(N,N)} \nabla_{\tilde{x}_i}\cdot\nabla_{\tilde{x}_j} - \frac{2m}{m+N}\sum_{i=1}^{N}\nabla_X\cdot\nabla_{\tilde{x}_i}.\end{aligned} \tag{1.3}$$

Thus we get the Hamiltonian

$$H = -\frac{1}{2(m+N)}\Delta_X - \frac{m+1}{2m}\sum_{i=1}^{N}\Delta_{\tilde{x}_i} - \frac{2}{2m}\sum_{\substack{(i,j)=(1,1)\\ i<j}}^{(N,N)} \nabla_{\tilde{x}_i}\cdot\nabla_{\tilde{x}_j} + \gamma\sum_{i=1}^{N}\delta(\tilde{x}_i), \tag{1.4}$$

which can be recast as

$$H = H_{\text{CM}} + \frac{m+1}{2m}H_{\text{rel}}, \tag{1.5}$$

with $H_{\text{CM}} = -\frac{1}{2(m+N)}\Delta_X$ the free centre of mass and the relative Hamiltonian given by

$$H_{\text{rel}} = -\sum_{i=1}^{N}\Delta_{\tilde{x}_i} - \frac{2}{m+1}\sum_{\substack{(i,j)=(1,1)\\ i<j}}^{(N,N)} \nabla_{\tilde{x}_i}\cdot\nabla_{\tilde{x}_j} + \tilde{\gamma}\sum_{i=1}^{N}\delta(\tilde{x}_i), \tag{1.6}$$

where $\tilde{\gamma} = \frac{2m}{m+1}\gamma$. Notice that the problem has now been split in two independent parts and thus we recognize the centre of mass part as the free particle which is solved by the Laplacian being essentially self adjoint on $C_c^\infty(\mathbb{R}^3)$ functions with self-adjoint extension Δ on $H^2(\mathbb{R}^3)$ where Δ acts in the distributional sense. The relative Hamiltonian on the other hand will be the main focus in the first part of these notes.

2 The $1+1$ case

We are now going to study different ways of rigorously defining the relative Hamiltonian (1.6) in the case of $N = 1$. The first method is easily implemented for $N = 1$ but is hard to generalize.

2.1 Self-adjoint extension

The first method we are going to study is that of self-adjoint extension. We thus restrict the formal Hamiltonian to a domain in which it is well defined. This could for example be $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. Notice since we have removed $\{0\}$ the δ-function has no support on this space and thus vanish. Therefore, we have the relative Hamiltonian

$$H_{\mathrm{rel}} = -\Delta|_{C_c^\infty(\mathbb{R}^3\setminus\{0\})}. \tag{2.1}$$

We now seek to extend this operator to a self-adjoint operator on a larger domain. This is possible since H_{rel} is symmetric and its closure, denoted $\dot{H}_{\mathrm{rel}}$ have deficiency indices $K_+ = K_- = 1$, with $K_\pm = \dim \operatorname{Ran}(H_{\mathrm{rel}} \pm iI)^\perp = \dim\ker(H_{\mathrm{rel}}^* \mp iI)$ where H_{rel}^* denotes the adjoint of H_{rel}. By definition on the adjoint we have that $\mathscr{D}(H_{\mathrm{rel}}^*) = \{f \in L^2(\mathbb{R}^3) |\, \langle f|H_{\mathrm{rel}}\cdot\rangle \text{ is bounded on } \mathscr{D}(H_{\mathrm{rel}})\}$, where the adjoint of the Laplacian acts as the Laplacian in the distributional sense. We determine first the closure of H_{rel}. This can be done by taking the adjoint twice. Notice that the domain of the adjoint is $\mathscr{D}(H_{\mathrm{rel}}^*) = \{f \in L^2(\mathbb{R}^3) |\, \langle f|\Delta\cdot\rangle \text{ is bounded on } C_c^\infty(\mathbb{R}^3 \setminus \{0\})\}$. This can be directly calculated to be

$$\mathscr{D}(H_{\mathrm{rel}}^*) = \{f \in H^{2,\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}) \cap L^2(\mathbb{R}^3) | \Delta f \in L^2(\mathbb{R}^3)\} \tag{2.2}$$

We emphasise that all elements in $\mathscr{D}(H_{\mathrm{rel}}^*)$ should be viewed as distributions in $H^{2,\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$. Therefore the requirement $\Delta f \in L^2(\mathbb{R}^3)$ does not simply restrict the domain to be $H^2(\mathbb{R}^3)$ as elements or their derivative (up to second order) can have singular behaviour at 0, e.g. δ-functions. Notice that $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ is dense in $L^2(\mathbb{R}^3 \setminus \{0\}) = L^2(\mathbb{R}^3)$ (only defined a.e). The domain of the double adjoint is then given by

$$\mathscr{D}(H_{\mathrm{rel}}^{**}) = \{f \in L^2(\mathbb{R}^3) |\, \langle H_{\mathrm{rel}}^* \cdot |f\rangle \text{ is bounded on } \mathscr{D}(H_{\mathrm{rel}}^*)\} = H_0^2(\mathbb{R}^3 \setminus \{0\}), \tag{2.3}$$

where Δ acts in the distributional sense and we have defined

$$H_0^2(\mathbb{R}^3\setminus\{0\}) = \left\{u \in L^2(\mathbb{R}^3) \;\middle|\; \Delta u \in L^2(\mathbb{R}^3) \text{ and } u(x) \to 0 \wedge \nabla u(x) \to 0 \text{ for } |x| \to 0 \vee |x| \to \infty\right\}. \tag{2.4}$$

Thus we have

$$\dot{H}_{\mathrm{rel}} = -\Delta, \qquad \mathscr{D}\left(\dot{H}_{\mathrm{rel}}\right) = H_0^2(\mathbb{R}^3 \setminus \{0\}). \tag{2.5}$$

The adjoint of $\dot{H}_{\mathrm{rel}}$ is simply given by $\dot{H}_{\mathrm{rel}}^* = H_{\mathrm{rel}}^*$, as the adjoint is already closed. Thus we are ready to find all self-adjoint extensions of H_{rel}. By the Krein theorem there exist self-adjoint extension if and only if $\dim\left(\operatorname{Ran}(H_{\mathrm{rel}} - iI)^\perp\right) = \dim\left(\operatorname{Ran}(H_{\mathrm{rel}} + iI)^\perp\right)$ or equivalently $\dim(\ker(H_{\mathrm{rel}}^* + iI)) = \dim(\ker(H_{\mathrm{rel}}^* - iI))$ thus we seek solutions of the equation

$$H_{\mathrm{rel}}^*\psi_\pm = \pm i\psi_\pm, \quad \psi_\pm \in \mathscr{D}(H_{\mathrm{rel}}^*). \tag{2.6}$$

The equation $-\Delta\psi_\pm = \pm i\psi_\pm$ has the unique solution

$$\psi(x)_\pm = \frac{e^{i\sqrt{\pm i}|x-y|}}{|x-y|}, \qquad x \in \mathbb{R}^3 \setminus \{y\}. \tag{2.7}$$

In order for this function to be in the domain of H^*_{rel} we need to choose $y = 0$. Thus we see that $\dim(\ker(H^*_{\text{rel}} + iI)) = \dim(\ker(H^*_{\text{rel}} - iI)) = 1$. By Krein's extension theorem for symmetric operators we have that there exist a one-parameter family of self-adjoint extensions of H_{rel}. Parametrizing the family by a complex phase we have the extensions

$$\mathscr{D}(H_{\text{rel},\theta}) = \left\{ h + c(\xi_+ + e^{i\theta}\xi_-) \middle| h \in \mathscr{D}\left(\dot{H}_{\text{rel}}\right),\ c \in \mathbb{C} \right\}, \tag{2.8}$$

where $\theta \in [0, 2\pi)$, $\xi_+ \in \ker(H^*_{\text{rel}} - iI)$, $\xi_- \in \ker(H^*_{\text{rel}} + iI)$ are fixed with $\|\xi_+\| = \|\xi_-\| = 1$, and where

$$H_{\text{rel},\theta}(h + c(\xi_+ + e^{i\theta}\xi_-)) = H^*_{\text{rel}}(h + c(\xi_+ + e^{i\theta}\xi_-)) = h + ic(\xi_+ - e^{i\theta}\xi_-). \tag{2.9}$$

Following the methods of [1], we however now that there is another characterization of these extensions. By decomposing the Hilbert space into spherical coordinates we obtain the decomposition

$$L^2(\mathbb{R}^3, d^3x) = L^2((0,\infty), r^2dr) \otimes L^2(S^2, d\Omega). \tag{2.10}$$

Furthermore by decomposing into spherical harmonics we have

$$L^2(\mathbb{R}^3, d^3x) = \bigoplus_{l=0}^{\infty} L^2((0,\infty), r^2dr) \otimes \left\langle Y_l^{-l}, Y_l^{-l+1}, ..., Y_l^0, ..., .Y_l^l \right\rangle. \tag{2.11}$$

Now using the unitary transformation $U : L^2((0,\infty), r^2dr) \to L^2((0,\infty), dr)$, defined by $Uf(r) = rf(r)$

$$L^2(\mathbb{R}^3, d^3x) = \bigoplus_{l=1}^{\infty} U^{-1} L^2((0,\infty), dr) \otimes \left\langle Y_l^{-l}, Y_l^{-l+1}, ..., Y_l^0, ..., .Y_l^l \right\rangle, \tag{2.12}$$

where $\langle ... \rangle$ denotes the span. Using the Laplacian in spherical coordinates

$$\Delta\phi = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j(\phi)) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\phi\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi\frac{\partial\phi}{\partial\varphi}) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\phi}{\partial^2\theta}, \tag{2.13}$$

with the usual notation g_{ij} the metric, g^{ij} the inverse metric, $g = \det(g_{ij})$ and where θ denotes the azimuthal angle and φ the zenith angle, it is straightforward to show that

$$\dot{H}_{\text{rel}} = \bigoplus_{l=0}^{\infty} U^{-1} h_l U \otimes \text{Id}_l \tag{2.14}$$

with Id_ℓ being the identity on $\left\langle Y_l^{-l}, Y_l^{-l+1}, ..., Y_l^0, ..., .Y_l^l \right\rangle$. Here we have defined

$$h_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \tag{2.15}$$

with the domains

$$\mathscr{D}(h_0) = \left\{u \in L^2((0,\infty),dr) | u, u' \in \mathrm{AC_{loc}}(0,\infty), u'' \in L^2((0,\infty),dr), u(0_+) = 0, u'(0_+) = 0\right\} \tag{2.16}$$

$$\mathscr{D}(h_l) = \left\{u \in L^2((0,\infty),dr) | u, u' \in \mathrm{AC_{loc}}(0,\infty), -u'' + l(l+1)r^{-2}u \in L^2((0,\infty),dr)\right\}, \quad l \geq 1 \tag{2.17}$$

Here $\mathrm{AC}(0,\infty)$ denotes the absolutely continuous functions on $(0,\infty)$, and $\mathrm{AC_{loc}}(0,\infty)$ denotes the locally absolutely continuous functions, *i.e.* AC on all compact intervals. Notice that for $l \geq 1$ the boundary conditions $u(0_+) = 0, u'(0_+) = 0$ are automatically satisfied by the requirement $-u'' + l(l+1)r^{-2}u \in L^2((0,\infty),dr)$ and continuity of u. According to [1], it is a standard result that h_l is self-adjoint for $l \geq 1$. However, it is not hard to see that h_0 has deficiency indices (1,1) and thus admits a one-parameter family of self-adjoint extensions. These extensions can all be characterized in terms of their self-adjoint boundary condition and are given by

$$h_{0,\alpha} = -\frac{d^2}{dr^2}, \tag{2.18}$$

with domain

$$\mathscr{D}(h_0) = \left\{u \in L^2((0,\infty),dr) | u, u' \in \mathrm{AC_{loc}}(0,\infty), u'' \in L^2((0,\infty),dr), -4\pi\alpha u(0_+) + u'(0_+) = 0\right\}, \tag{2.19}$$

with $\alpha \in (-\infty,\infty]$. The case $\alpha = \infty$ simply corresponds to the boundary condition $u(0+) = 0$, which simply implies $\lim_{|x|\to 0} |x|\psi(x) = 0$ for all $\psi \in \mathscr{D}(H_{\mathrm{rel}}^\infty)$. This is the usual Friedrich extension *i.e.* $H_{\mathrm{rel}}^\infty = -\Delta$ with $\mathscr{D}(H_{\mathrm{rel}}^\infty) = H^2(\mathbb{R}^3)$, the free particle. α can be related to θ from before by a simple computation: Let $f = h + c(\xi_+ + \mathrm{e}^{i\theta}\xi_-) \in \mathscr{D}(H_{\mathrm{rel},\theta})$ with $\xi_\pm = \frac{\mathrm{e}^{i\sqrt{\pm i}|x|}}{4\pi|x|} =$ then

$$\lim_{|x|\to 0} |x|f(x) = \frac{c}{4\pi}(1 + \mathrm{e}^{i\theta}), \qquad \lim_{|x|\to 0} \frac{d}{d|x|}(|x|f(x)) = \frac{ic}{4\pi}(\sqrt{i} + \mathrm{e}^{i\theta}\sqrt{-i}) \tag{2.20}$$

Thus we have

$$4\pi\alpha(1 + \mathrm{e}^{i\theta}) = i(\sqrt{i} + \mathrm{e}^{i\theta}\sqrt{-i}) = \sqrt{i}(i - \mathrm{e}^{i\theta}) = (\mathrm{e}^{i\frac{3}{4}} - \mathrm{e}^{i(\theta+\frac{1}{4})}) \tag{2.21}$$

from which it follows that

$$\begin{aligned}\alpha = \frac{(\mathrm{e}^{i\frac{3}{4}} - \mathrm{e}^{i(\theta+\frac{1}{4})})}{4\pi(1 + \mathrm{e}^{i\theta})} &= \frac{1}{4\pi}\frac{\mathrm{e}^{i(\theta+1)/2}(\mathrm{e}^{-i(\theta-\frac{1}{2})/2} - \mathrm{e}^{i(\theta-\frac{1}{2})/2})}{\mathrm{e}^{i\theta/2}(\mathrm{e}^{-i\theta/2} + \mathrm{e}^{i\theta/2})} = \frac{1}{4\pi}\frac{i(\mathrm{e}^{-i(\theta-\frac{1}{2})/2} - \mathrm{e}^{i(\theta-\frac{1}{2})/2})}{(\mathrm{e}^{-i\theta/2} + \mathrm{e}^{i\theta/2})} \\ &= \frac{1}{4\pi}\frac{\sin((\theta-\frac{1}{2})/2)}{\cos(\theta/2)} = \frac{1}{4\pi}\frac{-\cos(\theta/2)\sin(\frac{1}{4}) + \sin(\theta/2)\cos(\frac{1}{4})}{\cos(\theta/2)} = \frac{1}{4\sqrt{2}\pi}\left(\tan(\theta/2) - 1\right)\end{aligned} \tag{2.22}$$

Thereby we see that $\alpha \in \mathbb{R}$ for $\theta \in [0,\pi) \cup (\pi, 2\pi)$ and that $\alpha \to \infty$ when $\theta \uparrow \pi$. Now we study the the resolvent of these extensions i.e. $H^{\alpha}_{\mathrm{rel}} = U^{-1} h_{0,\alpha} U \otimes \mathrm{Id}_0 \oplus \left(\bigoplus_{l=1}^{\infty} U^{-1} h_l U \otimes \mathrm{Id}_\ell\right)$. To do this let us briefly summarize Krein's formula. In the following $\rho(O)$ denotes the resolvent set of the operator O.

Theorem 1 (Krein's formula, A.2 in [1])**.** *Let B and C be self-adjoint extensions of the densely defined, closed, and symmetric operator A on the Hilbert space H with deficiency indices $(1,1)$. Then their resolvant are related by:*

$$(B-z)^{-1} - (C-z)^{-1} = \lambda(z) \left\langle \phi(\bar{z}), \cdot \right\rangle \phi(z), \qquad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.23}$$

where $\lambda(z) \neq 0$ for $z \in \rho(B) \cap \rho(C)$ and λ, ϕ may be chosen to be analytic functions in $z \in \rho(B) \cap \rho(C)$. In fact, ϕ may be taken as

$$\phi(z) = \phi(z_0) + (z - z_0)(C-z)^{-1}\phi(z_0), \quad z \in \rho(C) \tag{2.24}$$

with $\phi(z_0)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$ being a solution of

$$A^* \phi(z_0) = z_0 \phi(z_0), \tag{2.25}$$

Choosing this ϕ, we furthermore have λ satisfying the equation

$$\lambda(z)^{-1} = \lambda(z')^{-1} - (z - z') \left\langle \phi(\bar{z}), \phi(z') \right\rangle, \qquad z, z' \in \rho(B) \cap \rho(C). \tag{2.26}$$

Proof. In order to prove this we remember the Krein extension theorem for densely defined, closed, symmetric operators. We have that the B and C are both of the form

$$\begin{aligned} \mathscr{D}(B) &= \{h + c(\xi_z + \mathrm{e}^{i\theta}\xi_{\bar{z}}) \mid h \in \mathscr{D}(A), c \in \mathbb{C}\}, \\ B(h + c(\xi_z + \mathrm{e}^{i\theta}\xi_{\bar{z}})) &= Ah + c(z\xi_z + \mathrm{e}^{i\theta}\bar{z}\xi_{\bar{z}}), \qquad \mathrm{Im}(z) \neq 0, \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} \mathscr{D}(C) &= \{h + c(\xi_z + \mathrm{e}^{i\omega}\xi_{\bar{z}}) \mid h \in \mathscr{D}(A),\ c \in \mathbb{C}\}, \\ C(h + c(\xi_z + \mathrm{e}^{i\omega}\xi_{\bar{z}})) &= Ah + c(z\xi_z + \mathrm{e}^{i\omega}\bar{z}\xi_{\bar{z}}). \qquad \mathrm{Im}(z) \neq 0 \end{aligned} \tag{2.28}$$

where $\xi_z \in \ker(A^* - z), \xi_{\bar{z}} \in \ker(A^* - \bar{z}), \omega \in [0, 2\pi)$ and $\theta \in [0, 2\pi)$ are fixed with $\|\xi_z\| = \|\xi_{\bar{z}}\| = 1$, Now for $z \in \rho(C)$, we know that $(C - z)$ has full range and thus for $x \in H$ we write $x = (C - z)y$. Assuming that $\mathrm{Im}(z) \neq 0$ we can write $y = h + \xi_z + \mathrm{e}^{i\omega}\xi_{\bar{z}} \in \mathscr{D}(C)$ where we have absorbed the $c \in \mathbb{C}$ into the ξ_z and $\xi_{\bar{z}}$. Consider now

$$\left((B-z)^{-1} - (C-z)^{-1}\right) x = (B-z)^{-1}(C-z)(h + \xi_z + \mathrm{e}^{i\omega}\xi_{\bar{z}}) - (h + \xi_z + \mathrm{e}^{i\omega}\xi_{\bar{z}}). \tag{2.29}$$

Clearly $(B - z)^{-1}(C - z)h = h$ since $h \in \mathscr{D}(A)$, and C and B are both extension of A. Thus we obtain

$$\left((B-z)^{-1}-(C-z)^{-1}\right)x=(B-z)^{-1}((\bar{z}-z)\mathrm{e}^{i\omega}\xi_{\bar{z}})-(\xi_z+\mathrm{e}^{i\omega}\xi_{\bar{z}}). \tag{2.30}$$

Since we have that $(B-z)(\xi_z+\mathrm{e}^{i\theta}\xi_{\bar{z}})=\mathrm{e}^{i\theta}(\bar{z}-z)\xi_{\bar{z}}$, with $\xi_z+\mathrm{e}^{i\theta}\xi_{\bar{z}}\in\mathscr{D}(B)$, we find that

$$\left((B-z)^{-1}-(C-z)^{-1}\right)x=(\mathrm{e}^{i(\omega-\theta)}-1)\xi_z,\qquad x=(C-z)(h+\xi_z+\mathrm{e}^{i\omega}\xi_{\bar{z}})=(C-z)h+(\bar{z}-z)\mathrm{e}^{i\omega}\xi_{\bar{z}}, \tag{2.31}$$

and we conclude

$$\left((B-z)^{-1}-(C-z)^{-1}\right)x=\frac{(\mathrm{e}^{-i\theta}-\mathrm{e}^{-i\omega})}{\bar{z}-z}\frac{\langle\xi_{\bar{z}},x\rangle}{\|\xi_{\bar{z}}\|^2}\xi_z=\lambda(z,\bar{z})\langle\phi(\bar{z}),x\rangle\phi(z) \tag{2.32}$$

where we have used that $(C-z)h=(A-z)h\in\operatorname{Ran}(A-z)\subset\ker(A^*-\bar{z})^{\perp}$, and the fact that by defining $\phi(z)=\phi(z_0)+(z-z_0)(C-z)^{-1}\phi(z_0)$, with $\phi(z_0)$ being a solution of $A^*\phi(z_0)=z_0\phi(z_0)$, we clearly have $\phi(z)\in\ker(A^*-z)$ such that $\phi(z)\parallel\xi_z$.
We have that λ is given by the formula

$$\lambda(z,\bar{z})=\frac{(\mathrm{e}^{-i\theta}-\mathrm{e}^{-i\omega})}{\bar{z}-z}\frac{\langle\phi(z)\xi_z\rangle}{\langle\phi(\bar{z}),\xi_{\bar{z}}\rangle\,\|\phi(z)\|^2} \tag{2.33}$$

Notice that the above calculation is for fixed z. Thus if we want to vary z, we get that θ and ω might depend on z as we may choose ξ_z and $\xi_{\bar{z}}$ differently at each z making a fixed θ or ω correspond to different extensions for each z. The fact that λ is analytic stems from the fact that all matrix elements of the resolvents are analytic in their resolvent sets and that we have chosen $\phi(z)$ such that $\langle\phi(\bar{z}),x\rangle$ is analytic for all $x\in H$. To show that λ satisfies (2.26) is simply a long computation and we refer to appendix A for the computation. □

Now we have two self-adjoint extension of $\dot{H}_{\text{rel}}$, namely H_{rel}^{∞} and H_{rel}^{α}. It is easily verified that by imposing $\alpha=\infty$ we obtain the Friedrich extension, given by (see above)

$$\mathscr{D}(H_{\text{rel}}^{\infty})=H^2(\mathbb{R}^3),\qquad H_{\text{rel}}^{\infty}=-\Delta. \tag{2.34}$$

We have already found the solution of $H_{\text{rel}}^*\phi(z)=z\phi(z)$ (although we only found it for $z=\pm i$) namely

$$\phi(z)(x)=\frac{\mathrm{e}^{i\sqrt{z}|x|}}{4\pi|x|},\qquad\operatorname{Im}(\sqrt{z})>0, \tag{2.35}$$

Furthermore it is a straightforward generalization of this result that the Green function of $(H_{\text{rel}}^{\infty}-z)$, *i.e.* the integral kernel of the resolvent $(H_{\text{rel}}^{\infty}-z)^{-1}$, is then

$$G_z(x,x')=\frac{\mathrm{e}^{i\sqrt{z}|x-x'|}}{4\pi|x-x'|}. \tag{2.36}$$

We immediately see that then

$$\left\langle \phi(\bar{z}), \phi(z') \right\rangle = \frac{1}{4\pi} \int_{(0,\infty)} dr e^{i(\sqrt{z'} - \overline{\sqrt{\bar{z}}})r} = \frac{1}{4\pi} \frac{-i}{\sqrt{z} - \sqrt{z'}}, \qquad \mathrm{Im}(\sqrt{z'}), \mathrm{Im}(\sqrt{\bar{z}}) > 0 \tag{2.37}$$

Remember that $\overline{\sqrt{\bar{z}}}|_{\mathrm{Im}(\sqrt{\bar{z}})>0} = \sqrt{z}|_{\mathrm{Im}(\sqrt{z})<0} = -\sqrt{z}|_{\mathrm{Im}(\sqrt{z})>0}$ so we have

$$\lambda(z)^{-1} - \lambda(z')^{-1} = \frac{i}{4\pi} \frac{z' - z}{\sqrt{z} + \sqrt{z'}} = \frac{i}{4\pi} (\sqrt{z'} - \sqrt{z}), \qquad \mathrm{Im}(z), \mathrm{Im}(z') > 0 \tag{2.38}$$

From which it follows that $\lambda(z) = (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1}$, for some constant $\kappa \in \mathbb{C}$. Furthermore we have from Krein's formula (Theorem 1) that

$$(H_{\mathrm{rel}}^{\alpha} - z)^{-1} = (H_{\mathrm{rel}}^{\infty} - z)^{-1} + (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1} \langle \phi(\bar{z}), \cdot \rangle \, \phi(z), \tag{2.39}$$

where we notice that by (2.24) we have

$$\begin{aligned} (\dot{H}_{\mathrm{rel}}^{*} - z)\phi(z) &= (\dot{H}_{\mathrm{rel}}^{*} - z)\phi(z_0) + (z - z_0)(\dot{H}_{\mathrm{rel}}^{*} - z)(H_{\mathrm{rel}}^{\infty} - z)^{-1}\phi(z_0) \\ &= (z_0 - z)\phi(z_0) + (z - z_0)\phi(z_0) = 0 \qquad z \in \rho(H_{\mathrm{rel}}^{\infty}) \end{aligned} \tag{2.40}$$

where we used that $H_{\mathrm{rel}}^{\infty}$ is a restriction of $\dot{H}_{\mathrm{rel}}^{*}$ such that $(\dot{H}_{\mathrm{rel}}^{*} - z)(H_{\mathrm{rel}}^{\infty} - z)^{-1} = \mathrm{Id}$. However from this we conclude that $\phi(z) = G_z(x, 0) = \frac{e^{i\sqrt{z}|x|}}{|x|}$, $\mathrm{Im}(\sqrt{z}) > 0$. Thereby we have

$$(H_{\mathrm{rel}}^{\alpha} - z)^{-1} = (H_{\mathrm{rel}}^{\infty} - z)^{-1} + (\kappa + i\sqrt{z})^{-1} \langle G_{\bar{z}}(*, 0), \cdot(*) \rangle \, G_z(\cdot, 0), \qquad z \in \rho(H_{\mathrm{rel}}^{\alpha}) \cap \rho(H_{\mathrm{rel}}^{\infty}), \tag{2.41}$$

where the $*$ refers to the integrated variable in the inner product.
In order to determine κ, we perform a simple calculation. Let $u \in \mathscr{D}(h_0^{\alpha})$ Then $\frac{1}{r} u Y_0^0 \in \mathscr{D}(H_{\mathrm{rel}}^{\alpha})$ and we have

$$(H_{\mathrm{rel}}^{\alpha} - z)\frac{1}{r} u Y_0^0 = \left(-\frac{1}{r}\frac{d^2 u(r)}{dr^2} - z\frac{1}{r}u(r) \right) Y_0^0 \tag{2.42}$$

Thus we have

$$\begin{aligned} u(0) &= \lim_{r \to 0} r \left((H_{\mathrm{rel}}^{\alpha} - z)^{-1} (H_{\mathrm{rel}}^{\alpha} - z) \frac{1}{r} u \right)(r) \\ &= \lim_{r \to 0} r 4\pi \left(\int_{(0,\infty)} dr r \left(-\frac{d^2 u}{dr^2} - z u \right) G_z(r, 0) + (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1} G_z(r, 0) \int dr r \overline{G_{\bar{z}}(r, 0)} \left(-\frac{d^2 u}{dr^2} - z u \right) \right) \end{aligned} \tag{2.43}$$

Notice that $\overline{G_{\bar{z}}(r, 0)} = G_z(r, 0)$ and that $r 4\pi G_z(r, 0) = e^{i\sqrt{z}r}$. By partial integration twice we have

$$\int_{(0,\infty)} dr \left(-e^{i\sqrt{z}r} \frac{d^2}{dr^2} u + u \frac{d^2}{dr^2} e^{i\sqrt{z}r} \right) = \frac{du}{dr}(0+) - i\sqrt{z} u(0+), \tag{2.44}$$

from which we get

$$u(0) = (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1} \frac{1}{4\pi} \left(\frac{du}{dr}(0+) - i\sqrt{z} u(0+) \right). \tag{2.45}$$

By imposing the boundary condition on u at 0 we obtain the equation for κ

$$1 = (\kappa - \frac{i}{4\pi}\sqrt{z})^{-1}\frac{1}{4\pi}\left(4\pi\alpha - i\sqrt{z}\right), \tag{2.46}$$

Thus that $\kappa = \alpha$ and we have the resolvent

$$(H^{\alpha}_{\text{rel}} - z)^{-1} = (H^{\infty}_{\text{rel}} - z)^{-1} + (\alpha - \frac{i}{4\pi}\sqrt{z})^{-1}\left\langle G_{\bar{z}}(*,0), \cdot(*)\right\rangle G_z(\cdot,0), \tag{2.47}$$

We are now ready to study the spectrum of the the operators $(H^{\alpha}_{\text{rel}})_{\{\alpha\in(-\infty,\infty]\}}$. Clearly $z \in \sigma(H^{\alpha}_{\text{rel}})$ if $z \in \sigma(-\Delta|_{C^{\infty}_c(\mathbb{R}^3)}) = [0,\infty)$. On the other hand we see that if $\alpha < 0$ then $z = -(4\pi\alpha)^2 \in \sigma(H^{\alpha}_{\text{rel}})$. Therefore the spectrum can be characterized as

$$\sigma(H^{\alpha}_{\text{rel}}) = \begin{cases} [0,\infty) & \text{if } \alpha \geq 0, \\ \{-(4\pi\alpha)^2\} \cup [0,\infty) & \text{if } \alpha < 0. \end{cases} \tag{2.48}$$

It is of course an exercise to show that no other points are in the spectrum. We refer to [1] for a short proof, and further classification of different parts of the spectrum, *i.e.* point-, singular continuous-, and absolute continuous spectrum. We note that for $\alpha < 0$ the point in the spectrum $\{-(4\pi\alpha)^2\}$ is an eigenvalue (*i.e.* a part of the point spectrum). Furthermore, we can actually, in the $\alpha < 0$ case, determine the eigenfunction corresponding to the eigenvalue $-(4\pi\alpha^2)$. To do this, notice that the domain of H^{α}_{rel} can be written as
$\mathscr{D}(H^{\alpha}_{\text{rel}}) = \{w(x) + (\alpha - \frac{i}{4\pi}k)^{-1}w(0)G_{k^2}(x,0) \mid w \in H^2(\mathbb{R}^3),\ k^2 \in \rho(H^{\alpha}_{\text{rel}})\}$ for $k \in \rho(H^{\alpha}_{\text{rel}}) \cap (H^{\alpha}_{\text{rel}})$.
This follows by the fact that

$$\mathscr{D}(H^{\alpha}_{\text{rel}}) = (H^{\alpha}_{\text{rel}} - k^2)^{-1}(H^{\infty}_{\text{rel}} - k^2)\mathscr{D}(H^{\infty}_{\text{rel}}), \tag{2.49}$$

where we have used that $\langle G_{\bar{z}}(x,0), (H^{\infty}_{\text{rel}} - z)w\rangle = \langle (H^{\infty}_{\text{rel}} - \bar{z})G_{\bar{z}}(x,0), w\rangle = \langle \delta_0, w\rangle = w(0)$. Notice that $w \in H^2(\mathbb{R}^3)$ is continuous, so $w(0)$ makes sense. We thus have the action of H^{α}_{rel}

$$(H^{\alpha}_{\text{rel}} - k^2)(w(x) + (\alpha - \frac{i}{4\pi}k)^{-1}w(0)G_{k^2}(x,0)) = (H^{\infty}_{\text{rel}} - k^2)w(x) = (-\Delta - k^2)w(x). \tag{2.50}$$

Now notice that if we fix $w \in H^2(\mathbb{R}^3)$ such that $w(0) = 1$ and we define $(x_n)_{(n\geq 1)}$ such that $x_n \to 0$ as $n \to \infty$, then $x_n w \to 0$ in $L^2(\mathbb{R}^3)$. Furthermore let $k_n^2 \to -(4\pi\alpha)^2$ such that $(\alpha - ik_n/(4\pi))\frac{1}{x_n} = 1$ for all $n \geq 1$, then we have

$$(H^{\alpha}_{\text{rel}} - k_n^2)(x_n w + (\alpha - \frac{i}{4\pi}k_n)^{-1}x_n G_{k_n^2}) = x_n(-\Delta - k^2)w \to 0 \qquad \text{as } n \to \infty, \tag{2.51}$$

with the notation G_{k^2} for $G_{k^2}(x,0)$. Thus we have that

$$\begin{aligned}\lim_{n\to\infty}\left(H^{\alpha}_{\text{rel}}(x_n w + (\alpha - \frac{i}{4\pi}k_n)^{-1}x_n G_{k_n^2})\right) &= \lim_{n\to\infty}\left(k_n^2(x_n w + (\alpha - \frac{i}{4\pi}k_n)^{-1}x_n G_{k_n^2})\right) \\ &= -(4\pi\alpha)^2 G_{-(4\pi\alpha)^2}.\end{aligned} \tag{2.52}$$

Where we have used that $k_n^2 I \to -(4\pi\alpha)^2 I$ in operator norm as $n \to \infty$ and that $(k_n^2 I)_{(n\geq 1)}$ is uniformly bounded. Furthermore, we used that $G_{k_n^2} \to G_{-(4\pi\alpha)^2}$ in $L^2(\mathbb{R}^3)$ for $\alpha < 0$. Thus we conclude that defining $\chi_n = x_n w + (\alpha - \frac{i}{4\pi} k_n)^{-1} x_n G_{k_n^2}$ we have that $(\chi_n)_{(n\geq 1)} \subset \mathscr{D}\left(H^\alpha_{\text{rel}}\right)$ converges in $L^2(\mathbb{R}^3)$ to $G_{-(4\pi\alpha)^2}$ and that $(H^\alpha_{\text{rel}}\chi_n)_{(n\geq 1)}$ converges in $L^2(\mathbb{R}^3)$ to $-(4\pi\alpha)^2 G_{-(4\pi\alpha)^2}$. By closedness of the self-adjoint operator H^α_{rel} we conclude that $G_{-(4\pi\alpha)^2} \in \mathscr{D}\left(H^\alpha_{\text{rel}}\right)$ and that

$$H^\alpha_{\text{rel}} G_{-(4\pi\alpha)^2} = -(4\pi\alpha)^2 G_{-(4\pi\alpha)^2}. \tag{2.53}$$

Thus $G_{-(4\pi\alpha)^2}$ is the eigenfunction corresponding to the bound state with energy $-(4\pi\alpha)^2$. We have thus constructed the Hamiltonian of the point-interaction in $3d$. We have seen that the parameter α, in some sense, controls the strength of the interaction, *i.e.* $\alpha = \infty$ is the free particle, and $\alpha < 0$ is the attractive point-interaction, since it has a bound state. By doing a analysis of the scattering theory of H^α_{rel} one finds that $-4\pi\alpha = \frac{1}{a}$, where a denotes the scattering length of the interaction [1].

2.2 Quadratic form

An alternative way of studying the point interaction is by the means of quadratic forms. In [3], the quadratic form, F_α describing a gas of point interacting fermions was obtained. It is a well-known result that if such a quadratic form is closed and bounded from below, then the corresponding operator is a bounded from below, self-adjoint operator. On the other hand it was proven in [3] that if the quadratic form F_α is not bounded from below, then the corresponding operator is not bounded from below and self-adjoint. The quadratic form was initially introduced by means of renormalization. On the other hand it is more clearly introduced as a rank-one perturbation of the free quadratic form. We write the rank-one perturbation as $\gamma \langle \phi, \cdot \rangle \phi$. Thus we imagine perturbing the Hamiltonian as $H = H_0 - \gamma \langle \phi, \cdot \rangle \phi$. For the point interaction we do this by simply projecting onto a ball $B_R(0)$, *i.e.* the ball of radius R centred at 0, in momentum space *i.e.*

$$\widehat{Hu} = \widehat{H_0 u} - \mathbb{1}_{B_R(0)} \frac{\gamma}{(2\pi)^3} \int_{B_R(0)} d^3p\ \hat{u}(p) = H_0 u - \mathbb{1}_{B_R(0)} \frac{\gamma}{(2\pi)^3} \int_{B_R(-k)} d^3p\ \hat{u}(k+p). \tag{2.54}$$

Thus we obtain the quadratic form

$$F^R_\gamma(\hat{u}) = \int_{\mathbb{R}^3} d^3k\ \left(\bar{\hat{u}}(k)(k^2)\hat{u}(k) - \frac{\gamma}{(2\pi)^3} \int_{B_R(-k)} d^3p\ \bar{\hat{u}}(k)\hat{u}(k+p) \right). \tag{2.55}$$

This can be rewritten in the following form

$$\begin{aligned} F^R_\gamma(u) = \int_{\mathbb{R}^3} d^3k\ \left(k^2+\mu\right) |\hat{u}(k) - \widehat{G\rho^R}(k)|^2 - \mu \|u\|^2_{L^2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} d^3k\ (k^2+\mu)|\widehat{G\rho^R}(k)|^2 \\ +2\text{Re} \int_{\mathbb{R}^3} d^3k\ \bar{\hat{u}}(k)(k^2+\mu)\widehat{G\rho^R}(k) - \int_{\mathbb{R}^3} d^3k\ \bar{\hat{u}}(k)\hat{\rho}^R(k), \end{aligned} \tag{2.56}$$

where $\mu > 0$ and we have defined

$$\hat{G}(k) = \frac{1}{k^2+\mu}, \qquad \hat{\rho}^R(k) = \gamma_R \mathbb{1}_{B_R(0)}(k) \int_{B_R(-k)} d^3p \ \hat{u}(k+p) = \mathbb{1}_{B_R(0)}(k)\xi_R, \\ \xi_R = \gamma_R \int_{B_R(0)} d^3p \ \hat{u}(p), \tag{2.57}$$

furthermore, $\widehat{G\rho^R}(k) = \hat{G}(k)\hat{\rho}^R(k)$ and $\gamma_R = \frac{\gamma}{(2\pi)^3}$, which we have allowed to depend on R, since it will need to be renormalized eventually. Now straightforward calculation shows that

$$\begin{aligned} &\overline{\int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)(k^2+\mu)\widehat{G\rho^R}(k)} = \int_{\mathbb{R}^3} d^3k \ \overline{\bar{\hat{u}}(k)\hat{\rho}^R(k)} = \gamma_R \int_{B_R(0)} d^3k \ \hat{u}(k) \int_{B_R(0)} d^3p \ \bar{\hat{u}}(p) \\ &= \gamma_R \int_{B_R(0)} d^3k \ \bar{\hat{u}}(k) \int_{B_R(0)} d^3p \ \hat{u}(p) = \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)\hat{\rho}^R(k) = \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)(k^2+\mu)\widehat{G\rho^R}(k), \end{aligned} \tag{2.58}$$

such that $2\mathrm{Re}\int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)(k^2+\mu)\widehat{G\rho^R}(k) = 2\int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)\hat{\rho}^R(k)$. Thereby we find the quadratic form

$$\begin{aligned} F_\gamma^R(u) &= \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{u}(k) - \widehat{G\rho^R}(k)|^2 - \mu\|u\|^2_{L^2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} d^3k \ (k^2+\mu)|\widehat{G\rho^R}(k)|^2 \\ &\qquad + \int_{\mathbb{R}^3} d^3k \ \bar{\hat{u}}(k)\hat{\rho}^R(k) \\ &= \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{u}(k) - \widehat{G\rho^R}(k)|^2 - \mu\|u\|^2_{L^2(\mathbb{R}^3)} - |\xi_R|^2 \int_{B_R(0)} d^3k \hat{G}(k) + \gamma_R^{-1}|\xi_R|^2. \end{aligned} \tag{2.59}$$

Now by computing the

$$\begin{aligned} \int_{B_R(0)} \hat{G} &= 4\pi \int_0^R dr \ \frac{r^2}{r^2+\mu} = 4\pi\sqrt{\mu}\int_0^{R/\sqrt{\mu}} dq \ \frac{q^2}{q^2+1} = 4\pi\sqrt{\mu}\left(\frac{R}{\sqrt{\mu}} - \int_0^{R/\sqrt{\mu}} dq \ \frac{1}{q^2+1}\right) \\ &= 4\pi\left(R - \sqrt{\mu}\arctan\left(\frac{R}{\sqrt{\mu}}\right)\right). \end{aligned} \tag{2.60}$$

Thereby we have the quadratic form

$$\begin{aligned} F_\gamma^R(u) = \int_{\mathbb{R}^3} d^3k \ &\left(k^2+\mu\right)|\hat{u}(k) - \widehat{G\rho^R}(k)|^2 - \mu\|u\|^2_{L^2(\mathbb{R}^3)} \\ &-|\xi_R|^2\left(4\pi R - 4\pi\sqrt{\mu}\arctan\left(\frac{R}{\sqrt{\mu}}\right) - \gamma_R^{-1}\right). \end{aligned} \tag{2.61}$$

Since we are interested in the limit $R \to \infty$ (corresponding to localizing the interaction to a point) we choose γ_R such that the divergence in R is cancelled. Choosing the coupling $\gamma_R^{-1} = 4\pi R + \alpha$ we obtain the final quadratic form

$$F_\alpha^R(u) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{w}_R(k)|^2 - \mu\|u\|^2_{L^2(\mathbb{R}^3)} + |\xi_R|^2\left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{R}{\sqrt{\mu}}\right)\right), \tag{2.62}$$

with $\hat{w}_R = \hat{u} - \widehat{G\rho^R}$. Notice, that the domain of this quadratic form is

$$\mathscr{D}\left(F_\alpha^R\right) = \left\{u \in L^2(\mathbb{R}^3) \mid w_R \in H^1(\mathbb{R}^3)\right\} \tag{2.63}$$

Heuristically, we can take the limit $R \to \infty$. This is done by noticing that for $w \in H^1(\mathbb{R}^3)$ we have that $\hat{w}(k) = \frac{\hat{f}(k)}{(|k|^2+1)^{\frac{1}{2}}}$ for some $f \in L^2(\mathbb{R}^3)$. By Hölder's inequality $(2,2)$ we thus see

$$\int_{B_R(0)} \hat{w} \leq \left(\int_{B_R(0)} \left|\frac{1}{|k|^2+1}\right|^2\right)^{\frac{1}{2}} \|f\|_2 \lesssim \sqrt{R} \tag{2.64}$$

where by $f(x) \lesssim g(x)$ we mean that there exist some constant $C \in \mathbb{R}$ such that $f(x) \leq Cg(x)$. Thus we see that the equation for ξ_R

$$\xi_R = \frac{1}{(4\pi R + \alpha)} \int_{B_R(0)} \left(\hat{w} + \widehat{G\rho^R}\right) = \frac{1}{(4\pi R + \alpha)} \int_{B_R(0)} \left(\hat{w} + \xi_R \hat{G}\right), \tag{2.65}$$

becomes in the limit $R \to \infty$ the equation $\xi_R = \xi_R$. Thus any choice of ξ_R is consistent and simply let it be a free parameter $\xi_R \to \xi \in \mathbb{C}$. We also see that in the limit $R \to \infty$ we have that $\arctan\left(R/\sqrt{\mu}\right) \to \frac{\pi}{2}$. Thus we get the quadratic form

$$F_\alpha(u) = \int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) |\hat{w}(k)|^2 - \mu \|u\|^2_{L^2(\mathbb{R}^3)} + |\xi|^2 \left(\alpha + 2\pi^2\sqrt{\mu}\right), \tag{2.66}$$

with domain

$$\mathscr{D}\left(F_\alpha(u)\right) = \left\{u \in L^2(\mathbb{R}^3) \mid \hat{u} = \hat{w} + \xi\hat{G}, \ w \in H^1(\mathbb{R}^3), \ \xi \in \mathbb{C}\right\}, \tag{2.67}$$

which matches the expression of [8] in the $N = M = 1$ case. Notice that the decomposition $u = w + \xi G$ is unique by the fact that $G \notin H^1(\mathbb{R}^3)$.

2.3 Hamiltonian from quadratic form

We are in this subsection going to construct the Hamiltonian for the point interactions from the quadratic form. This will serve both as an example of how to obtain the Hamiltonian given its quadratic form, but also as a motivation that the quadratic form given in the previous section is indeed equivalent to the the self-adjoint extension H^α_{rel}.
First we need to define a few properties of quadratic forms.

Definition 1. *We say a quadratic form on some Banach space $q : \mathscr{D}(q) \to \mathbb{R}$ is bounded from below if $q(v) \geq -c\|v\|^2$ for some $c > 0$.*

Definition 2. *We say that a quadratic form, $q : \mathscr{D}(q) \to \mathbb{R}$, which is bounded from below, $q(v,v) \geq -c\|v\|^2$, is closed if its domain, $\mathscr{D}(q)$, is a Banach space when equipped with the norm $\|v\|_q^2 = q(v,v) + C\|v\|^2$, where $C > c$.*

Notice that given a quadratic form $q : \mathscr{D}(q) \to \mathbb{R}$ we can always construct a symmetric sesquilinear from by

$$\begin{aligned} \mathrm{Re}(q(u,v)) &= \frac{1}{2}(q(u+v) - q(u) - q(v)), \\ \mathrm{Im}(q(u,v)) &= \frac{1}{2i}(q(u+iv) - q(u) - q(v)). \end{aligned} \tag{2.68}$$

where we abuse notation and use the symbol q for both the quadratic form and the sesquilinear form. This motivates the following proposition

Proposition 1. *A quadratic form on some Hilbert space H, $q : \mathscr{D}(q) \to \mathbb{R}$, which is bounded from below, $q(v) \geq -c\|v\|^2$, is closed if and only if its domain, $\mathscr{D}(q)$, is a Hilbert space when equipped with the inner product $\langle u, v\rangle_q = q(u,v) + C\langle u, v\rangle$, where $C > c$.*

Proof. Since the norm $\|\cdot\|_q$ is generated by $\langle\cdot,\cdot\rangle_q$ it is clear that this follows if we can show that $\langle u, v\rangle_q$ is in fact an inner product. Sesquilinearity is obvious by construction. Furthermore $\langle v, v\rangle_q = \|v\|_q^2 \geq 0$ since q is bounded from below, $q(v) \geq -c\|v\|^2$, and $\langle v, v\rangle_q = 0$ if and only if $v = 0$ follows from the fact that $\langle v, v\rangle_q \geq (C-c)\|v\|^2$. □

Notice that it follows from the fact that $\langle\cdot,\cdot\rangle_q$ is an inner product that $\|\cdot\|_q$ is in fact a norm. We start out by the quadratic form from the previous section

$$F_\alpha(u) = \int_{\mathbb{R}^3} d^3k\ \left(k^2+\mu\right)|\hat{w}(k)|^2 - \mu\|u\|^2_{L^2(\mathbb{R}^3)} + |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right), \tag{2.69}$$

with domain

$$\mathscr{D}(F_\alpha(u)) = \left\{u \in L^2(\mathbb{R}^3) \mid \hat{u} = \hat{w} + \xi\hat{G},\ w \in H^1(\mathbb{R}^3),\ \xi \in \mathbb{C}\right\}. \tag{2.70}$$

This quadratic form is closed and bounded from below, it is also clear that this quadratic form has a corresponding symmetric sesquilinear form, with domain
$\mathscr{D}(F_\alpha(\cdot,\cdot)) = \mathscr{D}(F_\alpha(\cdot)) \times \mathscr{D}(F_\alpha(\cdot))$, given by

$$F_\alpha(u,v) = \int_{\mathbb{R}^3} d^3k\ \left(k^2+\mu\right)\overline{\hat{w}(k)}\hat{h}(k) - \mu\langle u, v\rangle_{L^2(\mathbb{R}^3)} + \bar{\xi}\chi\left(\alpha + 2\pi^2\sqrt{\mu}\right), \tag{2.71}$$

where $u = w + \xi G$ and $v = h + \chi G$. The domain of the corresponding operator H_α is defined by

$$\mathscr{D}(H_\alpha) = \{u \in \mathscr{D}(F_\alpha) \mid F_\alpha(u,\cdot) \text{ is an } L^2(\mathbb{R}^3) \text{ bounded linear functional on } \mathscr{D}(F_\alpha)\}. \tag{2.72}$$

By density of $\mathscr{D}(F_\alpha)$ in $L^2(\mathbb{R}^3)$ and Riez representation theorem, we known that if $h \in \mathscr{D}(H_\alpha)$ then $F_\alpha(u,\cdot) = \langle x, \cdot\rangle$ and we define $H_\alpha u = x$. Clearly H_α is linear and symmetric by the very construction

$$\langle H_\alpha u, v\rangle = F_\alpha(u,v) = \overline{F_\alpha(v,u)} = \overline{\langle H_\alpha v, u\rangle} = \langle u, H_\alpha v\rangle \tag{2.73}$$

For $u, v \in \mathscr{D}(H_\alpha)$. Notice that

$$\begin{aligned} \mathscr{D}(H_\alpha^*) &= \{v \in \mathscr{D}(F_\alpha) \mid \langle H_\alpha \cdot, v\rangle \text{ is bounded on } \mathscr{D}(H_\alpha)\} \\ &= \{v \in L^2(\mathbb{R}^3) \mid F_\alpha(\cdot, v) \text{ is bounded on } \mathscr{D}(H_\alpha)\}. \end{aligned} \tag{2.74}$$

Assuming that $\mathscr{D}(H_\alpha)$ is dense in $\mathscr{D}(F_\alpha)$ and thus in $L^2(\mathbb{R}^3)$ we then have $\mathscr{D}(H_\alpha^*) = \mathscr{D}(H_\alpha)$ and the operator, H_α, is self-adjoint. It is a general fact that $\mathscr{D}(H_\alpha)$ is dense whenever F_α is closed and bounded from below ([5], Thm 12.18). Thus we are now ready to calculate the Hamiltonian of the quadratic form F_α. Notice that by the definition of $\mathscr{D}(H_\alpha)$ we must have that for $u \in \mathscr{D}(H_\alpha)$ and $(v_n)_{n\geq 1} \subset \mathscr{D}(F_\alpha)$ such that $v_n \to 0$ in $L^2(\mathbb{R}^3)$ it holds that $F_\alpha(v, u_n) \to 0$. Thus by writing $u = w + \xi G$ with $w \in H^1(\mathbb{R}^3)$ and $\xi \in \mathbb{C}$ and $v_n = h_n + \chi G \in \mathscr{D}(F_\alpha)$, with $h_n \in H^1(\mathbb{R}^3)$ such that $h_n \to -\chi G$ in $L^2(\mathbb{R}^3)$ we have

$$F(u, v_n) = \int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) \overline{\hat{w}(k)} \hat{h}_n(k) - \mu \left\langle u, v_n\right\rangle_{L^2(\mathbb{R}^3)} + \bar{\xi}\chi\left(\alpha + 2\pi^2\sqrt{\mu}\right). \tag{2.75}$$

We immediately see that for the first term to be $L^2(\mathbb{R}^3)$ bounded in h_n we must have that $w \in H^2(\mathbb{R}^3)$. Secondly since the first term is $L^2(\mathbb{R}^3)$ bounded (continuous) we must have

$$\int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) \overline{\hat{w}(k)} \hat{h}_n(k) \to -\int_{\mathbb{R}^3} d^3k \ \left(k^2 + \mu\right) \overline{\hat{w}(k)} \chi \hat{G}(k) = -\chi \int_{\mathbb{R}^3} d^3k \ \overline{\hat{w}(k)} \quad \text{as } n \to \infty. \tag{2.76}$$

The second term obviously goes to zero by continuity of the inner product. Thus we need to estimate $\int_{\mathbb{R}^3} \hat{h}$. This is done in the following lemma.

Lemma 1. *Let $w \in H^2(\mathbb{R}^3)$, then w is continuous and $\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3k \ \hat{w}(k) e^{ik\cdot x} = w(x)$.*

Proof. First that w is continuous follows from Sobolev's embedding theorem. Next notice that since $w \in H^2(\mathbb{R}^3)$ we must have that

$$\hat{w}(k) = \frac{\hat{f}(k)}{|k|^2 + 1}, \tag{2.77}$$

for some $\hat{f} \in L^2(\mathbb{R}^3)$. Thus we also see that $\hat{w}$ is clearly in $L_1(\mathbb{R}^3)$ by Hölder's inequality and the fact that $\frac{1}{|k|^2+1} \in L^2(\mathbb{R}^3)$ and therefore $\check{\hat{w}}$ is bounded and continuous. By Fourier's inversion theorem we have that $\check{\hat{w}} = w$ a.e, and by continuity of w, we conclude that $\check{\hat{w}} = w$. Since $\hat{w} \in L_1(\mathbb{R}^3)$ this amounts to

$$\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3k \ \hat{w}(k) e^{ik\cdot x} = w(x) \tag{2.78}$$

$\square$

Given the above lemma we clearly see that

$$\chi \int_{\mathbb{R}^3} d^3k \ \overline{\hat{w}(k)} = (2\pi)^{3/2} \chi \overline{w(0)}. \tag{2.79}$$

Thereby we find the condition on the domain

$$-(2\pi)^{3/2}\chi\overline{w(0)}+\bar{\xi}\chi(\alpha+2\pi^2\sqrt{\mu})=0 \tag{2.80}$$

correpsponding to the boundary condition $w(0)=(2\pi)^{-3/2}\xi(\alpha+2\pi^2\sqrt{\mu})$. Now turning to the action of the operator H_α it is easier to consider $H_\alpha+\mu$ since we then have

$$\begin{aligned}\langle(H_\alpha+\mu)u,v\rangle=F_\alpha(u,v)+\mu\langle u,v\rangle&=\int_{\mathbb{R}^3}d^3k\ \left(k^2+\mu\right)\overline{\hat{w}(k)}\hat{h}(k)+\bar{\xi}\chi\left(\alpha+2\pi^2\sqrt{\mu}\right)\\&=\int_{\mathbb{R}^3}d^3k\ \left(k^2+\mu\right)\overline{\hat{w}(k)}\hat{h}(k)+\chi\int_{\mathbb{R}^3}d^3k\overline{\hat{w}(k)}\\&=\int_{\mathbb{R}^3}d^3k\ \left(k^2+\mu\right)\overline{\hat{w}(k)}\hat{h}(k)+\chi\int_{\mathbb{R}^3}d^3k(k^2+\mu)\overline{\hat{w}(k)}\hat{G}(k)\\&=\langle(-\Delta+\mu)w,v\rangle\,,\end{aligned} \tag{2.81}$$

with $u=w+\xi G$ and $v=h+\chi G$ and where we used the boundary condition we found above in line 2.

We can therefore write down the Hamiltonian in the following manner:

$$\begin{gathered}\mathscr{D}\left(H_\alpha\right)=\left\{u\in L^2(\mathbb{R}^3)\mid u=w+\xi G,\ w\in H^2(\mathbb{R}^3),\ w(0)=(2\pi)^{-3/2}\xi(\alpha+2\pi^2\sqrt{\mu}),\ \xi\in\mathbb{C}\right\}\\(H_\alpha+\mu)u=(-\Delta+\mu)w\end{gathered} \tag{2.82}$$

This concludes how to obtain the Hamiltonian given the quadratic form. Notice that this expression also matches the one we found by self-adjoint extension. To see this, we have to notice that there is a bit a mismatch between the normalizations of the Green functions used in the two methods. We see this by computing

$$\begin{aligned}\hat{G}_{-\mu}(k)&=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}d^3xG_{-\mu}(x,0)\mathrm{e}^{-ik\cdot x}=\frac{2\pi}{4\pi(2\pi)^{3/2}}\int_{-1}^{1}d[\cos(\varphi)]\int_0^\infty drr\mathrm{e}^{-\sqrt{\mu}r}\mathrm{e}^{-i|k|r\cos(\varphi)}\\&=\frac{1}{(2\pi)^{3/2}}\int_0^\infty drr\frac{1}{(2\pi)^{3/2}}\frac{\mathrm{e}^{-\sqrt{\mu}r}\sin(|k|r)}{|k|r}=\frac{1}{(2\pi)^{3/2}}\frac{1}{2i|k|}\left[\frac{1}{i(i\sqrt{\mu}-|k|)}-\frac{1}{i(i\sqrt{\mu}+|k|)}\right]\\&=\frac{1}{(2\pi)^{3/2}}\frac{1}{2|k|}\left[\frac{i\sqrt{\mu}-|k|-i\sqrt{\mu}-|k|}{-\mu-|p^2|}\right]=\frac{1}{(2\pi)^{3/2}}\frac{1}{\mu+|k|^2}=\frac{1}{(2\pi)^{3/2}}\frac{1}{\mu+|k|^2}\end{aligned} \tag{2.83}$$

Thus we see that $G_{-\mu}=\frac{1}{(2\pi)^{3/2}}G$, and we get

$$\begin{aligned}\mathscr{D}\left(H_\alpha\right)&=\left\{u\in L^2(\mathbb{R}^3)\mid u=w+\xi G,\ w\in H^2(\mathbb{R}^3),\ w(0)=(2\pi)^{-3/2}\xi(\alpha+2\pi^2\sqrt{\mu}),\ \xi\in\mathbb{C}\right\}\\&=\left\{u\in L^2(\mathbb{R}^3)\mid u=w+(2\pi)^{3/2}(\alpha+2\pi^2\sqrt{\mu})^{-1}(2\pi)^{3/2}w(0)G_{-\mu},\ w\in H^2(\mathbb{R}^3)\right\}\\&=\left\{u\in L^2(\mathbb{R}^3)\mid u=w+\left(\frac{\alpha}{(2\pi)^3}+\frac{\sqrt{\mu}}{4\pi}\right)^{-1}w(0)G_{-\mu},\ w\in H^2(\mathbb{R}^3)\right\}\end{aligned} \tag{2.84}$$

We see that this exactly equal to the domain found in the previous section except for the fact that α has been replaced by $\alpha/(2\pi)^3$

$$\mathscr{D}\left(H_{\mathrm{rel}}^{\alpha}\right)=\left\{u\in L^2(\mathbb{R}^3)\mid u=w+\left(\alpha-i\frac{i\sqrt{\mu}}{4\pi}\right)^{-1}w(0)G_{-\mu},\ w\in H^2(\mathbb{R}^3)\right\}. \tag{2.85}$$

Therefore we conclude that $H_\alpha=H_{\mathrm{rel}}^{\frac{\alpha}{(2\pi)^3}}$.

3 Γ-convergence

In this section we are going to study the notion of Γ-convergence and apply it to the convergence problem of the quadratic form constructed in the previous section.

3.1 Introducing Γ-convergence

Let us first introduce a few definitions in order to study the Γ-convergence properties of F_α^R.

Definition 3 (Lower semicontinuity). *Let $F:X\to\mathbb{R}$ be a function on some topological space X. We say that F is lower semicontinuous at $x\in X$ if for every $t<F(x)$ there exist a neighbourhood $N_{x,t}$ such that $F(y)>t$ for all $y\in N_{x,t}$. We furthermore say that F is lower semicontinuous if it is lower semicontinuous at every point $x\in X$. In particular, if X is a normed space, we say that $F:X\to\mathbb{R}$ is norm lower semicontinuous if it is lower semicontinuous with respect to the norm topology.*

An equivalent formulation of lower semicontinuity is given by the following proposition

Proposition 2. *Let F be a function on a topological vector space X. Then F is lower semicontinuous at $x\in X$ if and only if*

$$\liminf_{z\to x}F(z):=\sup_{U\in\mathcal{N}_x}\inf_{z\in U}F(z)\geq F(x), \tag{3.1}$$

where $\mathcal{N}_x$ denotes the set of all open neighbourhoods of x.

Proof. "$\implies$": Choose any $t<F(x)$, then by lower semicontinuity of F there exist a neighbourhood U' of x such that $F(y)>t$ for all $y\in U'$ but then $\inf_{y\in U'}F(y)>t$. Thus the existence of such a U' implies that $\sup_{U\in\mathcal{N}_x}\inf_{z\in U}F(z)\geq t$. Since this was true for any $t<F(x)$ we conclude that $\sup_{U\in\mathcal{N}_x}\inf_{z\in U}F(z)\geq F(x)$
"$\impliedby$": Assume that there exists a $t<F(x)$ such that for all $U\in\mathcal{N}_x$ there exist a $y\in U$ with $F(y)\leq t$. Then $\inf_{z\in U}F(z)\leq t$ for all $U\in\mathcal{N}_x$ which implies that $\sup_{U\in\mathcal{N}_x}\inf_{z\in U}F(z)\leq t<F(x)$. Thus we have proven the contrapositive. □

Definition 4 (Γ-upper/-lower limit). *Given a topological space X, and some sequence of func-*

tions F_n on X, we define the Γ-upper and Γ-lower limits by

$$\begin{aligned} \Gamma\text{-}\limsup_{n\to\infty} F_n(x) &= \sup_{N_x\in\mathcal{N}_x} \limsup_{n\to\infty} \inf_{y\in N_x} F_n(y), \\ \Gamma\text{-}\liminf_{n\to\infty} F_n(x) &= \sup_{N_x\in\mathcal{N}_x} \liminf_{n\to\infty} \inf_{y\in N_x} F_n(y), \end{aligned} \tag{3.2}$$

where $\mathcal{N}_x$ denotes the collection of open neighbourhoods of x.

Definition 5. *(Γ-limit) Given a topological space X, and some sequence of function F_n on X, we say that F_n Γ-converges to the Γ-limit F if*

$$\Gamma\text{-}\liminf_{n\to\infty} F_n(x) = \Gamma\text{-}\limsup_{n\to\infty} F_n(x) = F(x), \quad \textit{for all } x\in X. \tag{3.3}$$

One immediate result of having Γ-convergence is lower semicontinuity of the limit, which is the following proposition

Proposition 3. *Let F_n be a sequence of functions on a topological space X. Then the Γ-lower and Γ-upper limits are both lower semicontinuous in the topology of X.*

Proof. We proof this for the Γ-lower limit, but the proof is equally valid for the upper limit by exchanging all $\liminf$ by $\limsup$. For the lower-limit this follows immediately by observing that For any $z\in X$ and $U\in\mathcal{N}_z$ we have that

$$\Gamma\text{-}\liminf_{n\to\infty} F_n(z) \geq \liminf_{n\to\infty} \inf_{y\in U} F_n(y).$$

It follows by the fact that U is an open neighbourhood of all its points that we then have

$$\inf_{z\in U} \Gamma\text{-}\liminf_{n\to\infty} F_n(z) \geq \liminf_{n\to\infty} \inf_{y\in U} F_n(y). \tag{3.4}$$

Now taking supremum on both sides gives us

$$\sup_{U\in\mathcal{N}_x} \inf_{z\in U} \Gamma\text{-}\liminf_{n\to\infty} F_n(z) \geq \sup_{U\in\mathcal{N}_x} \liminf_{n\to\infty} \inf_{y\in U} F_n(y) = \Gamma\text{-}\liminf_{n\to\infty} F_n(x). \tag{3.5}$$

By using Proposition 2 and the fact that the above inequality was for all $x\in X$ we conclude that $\Gamma\text{-}\liminf_{n\to\infty} F_n(x)$ is lower semicontinuous. □

As an obvious consequence of the above proposition we get the following corollary.

Corollary 1. *Let F_n be a sequence of functions on a topological space X, such that F_n Γ-converge to F. Then F is lower semicontinuous in the topology of X.*

Now we state an interesting result relating lower semicontinuity of quadratic form to them being bounded from below. A tool that will be of great importance when applying tools of Γ-convergence to prove stability of quantum mechanical systems. The following proposition and theorem is from lecture notes by Jan Philip Solovej.

Proposition 4 (Prop. 7.5 in [10]). *Let $F_D \to \mathbb{R} \cup \{\infty\}$ be a norm lower semicontinuous functional on a subspace D of a Hilbert space, H. If F satisfies $F(\alpha\phi) = |\alpha|^2 F(\phi)$ for all $\alpha \in \mathbb{R}$ and all $\phi \in D$,* i.e. *F is homogenous of degree 2, then F is bounded from below in the sense that there exist $M < \infty$ such that $F(\phi) \geq -M\|\phi\|^2$ for all $\phi \in D$.*

Proof. We consider the the set $S = \{h \in D \mid F(\phi) > -1\}$. By lower semi continuity of F we have that for every $h \in S$ there exist a neighbourhood $N_h \in \mathcal{N}_h$ such that $F(h') > -1$ for all $h' \in N_h$, *i.e.* all points of S are interior points. Thus S is open. By observing that $0 \in S$ we conclude that there exist some $\epsilon > 0$ such that the ball in D, $B_\epsilon(0) \subset D$ is contained in S. Thereby we know that for all $x \in D$ with $\|x\| < \epsilon$, we have $F(x) > -1$. For all $h \in D$ we therefore have $F(\epsilon h/2\|h\|) > -1$ which is equivalent to

$$F(h) > -\frac{4}{\epsilon^2}\|h\|^2. \tag{3.6}$$

□

Theorem 2 (Thm. 7.6 in [10]). *A quadratic form Q defined on a subspace $\mathscr{D}(Q)$ of a Hilbert space H is closable and bounded from below if and only if it is norm lower semicontinuous.*

Proof. Assume first that Q is bounded from below and closable. Then we can extend Q to a closed quadratic form on the set (Def. 7.4 in [10])

$$\bar{\mathscr{D}}(Q) = \{\phi \in H \mid \text{there exist a Cauchy sequence in } \mathscr{D}(Q) \text{ converging to } \phi\}.$$

Thus we may take Q to be closed. In that case we know that $\mathscr{D}(Q)$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_Q = Q(\cdot,\cdot) + M\langle\cdot,\cdot\rangle$, for some $M > 0$ such that $Q(v) > -M\|v\|$ for all $v \in H$, and where $Q(\cdot,\cdot)$ denotes the symmetric sesquilinear form associated to Q. Thus the norm on the $\mathscr{D}(Q)$ is $\|\cdot\|_Q = Q(\cdot) + M\|\cdot\|^2$. Assume now that we have some sequence $(x_n)_{(n\geq 1)} \subset \mathscr{D}(Q)$ converging to $x \in \mathscr{D}(Q)$. Let x_{n_j} denotes a subsequence such that

$$\liminf_{n\to\infty} Q(x_n) = \lim_{j\to\infty} Q(x_{n_j}).$$

Now if $\liminf_{n\to\infty} Q(x_n) = \infty$ we trivially have $\liminf_{n\to\infty} Q(x_n) \geq Q(x)$. On the other hand, by assuming $\liminf_{n\to\infty} Q(x_n) < \infty$ we may conclude that $Q(x_{n_j})$ is bounded. Thus $(x_{n_j})_{j\geq 1}$ is bounded in $\|\cdot\|_Q$ norm. By Alaoglu's theorem and the fact that $\mathscr{D}(Q)$ is a Hilbert space, we may conclude that any further subsequence $x_{n_{j_k}}$ contain a subsequence $x_{n_{j_{k_l}}}$ converging weakly in $\mathscr{D}(Q)$. However the limit is then bound to be x by the fact that the weak limit in $\mathscr{D}(Q)$ is also the weak limit in H and the weak limit in H is necessarily equal to the norm limit whenever it exists. Thus all subsequences, $x_{n_{j_k}}$, of x_{n_j}, contain a further subsequence $x_{n_{j_{k_l}}}$ converging to weakly to x in $\mathscr{D}(Q)$. Therefore we may conclude that x_{n_j} converge weakly to x. However, it is a known fact that $\liminf_{j\to\infty} \|v_j\| \geq \|v\|$ for any $v_j \rightharpoonup v$, as it follows directly from Cauchy

Schwartz. Using this we obtain

$$\begin{aligned}\liminf_{n\to\infty} Q(x_n) = \lim_{j\to\infty} Q(x_{n_j}) = \lim_{j\to\infty} \left(\|x_{n_j}\|_Q^2 - M\|x_{n_j}\|\right) \geq \liminf_{j\to\infty} \|x_{n_j}\|_Q^2 - M\|x\|^2 \\ \geq \||x\|_Q^2 - M\|x\|^2 = Q(x).\end{aligned} \tag{3.7}$$

Showing that Q is indeed norm lower semicontinuous (Proposition 2).
Now assume instead that Q is lower semicontinuous. Then we now from Proposition 4 that Q is bounded from below. On the other hand let $(x_n)_{n\geq 1}$ be a sequence such that $Q(x_n - x_m) \to 0$ for $n, m \to \infty$ and $x_n \to 0$. For any subsequence x_{n_j} we have Then $0 \leq Q(x_{n_j}) \leq \liminf_{m\to\infty} Q(x_{n_j} - x_m) \to 0$ for $j \to \infty$. Thus there exist a further subsequence such that $Q(x_{n_{j_k}}) \to 0$ for $k \to \infty$, and we conclude that the original sequence $Q(x_n)$ converge to 0. So Q is closable by definition of closability (Def. 7.2, [10]). □

See also [10], Thm 7.6 for a possibly even stronger result.
Finally in this subsection we state some important result from the book by Gianni Dal Maso [7].

Theorem 3 (Thm. 13.6(a), in [7])**.** *Let F_n be a sequence of lower semicontinuous positive semi-definite quadratic forms on a Hilbert space H and let F be a lower semicontinuous positive semi-definite quadratic form also on H. Let A_n and A denote the corresponding operators of F_n and F respectively. If F_n Γ-converges to F in the strong topology (norm topology), and in addition*

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n), \tag{3.8}$$

for all sequences x_n converging weakly to x. Then A_n converges to A in the strong resolvent sense, i.e. *$R_z(A_n)x \to R_z(A)x$ in norm for all $x \in H$ and all $z \in \rho(A)$, where $R_z(A) = (A - z)^{-1}$ denotes the resolvent of A.*

Notice that we can construct a positive semi-definite quadratic form from any bounded from below quadratic form, $q(v) \geq -m\|v\|^2$ by $Q = q + M\|\cdot\|^2$ where $M \geq m$. Thereby the result in Theorem 3 extends to bounded from below operators: Consider a bounded from below operator $F_n(v) \geq -m\|v\|^2$, and construct the positive semi-definite quadratic forms $\tilde{F}_n = F_n + m\|\cdot\|^2$ and $\tilde{F} = F + m\|\cdot\|^2$. If F_n and F satisfies the assumptions of Theorem 3, then $\tilde{F}_n$ and $\tilde{F}$ also satisfies them. To see this, notice that $m\|\cdot\|^2$ is a constant sequence and thus it is pointwise (continuously) convergent therefore by Prop. 6.20 in [7] we know that

$$\Gamma\text{-}\lim_{n\to\infty} \tilde{F}_n = \Gamma\text{-}\lim_{n\to\infty} F_n + m\|\cdot\|. \tag{3.9}$$

Furthermore, if $x_n \to x$ weakly, it also holds that

$$\liminf_{n\to\infty} \tilde{F}_n(x_n) \geq \liminf_{n\to\infty} F_n(x_n) + m \liminf_{n\to\infty} \|x_n\|^2 \geq \liminf_{n\to\infty} F_n(x_n) + m\|x\|^2 \geq \tilde{F}(x), \tag{3.10}$$

where we have used the basic property of lim inf that $\liminf_n (a_n + b_n) \geq \liminf_n a_n + \liminf_n b_n$, and that $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$ for $x_n \to x$ weakly. The last property follows from the Cauchy-Schwartz inequality: If $x_n \rightharpoonup x$ (converges weakly), then $|\langle x_n, x\rangle| \to \|x^2\|$, however $|\langle x_n, x\rangle| \leq \|x_n\|\|x\|$, so ultimately we have for all $\epsilon > 0$ there exist $k \geq 1$ such that $\|x_n\| \geq \|x\| - \epsilon$, but then $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$. Therefore we use Theorem 3 on $\tilde{F}$ and conclude that its corresponding operators $\tilde{A}_n$ converges to $\tilde{A}$ in the norm resolvent sense. Now using that $\tilde{A}_n = A_n + mI$ and $\tilde{A} = A + mI$, where A and A_n denotes the operators corresponding to F and F_n respectively, and the fact that if $R_z(A_n + mI)x \to R_z(A + mI)x$ in norm as $n \to \infty$, then $R_z(A_n)x = R_{z+m}(A_n + Im)x \to R_{z+m}(A + Im)x = R_z(A)$ in norm as $n \to \infty$, we get the following corollary

Corollary 2. *Let F_n be a sequence of lower semicontinuous bounded from below quadratic forms with a common lower bound, $F_n(v) \geq -m\|v\|^2$ on a Hilbert space H and let F be a lower semicontinuous bounded from below quadratic form with the same lower bound $F(v) \geq -m\|v\|^2$, on H. Let A_n and A denote the corresponding operators of F_n and F respectively. If F_n Γ-converges to F in the strong topology (norm topology), and in addition*

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n), \tag{3.11}$$

for all sequences x_n converging weakly to x. Then A_n converges to A in the strong resolvent sense, i.e. *$R_z(A_n)x \to R_z(A)x$ in norm for all $x \in H$ and all $z \in \rho(A)$, where $R_z(A) = (A - z)^{-1}$ denotes the resolvent of A.*

Another important result concerning Γ-convergence, it the classification of Γ-convergence in first countable spaces, such as for example normed spaces. The result can be characterized by the following proposition

Proposition 5 (Prop. 8.1 (e) and (f), in [7])**.** *Let X be a first countable space and F_n be a sequence of functions on X. Then F_n Γ-converges to the function F if and only the two following requirements are satisfied:*
(i) For all $x \in X$ and for every sequence x_n converging to x in X we have

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n); \tag{3.12}$$

(ii) For all $x \in X$ there exist a sequence $(x_n)_(n \geq 1) \subset X$ such that

$$F(x) = \lim_{n\to\infty} F_n(x_n). \tag{3.13}$$

Other important result concerning Γ-convergence are preservation of minimizers under Γ-convergence. These can be found in chapter 7 of [7].

3.2 Γ-convergence of F_α^R

We are in this subsection going to show that the quadratic forms, F_α^R that we discussed earlier, have some Γ-convergence properties, that will allow us to conclude on the convergence properties of their corresponding operators.

In the following we will apply the lemma:

Lemma 2. *Let $w \in H^1(\mathbb{R}^3)$, then*

$$\frac{1}{\sqrt{n}} \int_{B_n(0)} \hat{w} \to 0, \quad \text{as } n \to \infty, \tag{3.14}$$

where $\hat{w}$ denotes the Fourier transform of w. In other notation

$$\left| \int_{B_n(0)} \hat{w} \right| = \epsilon(n)\sqrt{n}, \tag{3.15}$$

for some ϵ-function, $\epsilon(n) \to 0$ as $n \to \infty$.

Proof. Notice that we have already seen that for $w \in H^1(\mathbb{R}^3)$ we have

$$\int_{B_n(0)} \hat{w} \lesssim \sqrt{n}, \tag{3.16}$$

which follows from Cauchy-Schwartz inequality. To improve the bound, we split the integral. Clearly $\hat{w}(k) = \frac{f(k)}{(k^2+1)^{1/2}}$ for some $f \in L^2(\mathbb{R}^3)$. Let $\epsilon_n = (\ln(n))^2/n^{3/2}$ and define the sets $B_n^{>} = B_n(0) \cap \{|f| > \epsilon_n\}$ and $B_n^{\leq} = B_n(0) \cap \{|f| \leq \epsilon_n\}$. Notice that $\epsilon_n \to 0$ as $n \to \infty$. By the usual triangle inequality we have

$$\left| \int_{B_n(0)} \hat{w} \right| \leq \left| \int_{B_n^{>}} \hat{w} \right| + \left| \int_{B_n^{\leq}} \hat{w} \right|. \tag{3.17}$$

We see that

$$\left| \int_{B_n^{\leq}} \hat{w} \right| \leq \|\mathbb{1}_{|f| \leq \epsilon_n} f\|_2 \sqrt{n}. \tag{3.18}$$

By dominated convergence theorem we now that $\|\mathbb{1}_{|f| \leq \epsilon_n} f\|_2 \to 0$, since $|\mathbb{1}_{|f| \leq \epsilon_n} f|^2 \to 0$ pointwise ($\epsilon_n \to 0$), and is dominated by integrable function $|f|^2$. Thereby we have

$$\left| \int_{B_n^{\leq}} \hat{w} \right| = \epsilon_1(n)\sqrt{n}, \tag{3.19}$$

for $\epsilon_1(n) \to 0$ as $n \to \infty$. On the other hand we use Hölder's inequality $(3/2, 3)$ on the other integral and obtain

$$\left| \int_{B_n^{>}} \hat{w} \right| \leq \int_{B_n^{>}} |\hat{w}| \leq \|\mathbb{1}_{|f| > \epsilon_n} f\|_{3/2} \|\mathbb{1}_{B_n(0)} (1 + k^2)^{-1/2}\|_3. \tag{3.20}$$

Now $\|\mathbb{1}_{B_n(0)} (1 + k^2)^{-1/2}\|_3 \sim (\ln(n))^{1/3}$. Furthermore, we have

$$\|\mathbb{1}_{|f| > \epsilon_n} f\|_{3/2}\| = \left(\int_{f > \epsilon_n} |f|^{3/2} \right)^{2/3} < \left(\frac{1}{\sqrt{\epsilon_n}} \int_{f > \epsilon_n} |f|^2 \right)^{2/3} = \frac{1}{\epsilon^{1/3}} \|f\|_2^{4/3}. \tag{3.21}$$

Thus we have

$$\left|\int_{B_n^>} \hat{w}\right| \lesssim \frac{1}{(\ln(n))^{1/3}} \|f\|_2^{4/3} \sqrt{n} = \epsilon_2(n)\sqrt{n}, \tag{3.22}$$

for some $\epsilon_2(n) \to 0$ as $n \to \infty$. Combining our two estimates gives us

$$\left|\int_{B_n(0)} \hat{w}\right| \lesssim (\epsilon_1(n) + \epsilon_2(n))\sqrt{n} \implies \left|\int_{B_n(0)} \hat{w}\right| = \epsilon(n)\sqrt{n}, \tag{3.23}$$

for some $\epsilon(n) \to 0$ as $n \to \infty$. This concludes the proof. □

We show that F_α^n where $n \in \mathbb{N}$ satisfies the assumptions of Theorem 3. By Proposition 5 we see that it suffices to show that:

1. For all $u \in \mathscr{D}(F_\alpha)$ there exist a sequence $(u_n)_{n\geq 1}$, such that $u_n \in \mathscr{D}(F_\alpha^n)$, u_n converges in $L^2(\mathbb{R}^3)$ to u and
$$F_\alpha(u) = \lim_{n\to\infty} F_\alpha^n(u_n). \tag{3.24}$$
2. For all $u \in \mathscr{D}(F_\alpha)$ and all $(u_n)_{n\geq 1}$, with $u_n \in \mathscr{D}(F_\alpha^n)$, converging weakly to u, we have
$$F_\alpha(u) \leq \liminf_{n\to\infty} F_\alpha^n(u_n). \tag{3.25}$$

Notice then that (2.) include both the weak lower bound on F_α as required by Theorem 3 but also the strong lower bound on F_α as required by Proposition 5, since all strongly converging sequences also are weakly convergent. Let us briefly remind ourselves that

$$F_\alpha^R(u) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{w}_R(k)|^2 - \mu\|u\|_{L^2(\mathbb{R}^3)}^2 + |\xi_R|^2\left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{R}{\sqrt{\mu}}\right)\right), \tag{3.26}$$

with $\hat{w}_R = \hat{u} - \widehat{G\rho^R}$, $\hat{\rho}^R = \xi_R \mathbb{1}_{B_R(0)}$, $\xi_R = (4\pi R + \alpha)^{-1}\int_{B_R(0)} \hat{u}$, and domain

$$\mathscr{D}\left(F_\alpha^R\right) = \left\{u \in L^2(\mathbb{R}^3) \mid w_R \in H^1(\mathbb{R}^3)\right\} = H^1(\mathbb{R}^3). \tag{3.27}$$

And

$$F_\alpha(u) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{w}(k)|^2 - \mu\|u\|_{L^2(\mathbb{R}^3)}^2 + |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right), \tag{3.28}$$

with domain

$$\mathscr{D}\left(F_\alpha(u)\right) = \left\{u \in L^2(\mathbb{R}^3) \mid \hat{u} = \hat{w} + \xi\hat{G},\ w \in H^1(\mathbb{R}^3),\ \xi \in \mathbb{C}\right\}, \tag{3.29}$$

Proposition 6. *Let F_α^n and F_α be defined as earlier. Then for all $u \in \mathscr{D}(F_\alpha)$ there exist $(u_n)_{(n\geq 1)}$ with $u_n \in \mathscr{D}(F_n)$ such that*

$$F_\alpha(u) = \lim_{n\to\infty} F_\alpha^n(u_n). \tag{3.30}$$

Proof. We prove this result by simply constructing the correct sequence u_n. For $u = w + \xi G$, with $w \in H^1(\mathbb{R}^3)$, let $u_n = w + \xi G_n$, where $\widehat{G_n} = \mathbb{1}_{B_n(0)}\hat{G}$. Then we clearly have

$$F_\alpha^n(u_n) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right) |\hat{w}_n(k)|^2 - \mu\|u_n\|^2_{L^2(\mathbb{R}^3)} + |\xi_n|^2 \left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{n}{\sqrt{\mu}}\right)\right), \quad (3.31)$$

with $\hat{w}_n = \hat{w} - (\xi_n - \xi)\mathbb{1}_{B_n(0)}\hat{G}$. Now it is obvious that $u_n \in H^1(\mathbb{R}^3)$ for all n and that $u_n \to u$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$. Furthermore, we see that

$$\xi_n = \frac{1}{4\pi n + \alpha}\int_{B_n(0)} \hat{w} + \xi\hat{G}, \quad (3.32)$$

and since $\int_{B_n(0)} \hat{w} \lesssim \sqrt{n}$, and $\int_{B_n(0)} \hat{G} \sim 4\pi n$ we see that $\xi_n \to \xi$ as $n \to \infty$. We can even say that $(\xi - \xi_n) \sim \frac{1}{4\pi n}\int_{B_n(0)} \hat{w}$. By lemma 2, we know that $\left|\frac{1}{4\pi n}\int_{B_n(0)} \hat{w}\right| = \epsilon(n)\frac{1}{\sqrt{n}}$ for some ϵ-function $\epsilon(n) \to 0$ as $n \to \infty$. Now expanding the expression for $F_\alpha^n(u_n)$ above we have

$$\begin{aligned} F_\alpha^n(u_n) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right) |\hat{w}(k)|^2 + |\xi_n - \xi|^2 \int_{B_n(0)} d^3k \ \left(k^2+\mu\right) |\hat{G}(k)|^2 \\ - 2\mathrm{Re}(\xi_n - \xi)\int_{B_n(0)} d^3k \ \left(k^2+\mu\right) \overline{\hat{w}(k)}\hat{G} \\ - \mu\|u_n\|^2_{L^2(\mathbb{R}^3)} + |\xi_n|^2\left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{n}{\sqrt{\mu}}\right)\right). \end{aligned} \quad (3.33)$$

Using that $\hat{G}(k) = \frac{1}{k^2+\mu}$ we have

$$|\xi_n - \xi|^2 \int_{B_n(0)} d^3k \ \left(k^2+\mu\right) |\hat{G}(k)|^2 \sim \frac{\epsilon(n)^2}{n}\int_{B_n(0)} d^3k \ \hat{G}(k) \sim \epsilon(n)^2 \to 0 \quad \text{as } n \to \infty. \quad (3.34)$$

Furthermore we know that

$$\left|(\xi_n - \xi)\int_{B_n(0)} d^3k \ \left(k^2+\mu\right) \overline{\hat{w}(k)}\hat{G}\right| = \left|(\xi_n - \xi)\int_{B_n(0)} d^3k \ \overline{\hat{w}(k)}\right| \sim \epsilon(n)^2 \to 0 \quad \text{as } n \to \infty. \quad (3.35)$$

Finally we have that $\|u_n\| \to \|u\|$ as $n \to \infty$ since $u_n \to u$ in $L^2(\mathbb{R}^3)$ and that

$$|\xi_n|^2\left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{n}{\sqrt{\mu}}\right)\right) \to |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right), \quad \text{as } n \to \infty, \quad (3.36)$$

where we have used that the limit of a product is the product of the limits whenever both limits exist. Thus we have in total

$$F_\alpha^n(u_n) \to \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right) |\hat{w}(k)|^2 - \mu\|u\|^2_{L^2(\mathbb{R}^3)} + |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right) = F_\alpha(u), \quad (3.37)$$

as we wanted. □

Proposition 7. *Let $(u_n)_{(n\geq 1)}$ be any sequence such that $u_n \in \mathscr{D}(F_\alpha^n) = H^1(\mathbb{R}^3)$ for all $n \geq 1$ and such that u_n converges weakly to u ($u_n \rightharpoonup u$) in $L^2(\mathbb{R}^3)$ as $n \to \infty$. Then we have the* *following inequality*

$$F_\alpha(u) \leq \liminf_{n\to\infty} F_\alpha^n(u_n). \tag{3.38}$$

Proof. Let u_n be as in the proposition above. Since the weak limit, u, is in $\mathscr{D}(F_\alpha)$ it is of the form $u = w + \xi G$ with $w \in H^1(\mathbb{R}^3)$ and $\xi \in \mathbb{C}$. On the other hand we then may without loss of generality take u_n to be of the form $u_n = w + \xi g_n$, with $g_n \in H^1(\mathbb{R}^3)$ such that $g_n \rightharpoonup G$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$. We then have

$$F_\alpha^n(u_n) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right) |\hat{w}_n(k)|^2 - \mu\|u_n\|^2_{L^2(\mathbb{R}^3)} + |\xi_n|^2 \left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{n}{\sqrt{\mu}}\right)\right), \tag{3.39}$$

where we have defined $\hat{w}_n(k) = \hat{w}(k) + \xi\hat{g}_n - \xi_n\hat{G}_n$, with $\hat{G}_n = \mathbb{1}_{B_n(0)}\hat{G}$, and $\xi_n = (4\pi n+\alpha)^{-1}\int_{B_n(0)} \hat{w} + \xi\hat{g}_n$. Now there exist a subsequence $F_\alpha^{n_j}(u_{n_j})$ that converges to $\liminf_{n\to\infty} F_\alpha^n(u_n)$. Expanding $F_\alpha^{n_j}(u_{n_j})$ we have

$$\begin{aligned} F_\alpha^{n_j}(u_{n_j}) = \int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right) \Big\{ |\hat{w}(k)|^2 + |\xi\hat{g}_{n_j}(k) - \xi_{n_j}\hat{G}_{n_j}(k)|^2 + 2\mathrm{Re}\left[(\xi\hat{g}_{n_j}(k) - \xi_{n_j}\hat{G}_{n_j}(k))\overline{\hat{w}(k)}\right]\Big\} \\ -\mu\|u_{n_j}\|^2_{L^2(\mathbb{R}^3)} + |\xi_{n_j}|^2 \left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{n_j}{\sqrt{\mu}}\right)\right). \end{aligned} \tag{3.40}$$

Since $\int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{f}(k)|^2 \cong \|f\|^2_{H^1(\mathbb{R}^3)}$, and $\mu > 0$ and $(n_j)_{j\in\mathbb{N}}$ can be chosen such that $\left(\alpha + 4\pi\sqrt{\mu}\arctan\left(\frac{n_j}{\sqrt{\mu}}\right)\right) > 0$, we see that either $\liminf F_\alpha^n(u_n) = \infty$ or $(\xi g_{n_j} - \xi_{n_j}G_{n_j})$ is $H^1(\mathbb{R}^3)$ norm bounded. In the first case the desired result is trivially true. In the second case we observe that if $h_j := (\xi g_{n_j} - \xi_{n_j}G_{n_j})_{j\geq 1}$ is $H^1(\mathbb{R}^3)$ norm bounded such that $(h_j)_{j\geq1} \subset B(0,M)$ for some $M > 0$. By Alaoglu's theorem and the fact that $H^1(\mathbb{R}^3)$ is a Hilbert space we then know that all subsequences h_{j_k} have a further subsequence $h_{j_{k_l}}$ that converge weakly in $H^1(\mathbb{R}^3)$ to some $h \in H^1(\mathbb{R}^3)$. However, if $h_{j_{k_l}}$ converge weakly in $H^1(\mathbb{R}^3)$ it also converges weakly in $L^2(\mathbb{R}^3)$ to the same limit, since the dual space of $H^1(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$ is $H_{-1}(\mathbb{R}^3) \supset L^2(\mathbb{R}^3)$. Thus we know that $h_{j_{k_l}}$ converges weakly in $L^2(\mathbb{R}^3)$. However, as we know that $g_n \rightharpoonup G$ in $L^2(\mathbb{R}^3)$ and $G_n \xrightarrow{\|\cdot\|_2} G$ so $G_n \rightharpoonup G$ in $L^2(\mathbb{R}^3)$ we conclude that in order for $h_{j_{k_l}} \rightharpoonup h$ in $L^2(\mathbb{R}^3)$ we must have that $\xi_{n_{j_{k_l}}} \to \chi$ for some χ and then $h = (\xi - \chi)G$. However notice that $G \notin H^1(\mathbb{R}^3)$. So $h \in H^1(\mathbb{R}^3)$ implies that $\chi = \xi$, such that $h_{j_{k_l}} \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$. But then we have shown that for all subsequences $(h_{j_k})_{k\geq1}$ of $(h_j)_{j\geq1}$, there exist a further subsequence, $h_{j_{k_l}}$ converging weakly to 0. Thereby $h_j \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$ and $\xi_{n_j} \to \xi$ as $j \to \infty$. Thus we have

$$\begin{aligned} \lim_{j\to\infty} F_\alpha^{n_j}(u_{n_j}) &= \lim_{j\to\infty} \left(\int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right) \Big\{ |\hat{w}(k)|^2 + |\xi\hat{g}_{n_j}(k) - \xi_{n_j}\hat{G}_{n_j}(k)|^2 \Big\} - \mu\|u_{n_j}\|^2_{L^2(\mathbb{R}^3)} \right) \\ &\qquad + |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right) \\ &\geq \lim_{j\to\infty} \left(\int_{\mathbb{R}^3} d^3k \ \left(k^2+\mu\right)|\hat{w}(k)|^2 + \mu(\|\xi g_{n_j} - \xi_{n_j}G_{n_j}\|^2_{L^2(\mathbb{R}^3)} - \|u_{n_j}\|^2_{L^2(\mathbb{R}^3)}) \right) \\ &\qquad + |\xi|^2\left(\alpha + 2\pi^2\sqrt{\mu}\right), \end{aligned} \tag{3.41}$$

where we in the second line threw away the positive term $\int_{\mathbb{R}^3} d^3k\, k^2|\xi\hat{g}_{n_j}(k)-\xi_{n_j}\hat{G}_{n_j}(k)|^2$. Now we see that

$$\begin{aligned}\|\xi g_{n_j}-\xi_{n_j}G_{n_j}\|^2-\|u_{n_j}\|^2=\|\xi g_{n_j}\|^2+\|\xi_{n_j}G_{n_j}\|^2-2\xi_{n_j}\bar{\xi}\mathrm{Re}\left\langle g_{n_j},G_{n_j}\right\rangle\\-\|w\|^2-\|\xi g_{n_j}\|^2-2\xi\mathrm{Re}\left\langle w,g_{n_j}\right\rangle,\end{aligned}\tag{3.42}$$

from which it follows that

$$\begin{aligned}\lim_{j\to\infty}\left(\|\xi g_{n_j}-\xi_{n_j}G_{n_j}\|^2-\|u_{n_j}\|^2\right)&=\|\xi G\|^2-\|w\|^2-2|\xi|^2\left\langle G,G\right\rangle-2\xi\mathrm{Re}\left\langle w,G\right\rangle\\&=-\|w\|^2-\|\xi G\|^2-2\xi\mathrm{Re}\left\langle w,G\right\rangle=-\|w+\xi G\|^2\\&=-\|u\|^2,\end{aligned}\tag{3.43}$$

where $\|\cdot\|=\|\cdot\|_{L^2(\mathbb{R}^3)}$, $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^3)}$ and we have used that for $f_n\xrightarrow{\|\cdot\|}f$ and $g_n\rightharpoonup g$ as $n\to\infty$ we have $\langle g_n,f_n\rangle\to\langle g,f\rangle$, which follows directly from norm boundedness of weakly convergent sequences. Collecting all the above we have

$$\lim_{j\to\infty}F_\alpha^{n_j}(u_{n_j})\geq\int_{\mathbb{R}^3}d^3k\ \left(k^2+\mu\right)|\hat{w}(k)|^2-\mu\|u\|^2_{L^2(\mathbb{R}^3)}+|\xi|^2\left(\alpha+2\pi^2\sqrt{\mu}\right)=F_\alpha(u),\tag{3.44}$$

which was the desired result. □

Now collecting the above results in Propositions 6 and 7 we find that F_α^n and F_α indeed satisfies the assumptions of Theorem 3 (Corollary 2) assuming that F_α^n admits a common lower bound (using that F_α^n is independent of μ, this is actually easy to see), and thus we may conclude that the corresponding operator H_α^n and H_α fulfil that $H_\alpha^n\to H_\alpha$ in the strong resolvent sense. This tells us that points in the spectrum cannot suddenly emerge under the convergence as the following proposition will show.

Proposition 8. *Let $(A_n)_{n\geq1}$ be a sequence of operators on a Hilbert space H and let A be an operator on H. Assume that $A_n\to A$ in the strong resolvent sense. If $\lambda\notin\sigma(A_n)$ for all $n\geq M_\lambda$ for some $M_\lambda>0$, then $\lambda\notin\sigma(A)$, where $\sigma(\cdot)$ denotes the spectrum.*

Proof. Let $R_\lambda(A_n)=(A_n-\lambda I)^{-1}$ denote the resolvents of A_n at λ. If there exist M_λ such that $\lambda\notin\sigma(A_n)$ for all $n\geq M_\lambda$ we know that $R_\lambda(A_n)\in\mathcal{B}(H)$ (bounded operators on H) for all $n\geq M_\lambda$. By the strong resolvent convergence we know that $R_\lambda(A_n)f$ converge to $R_\lambda(A)f$ as $n\to\infty$ for all $f\in H$. Thus we conclude that

$$\sup_{n\geq1}\left(\|R_\lambda(A_n)f\|\right)<\infty,\qquad\text{for all } f\in H.\tag{3.45}$$

By the uniform boundedness principle ([4], 5.13) we conclude that

$$\sup_{n\geq1}\left(||R_\lambda(A_n)||\right)<\infty.\tag{3.46}$$

Now estimating the operator norm of $R_\lambda(A)$ we find

$$\|R_\lambda(A)f\| < \|R_\lambda(A_{n_\epsilon(f)})f\| + \epsilon, \tag{3.47}$$

for some $n_\epsilon(f) \geq 1$ depending on ϵ and f. Thus taking supremum on both sides over all $f \in H$ with $\|f\| \leq 1$ we find

$$\begin{aligned}\|R_\lambda(A)\| = \sup\left(\|R_\lambda(A)f\| \mid f \in H,\ \|f\| \leq 1\right) &\leq \sup\left(\|R_\lambda(A_{n_\epsilon(f)})f\| \mid f \in H,\ \|f\| \leq 1\right) + \epsilon \\ &\leq \sup_{n\geq 1}\left(||R_\lambda(A_n)||\right) + \epsilon < \infty.\end{aligned} \tag{3.48}$$

Thereby $R_\lambda(A) \in \mathcal{B}(H)$ and $\lambda \notin \sigma(A)$. □

Another immediate consequence of Propositions 6 and 7 is that F_α^n Γ-converge to F_α in the norm topology. Thus we may conclude that F_α is a norm lower semicontinuous quadratic form, an by Theorem 2 we may conclude that F_α is closable and bounded from below.

4 The $N + 1$ case

We are in this section going to study the more general case of N fermions of one species and 1 of the other. We start by remembering that in this case we can split the formal Hamiltonian in two pieces; one regarding the centre of mass, and one regarding the N relative coordinates of the two species. This will not be possible in the $N + M$ case, as there are too many relative coordinates. Now one might seek to analyse this problem by constructing self-adjoint extensions of the Laplacian on some restricted domain, as we did in the $1 + 1$ case. However, this becomes tremendously more complicated. This can be realized by the fact that for each interaction, the coupling can end up depending on all other coordinates. Similar to the requirement of translational invariance in the $1 + 1$ case, which eliminated the possibility of the coupling depending on the centre of mass position, we thus need to impose certain symmetries on the system to end up with a Hamiltonian of the desired form. A sufficient requirement would probably be the that of locality and permutation invariance, such that all couplings are equal and must be independent all coordinates. However one way to avoid these complications, as we saw in section 2.2, is to consider a quadratic form instead. By this approach we can construct the quadratic form directly with the desired properties. Showing closability and boundedness from below of such a quadratic form, then allows us to conclude the existence of a self-adjoint and bounded from below Hamiltonian of such a system.

4.1 Quadratic form

Inspired by the approach in the $1 + 1$ case, we define the quadratic form in the $N + 1$ case, also by considering a limiting case of rank-one perturbations of the free Hamiltonian. Again we imagine the following family of Hamiltonians

$$H_\gamma^R = H_0 - \gamma_R \sum_{i=1}^N \phi_R^i \left\langle \phi_R^i, \cdot \right\rangle. \tag{4.1}$$

We choose the ϕ_R^i such that its Fourier transform is a ball of radius R in the i^{th} coordinate, $\hat{\phi}_R^i(k) = \mathbb{1}_{B_R(0)}(k_i)$. The corresponding quadratic forms thus become

$$\begin{aligned} F_\gamma^R(u) = \int_{\mathbb{R}^{3N}} \mathrm{d}k \Bigg[\overline{\hat{u}}(k) \left(\sum_{i=1}^N k_i^2 + \frac{2}{m+1} \sum_{\substack{(i,j)=(1,1) \\ i<j}}^{(N,N)} k_i \cdot k_j \right) \hat{u}(k) \\ -\gamma_R \sum_{i=1}^N \overline{\hat{u}}(k) \mathbb{1}_{B_R(0)}(k_i) \int_{B_R(-k_i)} \mathrm{d}p \ \hat{u}(k+p^i) \Bigg], \end{aligned} \tag{4.2}$$

where $p^i = (0,0,..., \underbrace{p}_{i^{\text{th}}}, 0, 0, ...)$. Introducing the following notation

$$\hat{G}(k) = \left(\sum_{i=1}^N k_i^2 + \frac{2}{m+1} \sum_{1 \le i < j \le N} k_i \cdot k_j + \mu \right)^{-1}, \quad \mu > 0 \tag{4.3}$$

where we note that $\sum_{i=1}^N k_i^2 + \frac{2}{m+1} \sum_{1\le i<j\le N} k_i \cdot k_j + \mu > 0$ since if $\sum_{1\le i<j\le N} k_i \cdot k_j$ is negative then $\sum_{i=1}^N k_i^2 + \frac{2}{m+1} \sum_{1\le i<j\le N} k_i \cdot k_j + \mu > (\sum_{i=1}^N k_i)^2 + \mu > 0$ for $m > 0$, and if $\sum_{1\le i<j\le N} k_i \cdot k_j$ is non-negative, then it is obvious. Thus $\hat{G}$ is well-defined. We also introduce

$$\hat{\rho}^R(k) = \gamma_R \sum_{i=1}^N \mathbb{1}_{B_R(0)}(k_i) \int_{B_R(-k_i)} \mathrm{d}p \ \hat{u}(k+p^i) := \sum_{i=1}^N \mathbb{1}_{B_R(0)}(k_i) \xi_i^R(\bar{k}^i) \tag{4.4}$$

where $\bar{k}^i = \{k_1, ..., k_{i-1}, k_{i+1}, ..., k_N\} \in \mathbb{R}^{3(N-1)}$. Here we defined

$$\xi_i^R(\bar{k}^i) = \gamma_R \int_{B_R(-k_i)} \mathrm{d}p \ \hat{u}(k+p^i). \tag{4.5}$$

We observe that if u is fermionic we have $\xi_i^R(\bar{k}^i) = (-1)^{i-1} \xi_1^R(\bar{k}^i) := (-1)^{i-1} \xi^R(\bar{k}^i)$. Thus we may write

$$\hat{\rho}^R(k) = \sum_{i=1}^N (-1)^{i-1} \mathbb{1}_{B_R(0)}(k_i) \xi^R(\bar{k}^i). \tag{4.6}$$

We notice also that $\xi^R : \mathbb{R}^{3(N-1)} \to \mathbb{C}$ is itself fermionic if u is.
The quadratic form $F_\gamma^R(u)$ can now be rewritten as

$$\begin{aligned} F_\gamma^R(u) = \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k)^{-1} |\hat{u}(k) - \widehat{\rho^R G}(k)|^2 - \mu \|u\|_{L^2(\mathbb{R}^{3N})} - \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k) \left| \hat{\rho}^R(k) \right|^2 \\ +2\mathrm{Re} \int_{\mathbb{R}^{3N}} \mathrm{d}k \overline{\hat{\rho}^R(k)} \hat{u}(k) - \int_{\mathbb{R}^{3N}} \mathrm{d}k \overline{\hat{u}(k)} \hat{\rho}^R(k), \end{aligned} \tag{4.7}$$

where $\widehat{\rho^R G}(k) = \hat{\rho}^R(k)\hat{G}(k)$. By a simple calculation we see that

$$\begin{aligned}\overline{\int_{\mathbb{R}^{3N}} \mathrm{d}k \overline{\hat{\rho}^R(k)}\hat{u}(k)} &= \gamma_R \sum_{i=1}^N \overline{\int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^i \left(\int_{B_R(0)} \mathrm{d}k_i \hat{u}(k)\right) \overline{\left(\int_{B_R(-k_i)} \mathrm{d}p\ \hat{u}(k+p^i)\right)}} \\ &= \int_{\mathbb{R}^{3N}} \mathrm{d}k \overline{\hat{\rho}^R(k)}\hat{u}(k) = \gamma_R^{-1} \sum_{i=1}^N \int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^i \left|\xi^R(\bar{k}^i)\right|^2 = \gamma_R^{-1} N \left\|\xi^R\right\|^2_{L^2(\mathbb{R}^{3(N-1)})}.\end{aligned} \tag{4.8}$$

Hence we conclude that $\int_{\mathbb{R}^{3N}} \mathrm{d}k\overline{\hat{\rho}^R(k)}\hat{u}(k)$ is real, such that

$$\begin{aligned}F_\gamma^R(u) = \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k)^{-1} |\hat{u}(k) - \widehat{\rho^R G}(k)|^2 - \mu \left\|u\right\|_{L^2(\mathbb{R}^{3N})} - \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k) \left|\hat{\rho}^R(k)\right|^2 \\ + \gamma_R^{-1} N \left\|\xi^R\right\|^2_{L^2(\mathbb{R}^{3(N-1)})}.\end{aligned} \tag{4.9}$$

Furthermore, we can rewrite the term

$$\begin{aligned}\int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k) \left|\hat{\rho}^R(k)\right|^2 &= \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k) \left|\sum_{i=1}^N (-1)^{i-1} \mathbb{1}_{B_R(0)}(k_i) \xi^R(\bar{k}^i)\right|^2 \\ &= \sum_{i=1}^N \int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^i \left(\int_{B_R(0)} \mathrm{d}k_i \hat{G}(k)\right) \left|\xi^R(\bar{k}^i)\right|^2 \\ &\quad - 2 \sum_{1\le i<j\le N} \int_{\mathbb{R}^{3(N-2)}} \mathrm{d}\bar{k}^{ij} \int_{B_R(0)} \mathrm{d}k_i \int_{B_R(0)} \mathrm{d}k_j \overline{\xi^R(k_i, \bar{k}^{ij})} G(k) \xi^R(k_j, \bar{k}^{ij}) \\ &= N \int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N \left(\int_{B_R(0)} \mathrm{d}k_N \hat{G}(k)\right) \left|\xi^R(\bar{k}^N)\right|^2 \\ &\quad - N(N-1) \int_{\mathbb{R}^{3(N-2)}} \mathrm{d}\bar{q} \int_{B_R(0)} \mathrm{d}s \int_{B_R(0)} \mathrm{d}t \overline{\xi^R(s,\bar{q})} \hat{G}(s,t,\bar{q}) \xi^R(t,\bar{q}).\end{aligned} \tag{4.10}$$

where $\bar{k}^{ij}$ is k with k_i and k_j removed, and we introduced $\bar{q} = \bar{k}^{N-1,N}$ in the last step. Furthermore, we used the symmetry of $\hat{G}$, and from the anti-symmetry of ξ^R that $\overline{\xi^R(\bar{k}^j)}\xi^R(\bar{k}^i) = (-1)^{i-1+j-2}\overline{\xi^R(k_i, \bar{k}^{ij})}\xi^R(k_j, \bar{k}^{ij})$ for $i<j$. Thus we need to calculate the integral

$$\int_{B_R(0)} \mathrm{d}k_N \hat{G}(k) = \int_{B_R(0)} \mathrm{d}k_N \left(\sum_{i=1}^N k_i^2 + \frac{2}{m+1} \sum_{1\le i<j\le N} k_i \cdot k_j + \mu\right)^{-1}. \tag{4.11}$$

Notice that this is of the form

$$\begin{aligned}\int_{B_R(0)} \mathrm{d}q \frac{1}{q^2 + q\cdot y + \alpha} &= \int_{B_R(0)} \mathrm{d}q \left(\frac{1}{q^2} - \frac{q\cdot y + \alpha}{(q^2 + q\cdot y + \alpha)q^2}\right) \\ &= 4\pi R - \int_{B_R(0)} \mathrm{d}q \frac{q\cdot y + \alpha}{(q^2 + q\cdot y + \alpha)q^2},\end{aligned} \tag{4.12}$$

with $q = k_N$, $y = \frac{2}{m+1}\sum_{i=1}^{N-1} k_i$, and $\alpha = \sum_{i=1}^{N-1} k_i^2 + \frac{2}{m+1}\sum_{1\le i<j\le N-1} k_i \cdot k_j + \mu$. Now the remaining integral can be estimated by the following way lemma.

Lemma 3. *Let $y \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$, and $R > 0$. Assume furthermore that $4\alpha > y^2$. Then we have*

$$\int_{B_R(0)} \mathrm{d}q \frac{q \cdot y + \alpha}{(q^2 + q \cdot y + \alpha)q^2} = \pi^2 \sqrt{4\alpha - y^2} + \mathcal{O}\left(\frac{|y|^2 + \alpha}{R}\right) + \mathcal{O}\left(\frac{|y|\left(|y|^2 + \alpha\right)}{R^2}\right). \tag{4.13}$$

However, it is still bounded in α, as can be seen from the $\alpha \to \infty$ limit, where we obtain

$$\int_{B_R(0)} \mathrm{d}q \frac{q \cdot y + \alpha}{(q^2 + q \cdot y + \alpha)q^2} \simeq 4\pi R. \tag{4.14}$$

Proof. Notice first that by Feynman parametrization we have (assuming $4\alpha > y^2$)

$$\frac{1}{(q^2 + q \cdot y + \alpha)q^2} = \int_0^1 \mathrm{d}t \frac{1}{(t(q^2 + q \cdot y + \alpha) + (1-t)q^2)^2} = \int_0^1 \mathrm{d}t \frac{1}{((q + ty/2)^2 - t^2y^2/4 + \alpha t)^2} \tag{4.15}$$

Thus we find

$$\begin{aligned} \int_{B_R(0)} \mathrm{d}q \frac{q \cdot y + \alpha}{(q^2 + q \cdot y + \alpha)q^2} &= \int_0^1 \mathrm{d}t \int_{B_R(0)} \mathrm{d}q \frac{(q + ty/2) \cdot y - ty^2/2 + \alpha}{((q + ty/2)^2 - t^2y^2/4 + \alpha t)^2} \\ &= \int_0^1 \mathrm{d}t \int_{B_R(ty/2)} \mathrm{d}\tilde{q} \frac{\tilde{q} \cdot y - ty^2/2 + \alpha}{(\tilde{q}^2 - t^2y^2/4 + \alpha t)^2} \end{aligned} \tag{4.16}$$

Notice that around the ball $B_R(ty/2)$ we can find a ball $B_{R+t|y|/2}(0)$. We have the simple geometric bound

$$\left| \int_{B_{R+t|y|/2}(0)} \mathrm{d}\tilde{q} \frac{\tilde{q} \cdot y}{(\tilde{q}^2 - t^2y^2/4 + \alpha t)^2} - \int_{B_R(ty/2)} \mathrm{d}\tilde{q} \frac{\tilde{q} \cdot y}{(\tilde{q}^2 - t^2y^2/4 + \alpha t)^2} \right| \leq C_1 \frac{t\,|y|^2 R^2}{R^3}, \tag{4.17}$$

for some $C_1 > 0$ independent of y and α and for $R \gg t\,|y|$. However, $\int_{B_{R+t|y|/2}(0)} \mathrm{d}\tilde{q} \frac{\tilde{q} \cdot y}{(\tilde{q}^2 - t^2y^2/4 + \alpha t)^2} = 0$ since the integrand is odd. Thus we can bound the integral

$$\left| \int_{B_R(ty/2)} \mathrm{d}\tilde{q} \frac{\tilde{q} \cdot y}{(\tilde{q}^2 - t^2y^2/4 + \alpha t)^2} \right| \leq C_1 \frac{|y|^2}{R}, \tag{4.18}$$

for $R \gg |y|$. Now considering instead the term

$$\int_0^1 \mathrm{d}t \int_{B_R(ty/2)} \mathrm{d}\tilde{q} \frac{ty^2/2 + \alpha}{(\tilde{q}^2 - t^2y^2/4 + \alpha t)^2}, \tag{4.19}$$

this actually converges as $R \to \infty$ by the dominated convergence theorem. Therefore, we may approximate it for $R \to \infty$ by

$$\int_0^1 \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}\tilde{q} \frac{-ty^2/2 + \alpha}{(\tilde{q}^2 - t^2y^2/4 + \alpha t)^2} = 4\pi \int_0^1 \mathrm{d}t \frac{\pi(\alpha - t^2y^2/2)}{4\sqrt{\alpha t - t^2y^2/4}}. \tag{4.20}$$

Changing variable $s = t|y|/2 - \alpha/|y|$ we find $\mathrm{d}s = \frac{|y|}{2}\,\mathrm{d}t$ such that

$$\int_0^1 \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}\tilde{q} \frac{-ty^2/2+\alpha}{(\tilde{q}^2 - t^2y^2/4 + \alpha t)^2} = \pi^2 \int_{-\alpha/|y|}^{|y|/2-\alpha/|y|} \mathrm{d}s \frac{2}{|y|} \frac{-|y|\,s}{\sqrt{\alpha^2/y^2 - s^2}} = \pi^2\sqrt{4\alpha - y^2}. \quad (4.21)$$

The error obtained by this approximation is a again straightforward to bound by geometrical considerations. Covering $B_R(ty/2)$ by the ball $B_{R+t|y|/2}(0)$ and integrating over this domain instead, we make at most an error $\epsilon_1 \leq C'|y|(|y|^2+\alpha)/R^2$. Now changing the domain of integration further to $\mathbb{R}^3$ we make at most an error $\epsilon_2 \leq C''(|y|^2+\alpha)/R$. Thus we may conclude that we make a total error of $\epsilon = \epsilon_1 + \epsilon_2 \leq C_2(|y|^2+\alpha)/R$ for $R \gg |y|$ and some C_2 independent of y and α. Thus combining (4.18), (4.21), and the bounds above we have

$$\left| \int_{B_R(0)} \mathrm{d}q \frac{q\cdot y + \alpha}{(q^2 + q\cdot y + \alpha)q^2} - \pi^2\sqrt{4\alpha - y^2} \right| \leq C\left(\frac{|y|^2+\alpha}{R} + \frac{|y|\,(|y|^2+\alpha)}{R^2} \right), \quad (4.22)$$

for some C independent of y and α. This proves the claim. □

Using Lemma 3 we obtain and the fact that $\alpha = \mathcal{O}\left(\left|\bar{k}^N\right|^2 + 1\right)$ and $|y| = \mathcal{O}\left(\left|\bar{k}^N\right|\right)$

$$\begin{aligned}
\int_{B_R(0)} \mathrm{d}k_N \hat{G}(k) &= 4\pi R - \pi^2 \sqrt{\left(4 - \frac{4}{(m+1)^2}\right)\sum_{i=1}^{N-1} k_i^2 + \left(4 - \frac{4}{m+1}\right)\frac{2}{m+1} \sum_{1\leq i<j\leq N-1} k_i\cdot k_j + 4\mu} \\
&\qquad + \mathcal{O}\left(\frac{\left|\bar{k}^N\right|^2+1}{R}\right) + \mathcal{O}\left(\frac{\left|\bar{k}^N\right|\left(\left|\bar{k}^N\right|^2+1\right)}{R^2}\right) \\
&= 4\pi R - 2\pi^2 \sqrt{\left(\frac{m(m+2)}{(m+1)^2}\right)\sum_{i=1}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{1\leq i<j\leq N-1} k_i\cdot k_j + \mu} \\
&\qquad + \mathcal{O}\left(\frac{\left|\bar{k}^N\right|^2+1}{R}\right) + \mathcal{O}\left(\frac{\left|\bar{k}^N\right|\left(\left|\bar{k}^N\right|^2+1\right)}{R^2}\right),
\end{aligned} \quad (4.23)$$

Therefore we define

$$\begin{aligned}
L^R(\bar{k}^N) &= 4\pi R - \int_{B_R(0)} \mathrm{d}k_N \hat{G}(k) \\
&= 2\pi^2 \left(\frac{m(m+2)}{(m+1)^2} \sum_{i=1}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{1\leq i<j\leq N-1} k_i\cdot k_j + \mu \right)^{1/2} \\
&\qquad + \mathcal{O}\left(\frac{\left|\bar{k}^N\right|^2+1}{R}\right) + \mathcal{O}\left(\frac{\left|\bar{k}^N\right|\left(\left|\bar{k}^N\right|^2+1\right)}{R^2}\right)
\end{aligned} \quad (4.24)$$

Remark: It follows from the definition of L^R that $L^R(\bar{k}^N) < 4\pi R$ for all $R > 0$. Furthermore, it also follows that $L^R(\bar{k}^N) \leq C\sqrt{\left|\bar{k}^N\right|^2+1}$ for some $C > 0$, which can be seen by the fact that $L^R(\bar{k}^N) = \mathbb{1}_{B_R(0)}(\bar{k}^N)L^R(\bar{k}^N) + \mathbb{1}_{B_R(0)^\complement}(\bar{k}^N)L^R(\bar{k}^N)$. From (4.24) we have $\mathbb{1}_{B_R(0)}(\bar{k}^N)L^R(\bar{k}^N) \leq C_1\mathbb{1}_{B_R(0)}\sqrt{\left|\bar{k}^N\right|^2+1}$, and since $L^R(\bar{k}^N) < 4\pi R$ we have $\mathbb{1}_{B_R(0)^\complement}(\bar{k}^N)L^R(\bar{k}^N) \leq 4\pi R\mathbb{1}_{B_R(0)^\complement} \leq$

$C_2 \mathbb{1}_{B_R(0)^\complement} \sqrt{\left|\bar{k}^N\right|^2 + 1}$.
We can actually get a closed form for L^R

Lemma 4. *Let $\hat{G}$ be defined as above and let $q = k_N$, $y = \frac{2}{m+1}\sum_{i=1}^{N-1} k_i$, and $\alpha = \sum_{i=1}^{N-1} k_i^2 + \frac{2}{m+1}\sum_{1\le i<j\le N-1} k_i \cdot k_j + \mu$, then*

$$\begin{aligned} L^R(\bar{k}^N) :=& 4\pi R - \int_{B_R(0)} \mathrm{d}k_N \hat{G}(k) \\ =& 4\pi R - \int_{B_R(0)} \mathrm{d}q \frac{1}{q^2 + q\cdot y + \alpha} \\ =& 2\pi \left[R - \frac{(2R^2 - y^2 + 2\alpha)}{4\,|y|} \ln\left(\frac{R(R+|y|)+\alpha}{R(R-|y|)+\alpha}\right) \right. \\ & \left. + \frac{1}{2}\sqrt{4\alpha - y^2}\left(\arctan\left(\frac{2R-|y|}{\sqrt{4\alpha-y^2}}\right) + \arctan\left(\frac{2R+|y|}{\sqrt{4\alpha-y^2}}\right)\right)\right] \end{aligned} \tag{4.25}$$

Proof. This follows from straighforward calculation (most easily done in Mathematica)

$$\begin{aligned} &\int_{B_R(0)} \mathrm{d}q \frac{1}{q^2 + q\cdot y + \alpha} = 2\pi \int_{-1}^{1} \mathrm{d}\cos(\theta) \int_0^\infty \mathrm{d}r r^2 \frac{1}{r^2 + \cos(\theta) r\,|y| + \alpha} \\ &= 2\pi \int_{-1}^{1} \mathrm{d}\cos(\theta) \left[\frac{1}{2}|y|\cos(\theta)\left(\ln(\alpha) - \ln\left(\alpha + R\,|y|\cos(\theta) + R^2\right)\right) \right. \\ &\quad \left. + \frac{\left(|y|^2\cos^2(\theta) - 2\alpha\right)\left(\arctan\left(\frac{|y|\cos(\theta)+2R}{\sqrt{4\alpha-|y|^2\cos^2(\theta)}}\right) - \arctan\left(\frac{|y|\cos(\theta)}{\sqrt{4\alpha-|y|^2\cos^2(\theta)}}\right)\right)}{\sqrt{4\alpha - |y|^2\cos^2(\theta)}} + R \right] \\ &= 2\pi \left[R + \frac{(2R^2 - y^2 + 2\alpha)}{4\,|y|} \ln\left(\frac{R(R+|y|)+\alpha}{R(R-|y|)+\alpha}\right) \right. \\ &\quad \left. - \frac{1}{2}\sqrt{4\alpha - y^2}\left(\arctan\left(\frac{2R-|y|}{\sqrt{4\alpha-y^2}}\right) + \arctan\left(\frac{2R+|y|}{\sqrt{4\alpha-y^2}}\right)\right)\right]. \end{aligned} \tag{4.26}$$

Thus the desired result follows. □

immediate consequence of this is that $L^R \ge 0$

Lemma 5. *Let L^R be defined as above. Then $L^R \ge 0$ for all $R \ge 0$.*

Proof 1. This follows from completing the square

$$q^2 + y\cdot q + \alpha = (q + y/2)^2 + \alpha - y^2/4. \tag{4.27}$$

Noting that $\alpha - y^2/4 > 0$ and the simple inequality for symmetric-decreasing rearrangements

$$\int fg \le \int f^* g^*, \tag{4.28}$$

where f^* denotes the symmetric-decreasing rearrangement of f ([6] Section 3.3 and 3.4). To see this notice that if f is itself symmetric and decreasing, then $f^* = f$, on the other hand if we define $f_t(x) := f(x-t)$ then $f_t^* = f^* = f$. Since $\mathbb{1}_{B_R(0)}$ is symmetric and decreasing and $\frac{1}{q^2+\alpha-y^2/4}$ also is symmetric-decreasing it follows that

$$\int_{B_R(0)} \mathrm{d}q \frac{1}{q^2+q\cdot y+\alpha} = \int_{B_R(0)} \mathrm{d}q \frac{1}{(q+y/2)^2+\alpha-y^2/4} \leq \int_{B_R(0)} \mathrm{d}q \frac{1}{q^2+\alpha-y^2/4} \leq 4\pi R. \tag{4.29}$$

□

Proof 2. We notice first that $L^0(\bar{k}^N) = 0$ furthermore,

$$L^R(\bar{k}^N) \to 2\pi^2 \left(\frac{m(m+2)}{(m+1)^2} \sum_{i=1}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{1\leq i<j\leq N-1} k_i \cdot k_j + \mu \right)^{1/2}, \tag{4.30}$$

pointwise as $R \to \infty$. Hence, for any $\bar{k}^N \in \mathbb{R}^{3(N-1)}$ we can find $\tilde{R} \geq 0$ such that $L^R(\bar{k}^N) \geq 0$ for all $R > \tilde{R}$. Now let us analyse L^R at any finite R. Since

$$\sqrt{4\alpha - y^2} \left(\arctan\left(\frac{2R - |y|}{\sqrt{4\alpha - y^2}} \right) + \arctan\left(\frac{2R + |y|}{\sqrt{4\alpha - y^2}} \right) \right) \geq 0, \tag{4.31}$$

we see that

$$L^R(\bar{k}^N) \geq 2\pi \left(R - \frac{(2R^2 - |y|^2 + 2\alpha)}{4\,|y|} \ln\left(\frac{R(R+|y|)+\alpha}{R(R-|y|)+\alpha} \right) \right) := f(R, y, \alpha). \tag{4.32}$$

We now show that $f \geq 0$ whenever $4\alpha \geq y$. We start by showing that $f \geq 0$ for $y^2 < 2\alpha$: To see this, we compute

$$\partial_R f(R,y,\alpha) = \frac{4R^4 + R^2\left(4\alpha - 3\,|y|^2\right) + \alpha\,|y|^2}{2\,(\alpha + R^2 - R\,|y|)\,(\alpha + R(R+|y|))} - \frac{R \ln\left(\frac{R(R+|y|)+\alpha}{R(R-|y|)+\alpha}\right)}{|y|} \tag{4.33}$$

Let R^* denote any extremum of $f(R,y,\alpha)$, *i.e.* $\partial_R f(R^*,y,\alpha) = 0$, then we clearly have

$$\ln\left(\frac{R^*(R^*+|y|)+\alpha}{R^*(R^*-|y|)+\alpha} \right) = \frac{|y|}{R^*} \frac{4(R^*)^4 + (R^*)^2\left(4\alpha - 3\,|y|^2\right) + \alpha\,|y|^2}{2\,(\alpha + (R^*)^2 - R^*\,|y|)\,(\alpha + R^*(R^*+|y|))} \tag{4.34}$$

Thereby we find

$$f(R^*,y,\alpha) = \frac{|y|^2\left((R^*)^2\left(8\alpha - 3\,|y|^2\right) + \alpha\left(|y|^2 - 2\alpha\right) + 2(R^*)^4 \right)}{8R^*\,(-R^*\,|y| + (R^*)^2 + \alpha)\,(R^*\,|y| + (R^*)^2 + z)}. \tag{4.35}$$

However, $\frac{|y|^2}{8R^*(-R^*|y|+(R^*)^2+\alpha)(R^*|y|+(R^*)^2+\alpha)} > 0$. So $f(R^*,y,\alpha) < 0$ if and only if $h(R^*,y,\alpha) := \left((R^*)^2\left(8\alpha - 3\,|y|^2\right) + \alpha\left(|y|^2 - 2\alpha\right) + 2(R^*)^4 \right) < 0$. We show that $f > 0$ for $y^2 > 2\alpha$. Notice that

$$h(R,y,\alpha) \geq 0 \tag{4.36}$$

for $2\alpha \leq |y|^2 \leq 8\alpha/3$. Furthermore, h takes the minimal value at $(R^*)^2 = (8\alpha - 3|y|^2)/4$ for $y^2 > 8\alpha/3$. Reinserting this into h we find

$$h\left(\sqrt{(8\alpha - 3|y|^2)/4}, y, \alpha\right) = -10\alpha^2 - \frac{9y^4}{8} + 7\alpha y^2 \tag{4.37}$$

which is a concave downward parabola in y^2 with discriminant

$$d = (7\alpha)^2 - 4 \cdot (9/8) \cdot 10\alpha^2 = (2\alpha)^2 \tag{4.38}$$

thus we find the $h\left(\sqrt{(8\alpha - 3|y|^2)/4}, y, \alpha\right)$ is zero at

$$y^2 = \frac{8(7\alpha \mp 2\alpha)}{18} = \begin{cases} 4\alpha \\ 20/9\alpha \end{cases} \tag{4.39}$$

thus in-between these two points, it must be positive, and since $8/3 > 20/9$, we conclude that $f(R^*, y, \alpha) > 0$ for $y^2 > 2\alpha$.
We know generalize this to $y \leq 4\alpha$: To see this we notice that $f(R, 0_+, \alpha) = 0$. On the other hand since $y^2 < 4\alpha$ we also know that $f(R, y, \alpha) > 0$ for $|y| \geq 2\alpha$. Now assume that there exist some $y > 0$ with $y \leq 4\alpha$ such that $f(R, y, \alpha) < 0$, this would imply that f has a minimum for some $4\alpha \geq y^* > 0$. This y^* would satisfy

$$\partial_y f(R, y, \alpha)|_{y=y^*} = \frac{\left(2\alpha + 2R^2 + (y^*)^2\right) \ln\ln\left(\frac{R(R+|y^*|)+\alpha}{R(R-|y^*|)+\alpha}\right) - \frac{2Ry^*\left(\alpha+R^2\right)\left(2\alpha+2R^2-(y^*)^2\right)}{(\alpha+R^2-Ry^*)(\alpha+R(R+y^*))}}{4(y^*)^2} = 0, \tag{4.40}$$

from which we conclude that

$$\ln\left(\frac{R(R+|y^*|)+\alpha}{R(R-|y^*|)+\alpha}\right) = \frac{2R|y^*|\left(\alpha + R^2\right)\left(2\alpha + 2R^2 - (y^*)^2\right)}{\left(2\alpha + 2R^2 + (y^*)^2\right)\left(\alpha + R^2 - R|y^*|\right)\left(\alpha + R(R + |y^*|)\right)}. \tag{4.41}$$

Inserting this back into $f(R, y^*, \alpha)$ we find that

$$f(R, y^*, \alpha) = R - \frac{R\left(\alpha + R^2\right)\left(-2\alpha - 2R^2 + (y^*)^2\right)^2}{4\left(\alpha + R^2 - R|y^*|\right)\left(\alpha + R(R + |y^*|)\right)\left(\alpha + R^2 + (y^*)^2/2\right)}. \tag{4.42}$$

It then follows from

$$\frac{1}{4}\left(-2\alpha - 2R^2 + (y^*)^2\right)^2 = (R^2 + \alpha)^2 + (y^*/2)^2 - (R^2 + \alpha)(y^*)^2 \leq (R^2 + \alpha)^2 - R^2(y^*)^2, \tag{4.43}$$

where we used that $(y^*)^2/4 \leq \alpha$, and from

$$\left(\alpha + R^2 - Ry^*\right)\left(\alpha + R(R + y^*)\right) = (R + \alpha)^2 - R^2(y^*)^2 \tag{4.44}$$

that

$$f(R, y^*, \alpha) > 0 \tag{4.45}$$

which contradict the assumption that there exist some y for which $f(R, y, \alpha) < 0$. Thus we conclude that $f(R, y, \alpha) \geq 0$ for all $y \in \mathbb{R}^{3(N-1)}$. □

From (4.10) we conclude that

$$\begin{aligned}\int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k) \left|\hat{\rho}^R(k)\right|^2 =& N 4\pi R \left\|\xi\right\|_{L^2(\mathbb{R}3(N-1))} - N \int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N L^R(\bar{k}^N) \left|\xi^R(\bar{k}^N)\right|^2 \\ & - N(N-1) \int_{\mathbb{R}^{3(N-2)}} \mathrm{d}\bar{q} \int_{B_R(0)} \mathrm{d}s \int_{B_R(0)} \mathrm{d}t \overline{\xi^R(s,\bar{q})} \hat{G}(s,t,\bar{q}) \xi^R(t,\bar{q}) \\ & = -N \left(T^R_{\text{diag}}(\xi^R) + T_{\text{off}}(\xi^R) - 4\pi R \left\|\xi\right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \right),\end{aligned} \tag{4.46}$$

where we in the last line defined

$$\begin{aligned} T^R_{\text{diag}}(\xi^R) &= \int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N L^R(\bar{k}^N) \left|\xi^R(\bar{k}^N)\right|^2, \\ T^R_{\text{off}}(\xi^R) &= (N-1) \int_{\mathbb{R}^{3(N-2)}} \mathrm{d}\bar{q} \int_{B_R(0)} \mathrm{d}s \int_{B_R(0)} \mathrm{d}t \overline{\xi^R(s,\bar{q})} \hat{G}(s,t,\bar{q}) \xi^R(t,\bar{q}). \end{aligned} \tag{4.47}$$

Thereby, using (4.9), we can write down the quadratic form

$$\begin{aligned} F^R_\gamma(u) = \int_{\mathbb{R}^{3N}} & \mathrm{d}k \hat{G}(k)^{-1} |\hat{u}(k) - \widehat{\rho^R G}(k)|^2 - \mu \left\|u\right\|^2_{L^2(\mathbb{R}^{3N})} \\ & + N \left(T^R_{\text{diag}}(\xi^R) + T^R_{\text{off}}(\xi^R) + (\gamma_R^{-1} - 4\pi R) \left\|\xi^R\right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \right). \end{aligned} \tag{4.48}$$

It is worth to mention that the sign of T_{off} was crucially dependent on the antisymmetry of u, and that for a bosonic system, T_{off} comes with opposite sign. Similarly to the $1+1$ case we see a singular behaviour for $R \to \infty$ unless we choose γ_R exactly to cancel the divergence. Thus we choose $\gamma_R = (4\pi R + \alpha)^{-1}$ and we end up with the quadratic form (slightly abusing notation by denoting the renormalized quadratic form by the same name as the non-renormalized one)

$$\begin{aligned} F^R_\alpha(u) = \int_{\mathbb{R}^{3N}} & \mathrm{d}k \hat{G}(k)^{-1} |\hat{u}(k) - \widehat{\rho^R G}(k)|^2 - \mu \left\|u\right\|^2_{L^2(\mathbb{R}^{3N})} \\ & + N \left(T^R_{\text{diag}}(\xi^R) + T^R_{\text{off}}(\xi^R) + \alpha \left\|\xi^R\right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \right). \end{aligned} \tag{4.49}$$

The domain of the quadratic form can also be determined, by requiring all terms to be well-defined as well as imposing the anti-symmetry of fermions. We see from the first term that we clearly need $u \in H^1(\mathbb{R}^{3N})$, however, from T_{diag} we see that we furthermore need $\xi_R = \gamma_R \int_{B_R(0)} \mathrm{d}k_N \hat{u}(k) \in H^{1/2}\left(\mathbb{R}^{3(N-1)}\right)$. Hence we have the domain

$$\mathscr{D}(F^R_\alpha) = \left\{ u \in H^1_{\text{as}}(\mathbb{R}^{3N}) \;\middle|\; \left(\int_{B_R(0)} \mathrm{d}k_N \hat{u}(k) \right)^\vee \in H^{1/2}_{\text{as}}\left(\mathbb{R}^{3(N-1)}\right) \right\}. \tag{4.50}$$

Now the following lemma will help us further simplifying this domain.

Lemma 6. *Let $w \in H^1(\mathbb{R}^3)$, then* $\left(\int_{B_R(0)} \mathrm{d}k_N \hat{w}(k)\right)^\vee \in H^1(\mathbb{R}^{3(N-1)})$.

Proof. Notice first that since $\left|\bar{k}^N\right|^2 + 1 \leq |k|^2 + 1$, we have that $\int \mathrm{d}\bar{k}^N \left(\left|\bar{k}^N\right|^2 + 1\right) |\hat{w}(k)|^2 \in L^1(\mathbb{R}^3)$ (by Tornelli), and thus $\left[\int \mathrm{d}\bar{k}^N \left(\left|\bar{k}^N\right|^2 + 1\right) |\hat{w}(k)|^2\right]^{1/2} \in L^2(\mathbb{R}^3)$. Now to prove the claim we need to show

$$\int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N \left(\left|\bar{k}^N\right|^2 + 1\right) \left|\int_{B_R(0)} \mathrm{d}k_N \hat{w}(k)\right|^2 < \infty. \tag{4.51}$$

By Minkowski's integral inequality we have

$$\int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N \left(\left|\bar{k}^N\right|^2 + 1\right) \left|\int_{B_R(0)} \mathrm{d}k_N \hat{w}(k)\right|^2 \leq \left(\int_{B_R(0)} \mathrm{d}k_N \left[\int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N \left(\left|\bar{k}^N\right|^2 + 1\right) |\hat{w}(k)|^2\right]^{1/2}\right)^2, \tag{4.52}$$

By Cauchy-Schwartz we then have

$$\begin{aligned} &\int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N \left(\left|\bar{k}^N\right|^2 + 1\right) \left|\int_{B_R(0)} \mathrm{d}k_N \hat{w}(k)\right|^2 \\ &\quad \leq \left\|\mathbb{1}_{B_R(0)}\right\|_2^2 \left\|\left[\int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N \left(\left|\bar{k}^N\right|^2 + 1\right) |\hat{w}(k)|^2\right]^{1/2}\right\|_2^2 < \infty. \end{aligned} \tag{4.53}$$

□

By lemma 6, we may conclude that

$$\mathscr{D}(F_\alpha^R) = \left\{u \in H^1_{\text{as}}(\mathbb{R}^{3N}) \;\middle|\; \left(\int_{B_R(0)} \mathrm{d}k_N \hat{u}(k)\right)^\vee \in H^{1/2}_{\text{as}}\left(\mathbb{R}^{3(N-1)}\right)\right\} = H^1_{\text{as}}(\mathbb{R}^{3N}), \tag{4.54}$$

and the domain is independent of R. We can also take the naïve $R \to \infty$ limit to obtain a candidate for a possible Γ-limit. We find the quadratic form

$$\begin{aligned} F_\alpha(u) &= \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k)^{-1} |\hat{u}(k) - \widehat{\rho G}(k)|^2 - \mu \|u\|_{L^2(\mathbb{R}^{3N})} \\ &\qquad + N\left(T_{\text{diag}}(\xi) + T_{\text{off}}(\xi) + \alpha \|\xi\|^2_{L^2(\mathbb{R}^{3(N-1)})}\right) \\ &= \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k)^{-1} |\hat{w}|^2 - \mu \|u\|_{L^2(\mathbb{R}^{3N})} + N\left(T_{\text{diag}}(\xi) + T_{\text{off}}(\xi) + \alpha \|\xi\|^2_{L^2(\mathbb{R}^{3(N-1)})}\right), \end{aligned} \tag{4.55}$$

where we have defined $\widehat{\rho G} = \sum_{i=1}^N (-1)^{i-1} \hat{G}(k) \xi(\bar{k}^i)$, and

$$\begin{aligned} T_{\text{diag}}(\xi) &= \int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N L(\bar{k}^N) \left|\xi(\bar{k}^N)\right|^2, \\ T_{\text{off}}(\xi) &= (N-1) \int_{\mathbb{R}^{3(N-2)}} \mathrm{d}\bar{q} \int_{\mathbb{R}^3} \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}t \overline{\xi(s, \bar{q})} \hat{G}(s, t, \bar{q}) \xi(t, \bar{q}). \end{aligned} \tag{4.56}$$

with

$$L(\bar{k}^N) = 2\pi^2 \left(\frac{m(m+2)}{(m+1)^2} \sum_{i=1}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{1\le i<j\le N-1} k_i \cdot k_j + \mu \right)^{1/2}. \tag{4.57}$$

The domain is

$$\mathscr{D}(F_\alpha) = \left\{ u \in L^2_{\mathrm{as}}(\mathbb{R}^{3N}) \;\middle|\; \hat{u} = \hat{w} + \widehat{\rho G},\ w \in H^1_{\mathrm{as}}(\mathbb{R}^{3N}),\ \xi \in H^{1/2}_{\mathrm{as}}(\mathbb{R}^{3(N-1)}) \right\}. \tag{4.58}$$

This exactly match the quadratic forms used in [3, 8]. Notice that the quadratic forms obtained in the $N+1$ case are direct generalizations of the $1+1$ case, *i.e.* we obtain the $1+1$ case by setting $N=1$ as we should. We note that $(\widehat{\rho G})^\vee \notin H^1(\mathbb{R}^{3N})$ (where $f^\vee$ is inverse Fourier transform of f) for $\xi \neq 0$, so the decomposition $\hat{u} = \hat{w} + \widehat{\rho G}$ is unique. Interestingly, we see as in the $1+1$ case that the domain changes in the $R \to \infty$ limit, even though it is the same for all finite R.

We now give some generalization of results from the $1+1$ case.

Lemma 7. *Let $w \in H^1(\mathbb{R}^{3N})$, then*

$$\frac{1}{\sqrt{n}} \int_{B_n(0)} \mathrm{d}k_N \hat{w}(k) \to 0, \text{ a.e.,} \quad \text{as } n \to \infty, \tag{4.59}$$

where $\hat{w}$ denotes the Fourier transform of w. More precisely

$$\frac{1}{\sqrt{n}} \left| \int_{B_n(0)} \mathrm{d}k_N \hat{w}(k) \right| = \epsilon(n)\sqrt{n} \left(g(\bar{k}^N) + g(\bar{k}^N)^{4/3} \right), \tag{4.60}$$

for some ϵ-function, $\epsilon(n) \to 0$ as $n \to \infty$, and some $g \in L^2(\mathbb{R}^{3(N-1)})$. Furthermore, we have

$$\frac{1}{\sqrt{n}} \left| \int_{B_n(0)} \mathrm{d}k_N \hat{w}(k) \right| \le g(\bar{k}^N). \tag{4.61}$$

In conclusion we have

$$\left\| \frac{1}{\sqrt{n}} \int_{B_n(0)} \mathrm{d}k_N \hat{w}(k) \right\|_2 \to 0, \text{ as } n \to \infty. \tag{4.62}$$

Proof. The proof of the first two statements is a straightforward generalization of the proof of lemma 2 noticing that $g(\bar{k}^N) = \|f(\bar{k}^N, \cdot)\|_{L_2(\mathbb{R}^3)} \in L^2(\mathbb{R}^{3(N-1)})$, for $f \in L^2(\mathbb{R}^{3N})$, by Tornelli's theorem. The last two statements simply follow by Cauchy-Schwartz and DCT. □

We also have the following lemma

Lemma 8. *Let $w \in H^1(\mathbb{R}^{3N})$, then*

$$\left\| \frac{1}{n} \int_{B_n(0)} \mathrm{d}k_N \hat{w}(k) \right\|_{H^{1/2}(\mathbb{R}^{3(N-1)})} \to 0, \quad \textit{for } n \to \infty. \tag{4.63}$$

Proof. Let $\hat{w}(k) = f(k)/(\sqrt{|k|^2+1})$, and let $g = \|f(\bar{k}^N, \cdot)\|_{L_2(\mathbb{R}^3)} \in L^2(\mathbb{R}^{3(N-1)})$. Then by

Cauchy-Schwartz we have

$$\frac{1}{n}\left|\int_{B_n(0)} \mathrm{d}k_N \hat{w}(k)\right| \le \frac{C}{\sqrt{n}}\sqrt{1-\frac{\sqrt{\left|\bar{k}^N\right|^2+1}}{n}\arctan\left(\frac{n}{\sqrt{\left|\bar{k}^N\right|^2+1}}\right)}g(\bar{k}^N). \tag{4.64}$$

Thus we may compute

$$\begin{aligned}
&\left\|\frac{1}{n}\int_{B_n(0)} \mathrm{d}k_N \hat{w}(k)\right\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})} \\
&\le \frac{C}{n}\int \mathrm{d}\bar{k}^N \sqrt{\left|\bar{k}^N\right|^2+1}\left(1-\frac{\sqrt{\left|\bar{k}^N\right|^2+1}}{n}\arctan\left(\frac{n}{\sqrt{\left|\bar{k}^N\right|^2+1}}\right)\right)\left|g(\bar{k}^N)\right|^2 \\
&\le \frac{C}{n}\int_{\left|\bar{k}^N\right|<n} \mathrm{d}\bar{k}^N \sqrt{\left|\bar{k}^N\right|^2+1}\left|g(\bar{k}^N)\right|^2 + \frac{C}{n}\int_{\left|\bar{k}^N\right|\ge n} \mathrm{d}\bar{k}^N \sqrt{\left|\bar{k}^N\right|^2+1}\frac{n^2}{(\left|\bar{k}^N\right|^2+1)}\left|g(\bar{k}^N)\right|^2.
\end{aligned} \tag{4.65}$$

Now notice that we have

$$\mathbb{1}_{B_n(0)}(\bar{k}^N)\frac{1}{n}\sqrt{\left|\bar{k}^N\right|^2+1}\left|g(\bar{k}^N)\right|^2 \le \sqrt{2}\left|g(\bar{k}^N)\right|^2 \tag{4.66}$$

and

$$\mathbb{1}_{B_n(0)}(\bar{k}^N)\frac{1}{n}\sqrt{\left|\bar{k}^N\right|^2+1}\left|g(\bar{k}^N)\right|^2 \to 0, \text{ a.e. for } n\to\infty. \tag{4.67}$$

We also have

$$\mathbb{1}_{(B_n(0))^\complement}(\bar{k}^N)\frac{n}{\sqrt{\left|\bar{k}^N\right|^2+1}}\left|g(\bar{k}^N)\right|^2 \le \left|g(\bar{k}^N)\right|^2 \tag{4.68}$$

and

$$\mathbb{1}_{(B_n(0))^\complement}(\bar{k}^N)\frac{n}{\sqrt{\left|\bar{k}^N\right|^2+1}}\left|g(\bar{k}^N)\right|^2 \to 0, \text{ a.e. for } n\to\infty. \tag{4.69}$$

Thus by DCT we may conclude that

$$\left\|\frac{1}{n}\int_{B_n(0)} \mathrm{d}k_N \hat{w}(k)\right\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})} \to 0, \quad \text{as } n\to\infty. \tag{4.70}$$

□

We also give the following lemmas, which will prove useful later

Lemma 9. *Let $N\ge 2$ and $\xi\in\mathcal{H}$, for $\mathcal{H}=L^2(\mathbb{R}^{3(N-1)})$ or $\mathcal{H}=H^{1/2}(\mathbb{R}^{3(N-1)})$, then we have the following bound*

$$\left\|\frac{1}{n}\int_{B_n(0)} \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\xi(\bar{k}^i)\right\|_{\mathcal{H}} \le C\left\|\xi\right\|_{\mathcal{H}}, \tag{4.71}$$

where $C>0$ depends only on N, m, and μ, for all $1\le i\le N$.

Proof. For $i = N$ this follows from the estimate

$$\hat{G}(k) \leq \frac{1}{\frac{m(m+N)}{(m+1)(m+N-1)} |k_N|^2 + \mu}, \tag{4.72}$$

which can be seen by a simply optimization. We minimize $\hat{G}(k)^{-1}$

$$\partial_{k_\alpha} \left(\sum_{i=1}^{N} k_i^2 + \frac{2}{m+1} \sum_{1 \leq i < j \leq N} k_i \cdot k_j \right) = 2k_\alpha + \frac{1}{m+1} \sum_{i \neq \alpha} k_i = 0, \text{ for all } \alpha \neq N. \tag{4.73}$$

Thus

$$\sum_{i=1}^{N} k_i^2 + \frac{2}{m+1} \sum_{1 \leq i < j \leq N} k_i \cdot k_j = \sum_{i=1}^{N} k_i \cdot \left(k_i + \frac{1}{m+1} \sum_{j \neq i} k_j \right) = k_N^2 + k_N \cdot \sum_{i=1}^{N-1} k_i. \tag{4.74}$$

Now by (4.73) we have

$$0 = \sum_{\alpha=1}^{N-1} \left(k_\alpha + \frac{1}{m+1} \sum_{i \neq \alpha} k_i \right) = \sum_{i=1}^{N-1} k_i + \frac{1}{m+1} \left((N-2) \sum_{i=1}^{N-1} k_i + (N-1) k_N \right) \tag{4.75}$$

from which we get

$$\sum_{i=1}^{N-1} k_i = -\frac{N-1}{N+m-1} k_N. \tag{4.76}$$

Inserting this into (4.74) we get

$$\sum_{i=1}^{N} k_i^2 + \frac{2}{m+1} \sum_{1 \leq i < j \leq N} k_i \cdot k_j = \frac{m(m+N)}{(m+1)(m+N-1)} |k_N|^2. \tag{4.77}$$

Thereby, we find

$$\left\| \frac{1}{n} \int_{B_n(0)} \mathrm{d}k_N \hat{G}(k) \xi(\bar{k}^N) \right\|_{\mathcal{H}} \lesssim \frac{1}{n} nC \|\xi\|_{\mathcal{H}} = C \|\xi\|_{\mathcal{H}}, \tag{4.78}$$

for some $C > 0$, where $\mathcal{H} = L^2(\mathbb{R}^{3(N-1)})$ or $\mathcal{H} = H^{1/2}(\mathbb{R}^{3(N-1)})$.
Now for $i \neq N$ we use the Cauchy-Schwartz inequality

$$\begin{aligned} \left\| \frac{1}{n} \int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i) \hat{G}(k) \xi(\bar{k}^i) \right\|_{\mathcal{H}}^2 \leq \frac{1}{n^2} \int \mathrm{d}\bar{k}^N \mathbb{1}_{B_n(0)}(k_i) g_{\mathcal{H}}^{-1}(\bar{k}^N) \left(\int \mathrm{d}k_N g_{\mathcal{H}}(\bar{k}^i) \left| \hat{G}(k) \right|^2 \right) \\ \times \left(\int \mathrm{d}k_N g_{\mathcal{H}}^{-1}(\bar{k}^i) \left| \xi(\bar{k}^i) \right|^2 \right), \end{aligned} \tag{4.79}$$

where

$$g_{\mathcal{H}}(\bar{k}^i) = \begin{cases} 1 & \text{for } \mathcal{H} = L^2(\mathbb{R}^{3(N-1)}), \\ 1/\sqrt{|\bar{k}^i|^2 + 1} & \text{for } \mathcal{H} = H^{1/2}(\mathbb{R}^{3(N-1)}). \end{cases} \tag{4.80}$$

To estimate this we use that

$$\hat{G}(k) \leq \frac{1}{Ak_i^2 + Ak_N^2 + B\left(2k_i \cdot k_N\right) + \mu} \tag{4.81}$$

where $A = 1 - \frac{N-2}{(m+1)(m+N-2)}$ and $B = \frac{1}{m+1} - \frac{N-2}{(m+1)(m+N-2)}$, so $0 < B < A < 1$. Similar to above this can be seen by optimizing $\hat{G}(k)^{-1}$ w.r.t k_α, for all $\alpha \neq i, N$:

$$0 = \partial_{k_\alpha} \hat{G}(k^*)^{-1} = 2k_\alpha^* + \frac{2}{m+1}\left(k_i + k_N + \sum_{l \neq \alpha, i, N} k_l^*\right). \tag{4.82}$$

Furthermore we have

$$\hat{G}(k)^{-1} = k_i^2 + k_N^2 + \frac{2}{m+1} k_i \cdot k_N + \sum_{\alpha \neq i, N} k_\alpha \cdot \left(k_\alpha + \frac{2}{m+1}\left(k_i + k_N + \frac{1}{2}\sum_{l \neq \alpha, i, N} k_l\right)\right) + \mu. \tag{4.83}$$

Using (4.82) we have

$$\hat{G}(k)^{-1} \geq k_i^2 + k_N^2 + \frac{2}{m+1} k_i \cdot k_N + \sum_{\alpha \neq i, N} k_\alpha^* \cdot \left(\frac{1}{m+1}\left(k_i + k_N\right)\right) + \mu. \tag{4.84}$$

Also using (4.82) we find

$$0 = \sum_{\alpha \neq i, N}\left(k_\alpha^* + \frac{1}{m+1}\left(k_i + k_N + \sum_{l \neq \alpha, i, N} k_l^*\right)\right) = \sum_{\alpha \neq i, N} k_\alpha^* + \frac{N-2}{m+1}(k_i + k_N) + \frac{N-3}{m+1}\sum_{\alpha \neq i, N} k_\alpha^*, \tag{4.85}$$

from which it follows that

$$\sum_{\alpha \neq i, N} k_\alpha^* = -\frac{N-2}{m+N-2}(k_i + k_N). \tag{4.86}$$

Inserting in (4.84) we find

$$\hat{G}(k)^{-1} \geq k_i^2 + k_N^2 + \frac{2}{m+1} k_i \cdot k_N - \left(\frac{N-2}{(m+1)(m+N-2)}\left(k_i + k_N\right)^2\right) + \mu \tag{4.87}$$

from which (4.81) follows. Using this in (4.79) together with the fact that for $\mathcal{H} = H^{1/2}(\mathbb{R}^{3(N-1)})$ we have

$$\mathbb{1}_{B_n(0)}(k_i) \frac{g_{\mathcal{H}}(\bar{k}^i)}{g_{\mathcal{H}}(\bar{k}^N)} = \mathbb{1}_{B_n(0)}(k_i) \frac{\sqrt{k_i^2 + K}}{\sqrt{k_N^2 + K}} \leq \mathbb{1}_{B_n(0)}(k_i) \frac{\sqrt{n^2 + K}}{\sqrt{k_N^2 + K}} \leq \mathbb{1}_{B_n(0)}(k_i)\left(1 + \frac{n}{|k_N|}\right), \tag{4.88}$$

and for $\mathcal{H} = L^2(\mathbb{R}^{3(N-1)})$ we have

$$\mathbb{1}_{B_n(0)}(k_i) \frac{g_{\mathcal{H}}(\bar{k}^i)}{g_{\mathcal{H}}(\bar{k}^N)} = \mathbb{1}_{B_n(0)}(k_i) \leq \mathbb{1}_{B_n(0)}(k_i)\left(1 + \frac{n}{|k_N|}\right), \tag{4.89}$$

we find that

$$\begin{aligned}
&\left(\int \mathrm{d}k_i\,\mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i)\left(1+\frac{n}{|k_N|}\right)\left|\hat{G}(k)\right|^2\right)\\
&\leq \int \mathrm{d}k_i\,\mathrm{d}k_N\left(1+\frac{n}{|k_N|}\right)\frac{\mathbb{1}_{B_n(0)}(k_i)}{\left(Ak_i^2+Ak_N^2+B\left(2k_i\cdot k_N\right)+\mu\right)^2}\\
&\leq C\int_{(0,n)}\mathrm{d}r\; r^2\int_{(0,\infty)}\mathrm{d}q\; q^2\left(1+\frac{n}{q}\right)\frac{1}{\left(Aq^2+Ar^2-2Bqr+\mu\right)^2}\\
&= C\int_{(0,n)}\mathrm{d}r\; r^2\int_{(0,\infty)}\mathrm{d}\tilde{q}\;\tilde{q}^2r^3\left(1+\frac{n}{\tilde{q}r}\right)\frac{1}{\left(A\tilde{q}^2r^2+Ar^2-2B\tilde{q}r^2+\mu\right)^2}\\
&\leq C'\int_{(0,n)}\mathrm{d}r\; r^2\left(\frac{1}{r}+\frac{n}{r^2}\right)\\
&\leq \tilde{C}n^2
\end{aligned} \tag{4.90}$$

where we defined $\tilde{q}=q/r$, and $C,\ C'$ and $\tilde{C}$ are positive constants depending only on $A,\ B$ and μ. On the other hand we also have

$$\int \mathrm{d}\bar{k}^{i,N}\left(\int \mathrm{d}k_N g_{\mathcal{H}}^{-1}(\bar{k}^i)\left|\xi(\bar{k}^i)\right|^2\right)=\|\xi\|_{\mathcal{H}}^2. \tag{4.91}$$

Therefore we conclude that

$$\left\|\frac{1}{n}\int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\xi(\bar{k}^i)\right\|_{\mathcal{H}}^2\leq \tilde{C}\,\|\xi\|_{\mathcal{H}}^2. \tag{4.92}$$

This concludes the proof. $\square$

In the case of $\xi\in H^{1/2}(\mathbb{R}^{3(N-1)})$ we can also show the result

Lemma 10. *Let $N\geq 2$ and $\xi\in H^{1/2}(\mathbb{R}^{3(N-1)})$, then we have the following limits*

$$\left\|\frac{1}{n}\int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\xi(\bar{k}^i)\right\|_{L^2(\mathbb{R}^{3(N-1)})}^2=\mathcal{O}\left(\frac{1}{n}\right)\to 0,\ \textit{for } i\neq N \textit{ and } n\to\infty, \tag{4.93}$$

and

$$\left\|\xi(\bar{k}^i)-\frac{1}{4\pi n}\int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\xi(\bar{k}^i)\right\|_{L^2(\mathbb{R}^{3(N-1)})}^2=\mathcal{O}\left(\frac{1}{n}\right)\to 0,\ \textit{for } i=N \textit{ and } n\to\infty, \tag{4.94}$$

Proof. For $i \neq N$ this can be seen by

$$
\begin{aligned}
&\left\| \frac{1}{n} \int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i) \hat{G}(k) \xi(\bar{k}^i) \right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \\
&\leq \frac{1}{n^2} \int \mathrm{d}\bar{k}^N \mathbb{1}_{B_n(0)}(k_i)(\bar{k}^N) \left(\int \mathrm{d}k_N \frac{1}{\sqrt{\left|\bar{k}^i\right|^2 + 1}} \left|\hat{G}(k)\right|^2 \right) \left(\int \mathrm{d}k_N \sqrt{\left|\bar{k}^i\right|^2 + 1} \left|\xi(\bar{k}^i)\right|^2 \right) \\
&\leq \frac{1}{n^2} \int \mathrm{d}\bar{k}^N \mathbb{1}_{B_n(0)}(k_i)(\bar{k}^N) \left(\int \mathrm{d}k_N \frac{1}{|k_N|} \frac{1}{(Ak_N^2 + Ak_i^2 + 2Bk_N \cdot k_i + \mu)^2} \right) \left(\int \mathrm{d}k_N \sqrt{\left|\bar{k}^i\right|^2 + 1} \left|\xi(\bar{k}^i)\right|^2 \right) \\
&\leq \frac{1}{n^2} \int_0^n \mathrm{d}r r^2 \left(\int \mathrm{d}q \, q \frac{1}{(Aq^2 + Ar^2 - 2Bqr)^2} \right) \|\xi\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})} \\
&= \frac{1}{n^2} \int_0^n \mathrm{d}r r^2 \left(\int \mathrm{d}\tilde{q} \, \tilde{q} r^2 \frac{1}{(A\tilde{q}^2 r^2 + Ar^2 - 2B\tilde{q} r^2)^2} \right) \|\xi\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})} \\
&\leq \frac{1}{n^2} \int_0^n \mathrm{d}r C \|\xi\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})} \\
&= \frac{1}{n} C \|\xi\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})} \to 0, \text{ as } n \to \infty.
\end{aligned}
\tag{4.95}
$$

where we used Cauchy-Schwartz in the second line, bounded $\hat{G}$ in the third line and changed variables $q = \tilde{q}r$ in the fifth line. Furthermore, $A = 1 - \frac{N-2}{(m+1)(m+N-2)}$ and $B = \frac{1}{m+1} - \frac{N-2}{(m+1)(m+N-2)}$, so $0 < B < A < 1$ as above.

For $i = N$, we proceed by

$$
\begin{aligned}
&\left\| \xi(\bar{k}^i) - \frac{1}{4\pi n} \int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i) \hat{G}(k) \xi(\bar{k}^i) \right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \\
&\quad = \int \mathrm{d}\bar{k}^N \left| \frac{1}{4\pi n} \left(4\pi n - \int_{B_n(0)} \mathrm{d}k_N \hat{G}(k) \right) \xi(\bar{k}^N) \right|^2 \\
&\quad = \int \mathrm{d}\bar{k}^N \left| \frac{1}{4\pi n} L^n(\bar{k}^N) \xi(\bar{k}^N) \right|^2 .
\end{aligned}
\tag{4.96}
$$

Now remember that $L^n(\bar{k}^N) < 4\pi n$, but we also have from the definition of L^n, (4.24), that

$$
L^n(\bar{k}^N) = 2\pi^2 \left(\frac{m(m+2)}{(m+1)^2} \sum_{i=1}^{N-1} k_i^2 + \frac{2m}{(m+1)^2} \sum_{1 \leq i < j \leq N-1} k_i \cdot k_j + \mu \right)^{1/2} \mathcal{O}(1) \leq CL(\bar{k}^N),
\tag{4.97}
$$

for $\left|\bar{k}^N\right| < n$. Thus we may estimate

$$
\begin{aligned}
\int \mathrm{d}\bar{k}^N \left| \frac{1}{4\pi n} L^n(\bar{k}^N) \xi(\bar{k}^N) \right|^2 &\leq \frac{C}{4\pi n} \int_{\left|\bar{k}^N\right| \leq n} \mathrm{d}\bar{k}^N L(\bar{k}^N) \left|\xi(\bar{k}^N)\right|^2 + \int_{\left|\bar{k}^N\right| > n} \left|\xi(\bar{k}^N)\right|^2 \\
&\leq \frac{C'}{4\pi n} \|\xi\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})} + \frac{1}{n} \int_{\left|\bar{k}^N\right| > n} \sqrt{\left|\bar{k}^N\right|^2 + 1} \left|\xi(\bar{k}^N)\right|^2 \\
&\leq \frac{\tilde{C}}{n} \|\xi\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})} \to 0 \text{ as } n \to \infty,
\end{aligned}
\tag{4.98}
$$

where we used that $\left(\int_{|\bar{k}^N|<n} \mathrm{d}\bar{k}^N L(\bar{k}^N) \left|\xi(\bar{k}^N)\right|^2\right)^{1/2} \equiv \|\xi\|_{H^{1/2}(\mathbb{R}^{3(N-1)})}$. This concludes the proof. □

This result can actually be made even stronger which is captured in the following lemma

Lemma 11. *Let $N \geq 2$ and $\xi \in H^{1/2}(\mathbb{R}^{3(N-1)})$, then we have the following limits*

$$\left\| \frac{1}{n} \int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i) \hat{G}(k) \xi(\bar{k}^i) \right\|^2_{L^2(\mathbb{R}^{3(N-1)})} = o\left(\frac{1}{n}\right) \to 0, \text{ for } i \neq N \text{ and } n \to \infty, \tag{4.99}$$

and

$$\left\| \xi(\bar{k}^i) - \frac{1}{4\pi n} \int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i) \hat{G}(k) \xi(\bar{k}^i) \right\|^2_{L^2(\mathbb{R}^{3(N-1)})} = o\left(\frac{1}{n}\right) \to 0, \text{ for } i = N \text{ and } n \to \infty, \tag{4.100}$$

where $o\left(\frac{1}{n}\right) = \epsilon(n)\mathcal{O}(\frac{1}{n})$ for some function $\epsilon(n) \to$ as $n \to \infty$.

Proof. The proof is a slight generalization of the proof of lemma 10. For $i \neq N$ we consider

$$\left\| \frac{1}{n} \int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i) \hat{G}(k) \xi(\bar{k}^i) \right\|^2_{L^2(\mathbb{R}^{3(N-1)})} = \int \mathrm{d}\bar{k}^{N,i} \int_{B_n(0)} \mathrm{d}k_i \left| \frac{1}{n} \int \mathrm{d}k_N \hat{G}(k) \xi(\bar{k}^i) \right|^2. \tag{4.101}$$

Therefore, we consider

$$\int_{B_n(0)} \mathrm{d}k_i \left| \frac{1}{n} \int \mathrm{d}k_N \hat{G}(k) \xi(\bar{k}^i) \right|^2 = \int_{B_n(0)} \mathrm{d}k_i \frac{1}{n^2} \left| \int_{M_n} \mathrm{d}k_N \hat{G}(k) \xi(\bar{k}^i) + \int_{M_n^{\complement}} \mathrm{d}k_N \hat{G}(k) \xi(\bar{k}^i) \right|^2, \tag{4.102}$$

with $M_n = \left\{ \left(\left|\bar{k}^i\right|^2 + 1\right)^{1/4} \left|\xi(\bar{k}^i)\right| < \varepsilon_n \right\}$ for some sequence $\varepsilon_n \to 0$ as $n \to \infty$. We now use Hölder's inequality, $(2,2)$ on the first term and $(3,3/2)$ on the second term

$$\begin{aligned}
&\int_{B_n(0)} \mathrm{d}k_i \left| \frac{1}{n} \int \mathrm{d}k_N \hat{G}(k) \xi(\bar{k}^i) \right|^2 \\
&\leq \int_{B_n(0)} \mathrm{d}k_i \frac{1}{n^2} \left| \left(\int_{M_n} \mathrm{d}k_N \frac{1}{\sqrt{\left|\bar{k}^i\right|^2 + 1}} \hat{G}(k)^2 \right)^{1/2} \left(\int_{M_n} \mathrm{d}k_N \sqrt{\left|\bar{k}^i\right|^2 + 1} \left|\xi(\bar{k}^i)\right|^2 \right)^{1/2} \right. \\
&\qquad \left. + \left(\int_{M_n^{\complement}} \mathrm{d}k_N \frac{1}{\left(\left|\bar{k}^i\right|^2 + 1\right)^{3/4}} \hat{G}(k)^3 \right)^{1/3} \left(\int_{M_n^{\complement}} \mathrm{d}k_N \left(\left(\left|\bar{k}^i\right| + 1\right)^{1/4} \left|\xi(\bar{k}^i)\right| \right)^{3/2} \right)^{2/3} \right|^2.
\end{aligned} \tag{4.103}$$

Now notice that

$$\left(\int_{M_n} \mathrm{d}k_N \frac{1}{\sqrt{\left|\bar{k}^i\right|^2 + 1}} \hat{G}(k)^2 \right)^{1/2} \leq \frac{C_1}{|k_i|}, \tag{4.104}$$

for some $C_1 > 0$, and by DCT we have

$$\left(\int_{M_n} \mathrm{d}k_N \sqrt{\left|\bar{k}^i\right| + 1} \left|\xi(\bar{k}^i)\right|^2 \right)^{1/2} = \epsilon_1(n) f(\bar{k}^{N,i}) \tag{4.105}$$

For $f(\bar{k}^{i,N}) = \left(\int \mathrm{d}k_N \sqrt{\left|\bar{k}^i\right|^2+1}\left|\xi(\bar{k}^{i,N},k_N)\right|^2\right)^{1/2} \in L^2(\mathbb{R}^{3(N-2)})$, and $\epsilon(n) \to 0$ as $n \to \infty$. Furthermore, we have

$$\left(\int_{M_n^\complement} \mathrm{d}k_N \frac{1}{\left(\left|\bar{k}^i\right|^2+1\right)^{3/4}}\hat{G}(k)^3\right)^{1/3} \leq \frac{C_2}{|k_i|^{3/2}}, \tag{4.106}$$

for some $C_2 > 0$, and

$$\left(\int_{M_n^\complement} \mathrm{d}k_N \left(\left(\left|\bar{k}^i\right|+1\right)^{1/4}\left|\xi(\bar{k}^i)\right|\right)^{3/2}\right)^{2/3} \leq \frac{1}{\varepsilon_n^{1/3}} f(\bar{k}^{N,i})^{4/3}. \tag{4.107}$$

Combining everything, we thus obtain

$$\begin{aligned}
&\int_{B_n(0)} \mathrm{d}k_i \left|\frac{1}{n}\int \mathrm{d}k_N \hat{G}(k)\xi(\bar{k}^i)\right|^2 \\
&\leq \frac{1}{n^2}\epsilon_1(n)^2 C_1^2 \left(\int_{B_n(0)} \mathrm{d}k_i \frac{1}{|k|_i^2}\right)|f|^2 + \frac{1}{n^2}\frac{1}{\varepsilon_n^{2/3}} C_2^2 \left(\int_{B_n(0)} \mathrm{d}k_i \frac{1}{|k|_i^3}\right)(|f|^2)^{4/3} \\
&\quad + \frac{2}{n^2}\epsilon_1(n)\frac{1}{\varepsilon_n^{1/3}} C_1 C_2 \left(\int_{B_n(0)} \mathrm{d}k_i \frac{1}{|k|_i^{5/2}}\right)(|f|^2)^{7/6}.
\end{aligned} \tag{4.108}$$

Choosing $\varepsilon_n = 1/\ln(n)^3$ we see that

$$n\int_{B_n(0)} \mathrm{d}k_i \left|\frac{1}{n}\int \mathrm{d}k_N \hat{G}(k)\xi(\bar{k}^i)\right|^2 \leq \frac{C}{n}\left(\epsilon_1(n)^2 n\,|f|^2 + \ln(n)^3(|f|^2)^{4/3} + \epsilon_1(n)\ln(n)\sqrt{n}(|f|^2)^{7/6}\right), \tag{4.109}$$

from which is follows that

$$n\int_{B_n(0)} \mathrm{d}k_i \left|\frac{1}{n}\int \mathrm{d}k_N \hat{G}(k)\xi(\bar{k}^i)\right|^2 \to 0 \text{ a.e. as } n \to \infty. \tag{4.110}$$

On the other hand by doing the above analysis over but with $M_n = \mathbb{R}^3$ we find

$$n\int_{B_n(0)} \mathrm{d}k_i \left|\frac{1}{n}\int \mathrm{d}k_N \hat{G}(k)\xi(\bar{k}^i)\right|^2 \leq C_1 |f|^2. \tag{4.111}$$

Hence, we conclude by DCT that

$$n\left\|\frac{1}{n}\int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\xi(\bar{k}^i)\right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \to 0 \text{ as } n \to \infty, \tag{4.112}$$

which is the desired result.

For $i = N$, consider

$$n\left\|\xi(\bar{k}^i) - \frac{1}{4\pi n}\int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\xi(\bar{k}^i)\right\|^2_{L^2(\mathbb{R}^{3(N-1)})} = \int \mathrm{d}\bar{k}^N \left|\frac{1}{4\pi\sqrt{n}}L^n(\bar{k}^N)\xi(\bar{k}^N)\right|^2. \tag{4.113}$$

Furthermore, we have

$$\begin{aligned}\left|\frac{1}{4\pi\sqrt{n}}L^n(\bar{k}^N)\xi(\bar{k}^N)\right|^2 &\leq C_1\mathbb{1}_{\{|\bar{k}^N|\leq n\}}L(\bar{k}^N)\left|\xi(\bar{k}^N)\right|^2 + C_2\mathbb{1}_{\{|\bar{k}^N|>n\}}\sqrt{\left|\bar{k}^N\right|^2+1}\left|\xi(\bar{k}^N)\right|^2\\ &\leq C_3\sqrt{\left|\bar{k}^N\right|^2+1}\left|\xi(\bar{k}^N)\right|^2\end{aligned} \tag{4.114}$$

where we in the first term used that $L^n(\bar{k}^N) \sim L(\bar{k}^N) \sim \left|\bar{k}^N\right|$ for $\bar{k}^N \leq n$, and in the second term we used that $L^n < 4\pi n$ and that $\sqrt{\left|\bar{k}^N\right|^2+1} > n$ for $|k_N| > n$. In the second inequality we used that $L(\bar{k}^N) \leq C\sqrt{\left|\bar{k}^N\right|^2+1}$ for some $C > 0$. Clearly the right hand side is an $L^1(\mathbb{R}^{3(N-1)})$ function, and thus we conclude that $\left|\frac{1}{4\pi\sqrt{n}}L^n(\bar{k}^N)\xi(\bar{k}^N)\right|^2$ is dominated by an $L^1(\mathbb{R}^{3(N-1)})$ function. We also note that $\left|\frac{1}{4\pi\sqrt{n}}L^n(\bar{k}^N)\xi(\bar{k}^N)\right|^2$ converges pointwise a.e. to zero. Hence, by DCT we then have

$$n\left\|\xi(\bar{k}^i) - \frac{1}{4\pi n}\int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\xi(\bar{k}^i)\right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \to 0 \text{ as } n \to \infty. \tag{4.115}$$

which is the desired result. □

The following lemma from [8] will also prove useful below

Lemma 12 (Lemma 1 in [8])**.** *The operator σ on $L^2(\mathbb{R}^3)$ with integral kernel*

$$\sigma(s,t) = (s^2+1)^{(\beta-1)/4}\frac{1}{s^2+t^2+\lambda s\cdot t+1}(t^2+1)^{-(\beta+1)/4}, \tag{4.116}$$

is bounded for all $-2 < \lambda < 2$, and $-2 < \beta < 2$.

This lemma generalizes to higher dimensions and we actually get the lemma

Lemma 13. *Let $i, j \in \{1, ..., N\}$ and $i \neq j$. The operator $\sigma_G^j : L^2(\mathbb{R}^{3(N-1)}) \to L^2(\tilde{\mathbb{R}}^{3(N-1)})$ given by*

$$\sigma_G^j : f(\bar{k}^i) \mapsto h(\bar{k}^j) = \int \mathrm{d}k_j \frac{1}{(\left|\bar{k}^j\right|^2+1)^{1/4}}\hat{G}(k)\frac{1}{(\left|\bar{k}^i\right|^2+1)^{1/4}}f(\bar{k}^i), \tag{4.117}$$

is bounded.

Proof. We simply make the following estimate

$$
\begin{aligned}
\left\|\sigma_G^j f\right\|_2^2 &= \left\|\int \mathrm{d}k_j \frac{1}{(\left|\bar{k}^j\right|^2+1)^{1/4}} \hat{G}(k) \frac{1}{(\left|\bar{k}^i\right|^2+1)^{1/4}} f(\bar{k}^i)\right\|_2^2 \\
&= \int \mathrm{d}\bar{k}^j \left|\int \mathrm{d}k_j \frac{1}{(\left|\bar{k}^j\right|^2+1)^{1/4}} \hat{G}(k) \frac{1}{(\left|\bar{k}^i\right|^2+1)^{1/4}} f(\bar{k}^i)\right|^2 \\
&\leq \int \mathrm{d}k_i \int \mathrm{d}\bar{k}^{i,j} \left|\int \mathrm{d}k_j \frac{1}{(|k_i|^2+1)^{1/4}} \frac{1}{Ak_i^2+Ak_j^2+2Bk_i\cdot k_j+\mu} \frac{1}{(|k_j|^2+1)^{1/4}} f(\bar{k}^i)\right|^2,
\end{aligned}
\tag{4.118}
$$

where we know from above that $0 < B < A < 1$. By a simple application of Minkowski's integral inequality we have

$$
\begin{aligned}
&\left\|\sigma_G^j f\right\|_2^2 \\
&\leq \int \mathrm{d}k_i \left(\int \mathrm{d}k_j \frac{1}{(|k_i|^2+1)^{1/4}} \frac{1}{Ak_i^2+Ak_j^2+2Bk_i\cdot k_j+\mu} \frac{1}{(|k_j|^2+1)^{1/4}} \left(\int \mathrm{d}\bar{k}^{i,j} \left|f(\bar{k}^i)\right|^2\right)^{1/2}\right)^2.
\end{aligned}
\tag{4.119}
$$

By Tornelli's theorem $g(k_j) := \left(\int \mathrm{d}\bar{k}^{i,j} \left|f(\bar{k}^i)\right|^2\right)^{1/2} \in L^2(\mathbb{R}^3)$ with $\|g\|_{L^2(\mathbb{R}^3)} = \|f\|_{L^2(\mathbb{R}^{3(N-1)})}$. Thus we have, by lemma 12 with $\beta = 0$ that

$$
\left\|\sigma_G^j f\right\|_2^2 \leq C\|\sigma\|^2 \|g\|^2_{L^2(\mathbb{R}^3)} = C' \|f\|^2_{L^2(\mathbb{R}^{3(N-1)})}
\tag{4.120}
$$

□

Remark: It is worth pointing out, that this is what makes T_{off} well-defined. Indeed it implies if we have $\chi, \xi \in H^{1/2}(\mathbb{R}^{3(N-1)})$, that

$$
\int \mathrm{d}k \overline{\chi(\bar{k}^j)} \hat{G}(k) \xi(\bar{k}^i) \leq \sqrt{C'} \|\chi\|_{H^{1/2}(\mathbb{R}^{3(N-1)})} \|\xi\|_{H^{1/2}(\mathbb{R}^{3(N-1)})}.
\tag{4.121}
$$

In particular

$$
T_{\text{off}}(\xi) = (N-1)\int \mathrm{d}k \overline{\xi(\bar{k}^2)} \hat{G}(k) \xi(\bar{k}^1) \leq \sqrt{C'}(N-1)\|\xi\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})}
\tag{4.122}
$$

We now present a result generalizing Proposition 6

Proposition 9. *Let F_α^n and F_α be defined as above. Then for all $u \in \mathscr{D}(F_\alpha)$ there exist $(u_n)_{(n\geq 1)}$ with $u_n \in \mathscr{D}(F_n)$ such that*

$$
F_\alpha(u) = \lim_{n\to\infty} F_\alpha^n(u_n).
\tag{4.123}
$$

Proof. Similar to the proof of Proposition 6 we simply construct the wanted sequence. Let $u = w + \rho G \in \mathscr{D}(F_\alpha)$, where $\rho G = \left(\sum_{i=1}^N (-1)^{i-1} \hat{G}(k)\xi(\bar{k}^i)\right)^\vee$. We then choose the sequence

$u_n = w + (\widehat{\rho^n G})^\vee$, *i.e.*

$$\hat{u}_n(k) = \hat{w}(k) + \sum_{i=1}^{N} (-1)^{i-1} \mathbb{1}_{B_n(0)}(k_i) \hat{G}(k) \xi(\bar{k}^i). \tag{4.124}$$

To do this we need to show that for $\xi \in H^{1/2}_{\mathrm{as}}(\mathbb{R}^{3(N-1)})$ we have $(\mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\xi(\bar{k}^i))^\vee \in H^1_{\mathrm{as}}(\mathbb{R}^{3N})$. However, this actually follows from calculations we have previously made

$$\left\|(\mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\xi(\bar{k}^i))^\vee\right\|^2_{H^1(\mathbb{R}^{3N})} \equiv \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k)^{-1} \mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)^2 |\xi(\bar{k}^i)|^2, \tag{4.125}$$

where we use $\equiv$ to denote equivalence of norms By symmetry of $\hat{G}$ we may rewrite this as

$$\int_{\mathbb{R}^{3N}} \mathrm{d}k \mathbb{1}_{B_n(0)}(k_N)\hat{G}(k) \left|\xi(\bar{k}^N)\right|^2 = \int_{\mathbb{R}^{3(N-1)}} \mathrm{d}\bar{k}^N (4\pi n - L^n(\bar{k}^N)) \left|\xi(\bar{k}^N)\right|^2 \le 4\pi n \left\|\xi\right\|^2_{L^2(\mathbb{R}^{3(N-1)})}, \tag{4.126}$$

for some $C > 0$. Now we see that

$$\begin{aligned}
\xi^n(\bar{k}^N) &= \frac{1}{4\pi n + \alpha} \int_{B_n(0)} \mathrm{d}k_N \left(\hat{w}(k) + \sum_{i=1}^{N} (-1)^{i-2+N} \mathbb{1}_{B_n(0)}(k_i) \hat{G}(k) \xi(\bar{k}^i) \right) \\
&= \frac{1}{4\pi n + \alpha} \left(\left(\int_{B_n(0)} \mathrm{d}k_N \hat{w}(k) \right) + \sum_{i=1}^{N-1} (-1)^{i-2+N} \mathbb{1}_{B_n(0)}(k_i) \left\langle \hat{G}(\bar{k}^N, \cdot), \xi(\bar{k}^{iN}, \cdot) \right\rangle \right. \\
&\qquad \left. + \left(4\pi n - L^n(\bar{k}^N)\right) \xi(\bar{k}^N) \right) \to \xi(\bar{k}^N), \text{ a.e. as } n \to \infty.
\end{aligned} \tag{4.127}$$

We can actually furthermore show norm L^2-convergence of ξ^n. By lemmas 7 and 11 we have

$$\begin{aligned}
\|\xi^n - \xi\|_{L^2(\mathbb{R}^{3(N-1)})} &\le \frac{1}{4\pi n + \alpha} \left(\left\| \int_{B_n(0)} \mathrm{d}k_N \hat{w}(k) \right\|_{L^2(\mathbb{R}^{3(N-1)})} \right. \\
&\quad + \sum_{i=1}^{N-1} \left\| \int \mathrm{d}k_N \left(\mathbb{1}_{B_n(0)}(k_i) \hat{G}(k) \xi(\bar{k}^i) \right) \right\|_{L^2(\mathbb{R}^{3(N-1)})} \\
&\quad \left. + \left\| \int \mathrm{d}k_N \mathbb{1}_{B_n(0)}(k_N) \hat{G}(k) \xi(\bar{k}^N) - (4\pi n + \alpha) \xi(\bar{k}^N) \right\|_{L^2(\mathbb{R}^{3(N-1)})} \right) \\
&= o\left(\frac{1}{\sqrt{n}}\right) \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{4.128}$$

We are now ready to show the convergence of $F^n_\alpha(u^n)$. We have

$$F^n_\alpha(u^n) = \int \mathrm{d}k \hat{G}(k)^{-1} \left|\hat{u}_n - \hat{\rho}^n \hat{G}\right|^2 - \mu \left\|u_n\right\|^2_2 + N \left(T^n_{\mathrm{diag}}(\xi^n) + T^n_{\mathrm{off}}(\xi^n) + \alpha \left\|\xi^n\right\|^2_{L^2(\mathbb{R}^{3(N-1)})} \right) \tag{4.129}$$

The terms $N\alpha \left\|\xi^n\right\|^2_{L^2(\mathbb{R}^{3(N-1)})}$ and $-\mu \left\|u^n\right\|^2_2$ converge as we want by the convergence of ξ^n. Let us therefore write out

$$\int \mathrm{d}k\hat{G}(k)^{-1}\left|\hat{u}_n-\hat{\rho}^n\hat{G}\right|^2=\int \mathrm{d}k\hat{G}(k)^{-1}\Bigg\{|\hat{w}(k)|^2+2\mathrm{Re}\hat{w}(k)\left(\sum_{i=1}^N(-1)^{i-1}\mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\left(\xi(\bar{k}^i)-\xi^n(\bar{k}^i)\right)\right)$$
$$+\sum_{i,j}(-1)^{i+j}\mathbb{1}_{B_n(0)}(k_i)\mathbb{1}_{B_n(0)}(k_j)\hat{G}(k)^2\left(\overline{\xi(\bar{k}^i)}-\overline{\xi^n(\bar{k}^i)}\right)\left(\xi(\bar{k}^j)-\xi^n(\bar{k}^j)\right)\Bigg\}. \tag{4.130}$$

The first term we want to keep. The second term goes to zero, by (4.128) and Lemma 7. The third term on the other hand, cancels $N\left(T^n_{\mathrm{diag}}(\xi^n)+T^n_{\mathrm{off}}(\xi^n)\right)$ and replaces it by $N\left(T_{\mathrm{diag}}(\xi)+T_{\mathrm{off}}(\xi)\right)$, by the fact that

$$\int \mathrm{d}k\overline{\xi}(\bar{k}^i)\hat{G}(k)\xi^n(\bar{k}^j)\to\int \mathrm{d}k\overline{\xi}(\bar{k}^i)\hat{G}(k)\xi(\bar{k}^j). \tag{4.131}$$

To see this, notice that by lemma 8 and 9 we now that ξ^n is $H^{1/2}(\mathbb{R}^{3(N-1)})$ bounded. Thus for any subsequence, ξ^{n_j} there exist a further subsequence $\xi^{n_{j_k}}$ such that $\xi^{n_{j_k}}\rightharpoonup\chi$ in $H^{1/2}(\mathbb{R}^{3(N-1)})$ for some $\chi\in H^{1/2}(\mathbb{R}^{3(N-1)})$. However, then $\xi^{n_{j_k}}\rightharpoonup\chi$ in $L^2(\mathbb{R}^{3(N-1)})$, but since we know $\xi^{n_{j_k}}\to\xi$ in $L^2(\mathbb{R}^{3(N-1)})$, we conclude that $\chi=\xi$. Therefore, we infer that $\xi^n\rightharpoonup\xi$ in $H^{1/2}(\mathbb{R}^{3(N-1)})$. By lemma 13 (the remark below) we know that for $i\neq j$ $\phi\mapsto\int \mathrm{d}k\overline{\xi}(\bar{k}^i)\hat{G}(k)\phi(\bar{k}^j)$ is a bounded linear functional on $H^{1/2}(\mathbb{R}^{3(N-1)})$ for $\xi\in H^{1/2}(\mathbb{R}^{3(N-1)})$. Hence we conclude that

$$\int \mathrm{d}k\overline{\xi}(\bar{k}^i)\hat{G}(k)\xi^n(\bar{k}^j)\to\int \mathrm{d}k\overline{\xi}(\bar{k}^i)\hat{G}(k)\xi(\bar{k}^j),\quad \text{for } i\neq j. \tag{4.132}$$

Knowing this it is straightforward to show that

$$\int \mathrm{d}k\mathbb{1}_{B_n(0)}(k_i)\mathbb{1}_{B_n(0)}(k_i)\overline{\xi}(\bar{k}^i)\hat{G}(k)\xi^n(\bar{k}^j)\to\int \mathrm{d}k\overline{\xi}(\bar{k}^i)\hat{G}(k)\xi(\bar{k}^j),\quad \text{for } i\neq j, \tag{4.133}$$

since for $i\neq j$ and $\phi,\xi\in H^{1/2}(\mathbb{R}^{3(N-1)})$, $(\xi,\phi)\mapsto\int \mathrm{d}k\overline{\xi}(\bar{k}^i)\hat{G}(k)\phi(\bar{k}^j)$ is a bounded sesquilinear form on $H^{1/2}(\mathbb{R}^{3(N-1)})$, and since $\mathbb{1}_{B_n(0)}(k_j)\xi(\bar{k}^i)\to\xi(\bar{k}^i)$ in $H^{1/2}(\mathbb{R}^{3(N-1)})$ norm, and $\mathbb{1}_{B_n(0)}(k_i)\xi^n(\bar{k}^j)\rightharpoonup\xi(\bar{k}^j)$ in $H^{1/2}(\mathbb{R}^{3(N-1)})$ which is are simple consequences of DCT. Thus we find by antisymmetry of ξ^n and ξ

$$\begin{aligned}\int \mathrm{d}k\sum_{\substack{i,j\\ i\neq j}}(-1)^{i+j}\mathbb{1}_{B_n(0)}(k_i)\mathbb{1}_{B_n(0)}(k_j)\hat{G}(k)\left(\overline{\xi(\bar{k}^i)}-\overline{\xi^n(\bar{k}^i)}\right)\left(\xi(\bar{k}^j)-\xi^n(\bar{k}^j)\right)+NT^n_{\mathrm{off}}(\xi^n)\\ \to-\int \mathrm{d}k\sum_{\substack{i,j\\ i\neq j}}(-1)^{i+j}\mathbb{1}_{B_n(0)}(k_i)\mathbb{1}_{B_n(0)}(k_j)\hat{G}(k)\overline{\xi(\bar{k}^i)}\xi(\bar{k}^j)\\ =N(N-1)\int \mathrm{d}k\overline{\xi(s,\bar{q})}\hat{G}(s,t,\bar{q})\xi(t,\bar{q})=NT_{\mathrm{off}}(\xi)\end{aligned} \tag{4.134}$$

For $i = N$ on the other hand we have

$$\begin{aligned}&\int \mathrm{d}k \mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\left(\overline{\xi(\bar{k}^i)} - \overline{\xi^n(\bar{k}^i)}\right)\left(\xi(\bar{k}^i) - \xi^n(\bar{k}^i)\right)\\ &\quad = \int \mathrm{d}\bar{k}^i(4\pi n - L^n(\bar{k}^i))\left|\xi(\bar{k}^i) - \xi^n(\bar{k}^i)\right|^2.\end{aligned} \tag{4.135}$$

By lemma 11 the first term converges to zero. The second term on the other hand can be written as

$$\begin{aligned}-\int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\left|\xi(\bar{k}^i) - \xi^n(\bar{k}^i)\right|^2 = \overbrace{-\int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\left|\xi^n(\bar{k}^i)\right|^2}^{-T^n_{\mathrm{diag}}(\xi^n)} - \int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\left|\xi(\bar{k}^i)\right|^2\\ +2\mathrm{Re}\int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\overline{\xi(\bar{k}^i)}\xi^n(\bar{k}^i).\end{aligned} \tag{4.136}$$

Similar to above in (4.114) we have

$$L^n(\bar{k}^N) \leq C_1 \mathbb{1}_{B_n(0)}(\bar{k}^N)L(\bar{k}^N) + C_2\mathbb{1}_{B_n(0)^\complement}(\bar{k}^N)\sqrt{\left|\bar{k}^N\right|^2+1} \leq C\sqrt{\left|\bar{k}^N\right|^2+1}, \tag{4.137}$$

and $L^n \to L$ pointwise a.e. as $n \to \infty$. By DCT we infer that

$$\int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\left|\xi(\bar{k}^i)\right|^2 \to \int \mathrm{d}\bar{k}^i L(\bar{k}^i)\left|\xi(\bar{k}^i)\right|^2. \tag{4.138}$$

Furthermore, we have

$$\int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\overline{\xi(\bar{k}^i)}\xi^n(\bar{k}^i) = \int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\overline{\xi(\bar{k}^i)}\xi(\bar{k}^i) + \int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\overline{\xi(\bar{k}^i)}(\xi^n(\bar{k}^i) - \xi(\bar{k}^i)). \tag{4.139}$$

Since we know that $|\xi^n - \xi| \rightharpoonup 0$ in $H^{1/2}(\mathbb{R}^{3(N-1)})$ (Similar argument to the one above for $\xi_n \rightharpoonup \xi$, just observe that $|\xi^n - \xi|$ is $H^{1/2}(\mathbb{R}^{3(N-1)})$ bounded and converge to zero in $L^2(\mathbb{R}^{3(N-1)})$ is). Thus we know

$$\begin{aligned}&\left|\int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\overline{\xi(\bar{k}^i)}(\xi^n(\bar{k}^i) - \xi(\bar{k}^i))\right| \leq \int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\left|\xi(\bar{k}^i)\right|\left|\xi^n(\bar{k}^i) - \xi(\bar{k}^i)\right|\\ &\leq C\int \mathrm{d}\bar{k}^i\sqrt{\left|\bar{k}^i\right|^2+1}\left|\xi(\bar{k}^i)\right|\left|\xi^n(\bar{k}^i) - \xi(\bar{k}^i)\right| = \langle|\xi|, |\xi^n - \xi|\rangle_{H^{1/2}(\mathbb{R}^{3(N-1)})} \to 0.\end{aligned} \tag{4.140}$$

Combining with (4.138) and (4.139) we obtain

$$-\int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\left|\xi(\bar{k}^i) - \xi^n(\bar{k}^i)\right|^2 + \int \mathrm{d}\bar{k}^i L^n(\bar{k}^i)\left|\xi^n(\bar{k}^i)\right|^2 \to \int \mathrm{d}\bar{k}^i L(\bar{k}^i)\left|\xi(\bar{k}^i)\right|^2 \tag{4.141}$$

Combining all the above, we find

$$\left(\int \mathrm{d}k\mathbb{1}_{B_n(0)}(k_i)\hat{G}(k)\left(\overline{\xi(\bar{k}^i)} - \overline{\xi^n(\bar{k}^i)}\right)\left(\xi(\bar{k}^i) - \xi^n(\bar{k}^i)\right)\right) + T^n_{\mathrm{diag}}(\xi^n) \to T_{\mathrm{diag}}(\xi) \quad \text{as } n \to \infty. \tag{4.142}$$

Combining (4.134) and (4.142) we find

$$\int \mathrm{d}k\hat{G}(k)^{-1}\left|\hat{u}_n-\hat{\rho}^n\hat{G}\right|^2+N(T^n_{\text{diag}}(\xi^n)+T^n_{\text{off}}(\xi^n))\to\int \mathrm{d}k\hat{G}(k)^{-1}\left|\hat{w}(k)\right|^2+N(T_{\text{diag}}(\xi)+T_{\text{off}}(\xi)), \tag{4.143}$$

as $n\to\infty$, from which it follows that

$$F^n_\alpha(u_n)\to F_\alpha(u), \qquad \text{as } n\to\infty \tag{4.144}$$

□

Remark: It is worth pointing out that we actually circumvent showing that $T^n_{\text{off}}(\xi^n)\to T_{\text{off}}(\xi)$. Although this might be true, showing it probably requires (it would at least be sufficient) that $\xi^n\to\xi$ in $H^{1/2}(\mathbb{R}^{3(N-1)})$, which we have not been able to show. However, $|\xi^n-\xi|\rightharpoonup 0$ in $H^{1/2}(\mathbb{R}^{3(N-1)})$ suffices in showing convergence of $F^n_\alpha(u_n)$ as can be seen from the proof above.

Lower bound

Next step: Show uniform lower bound of F^n_α, since combining with the upper bound of Proposition 9 gives lower bound of F_α. We want to show that $\left(T^R_{\text{diag}}(\xi^R)+T^R_{\text{off}}(\xi^R)+\alpha\left\|\xi^R\right\|^2_{L^2(\mathbb{R}^{3(N-1)})}\right)$ is positive under certain requirements of the mass ratio, m and for certain choice of μ. If μ can be chosen independently of R, this indeed provides a uniform lower bound, as we then have $F^n_\alpha(u)\geq-\mu\left\|u\right\|^2_2$.
Since we have reduced the problem to working with F^R_α at finite R. It is convenient to use the expression for F^R_α given in (4.2)

$$F^R_\alpha(u)=\int \mathrm{d}k\hat{G}(k)^{-1}\left|\hat{u}(k)\right|^2-\mu\left\|u\right\|^2_2-\frac{1}{4\pi R+\alpha}\sum_{i=1}^N\int \mathrm{d}\bar{k}^i\left|\int_{B_R(0)}\mathrm{d}k_i\hat{u}(k)\right|^2. \tag{4.145}$$

Defining $\hat{\phi}(k)=\sqrt{\hat{G}(k)^{-1}}\hat{u}(k)$ this can be rewritten

$$F^R_\alpha(u)=\left\|\phi\right\|^2_2-\mu\left\|u\right\|^2_2-\frac{1}{4\pi R+\alpha}\sum_{i=1}^N\int \mathrm{d}\bar{k}^i\left|\int_{B_R(0)}\mathrm{d}k_i\sqrt{\hat{G}(k)}\hat{\phi}(k)\right|^2. \tag{4.146}$$

Thus if we can show that $\frac{1}{4\pi R+\alpha}\sum_{i=1}^N\int \mathrm{d}\bar{k}^i\left|\int_{B_R(0)}\mathrm{d}k_i\sqrt{\hat{G}(k)}\hat{\phi}(k)\right|^2\leq\left\|\phi\right\|^2$ for all $R>0$, we find that $F^R_\alpha(u)\geq-\mu\left\|u\right\|^2_2$ and F^R_α is uniformly bounded from below.
Notice that $\frac{1}{4\pi R+\alpha}\sum_{i=1}^N\int \mathrm{d}\bar{k}^i\left|\int_{B_R(0)}\mathrm{d}k_i\sqrt{\hat{G}(k)}\hat{\phi}(k)\right|^2\leq\left\|\phi\right\|^2_2$ depends on the choice μ, thus we

need only prove that there exist a finite μ such that the bound holds. We now use the bound

$$\begin{aligned}\left|\int_{B_R(0)} \mathrm{d}k_i \sqrt{\hat{G}(k)}\hat{\phi}(k)\right|^2 &= \left|\int_{B_R(0)} \mathrm{d}k_i h(k) \left(\hat{G}(k)\right)^{1/4} h(k)^{-1} \left(\hat{G}(k)\right)^{1/4} \hat{\phi}(k)\right|^2 \\ &\leq \left(\int_{B_R(0)} \mathrm{d}k_i h(k)^{-2} \hat{G}(k)^{1/2}\right)\left(\int_{B_R(0)} \mathrm{d}k_i h(k)^2 \hat{G}(k)^{1/2} \left|\hat{\phi}(k)\right|^2\right) \\ &\leq \end{aligned} \tag{4.147}$$

Choosing $h(k) = \sqrt{\frac{k_i^2}{k_j^2}}$ for $j \neq i$ we find

$$\begin{aligned}\left|\int_{B_R(0)} \mathrm{d}k_i \sqrt{\hat{G}(k)}\hat{\phi}(k)\right|^2 &\leq \left(\int_{B_R(0)} \mathrm{d}k_i \frac{k_j^2}{k_i^2} \hat{G}(k)^{1/2}\right)\left(\int_{B_R(0)} \mathrm{d}k_i \frac{k_i^2}{k_j^2} \hat{G}(k)^{1/2} \left|\hat{\phi}(k)\right|^2\right) \\ &\leq \sup_{k_j}\left(\int_{B_R(0)} \mathrm{d}k_i \frac{k_j^2}{k_i^2} \hat{G}(k)^{1/2}\right) \sup_{k_i} \frac{k_i^2}{k_j^2} \hat{G}(k)^{1/2} \left(\int_{B_R(0)} \mathrm{d}k_i \left|\hat{\phi}(k)\right|^2\right)\end{aligned} \tag{4.148}$$

Problem with factor N from the sum.

INSTEAD: Let $\mathscr{D}(F_\alpha^n) \ni u_n \to u \in \mathscr{D}(F_\alpha)$ in $L^2(\mathbb{R}^{3N})$ we then show that

$$F_\alpha^n(u_n) \geq -M \|u_n\|_2^2, \tag{4.149}$$

for some $M > 0$ independent of $n \geq 1$.

OR SHOW Γ-lower bound

$$\liminf_{n\to\infty} F_\alpha^n(u_n) \geq F_\alpha(u). \tag{4.150}$$

If $\liminf_{n\to\infty} F_\alpha^n(u_n) = \infty$ we are done. Assume therefore that $\liminf_{n\to\infty} F_\alpha^n(u_n) < \infty$. Assume, by possibly passing to a subsequence that $F_\alpha^n(u_n) \to \liminf_{n\to\infty} F_\alpha^n(u_n)$. We may write $\hat{u} = \hat{w} + \hat{G}\hat{\rho}$, with $w \in H^1_{\mathrm{as}}(\mathbb{R}^{3N})$ and $\rho = \sum_{i=1}^N (-1)^{i+1}\xi$, with $\xi \in H^{1/2}_{\mathrm{as}}(\mathbb{R}^{3(N-1)})$. Since $\mathscr{D}(F_\alpha^n) = H^1_{\mathrm{as}}(\mathbb{R}^{3N})$ for all $n \in \mathbb{N}$ we may also write $\hat{u}_n = \hat{w} + \hat{G}\hat{\zeta}^n$ with $\zeta^n \in H^{-1}_{\mathrm{as}}(\mathbb{R}^{3N})$ and $\hat{G}\hat{\zeta}^n \to \hat{G}\hat{\rho}$ in $L^2(\mathbb{R}^{3N})$, *i.e.* $\zeta^n \to \rho$ in $H^{-2}(\mathbb{R}^{3N})$. Thus we have

$$F_\alpha^n(u^n) = \int \mathrm{d}k \hat{G}(k)^{-1} \left|\hat{w} + (\hat{\zeta}^n - \hat{\rho}^n)\hat{G}\right|^2 - \mu \|u_n\|_2^2 + N\left(T^n_{\mathrm{diag}}(\xi^n) + T^n_{\mathrm{off}}(\xi^n) + \alpha \|\xi^n\|^2_{L^2(\mathbb{R}^{3(N-1)})}\right) \tag{4.151}$$

With $\xi^n(\bar{k}^N) = \frac{1}{4\pi n + \alpha}\int_{B_n(0)} \mathrm{d}k_N \hat{u}_n(k) = \frac{1}{4\pi n+\alpha}\int_{B_n(0)} \mathrm{d}k_N \left(\hat{w}(k) + \hat{G}(k)\zeta^n(k)\right)$, and $\rho^n(k) = \sum_{i=1}^N (-1)^{i+1}\xi^n(\bar{k}^i)$. Knowing that

$$L^n(\bar{k}^N) \geq \pi\sqrt{4\beta - y^2}\left(\arctan\left(\frac{2n - |y|}{\sqrt{4\beta - y^2}}\right) + \arctan\left(\frac{2n + |y|}{\sqrt{4\beta - y^2}}\right)\right), \tag{4.152}$$

where $y = \frac{2}{m+1}\sum_{i=1}^{N-1} k_i$, and $\beta = \sum_{i=1}^{N-1} k_i^2 + \frac{2}{m+1}\sum_{1\leq i<j\leq N-1} k_i \cdot k_j + \mu$ (see the proof of lemma 5). On the other hand we also have

$$L^n(\bar{k}^N) \geq 2\pi\left[n - \frac{(2n^2 - y^2 + 2\beta)}{4\,|y|}\ln\left(\frac{n(n+|y|)+\beta}{n(n-|y|)+\beta}\right)\right], \tag{4.153}$$

So for $|y| > 2n$ we have

$$L^n(\bar{k}^N) \geq 2\pi\left[n - \frac{(2n^2 - y^2 + 2\beta)}{4\,|y|}\frac{2n\,|y|}{n^2 - n\,|y| + \beta}\right] \geq 2\pi\left[n - \frac{4n\,|y|}{4\,|y|}\right]. \tag{4.154}$$

We consider $L^n(\bar{k}^N)/\,|y|$ and define $x = n/\,|y|$, and $\gamma = 4\beta/y^2$

$$\begin{aligned} l(x,\gamma) := L^n(\bar{k}^N)/\,|y| = \pi\sqrt{\gamma-1}\left(\arctan\left(\frac{2x-1}{\sqrt{\gamma-1}}\right) + \arctan\left(\frac{2x+1}{\sqrt{\gamma-1}}\right)\right) \\ +2\pi\left[x - \frac{(2x^2 - 1 + \frac{\gamma}{2})}{4}\ln\left(\frac{x(x+1)+\frac{\gamma}{4}}{x(x-1)+\frac{\gamma}{4}}\right)\right] \end{aligned} \tag{4.155}$$

with

$$\frac{\partial}{\partial x} l(x,\gamma) = 2\pi\left(2 - x\ln\left(\frac{x(x+1)+\gamma/4}{x(x-1)+\gamma/4}\right)\right) = 2\pi\left(2 - x\ln\left(1 + \frac{2x}{x(x-1)+\gamma/4}\right)\right) \tag{4.156}$$

Notice that l has no maximum in x if $\gamma \geq 2$ since

$$\frac{\partial}{\partial x} l(x,\gamma) \geq 2\pi\left(2 - x\frac{\frac{2x}{x(x-1)+\gamma/4}}{\sqrt{1 + \frac{2x}{x(x-1)+\gamma/4}}}\right), \tag{4.157}$$

where we used the inequality $\ln(1+x) \geq \frac{x}{\sqrt{1+x}}$ for $x \geq 0$. And for $\gamma \geq 2$ we have

$$2 - x\frac{\frac{2x}{x(x-1)+\gamma/4}}{\sqrt{1 + \frac{2x}{x(x-1)+\gamma/4}}} > 0. \tag{4.158}$$

Numerics suggest that $l(x,\gamma)$ has no maximum in x for $\gamma \geq 4/3$. The following results will be useful

Lemma 14. *Let β and y be defined as above, and define $\gamma = 4\beta/y^2$. Then*

$$\gamma = (m+1) + \frac{m(m+1)\sum_{i=1}^{N-1} k_i^2 + (m+1)^2\mu}{\left(\sum_{i=1}^{N-1} k_i\right)^2}. \tag{4.159}$$

In particular $\gamma \geq m + 1 + \frac{m(m+1)}{N-1} > 1 + m$

Proof. We immediately have

$$4\beta = 4\sum_{i=1}^{N} k_i^2 + 4\frac{2}{m+1}\sum_{1\leq i<j\leq N-1} k_i \cdot k_j + 4\mu, \tag{4.160}$$

and

$$(m+1)y^2 = \frac{4}{m+1}\sum_{i=1}^{N-1} k_i^2 + 4\frac{2}{m+1}\sum_{1\leq i<j\leq N-1} k_i\cdot k_j. \tag{4.161}$$

Thus we find

$$4\beta-(m+1)y^2 = \frac{4m}{m+1}\sum_{i=1}^{N-1} k_i^2 + 4\mu, \tag{4.162}$$

which is equivalent to

$$\frac{4\beta}{y^2} = (m+1) + \frac{m(m+1)\sum_{i=1}^{N-1} k_i^2 + (m+1)^2\mu}{\left(\sum_{i=1}^{N-1} k_i\right)^2} \geq m+1+\frac{m(m+1)}{N-1} > m+1 \tag{4.163}$$

where we in the first inequality used that $\frac{m(m+1)\sum_{i=1}^{N-1} k_i^2+(m+1)^2\mu}{\left(\sum_{i=1}^{N-1} k_i\right)^2}$ attain its minimum when all k_i are equal, and their magnitude to to infinity. □

Lemma 15. *Let $L^R(y,\beta)$ be defined as in lemma 4, where y and β are defined as above, and let $\gamma = 4\beta/y^2$. Then L^R is a concave function of R whenever $\gamma \geq 4/3$ (Analytic proof for $\gamma\geq 2$, numerics suggest for $\gamma \geq 4/3$). On the other hand $L^R(y,\beta)$ is a concave function of R for any $\gamma > 1$ and $R \leq y/2$.*

Proof. This follows from the simple calculation of the second derivative

$$\frac{\partial^2 L^R(y,\beta)}{\partial R^2} = 2\pi\left(\frac{2R\left(R^2-\beta\right)}{(\beta+R(R-|y|))\,(\beta+R(R+|y|))} - \frac{\ln\left(\frac{\beta+R(R+|y|)}{\beta+R(R-|y|)}\right)}{|y|}\right) \tag{4.164}$$

Numerics suggest that $\frac{\partial^2 L^R(y,\beta)}{\partial R^2} \leq 0$ for $\gamma \geq 4/3$ and the result follows. Furthermore, since $\ln\left(\frac{\beta+R(R+|y|)}{\beta+R(R-|y|)}\right) = \ln\left(1+\frac{2R|y|}{\beta+R(R-|y|)}\right) \geq \frac{2\left(\frac{2R|y|}{\beta+R(R-|y|)}\right)}{2+\frac{2R|y|}{\beta+R(R-|y|)}}$ we have

$$\begin{aligned}\frac{1}{2\pi}\frac{\partial^2 L^R(y,\gamma y^2/4)}{\partial R^2} &\leq \frac{2R\left(R^2-\gamma y^2/4\right)}{(\gamma y^2/4+R(R-|y|))\,(\gamma y^2/4+R(R+|y|))} - \frac{1}{|y|}\frac{2\left(\frac{2R|y|}{\gamma y^2/4+R(R-|y|)}\right)}{2+\frac{2R|y|}{\gamma y^2/4+R(R-|y|)}}\\ &= -\frac{16R\left(\gamma^2\,|y|^4+4(\gamma-2)R^2y^2\right)}{(\gamma y^2+4R^2)\,(\gamma y^2+4R^2-4R\,|y|)\,(\gamma y^2+4R^2+4R\,|y|)} \leq 0.\end{aligned} \tag{4.165}$$

for any $R>0$ $\gamma\geq 2$ *or* for any $\gamma>1$ and $R\leq |y|/2$. □

Corollary 3. *Let L^R, y, β and γ be defined as above, then for $R\leq |y|/2$ we have*

$$L^R(y,\gamma y^2/4) \geq \frac{\pi}{2}\left((\gamma-1)\ln\left(\frac{\gamma-1}{\gamma+3}\right) + 4\sqrt{\gamma-1}\arctan\left(\frac{2}{\sqrt{\gamma-1}}\right)+4\right)R \tag{4.166}$$

Proof. This follows from lemma 15 and the fact that $L^0=0$ and

$$\frac{2}{|y|}L^{|y|/2}(y,\gamma y^2/4) = \frac{\pi}{2}\left((\gamma-1)\ln\left(\frac{\gamma-1}{\gamma+3}\right) + 4\sqrt{\gamma-1}\arctan\left(\frac{2}{\sqrt{\gamma-1}}\right)+4\right) \tag{4.167}$$

□

We now generalize the proof of Moser and Seiringer [8].....

$$
\begin{aligned}
T^n_{\text{diag}}(\xi^n) &= \int \mathrm{d}\bar{k}^N L^n(\bar{k}^N) \left|\xi^n(\bar{k}^N)\right|^2, \\
T^n_{\text{off}}(\xi^n) &= (N-1)\int_{\mathbb{R}^{3(N-2)}} \mathrm{d}\bar{q} \int_{B_n(0)} \mathrm{d}s \int_{B_n(0)} \mathrm{d}t \overline{\xi^n(s,\bar{q})}\hat{G}(s,t,\bar{q})\xi^n(t,\bar{q})
\end{aligned}
\tag{4.168}
$$

Define $\phi^n = \sqrt{L^n}\xi^n$ such that $T^n_{\text{diag}}(\xi^n) = \|\phi^n\|_2^2$. Furthermore, we have $T^n_{\text{off}}(\xi^n) = (N-1)\int_{\mathbb{R}^{3(N-2)}} \mathrm{d}\bar{q} \int_{\mathbb{R}^3} \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}t \overline{\phi^n(s,\bar{q})}\hat{J}^n(s,t,\bar{q})\phi^n(t,\bar{q})$, with $\hat{J}^n(s,t,\bar{q}) = \mathbb{1}_{B_n(0)}(s)\mathbb{1}_{B_n(0)}(t)\frac{1}{\sqrt{L^n(s,\bar{q})}}\hat{G}(s,t,\bar{q})\frac{1}{\sqrt{L^n(t,\bar{q})}}$.

Conjecture: Stability seems to be related to weather L^R has a maximum in R or not (Stability if no maximum). Numerics suggest that such a maximum does not exists for $\frac{4\beta}{y^2}-1 > 1/3$. At least there is no maximum for $\frac{4\beta}{y^2}-1 \geq 0.33334$ and there is a maximum for $\frac{4\beta}{y^2}-1 \leq 0.33333$.

Lemma 16. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathscr{D}(F^n_\alpha) = H^1(\mathbb{R}^{3N})$ such that $u_n \to u \in \mathscr{D}(F_\alpha)$ in $L^2(\mathbb{R}^{3N})$ then*

$$
\liminf_{n\to\infty} F^n_\alpha(u_n/n) \geq 0 \tag{4.169}
$$

Proof. Remember that

$$
F^n_\alpha(u_n/n) = \frac{1}{n^2}\int \mathrm{d}k\hat{G}(k)^{-1}\left|\hat{u}_n(k)\right|^2 - \frac{1}{n^2}\mu\left\|u_n\right\|_2^2 - \frac{1}{n^2}\frac{1}{4\pi n+\alpha}\sum_{i=1}^N \int \mathrm{d}\bar{k}^i \left|\int_{B_n(0)} \mathrm{d}k_i \hat{u}_n(k)\right|^2. \tag{4.170}
$$

Thus we see that $\liminf_{n\to\infty} F^n_\alpha(u_n/n) \geq \liminf_{n\to\infty}\left(-\frac{1}{n^2}\frac{1}{4\pi n+\alpha}\sum_{i=1}^N \int \mathrm{d}\bar{k}^i \left|\int_{B_n(0)} \mathrm{d}k_i\hat{u}_n(k)\right|^2\right)$. By Cauchy-Schwartz it follows that

$$
\liminf_{n\to\infty} F^n_\alpha(u_n/n) \geq -\liminf_{n\to\infty}\left(N\frac{4/3\pi n^3}{n^2(4\pi n+\alpha)}\left\|u_n\right\|_2^2\right) \geq -\frac{N}{3}\left\|u\right\|_2^2. \tag{4.171}
$$

By possibly passing to a subsequence we may assume that

$$
\lim_{n\to\infty} F^n_\alpha(u_n/n) = \liminf_{n\to\infty} F^n_\alpha(u_n/n). \tag{4.172}
$$

Thus we immediately see that either $\liminf_{n\to\infty} F^n_\alpha(u_n/n) = \infty$ or u_n/n is $H^1(\mathbb{R}^{3N})$ norm bounded. In the first case we are done. Therefore assume that $\|u_n/n\|_{H^1(\mathbb{R}^{3N})}$ is bounded. Then we know that there by possibly passing to a subsequence that $u_n/n \rightharpoonup \chi$ in $H_1(\mathbb{R}^{3N})$. But then $u_n/n \rightharpoonup \chi$ in $L^2(\mathbb{R}^{3N})$ but we already know that $u_n/n \rightharpoonup 0$ in $L^2(\mathbb{R}^{3N})$ so we conclude that $\chi = 0$. Now write $\hat{u} = \hat{w} + \hat{\rho}\hat{G}$ with $w \in H^1(\mathbb{R}^{3N})$ and $\hat{\rho} = \sum_{i=1}^N (-1)^{(i-1)}\xi(\bar{k}^i)$, and $\xi \in H^{1/2}(\mathbb{R}^{3(N-1)})$ then we may write $\hat{u}_n = \hat{w} + \hat{\zeta}^n\hat{G}$, where $\zeta^n \to \rho$ in $H^{-2}(\mathbb{R}^{3N})$. Then we have $\left(\hat{\zeta}^n\hat{G}/n\right)^{\vee} \rightharpoonup 0$ in $H^1(\mathbb{R}^{3N})$ or equivalently $\zeta^n/n \rightharpoonup 0$ in $H^{-1}(\mathbb{R}^{3N})$ then we have

$$F_\alpha^n(u^n/n) = \frac{1}{n^2}\int \mathrm{d}k\hat{G}(k)^{-1}\left|\hat{w}+(\hat{\zeta}^n-\hat{\rho}^n)\hat{G}\right|^2 - \mu\left\|u_n/n\right\|_2^2 + N\left(T^n_{\mathrm{diag}}(\xi^n)+T^n_{\mathrm{off}}(\xi^n)+\alpha\left\|\xi^n\right\|^2_{L^2(\mathbb{R}^{3(N-1)})}\right) \tag{4.173}$$

With $\xi^n(\bar{k}^N) = \frac{1}{n(4\pi n+\alpha)}\int_{B_n(0)}\mathrm{d}k_N\hat{u}_n(k) = \frac{1}{n(4\pi n+\alpha)}\int_{B_n(0)}\mathrm{d}k_N\left(\hat{w}(k)+\hat{G}(k)\zeta^n(k)\right)$, and $\rho^n(k) = \sum_{i=1}^N(-1)^{i+1}\mathbb{1}_{B_n(0)}(k_i)\xi^n(\bar{k}^i)$. NOT FINISHED □

4.2 Uniform lower bound in the $2+1$ case

We will in this section study the $2+1$ case. This will serve mostly as a transition from $1+1$ to $N+1$, as this case is simpler than $N+1$ but indeed already shows many of the difficulties of the $N+1$ case such as a mass-region of instability. The quadratic forms of interest are just special cases of the $N+1$ forms and can be summarized as follows: At finite radius of the rank-one perturbations the quadratic forms can be written as

$$F_\alpha^R(u) = \int \mathrm{d}k\hat{G}(k)^{-1}\left|\hat{u}(k)\right|^2 - \mu\left\|u\right\|_2^2 - \frac{1}{4\pi R+\alpha}\sum_{i=1}^2\int \mathrm{d}\bar{k}^i\left|\int_{B_R(0)}\mathrm{d}k_i\hat{u}(k)\right|^2, \tag{4.174}$$

with domain

$$\mathscr{D}\left(F_\alpha^R\right) = H^1(\mathbb{R}^6). \tag{4.175}$$

Equivalently they can be written

$$\begin{aligned} F_\alpha^R(u) = &\int_{\mathbb{R}^{3N}}\mathrm{d}k\hat{G}(k)^{-1}|\hat{u}(k)-\widehat{\rho^R G}(k)|^2 - \mu\left\|u\right\|_{L^2(\mathbb{R}^6)} \\ &+ 2\left(T^R_{\mathrm{diag}}(\xi^R)+T^R_{\mathrm{off}}(\xi^R)+\alpha\left\|\xi^R\right\|^2_{L^2(\mathbb{R}^3)}\right), \end{aligned} \tag{4.176}$$

where $\xi^R(k_1) = \frac{1}{4\pi R+\alpha}\int_{B_R(0)}\mathrm{d}k_2\hat{u}(k_1,k_2)$ and

$$\begin{aligned} T^R_{\mathrm{diag}}(\xi^R) &= \int \mathrm{d}k_1 L^R(k_1)\left|\xi^R(k_1)\right|^2, \\ T^R_{\mathrm{off}}(\xi^R) &= \int_{B_R(0)}\mathrm{d}s\int_{B_R(0)}\mathrm{d}t\overline{\xi^R(s)}\hat{G}(s,t)\xi^R(t), \\ L^R(k) &= 2\pi\left[\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}\left(\arctan\left(\frac{-\frac{|k|}{m+1}+R}{\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}}\right)+\arctan\left(\frac{\frac{|k|}{m+1}+R}{\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}}\right)\right)\right. \\ &\qquad \left. R+\frac{\left(m(m+2)\left(k^2+\mu\right)-k^2+\mu+(m+1)^2R^2\right)\ln\left(\frac{-\frac{2R|k|}{m+1}+k^2+\mu+R^2}{\frac{2R|k|}{m+1}+k^2+\mu+R^2}\right)}{4(m+1)\left|k\right|}\right], \\ \hat{G}(s,t) &= \left(s^2+t^2+\frac{2}{m+1}s\cdot t+\mu\right)^{-1}. \end{aligned} \tag{4.177}$$

The limit quadratic form can be written

$$F_\alpha(u) = \int_{\mathbb{R}^{3N}} \mathrm{d}k \hat{G}(k)^{-1} |\hat{w}(k)|^2 - \mu \|u\|_{L^2(\mathbb{R}^6)} + 2\left(T_{\mathrm{diag}}(\xi) + T_{\mathrm{off}}(\xi) + \alpha \|\xi\|^2_{L^2(\mathbb{R}^3)}\right), \tag{4.178}$$

with domain

$$\mathscr{D}(F_\alpha) = \left\{u \in L^2(\mathbb{R}^6) | \hat{u}(k) = \hat{w}(k) + \hat{G}(k)\left(\xi(k_1) - \xi(k_2)\right),\ w \in H^1(\mathbb{R}^6),\ \xi \in H^{1/2}(\mathbb{R}^3)\right\}, \tag{4.179}$$

and where

$$\begin{aligned} T_{\mathrm{diag}}(\xi) &= \int \mathrm{d}k_1 L(k_1) \left|\xi(k_1)\right|^2, \\ T_{\mathrm{off}}(\xi) &= \int_{\mathbb{R}^3} \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}t \overline{\xi(s)} \hat{G}(s,t) \xi(t), \\ L(k) &= 2\pi^2 \sqrt{\frac{k^2 m(m+2)}{(m+1)^2} + \mu}. \end{aligned} \tag{4.180}$$

We now show a uniform lower bound of F_α^R in the $2+1$ case for $m \geq 0.72$. One way to show boundedness from below, is through the result of Moser and Seiringer [8]. We know that for $\xi \in H^{1/2}(\mathbb{R}^3)$ we have $T_{\mathrm{off}} \geq -T_{\mathrm{diag}}(\xi)$ for $m \geq 0.36$. Since, $(\xi^R \mathbb{1}_{B_R(0)})^\vee \in H^1(\mathbb{R}^3) \subset H^{1/2}(\mathbb{R}^3)$ and $T^R_{\mathrm{off}}(\xi^R) = T_{\mathrm{off}}(\xi^R \mathbb{1}_{B_R(0)})$, the problem at finite $R > 0$ reduces to showing $T^R_{\mathrm{diag}}(\xi^R) - T_{\mathrm{diag}}(\xi^R \mathbb{1}_{B_R(0)}) \geq -M \|u\|_2^2$. A step on the way is provided by the lemma

Lemma 17. *Let $L(k)$ and $L^R(k)$ be defined as above, then $2L^R(k) \geq L(k)$ for $|k| \leq R$ and $R \geq (1+m)\sqrt{\mu}$.*

Proof. Notice first that by lemma 28 in appendix C, $L^R(k)/L(k)\big|_{R \geq (1+m)\sqrt{\mu}}$ is a non-increasing function of $|k|$. Therefore, we need only consider the case $|k| = R$. In this case we have

$$\begin{aligned} L^R(R) = 2\pi \Bigg(R &- \frac{\left(\mu + m(m+2)\left(\mu + 2R^2\right)\right) \ln\left(1 + \frac{4R^2}{(m+1)\mu + 2mR^2}\right)}{4(m+1)R} \\ &+ \sqrt{\mu + \frac{m(m+2)R^2}{(m+1)^2}} \Bigg[\arctan\left(\frac{mR}{(m+1)\sqrt{\mu + \frac{m(m+2)R^2}{(m+1)^2}}}\right) \\ &+ \arctan\left(\frac{(m+2)R}{(m+1)\sqrt{\mu + \frac{m(m+2)R^2}{(m+1)^2}}}\right)\Bigg]\Bigg), \\ L(R) = 2\pi^2 &\sqrt{\mu + \frac{m(m+2)R^2}{(m+1)^2}}. \end{aligned} \tag{4.181}$$

We then notice that $L^R(R)/L(R)\big|_{R \geq (1+m)\sqrt{\mu}}$ is an increasing function of R, which is verified by finding the derivative

$$\frac{\partial}{\partial R}(L^R(R)/L(R)) = \frac{\mu\left(\mu(m+1)\ln\left(\frac{\mu + \frac{2(m+2)R^2}{m+1}}{\mu + \frac{2mR^2}{m+1}}\right) + 4R^2\right)}{4\pi R^2\left(\mu + \frac{m(m+2)R^2}{(m+1)^2}\right)^{3/2}} \tag{4.182}$$

Thus we find that

$$
\begin{aligned}
L^R(k)/L(k)|_{R\geq(1+m)\sqrt{\mu}} \geq h(m,\mu) &:= L^{(1+m)\sqrt{\mu}}((1+m)\sqrt{\mu})/L((1+m)\sqrt{\mu}) \\
&= -\frac{\sqrt{\mu}\left(\left(2m^2+4m+1\right)\ln\left(\frac{\mu(2m^2+6m+5)}{\mu(2m^2+2m+1)}\right)+4(m+1)\right)}{4\pi\sqrt{\mu(m+1)^2}} \\
&\quad+\frac{1}{\pi}\left(\arctan\left(\frac{\sqrt{\mu}m}{\sqrt{\mu(m+1)^2}}\right)+\arctan\left(\frac{\sqrt{\mu}(m+2)}{\sqrt{\mu(m+1)^2}}\right)\right)
\end{aligned} \tag{4.183}
$$

This is a non-increasing function of m, as is verified by considering the derivative

$$
\frac{\partial}{\partial m}h(m,\mu) = \frac{\mu^{3/2}(m+1)\left(-\left(2m^2+4m+3\right)\ln\left(\frac{2m^2+6m+5}{2m^2+2m+1}\right)+4(m+1)\right)}{4\pi\left(\mu(m+1)^2\right)^{3/2}}. \tag{4.184}
$$

Using the inequality $\ln(1+x)\geq\frac{2x}{2+x}$ we find

$$
\frac{\partial}{\partial m}h(m,\mu) \leq \frac{\mu^{3/2}(m+1)\left(4(m+1)-\frac{2(4m+4)(2m(m+2)+3)}{(2m(m+1)+1)\left(\frac{4m+4}{2m(m+1)+1}+2\right)}\right)}{4\pi\left(\mu(m+1)^2\right)^{3/2}} = 0. \tag{4.185}
$$

Knowing that $L^R(k)/L(k)|_{R\geq(1+m)\sqrt{\mu}} \geq h(m,\mu)$ and that $h(m,\mu)$ is non-increasing, we need only consider the limit $L^R(k)/L(k)\big|_{R\geq(1+m)\sqrt{\mu}} \geq \lim_{m\to\infty}(h(m,\mu))=\frac{1}{2}$. ☐

Boundedness from below now follows in the mass region where $T_{\mathrm{off}} > -\frac{1}{2}T_{\mathrm{diag}}$. In [8] it was shown that $T_{\mathrm{off}} \geq -\Lambda(m)T_{\mathrm{diag}}$ for a complicated function $\Lambda(m)$. It was shown that stability holds whenever $\Lambda(m)<1$ which is true for $m\geq 0.36$. Furthermore, it was shown that $\Lambda_1(m)\geq 2\Lambda(m)$ for some function $\Lambda_1(m)$ which satisfies $\Lambda_1(m)<1$ for $m\geq 0.72$. Thus we conclude that $T_{\mathrm{off}} > -\frac{1}{2}T_{\mathrm{diag}}$ for $m\geq 0.72$.

Proposition 10. *Let F_α^R be defined as above and let $m\geq 0.72$. Then the sequence $(F_\alpha^n)_{n\geq(1+m)\sqrt{\mu}}$ is uniformly bounded from below.*

Proof. The proof is based on the result by Moser and Seiringer [8]. We notice that $T_{\mathrm{off}}^R(\xi^R) = T_{\mathrm{off}}(\xi^R \mathbb{1}_{B_R(0)})$ and that $T_{\mathrm{diag}}^R(\xi^R) \geq T_{\mathrm{diag}}^R(\xi^R\mathbb{1}_{B_R(0)})$. Thus, we have by lemma 17

$$
\begin{aligned}
F_\alpha^n(u) \geq& \int_{\mathbb{R}^{3N}} \mathrm{d}k\hat{G}(k)^{-1}|\hat{u}(k)-\widehat{\rho^n G}(k)|^2 - \mu\|u\|_{L^2(\mathbb{R}^6)} \\
&+2\Bigg(T_{\mathrm{diag}}^n(\xi^n\mathbb{1}_{B_n(0)^\complement}) + T_{\mathrm{diag}}^n(\xi^n\mathbb{1}_{B_n(0)}) - \frac{1}{2}T_{\mathrm{diag}}(\xi^n\mathbb{1}_{B_n(0)}) + \frac{1}{2}T_{\mathrm{diag}}(\xi^n\mathbb{1}_{B_n(0)}) \\
&\quad + T_{\mathrm{off}}(\xi^n\mathbb{1}_{B_n(0)}) + \alpha\|\xi^n\|_{L^2(\mathbb{R}^3)}^2\Bigg) \\
\geq& -\mu\|u\|_2^2 + 2\left(T_{\mathrm{diag}}^n(\xi^n\mathbb{1}_{B_n(0)^\complement}) + \frac{1}{2}T_{\mathrm{diag}}(\xi^n\mathbb{1}_{B_n(0)}) + T_{\mathrm{off}}(\xi^n\mathbb{1}_{B_n(0)}) + \alpha\|\xi^n\|_{L^2(\mathbb{R}^3)}^2\right).
\end{aligned} \tag{4.186}
$$

It is known that for $m \geq 0.72$ we have $\frac{1}{2}T_{\mathrm{diag}}(\xi^n \mathbb{1}_{B_n(0)})+T_{\mathrm{off}}(\xi^n \mathbb{1}_{B_n(0)}) \geq \pi^2(1-2\Lambda(m))\sqrt{\mu}\left\|\xi \mathbb{1}_{B_n(0)}\right\|_2^2 > 0$. On the other hand, since $L^R(k)$ is a concave function of R for $m \geq 1/3$, so we have

$$\begin{aligned} L^n(k) &\geq \frac{1}{k}L^k(k)n \\ &= 2\pi n\Bigg\{1+\frac{\left(m(m+2)\left(2k^2+\mu\right)+\mu\right)\ln\left(1-\frac{4k^2}{2k^2(m+2)+\mu+\mu m}\right)}{4k^2(m+1)}+\frac{\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}}{k} \\ &\quad\times\left[\arctan\left(\frac{km}{(m+1)\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}}\right)+\arctan\left(\frac{k(m+2)}{(m+1)\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}}\right)\right]\Bigg\} \end{aligned} \tag{4.187}$$

for $|k| \geq n$. Notice first that

$$\lim_{\mu\to\infty}\frac{1}{k}L^k(k)n = 4\pi n. \tag{4.188}$$

On the other hand we have

$$\begin{aligned} \lim_{\mu\to 0}\frac{1}{k}L^k(k)n =& \frac{\pi n}{m+1}\Bigg(2(m+1)-m(m+2)\ln\left(\frac{m+2}{m}\right) \\ &+2\sqrt{m}\sqrt{m+2}\left[\arctan\left(\frac{1}{\sqrt{\frac{m}{m+2}}}\right)+\arctan\left(\sqrt{\frac{m}{m+2}}\right)\right]\Bigg). \end{aligned} \tag{4.189}$$

Using Shafer's inequality $\arctan(x) \geq \frac{3x}{1+2\sqrt{1+x^2}}$, [9], and the logarithmic inequality $\ln(1+x) \leq \frac{x}{\sqrt{1+x}}$ we find

$$\lim_{\mu\to 0}\frac{1}{k}L^k(k)n \geq \frac{2\pi}{m+1}\left(\frac{3m}{2\sqrt{2}\sqrt{\frac{m+1}{m+2}}+1}+m+\frac{3(m+2)}{2\sqrt{\frac{2}{m}+2}+1}+\sqrt{m+2}\sqrt{m}+1\right)n \geq 2\pi n. \tag{4.190}$$

Thereby, we need only show that $\frac{1}{k}L^k(k)n \geq 2\pi n$ for any potential minimum in μ. Differentiating (4.187) w.r.t μ we find

$$\begin{aligned} \frac{\partial}{\partial\mu}\left(\frac{1}{k}L^k(k)n\right) &= \frac{\pi}{2k^2}\Bigg\{(m+1)\ln\left(1-\frac{4k^2}{2k^2(m+2)+\mu+\mu m}\right)+\frac{2k}{\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}} \\ &\quad\times\left[\arctan\left(\frac{km}{(m+1)\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}}\right)+\arctan\left(\frac{k(m+2)}{(m+1)\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}}\right)\right]\Bigg\}. \end{aligned} \tag{4.191}$$

Thus we find that at any potential minimum in μ, say at $\mu*$, we have

$$\begin{aligned} &\left[\arctan\left(\frac{km}{(m+1)\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu^*}}\right)+\arctan\left(\frac{k(m+2)}{(m+1)\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu^*}}\right)\right] = \\ &\qquad -(m+1)\frac{\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu^*}}{2k}\ln\left(1-\frac{4k^2}{2k^2(m+2)+\mu^*+\mu^*m}\right) \end{aligned} \tag{4.192}$$

Inserting this back into (4.187) we find

$$L^n(k)|_{\mu=\mu^*} \geq 2\pi n \left\{1 - \frac{\mu^*(m+1)}{4k^2} \ln\left(1 - \frac{4k^2}{2k^2(m+2)+\mu^*+\mu^* m}\right)\right\} \geq 2\pi n, \qquad (4.193)$$

for $|k| \geq n$. Now knowing that $\frac{1}{k}L^k(k)n \geq 2\pi n$ we easily see that

$$F_\alpha^n(u) \geq -\mu \|u\|_2^2 + 2\left(2\pi n \left\|\xi^n \mathbb{1}_{B_n(0)^\complement}\right\|_2^2 + \pi^2(1-2\Lambda(m))\sqrt{\mu}\left\|\xi \mathbb{1}_{B_n(0)}\right\|_2^2 + \alpha\|\xi\|_2^2\right). \quad (4.194)$$

And since $n \geq (1+m)\sqrt{\mu}$ we find

$$F_\alpha^n(u) \geq -\mu\|u\|_2^2 + 2\left(\pi^2(1-2\Lambda(m))\sqrt{\mu}\|\xi\|_2^2 + \alpha\|\xi\|_2^2\right), \qquad (4.195)$$

where we used that $2\pi(1+m) > \pi^2$ for $m \geq 0.72$. From this it follows, by choosing $\mu = -\alpha^2/(\pi^2(1-2\Lambda(m)))^2$ for $\alpha < 0$ and $\mu \to 0$ for $\alpha \geq 0$, that

$$F_\alpha^n(u) \geq \begin{cases} -\alpha^2/(\pi^2(1-2\Lambda(m)))^2 & \text{for } \alpha < 0, \\ 0 & \text{for } \alpha \geq 0. \end{cases} \qquad (4.196)$$

Since the bound is independent of n, we conclude that the sequence is uniformly bounded from below. □

Notice that the bound is not sharp, since the binding energy of a single δ-potential is $-\alpha^2/(2\pi^2)^2$ and $\Lambda(m) \to 0$ as $m \to \infty$. The following simple lemma will be useful below

Lemma 18. *Let $f_n : A \to \mathbb{C}$ be a sequence of functions. If $\Delta f_n : A \times A \to \mathbb{C}$ defined by $\Delta f_n : (x,y) \mapsto f_n(x) - f_n(y)$ converges pointwise to $h(x,y)$, then h is antisymmetric and furthermore, $h(x,y) = f(x) - f(y)$ for some function f.*

Proof. Notice first that $h(x,y) = f(x) - f(y)$ for some function f, if and only if $h(x,y) = h(x,0) - h(y,0)$. Now if $f_n(x) - f_n(y) \to h(x,y)$ as $n \to \infty$ it is obvious that h is antisymmetric. Furthermore, we have $h(x,y) = \lim_{n\to\infty}(f_n(x) - f_n(y)) = \lim_{n\to\infty}(f_n(x) - f(0) - f_n(y) + f(0)) = h(x,0) - h(y,0)$, which proves the claim. □

We now proceed to show the Γ-lower bound for the 2+1 case

Theorem 4. *Let $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$. Furthermore, let $m \geq 0.72$ then*

$$\liminf_{n\to\infty} F_\alpha^n(u_n) \geq F_\alpha(u). \qquad (4.197)$$

Proof. By possibly restricting to a subsequence we may assume $F_\alpha^n(u_n) \to \liminf_{n\to\infty} F_\alpha^n(u_n)$, as $n \to \infty$. Since $m \geq 0.72$ we know that F_α^n is uniformly bounded from below, and we know that $\|u_n\|_2$ is bounded by weak convergence and the uniform boundedness principle, we may conclude that either $\liminf_{n\to\infty} F_\alpha^n(u_n) = \infty$ or $u_n - \rho^n G$ is $H^1(\mathbb{R}^6)$ bounded where $\widehat{\rho^n G}(k) = (\mathbb{1}_{B_n(0)}(k_1)\xi^n(k_2) - \mathbb{1}_{B_n(0)}(k_2)\xi^n(k_1))\hat{G}(k)$. In the first case, the theorem in obvious.

Therefore, assume that $u_n - \rho^n G$ is $H^1(\mathbb{R}^6)$ bounded. By Alaoglu's theorem we infer, by possibly passing to a subsequence that $u_n - \rho^n G \rightharpoonup \chi$ for some $\chi \in H^1(\mathbb{R}^6)$. Since $H^1(\mathbb{R}^6) \subset L^2(\mathbb{R}^6)$ it is clear that $u_n - \rho^n G \rightharpoonup \chi$ in $L^2(\mathbb{R}^6)$. Since we already know that $u_n \to u$ in $L^2(\mathbb{R}^6)$ we conclude that $\rho^n \hat{G} \rightharpoonup u - \chi$ in $L^2(\mathbb{R}^6)$. Writing

$$\hat{u}(k) = \hat{w}(k) + (\xi(k_2) - \xi(k_1))\hat{G}(k) \tag{4.198}$$

for $\xi \in H^{1/2}(\mathbb{R}^3)$ and

$$\hat{u}_n(k) = \hat{w}(k) + (\xi(k_2) - \xi(k_1))\hat{g}_n(k) + \hat{h}_n(k), \tag{4.199}$$

with $\wedge\xi\hat{g}_n \to \wedge\xi\hat{G}$ and $h_n \to 0$ in $L^2(\mathbb{R}^6)$, where $\wedge\xi(k) := \xi(k_2) - \xi(k_1)$.
Convergence $\rho^n G \rightharpoonup \zeta$ in $L^2(\mathbb{R}^6)$, means that $\rho^n \rightharpoonup \eta$ in $H^{-2}(\mathbb{R}^6)$ such that $\rho^n G \rightharpoonup \eta G$ in $L^2(\mathbb{R}^6)$. Thus we have $u_n - \rho^n G \rightharpoonup w + ((\wedge\xi - \hat{\eta})\hat{G})^\vee = \chi \in H^1(\mathbb{R}^6)$.

Notice now that weak H^1-convergence of $u_n - \rho^n G$ gives strong L^2-convergence on all bounded open domains Ω, by the Rellich-Kondrachov theorem, since it states that $H^1(\Omega) \subset\subset L^2(\Omega)$, *i.e.* compact embedding. Thus for all balls of radius r, $\Omega_r = B_r(0)$, we have that $\rho^n G|_{\Omega_r} \to \zeta|_{\Omega_r}$ in $L^2(\Omega_r)$ (since weak convergence of x_n implies norm convergence of Tx_n for T compact, and $\mathbb{I}$ is compact operator from H^1 to L^2).
Thus, $(u_n - \rho^n G)\mathbb{1}_{\Omega_r}$ converges to $(w + ((\wedge\xi - \hat{\eta})\hat{G})^\vee)\mathbb{1}_{\Omega_r}$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$. By Cantors diagonalization argument (choosing the diagonal sequence) we may, by possibly passing to a subsequence, assume that $u_n - \rho^n G \to w + ((\wedge\xi - \hat{\eta})\hat{G})^\vee$ pointwise a.e.

Therefore we also have by continuity of the Fourier transform that

$$(\rho^n G\mathbb{1}_{\Omega_r})^\wedge \to (\zeta\mathbb{1}_{\Omega_r})^\wedge, \text{ as } n \to \infty. \tag{4.200}$$

Now for every $r \geq 1$ we may choose $n_r \geq 1$ satisfying $n_1 < n_2 < ...$, *i.e.* $\rho^{n_r} G$ forms a subsequence of $\rho^n G$, such that

$$\left\|(\rho^{n_r} G\mathbb{1}_{\Omega_r})^\wedge - (\zeta\mathbb{1}_{\Omega_r})^\wedge\right\| < \frac{1}{r}. \tag{4.201}$$

But then we have

$$\left\|\widehat{\rho^{n_r} G} - \hat{\zeta}\right\| \leq \left\|\widehat{\rho^{n_r} G} - (\rho^{n_r} G\mathbb{1}_{\Omega_r})^\wedge\right\| + \left\|(\rho^{n_r} G\mathbb{1}_{\Omega_r})^\wedge - (\zeta\mathbb{1}_{\Omega_r})^\wedge\right\| + \left\|(\zeta\mathbb{1}_{\Omega_r})^\wedge - \hat{\zeta}\right\|. \tag{4.202}$$

Looking at the term $\left\|\widehat{\rho^{n_r} G} - (\rho^{n_r} G\mathbb{1}_{\Omega_r})^\wedge\right\|$

$$\left|\widehat{\rho^n G}(k) - \zeta(k)\right| = \left|\widehat{\rho^n G}(k) - (\rho^n G\mathbb{1}_{\Omega_r})^\wedge(k) + (\rho^n G\mathbb{1}_{\Omega_r})^\wedge(k) + (\zeta\mathbb{1}_{\Omega_r})^\wedge(k) - (\zeta\mathbb{1}_{\Omega_r})^\wedge(k) - \hat{\zeta}(k)\right| \tag{4.203}$$

Thus, by possibly passing to a subsequence, we may assume that $(\rho^n G\mathbb{1}_{\Omega_r})^\wedge$ converges

pointwise to $(\zeta\mathbb{1}_{\Omega_r})^\wedge$, but since $(\zeta\mathbb{1}_{\Omega_r})^\wedge$ converges to $\hat{\zeta}$ in $L^2(\mathbb{R}^6)$, we may, by passing to a further subsequence, assume that $(\zeta\mathbb{1}_{\Omega_r})^\wedge$ converges pointwise to $\hat{\zeta}$ as $r\to\infty$. we see that...
Notice that

$$\lim_{r\to\infty}\lim_{n\to\infty}(\rho^n G\mathbb{1}_{\Omega_r})^\wedge=\widehat{\zeta}, \tag{4.204}$$

and on the other hand we have

$$\lim_{n\to\infty}\lim_{r\to\infty}(\rho^n G\mathbb{1}_{\Omega_r})^\wedge=\lim_{n\to\infty}\widehat{\rho^n G} \tag{4.205}$$

Notice that $\rho^n G\to\eta G$ in $L^2(\mathbb{R}^6)$ means that, by possibly passing to a subsequence, we may assume that $\hat{\rho}^n(k)\hat{G}(k)\to\hat{\eta}(k)\hat{G}(k)$ as $n\to\infty$, *i.e.* pointwise convergence. By lemma 18 we then know that $\hat{\eta}(k)=f(k_2)-f(k_1)$ for some function k. Now use that $(g(k_2)-g(k_1))\hat{G}\notin H^1(\mathbb{R}^6)$ for $g(k)\neq$ const.

Use that weak H^1 convergence gives pointwise convergence of a subsequence (and understand this statement).

□

4.3 Uniform lower bound in the $N+1$ case

We will now generalize the results from the previous subsection. We split T^n_{off} in two, namely

$$\begin{aligned}T^n_{\text{off}}(\xi^n)=&(N-1)\int_{B_n(0)^{3(N-2)}}\mathrm{d}\bar{q}\int_{B_n(0)}\mathrm{d}s\int_{B_n(0)}\mathrm{d}t\overline{\xi^n(s,\bar{q})}\hat{G}(s,t,\bar{q})\xi^n(t,\bar{q})\\&+(N-1)\int_{(B_n(0)^{3(N-2)})^\complement}\mathrm{d}\bar{q}\int_{B_n(0)}\mathrm{d}s\int_{B_n(0)}\mathrm{d}t\overline{\xi^n(s,\bar{q})}\hat{G}(s,t,\bar{q})\xi^n(t,\bar{q})\end{aligned} \tag{4.206}$$

NEW IDEA Try uniform boundedness, operators corresponding to F^n_α are bounded pointwise, so they are uniformly bounded and so are the quadratic forms???

The problem is analogous to the problem in the $2+1$ case, except for the substitution $k\to\left|\sum_{i=1}^{N-1}k_i\right|$ and $\mu\to\mu-$.

Use that $\xi^R\in H^1$ so we have better large q behavior than Moser and Seiringer, and remember that Moser and Seiringer use symmetry of ξ.
Try cutting down to balls of radius n/N.

Notice that

$$\begin{aligned}
&\left|\int_{(B_r(0))^\complement} \mathrm{d}s \int_{(B_r(0))^\complement} \mathrm{d}t \overline{\xi^n(s,\bar{q})}\hat{G}(s,t,\bar{q})\xi^n(t,\bar{q})\right| \\
&\leq A \int_{(B_r(0))^\complement} \mathrm{d}s \int_{(B_r(0))^\complement} \mathrm{d}t \frac{1}{\sqrt{s^2+1}} \frac{1}{s^2+t^2+1} \frac{1}{\sqrt{t^2+1}} \left|\overline{f^n(s,\bar{q})}\right| |f^n(t,\bar{q})| \\
&\leq A \int_{(B_r(0))^\complement} \mathrm{d}s \int_{(B_r(0))^\complement} \mathrm{d}t \frac{1}{s^2+1} \frac{1}{t^2+1} |f^n(s,\bar{q})| |f^n(t,\bar{q})| \\
&\leq A \left(\int_{(B_r(0))^\complement} \mathrm{d}s \frac{1}{(s^2+1)^2}\right)^{1/2} \left(\int_{(B_r(0))^\complement} \mathrm{d}t \frac{1}{(t^2+1)^2}\right)^{1/2} \int \mathrm{d}u\, |f^n(u,\bar{q})|^2 \\
&\leq 4\pi A \frac{1}{r} |f^n(u,\bar{q})|^2
\end{aligned} \tag{4.207}$$

Lemma 19. *Let $(\xi^n)^\vee \in H^1(\mathbb{R}^{3(N-1)})$ then,*

$$\int_{(B_n(0)^{3(N-2)})^\complement} \mathrm{d}\bar{q} \int_{B_n(0)} \mathrm{d}s \int_{B_n(0)} \mathrm{d}t \overline{\xi^n(s,\bar{q})}\hat{G}(s,t,\bar{q})\xi^n(t,\bar{q}) \leq C\ln(n) \|\xi^n\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})}. \tag{4.208}$$

Proof. Notice first that $\xi^n(t,\bar{q}) = \frac{1}{(t^2+\bar{q}^2+1)^{1/4}} f^n(t,\bar{q})$ for some $f^n \in L^2(\mathbb{R}^{3(N-1)})$ such that $\|f^n\|_2 = \|(\xi^n)^\vee\|_{H^{1/2}(\mathbb{R}^{3(N-1)})}$, and notice then that

$$\begin{aligned}
&\left|\int_{B_n(0)} \mathrm{d}s \int_{B_n(0)} \mathrm{d}t \overline{\xi^n(s,\bar{q})}\hat{G}(s,t,\bar{q})\xi^n(t,\bar{q})\right| \\
&\leq A \int_{B_n(0)} \mathrm{d}s \int_{B_n(0)} \mathrm{d}t \frac{1}{(s^2+1)^{1/4}} \frac{1}{s^2+t^2+1} \frac{1}{(t^2+1)^{1/4}} \left|\overline{f^n(s,\bar{q})}\right| |f^n(t,\bar{q})| \\
&\leq A \int_{B_n(0)} \mathrm{d}s \int_{B_n(0)} \mathrm{d}t \frac{1}{(s^2+1)^{3/4}} \frac{1}{(t^2+1)^{3/4}} |f^n(s,\bar{q})| |f^n(t,\bar{q})| \\
&\leq A \left(\int_{B_n(0)} \mathrm{d}s \frac{1}{(s^2+1)^{3/2}}\right)^{1/2} \left(\int_{B_n(0)} \mathrm{d}t \frac{1}{(t^2+1)^{3/2}}\right)^{1/2} \int \mathrm{d}u\, |f^n(u,\bar{q})|^2 \\
&= 4\pi A \left(\ln\left(n+\sqrt{1+n^2}\right) - \frac{n}{\sqrt{1+n^2}}\right) \|f^n(\cdot,\bar{q})\|_2^2 \\
&\leq 4\pi A \ln(n) \left\|(\xi^n)^\vee\right\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})},
\end{aligned} \tag{4.209}$$

where $A>0$ is some constant. Here we used (4.81) in line two, and Cauchy-Schwartz in line four. ∎

Lemma 20. *Let L^R_{N-1} be defined as above, where N simply denotes the number of particles. Now fix $k_1,...,k_M$ for $M<N-1$, then $L^R_{N-1}(k_1,...,k_{N-1}) \geq L^R_M(k_1,...,k_M)$.*

Proof. Notice first that $\alpha_{N-1} = \sum_{i=1}^{N-1} k_i^2 + \frac{2}{m+1}\sum_{1\leq i<j\leq N-1} k_i\cdot k_j + \mu$ and $y_{N-1} = \left|\frac{2}{m+1}\sum_{=1}^{N-1} k_i\right|$, thus we may write

$$\alpha_M = \frac{(m+1)}{4} y_M^2 + \frac{m}{m+1}\sum_{i=1}^{M} k_i^2 + \mu := \frac{(m+1)}{4} y_M^2 + r_M. \tag{4.210}$$

Clearly $r_M := \frac{m}{m+1}\sum_{i=1}^M k_i^2 + \mu$ is an increasing function of M. Now we simply show that $L^R(\alpha_M, y_M) = L^R\left(\frac{(m+1)}{4}y_M^2 + r_M, y_M\right)$ is an increasing function of r_M. Consider therefore, the derivative

$$
\begin{aligned}
&\frac{\partial}{\partial r}L^R\left(\frac{(m+1)}{4}y^2 + r, y\right)\\
&= \pi\left(-\frac{\ln\left(\frac{8Ry}{(m+1)y^2+4(r+R^2)-4Ry}+1\right)}{y} - \frac{2\left(\arctan\left(\frac{y-2R}{\sqrt{my^2+4r}}\right) - \arctan\left(\frac{2R+y}{\sqrt{my^2+4r}}\right)\right)}{\sqrt{my^2+4r}}\right)
\end{aligned}
\tag{4.211}
$$

To show that $\frac{\partial}{\partial r}L^R\left(\frac{(m+1)}{4}y^2 + r, y\right) \geq 0$ notice that

$$
\frac{\partial}{\partial R}\frac{\partial}{\partial r}L^R\left(\frac{(m+1)}{4}y^2 + r, y\right) = \frac{64\pi R^2}{8y^2\left((m+1)r + (m-1)R^2\right) + (m+1)^2y^4 + 16\left(r+R^2\right)^2} \geq 0.
\tag{4.212}
$$

Thus we have $\frac{\partial}{\partial r}L^R\left(\frac{(m+1)}{4}y^2 + r, y\right) \geq \lim_{R\to 0}\frac{\partial}{\partial r}L^R\left(\frac{(m+1)}{4}y^2 + r, y\right) = 0$. This proves that $L^R\left(\frac{(m+1)}{4}y_M^2 + r_M, y_M\right)$ is an increasing function of r_M. Now imagine adding one more particle. In this case y may change according to $y \to y + x$, $x \in [-y, \infty)$ and r also change according to $r \to \tilde{r} \geq r + \frac{m(m+1)}{4}x^2$. DOES NOT WORK (LEMMA NOT TRUE) $\square$

Proposition 11. *Let $A_n : H^1(\mathbb{R}^{3N}) \to H^{1/2}(\mathbb{R}^{3(N-1)})$, be defined by*

$$
A_n : u \mapsto \xi^n = \frac{1}{4\pi n + \alpha}\int_{B_n(0)} \mathrm{d}k^N u(\cdot, k^N),
\tag{4.213}
$$

then $\sup_{n\geq 1}\|A_n\| < \infty$ and we have that $\|\xi^n\|_{H^{1/2}(\mathbb{R}^{3(N-1)})} \leq C\|u\|_{H^1(\mathbb{R}^{3N})}$, for some $C > 0$ and all $n \geq 1$.

Proof. By the proof of lemma 6 and by lemma 8, we know that A_n are all bounded, and that $\sup_{n\geq 1}\|A_n u\| < \infty$, it then follows from the uniform boundedness principle that $\sup_{n\geq 1}\|A_n\| < \infty$, which proves the desired result. $\square$

Lemma 21. *Let $N > 2$ and let $\alpha = \sum_{i=1}^{N-1} k_i^2 + \frac{2}{m+1}\sum_{1\leq i<j\leq N-1} k_i \cdot k_j + \mu$, $y = \frac{2}{m+1}\left|\sum_{i=1}^{N-1} k_i\right|$ and $\mu > 0$ then $2L^R(y,\alpha)/L(y,\alpha) \geq 1$ for $|k_i| \leq R/(N-1)$ and $R \geq \frac{1+m}{m}\frac{N-1}{N-2}\sqrt{\mu}$, or for $\alpha < R^2$.*

Proof. We start by noticing that $L^R(y,\alpha)/L(y,\alpha)$ is an non-increasing function of α, which is verified by finding the derivative

$$
\frac{\partial}{\partial\alpha}\left(\frac{L^R(y,\alpha)}{L(y,\alpha)}\right) = -\frac{4Ry - 2\left(R^2-\alpha\right)\ln\left(1 + \frac{4Ry}{\alpha+R^2-Ry}\right)}{\pi y\left(4\alpha - y^2\right)^{3/2}},
\tag{4.214}
$$

Notice that $R^2 \geq \alpha$, as the maximal value α can take is

$$
\alpha = R^2 - \frac{m}{1+m}\frac{(N-1)(N-2)}{(N-1)^2}R^2 + \mu = \left(1 - \frac{m}{1+m}\frac{N-2}{N-1}\right)R^2 + \mu \leq R^2.
\tag{4.215}
$$

Differentiating once again we find

$$\frac{\partial^2}{\partial \alpha^2}\left(\frac{L^R(y,\alpha)}{L(y,\alpha)}\right)=\frac{2\left(2\alpha-6R^2+y^2\right)\ln\left(\frac{4Ry}{\alpha+R^2-Ry}+1\right)+\frac{8Ry\left(3R^4+6R^3y+2R^2\left(\alpha-4y^2\right)+6\alpha Ry+\alpha\left(7\alpha-y^2\right)\right)}{(\alpha+R^2-Ry)(\alpha+R^2+3Ry)}}{\pi y\left(4\alpha-y^2\right)^{5/2}} \tag{4.216}$$

thus we see that assuming $\frac{\partial^2}{\partial \alpha^2}\left(\frac{L^R(y,\alpha)}{L(y,\alpha)}\right)|_{\alpha=\alpha^*}=0$ we have

$$\ln\left(\frac{4Ry}{\alpha^*+R^2-Ry}+1\right)=-\frac{\frac{8Ry\left(3R^4+6R^3y+2R^2\left(\alpha^*-4y^2\right)+6\alpha^* Ry+\alpha^*\left(7\alpha^*-y^2\right)\right)}{(\alpha^*+R^2-Ry)(\alpha^*+R^2+3Ry)}}{2\left(2\alpha^*-6R^2+y^2\right)} \tag{4.217}$$

inserting back into $\frac{\partial}{\partial \alpha}\left(\frac{L^R(y,\alpha)}{L(y,\alpha)}\right)$ we find

$$\frac{\partial}{\partial \alpha}\left(\frac{L^R(y,\alpha)}{L(y,\alpha)}\right)|_{\alpha=\alpha^*}=-\frac{4R\left(3(\alpha^*)^2+R^2\left(3R^2+2Ry-3y^2\right)+2\alpha^*R(y-R)\right)}{\pi\sqrt{4\alpha^*-y^2}\left(-2\alpha^*+6R^2-y^2\right)\left(\alpha^*+R^2-Ry\right)\left(\alpha^*+R^2+3Ry\right)}<0. \tag{4.218}$$

and since $\frac{\partial}{\partial \alpha}\left(\frac{L^R(y,\alpha)}{L(y,\alpha)}\right)<0$ for $\alpha\to\mu$ (which implies $y\to 0$) and for $\alpha\to R^2$ we conclude that $\frac{\partial}{\partial \alpha}\left(\frac{L^R(y,\alpha)}{L(y,\alpha)}\right)<0$ for any $\mu\leq\alpha\leq R^2$ and we have

$$\frac{L^R(y,\alpha)}{L(y,\alpha)}\geq\frac{L^R(y,R^2)}{L(y,R^2)}. \tag{4.219}$$

Now computing $\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)$ we find

$$\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)=\frac{2R\left(R\left(4R^2-y^2\right)\ln\left(\frac{2y}{2R-y}+1\right)-4R^2y+2y^3\right)}{\pi y^2\left(4R^2-y^2\right)^{3/2}}. \tag{4.220}$$

Using that $\ln(1+x)\geq\frac{x}{\sqrt{1+x}}$ we get

$$\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)\geq\frac{4R\left(R\sqrt{4R^2-y^2}-2R^2+y^2\right)}{\pi y\left(4R^2-y^2\right)^{3/2}} \tag{4.221}$$

now using the reverse triangle inequality $\sqrt{a^2-b^2}\geq||a|-|b||$, we find

$$\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)\geq\frac{4R\left(R(2R-y)-2R^2+y^2\right)}{\pi y\left(4R^2-y^2\right)^{3/2}}=\frac{4R\left(-Ry+y^2\right)}{\pi y\left(4R^2-y^2\right)^{3/2}}, \tag{4.222}$$

so for $y\geq R$ we clearly have $\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)\geq 0$. For $y<R$ consider instead

$$\frac{\partial^2}{\partial y^2}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)=\frac{4R\left(8R^4-4R^2y^2+3Ry^2\sqrt{4R^2-y^2}-4R^3\sqrt{4R^2-y^2}+2y^4\right)}{\pi y^2\left(4R^2-y^2\right)^{5/2}}. \tag{4.223}$$

at any extremum of $\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)$, say y^*, where $\frac{\partial^2}{\partial y^2}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)|_{y=y^*}=0$ we thus conclude that

$$\sqrt{4R^2-(y^*)^2}=\frac{2\left(4R^4-2R^2(y^*)^2+(y^*)^4\right)}{R\left(4R^2-3(y^*)^2\right)},\quad y^*\neq\frac{2}{\sqrt{3}}R. \tag{4.224}$$

Thus we see that

$$\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)|_{y=y^*}=\frac{4Ry^*\left(6R^2-(y^*)^2\right)}{\pi\left(4R^2-3(y^*)^2\right)\left(4R^2-(y^*)^2\right)^{3/2}}, \tag{4.225}$$

from which it is clear that $\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)|_{y=y^*}\geq 0$ for $y^*<R$. Combining this with the fact that $\lim_{y\to 0}\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)=0$ and that $\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)\geq 0$ for $y\geq R$ we conclude that $\frac{\partial}{\partial y}\left(\frac{L^R(y,R^2)}{L(y,R^2)}\right)|_{y=y^*}\geq 0$ for all $y<2R$. Thus we have

$$\frac{L^R(y,\alpha)}{L(y,\alpha)}\geq\frac{L^R(y,R^2)}{L(y,R^2)}\geq\lim_{y\to 0}\frac{L^R(y,R^2)}{L(y,R^2)}=1/2 \tag{4.226}$$

which concludes the proof. □

NEW IDEA (USE UNIRFORM BOUNDEDNESS PRINCIPLE):

Lemma 22. *Let $T^n_{diag/off}$ be defined as above. Then for each $n\geq 1$, we have that $T^n_{diag}+T^n_{off}$ is closed/closable and bounded from below.*

Proof. Needs proof □

Thus to $T^n_{\text{diag}}+T^n_{\text{diag}}$ we may assign an operator Γ^n.

Lemma 23. *Let Γ^n be defined as above. Then for each $n\geq 1$ the negative part of Γ^n, denoted by Γ^n_-, is a bounded operator.*

Proof. This follows from the fact that for each $n\geq 1$, Γ^n is bounded from below and the spectral theorem. □

Lemma 24. *Let Γ^n_- be the negative part of Γ^n defined above. If Γ^n is uniformly bounded from below, then $\Gamma^n_- u\rightharpoonup 0$ as $n\to\infty$, for all $u\in L^2(\mathbb{R}^{3N})$.*

Proof. Since Γ^n is uniformly bounded from below, we know that Γ^n_- (the negative part of Γ^n) is uniformly bounded. Now if $u\in H^1(\mathbb{R}^{3N})$, we know that as a quadratic form we have $|\langle u,\Gamma^n u\rangle|\leq C\left\|\xi^n\right\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})}$ for some $C>0$ independent of u, where $\xi^n(\bar{k}^N)=\frac{1}{4\pi n+\alpha}\int_{B_n(0)}\mathrm{d}k_N\hat{u}(k)$. Furthermore, by lemma 8, it is known that $\|\xi^n\|_{H^{1/2}(\mathbb{R}^{3(N-1)})}\to 0$ as $n\to\infty$. Thus we see that as a sesquiliniear form we have $\langle v,\Gamma^n u\rangle\to 0$ for all $u,v\in H^1(\mathbb{R}^{3N})$, since $\langle u+iv,\Gamma^n(u+iv)\rangle\to 0$ and $\langle u+v,\Gamma^n(u+v)\rangle\to 0$. From this we conclude that $\left\langle v,\Gamma^n_- u\right\rangle\to 0$ for all $v,u\in L^2(\mathbb{R}^{3N})$. So $\Gamma^n_- u\rightharpoonup 0$ as $n\to\infty$ since $H^1(\mathbb{R}^{3N})$ is dense in $L^2(\mathbb{R}^{3N})$ and $\left\|\Gamma^n_-\right\|$ is uniformly bounded. Hence $\Gamma^n_- u\rightharpoonup 0$ for all $u\in L^2(\mathbb{R}^{3N})$. □

Lemma 25. *If $\Gamma_-^n u \rightharpoonup \chi_u$ for all $u \in L^2(\mathbb{R}^{3N})$, then Γ_-^n is uniformly bounded. In particular, is Γ^n uniformly bounded from below and $\chi_u = 0$.*

Proof. By the uniform boundedness principle we know that $\|\Gamma_-^n u\|$ is bounded if $\Gamma_-^n u \rightharpoonup \chi_u$. Thus by the uniform boundedness principle we again know that $\sup_{n\geq 1}\|\Gamma_-^n\| < \infty$. It clearly follows that Γ^n is uniformly bounded from below, and by lemma 24 it follows that $\chi_u = 0$. □

Proposition 12. *Γ^n is uniformly bounded from below if and only if $\Gamma_-^n u \rightharpoonup 0$ for all $u \in L^2(\mathbb{R}^{3N})$.*

Proof. This follows from lemma 24 and 25. □

Lemma 26. *Let Γ_-^n be the negative part of Γ^n defined above. If as a sesquilinear form we have $\langle v, \Gamma^n u\rangle \to 0$ as $n \to \infty$ for all $v, u \in H^1(\mathbb{R}^{3N})$, then $\langle v, \Gamma_-^n u\rangle \to 0$ as $n \to \infty$.*

Proof. Notice that $\operatorname{Ran}(\Gamma_-^n) \subset \ker\Gamma_+^n$, and thus $\overline{\operatorname{Ran}(\Gamma_-^n)} \subset \ker\Gamma_+^n$. Now decompose $u = u_1 + u_2$ with $u_1 \in \overline{\operatorname{Ran}(\Gamma_-^n)}$ and $u_2 \in \ker\Gamma_-^n$, so $\Gamma_-^n u = \Gamma_-^n u_1$, furthermore, any $v \in L^2(\mathbb{R}^{3N})$ may be decomposed similarly $v = v_1 + v_2$ with $v_1 \in \overline{\operatorname{Ran}(\Gamma_-^n)}$ and $v_2 \in \ker\Gamma_-^n$. Thus $\langle v, \Gamma_-^n u\rangle = \langle v_1, \Gamma_-^n u_1\rangle = -\langle v_1, \Gamma^n u\rangle \to 0$ as $n \to \infty$. So $\Gamma_-^n u \rightharpoonup 0$ as $n \to \infty$. check this proof w.r.t domains etc □

From proposition 12 we see that uniform lower boundedness of Γ^n is equivalent to showing pointwise weak convergence of Γ_-^n (which will in fact automatically be pointwise weak convergence to 0 then).

IDEA: Maybe, F_α^n need only be uniformly bounded from below on $\mathscr{D}(F_\alpha) \subset L^2(\mathbb{R}^{3N})$ to show boundedness from below of F_α. Then maybe use the above classification of proposition 12, if it generalizes to subsets of L^2.

4.4 Rank-one perturbation

We analyze in this subsection the simpler problem of perturbing the free one-body Hamiltonian by a rank one perturbation, like the one used above, in order to understand better the sequence F_α^n. This corresponds to removing the term from the Hamiltonian given by $\frac{-2}{m+1}\sum_{i<j}^N \nabla_i \cdot \nabla_j$, rendering the Hamiltonian non-interacting, *i.e.* of the form $H_\alpha^n = \oplus_{i=1}^N H_i$, with $H_i = -\Delta - \gamma_\alpha \langle \mathbb{1}_{B_n}, \hat{\cdot}\rangle \mathbb{1}_{B_n}^\vee$. It is our hope that fully analysing this special case ($m \to \infty$) will help solve the more general problem, *e.g.* by perturbation theory.

To begin with, we notice that we need only analyze the one-body Hamiltoninan. Furthermore, we notice that one may split the degrees of freedom, such that only only a certain subspace of the Hilbert space ($L^2(\mathbb{R}^3)$) is affected by the rank-one perturbation. This is done in the following way. We may factor $L^2(\hat{\mathbb{R}}^3)$ (momentum space) according to a angular momentum decomposition

$$L^2(\hat{\mathbb{R}}^3) = L^2(\mathbb{R}_+, r^2\,\mathrm{d}r) \otimes L^2(S^2, \mathrm{d}\Omega) = \oplus_{l=0}^\infty L^2(\mathbb{R}_+, r^2\,\mathrm{d}r) \otimes \left\langle (Y_l^m)_{m=-l}^l \right\rangle := \oplus_{l=0}^\infty \mathcal{H}_l, \quad (4.227)$$

where Y_l^m are the spherical harmonics. Since Y_0^0 is the constant function, it is clear that $\langle \mathbb{1}_{B_n}, \hat{\cdot} \rangle \mathbb{1}_{B_n}^\vee$ acts the zero operator on $\mathcal{H}_l$ for all $l > 0$. Thus $H_\alpha|_{\mathcal{H}_l} = k^2 = M_{r^2}$ for $l > 0$. On the other hand, for $l = 0$ the rank-one perturbation cannot be ignored. In this case, we study the generalized eigenvalue equation of the radial function (since the angular part is unique, the constant function). In the following $k \in \mathbb{R}$ denoted a scalar (the magnitude of the momentum), even though we usually denote the three-momentum by k. The generalized eigenvalue equation for $l = 0$ reads

$$k^2 \hat{f}(k) - 4\pi\gamma_\alpha \int_0^n \hat{f}(r) r^2 \, \mathrm{d}r \mathbb{1}_{[0,n)}(k) = \lambda \hat{f}(k), \tag{4.228}$$

where the 4π comes from the normalization of $\mathbb{1}_{B_n} = \sqrt{4\pi}\mathbb{1}_{[0,n)}(k) \otimes Y_0^0$. For $\lambda > n^2$ this equation may be solved by the standard solutions of the generalized eigenvalue problem for the free Hamiltonian. For $\lambda < n^2$ on the other hand, these solutions are no longer sufficient. In order to solve the generalized eigenvalue equation for $\lambda < n$, we consider first the second term in the equation, we may define

$$-4\pi\gamma_\alpha \int_0^n \hat{f}(r) r^2 \, \mathrm{d}r := A. \tag{4.229}$$

Substituting this back into (4.228) we find

$$(k^2 - \lambda)\hat{f}(k) = -A\mathbb{1}_{[0,n)}, \qquad k < n. \tag{4.230}$$

and

$$(k^2 - \lambda)\hat{f}(k) = 0, \qquad k \geq n. \tag{4.231}$$

We seek distributional solutions to this equation. The homogenous solutions are the usual free solutions $\hat{f}(k) = \delta(k^2 - \lambda) = \frac{1}{2k}\delta(k - \sqrt{\lambda})$ (where we discarded $\frac{1}{2k}\delta(k + \sqrt{\lambda}) = 0$ on $\mathbb{R}_+$). The particular solution on the other hand, may be found

$$\hat{f}(k) = \mathcal{P}\frac{-A\mathbb{1}_{[0,n)}(k)}{k^2 - \lambda}, \tag{4.232}$$

where $\mathcal{P}$ denotes the Cauchy principal value. Thus in order to have consistency we need only that the solution satisfies (4.229). Notice that

$$\hat{f}(k) = \mathcal{P}\frac{-A}{k^2 - \lambda}\mathbb{1}_{[0,n)}(k) + \frac{B}{\sqrt{\lambda}}\delta(k - \sqrt{\lambda}), \tag{4.233}$$

By adjusting B appropriately is satisfies (4.229). Thus by fixing B such that (4.229) is satisfied, $(\hat{f}, \lambda)$ is the solution to the generalized eigenvalue problem for this Hamiltonian. Thus we calculate

$$\begin{aligned} A &= -4\pi\gamma_\alpha \int_0^n \mathrm{d}r r^2 \left(\mathcal{P}\frac{-A}{r^2 - \lambda} + \frac{B}{\sqrt{\lambda}}\delta(r - \sqrt{\lambda}) \right) \\ &= -\frac{1}{n + \frac{\alpha}{4\pi}} \left(-A \left[n - \sqrt{\lambda}\operatorname{arctanh}\left(\frac{\sqrt{\lambda}}{n} \right) \right] + \sqrt{\lambda}B \right) \end{aligned} \tag{4.234}$$

Thus setting $B = -A\left(\operatorname{arctanh}\left(\frac{\sqrt{\lambda}}{n}\right) + \frac{\alpha}{4\pi\sqrt{\lambda}}\right)$, we have consistency. Thereby we conclude that

$$\hat{f}_\lambda(k) = -A\left(\mathcal{P}\frac{1}{k^2-\lambda}\mathbb{1}_{[0,n)}(k) + \frac{\left(\operatorname{arctanh}\left(\frac{\sqrt{\lambda}}{n}\right) + \frac{\alpha}{4\pi\sqrt{\lambda}}\right)}{\sqrt{\lambda}}\delta(k-\sqrt{\lambda})\right), \quad \lambda < n^2, \tag{4.235}$$

satisfies the generalized eigenvalue equation (4.228). Here A need to be determined such that $\hat{f}_\lambda$ is properly normalized. That $\hat{f}_\lambda$ and $\hat{f}_\mu$ are orthogonal for $\lambda \neq \mu$ is seen by the following computation

$$\mathcal{P}\int_0^n \frac{1}{k^2-\lambda}\frac{1}{k^2-\mu}k^2\,\mathrm{d}k = \frac{\sqrt{\mu}\operatorname{arctanh}\left(\frac{\sqrt{\mu}}{n}\right) - \sqrt{\lambda}\operatorname{arctanh}\left(\frac{\sqrt{\lambda}}{n}\right)}{\lambda-\mu}, \tag{4.236}$$

and

$$\int_0^n \frac{\left(\operatorname{arctanh}\left(\frac{\sqrt{\lambda}}{n}\right) + \frac{\alpha}{4\pi\sqrt{\lambda}}\right)}{\sqrt{\lambda}}\delta(k-\sqrt{\lambda})\frac{1}{k^2-\mu}k^2\,\mathrm{d}k = \frac{\left(\sqrt{\lambda}\operatorname{arctanh}\left(\frac{\sqrt{\lambda}}{n}\right) + \frac{\alpha}{4\pi}\right)}{\lambda-\mu}, \tag{4.237}$$

$$\int_0^n \frac{\left(\operatorname{arctanh}\left(\frac{\sqrt{\mu}}{n}\right) + \frac{\alpha}{4\pi\sqrt{\mu}}\right)}{\sqrt{\mu}}\delta(k-\sqrt{\mu})\frac{1}{k^2-\lambda}k^2\,\mathrm{d}k = -\frac{\left(\sqrt{\mu}\operatorname{arctanh}\left(\frac{\sqrt{\mu}}{n}\right) + \frac{\alpha}{4\pi}\right)}{\lambda-\mu}, \tag{4.238}$$

Collecting there results, it is easily verified that

$$\int_0^\infty \overline{\hat{f}_\lambda}(k)\hat{f}_\mu(k)k^2\,\mathrm{d}k = 0, \quad \lambda \neq \mu. \tag{4.239}$$

A similar calculation yields

$$\mathcal{P}\int_0^n \left(\frac{1}{k^2-\lambda}\right)^2 k^2\,\mathrm{d}k = \frac{(n^2-\lambda)\operatorname{arctanh}\left(\frac{n}{\sqrt{\lambda}}\right) + \sqrt{\lambda}n}{2\sqrt{\lambda}\,(\lambda-n^2)}, \tag{4.240}$$

Since $(f_\lambda)_{\lambda\in[0,n)}$ are supported in $[0,n)$ and $(f_\lambda)_{\lambda\geq n}$ are supported in $[n,\infty)$, we expect that $(f_\lambda)_{\lambda\in[0,n)}$ span all L^2-functions that are supported in $[0,n)$. To see that notice that for $g \in C^\infty([0,n))$ (notice also that that A can depend on λ, $A = A_\lambda$)

$$\begin{aligned}
&\int_0^n f_\lambda(k')\int_0^n f_\lambda(k)g(k)k^2\,\mathrm{d}k\frac{\sqrt{\lambda}}{2}\,\mathrm{d}\lambda \\
&= \int_0^n A_\lambda^2 f_\lambda(k')\left(\mathcal{P}\int_0^n \frac{g(k)k^2}{k^2-\lambda}\,\mathrm{d}k + \left(\sqrt{\lambda}\operatorname{arctanh}\left(\frac{\sqrt{\lambda}}{n}\right) + \frac{\alpha}{4\pi}\right)g(\sqrt{\lambda})\right)\frac{\sqrt{\lambda}}{2}\,\mathrm{d}\lambda \\
&= \mathbb{1}_{[0,n)}(k')\Bigg(\mathcal{P}\int_0^n \mathcal{P}\int_0^n \frac{A_{\mu^2}^2\mu^2k^2g(k)}{(k'^2-\mu^2)(k^2-\mu^2)}\,\mathrm{d}k\,\mathrm{d}\mu \\
&\quad + \mathcal{P}\int_0^n A_{\mu^2}^2\frac{g(\mu)\mu^2}{k'^2-\mu^2}\left(\mu\operatorname{arctanh}(\mu/n) + \frac{\alpha}{4\pi}\right)\mathrm{d}\mu \\
&+ A_{k'^2}^2\left(k'\operatorname{arctanh}(k'/n) + \frac{\alpha}{4\pi}\right)k'^2\left(\mathcal{P}\int_0^n \frac{k^2g(k)}{(k^2-k'^2)}\,\mathrm{d}k + \left(k'\operatorname{arctanh}(k'/n) + \frac{\alpha}{4\pi}\right)g(k')\right)\Bigg),
\end{aligned} \tag{4.241}$$

where we changed coordinates $\lambda = \mu^2$. Now it is non-trivial that the integrations in the first term can be interchanged. But using the Sokhotski–Plemelj theorem we see that

$$\mathcal{P}\int_0^n \mathcal{P}\int_0^n \frac{\mu^2 k^2 g(k)}{(k'^2-\mu^2)(k^2-\mu^2)}\,\mathrm{d}k\,\mathrm{d}\mu = ... \tag{4.242}$$

We notice that the principal value $\mathcal{P}(\frac{1}{k^2-\mu^2})$ is anti-symmetric in k and μ. Thus in order to have the correct normalization, *i.e.* $\int f_{\mu^2}(k')f_{\mu^2}(k)\,\mathrm{d}\Omega_\mu = \frac{\delta(k-k')}{k}$, where Ω_μ denotes the measure on the spectral parameter μ, and $\int f_{\mu^2}(k)f_{\nu^2}(k)k^2\,\mathrm{d}k = \delta_{\Omega_\mu}(\mu-\nu)$, where δ_{Ω_μ} is defined such that $\int \delta_{\Omega_\mu}(\mu)\,\mathrm{d}\Omega_\mu = 1$ (Think about these normalizations, what about the free case.), we expect a normalization constant A_μ that depends on μ. One example of this is to consider the constant function on $\mathbb{1}_{[0,n)}$, where

$$\int_0^n f_\mu^2(k)k^2\,\mathrm{d}k = (4\pi\gamma_\alpha)^{-1}\,\mathbb{1}_{[0,n)} = \left(n+\frac{\alpha}{4\pi}\right)\mathbb{1}_{[0,n)}. \tag{4.243}$$

but the inverse transform (without a normalization constant depending on μ) will clearly not give $\mathbb{1}_{[0,n)}$ back, since the first term in the generalized eigenfunctions is anti-symmetric. Thus since the transform from k to μ returned $\mathbb{1}_{[0,n)}$ the same cannot be true for the inverse, in fact we have with normalization constant $A_\mu = 1$ and the naïve measure $\mathrm{d}\Omega_\mu = \mu^2\,\mathrm{d}\mu$ that

$$\int_0^n f_{\mu^2}(k')\int_0^n f_\mu^2(k)k^2\,\mathrm{d}k\mu^2\,\mathrm{d}\mu = \left(-n+\frac{\alpha}{4\pi}+2k'\operatorname{arctanh}(k'/n)\right)\left(n+\frac{\alpha}{4\pi}\right) \tag{4.244}$$

Thus we need to find A_μ and $\mathrm{d}\Omega_\mu$ that gives us the correct normalizations.
Idea: Assuming that $\int_0^n f_{\mu^2}(k)f_{\nu^2}(k)k^2\,\mathrm{d}k = C\delta(\mu-\nu)$ we have, if we can exchange integrals,

$$\int_0^n f_{\nu^2}(k)\left(\int_0^n \frac{1}{A_\mu}f_{\mu^2}(k)\mu^2\,\mathrm{d}\mu\right)k^2\,\mathrm{d}k = \frac{C}{A_\nu} \tag{4.245}$$

As seen above we can solve the generalized eigenvalue, problem, (4.228), for the Hamiltonian $H_\alpha|_{\mathcal{H}_0}$, with eigenvalue $\mu^2 < n^2$ by the functions

$$f_{\mu^2}(k) = A_\mu\left(\mathcal{P}\frac{1}{k^2-\mu^2}\mathbb{1}_{[0,n)}(k) + \frac{\left(\operatorname{arctanh}\left(\frac{\mu}{n}\right)+\frac{\alpha}{4\pi\mu}\right)}{\mu}\delta(k-\mu)\right), \quad \mu^2 < n^2. \tag{4.246}$$

The normalizationm A_μ, is left unspecified at this point, however the following theorems motivates a specific choice

Theorem 5. *Let f_{μ^2} denote the functions in (4.246). Let further A_μ be defined by*

$$A_\mu := 4\pi\Bigg(\alpha^2 + 8i\pi^3\mu^2\log\left(\frac{4n^2}{n^2-\mu^2}\right) + 8\pi\alpha\mu\,\mathrm{arctanh}\left(\frac{\mu}{n}\right) + 32\pi^2\mu^2\,\mathrm{arctanh}\left(\frac{\mu}{n}\right)^2$$
$$+8\pi^2\mu n\log\left(\frac{\mu+n}{n-\mu}\right) - 16\pi^2\mu n\,\mathrm{arctanh}\left(\frac{\mu}{n}\right) \tag{4.247}$$
$$+8\pi^2\mu^2\mathrm{Li}_2\left(\frac{2n}{n-\mu}\right) + 8\pi^2\mu^2\mathrm{Li}_2\left(\frac{2n}{n+\mu}\right)\Bigg)^{-1/2},$$

then we have the following inversion theorems

$$\begin{aligned}\int_0^n f_{\mu^2}(k)\int_0^n f_{\nu^2}(k)g(\nu)\nu^2\,\mathrm{d}\nu k^2\,\mathrm{d}k = g(\mu)\\ \int_0^n f_{\nu^2}(k')\int_0^n f_{\nu^2}(k)h(k)k^2\,\mathrm{d}k\nu^2\,\mathrm{d}\nu = h(k')\end{aligned} \tag{4.248}$$

almost everywhere, for any function $g\in L^2([0,n))$.

Proof. (We have numerical evidence.) Notice first, if there is an A_μ such that the theorem is true we must have

$$\int_0^n \frac{1}{A_\nu}f_{\nu^2}(k)\left(\int_0^n \frac{1}{A_\mu}f_{\mu^2}(k)\mu^2\,\mathrm{d}\mu\right)k^2\,\mathrm{d}k = \frac{1}{A_\nu^2}, \tag{4.249}$$

however, this is an equation for A_ν since $\frac{1}{A_\mu}f_{\mu^2}(k)$ is known and independent of A_μ. Solving this equation we find exactly A_μ as defined in (4.247). Now existence of A_μ follows from Theorem 2.2 in [2] □

4.4.1 Many-body rank-one pair interactions

Given an explicit diagonalization of the one particle rank-one perturbation, we are ready to analyze the the many-body Hamiltonian with rank-one pair interactions, given by

$$H = -\sum_{i=1}^N \Delta_i - \frac{2}{m+1}\sum_{i<j}\nabla_i\cdot\nabla_j - \gamma_\alpha\sum_{i=1}^N \langle \mathbb{1}_{B_n},\hat{\cdot}\rangle\,\mathbb{1}_{B_n}^\vee. \tag{4.250}$$

or in momentum space

$$H = \sum_{i=1}^N k_i^2 + \frac{2}{m+1}\sum_{i<j}k_i\cdot k_j - \gamma_\alpha\sum_{i=1}^N \langle \mathbb{1}_{B_n},\cdot\rangle\,\mathbb{1}_{B_n}. \tag{4.251}$$

For simplicity, we assume to begin with that $\alpha\geq 0$ such that the one-particle system has no bound states. We know from the one-particle case that there exist a unitary operator U such that

$$UH_0U^\dagger = \sum_{i=1}^N \mu_i^2, \qquad H_0 = \sum_{i=1}^N k_i^2 - \gamma_\alpha\sum_{i=1}^N \langle \mathbb{1}_{B_n},\cdot\rangle\,\mathbb{1}_{B_n}. \tag{4.252}$$

Furthermore, we know that that the unitary U in each coordinate act trivially one the orthogonal complement of spherical symmetric functions, supported on the ball B_n and $\mu_i = k_i$ in this case. Now let Q_i denote the projection onto this orthogonal complement in the i^{th} coordinate,*i.e.*

$$Q_i = P_M, \qquad M = \{\psi \in L^2(\mathbb{R}^3) : \psi = f \otimes Y_0^0,\ \text{supp}(f) \subset \mathbb{1}_{\overline{B_n}}\}^{\perp} \tag{4.253}$$

and let $P_i = 1 - Q_i$. Now notice that if we define $P = \sum_{i=1}^N I^{\otimes i-1} \otimes P_i \otimes I^{\otimes N-i+1}$ and $Q = \sum_{i=1}^N I^{\otimes i-1} \otimes Q_i \otimes I^{\otimes N-i+1}$ we have $P + Q = I^{\otimes N}$ and

$$PH_0Q = QH_0P = 0, \tag{4.254}$$

so we have

$$H_0 = (P+Q)H_0(P+Q) = PH_0P + QH_0Q = PU^\dagger \sum_{i=1}^N \mu_i^2 UP + Q\sum_{i=1}^N k_i^2 Q. \tag{4.255}$$

Now for the remaining terms we have

$$\begin{aligned} \frac{2}{m+1}\sum_{i<j} k_i \cdot k_j &= \frac{2}{m+1}\sum_{i<j}(P_i+Q_i)(P_j+Q_j)(k_i\cdot k_j)(P_i+Q_i)(P_j+Q_j) \\ &= \frac{2}{m+1}\sum_{i<j}(P_i+Q_i)k_i(P_i+Q_i)\cdot(P_j+Q_j)k_j(P_J+Q_j). \end{aligned} \tag{4.256}$$

Now remember that the operator k_i resembles an electric dipole, and thus selection rules are known for computing transitionvalues $\langle l, m|k|l', m'\rangle$. These are exactly that $\Delta l = \pm 1$ (conservation of spin, photon spin is 1) and $\Delta m = 0, \pm 1$. As a consequence of this we know that $P_i k_i P_i = 0$ for all $i = 1, ..., N$. Hence, we may write

$$\frac{2}{m+1}\sum_{i<j} k_i\cdot k_j = QQQQ+PQQQ+QPQQ+QQPQ+QQQP+PPQQ+PQQP+QPPQ+QQPP. \tag{4.257}$$

where *e.g.* $PQQP = \frac{2}{m+1}\sum_{i<j} P_iQ_jk_i\cdot k_jQ_iP_j$. These terms, may be collected to simplify the expression greatly. To see this, we write out all the terms:

$$\begin{aligned} PQQP &= \frac{2}{m+1}\sum_{i<j} P_ik_iQ_i\cdot Q_jk_jP_j, \\ QPPQ &= \frac{2}{m+1}\sum_{i<j} Q_ik_iP_i\cdot P_jk_jQ_j, \\ PPQQ &= \frac{2}{m+1}\sum_{i<j} P_ik_iQ_i\cdot P_jk_jQ_j = \frac{1}{m+1}\left(\sum_{i=1}^N P_ik_iQ_i\right)^2, \\ QQPP &= \frac{2}{m+1}\sum_{i<j} Q_ik_iP_i\cdot Q_jk_jP_j = \frac{1}{m+1}\left(\sum_{i=1}^N Q_ik_iP_i\right)^2 \end{aligned} \tag{4.258}$$

Collecting these terms, we see that we have

$$\begin{aligned} PQQP + QPPQ + PPQQ + QQPP &= \frac{1}{m+1}\left(\sum_{i=1}^N Q_ik_iP_i + \sum_{i=1}^N P_ik_iQ_i\right)^2 \\ &\quad - \frac{1}{m+1}\left(\sum_{i=1}^N Pk_iQ_ik_iP_i + \sum_{i=1}^N Q_ik_iP_ik_iQ_i\right) \\ &= \frac{1}{m+1}\left(\sum_{i=1}^N Q_ik_iP_i + \sum_{i=1}^N P_ik_iQ_i\right)^2 \\ &\quad - \frac{1}{m+1}\left(\sum_{i=1}^N Pk_i^2P_i + \underbrace{\underbrace{\sum_{i=1}^N Q_ik_iP_ik_iQ_i}_{=\sum_{i=1}^N k_iP_ik_i}}_{=\sum_{i=1}^N k_i^2 - \sum_{i=1}^N k_iQ_ik_i}\right) \end{aligned} \tag{4.259}$$

Furthermore, we have

$$\begin{aligned} PQQQ + QPQQ &= \frac{2}{m+1}\sum_{i<j}\left(P_ik_iQ_i\cdot Q_jk_jQ_j + Q_ik_iQ_i\cdot P_jk_jQ_j\right) \\ &= \frac{2}{m+1}\left(\sum_{i=1}^N P_ik_iQ_i\right)\cdot\left(\sum_{j=1}^N Q_jk_jQ_j\right) - \frac{2}{m+1}\underbrace{\sum_{i=1}^N P_ik_iQ_ik_iQ_i}_{=\sum_{i=1}^N P_ik_i^2Q_i=0} \\ &= \frac{2}{m+1}\left(\sum_{i=1}^N P_ik_iQ_i\right)\cdot\left(\sum_{j=1}^N Q_jk_jQ_j\right), \end{aligned} \tag{4.260}$$

and similarly

$$\begin{aligned} QQQP + QQPQ &= \frac{2}{m+1}\sum_{i<j}\left(Q_ik_iQ_i\cdot Q_jk_jP_j + Q_ik_iP_i\cdot Q_jk_jQ_j\right) \\ &= \frac{2}{m+1}\left(\sum_{i=1}^N Q_ik_iP_i\right)\cdot\left(\sum_{j=1}^N Q_jk_jQ_j\right) - \frac{2}{m+1}\underbrace{\sum_{i=1}^N Q_ik_iQ_ik_iP_i}_{=\sum_{i=1}^N Q_ik_i^2P_i=0} \\ &= \frac{2}{m+1}\left(\sum_{i=1}^N P_ik_iQ_i\right)\cdot\left(\sum_{j=1}^N Q_jk_jQ_j\right). \end{aligned} \tag{4.261}$$

Collecting all terms with three Qs we have

$$PQQQ + QPQQ + QQPQ + QQQP = \frac{2}{m+1}\left(\sum_{i=1}^N Q_ik_iQ_i\right)\cdot\left(\sum_{i=1}^N P_ik_iQ_i + \sum_{i=1}^N Q_ik_iP_i\right). \tag{4.262}$$

And finally, noting that

$$QQQQ = \frac{1}{m+1}\left(\sum_{i=1}^{N} Q_i k_i Q_i\right)^2 - \frac{1}{m+1}\sum_{i=1}^{N} Q_i k_i Q_i k_i Q_i \tag{4.263}$$

collecting all terms we have

$$\begin{aligned}
&QQQQ + PQQQ + QPQQ + QQPQ + QQQP + PQQP + QPPQ + PPQQ + QQPP \\
&= \frac{1}{m+1}\left[\left(\sum_{i=1}^{N} Q_i k_i Q_i\right) + \left(\sum_{i=1}^{N} P_i k_i Q_i + \sum_{i=1}^{N} Q_i k_i P_i\right)\right]^2 - \frac{1}{m+1}\left(\sum_{i=1}^{N} P_i k_i^2 P_i + \sum_{i=1}^{N} Q_i k_i^2 Q_i\right) \\
&= \frac{1}{m+1}\left[\left(\sum_{i=1}^{N} Q_i k_i Q_i\right) + \left(\sum_{i=1}^{N} P_i k_i + \sum_{i=1}^{N} k_i P_i\right)\right]^2 - \frac{1}{m+1}\sum_{i=1}^{N} k_i^2 \\
&= \frac{1}{m+1}\left[\begin{pmatrix} 0 & \sum_i k_i \\ \sum_i k_i & \sum_i k_i \end{pmatrix}^2 - \begin{pmatrix} \sum_i k_i^2 & 0 \\ 0 & \sum_i k_i^2 \end{pmatrix}\right] \\
&= \frac{1}{m+1}\begin{pmatrix} (\sum_i k_i)^2 - \sum_i k_i^2 & (\sum_i k_i)^2 \\ (\sum_i k_i)^2 & 2(\sum_i k_i)^2 - \sum_i k_i^2 \end{pmatrix}
\end{aligned} \tag{4.264}$$

4.5 Second quantization

We are in this subsection gonna proceed by writing up the problem in the second quantization formalism. It is the hope, that this will better enable us to locate the relevant terms in the Hamiltonian for proving a uniform lower bound of the rank-one-perturbation with increasing radius.

The Hamiltonian can be written

$$H = \sum_{i=1}^{N} k_i^2 + \frac{2}{m+1}\sum_{i<j} k_i \cdot k_j - \gamma_R \sum_{i=1}^{N} \left\langle \hat{\phi}_i | \cdot \right\rangle \hat{\phi}_i. \tag{4.265}$$

We may rewrite this as

$$H = \frac{m}{m+1}\sum_{i=1}^{N} k_i^2 + \frac{1}{m+1}\left(\sum_{i=1}^{N} k_i\right)^2 - \gamma_R \sum_{i=1}^{N} \left\langle \hat{\phi}_i | \cdot \right\rangle \hat{\phi}_i. \tag{4.266}$$

We know from section 4.4 that $H_1 := \frac{m}{m+1}k^2 - \gamma_R \left\langle \hat{\phi} | \cdot \right\rangle \hat{\phi}$ have a spectrum with (at most) one negative eigenvalue, and continuous spectrum consisting of $(0, \infty)$. Thus we may write H as

$$H = \sum_i \varepsilon_i c_i^\dagger c_i + \frac{1}{m+1}\sum_{m=1}^{3}\left(\sum_{ij} a_{ij}^m c_i^\dagger c_j\right)^2 \tag{4.267}$$

where $a_{ij}^m = \langle \psi_i | k^m | \psi_j \rangle$, and m is just the vector index of $k = (k^1, k^2, k^3) := (k_x, k_y, k_z)$ and ψ_i are the (generalized) eigenfunctions of H_1, and ε_i are the generalized eigenvalues of ψ_i. Further-

more, as a consequence of the fact that $\sigma(\frac{m+1}{m}H_1) = \begin{cases} \{-\lambda\} \cup (0,\infty), & \text{for } \frac{m+1}{m}\gamma_R = \frac{m+1}{m}\frac{1}{4\pi R+\alpha} > \frac{1}{4\pi R} \\ (0,\infty), & \text{for } \frac{m+1}{m}\gamma_R \leq \frac{1}{4\pi R}. \end{cases}$,
we index ε_i with $i=0$ for the ground state energy if there is any and otherwise we just put $\varepsilon_0 = 0$. Thus we know that $\varepsilon_i > 0$ for $i > 0$. Furthermore, we know that ψ_0 us sperically symmetric, as was seen in section 4.4 or by the fact that H_1 is sperically symmetric. Therefore $a_{00}^m = \langle \psi_0 | k^m | \psi_0 \rangle = 0$ by simple selection rules. We thus have

$$H = \sum_i \varepsilon_i c_i^\dagger c_i + \frac{1}{m+1} \sum_{m=1}^{3} \sum_{i,j,k,l} a_{ij}^m a_{kl}^m c_i^\dagger c_j c_k^\dagger c_l \tag{4.268}$$

we analyze the last term, by splitting it in the cases where $\{i,j,k,l\}$ contains $0,1,2,3$ and 4 0s.

Four 0s

In the case where $\{i,j,k,l\} = \{0,0,0,0\}$ we have

$$\sum_{m=1}^{3} a_{00}^m a_{00}^m c_0^\dagger c_0 c_0^\dagger c_0 = 0, \tag{4.269}$$

since $a_{00}^m = 0$.

Three 0s

If three of i,j,k,l are 0, we see that

$$a_{ij}^m a_{kl}^m c_i^\dagger c_j c_k^\dagger c_l = 0 \tag{4.270}$$

since either $i,j = 0$ *or* $k,l = 0$, and then $a_{ij}^m = 0$ *or* $a_{kl}^m = 0$.

Two 0s

In the cases where $i = j = 0$ (or $k = l = 0$) we clearly get no contribution as

$$a_{ij}^m a_{kl}^m c_i^\dagger c_j c_k^\dagger c_l = a_{00}^m a_{kl}^m c_0^\dagger c_0 c_k^\dagger c_l = 0. \tag{4.271}$$

In the cases where $i = l = 0$ we get

$$a_{ij}^m a_{kl}^m c_i^\dagger c_j c_k^\dagger c_l = a_{0j}^m a_{k0}^m c_0^\dagger c_j c_k^\dagger c_0 = N_0 a_{0j}^m a_{k0}^m c_j c_k^\dagger = N_0 a_{0j}^m a_{k0}^m (\delta_{jk} - c_k^\dagger c_j). \tag{4.272}$$

In the case where $j = k = 0$ we get

$$a_{i0}^m a_{0l}^m c_i^\dagger c_0 c_0^\dagger c_l = a_{i0}^m a_{0l}^m c_i^\dagger c_l (1 - N_0), \tag{4.273}$$

where we defined $N_0 = c_0^\dagger c_0$. If $i = k = 0$ we get

$$a_{ij}^m a_{kl}^m c_i^\dagger c_j c_k^\dagger c_l = a_{0j}^m a_{0l}^m c_0^\dagger c_j c_0^\dagger c_l = -a_{0j}^m a_{0l}^m c_0^\dagger c_0^\dagger c_j c_l = 0, \tag{4.274}$$

since $(c^\dagger)^2 = 0$. Similarly we have for $j = l = 0$

$$a_{ij}^m a_{kl}^m c_i^\dagger c_j c_k^\dagger c_l = a_{i0}^m a_{k0}^m c_i^\dagger c_0 c_k^\dagger c_0 = -a_{i0}^m a_{k0}^m c_i^\dagger c_0 c_0 c_k^\dagger = 0, \tag{4.275}$$

since $c_0^2 = 0$. Hence we see that the only contributing terms with two 0s are $j = k = 0$ and $i = l = 0$.

One 0

If $i = 0$ then we have

$$\begin{aligned} 0\#\#\# = \sum_{m=1}^{3} \sum_{j,k,l} a_{0j}^m a_{kl}^m c_0^\dagger c_j c_k^\dagger c_l &= \sum_{m=1}^{3} \sum_{j,k,l} a_{0j}^m a_{kl}^m c_0^\dagger (\delta_{jk} - c_k^\dagger c_j) c_l \\ &= \sum_{m=1}^{3} \left(\sum_{k,l} a_{kl}^m c_k^\dagger c_l \right) \left(\sum_{j} a_{0j}^m c_0^\dagger c_j \right) + \sum_{m=1}^{3} \sum_{j,l} a_{0j}^m a_{jl}^m c_0^\dagger c_l. \end{aligned} \tag{4.276}$$

If $l = 0$ we have

$$\begin{aligned} \#\#\#0 = \sum_{m=1}^{3} \sum_{i,j,k} a_{ij}^m a_{k0}^m c_i^\dagger c_j c_k^\dagger c_0 &= \sum_{m=1}^{3} \sum_{i,j,k} a_{ij}^m a_{k0}^m c_i^\dagger (\delta_{jk} - c_k^\dagger c_j) c_0 \\ &= \sum_{m=1}^{3} \left(\sum_{k} a_{k0}^m c_k^\dagger c_0 \right) \left(\sum_{i,j} a_{ij}^m c_i^\dagger c_j \right) + \sum_{m=1}^{3} \sum_{i,j} a_{ij}^m a_{j0}^m c_i^\dagger c_0. \end{aligned} \tag{4.277}$$

If $j = 0$ we have

$$\#0\#\# = \sum_{m=1}^{3} \sum_{i,k,l} a_{i0}^m a_{kl}^m c_i^\dagger c_0 c_k^\dagger c_l = \sum_{m=1}^{3} \left(\sum_{i} a_{i0}^m c_i^\dagger c_0 \right) \left(\sum_{k,l} a_{kl}^m c_k^\dagger c_l \right). \tag{4.278}$$

If $k = 0$ we have

$$\#\#0\# = \sum_{m=1}^{3} \sum_{i,j,l} a_{ij}^m a_{0l}^m c_i^\dagger c_j c_0^\dagger c_l = \sum_{m=1}^{3} \left(\sum_{i,j} a_{ij}^m c_i^\dagger c_j \right) \left(\sum_{l} a_{0l}^m c_0^\dagger c_l \right). \tag{4.279}$$

Now using the inequality $\left|A^\dagger B + B^\dagger A\right| \leq \frac{1}{\epsilon} A^\dagger A + \epsilon B^\dagger B$ on $0\#\#\# + \#\#\#0 + \#0\#\# + \#\#0\#$ we find

$$\text{one 0 terms} \geq -\frac{2}{\epsilon} \sum_{m=1}^{3} \sum_{i,j} a_{i0}^m a_{0j}^m c_i^\dagger \underbrace{c_0 c_0^\dagger}_{(1-N_0)} c_j - 2\epsilon \sum_{m=1}^{3} \sum_{i,j,k,l} a_{ij}^m a_{kl}^m c_i^\dagger c_j c_k^\dagger c_l + \sum_{m=1}^{3} \sum_{i,j} \left(a_{0j}^m a_{ji}^m c_0^\dagger c_i + a_{ij}^m a_{j0}^m c_i^\dagger c_0 \right) \tag{4.280}$$

No 0s

If all indices are non-zero we have

$$\#\#\#\# = \sum_{m=1}^{3} \sum_{i,j,k,l} a_{ij}^m a_{kl}^m c_i^\dagger c_j c_k^\dagger c_l = \left(\sum_{i,j>0} \overline{a}_{ij} c_i^\dagger c_j \right)^2, \tag{4.281}$$

where $\overline{a}_{ij}$ is a three-dimensional vector for each i, j.

Collecting the terms

Notice that we may conclude that the term comming from having no 0s is positive since it is a sum of squares of self-adjoint operators $\sum_{i,j>0} a_{ij}^m c_i^\dagger c_j$, $m = 1, 2, 3$.
The terms comming from two 0s may be collected to give

$$\begin{aligned} \text{two 0s} &= \sum_{m=1}^{3} \sum_{j,k} \Big(N_0 a_{0j}^m a_{k0}^m (\delta_{jk} - c_k^\dagger c_j) + a_{k0}^m a_{0j}^m c_k^\dagger c_j (1 - N_0) \Big) \\ &= \sum_{m=1}^{3} \sum_{j,k} \Big((1 - 2N_0) a_{0j}^m a_{k0}^m c_k^\dagger c_j \Big) + N_0 \sum_{m=1}^{3} \sum_{i} a_{0i}^m a_{i0}^m \end{aligned} \tag{4.282}$$

Now remember that $a_{0i} = \langle 0|k|i\rangle$. Thus $\sum_{i>0} a_{0i}^m a_{i0}^m = \langle 0|k^m|i\rangle \langle i|k^m|0\rangle = \left\langle 0|(k^m)^2|0\right\rangle$. Thus we see that the coefficient of N_0 in H is $\left\langle 0|\frac{m}{m+1}\sum_i k_i^2 + \frac{1}{m+1}\sum_i k_i^2 - \gamma_R \sum_i \left\langle \hat{\phi}_i, \cdot \right\rangle \hat{\phi}_i|0\right\rangle = \left\langle 0|\sum_i k_i^2 - \gamma_R \sum_i \left\langle \hat{\phi}_i, \cdot \right\rangle \hat{\phi}_i|0\right\rangle \geq -\lambda$ for some λ independent of R. The last inequality follows direct diagonalization of the operator $\sum_i k_i^2 - \gamma_R \sum_i \left\langle \hat{\phi}_i, \cdot \right\rangle \hat{\phi}_i$, or in fact by just finding it eigenvalue, if it exists. Now for the remaining term comming from two 0s we notice that $0 \leq (1 - 2N_0) \leq 1$ and that

$$a_{0j}^m a_{i0}^m = \langle i|k^m|0\rangle \langle 0|k^m|j\rangle = \langle i|k^m P_0 k^m|j\rangle, \tag{4.283}$$

where $P_0 = |0\rangle\langle 0|$ is the projector onto the ground state. Thus we see that $\sum_{m=1}^{3}\sum_{j,k}\left(a_{0j}^m a_{k0}^m c_k^\dagger c_j\right)$ is the second quantization of the operator $\sum_{m=1}^{3} k^m P_0 k^m$, which is positive. Thus we clearly have

$$(1 - 2N_0) \sum_{m=1}^{3} \sum_{j,k} \left(a_{0j}^m a_{k0}^m c_k^\dagger c_j \right) \geq - \sum_{m=1}^{3} \sum_{j,k} \left(a_{0j}^m a_{k0}^m c_k^\dagger c_j \right) \tag{4.284}$$

where the righthand side is simply the second quantization of the one-particle operator $\sum_{m=1}^{3} k^m P_0 k^m$. We now want to estimate $P_0^\perp H_1 P_0^\perp - \frac{1}{m+1}\sum_{m=1}^{3} k^m P_0 k^m$, were P_0 is the orthogonal projection onto the groundstate of H_1. For now we assume that $m \geq 1$ such that $m - 1 \geq 0$. Using that $\sum_{m=1}^{3} k^m P_0 k^m \leq k^2$, we find

$$H_1 - \frac{1}{m+1} \sum_{m=1}^{3} k^m P_0 k^m \geq \frac{m-1}{m+1} k^2 - \gamma_R \left\langle \hat{\phi}_R, \cdot \right\rangle \hat{\phi}_R. \tag{4.285}$$

Now notice that $\left(H_1 - \frac{1}{m+1}\sum_{m=1}^{3} k^m P_0 k^m\right)|0\rangle = H_1|0\rangle = -\lambda|0\rangle$, since $k^m P_0 k^m|0\rangle = 0$ by spherical symmetry of $|0\rangle$. On the contrary by the min-max principle, we bound the next element in the spectrum, by

$$\begin{aligned}\lambda_1 &= \min_{\psi,\varphi\in H^2(\mathbb{R}^3)} \max_{u\in\operatorname{span}(\psi,\varphi)} \left(\left\langle \hat{u}\middle| \left(H_1 - \frac{1}{m+1}\sum_{m=1}^{3} k^m P_0 k^m\right)\hat{u}\right\rangle\right) \\ &\geq \min_{\psi,\varphi\in H^2(\mathbb{R}^3)} \max_{u\in\operatorname{span}(\psi,\varphi)} \left(\left\langle \hat{u}\middle| \left(\frac{m-1}{m+1}k^2 - \gamma_R\left\langle\hat{\phi}_R,\cdot\right\rangle\hat{\phi}_R\right)\hat{u}\right\rangle\right) \geq 0.\end{aligned} \tag{4.286}$$

Notice that it is a general fact that a rank one perturbation of a non-negative operator, has at most one negative eigenvalue, as can be seen by the min-max principle, since one can always find vector in any two-dimensional subspace that is orthogonal to the rank-one perturbation. By the above we conclude that $|0\rangle$ is the unique groundstate of $H_1 - \frac{1}{m+1}\sum_{m=1}^{3} k^m P_0 k^m$ with eigenvalue $-\lambda$. Now it is clear that $k^m P_0 k^m = P_0^\perp k^m P_0 k^m P_0^\perp$, so we get

$$P_0^\perp\left(H_1 - \frac{1}{m+1}\sum_{m=1}^{3} k^m P_0 k^m\right)P_0^\perp = P_0^\perp H_1 P_0^\perp - \frac{1}{m+1}\sum_{m=1}^{3} k^m P_0 k^m \geq 0. \tag{4.287}$$

Now we may improve the inequality $\sum_{m=1}^{3} k^m P_0 k^m \leq k^2$. Let $v \in L^2((0,\infty)\times S^2) \cong L^2(\mathbb{R}^3)$ be arbitrary and notice that, using regular selection rules for the eletric dipole, we have

$$\left\langle v\middle|\left(\sum_{m=1}^{3} k^m P_0 k^m\right)v\right\rangle = \left\langle P_{l=1}v\middle|\left(\sum_{m=1}^{3} k^m P_0 k^m\right)P_{l=1}v\right\rangle \tag{4.288}$$

where $P_{l=1}$ is the orthogonal projection onto the angular subspace spanned by $(Y_1^m)_{m=-1}^{1}$. Now we use the following relations

$$\left\langle Y_0^0|\hat{\omega}|Y_1^m\right\rangle = \begin{pmatrix} a \\ ima \\ 0\end{pmatrix} \qquad m = \pm 1 \tag{4.289}$$

and

$$\left\langle Y_0^0|\hat{\omega}|Y_1^0\right\rangle = \begin{pmatrix} 0 \\ 0 \\ b\end{pmatrix}. \tag{4.290}$$

Here a and b are given by

$$\begin{aligned} a &= \int_{S^2} Y_0^0(\theta,\phi)\sin(\theta)\cos(\phi)Y_1^m(\theta,\phi)\,\mathrm{d}\Omega \\ b &= \int_{S^2} Y_0^0(\theta,\phi)\cos(\theta)Y_1^0(\theta,\phi)\,\mathrm{d}\Omega,\end{aligned} \tag{4.291}$$

where $\phi \in [0, 2\pi)$ is the azimuth. In conclusion we have $\left\langle Y_1^{m'}|\hat{\omega}|Y_0^0\right\rangle \cdot \left\langle Y_0^0|\hat{\omega}|Y_1^m\right\rangle = 0$ when $m \neq m'$ and

$$\left\langle Y_1^m|\hat{\omega}|Y_0^0\right\rangle \cdot \left\langle Y_0^0|\hat{\omega}|Y_1^m\right\rangle = \begin{cases} 2\left|a\right|^2 & \text{for } m = \pm 1, \\ |b|^2 & \text{for } m = 0 \end{cases}. \tag{4.292}$$

Now notice that

$$\left\langle v|k^2 v\right\rangle = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^\infty \left\langle v|k,l,m\right\rangle \left\langle k,l,m|k^2 v\right\rangle k^2 \,\mathrm{d}k \tag{4.293}$$

where $\langle k,l,m|v\rangle = \int_{S^2} Y_l^m(\theta,\phi) v(k,\theta,\phi)\,\mathrm{d}\Omega$. Furthermore, we have

$$\begin{aligned}\left\langle v\middle|\left(\sum_{m=1}^{3} k^m P_0 k^m\right) v\right\rangle &= \left\langle P_{l=1}v\middle|\left(\sum_{m=1}^{3} k^m P_0 k^m\right) P_{l=1}v\right\rangle \\ \leq \sum_{m=-1}^{1}\sum_{m'=-1}^{1}\int_0^\infty & \left\langle v|k,1,m\right\rangle \left\langle Y_1^m|\hat{\omega}|Y_0^0\right\rangle \cdot \left\langle Y_0^0|\hat{\omega}|Y_1^{m'}\right\rangle \left\langle k,1,m'|k^2 v\right\rangle k^2\,\mathrm{d}k \\ &\leq \max(2\left|a\right|^2, |b|^2)\sum_{m=-1}^{1}\int_0^\infty \left\langle v|k,1,m\right\rangle \left\langle k,1,m|k^2 v\right\rangle k^2\,\mathrm{d}k \\ &\leq \max(2\left|a\right|^2, |b|^2)\left\langle v|k^2 v\right\rangle.\end{aligned} \tag{4.294}$$

We conclude that $\sum_{m=1}^{3} k^m P_0 k^m \leq \max(2\left|a\right|^2, |b|^2)k^2$. (FIX INDICES... m used multiple times)

Notice that straightforward computation shows $2\left|a\right|^2 = |b|^2 = \frac{1}{3}$. Hence we have $\sum_{m=1}^{3} k^m P_0 k^m \leq \frac{1}{3}k^2$. Going back to (4.285) we see that we instead may conclude

$$H_1 - \frac{1}{m+1}\sum_{m=1}^{3} k^m P_0 k^m \geq \frac{m - 1/3}{m+1}k^2 - \gamma_R\left\langle\hat{\phi}_R, \cdot\right\rangle\hat{\phi}_R. \tag{4.295}$$

and by the same argument as above, we conclude that

$$P_0^\perp\left(H_1 - \frac{1}{m+1}\sum_{m=1}^{3} k^m P_0 k^m\right)P_0^\perp = P_0^\perp H_1 P_0^\perp - \frac{1}{m+1}\sum_{m=1}^{3} k^m P_0 k^m \geq 0. \tag{4.296}$$

holds for $m \geq 1/3$.

The term comming from one 0 may be estimated by Cauchy-Schwartz inequality

$$\left|\sum_{m=1}^{3}\sum_{i,j} a_{ij}^m a_{j0}^m c_i^\dagger c_0\right| \leq \sum_{m=1}^{3}\sum_{j}\left(\sum_i a_{ij}^m c_i^\dagger\right) a_{j0}^m c_0 \tag{4.297}$$

$$\|0\#\#\# + \#0\#\# + \#\#0\# + \#\#\#0\| \leq \|2(0\#\#\# + \#\#\#0)\| + \sum_{m=1}^{3}\sum_{i,j} a_{ij}^m a_{j0}^m \left\|c_i^\dagger c_i\right\|^{1/2}\left\|c_0^\dagger c_0\right\|^{1/2} \tag{4.298}$$

Again we may use $\sum_{j>0} a_{ij}^m a_{j0}^m = \langle 0|k^m|j\rangle \langle j|k^m|0\rangle = \langle i|(k^m)^2|0\rangle$.

$$\left(\sum_i k_i\right)\cdot\left(\sum_i (P_0^\perp)_i k_i (P_0^\perp)_i\right) + \left(\sum_i (P_0^\perp)_i k_i (P_0^\perp)_i\right)\cdot\left(\sum_i k_i\right) - \left(\sum_i (P_0^\perp)_i k_i (P_0^\perp)_i\right)^2 \tag{4.299}$$

4.6 Collecting all terms part 2

Collecting all terms and using estimate (4.280) with $\epsilon = 1/2$, we see that

$$H \geq \sum_i \tilde{\varepsilon}_i N_i - \frac{3}{m+1}\left(1-\frac{2}{3}N_0\right)\sum_{m=1}^{3}\sum_{i,j>0} a_{i0}^m a_{0j}^m c_i^\dagger c_j + \frac{1}{m+1}\sum_{m=1}^{3}\sum_{i,j>0}\left(a_{0j}^m a_{ji}^m c_0^\dagger c_i + a_{ij}^m a_{j0}^m c_i^\dagger c_0\right) \tag{4.300}$$

where $\tilde{\varepsilon}_0 = \left\langle 0 \middle| k^2 - \gamma_R \left\langle \hat{\phi}_R, \cdot \right\rangle \hat{\phi}_R \middle| 0 \right\rangle$, and $\tilde{\varepsilon}_i = \left\langle i \middle| \frac{m}{m+1}k^2 - \gamma_R \left\langle \hat{\phi}_R, \cdot \right\rangle \hat{\phi}_R \middle| i \right\rangle$ for $i > 0$. Consider now a trial function of the form $\psi = \psi_0 + \psi_1$, where $N_0\psi_0 = 0$ and $N_0\psi_1 = \psi_1$. We then clearly have

$$\begin{aligned}\langle \psi | H\psi\rangle \geq & \sum_{i>0} \tilde{\varepsilon}_i \langle \psi_0 | N_i \psi_0\rangle - \frac{3}{m+1}\left\langle \psi_0 \middle| \sum_{m=1}^{3}\sum_{i,j>0} a_{i0}^m a_{0j}^m c_i^\dagger c_j \ \psi_0 \right\rangle \\ & + \sum_{i\geq 0} \tilde{\varepsilon}_i \langle \psi_1 | N_i \psi_1\rangle - \frac{1}{m+1}\left\langle \psi_1 \middle| \sum_{m=1}^{3}\sum_{i,j>0} a_{i0}^m a_{0j}^m c_i^\dagger c_j \ \psi_1 \right\rangle \\ & + \frac{2}{m+1}\mathrm{Re}\left[\left\langle \psi_0 \middle| \sum_{m=1}^{3}\sum_{i,j>0} a_{0j}^m a_{ji}^m c_0^\dagger c_i \ \psi_1 \right\rangle\right]\end{aligned} \tag{4.301}$$

Use (4.267) for the ψ_0 part instead. Then $\langle \psi_0 | H\psi_0\rangle \geq 0$. Therefore we have

$$\langle \psi | H\psi\rangle \geq \langle \psi_1 | H\psi_1\rangle + \frac{2}{m+1}\mathrm{Re}\left[\left\langle \psi_0 \middle| \sum_{m=1}^{3}\left(\sum_{ij} a_{ij}^m c_i^\dagger c_j\right)^2 \psi_1 \right\rangle\right]. \tag{4.302}$$

4.6.1 Collecting all terms part 3

We now try doing the same computation as above but now splitting the wave function in the following way $\psi = \psi_0 + \psi_1$, where $N_0\psi_0 = 0$ and $N_0\psi_1 = \psi_1$. We then use individual bounds for the ψ_0 and ψ_1 parts. Notice first that

$$\langle \psi | H\psi\rangle = \langle \psi_0 | H\psi_0\rangle + \langle \psi_1 | H\psi_1\rangle + 2\mathrm{Re}\left[\langle \psi_1 | H\psi_0\rangle\right]. \tag{4.303}$$

Since $\sum_i \epsilon_i N_i$ commutes with N_0 we see that $\langle \psi_1 | (\sum_i \epsilon_i N_i)\psi_0\rangle = 0$. In fact, we see that only terms from $\frac{1}{m+1}\sum_{m=1}^{3}\left(\sum_{i,j} a_{ij}^m c_i^\dagger c_j\right)$ containing exactly one $c_0^\dagger$ can contribute in this term.

Thus we conclude that

$$2\left|\mathrm{Re}\left[\langle\psi_1|H\psi_0\rangle\right]\right| \geq \frac{2}{m+1}\left|\left\langle \psi_i\right| \sum_{m=1}^{3}\sum_{i,j,k} a \right\rangle\right| \tag{4.304}$$

NEW IDEA: Utilize that the basis $(|i\rangle)_{i\geq 0}$ can be chosen arbitrarily and collect all bounds on $\sum_{m=1}^{3}\left(\sum_{i,j} a_{ij}^m c_i^\dagger c_k\right)^2$. Recognize the resulting operator as second quantized version of some single particle operator. use

$$\begin{aligned}\sum_{m=1}^{3}\sum_{i,j>0} a_{i0}^m a_{0j}^m c_i^\dagger c_j + \sum_{m=1}^{3}\sum_{i,j>0}\left(a_{0j}^m a_{ji}^m c_0^\dagger c_i + a_{ij}^m a_{j0}^m c_i^\dagger c_0\right)& \\ = \mathrm{d}\Gamma\left(\sum_{m=1}^{3} P_0^\perp k^m P_0 k^m P_0^\perp + P_0^\perp k^2 P_0 + P_0 k^2 P_0^\perp\right)& \\ = \mathrm{d}\Gamma\left(k^2 - P_0^\perp k P_0^\perp k P_0^\perp - P_0 k^2 P_0\right)&\end{aligned} \tag{4.305}$$

Theorem 6. *Let H be given as above, and let $m \geq 1$, then $H \geq -\lambda$ for some $\lambda > 0$ independent of R.*

Proof. Notice first that

$$\begin{aligned}H =& \mathrm{d}\Gamma\left(\frac{m}{m+1}k^2 - \gamma_R\left|\phi_R\right\rangle\left\langle\phi_R\right|\right) + \mathrm{d}\Gamma\left(\frac{1}{m+1}P_0 k^2 P_0\right) \\ &+ (1-2N_0)\,\mathrm{d}\Gamma\left(\frac{1}{m+1}\sum_{\mu=1}^{3} k^\mu P_0 k^\mu\right) \\ &+ \frac{2}{m+1}\left[\sum_{\mu=1}^{3}\left(\sum_{i,j>0} a_{ij}^\mu c_i^\dagger\right)\left(\sum_{l>0} a_{0l}^\mu c_0^\dagger c_l\right) + \left(\sum_{l>0} a_{l0}^\mu c_l^\dagger c_0\right)\left(\sum_{i,j>0} a_{ij}^\mu c_i^\dagger\right)\right] \\ &+ \frac{1}{m+1}\sum_{\mu=1}^{3}\left(\sum_{i,j>0} a_{ij}^\mu a_{j0}^\mu c_i^\dagger c_0 + \sum_{i,j>0} a_{0j}^\mu a_{ji}^\mu c_0^\dagger c_i\right) \\ &+ \frac{1}{m+1}\sum_{\mu=1}^{3}\left(\sum_{i,j>0} a_{ij}^\mu c_i^\dagger c_j\right)^2\end{aligned} \tag{4.306}$$

where the states $(|i\rangle)_{i\geq 0}$ form a basis in single particle Hilbert space and $a_{ij}^\mu = \langle i|k^\mu|j\rangle$. By Cauchy-Schwarz in the form $A^\dagger B + B^\dagger A \geq -\left(\frac{1}{\epsilon}A^\dagger A + \epsilon B^\dagger B\right)$, with $\epsilon = \frac{1}{2}$ used in the fourth term we have

$$
\begin{aligned}
H \geq & \, \mathrm{d}\Gamma\left(\frac{m}{m+1}k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|\right) + \mathrm{d}\Gamma\left(\frac{1}{m+1}P_0 k^2 P_0\right) \\
& + (1-2N_0)\,\mathrm{d}\Gamma\left(\frac{1}{m+1}\sum_{\mu=1}^{3} k^\mu P_0 k^\mu\right) \\
& - \frac{4}{m+1}(1-N_0)\sum_{\mu=1}^{3}\sum_{i,j>0} a^\mu_{i0} a^\mu_{0j} c_i^\dagger c_j \\
& + \frac{1}{m+1}\sum_{\mu=1}^{3}\left(\sum_{i,j>0} a^\mu_{ij} a^\mu_{j0} c_i^\dagger c_0 + \sum_{i,j>0} a^\mu_{0j} a^\mu_{ji} c_0^\dagger c_i\right) \\
= & \, \mathrm{d}\Gamma\left(\frac{m}{m+1}k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|\right) + \mathrm{d}\Gamma\left(\frac{1}{m+1}P_0 k^2 P_0\right) \\
& - \left(3 - \frac{2}{3}N_0\right)\mathrm{d}\Gamma\left(\frac{1}{m+1}\sum_{\mu=1}^{3} k^\mu P_0 k^\mu\right) \\
& + \frac{1}{m+1}\,\mathrm{d}\Gamma\left(P_0^\perp k^2 P_0 + P_0 k^2 P_0^\perp\right) \\
= & \, \mathrm{d}\Gamma\left(k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|\right) - (3-2N_0)\,\mathrm{d}\Gamma\left(\frac{1}{m+1}\sum_{\mu=1}^{3} k^\mu P_0 k^\mu\right) - \frac{1}{m+1}\,\mathrm{d}\Gamma\left(P_0^\perp k^2 P_0^\perp\right).
\end{aligned}
\tag{4.307}
$$

Using that $0 \leq \sum_{\mu=1}^{3} k^\mu P_0 k^\mu \leq \frac{1}{3}P_0^\perp k^2 P_0^\perp$, we find

$$
H \geq \mathrm{d}\Gamma\left(k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|\right) - \mathrm{d}\Gamma\left(\frac{1}{m+1}P_0^\perp k^2 P_0^\perp\right) - \frac{1}{m+1}\,\mathrm{d}\Gamma\left(P_0^\perp k^2 P_0^\perp\right). \tag{4.308}
$$

Choosing $|0\rangle$ to be the ground state of $H_0 = k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|$ (if it exists), we see that

$$
H \geq \mathrm{d}\Gamma\left(P_0\left[k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|\right]P_0 + P_0^\perp\left[\frac{m-1}{m+1}k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|\right]P_0^\perp\right) \tag{4.309}
$$

We therefore analyze the single particle operator

$$
H_1' = P_0\left[k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|\right]P_0 + P_0^\perp\left[\frac{m-1}{m+1}k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|\right]P_0^\perp. \tag{4.310}
$$

Clearly $|0\rangle$ is an eigenstate of H_1', with eigenvalue $-\lambda_1'$ for some $\lambda_1' > 0$ which is known to be bounded from below by some negative constant $-\lambda$ independent of R. But then H_1' can have no more negative eigenvalues, since there would necessarily be orthogonal to $|0\rangle$. This can be seen since the one and only eigenstate of $\frac{m-1}{m+1}k^2 - \gamma_R \left|\phi_R\right\rangle\left\langle\phi_R\right|$ is known when $m \geq 1$, and it is not orthogonal to $|0\rangle$. Thus we conclude that the rest of the spectrum satisfies $\sigma(H_1') \subset \{-\lambda_1'\} \cup [0, \infty)$. For the the many-body Hamiltonian H definined on the fermionic Fock-space, we thus conclude that $H \geq -\lambda$ □

Remark It might be possible to improve the bound on m to $m \geq 1/3$, as this would be the bound for $N_0 = 1$.

Lemma 27. *If $k^2 - \gamma_R |\phi_R\rangle \langle\phi_R|$ has a ground state, and P_0 is the orthogonal projection onto this groundstate, then for all $\xi > 0$ $A = P_0^\perp \xi k^2 - \gamma_R |\phi_R\rangle \langle\phi_R| P_o^\perp \geq 0$*

Proof. Notice first that since A is a rank-one perturbation of a positive operator, it can have at most one negative eigenvalue. Assume that ψ is and eigenstate of A with eigenvalue $-\lambda^2$ with $\lambda > 0$. Then $P_0^\perp \psi = \psi$, and $(\xi k^2 - \gamma_R |\phi_R\rangle \langle\phi_R|)\psi = -\lambda^2\psi + \beta\psi_0$. Equivalently, we have up to a normalization constant

$$\psi(k) = \frac{\gamma_R C\phi_R(k) + \beta\psi_0}{\xi k^2 + \lambda^2},$$

where C and β are determined by the self-consistent conditions

$$C = \langle\phi_R|\psi\rangle, \qquad \langle\psi_0|\psi\rangle = 0. \tag{4.311}$$

Since we know that $\psi_0 \sim \frac{1}{k^2+\lambda_0^2}$ we may absorb the nomalization in β, and we get

$$C = \frac{4\pi}{1 - 4\pi\gamma_R \int_0^R \frac{1}{\xi k^2+\lambda^2} k^2 \,\mathrm{d}k}\beta \int_0^R \frac{1}{\xi k^2 + \lambda^2} \frac{1}{k^2 + \lambda_0^2} k^2 \,\mathrm{d}k$$

and

$$0 = \gamma_R 4\pi C \int_0^R \frac{1}{\xi k^2 + \lambda^2} \frac{1}{k^2 + \lambda_0^2} k^2 \,\mathrm{d}k + 4\pi\beta \int_0^R \frac{1}{\xi k^2 + \lambda^2} \left(\frac{1}{k^2 + \lambda_0^2}\right)^2 k^2 \,\mathrm{d}k. \tag{4.312}$$

Computing all integrals and using the relation $\arctan\left(\frac{R}{\lambda_0}\right) = -\frac{\alpha}{4\pi\lambda_0}$ $(\alpha < 0)$, we find the equivalent equation

$$\ldots \tag{4.313}$$

□

A λ relation in Krein formula (2.26)

To show the relation of $\lambda(z, \bar{z})$ we use the following properties established in the proof in the main text. We have that

$$(B - z)^{-1} - (C - z)^{-1} = \lambda(z) \langle\phi(\bar{z}), \cdot\rangle \phi(z), \tag{A.1}$$

where we have already used that $\lambda(z, \bar{z}) = \lambda(z)$, and where we have defined

$$\phi(z) = \phi(z_0) + (z - z_0)(C - z)^{-1}\phi(z_0), \tag{A.2}$$

for some $\phi(z_0)$ satisfying $A^*\phi(z_0) = z_0\phi(z_0)$.

In the following we will switch to bra-ket notation to simplify the calculations and ease the notation. Now let us consider the relation

$$(B-z)^{-1} = (C-z)^{-1} + \lambda(z) \left|\phi(z)\right\rangle \left\langle\phi(\bar{z})\right| . \tag{A.3}$$

By multiplying this relation with itself, but with z' instead of z we get

$$\begin{aligned}(B-z)^{-1}(B-z')^{-1} = (C-z)^{-1}(C-z')^{-1} + \lambda(z) \left|\phi(z)\right\rangle \left\langle\phi(\bar{z})\right| (C-z')^{-1} \\ +\lambda(z')(C-z)^{-1} \left|\phi(z')\right\rangle \left\langle\phi(\bar{z}')\right| + \lambda(z)\lambda(z') \left|\phi(z)\right\rangle \left\langle\phi(\bar{z}), \phi(z')\right\rangle \left\langle\phi(\bar{z}')\right| .\end{aligned} \tag{A.4}$$

Now using that $(B-z)^{-1} - (B-z')^{-1} = (z-z')(B-z)^{-1}(B-z')^{-1}$ and the same relation for C we get, by multiplying through with $(z-z')$ that

$$\begin{aligned}&(B-z)^{-1} - (B-z')^{-1} - (C-z)^{-1} + (C-z')^{-1} = \\ &\quad (z-z')\lambda(z) \left|\phi(z)\right\rangle \left\langle\phi(\bar{z})\right| (C-z')^{-1} + (z-z')\lambda(z')(C-z)^{-1} \left|\phi(z')\right\rangle \left\langle\phi(\bar{z}')\right| \\ &\qquad\qquad + (z-z')\lambda(z)\lambda(z') \left|\phi(z)\right\rangle \left\langle\phi(\bar{z}), \phi(z)'\right\rangle \left\langle\phi(\bar{z}')\right| .\end{aligned} \tag{A.5}$$

Using again the relation (A.1) we obtain

$$\begin{aligned}&\lambda(z) \left|\phi(z)\right\rangle \left\langle\phi(\bar{z})\right| - \lambda(z') \left|\phi(z')\right\rangle \left\langle\phi(\bar{z}')\right| = \\ &\quad (z-z')\lambda(z) \left|\phi(z)\right\rangle \left\langle\phi(\bar{z})\right| (C-z')^{-1} + (z-z')\lambda(z')(C-z)^{-1} \left|\phi(z')\right\rangle \left\langle\phi(\bar{z}')\right| \\ &\qquad\qquad + (z-z')\lambda(z)\lambda(z') \left|\phi(z)\right\rangle \left\langle\phi(\bar{z}), \phi(z)'\right\rangle \left\langle\phi(\bar{z}')\right| ,\end{aligned} \tag{A.6}$$

from which we obtain by simple rearrangement

$$\begin{aligned}\lambda(z) \left|\phi(z)\right\rangle \left\langle\phi(\bar{z})\right| \left(I - (z-z')(C-z')^{-1}\right) - \lambda(z') \left(I - (z'-z)(C-z)^{-1}\right) \left|\phi(z')\right\rangle \left\langle\phi(\bar{z}')\right| = \\ (z-z')\lambda(z)\lambda(z') \left|\phi(z)\right\rangle \left\langle\phi(\bar{z}), \phi(z)'\right\rangle \left\langle\phi(\bar{z}')\right| .\end{aligned} \tag{A.7}$$

Notice now that

$$\left(I - (z-z')(C-z')^{-1}\right) = (C-z')^{-1}(C-z), \quad \text{and} \quad \left(I - (z'-z)(C-z)^{-1}\right) = (C-z')(C-z)^{-1}. \tag{A.8}$$

Now clearly by (A.2) we have

$$(C-z')(C-z)^{-1}\phi(z') = (C-z')(C-z)^{-1}\phi(z_0) + (z'-z_0)(C-z)^{-1}\phi(z_0), \tag{A.9}$$

where we have used that $(C-z')(C-z)^{-1}h = (C-z)^{-1}(C-z')h$ whenever $h \in \mathscr{D}\left((C-z')\right)$. By using that $(C-z')(C-z)^{-1} = I + (z-z')(C-z)^{-1}$ we obtain

$$(C-z')(C-z)^{-1}\phi(z') = \phi(z_0) + (z-z_0)(C-z)^{-1}\phi(z_0) = \phi(z). \tag{A.10}$$

By a similar computation we have

$$(C-z)(C-z')^{-1}\phi(z)=\phi(z_0)+(z'-z_0)(C-z')^{-1}\phi(z_0)=\phi(z'), \tag{A.11}$$

and thus we obtain

$$\begin{aligned}\lambda(z)\left|\phi(z)\right\rangle\left\langle\phi(\bar z)\right|\left(I-(z-z')(C-z')^{-1}\right)-\lambda(z')\left(I-(z'-z)(C-z)^{-1}\right)\left|\phi(z')\right\rangle\left\langle\phi(\bar z')\right|=\\ \lambda(z)\left|\phi(z)\right\rangle\left\langle\left(I-(\bar z-\bar z')(C-\bar z')^{-1}\right)\phi(\bar z)\right|-\lambda(z')\left|\left(I-(z'-z)(C-z)^{-1}\right)\phi(z')\right\rangle\left\langle\phi(\bar z')\right|=\\ \lambda(z)\left|\phi(z)\right\rangle\left\langle\phi(\bar z')\right|-\lambda(z')\left|\phi(z)\right\rangle\left\langle\phi(\bar z')\right|=(z-z')\lambda(z)\lambda(z')\left|\phi(z)\right\rangle\left\langle\phi(\bar z),\phi(z)'\right\rangle\left\langle\phi(\bar z')\right|.\end{aligned} \tag{A.12}$$

Observing the last line in the above calculation we observe that we have the relation

$$\lambda(z)-\lambda(z')=\lambda(z)\lambda(z')(z-z')\left\langle\phi(\bar z),\phi(z')\right\rangle, \tag{A.13}$$

which is equivalent to the relation

$$\lambda(z)^{-1}-\lambda(z')^{-1}=-(z-z')\left\langle\phi(\bar z),\phi(z')\right\rangle. \tag{A.14}$$

This proves equation (2.26).

B Schur test

We prove that given an operator with symmetric integral kernel $\sigma(s,t)$ we have

$$\|\sigma\|\leq\sup_t\left(h(t)\int \mathrm{d}s\frac{\sigma(s,t)}{h(s)}\right). \tag{B.1}$$

for any positive function h.

Proof. First notice that that by the regular Cauchy-Schwartz we have

$$\begin{aligned}\left|\int \mathrm{d}t\sigma(s,t)f(t)\right|^2&=\left|\int \mathrm{d}t\left(\frac{h(s)}{h(t)}\right)^{1/2}\sigma(t,s)^{1/2}\left(\frac{h(t)}{h(s)}\right)^{1/2}\sigma(s,t)^{1/2}f(t)\right|^2\\ &\leq\left(\int \mathrm{d}t\left|\sigma(t,s)\right|\frac{h(s)}{h(t)}\right)\left(\int \mathrm{d}t\left|\sigma(s,t)\right|\frac{h(t)}{h(s)}\left|f(t)\right|^2\right)\end{aligned} \tag{B.2}$$

where we used symmetry of σ in the first line, Cauchy-Schwartz in the second line Now integrating w.r.t s we find

$$\begin{aligned}
\|\sigma\|^2 \|f\|_2^2 &\leq \int \mathrm{d}s \left| \int \mathrm{d}t \sigma(s,t) f(t) \right|^2 \\
&\leq \int \mathrm{d}s \left[\left(\int \mathrm{d}t \, |\sigma(t,s)| \frac{h(s)}{h(t)} \right) \left(\int \mathrm{d}t \, |\sigma(s,t)| \frac{h(t)}{h(s)} |f(t)|^2 \right) \right] \\
&\leq \sup_s \left\{ h(s) \left(\int \mathrm{d}t \, |\sigma(t,s)| \frac{1}{h(t)} \right) \right\} \left(\int \mathrm{d}s \int \mathrm{d}t \, |\sigma(s,t)| \frac{h(t)}{h(s)} |f(t)|^2 \right) \\
&= \sup_s \left\{ h(s) \left(\int \mathrm{d}t \, |\sigma(t,s)| \frac{1}{h(t)} \right) \right\} \left(\int \mathrm{d}t \int \mathrm{d}s \left[|\sigma(s,t)| \frac{h(t)}{h(s)} \right] |f(t)|^2 \right) \\
&\leq \sup_s \left\{ h(s) \left(\int \mathrm{d}t \, |\sigma(t,s)| \frac{1}{h(t)} \right) \right\} \sup_t \left\{ h(t) \left(\int \mathrm{d}s \, |\sigma(s,t)| \frac{1}{h(s)} \right) \right\} \|f\|_2^2 \\
&= \left[\sup_t \left\{ h(t) \left(\int \mathrm{d}s \, |\sigma(s,t)| \frac{1}{h(s)} \right) \right\} \right]^2 \|f\|_2^2 ,
\end{aligned} \tag{B.3}$$

where we used Fubini's theorem in the fourth line and Hölder's inequality $(1, \infty)$ in the firth line. Thus the claim follows. □

C Monotonicity of $L^R(k)/L(k)$ in the $2+1$ case

Lemma 28. *Let*

$$\begin{aligned}
L^R(k) =& 2\pi \left[\sqrt{\frac{k^2 m(m+2)}{(m+1)^2} + \mu} \left(\arctan\left(\frac{-\frac{|k|}{m+1} + R}{\sqrt{\frac{k^2 m(m+2)}{(m+1)^2} + \mu}} \right) + \arctan\left(\frac{\frac{|k|}{m+1} + R}{\sqrt{\frac{k^2 m(m+2)}{(m+1)^2} + \mu}} \right) \right) \right. \\
& \left. R + \frac{\left(m(m+2)\left(k^2+\mu\right) - k^2 + \mu + (m+1)^2 R^2\right) \ln\left(\frac{-\frac{2R|k|}{m+1} + k^2 + \mu + R^2}{\frac{2R|k|}{m+1} + k^2 + \mu + R^2} \right)}{4(m+1)|k|} \right], \\
L(k) =& 2\pi^2 \sqrt{\frac{k^2 m(m+2)}{(m+1)^2} + \mu}.
\end{aligned} \tag{C.1}$$

then $L^R(k)/L(k)$ is a non-increasing function k.

Proof. We first compute

$$\begin{aligned}
\frac{\partial}{\partial |k|} \frac{L^R(k)}{L(k)} =& \frac{-8k^3 m(m+2)R - 4k\mu(m+1)^2 R}{4\pi k^2 \sqrt{\frac{k^2 m(m+2)}{(m+1)^2} + \mu} \left(m(m+2)\left(k^2+\mu\right) + \mu\right)} \\
&+ \frac{(m+1)\left(k^2\left(\mu + m(m+2)\left(\mu + 2R^2\right)\right) + \mu(m+1)^2\left(\mu + R^2\right)\right) \ln\left(\frac{k^2 + \frac{2kR}{m+1} + \mu + R^2}{k^2 - \frac{2kR}{m+1} + \mu + R^2} \right)}{4\pi k^2 \sqrt{\frac{k^2 m(m+2)}{(m+1)^2} + \mu} \left(m(m+2)\left(k^2+\mu\right) + \mu\right)}
\end{aligned} \tag{C.2}$$

using $\ln(1+x) \leq \frac{x}{\sqrt{1+x}}$, we find

$$\begin{aligned}\frac{\partial}{\partial |k|}\frac{L^R(k)}{L(k)} &\leq g(R,k,\mu) \\ &:= \frac{R\left(\frac{(m+1)\left(k^2\left(\mu+m(m+2)\left(\mu+2R^2\right)\right)+\mu(m+1)^2\left(\mu+R^2\right)\right)}{(k^2(m+1)-2kR+(m+1)(\mu+R^2))\sqrt{\frac{4kR}{k^2(m+1)-2kR+(m+1)\left(\mu+R^2\right)}+1}} - 2k^2m(m+2) - \mu(m+1)^2\right)}{\pi k\sqrt{\frac{k^2m(m+2)}{(m+1)^2}+\mu}\,(m(m+2)\,(k^2+\mu)+\mu)}.\end{aligned} \tag{C.3}$$

We immediately notice that

$$f(R) := \frac{(m+1)\left(k^2\left(\mu+m(m+2)\left(\mu+2R^2\right)\right)+\mu(m+1)^2\left(\mu+R^2\right)\right)}{(k^2(m+1)-2kR+(m+1)\,(\mu+R^2))\sqrt{\frac{4kR}{k^2(m+1)-2kR+(m+1)(\mu+R^2)}+1}} \tag{C.4}$$

is an increasing function of R, which is easily verified by find the derivative

$$\begin{aligned}\frac{\partial}{\partial R}f(R) &= \frac{4k^2R\left(k^2m(m+2)+\mu(m+1)^2\right)\left(k^2+\frac{m^2+2m-1}{(m+1)^2}R^2+\mu\right)}{\left(k^2-\frac{2kR}{(m+1)}+(\mu+R^2)\right)^2\left(k^2+\frac{2kR}{(m+1)}+(\mu+R^2)\right)\sqrt{\frac{4kR}{k^2(m+1)-2kR+(m+1)(\mu+R^2)}+1}} \\ &> 0.\end{aligned} \tag{C.5}$$

Thus $g(R,k,\mu)$ is also an increasing function of R and we have

$$g(R,k,\mu) \leq \lim_{R\to\infty} g(R,k,\mu) = 0. \tag{C.6}$$

Thereby we find

$$\frac{\partial}{\partial |k|}\frac{L^R(k)}{L(k)} \leq 0, \tag{C.7}$$

and $\frac{L^R(k)}{L(k)}$ is non-increasing. □

References

[1] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Theoretical and Mathematical Physics, Springer Berlin Heidelberg, 2012.

[2] I.U.M. Berezanskii, *Expansions in eigenfunctions of selfadjoint operators*, Translations of mathematical monographs, American Mathematical Society, 1968.

[3] Domenico Finco and Alessandro Teta, *Quadratic forms for the fermionic unitary gas model*, Reports on Mathematical Physics **69** (2012), no. 2, 131 – 159.

[4] Gerald B. Folland, *Real analysis:*, 2nd ed. ed., Wiley,, New York:, 1999.

[5] G. Grubb, *Distributions and operators*, Graduate Texts in Mathematics, Springer New York, 2008.

[6] E.H. Lieb, M. Loss, M.A. LOSS, and American Mathematical Society, *Analysis*, Crm Proceedings & Lecture Notes, American Mathematical Society, 2001.

[7] G.D. Maso, *An introduction to [gamma]-convergence*, Progress in nonlinear differential equations and their applications, Birkhäuser, 1993.

[8] Thomas Moser and Robert Seiringer, *Stability of a fermionic n + 1 particle system with point interactions*, Communications in Mathematical Physics **356** (2017), no. 1, 329–355.

[9] Feng Qi, Shi-Qin Zhang, and Bai-Ni Guo, *Sharpening and generalizations of shafer's inequality for the arc tangent function*, Journal of Inequalities and Applications **2009** (2009), no. 1, 930294.

[10] J. P. Solovej, *Rigorous stability theory of quantum mechanics*, unpublished, 1997.