Mandatory assignment 2 - FunkAn

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Problem 1

a)

We want to show that $f_N \to 0$ weakly for $n \to \infty$. We have by Riezs Fischer theorem that every bounded linear functional over the Hilbert space H is of the form $F(x) = \langle x, y \rangle$, where $y = \sum_{i=1}^{\infty} y_i e_i \in H$, $y_i \in H$. So we have:

$y_i \in \mathbb{C}$

$$F(f_N) = \langle f_N, y \rangle = \langle N^{-1} \sum_{n=1}^{N^2} e_n, \sum_{i=1}^{\infty} y_i e_i \rangle = N^{-1} \sum_{n=1}^{N^2} \sum_{i=1}^{\infty} y_i \langle e_n, e_i \rangle = N^{-1} \sum_{n=1}^{N^2} y_n$$

We have that F is bounded, so $F(f_N) < \infty$, so $N^{-1} \sum_{n=1}^{N^2} y_n < \infty$

Not well-defined. You need some $|\cdot|$ for this, since in general $F(f_N) \in \mathbb{C}$.

$$\left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} y_n \right)^2 \le \left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} |y_n| \right)^2 = \left(\sum_{n=1}^{N} \frac{1}{\sqrt{N}} |y_n| \right)^2 \le \sum_{n=1}^{N} \left(\frac{1}{\sqrt{N}} \right)^2 \sum_{n=1}^{N} |y_n|^2 = \sum_{n=1}^{N} |y_n|^2$$

Where the first inequality comes from the triangle inequality, and the seccond inequality comes from Cauchy Schwarz inequality. From this we then get:

$$\left| \frac{1}{\sqrt{N}} \sum_{n=1}^{N} y_n \right| \le \left(\sum_{n=1}^{N} |y_n|^2 \right)^{\frac{1}{2}} < \infty$$

For all $N \ge 1$, since $(y_n)_{n \ge 1} \in \ell_2(\mathbb{N})$. Since $\sum_{n=1}^{N} |y_n|^2 < \infty$ we have the existence of a constant $C \in \mathbb{K}$ such that: [Why is this true?

No, this requires $\sum_{n=1}^{\infty} |y_n|^2 < \infty$, which you have already stated without proof.

$$\sum_{n=1}^{N} |y_n|^2 \to C, \quad n \to \infty$$

This implies for all $\varepsilon > 0$ there exists M such that $\sum_{n=M+1}^{\infty} |y_n|^2 < \varepsilon$. This implies that for any constant $K \ge 1$, we have $\sum_{n=M+1}^{K+M} |y_n|^2 < \varepsilon$. Now if we let $N \ge \frac{C^2}{\varepsilon^2}$, then we'll have:

Where did the square go?

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{M} |y_n| \le \frac{\varepsilon}{C} C = \varepsilon$$

Now we have from Triangle inequality and Cauchy Schwarz:

$$\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n\right| = \left|\frac{1}{\sqrt{N}}\right|\left|\sum_{n=1}^{N} y_n\right| \leq \frac{1}{\sqrt{N}}\sum_{n=1}^{N}|y_n| = \frac{1}{\sqrt{N}}\sum_{n=1}^{M}|y_n| + \frac{1}{\sqrt{N}}\sum_{n=M+1}^{N}|y_n| \leq \varepsilon + \frac{1}{\sqrt{N}}\sum_{n=M+1}^{N}|y_n|$$

$$= \varepsilon + \sum_{n=M+1}^{N}\frac{1}{\sqrt{N}}|y_n| \leq \varepsilon + \sum_{n=M+1}^{N+M}\frac{1}{\sqrt{N}}|y_n| = \varepsilon + \sqrt{\left(\sum_{n=M+1}^{N+M}\frac{1}{\sqrt{N}}|y_n|\right)^2} \leq \varepsilon + \sqrt{\sum_{n=M+1}^{N+M}\frac{1}{N}\sum_{n=M+1}^{N+M}|y_n|^2}$$

$$= \varepsilon + \sqrt{N\frac{1}{N}\sum_{n=M+1}^{N+M}|y_n|^2} = \varepsilon + \sqrt{\sum_{n=M+1}^{N+M}|y_n|^2} < \varepsilon + \sqrt{\varepsilon}$$

Hence:

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N} y_n \to 0, \qquad N \to \infty$$

This implies that:

Very convoluted but OK

$$\frac{1}{N}\sum_{n=1}^{N^2} y_n \to 0, \quad N \to \infty$$

This formulation is unclear.

So we have $\lim_{N\to\infty} F(f_N) = \lim_{N\to\infty} N^{-1}\sum_{n=1}^{N^2} y_n = 0$. Since F is bounded, we have that F is continuous, so we have that $f_N \to 0$ weakly for $N \to \infty$.

(✓)

Now we want to compute $\|f_N\|$, we note that $\|e_n\| = 1$:

$$\|f_N\|^2 = \|N^{-1}\sum_{n=1}^{N^2} e_n\|^2 = |N^{-1}|^2\|\sum_{n=1}^{N^2} e_n\|^2 = N^{-2}\sum_{n=1}^{N^2}\|e_n\|^2 = N^{-2}\sum_{n=1}^{N^2} 1^2 = N^{-2}N^2 = 1$$

Why is this true?

(✓)

So $\|f_N\| = 1$.

b)

We want to argue that $K = \overline{co\{f_N : N \geq 1\}}^{\|\cdot\|}$ is weakly compact. We have by definition 7.7 that $co\{f_N : N \geq 1\}$ is convex, so we have by theorem 5.7 that:

$$K = \overline{co\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{co\{f_N : N \geq 1\}}^{\tau_w}$$

This implies that K is weakly closed. We now consider the closed unit ball $\overline{B}_{H^*}(0,1) \subset H^*$. H is a Hilbert space, hence a normed vector space, so we have by Alaouglu's theorem that $\overline{B}_{H^*}(0,1) = \{f \in H^* : \|f\| \leq 1\}$ is compact in the w^*-topology. Since H is a Hilbert space, we have from proposition 2.10 that H is reflexive. Since H is a Hilbert space, it is a Banach space as well, so we have from theorem 5.9 $\tau_{w^*} = \tau_w$ for H^*. From this we can conclude that $\overline{B}_{H^*}(0,1)$ is compact in the w-topology, hence $\overline{B}_{H^*}(0,1)$ is weakly compact.

This is an antilinear isomorphism!

By Riesz theorem we have that every element in H^* has the form $F_y = \langle \cdot, y \rangle$ with $y \in H^*$. So we have an isomorphism from H^* to H. Since $B_{H^*} \subset H^*$, and $B_H \subset H$, then there is an isomorphism from $\overline{B}_{H^*}(0,1)$ to $\overline{B}_H(0,1)$. This concludes that $K \subset \overline{B}_H(0,1)$ is a weakly closed subset of a weakly compact set, so K is weakly compact itself, which is what we wanted to show. Furthermore since K is weakly closed, and $f_N \to 0$ then we have that $0 \in K$. ✓

weakly!

c)

To start with we want to show that 0 is an extreme point in K. We start by noting that $K \subset H$ is a non-empty convex weakly compact subset. For any $n \in \mathbb{N}$ we consider:

what is k?

$$h_n = \langle \cdot, -e_n \rangle \in H^*$$

Which is a linear continuos linear functional. Note that $h_n(k) \subset \mathbb{R}$, and set:

$$C = \sup_{n\in\mathbb{N}}\{\langle x, e_n\rangle |\ x \in K\} \overset{?}{=} \sup_{n\in\mathbb{N}}\{-\langle x, e_n\rangle | x \in K\}$$

We have that $C \leq 0$ since for $x \in K$ we have that $x \geq 0$, and $0 \in K$.? By lemma 7.5 we have that:

$$F_n := \{x \in K | Re\langle x, -e_n\rangle = 0\} \neq \emptyset$$

Is a compact face of K for all $n \in \mathbb{N}$. We have that $0 \in F_n$ for all $n \in \mathbb{N}$, so:

$$0 \in \bigcap_{n=1}^{\infty} F_n \neq \emptyset$$

so:

$$\{0\} \subset \bigcap_{n=1}^{\infty} F_n$$

Now we take $x \in \bigcap_{n=1}^{\infty} F_n$, then we'll have $\langle x, -e_n \rangle = 0$ for all $n \in \mathbb{N}$, and the only element for which it holds $\langle x, -e_n \rangle = 0$ for all $n \in \mathbb{N}$ is 0, so $x = 0$, hence $x \in \{0\}$, so $\bigcap_{n=1}^{\infty} F_n \subset \{0\}$, so $\bigcap_{n=1}^{\infty} F_n = \{0\}$. By remark 7.4(3) we have that $\bigcap_{n=1}^{\infty} F_n = \{0\}$ is a face of K, since F_n is a face of K for all $n \geq 1$ (Lemma 7.5). Since we have from problem 1b, that $0 \in K$, and $\{0\}$ is a face of K, then we have from remark 7.4 (1) that $0 \in Ext(K)$, i.e 0 is an extreme point in K. ✓

Now we want to show that f_N is an extreme point in K. To do so we start by fixing $N \geq 1$, and we suppose that $f_N = \alpha x_1 + (1-\alpha)x_2$ for $x_1, x_2 \in K$, $\quad 0 < \alpha < 1$. We have from problem 1a that $\|f_N\|^2$. We consider:

$$1 = \langle f_N, f_N \rangle = \langle \alpha x_1 + (1-\alpha)x_2, f_N \rangle = \alpha\langle x, f_N \rangle + \langle 1-\alpha \rangle \langle x_2, f_N \rangle$$

This implies:

Unclear what you mean here?

$$0 = \alpha\langle x, f_N \rangle + \langle 1-\alpha \rangle \langle x_2, f_N \rangle - 1 = \alpha\langle x, f_N \rangle + (1-\alpha)\langle x_2, f_N \rangle - 1 + \alpha - \alpha$$
$$= \alpha(\langle x_1, f_N \rangle - 1) + (1-\alpha)(\langle x_2, f_N \rangle - 1)$$

Since $0 < \alpha < 1$, and $\langle x_1, f_N \rangle, \langle x_2, f_N \rangle \geq 0$ (since $f_N = \alpha x_1 + (1-\alpha)x_2 \geq 0$, and $x_1, x_2 \in K$, so $x_1, x_2 \geq 0$). From this we have that:

Where do these (in)equalities come from?

$$0 \leq \langle x_1, f_N \rangle \leq 1$$
$$0 \leq \langle x_2, f_N \rangle \leq 1$$

So we have:

$$\langle x_1, f_N \rangle = 1$$
$$\langle x_2, f_N \rangle = 1$$

To show that $f_N \in Ext(K)$, we want to show that $x_1 = x_2 = f_N$.

$$1 = \|\langle x_1, f_N \rangle\| \leq \|x_1\|\|f_N\| = \|x_1\|$$

Since $x_1 \in K \subset \overline{B}_H(0,1)$, then we have $\|x_1\| \leq 1$. Thus:

$$\|\langle x_1, f_N \rangle\| = \|x_1\|\|f_N\|$$

So we have that x_1, and f_N are linearly independent. So $x_1 = \lambda f_N$ for a scalar λ. Then:

$$1 = \langle x_1, f_N \rangle = \langle \lambda f_N, f_N \rangle = \lambda \langle f_N, f_N \rangle = \lambda \|f_N\|^2 = \lambda$$

So we have that $x_1 = \lambda f_N = 1 \cdot f_N = f_N$. We shoe similarily $x_2 = f_N$, then we've shown $x_1 = x_2 = f_N$, hence $f_N \in Ext(K)$.

d)

We note that $K = \overline{co\{f_N|N \geq 1\}}^{\tau_w}$ is a non-empty convex compact subset of H, and H is LCTVS, so by Milman (theorem 7.9) we have that $Ext(K) \subset \overline{\{f_N|N \geq 1\}}^{\tau_w}$, because $\{f_N|N \geq 1\} \subseteq K$. Furthermore by problem 1c we have that $\{f_N|N \geq 1\} \subseteq Ext(K)$, and $\{0\} \subseteq Ext(K)$, so:

$$\{f_N|N \geq 1\} \cup \{0\} \subseteq Ext(K)$$

So we have $\{f_N|N \geq 1\} \cup \{0\} \subseteq \overline{\{f_N|N \geq 1\}}^{\tau_w}$.

H is a Hilbert space, so it is a normed vector space, so it is ismetrizable, hence $\{f_N|N \geq 1\} \subset H$ is metrizable, so $\{f_N|N \geq 1\}$ is first countable, so it is sufficient to consider sequences in $\{f_N|n \geq 1\}$ instead of nets. We suppose that $(x_n)_{n\geq 1}$ in $\{f_N|N \geq 1\}$ are converging weakly to $x \in \overline{\{f_N|N \geq 1\}}^{\tau_w}$. Then we'll have that for some $N \geq 1$, each $x_i = f_N$, which implies that each $x_1 = f_N$, and $x_2 = f_N$ for some $N \geq 1$, so $x = f_N$, or $x = 0$. Hence:

Why should the sequence be constant?

This is badly explained.

$$Ext(K) \subseteq \overline{\{f_N|N \geq 1\}}^{\tau_w} = \{f_N|N \geq 1\} \cup \{0\}$$

And by problem 1c since 0 and f_N are extreme points then $\{f_N|N \geq 1\} \cup \{0\} \subseteq Ext(K)$, so $Ext(K) = \{f_N|N \geq 1\} \cup \{0\}$. Hence there are no other extreme points than f_N, and 0 in K.

(✓)

Problem 2

a)

We want to show:

$$x_n \to x \text{ weakly for } n \to \infty \Rightarrow Tx_n \to Tx \text{ weakly for } n \to \infty$$

We start by assuming $x_n \to x$ weakly for $n \to \infty$. Because x_n is a sequence it has the propperties of a net, because a net is a generelazation of a sequence, so we have from HW4 problem 2a that for all $f \in X^*$

$$f(x_n) \to f(x) \Leftrightarrow x_n \to x \text{ weakly}$$

So if we now take $g \in Y^*$ then since $g \in \mathcal{L}(Y, \mathbb{K})$, and $T \in \mathcal{L}(X, Y)$, then we have that:

$$g \circ T \in \mathcal{L}(X, \mathbb{K}) = X^*$$

So since $x_n \to x$ weakly we'll have from problem 2a

$$g \circ T(x_n) \to g \circ T(x)$$

.

This is the same as

$$g(Tx_n) \to g(Tx)$$

So now since $g(Tx_n) \to g(Tx)$ where $g \in Y^*$ we again have from problem 2a that $Tx_n \to Tx$ weakly, where $Tx \in Y$. ✓

b)

We want to show:

$$x_n \to x \text{ weakly } \Rightarrow \|Tx_n - Tx\| \to 0 \text{ for } n \to \infty$$

We start by assuming $x_n \to x$ weakly, and we want to show $\|Tx_n - Tx\| \to 0$ for $n \to \infty$ by contradiction. So we assume $\|Tx_n - Tx\| \nrightarrow 0$ for $n \to \infty$. This implies that there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ s.t:

What is ε?

$$\|Tx_{n_k} - Tx\| > \varepsilon \text{ for all } k \in \mathbb{N}$$

Why is $(x_{n_k})_{k\in\mathbb{N}}$ a bounded sequence? And bounded seq. need not have conv. subsequences.

The subsequence $(Tx_{n_{k_i}})$ need not converge in $T(x)$.

We assumes that $x_n \to x$ weakly for $n \to \infty$, so we'll have $x_{n_k} \to x$ weakly for $k \to \infty$. We have as well that $(x_{n_k})_{k \in \mathbb{N}}$ is bounded. This implies that $(x_{n_k})_{k \in \mathbb{N}}$ has a convergent subsequence $(x_{n_{k_i}})_{i \in \mathbb{N}}$ s.t $\|Tx_{n_{k_i}} - Tx'\| \to 0$ by proposition 8.2 (4)because X and Y are Banach spaces. Since $x_{n_k} \to x$ weakly we have by problem 2a $Tx_{n_k} \to Tx$ weakly. This implies $Tx_{n_{k_i}} \to Tx$ for $i \to \infty$, which is hte same as $\|Tx_{n_{k_i}} - Tx\| \to 0$ for $i \to \infty$, since X and Y are Banach spaces. This is a contradiction since we had $\|Tx_{n_k} - Tx\| > \varepsilon$, hence $\|Tx_n - Tx\| \to 0$ for $n \to \infty$ Why?

Elaborate more!

c)

This sentence makes no sense.

We want to show $T \in \mathcal{K}(H, Y)$, so we need to show that $T : H \to Y$ is compact, and that $T : H \to Y$ is linear since $T \in \mathcal{L}(H, Y)$, so we just need to show that $T : H \to Y$

At this point, $(x_n)_{n\geq 1}$ is not defined

is compact. We do this by contradiction, so we assume T is not compact, so we have by proposition 8.2 (2) that $T(\overline{B}_H(0,1))$ is not totally bounded. This implies that there exists $\delta > 0$ s.t $T(\overline{B}_H(0,1))$ is not contained in the union of finitely many open balls radius δ. Now we want to show that there exists $\delta > 0$ and a sequence $(x_n)_{n\geq 1}$ in the closed unit ball of H such that $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$. We do this by taking $x_1 \in (x_n)_{n\geq 1} \subset \overline{B}_H(0,1)$, where $\overline{B}_H(0,1)$ is the closed unit ball of H. We assume that $x_2, x_3, ..., x_n \in \overline{B}_H(0,1)$. We look at the set:

$$P := T\left(\overline{B}_H(0,1)\right) \cap \left(\bigcup_{i=1}^{n} B_Y(Tx_i,\delta)\right)^c$$

We have $P \neq \emptyset$. If this wasn't the case then:

$$T\left(\overline{B}_H(0,1)\right) \subset \left(\bigcup_{i=1}^{n} B_Y(Tx_i,\delta)\right)$$

$T(\overline{B_H}(0,1))$

And this would contradict with the fact that $\overline{B}_H(0,1)$ is not totally bounded. So we have $P \neq \emptyset$. We now take $x_{n+1} \in \overline{B}_H(0,1)$ s.t $Tx_{n+1} \in P$. This implies that $Tx_{n+1} \in (\bigcup_{i=1}^{n} B_Y(Tx_i,\delta))^c$, hence $Tx_{n+1} \notin B_Y(Tx_i,\delta)$ for all i. So we'll have $\|Tx_{n+1} - Tx_i\| \geq \delta$ for all $i \leq n$. If we continue this then we have shown that there exists $\delta > 0$ and a sequence in the closed unit ball of H s.t $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$. We have that H is reflexive, because it is a Hilbert space, then we have by theorem 6.3 that $\overline{B}_H(0,1)$ is weakly compact. This implies that every sequence has a weakly convergent subsequence $\|Tx_n - Tx_m\| \geq \delta$ which we showed earlier for all $n \neq m$. Then $\|Tx_{n_k} - Tx\| \nrightarrow 0$ as $k \to \infty$. This gives us a contradiction hence T is compact.

what do you mean here?

Why does it hold for seq. instead of nets?

(✓)

d)

We let $(x_n)_{n\geq 1} \in \ell_2(\mathbb{N})$, and we suppose that $x_n \to x$ weakly as $n \to \infty$ then problem 2a implies $Tx_n \to Tx$ weakly in $\ell_1(\mathbb{N})$, because $Tx \in \ell_1(\mathbb{N})$. Remark 5.3 then implies that $\|Tx_n - Tx\| \to 0$. We have that $\ell_2(\mathbb{N})$ is an infinite dimensional Hilbert space, and we have from HW4 problem 4a that $\ell_2(\mathbb{N})$ is separable, and we have that $\ell_1(\mathbb{N})$ is an infinite Banach space, so we have from problem 2c that $T : \ell_2(\mathbb{N}) \to \ell_1(\mathbb{N})$ is compact. ✓

e)

We start by assuming $T \in \mathcal{K}(X,Y)$. We show by contradiction that T is not onto. So we assume that T is onto, and then we have from the open-mapping theorem, that T is open. Then from page 18 in the lecture notes we have since X and Y are normed vector spaces (because they are Banach spaces), $T : X \to Y$ is a linear map, because $(T \in \mathcal{K}(X,Y))$ then we have that there exists $r > 0$ such that $B_Y(0,r) \subset T(B_X(0,1))$.

We have that closure is inclusion preserving, so we have that $\overline{B_Y(0,r)} \subset \overline{T(B_X(0,1))}$. Since T is a compact operator we have that $\overline{T(B_X(0,1))}$ is compact, then we'll have that the closed subset $\overline{B_Y(0,r)}$ of the compact set $\overline{T(B_X(0,1))}$ is compact. Now we look at different values for r:

For $r = 1$ we have $\overline{B_Y(0,r)} = \overline{B_Y(0,1)}$. Since Y is infinite dimensional, then we have by Riezs lemma, that $\overline{B_Y(0,1)}$ is not compact. So we have that $\overline{B}_Y(0,r)$ is not compact for $r = 1$. A result of Mandatory 1.

For $r > 1$ we have that $\overline{B_Y(0,1)} \subset \overline{B_Y(0,r)}$. Since $\overline{B_Y(0,1)}$ is not compact, then $\overline{B_Y(0,r)}$ is not compact, so we have that $\overline{B_Y(0,r)}$ is not compact for $r > 1$ either

For $r < 1$ we consider the map $f : Y \to Y$ defined by $x \mapsto \frac{1}{r}x$. This is a continuous map. We have $f(\overline{B_Y(0,r)}) = \overline{B_Y(0,1)}$, and we have that $\overline{B_Y(0,1)}$ is not compact, so we have that $f(\overline{B_Y(0,r)})$ is not compact, which implies $\overline{B_Y(0,r)}$ is not compact, since the image over a compact set is compact. So now we have that $\overline{B_Y(0,r)}$ is not compact for $r < 1$, so we have a contradiction, hence $T \in \mathcal{K}(X,Y)$ is onto. ✓

f)

We start by justifying that M is self-adjoint, which we do by considering the inner-product on H. We start by noticing that $\bar{t} = t$: What are f, g?

$$\langle Mf, g\rangle = \int_{[0,1]} Mf(t)\overline{g(t)}\, dm(t) = \int_{[0,1]} tf(t)\overline{g(t)}\, dm(t) = \int_{[0,1]} f(t)\bar{t}\overline{g(t)}\, dm(t) = \int_{[0,1]} f(t)\overline{tg(t)}\, dm(t)$$
$$= \int_{[0,1]} f(t)\overline{Mg(t)}\, dm(t) = \langle f, Mg\rangle$$

From this we see that $M = M^*$, so M is self-adjoint. ✓

Now we want to show that M is not compact. We do this by contradiction. So we assume that M is compact. We just showed that M is self-adjoint, and we have from HW 4 problem 4a that $L_2([0,1], m)$ is separable, and we have that it is an infinite dimensional Hilbert space, so we have from theorem 10.1 in the lecture notes that H has an ONB $(e_n)_{n\geq 1}$ consisting of eigenvectors for M with corresponding eigenvalues $\lambda_n \in \mathbb{R}$. On the other hand we have from HW 6 problem 3a that M has no eigenvalues, so we have a contradiction, hence M is not compact. ✓

Problem 3

a)

We have that $[0, 1]$ is a compact Hausdorff topological space. The Lebesgue measure on $[0, 1]$ is a measure on the Borel σ-algebra, so it is a finite Borel measure. By definition of K, it is seen that K i continuous on $[0, 1] \times [0, 1]$, so it applies that $K \in C([0, 1] \times [0, 1])$, so we have by theorem 9.6 that $T : H \to H$ is a compact operator. ?

Checks at least $s \to t$

only if $T = T_K$ (in fact $T = T_{\tilde{K}}$ for $\tilde{K}(s,t) = K(t,s)$)

b)

To show $T^* = T$, we want to show $\langle Tf, g\rangle = \langle f, Tg\rangle$. We notice $K(s, t) = K(t, s) = \overline{K(s, t)}$:

$$\langle Tf, g\rangle = \int_{[0,1]} Tf(s)\overline{g(s)}\ dm(s) = \int_{[0,1]}\int_{[0,1]} K(s, t)f(t)\ dm(t)\overline{g(s)}\ dm(s)$$

$$= \int_{[0,1]}\int_{[0,1]} K(t, s)f(t)\overline{g(s)}\ dm(t)\ dm(s) = \int_{[0,1]}\int_{[0,1]} \overline{K(s, t)}f(t)\overline{g(s)}\ dm(t)\ dm(s)$$

Now we have by Fubini-Tonelli theorem: why is it justified?

$$\langle Tf, g\rangle = \int_{[0,1]}\int_{[0,1]} \overline{K(s, t)}f(t)\overline{g(s))}\, dm(t)\, dm(s) = \int_{[0,1]}\int_{[0,1]} \overline{K(s, t)}f(t)\overline{g(s))}\, dm(s)\, dm(t)$$

$$= \int_{[0,1]}\int_{[0,1]} \overline{K(s, t)g(s))}\ dm(s)\ f(t)dm(t) = \int_{[0,1]} \overline{Tg(t)}f(t)\ dm(t) = \langle f, Tg\rangle$$

Now we've shown that $T^* = T$.

c)

We have by linearity of Lebesgue integrals:

$$(Tf)(s) = \int_{[0,1]} K(s, t)f(t)\ dm(t) = \int_{[0,s]} K(s, t)f(t)\ dm(t) + \int_{[s,1]} K(s, t)f(t)\ dm(t)$$

$$= \int_{[0,s]} (1-s)tf(t)\ dm(t) + \int_{[s,1]} (1-t)sf(t)\ dm(t) = (1-s)\int_{[0,s]} tf(t)\ dm(t) + s\int_{[s,1]} (1-t)f(t)\ dm(t)$$

Now we want that Tf is continuous. We have that $f \in L_2([0, 1], m)$, so we have that:

$$\left(\int_{[0,1]} |f|^2 \, dm(t)\right)^{\frac{1}{2}} = \|f\|_2 < \infty$$

And since $t, s < \infty$:

$$(1-s) \int_{[0,s]} tf(t) \, dm(t) < \infty$$

And:

This does not show Tf bounded

$$s \int_{[s,1]} (1-t)f(t) \, dm(t) < \infty$$

This is for linear operators not (non-linear) functions.

So we have that Tf is bounded and then by proposition 1.10 we have that Tf is continuous. Now we want to show $(Tf)(0) = (Tf)(1) = 0$. We start by showing $(Tf)(0) = 0$:

$$(Tf)(0) = (1-0) \int_{[0,0]} tf(t) \, dm(t) + 0 \cdot \int_{[0,1]} (1-t)f(t) \, dm(t) = \int_{[0,0]} tf(t) \, dm(t) = 0$$

Now we want to show $Tf(1) = 0$:

$$(Tf)(1) = (1-1) \int_{[0,1]} tf(t) \, dm(t) + 1 \cdot \int_{[1,1]} (1-t)f(t) \, dm(t) = \int_{[1,1]} (1-t)f(t) \, dm(t) = 0$$

So we have $(Tf)(0) = (Tf)(1) = 0$.

Problem 4

a)

We start by noting $e^{-x^2} = e^{-\|x\|^2}$. We have from HW7 problem 1, that $e^{-\|x\|^2} \in \mathscr{S}(\mathbb{R})$, so $e^{-x^2} \in \mathscr{S}(\mathbb{R})$, then we have by problem 1d that:

$$S_{\sqrt{2}}(e^{-x^2}) = e^{-\left(\frac{x}{\sqrt{2}}\right)^2} = e^{-\frac{x^2}{2}} \in \mathscr{S}(\mathbb{R})$$

So we have by HW7 Problem 1a that:

$$g_k(x) = x^k e^{-\frac{x^2}{2}} \in \mathscr{S}(\mathbb{R})$$

Now we want to compute $\mathcal{F}(g_k)$ for $k = 0, 1, 2, 3$. We start by fixing k. We have that $g_k \in \mathscr{S}(\mathbb{R})$, so we have from HW7 problem 1c that $g_k \in L_1(\mathbb{R})$, so we have by definition 11.1 that:

$$\mathcal{F}(g_k) = \hat{g}_k$$

We let $\phi(x) = e^{\frac{x^2}{2}}$. This is integrable, furthermore we have that g_k is integrable, so we get:

$$\mathcal{F}(g_k)(\xi) = \hat{g}_k(\xi) = (g_k)\hat{}(\xi) = (x^k\phi)\hat{}(\xi)$$

Since $\phi(x) = e^{-\frac{x^2}{2}} L_(\mathbb{R})$, and $x^k e^{-\frac{x^2}{2}} \in L_1(\mathbb{R})$, then we have by proposition 11.4 (d) (note $|k| = k$ since $k \geq 0$):

$$(x^k\phi)\hat{}(\xi) = i^{|k|}\left(\partial^k\hat{\phi}\right)(\xi) = i^k\left(\partial^k\hat{\phi}\right)(\xi) = i^k(\partial^k\phi)(\xi)$$

So:

$$\mathcal{F}(g_0)(\xi) = i^0(\partial^0\phi)(\xi) = \phi(\xi) = e^{-\frac{\xi^2}{2}}$$
$$\mathcal{F}(g_1)(\xi) = i^1(\partial^1\phi)(\xi) = -i\xi e^{-\frac{\xi^2}{2}}$$
$$\mathcal{F}(g_2)(\xi) = i^2(\partial^2\phi)(\xi) = e^{-\frac{\xi^2}{2}} - \xi^2 e^{-\frac{\xi^2}{2}}$$
$$\mathcal{F}(g_3)(\xi) = i^3(\partial^3\phi)(\xi) = i^3\left(\xi e^{-\frac{\xi^2}{2}} + 2\xi e^{-\frac{\xi^2}{2}} - \xi^3 e^{-\frac{\xi^2}{2}}\right) = i\left(-3\xi e^{-\frac{\xi^2}{2}} + \xi^3 e^{-\frac{\xi^2}{2}}\right)$$

b)

For $h_0 \in \mathscr{S}(\mathbb{R})$ we need to have that $\mathcal{F}(h_0) = i^0 h_0$. we have that:

$$\mathcal{F}(g_0)(\xi) = e^{-\frac{\xi^2}{2}} = i^0 g_0$$

So if we let $h_0 = g_0$ we'll have $\mathcal{F}(h_0) = i^0 h_0$.

For $h_1 \in \mathscr{S}(\mathbb{R})$ we need to have that $\mathcal{F}(h_1) = i^1 h_1$. From a) we have:

$$\mathcal{F}(g_3)(\xi) = i\left(-3\xi e^{-\frac{\xi^2}{2}} + \xi^3 e^{-\frac{\xi^2}{2}}\right) = i(-3g_1(\xi) + g_3(\xi))$$

Now by the linearity of the Fourier transform we have:

$$\mathcal{F}(g_3-\frac{3}{2}g_1)(\xi) = \mathcal{F}(g_3)(\xi)-\frac{3}{2}\mathcal{F}(g_1)(\xi) = i(-3g_1(\xi)+g_3(\xi))+\frac{3}{2}i\xi e^{-\frac{\xi^2}{2}} = i(-3g_1(\xi)+g_3(\xi)+\frac{3}{2}g_1(\xi))$$

$$= i(-\frac{3}{2}g_1(\xi) + g_3(\xi))$$

So if we let $h_1 = g_3 - \frac{3}{2}g_1$, then we'll have that $\mathcal{F}(h_1) = i^1 h_1$

For $h_2 \in \mathscr{S}(\mathbb{R})$ we need to have that $\mathcal{F}(h_2) = i^2 h_3$. We have from a) that:

$$\mathcal{F}(g_2)(\xi) = e^{-\frac{\xi^2}{2}} - \xi^2 e^{-\frac{\xi^2}{2}} = g_0(\xi) - g_2(\xi)$$

Now by the linearity of the Fourier transform we have:

$$\mathcal{F}(g_2 - \frac{1}{2}g_0 9(\xi) = \mathcal{F}(g_2)(\xi) - \frac{1}{2}\mathcal{F}(g_0)(\xi) = g_0(\xi) - g_2(\xi) - \frac{1}{2}\mathcal{F}(g_0)(\xi) = \frac{1}{2}g_0(\xi) - g_2(\xi)$$

$$= -\left(g_2(\xi) - \frac{1}{2}g_0(\xi)\right) = i^2\left(g_2(\xi) - \frac{1}{2}g_0(\xi)\right)$$

So if we let $h_2 = g_2 - \frac{1}{2}g_0$, then we'll have that $\mathcal{F}(h_2) = i^2 h_2$.

For $h_3 \in \mathscr{S}(\mathbb{R})$ we need to have that $\mathcal{F}(h_3) = i^3 h_3$. we have from a):

$$\mathcal{F}(g_1)(\xi) = -i\xi e^{-\frac{\xi^2}{2}} = -ig_1(\xi) = i^3 g_1(\xi)$$

So if we let $h_3 = g_1$ then we'll have $\mathcal{F}(h_3) = i^3 h_3(\xi)$.

c)

From HW7 Problem 1c we have that $\mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$, so we have that $f, \hat{f} \in L_1(\mathbb{R})$, since $f \in \mathscr{S}(\mathbb{R})$, this as well implies from corollary 12.12 (iii) that $\mathcal{F}^*(\mathcal{F}(f)) = \mathcal{F}(\mathcal{F}^*(f)) = f$, so we have:

$$\mathcal{F}^2(f)(\xi) = \mathcal{F}(\mathcal{F}(f)(\xi) = \mathcal{F}(\hat{f})(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\hat{f}(x)\ dm(x)$$

We now consider:

$$(S_{-1}f)(\xi) = f\left(\frac{\xi}{-1}\right) = f(-\xi) = \mathcal{F}^*(\mathcal{F}(f))(-\xi) = \mathcal{F}^*(\hat{f})(-\xi)\ dm(x)$$
$$= \int_{\mathbb{R}} e^{-ix\xi}\hat{f}(x)\ dm(x) = \mathcal{F}^2(f)(\xi)$$

At last we consider:

$$(\mathcal{F}^4 f)(x) = \mathcal{F}^2((\mathcal{F}^2 f))(x) = \mathcal{F}^2(S_{-1}f)(x) = \mathcal{F}^2(f)(-x) = (S_{-1}f)(-x) = f(x)$$

As desired.

d)

We start by assuming $f \in \mathscr{S}(\mathbb{R})$ is non-zero, and $\mathcal{F}(f) = \lambda f$, in c) we have shown $\mathcal{F}^4(f) = f$, so we get:

$\mathcal{F}^4(f) = \lambda^4 f$ $\mathcal{F}$ linear

$$(\lambda f)^4 = f \Rightarrow \lambda^4 f^4 = f \Rightarrow \lambda^4 = \frac{f}{f^4}$$

f need not be non-zero everywhere!

Now we consider:

$$f^2 = \mathcal{F}^8(f) = \mathcal{F}^4(\mathcal{F}^4(f)) = \mathcal{F}^4(f) = f$$

$\mathcal{F}^8(f) = \mathcal{F}^4(\mathcal{F}^4(f)) = \mathcal{F}^4(f) = f$

So we get:

$$\lambda^4 = \frac{f}{f^4} = \frac{f}{f^2 f^2} = \frac{f}{f^2} = \frac{f}{f} = 1$$

For this to be satisfied we must have that $\lambda \in \{1, -1, i, -i\}$. Since when $\mathcal{F}(f) = \lambda f$ then λ is precisely either $i - i, -1$ or 1, then we have that the eigenvalues for $\mathcal{F}$ are precisely $\{1, -1, i, -i\}$. you need b to conclude this!

Problem 5

We want to show that $supp(\mu) = [0, 1]$. We have by definition $supp(\mu) = N^c$, where N is the union of all open subsets $U \subseteq [0, 1]$. We have by HW8 problem 3a, that $\mu(N) = 0$, so if we can show $\mu([0, 1]^c) = 0$, then we'll have $supp(\mu) = [0, 1]$. We have:

This is false; this would only show that $\mu(\emptyset) = 0$ so $\emptyset \subseteq N$, which is trivial.

$$\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$$

Where $\delta_{x_n}([0,1]^c) = 0$, if $x_n \notin [0,1]^c$, and $\delta_{x_n}([0,1]^c) = 1$, if $x_n \in [0,1]^c$. So we have $\delta_{x_n}([0,1]^c) = 0$, because we have that μ is measure one $[0,1]$, so $x_n \in [0,1]$ for all $n \geq 1$, so $\mu([0,1]^c) = 0$, hence $supp(\mu) = [0,1]$. 0/0