Homework for week 1, FunkAn 2018/19

Problem 1: Justify (briefly) that the following are Banach spaces over $\mathbb{R}$:

(a) $\mathbb{R}$ with the norm given by $\|x\|: = |x|$, for all $x \in \mathbb{R}$.

(b) $\mathbb{R}^2$ with any of the following norms

$$\|(x,y)\|_\infty : = \max\{|x|, |y|\} \qquad \|(x,y)\|_2 : = \sqrt{|x|^2 + |y|^2}$$
$$\|(x,y)\|_1 : = |x| + |y| \,, \quad x, y \in \mathbb{R}.$$

Problem 2:

(a) If X is a normed vector space, then the closure of any subspace of X is again a subspace. (Exercise 5, Section 5.1, Folland.)

(b) Every closed subspace of a Banach space is a Banach space.

(c) Show that $c(\mathbb{N})$ (the space of convergent sequences, defined in Lecture 1) is a closed subspace of $l_\infty(\mathbb{N})$, and hence a Banach space.

Problem 3: As usual, $\mathbb{K}$ denotes the field $\mathbb{R}$ of real numbers or the complex field $\mathbb{C}$. Let $c_c(\mathbb{N}) : = \{(x_n)_{n\geq 1} \subset \mathbb{K} : \{n \geq 1 : x_n \neq 0\} \text{ is finite}\}$. Recall also that for $(x_n)_{n\geq 1} \in c_c(\mathbb{N})$ we have defined

$$\begin{aligned}
\|(x_n)_{n\geq 1}\|_\infty &: = \sup\{|x_n| : n \geq 1\}, \\
\|(x_n)_{n\geq 1}\|_2 &: = \Big(\sum_{n=1}^{\infty} |x_n|^2 \Big)^{1/2}, \\
\|(x_n)_{n\geq 1}\|_1 &: = \sum_{n=1}^{\infty} |x_n| \,.
\end{aligned}$$

(a) Verify that these are all norms on $c_c(\mathbb{N})$.

(b) Show that $c_c(\mathbb{N})$ is <u>not</u> complete with respect to any of the three norms. In other words, for each of the three norms you should find a Cauchy sequence in $c_c(\mathbb{N})$ with no limit inside $c_c(\mathbb{N})$.

(c) Show that $c_c(\mathbb{N})$ is dense in $c_0(\mathbb{N})$ with respect to $\|\cdot\|_\infty$, respectively that $c_c(\mathbb{N})$ is dense in $l_2(\mathbb{N})$ with respect to $\|\cdot\|_2$, and that $c_c(\mathbb{N})$ is dense in $l_1(\mathbb{N})$ with respect to $\|\cdot\|_1$.

Problem 4: The purpose of this exercise is to show that the dual space of $c_0(\mathbb{N})$ can be identified with $\ell_1(\mathbb{N})$, in short $c_0(\mathbb{N})^* \cong \ell_1(\mathbb{N})$ (isometrically isomorphic).

Let $x = (x_n)_{n\geq 1} \in \ell_1(\mathbb{N})$ be given and define $f_x \colon c_0(\mathbb{N}) \to \mathbb{C}$ by

$$f_x(y) = \sum_{n=1}^{\infty} x_n y_n, \quad \text{for all } y = (y_n)_{n\geq 1} \in c_0(\mathbb{N}). \tag{1}$$

(a) Argue that the sum in (1) is absolutely convergent, i.e. that $\sum_{n=1}^{\infty} |x_n y_n| < \infty$, and hence f_x is well-defined.

(b) Show that f_x is a linear functional on $c_0(\mathbb{N})$ and that its norm satisfies $\|f_x\| \leq \|x\|_1$.

(c) To prove $\|f_x\| \geq \|x\|_1$, apply f_x to appropriate $y \in c_0(\mathbb{N})$ such that

$$f_x(y) = \sum_{n=1}^{k} |x_n|,$$

and take the supremum over $k \in \mathbb{N}$.

(d) Let $f \in c_0(\mathbb{N})^*$ be arbitrary. Construct $x \in \ell_1(\mathbb{N})$ so $f = f_x$.

(e) Let $T \colon \ell_1(\mathbb{N}) \to c_0(\mathbb{N})^*$ be defined by $T(x) = f_x$, $x \in \ell_1(\mathbb{N})$. Show that T is a linear, norm-preserving bijection. (Hint: most of this follows from what you have done so far)

If you have completed all the steps above, you have proved that $c_0(\mathbb{N})^* \cong l_1(\mathbb{N})$ (isometrically isomorphic).

Problem 5: Show that $l_1(\mathbb{N})$ is a Banach space.

If $x = (x_n)_{n\geq 1} \in \ell_\infty(\mathbb{N})$ is given, define $f_x \colon \ell_1(\mathbb{N}) \to \mathbb{C}$ by

$$f_x(y) = \sum_{n=1}^{\infty} x_n y_n, \quad \text{for all } y = (y_n)_{n\geq 1} \in \ell_1(\mathbb{N}).$$

Follow the steps of Problem 4 to show that $(l_1(\mathbb{N}))^* \cong l_\infty(\mathbb{N})$ (isometrically isomorphic). Furthermore, show that for every $1 < p < \infty$, $l_p(\mathbb{N})$ is a Banach space and $(l_p(\mathbb{N}))^* \cong l_q(\mathbb{N})$ (isometrically isomorphic), where $1/p + 1/q = 1$. In this part, it may be useful to recall Hölder's inequality:

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|x\|_p \|y\|_q$$

for all $(x_n)_{n\geq 1} = x \in \ell_p(\mathbb{N})$ and $(y_n)_{n\geq 1} = y \in \ell_q(\mathbb{N})$.