Mandatory assignment, FunkAn 2

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Problem 1

Let H be an infinite dimensional seperable Hilbert space with orthonormal basis $(e_n)_{n\geq 1}$. Set $f_N = N^{-1}\sum_{n=1}^{N^2} e_n$ for all $N \geq 1$.

(a) Show that $f_N \to 0$ weakly, as $N \to \infty$ while $\|f_N\| = 1$ for all $N \geq 1$.

Since e_n is a basis for H it follows that $f_N \in H$ for all $N \geq 1$.
Now let $F_n : H \to \mathbb{C}$ be any linear bounded functional. By Riesz' representation thm. there exist $h = \sum_{n=1}^{\infty} \alpha_n e_n \in H$ s.t. $F_n(x) = \langle x, h\rangle$. Lets consider this

$$\begin{aligned} F_n(f_N) &= \langle N^{-1}\sum_{n=1}^{N^2} e_n, \sum_{n=1}^{\infty} \alpha_n e_n\rangle \\ &= N^{-1}\sum_{n=1}^{N^2} \langle e_n, \sum_{n=1}^{\infty} \alpha_n e_n\rangle \\ &= N^{-1}\sum_{n=1}^{N^2} \alpha_n \end{aligned}$$

By def. of weak convergence we want to show that $\frac{1}{\sqrt{N}}\sum_{n=1}^{N} \alpha_n \to 0$ as $n \to \infty$.
Now, by using both the triangle inequality and Cauchy-Schwarz' inequality we obtain that

$$\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} \alpha_n\right)^2 \leq \left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} |\alpha_n|\right)^2 \leq \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{N}}\right)^2 \sum_{n=1}^{N} |\alpha_n|^2 = \sum_{n=1}^{N} |\alpha_n|^2$$

Since $(\alpha_n)_{n\geq 1} \in \ell_2(\mathbb{N})$ by Riesz' representation thm. we now obtain, by def. of $\ell_2(\mathbb{N})$ that

$$\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N} \alpha_n\right| \leq \left(\sum_{n=1}^{N} |\alpha_n|^2\right)^{1/2} < \infty \quad \text{for all } N \geq 1$$

Since $\sum_{n=1}^{N} |\alpha_n|^2 < \infty$ there exist a $C \in \mathbb{C}$ s.t. $\sum_{n=1}^{N} |\alpha_n|^2 \to C$ when $n \to \infty$.
For all $\varepsilon > 0$ there exist m s.t. $\sum_{n=m+1}^{\infty} |\alpha_n|^2 < \varepsilon$. This shows that for any constant $K \geq 1$ $\sum_{n=m+1}^{K+m} |\alpha_n|^2 < \varepsilon$ holds. Now for $N \geq \frac{C^2}{\varepsilon^2}$ we have that

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{m} |\alpha_n| \leq \frac{\varepsilon}{C} \cdot C = \varepsilon$$

Now we can use Cauchy Schwarz' inequality and obtain

$$\begin{aligned}
\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\alpha_n\right| &\leq \frac{1}{\sqrt{N}}\sum_{n=1}^{N}|\alpha_n| \\
&= \frac{1}{\sqrt{N}}\sum_{n=1}^{m}|\alpha_n| + \frac{1}{\sqrt{N}}\sum_{n=m+1}^{N}|\alpha_n| \\
&\leq \varepsilon + \frac{1}{\sqrt{N}}\sum_{n=m+1}^{N+m}|\alpha_n| \\
&\leq \varepsilon + \sqrt{\left(\sum_{n=m+1}^{N+m}\frac{1}{N}\right)\left(\sum_{n=m+1}^{N+m}|\alpha_n|^2\right)} \\
&= \varepsilon + \sqrt{1\cdot\left(\sum_{n=m+1}^{N+m}|\alpha_n|^2\right)} \\
&< \varepsilon + \sqrt{\varepsilon}
\end{aligned}$$

This shows that $\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\alpha_n\right| \to 0$ as $N \to \infty$ which implies that $\left|\frac{1}{N}\sum_{n=1}^{N^2}\alpha_n\right| \to 0$ as $N \to \infty$. We now obtain that $\lim_{n\to\infty}\frac{1}{N}\sum_{n=1}^{N^2}\alpha_n = 0$, but $\lim_{n\to\infty}\frac{1}{N}\sum_{n=1}^{N^2}\alpha_n = \lim_{n\to\infty}F_n(f_N)$. Since F is bounded, hence continuous we have now obtained the desired, that $f_N \to 0$ weakly as $N \to \infty$.

Now lets compute $\|f_N\|$.

$$\begin{aligned}
\|f_N\|^2 = \|N^{-1}\sum_{n=1}^{N^2}e_n\|^2 &= |N^{-1}|^2\|\sum_{n=1}^{N^2}e_n\|^2 \\
= N^{-2}\|\sum_{n=1}^{N^2}e_n\|^2 &= N^{-2}\sum_{n=1}^{N^2}\|e_n\|^2 \\
= N^{-2}\sum_{n=1}^{N^2}1^2 &= N^{-2}N^2 \\
= 1 &
\end{aligned}$$

This shows that $\|f_N\| = 1$ for all $N \leq 1$. $\square$

Let K be the norm closure of co$\{f_n : N \geq 1\}$.

(b) Argue that K is weakly compact, and that $0 \in K$.

We have that $K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|}$, and since $\text{co}\{f_N : N \geq 1\}$ is convex by definition of the convex hull we obtain, by thm. 5.7, that

$$K = \overline{\text{co}\{f_N : N \geq 1\}}^{\|\cdot\|} = \overline{\text{co}\{f_N : N \geq 1\}}^{\tau_w}$$

i.e. that the norm and the weak closure coincide. This shows that K is weakly closed. Since K is weakly closed, and since we showed in (a) that $f_N \to 0$ weakly as $N \to \infty$, then $0 \in K$.

Now lets consider the unit ball $\overline{B_{H^*}(0,1)} \subset H^*$.
By Alaoglu's thm. we know that $\overline{B_{H^*}(0,1)}$ is compact in the w^*-topology. Since H is a Hilbert space it follows by prop. 2.10 that it is a reflexive Banach space. By thm. 5.9 and the topologies on H^* we obtain that $\tau_w = \tau_{w^*}$ and thereby we get that $\overline{B_{H^*}(0,1)}$ is weakly compact.
By Riesz' representation thm. we have that for every $y \in H$ every element in H^* is given by $F_y = \langle \cdot, y\rangle$. This shows that we have an isomorphism from H^* to H, which sends F_y to y. Then we have an isomorphism between $\overline{B_{H^*}(0,1)}$ and $\overline{B_H(0,1)}$, why $\overline{B_H(0,1)}$ also is weakly compact. Since $K \subseteq \overline{B_H(0,1)}$ we now obtain that K, the weakly closed set, is a subset of a weakly compact set, hence K is weakly compact. □

(c) Show that 0, as well as each f_N, $N \geq 1$ are extreme points in K.

By def. 7.1 we obtain that

$$\text{Ext}(K) = \{x \in K \mid x = \alpha x_1 + (1-\alpha)x_2 \text{ implies } x_1 = x_2 = x,\ x_1, x_2 \in K,\ 0 < \alpha < 1\}$$

Lets first show that $0 \in \text{Ext}(K)$.
Note that by def. $K \subseteq H$ is a non-empty convex compact subset. Lets consider the continuous linear functional $G_n = \langle \cdot, -e_n\rangle \in H^*$ for any $n \in \mathbb{N}$. Note that $G_n(K) \subseteq \mathbb{R}$. Now let

$$C = \sup_n\{\langle x, -e_n\rangle \mid x \in K\} = \sup_n\{-\langle x, e_n\rangle \mid x \in K\}$$

Since $x \in K$ we know that $x \geq 0$, and we furthermore have that $0 \in K$, why we obtain that $-\langle x, e_n\rangle \leq 0$ for $x \in K$. We can now use lemma 7.5, why we get that $F_n := \{x \in K \mid \text{Re}\langle x, -e_n\rangle = 0\} \neq \emptyset$ is a compact face of K for all $n \in \mathbb{N}$.
We have that $0 \in F_n$ for all $n \in \mathbb{N}$ why $0 \in \cap_{n=1}^\infty F_n \neq \emptyset$. Since the only element which is orthogonal on all elements e_n is zero we obtain

$$\bigcap_{n=1}^\infty F_n = \{x \in K \mid \text{Re}\langle x, -e_n\rangle = 0,\ \forall n \in \mathbb{N}\} = \{0\}$$

Now we can use remark 7.4(3) to say that $\cap_{n=1}^\infty F_n = \{0\}$ is a face of K and by applying remark 7.4(1) we have now reached that $0 \in \text{Ext}(K)$ as desired.

Now lets show that $f_N \in \text{Ext}(K)$.
Lets fix $N \geq 1$ and suppose that $f_N = \alpha x_1 + (1-\alpha)x_2$ for $x_1, x_2 \in K$ and $0 < \alpha < 1$. We know that $1 = \|f_N\|^2 = f_N, f_N\rangle$. Now consider

$$\begin{aligned} 1 = \langle f_N, f_N\rangle &= \langle \alpha x_1 + (1-\alpha)x_2, f_N\rangle \\ &= \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle \end{aligned}$$

this implies that

$$\begin{aligned} 0 &= \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle - 1 \\ &= \alpha\langle x_1, f_N\rangle + (1-\alpha)\langle x_2, f_N\rangle - (\alpha + (1-\alpha)) \\ &= \alpha(\langle x_1, f_N\rangle - 1) + (1-\alpha)(\langle x_2, f_N\rangle - 1) \end{aligned}$$

since $0 < \alpha < 1$ and $\langle x_1, f_N\rangle$, $\langle x_2, f_N\rangle \geq 0$ we can see that $0 \leq \langle x_i, f_N\rangle \leq 1$ for $i = 1, 2$. But by what we just found this shows that $\langle x_1, f_N\rangle = 1 = \langle x_1, f_N\rangle$.
Now we wanna show that $x_1 = x_2 = f_N$, since it would then follow that $f_N \in \mathrm{Ext}(K)$. That $x_1 = f_N$ and that $x_2 = f_N$ is found with the same approach, why I will only show that $x_1 = f_N$.
See that

$$1 = \|\langle x_1, f_N\rangle\| \leq \|x_1\|\|f_N\| = \|x_1\|$$

by Cauchy-Schwarz. Since $x_1 \in K \subseteq \overline{B_H(0,1)}$, then $\|x_1\| \leq 1$. This shows that

$$1 = \|\langle x_1, f_N\rangle\| = \|x_1\|\|f_N\| = \|x_1\|$$

Then F_N and x_1 are linealy dependent, why $x_1 = \lambda f_N$ for a scalar λ. Then it follows that

$$1 = \langle \lambda f_N, f_N\rangle = \lambda\langle f_N, f_N\rangle = \lambda\|f_N\|^2 = \lambda$$

which shows that $x_1 = f_N$ why $f_N \in \mathrm{Ext}(K)$ for all $N \geq 1$. □

(d) Are there any other extreme points in K?

See that $K = \overline{\mathrm{co}\{f_N \mid N \geq 1}^{\tau_w}$ is a non-empty convex subset for H. By Milmans thm. we get that $\mathrm{Ext}(K) \subseteq \overline{\{f_N \mid N \geq 1\}}^{\tau_w}$.
By (c) we now obtain that $\{f_N \mid N \geq 1\} \cup \{0\} \subseteq \overline{\{f_N \mid N \geq 1\}}^{\tau_w}$.
Since H is a normed space it is metrizable and then $\{f_N \mid N \geq 1\}$ is also metrizable. This shows that $\{f_N \mid N \geq 1\}$ is first countable and it is then enough to consider sequences in $\{f_N \mid N \geq 1\}$ instead of nets.
Now lets assume that $(x_n)_{n\geq 1}$ is a sequence in $\{f_N \mid N \geq 1\}$ which converges weakly to $x \in \overline{\{f_N \mid N \geq 1\}}^{\tau_w}$. It then follows that each $x_i = f_N$ for some $N \geq 1$, why x is equal to some F_N or to zero. We then obtain that

$$\mathrm{Ext}(K) \subseteq \overline{\{f_N \mid N \geq 1\}}^{\tau_w} = \{f_N \mid N \geq 1\} \cup \{0\}$$

And since we by (c) have that

$$\{f_N \mid N \geq 1\} \cup \{0\} \subseteq \mathrm{Ext}(K)$$

we can conclude that $\mathrm{Ext}(K) = \{f_N \mid N \geq 1\} \cup \{0\}$ why there are no other extreme points in K. □

Problem 2

Let X and Y be infinite dimensional Banach spaces.

(a) Let $T \in \mathcal{L}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $Tx_n \to Tx$ weakly, as $n \to \infty$.

Assume that $x_n \to x$ weakly as $n \to \infty$ for $x \in X$. From HW 4 problem 2 we know that this holds if and only if $Fx_n \to Fx$ for all $F \in X^*$. I can use this problem since a net is said to be a more general case of a sequence.

Now lets take $G \in Y^*$, then we obtain that the decomposition $G \circ T \in X^*$, why $(G \circ T)(x_n) \to (G \circ T)(x)$ as $n \to \infty$ for all $G \in Y^*$. But this means exactly what we wanted to show, that $Tx_n \to Tx$ weakly as $n \to \infty$. $\square$

(b) Let $T \in \mathcal{K}(X, Y)$. For a sequence $(x_n)_{n\geq 1}$ in X and $x \in X$, show that $x_n \to x$ weakly, as $n \to \infty$, implies that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$.

Assume that $x_n \to x$ weakly as $n \to \infty$ for $x \in X$. Lets assume by contradiction that $\|Tx_n - Tx\| \nrightarrow 0$ as $n \to \infty$. Then there exist a subsequence $(x_{n_i})_{i\geq 1}$ and $\varepsilon > 0$ s.t. $\|Tx_{n_i} - Tx\| > \varepsilon$ for all $i \geq 1$.
Since $x_n \to x$ weakly as $n \to \infty$, we get that $x_{n_i} \to x$ weakly as $n \to \infty$ as well. We obtain that $(x_{n_i})_{i\geq 1}$ is bounded, which means that it has a subsequence $(x_{n_{i_k}})_{k\geq 1}$ which fulfills that $\|Tx_{n_{i_k}} - Tx'\| \to 0$ as $k \to \infty$ for some $x' \in X$. We can now use (a) to say that $Tx_{n_i} \to Tx$ weakly as $i \to \infty$ since $x_{n_i} \to x$ weakly as $i \to \infty$, but then it also holds that $Tx_{n_{i_k}} \to Tx$ weakly as $k \to \infty$. If something converges by norm to something, then it will also converge weakly to the same, why we must obtain that $Tx' = Tx$ which shows that $\|Tx_{n_{i_k}} - Tx\| \to 0$ as $k \to \infty$. However this is a contradiction to what we found earlier, that $\|Tx_{n_i} - Tx\| > \varepsilon$ for all $i \geq 1$, why we have reached a contradiction and can conclude that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. $\square$

(c) Let H be a separable infinite dimensional Hilbert Space. If $T \in \mathcal{L}(H, Y)$ satisfies that $\|Tx_n - Tx\| \to 0$, as $n \to \infty$, whenever $(x_n)_{n\geq 1}$ is a sequence in H converging weakly to $x \in H$, then $T \in \mathcal{K}(H, Y)$.

Lets assume by contradiction that T is *not* compact (i.e. $T \notin K(H, Y)$), but by prop. 8.2 this holds if and only if the closed unit ball $T(\bar{B}_H(0, 1))$ is *not* totally bounded, and by def. this means that there exist $\delta > 0$ s.t. every finite union of open balls with radius δ does not cover $T(\bar{B}_H(0, 1))$.
Now lets take an $x_1 \in \bar{B}_H(0, 1)$ where $x_1 \in (x_n)_{n\geq 1} \subset \bar{B}_H(0, 1)$. Assume that $x_2, x_3, ..., x_n$ are satisfying that $\|Tx_q - Tx_r\| \geq \delta$ for all $1 < q, r \leq n$ and $q \neq r$. Now lets define the set

$$M := T(\bar{B}_H(0, 1) \cap (\cup_{i=1}^n B_Y(Tx_i, \delta))^C$$

Observe that $M \neq \emptyset$, since $T(\bar{B}_H(0, 1))$ is *not* totally bounded, why we obtain that $T(\bar{B}_H(0, 1)) \subset (\cup_{i=1}^n B_Y(Tx_i, \delta))^C$.
Now lets take $x_{n+1} \in \bar{B}_H(0, 1)$ s.t. we obtain $Tx_{n+1} \in M$, thereby we also get that $Tx_{n+1} \in (\cup_{i=1}^n B_Y(Tx_i, \delta))^C$ and following this also that $Tx_{n+1} \notin B_Y(Tx_i, \delta)$ for any i. This shows that $\|Tx_{n+1} - Tx_i\| \geq \delta$ for all $i \leq n$. We can continue this process, thereby obtaining a sequence $(x_n)_{n\geq 1}$ s.t. $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$.
By prop. 2.10 H is reflexive, why $\bar{B}_H(0, 1)$ is weakly compact by thm. 6.3. This shows that every sequence has a weakly convergent subsequence $(x_{n_k})_{k\geq 1}$. Since we found that $\|Tx_n - Tx_m\| \geq \delta$ for all $n \neq m$ we will then obtain that $\|Tx_{n_k} - Tx\| \geq \delta$, hence that $\|Tx_{n_k} - Tx\| \nrightarrow 0$ as $k \to \infty$, since we assumed that $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. This is a contradiction, why T must be compact, i.e. $T \in K(H, Y)$. $\square$

(d) Show that each $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact.

Take $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ and let $(x_n)_{n\geq 1} \in \ell_2(\mathbb{N})$. Suppose further that $x_n \to x$ weakly as $n \to \infty$. By (a) this implies that $Tx_n \to Tx$ weakly in $\ell_1(\mathbb{N})$ as $n \to \infty$. Using remark 5.3 this holds if and only if $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. Now we can use (c) (since $\ell_2(\mathbb{N})$ by def. is a infinite dimensional Hilbert space, and by HW4 problem 4 also separable) to conclude that $T \in \mathcal{L}(\ell_2(\mathbb{N}), \ell_1(\mathbb{N}))$ is compact. $\square$

(e) Show that <u>no</u> $T \in \mathcal{K}(X, Y)$ is onto.

Suppose that $T \in \mathcal{L}(X, Y)$ is compact and onto, thereby surjective and by the Open mapping thm. also open. Since X, Y are normed vector spaces and T is open we get (by p. 18 of the lecture notes) that there exist $r > 0$ s.t. $B_Y(0, r) \subset T(B_X(0, 1))$, hence also that $\overline{B_Y(0, r)} \subset \overline{T(B_X(0, 1))}$ (since closure preserves inclusion). Since T is a compact operator $\overline{T(B_X(0, 1))}$ is compact and it also follows that $\overline{B_Y(0, r)}$ is compact. Now lets consider different values of r.

- $r = 1$
  Then it follows that $\overline{B_Y(0, r)} = \overline{B_Y(0, 1)}$, and since $\overline{B_Y(0, r)}$ is compact so is $\overline{B_Y(0, 1)}$. But since Y is an infinite-dimensional normed space it follows from Riesz's lemma that $\overline{B_Y(0, 1)}$ cannot be compact, why we have reached a contradiction.

- $r > 1$
  Then $\overline{B_Y(0, 1)}$ is a closed set of the compact set $\overline{B_Y(0, r)}$, hence compact as well, but with the same argument as before this is a contradiction.

- $r < 1$
  Lets consider the map $g : Y \to Y$ given by $x \mapsto \frac{1}{r}x$, which is continuous. We know that the image under a continuous function of a compact set is compact, why we obtain that $g(\overline{B_Y(0, 1)}) = \overline{B_Y(0, 1)}$ is compact, which again is a contradiction.

So we have now showed that $\overline{B_Y(0, r)}$ is not compact for any r, which is a contradiction, hence no $T \in \mathcal{K}(X, Y)$ is onto. $\square$

(f) Let $H = L_2([0, 1], m)$, and consider the operator $M \in \mathcal{L}(H, H)$ given by $Mf(t) = tf(t)$, for $f \in H$ and $t \in [0, 1]$. Justify that M is self-adjoint, but not compact.

First lets show that M is self-adjoint.
Observe that $t = \bar{t}$ since t only has real values. Now lets consider the inner product on

H.

$$
\begin{aligned}
\langle Mf, g\rangle &= \int_0^1 Mf(t)g(\bar{t})dm(t) \\
&= \int_0^1 tf(t)g(\bar{t})dm(t) \\
&= \int_0^1 f(t)tg(\bar{t})dm(t) \\
&= \int_0^1 f(t)tg(t)dm(t) \\
&= \int_0^1 f(t)Mg(t)dm(t) \\
&= \langle f, Mg\rangle
\end{aligned}
$$

Where I have used p. 56 of the lecture notes.
This shows that $M = M^*$ and by def. that it is self-adjoint.

Now lets justify that M is not compact.
Lets assume by contradiction that M is compact. We have furthermore just showed that M is self-adjoint. H is by HW 4 problem 4 seperable and we also know that it is infinite-dimensional, so thm. 10.1 implies that H has an othonormal basis consisting of eigenvectors for M with corresponding eigenvalues. In HW 6 problem 3 we proved that M has no eigenvalues, why we have reached a contradiction, which shows that M is compact. $\square$

Problem 3

Consider the Hilbert space $H = L_2([0,1], m)$, where m is the Lebesgue measure. Define $K : [0,1] \to \mathbb{R}$ by

$$
K(s,t) = \begin{cases} (1-s)t, & \textbf{if } 0 \le t \le s \le 1, \\ (1-t)s, & \textbf{if } 0 \le s \le t \le 1, \end{cases}
$$

and consider $T \in \mathcal{L}(H, h)$ defined by

$$
(Tf)(s) = \int_{[0,1]} K(s,t)f(t)dm(t), \quad s \in [0,1], \;\; f \in H
$$

(a) Justify that T is compact.

Note that $[0,1]$ is in $\mathbb{R}$ hence a compact Hausdorff topological space. Furthermore K is, by how it is defined, continuous on $[0,1] \times [0,1]$, hence $K \in C([0,1] \times [0,1])$. At last, see that since m is the Lebesgue measure it is a finite Borel measure on $[0,1]$. Now we can use thm. 9.6 to conclude tha T is compact. $\square$

after noting that $T = T_{\tilde{K}}$ for $\tilde{K}(s,t) = K(t,s)$

(b) Show that $T = T^*$.

Observe that $K(s,t) = K(t,s)$ always. Now lets consider the inner product on H.

$$\begin{aligned}
\langle Tf, g\rangle &= \int_{[0,1]} Tf(s)\overline{g(s)}\mathrm{dm}(s) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(s,t)f(t)\mathrm{dm}(t)\right)\overline{g(s)}\mathrm{dm}(s) \\
&= \int_{[0,1]\times[0,1]} K(s,t)f(t)\overline{g(s)}\mathrm{dm}(s,t) \\
&= \int_{[0,1]\times[0,1]} K(t,s)f(t)\overline{g(s)}\mathrm{dm}(s,t) \\
&= \int_{[0,1]\times[0,1]} K(t,s)\overline{g(s)}f(t)\mathrm{dm}(s,t) \\
&= \int_{[0,1]} \left(\int_{[0,1]} K(t,s)\overline{g(s)}\mathrm{dm}(s)\right) f(t)\mathrm{dm}(t) \\
&= \int_{[0,1]} \overline{Tg(t)}f(t)\mathrm{dm}(t) \\
&= \langle f, Tg\rangle
\end{aligned}$$

K real

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Here it is assumed that $K \in L_2(X \times Y)$

Where I have used p. 56 of the lecture notes and Fubini-Tonelli's thm. twice. This shows that $T = T^*$, hence self-adjoint. □

So this needs to be shown.

(c) Show that

$$(Tf)(s) = (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t), \quad s \in [0,1], \; f \in H.$$

Use this to show that Tf is continuous on $[0,1]$, and that $(Tf)(0) = (Tf)(1) = 0$.

First lets look at $(Tf)(s)$

$$\begin{aligned}
(Tf)(s) &= \int_{[0,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} K(s,t)f(t)dm(t) + \int_{[s,1]} K(s,t)f(t)dm(t) \\
&= \int_{[0,s]} (1-s)tf(t)dm(t) + \int_{[s,1]} (1-t)sf(t)dm(t) \\
&= (1-s)\int_{[0,s]} tf(t)dm(t) + s\int_{[s,1]} (1-t)f(t)dm(t)
\end{aligned}$$

✓

This follows by linearity of integrals and furthermore that $s \in [0,1]$.

Lets use this to show that Tf is continuous.
By prop. 1.10 Tf is continuous if it is bounded. Lets show this by looking at each integral

N that is for linear operator. Tf is function (not necessarily linear)

separately.
By def. of $L_2([0,1],m)$ we obtain that

$$\left(\int_{[0,1]} |f(t)|^2 dm(t)\right)^{1/2} < \infty.$$

Since $s \in [0,1]$ this also shows that

$$(1-s)\left(\int_{[0,s]} t|f(t)|^2 dm(t)\right)^{1/2} < \infty$$

and at last that

$$(1-s)\int_{[0,s]} tf(t)dm(t) < \infty.$$

The exact same can be done for the other part of (Tf)(s) why we could obtain

$$s\int_{[s,1]} (1-t)f(t)dm(t) < \infty$$

which shows that Tf is bounded on $[0,1]$, hence continuous.

Now lets show that $(Tf)(0) = (Tf)(1) = 0$.
First notice that

$$\begin{aligned}
(Tf)(0) &= (1-0)\int_{[0,0]} tf(t)dm(t) + 0\int_{[0,1]} (1-t)f(t)dm(t) \\
&= \int_{[0,0]} tf(t)dm(t) \\
&= 0
\end{aligned}$$

And now that

$$\begin{aligned}
(Tf)(1) &= (1-1)\int_{[0,1]} tf(t)dm(t) + 1\int_{[1,1]} (1-t)f(t)dm(t) \\
&= \int_{[1,1]} (1-t)f(t)dm(t) \\
&= 0
\end{aligned}$$

Hence $(Tf)(0) = (Tf)(1) = 0$. ✓ □

Problem 4

Consider the Schwartz space $\mathscr{S}(\mathbb{R})$ and view the Fourier transform as a linear map $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$.

(a) For each integer $k \geq 0$, set $g_k(x) = x^k e^{-x^2/2}$, for $x \in \mathbb{R}$.
Justify that $g_k \in \mathscr{S}(\mathbb{R})$, for all integers $k \geq 0$.
Compute $\mathcal{F}(g_k)$, for $k = 0, 1, 2, 3$.

First lets justify that $g_k \in \mathscr{S}(\mathbb{R})$ for all integers $k \geq 0$.
By HW 7 problem 1 we obtain that $e^{-x^2} \in \mathscr{S}(\mathbb{R})$, and then for $a = \sqrt{2} \in \mathbb{R} \setminus \{0\}$ that $S_{\sqrt{2}} e^{-x^2} \in \mathscr{S}(\mathbb{R})$. By p. 62 in the lecture notes we obtain $S_{\sqrt{2}} e^{-x^2} = e^{-x^2/2} \in \mathscr{S}(\mathbb{R})$.
By applying HW 7 problem 1 again we have obtained $g_k \in \mathscr{S}(\mathbb{R})$ as desired.

Now lets compute $\mathcal{F}(g_k)$ for $k = 0, 1, 2, 3$.
Let $\varphi(x) := e^{-x^2/2}$ and note that this is integrable. See also that $x^k e^{-x^2/2}$ is integrable.
Note that $\varphi(x) = \hat{\varphi}(x)$ by prop. 11.4 for $n = 1$. Using this and prop. 11.3 we obtain that

$$\begin{aligned}
\mathcal{F}(g_k)(\xi) &= \hat{g}_k(\xi) \\
&= (g_k)^\wedge(\xi) \\
&= (x^k \varphi)^\wedge \xi \\
&= i^k (\partial^k \hat{\varphi})(\xi) \\
&= i^k (\partial^k \varphi)(\xi)
\end{aligned}$$

And we obtain:

<u>$k = 0$</u>.
$\mathcal{F}(g_0)(\xi) = i^0 (\partial^0 \varphi)(\xi) = e^{-\xi^2/2}$

<u>$k = 1$</u>.
$\mathcal{F}(g_1)(\xi) = i^1 (\partial^1 \varphi)(\xi) = -i\xi e^{-\xi^2/2}$

<u>$k = 2$</u>.
$\mathrm{F}(\mathrm{g}_2)(\xi) = i^2 (\partial^2 \varphi)(\xi) = i^2 e^{-\xi^2/2}(\xi^2 - 1) = e^{-\xi^2/2} - \xi^2 e^{-\xi^2/2}$

<u>$k = 3$</u>.
$\mathcal{F}(g_3)(\xi) = i^2 (\partial^3 \varphi)(\xi) = i^3 \xi e^{-\xi^2/2}(3 - \xi^2) = i\xi^3 e^{-\xi^2/2} - 3i\xi e^{-\xi^2/2}$ $\square$

(b) Find non-zero functions $h_k \in \mathscr{S}(\mathbb{R})$ such that $\mathcal{F}(h_k) = i^k h_k$, for $k = 0, 1, 2, 3$.

For non-zero $h_0 \in \mathscr{S}(\mathbb{R})$ we wanna show that $\mathcal{F}(h_0) = i^0 h_0 = h_0$.
Lets compute $\mathcal{F}(g_0(\xi))$.

$$\mathcal{F}(g_0(\xi)) = e^{-\xi^2/2} = g_0(\xi)$$

So for $h_0 = g_0$ we obtain $\mathcal{F}(h_0) = h_0$ as desired.

For non-zero $h_1 \in \mathscr{S}(\mathbb{R})$ we wanna show that $\mathcal{F}(h_1) = i^1 h_1 = ih_1$.
Notice that

$$\mathcal{F}(g_3)(\xi) = i\xi^3 e^{-\xi^2/2} - 3i\xi e^{-\xi^2/2} = i(g_3(\xi) - 3g_1(\xi))$$

Now lets compute $\mathcal{F}(g_3(\xi) - \frac{3}{2} g_1(\xi))$.

$$\begin{aligned}
\mathcal{F}(g_3(\xi) - \frac{3}{2} g_1(\xi)) &= \mathcal{F}(g_3(\xi)) - \frac{3}{2} \mathcal{F}(g_1(\xi)) \\
&= i(g_3(\xi) - 3g_1(\xi)) + \frac{3}{2} i\xi e^{-\xi^2/2} \\
&= i(g_3(\xi) - \frac{3}{2} g_1(\xi))
\end{aligned}$$

Why we obtain $\mathcal{F}(h_1) = ih_1$ for $h_1 = g_3 - \frac{3}{2}g_1$.

For non-zero $h_2 \in \mathscr{S}(\mathbb{R})$ we want to show that $\mathcal{F}(h_2) = i^2 h_2 = -h_2$.
First notice that

$$\mathcal{F}(g_2)(\xi) = e^{-\xi^2/2} - \xi^2 e^{-\xi^2/2} = g_0(\xi) - g_2(\xi)$$

Lets compute $\mathcal{F}(g_2(\xi) - \frac{1}{2}g_0(\xi))$.

$$\begin{aligned}\mathcal{F}(g_2(\xi) - \frac{1}{2}g_0(\xi)) &= \mathcal{F}(g_2(\xi)) - \frac{1}{2}\mathcal{F}(g_0(\xi)) \\ &= g_0(\xi) - g_2(\xi) - \frac{1}{2}g_0(\xi) \\ &= -g_2(\xi) + \frac{1}{2}g_0(\xi) \\ &= -(g_2(\xi) - \frac{1}{2}g_0(\xi))\end{aligned}$$

Which shows that $\mathcal{F}(h_2) = -h_2$ for $h_1 = g_2 - \frac{1}{2}g_0$.

For non-zero $h_3 \in \mathscr{S}(\mathbb{R})$ we want to show that $\mathcal{F}(h_3) = i^3 h_3 = -ih_2$.
Lets notice that

$$\mathcal{F}(g_1)(\xi) = -i^{-\xi^2/2} = -ig_1(\xi)$$

typo

Why we have obtained that $\mathcal{F}(h_3) = -ih_3$ when $h_3 = g_1$. ✓ $\square$

(c) Show that $\mathcal{F}^4(f) = f$, for all $f \in \mathscr{S}(\mathbb{R})$.

Lets compute $\mathcal{F}^2(f)$

$$\begin{aligned}\mathcal{F}^2(f(\xi)) &= \mathcal{F}(\mathcal{F}(f(\xi)) = \mathcal{F}(\hat{f}(\xi)) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(x)e^{-ix\xi}\mathrm{dx}\end{aligned}$$

Where I have used def. 11.1, which I can since HW 7 problem 1 states that $\mathscr{S}(\mathbb{R}) \subset L_1(\mathbb{R})$ why $f \in L_1(\mathbb{R})$.
Now lets define $T(f) = S_{-1}(f)$ which by Hw 7 problem 1 is in $\mathscr{S}(\mathbb{R})$ since $f \in \mathscr{S}(\mathbb{R})$.
Now observe that

$$T^2 f(x) = T(Tf(x)) = T(S_{-1}f(x)) = (Tf(-x)) = S_{-1}f(-x) = f(x)$$

Where we have used p. 62 in the lecture notes. This shows that $T^2 = Id$.
Furthermore see that

$$\begin{aligned}Tf(\xi) &= f(-\xi) \\ &= \mathcal{F}^*(\mathcal{F}(f(-\xi))) \\ &= \mathcal{F}^*(\hat{f}(-\xi)) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(x)e^{-ix\xi}\mathrm{dx} \\ &= \mathcal{F}^2(f(\xi))\end{aligned}$$

So now we have obtained the desired since

$$\mathcal{F}^4(f) = \mathcal{F}^2(\mathcal{F}^2(f)) = T^2(f) = f.$$

$\square$

(d) Use (c) to show that if $f \in \mathscr{S}(\mathbb{R})$ is non-zero and $\mathcal{F}(f) = \lambda f$, for some $\lambda \in \mathbb{C}$, then $\lambda \in \{1, i, -1, -i\}$. Conclude that the eigenvalues of $\mathcal{F}$ precisely are $\{1, i, -1, -i\}$.

Assume $f \in \mathscr{S}(\mathbb{R})$ is non-zero. To show that $\lambda \in \{1, i, -1, -i\}$ it suffices to show that $\lambda^4 = 1$.
Let $\mathcal{F}(f) = \lambda f$. This would imply that $\lambda^4 f^4 = \mathcal{F}^4(f) = f$ (by (c)), and moreover that $\lambda^4 = \frac{f}{f^4}$. ← f need not be non-zero everywhere! ← where does this come from ?
By (c) we furthermore obtain that

$$\text{No} \qquad f^2 = \mathcal{F}^8(f) = \mathcal{F}^4(\mathcal{F}^4(f)) = \mathcal{F}^4(f) = f$$

$\mathcal{F}^4(f) = f$

$f^2 = (\mathcal{F}^4(f))^2$

why

$$f^4 = (f^2)^2 = f^2 = f$$

Then we obtain

$$\lambda^4 = \frac{f}{f^4} = \frac{f}{f} = 1$$

And we have obtained the desired that $\lambda \in \{1, i, -i - i\}$.
Since these values for λ are the only that satisfy $\mathcal{F}(f) = \lambda(f)$, the eigenvalues of $\mathcal{F}$ are precisely $\{1, i, -1, -i\}$. you have not shown that $\{1, i, -i, -1\}$ are actually eigenvalues $\square$

Problem 5

Let $(x_n)_{n\geq 1}$ be a dense subset of $[0,1]$ and consider the Radon measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\delta_{x_n}$ on $[0,1]$. Show that supp$(\mu) = [0,1]$.

Using HW 8 problem 3 we have to show that $\mu([0,1]^C) = 0$.
First lets look at the Dirac mass:

$$\delta_{x_n}([0,1]^C) = \begin{cases} 0 & , x_n \in [0,1] \\ 1 & , x_n \notin [0,1] \end{cases}$$

So we obtain

$$\mu([0,1]^C) = \sum_{n=1}^{\infty} 2^{-1}\delta_{x_n}([0,1]^C) = 0$$

since μ is defined on $[0,1]$ where δ_{x_n} is exactly 0. Now we have obtained, by HW 8 problem 3, that

$$\text{supp}(\mu) = [0,1]$$

as desired. $\square$