weberknecht – a One-Sided Crossing Minimization solver

Johannes Rauch
Ulm University, Institute of Optimization and Operations Research, Germany

Abstract

We describe the implementation of the exact solver weberknecht[1] and the heuristic solver weberknecht_h for the ONE-SIDED CROSSING MINIMIZATION optimization problem.

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1 Introduction and preliminaries

An instance $(G = (A\dot{\cup}B, E), \pi_A)$ of ONE-SIDED CROSSING MINIMIZATION is a bipartite graph G with n vertices, bipartition sets A and B, and a linear ordering π_A. The goal is to find a linear ordering π_B of B that minimizes the number of crossing edges if the graph were to be drawn in the plane such that the vertices of A and B are on two distinct parallel lines, respectively, the order of the vertices of A and B on the lines is consistent with the linear orderings π_A and π_B, respectively, and the edges are drawn as straight lines. ONE-SIDED CROSSING MINIMIZATION was the Parameterized Algorithms and Computational Experiments Challenge 2024[2].

We may assume that $A = [n_0] := \{1, \dots, n_0\}$ and $B = \{n_0 + 1, \dots, n_0 + n_1\}$ for some positive integers n_0 and n_1. We think of π_A and π_B as bijections $A \to [n_0]$ and $B \to [n_1]$, respectively. With this we can view ONE-SIDED CROSSING MINIMIZATION as a purely combinatorial problem, that is, the edges a_1b_1 and a_2b_2 of G with $a_1, a_2 \in A$ and $b_1, b_2 \in B$ cross if and only if

$$(\pi_A(a_1) < \pi_A(a_2) \text{ and } \pi_B(b_1) > \pi_B(b_2)) \text{ or } (\pi_A(a_1) > \pi_A(a_2) \text{ and } \pi_B(b_1) < \pi_B(b_2)),$$

and we wish to minimize the number of crossing edges. If $\pi_B(u) < \pi_B(v)$ for $u, v \in B$, we say that u is ordered before v, or u is to the left of v.

Let $c_{u,v}$ denote the number of crossings of edges incident to $u, v \in B$ with $u \neq v$ if $\pi_B(u) < \pi_B(v)$. A mixed-integer program for ONE-SIDED CROSSING MINIMIZATION is given by

$$\begin{aligned} &\text{minimize} \sum_{\substack{u,v \in B \\ u<v}} (c_{u,v} - c_{v,u}) \cdot x_{u,v} + \sum_{\substack{u,v \in B \\ u<v}} c_{v,u} \\ &\text{subject to } 0 \le x_{u,v} + x_{v,w} - x_{u,w} \le 1 \quad \text{for all } u, v, w \in B, u < v < w, \\ &\qquad\qquad\qquad\qquad x_{u,v} \in \{0, 1\} \quad \text{for all } u, v \in B, u < v. \end{aligned} \tag{P_I}$$

[1] Weberknecht is the german name for the harvestman spider. It is a composite word consisting of the words Weber = weaver and Knecht = workman.

[2] https://pacechallenge.org/2024/

We refer to the article of Jünger and Mutzel [7] for a detailed derivation of the mixed-integer program (P_I). Observe that u is ordered before v if and only if $x_{u,v} = 1$ for $u, v \in B$ with $u < v$.

For more information on One-Sided Crossing Minimization and graph drawing in general we refer to the handbook of graph drawing and visualization of Tamassia [9].

2 Overview

We describe the general modus operandi of weberknecht(_h), which are written in C++ and available on GitHub[3]. First, the exact solver weberknecht runs the uninformed and improvement heuristics described in Section 3. Then it applies the data reduction rules described in Section 4. Last, it solves a reduced version of the mixed-integer program (P_I) associated to the input instance with a custom branch and bound and cut algorithm described in Section 5. The heuristic solver weberknecht_h only runs the uninformed and improvement heuristics (except the local search heuristic).

3 Heuristics

We distinguish between uninformed and informed heuristics, which build a solution from the ground up, and improvement heuristics, which try to improve a given solution. Due to the reduction rules we may assume from here that there are no isolated vertices in G.

Uninformed Heuristics. The uninformed heuristics order the vertices of B such that the *scores* $s(v)$ of vertices $v \in B$ is non-decreasing.

- In the *barycenter heuristic*, we set $s(v) = \frac{1}{d_G(v)} = \sum_{u \in N_G(v)} u$ (recall that $A = [n_0]$). Eades and Wormald [4] proved that this method has an $\mathcal{O}(\sqrt{n})$ approximation factor, which is best possible up to a constant factor under certain assumptions.
- Let $d = d_G(v)$ and let $w_0, \dots, w_{d-1}$ be the neighbors of v in G with $w_0 < \cdots < w_{d-1}$. In the *median heuristic*, the score of v is $s(v) = w_{(d-1)/2}$ if d is odd and $s(v) = (w_{d/2-1} + w_{d/2})/2$ if d is even. Eades and Wormald [4] proved that this method is a 3-approximation algorithm.
- In the *probabilistic median heuristic*, we draw a value x from $[0.0957, 0.9043]$ uniformly at random, and the score of v is then $s(v) = w_{\lfloor x \cdot d \rfloor}$. This is essentially the approximation algorithm of Nagamochi [8], which has an approximation factor of 1.4664 in expectancy.

Informed Heuristics. The informed heuristics get a fractional solution of the linear program relaxation of (P_I) as an additional input.

- The *sort heuristic* works like a uninformed heuristics. The score for vertex $v \in B$ is $s(v) = \sum_{u \in B, u < v} x_{u,v} + \sum_{u \in B, v < u} (1 - x_{v,u})$.
- Classical *randomized rounding heuristic*.
- *Relaxation induced neighborhood search* [1].

Improvement Heuristics. Assume that $\pi_B = u_1 u_2 \dots u_{n_1}$ is the current best solution.

- The *shift heuristic* that Grötschel et al. [5] describe tries if shifting a single vertex improves the current solution.
- In the *local search heuristic*, we try to solve a reduced version of (P_I) to optimality, where we only add variables x_{u_i,u_j} with $|i - j| < w$ for some parameter w.

[3] https://github.com/johannesrauch/PACE-2024/

Figure 1

4 Data reduction

The solver weberknecht implements the following data reduction rules.

- Vertices of degree zero in B are put on the leftmost positions in the linear ordering π_B. Note that there is an optimal solution with exactly these positions for the isolated vertices in B.
- Let l_v (r_v) be the neighbor of $v \in B$ in G that minimizes (maximizes) π_A, respectively. Dujmović and Whitesides [3] noted that, if there exists two nonempty sets $B_1, B_2 \subseteq B$ and a vertex $q \in A$ such that for all $v \in B_1$ we have that $\pi_A(r_v) \leq \pi_A(q)$, and for all $v \in B_2$ we have that $\pi_A(q) \leq \pi_A(l_v)$, then the vertices of B_1 appear before the vertices of B_2 in an optimal solution. In this case we can split the instance into two subinstances.
- Dujmović and Whitesides [3] proved that, if π_B is an optimal solution, and $c_{u,v} = 0$ and $c_{v,u} > 0$, then $\pi_B(u) < \pi_B(v)$.
- Dujmović et al. [2] described a particular case of the next reduction rule. Let $c_{u,v} < c_{v,u}$. We describe the idea with the example in Figure 1. Imagine that we draw some edge $x_i y_j$ into Figure 1. If the number of edges crossed by $x_i y_j$ on the left side is at most the number of edges crossed by $x_i y_j$ on the right side for all edges of the form $x_i y_j$, then we have $\pi_B(u) < \pi_B(v)$ in any optimal solution π_B: Otherwise we could improve the solution by simply exchanging the positions of u and v. Note that this reduction rule is only applicable if $d_G(u) = d_G(v)$ as witnessed by $x_2 y_1$ and $x_2 y_k$ ($k = 5$ here).
- The value $\ell b = \sum_{u,v \in B, u<v} \min(c_{u,v}, c_{v,u})$ is a lower bound on the number of crossings of an optimal solution. Suppose that we have already computed a solution with ub crossings. Then, if $c_{u,v} \geq ub - \ell b$ for some $u, v \in B$, it suffices to only consider orderings π_B with $\pi_B(u) > \pi_B(v)$ for the remaining execution.
- After the execution of the described reduction rules, some variables $x_{u,v}$ of (P_I) have a fixed value due to the constraints of (P_I).

5 Branch and bound

The solver weberknecht implements a rudimentary branch and bound algorithm. We use HiGHS [6] only as a linear program solver, and not as a mixed-integer program solver, since the mixed-integer program solver does not (yet) implement lazy constraints. To avoid adding all $\Theta(n^3)$ constraints, we solve the linear program relaxation of (P_I) as follows.

1. Create a linear program (P) with the objective function of (P_I) and no constraints.
2. Solve (P).
3. If the current solution violates constraints of (P_I), add them to (P) and go to 2.

Let ub denote the number of crossings of the current best solution. Then, until we have a optimal solution, weberknecht executes the following.

1. Solve (P) with the method described above.
2. If (P) is infeasible, backtrack.

3. If the rounded objective value of P is at least ub, backtrack.
4. If the current solution of (P) is integral, update the best solution and backtrack.
5. Run informed heuristics and branch.

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