Strong Kleene Supervaluation: Restricted Quantification and Material Implication.

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Truth, Vagueness, and Indeterminacy

European Research Council
Established by the European Commission

ERC-2018-STG 803684 TRUST

University of BRISTOL

Aims and Outline

- Restricted quantification and material implication under semantic indeterminacy;
- Sorites, Vagueness, and Restricted Quantification
- New truth conditions (strong Kleene supervaluation);
- Restricted quantification and Curry's paradox;
- Show how to construct naive truth theories for strong Kleene supervaluation;
- Outlook: Conditionals, theories, and more.

Semantic Indeterminacy and Conditional Reasoning

Challenges

- S Vagueness: Sorites series/paradox
- C Naive truth: Curry paradox
- P Presuppositional vs non-presuppositional readings/truth-conditions

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- S, C, and P provide prima facie support for partial interpretations/models.
- Focus on S and C.
- Focus on the determiner Every
  - $\forall x\varphi := \mathrm{Every}_x(\top, \varphi)$;
  - $\varphi \to \psi := \mathrm{Every}_x(\varphi, \psi)$ with $x \notin \mathrm{FV}(\varphi \wedge \psi)$.

Vagueness

Every and Sorites

A Sorites series

Premiss 1 Pa_0.

Premiss 2 $\mathsf{Every}_x(Pa_x, Pa_{x+1})$.

Conclusion $\forall x(Pa_x)$

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Conclusion $\forall x(Pa_x)$

- Either a premiss is false or the inference unsound.
- Against partial interpretation of P.
- What is the logic/semantics of Every_x?

Two approaches to Partiality

Classical supervaluation

- Quantification over (classical) admissible precisification;
- Vindicates all first-order logical truths.

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- Logic/semantics of restricted quantification remains to be determined…

- SV provides a logic/semantics for Every
- If on accepts that S, C, and P introduce partiality, one should not try to vindicate classical tautologies.

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(a) Every red ball is red.

(b) Every red ball is Jacky's.

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Theoretical Motivation

Indefinite extensibility: stability under domain extensions.

Toward Restricted Quantification in Partial Logics

- Non-instance based, “generic” understanding
- extensibility of the local domain
- Determiners as relations/concepts of first-order concepts
  - intensional [non-modal] understanding of concepts?
  - functions from interpretations to the domain

Theoretical Motivation

Semantic indeterminacy and partiality: stability under semantic precisifications/local domain extensions (“Monotonicity”,)

Strong Kleene Supervaluation

Supervaluation structure $\mathfrak{M}$

A tuple (D, X, H) such that $D \neq \emptyset$ and

- X is a set of partial (strong Kleene) interpretations such that for all $I, J \in X$ and all closed terms t
  - $J(t) = I(t)$
- $H \subseteq X \times X$ such that
  - H is transitive
  - if $(I, J) \in H$, then $I \leq J$.

Truth relative to an Interpretation

Let $J \in X$ and $\|\chi\|_x^{J,\beta} = \{d \in D \mid \mathfrak{M}, J \Vdash \varphi[\beta(x:d)]\}$.:

$$\begin{array}{ll} \mathfrak{M}, J \Vdash \text{Every}_x(\varphi, \psi)[\beta] & \text{iff } \forall J'((J, J') \in H \Rightarrow \|\varphi\|_x^{J',\beta} \subseteq \|\psi\|_x^{J',\beta}) \\ \mathfrak{M}, J \Vdash \neg\text{Every}_x(\varphi, \psi)[\beta] & \text{iff } \|\varphi\|_x^{J,\beta} \cap \|\neg\psi\|_x^{J,\beta}) \neq \emptyset \end{array}$$

- strong Kleene truth for remaining clauses.

Every as a higher-order concept/relation

- Every as a relation on functions from interpretations to the domain;
  - $\mathrm{Dom}_{\mathfrak{M}} := \{f \mid f \in {}^{X}D\}$
  - $\mathrm{Every}_{\mathfrak{M}} \subseteq \mathrm{Dom}_{\mathfrak{M}} \times \mathrm{Dom}_{\mathfrak{M}}$

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- $(f, g) \in \mathrm{Every}_{\mathfrak{M}}$ iff $f(J) \subseteq g(J)$ for all $J \in X$.
- $(f, g) \in \mathrm{Some}_{\mathfrak{M}}$ iff $f(J) \cap g(J) \neq \emptyset$ for all $J \in X$.

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- For φ set $\|\varphi\|^{\beta}_{v} \in \mathrm{Dom}(\mathfrak{M})$ such that for all $J \in X$

$$\|\varphi\|^{\beta}_{v}(J) := \{a \in D \mid \mathfrak{M}, J \Vdash \varphi[\beta(x : a)]\}.$$

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Higher-Order Truth Condition

$$\begin{array}{ll} \mathfrak{M}, J \Vdash \mathrm{Every}_x(\varphi, \psi)[\beta] & \text{iff } (\|\varphi\|^{\beta}_{x}.\|\psi\|^{\beta}_{x}) \in \mathrm{Every}_{\mathfrak{M}_J} \\ \mathfrak{M}, J \Vdash \neg\mathrm{Every}_x(\varphi, \psi)[\beta] & \text{iff } (\|\varphi\|^{\beta}_{x}, \|\neg\psi\|^{\beta}_{x}) \in \mathrm{Some}_{\mathfrak{M}_J} \end{array}$$

Taking Stock

Logic

- Corresponds to Nelson logic (N3);
- Some flexibility:
  - Use fde-style semantics: N4, Hype (QN^*),...
  - Strengthening of tc for Every to allow for contraposition
- Kripke frames for intuitionistic logic

Data

(a) ‘Every red ball is red.’ true at all interpretation.

(b) ‘Every red ball is Jacky’s.’ true if all borderline red balls belong Jacky.

Truth

A terminological primer

Naivity

A truth theory Th is called naive iff for all sentences φ

$$\varphi \in \text{Th iff } T\ulcorner\varphi\urcorner \in \text{Th}.$$

Transparency

A truth theory Th is called transparent iff for all sentences φ,ψ

$$\psi(\varphi/p) \in \text{Th iff } \psi(T\ulcorner\varphi\urcorner/p) \in \text{Th}.$$

Every and Curry

Quantified Curry

Let κ be the sentence

$$\mathsf{Every}_x(x = \ulcorner\kappa\urcorner \wedge \mathrm{T}x, x \neq x)$$

- Curry's paradox main obstacle for conditionals/RQ in truth theories.
- Orthodox TC: κ is true iff κ is not in the interpretation of the truth predicate.
- No naive truth models with orthodox TCs
- Strong Kleene supervaluation: κ cannot be stably true or untrue under semantic precisification/local domain extensions

More on Curry

We cannot have

- Transparency, structural rules, and deduction theorem
- Transparency + MP + $\rightarrow$-contraction + $\rightarrow$-reflexivity

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Logicality: Truth vs Conditional

- **Logicality of $\rightarrow$**: Conditional defined relative to a model class also containing non-naive truth models.
- **Logicality of truth**: Conditional defined relative to naive truth models only; loss of crucial logical properties of $\rightarrow$.

- We opt for the logicality of $\rightarrow$;
- Construct naive truth model that sees non-naive models.

Interpreting Truth

Expand supervaluation structure $\mathfrak{M} = (D, X, H)$ for $\mathcal{L}$ to an supervaluation structure for $\mathcal{L}_{\mathrm{T}}$

Assumptions

- $\mathcal{L}$ extends the language of some syntax theory $\mathcal{L}_S$, e.g., the language of arithmetic;
- $\mathcal{L}$ contains names of all elements of D;
- for all $\varphi \in \mathcal{L}_S$; $J, J' \in X$ and assignments β.
  - $\mathfrak{M}, J \Vdash \varphi[\beta]$ iff $\mathfrak{M}, J' \Vdash \varphi[\beta]$
  - $\mathfrak{M}, J \Vdash \varphi \vee \neg\varphi[\beta]$

Valuation on $\mathfrak{M}$

Function that assigns an interpretation to the truth predicate relative to a world and an interpretation:

- $f : X \to \mathcal{P}(\mathrm{Sent})$

Admissible Valuations

Not all valuations are equally good. A valuation f is admissible on $\mathfrak{M} = (D, X, H)$ iff

- f is consistent, i.e., if for all $J \in X$ and $\varphi \in \mathcal{L}_{\mathrm{T}}$:

$$\varphi \notin f(J) \text{ or } \neg\varphi \notin f(J);$$

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- for all $J, J' \in X$, if $(J, J') \in H$, then $f(J) \subseteq f(J')$.

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Truth Interpretation

Let $J \in X$ and f an admissible valuation, then J_f is a called a truth-interpretation for the language $\mathcal{L}_{\mathrm{T}}$:

$$J_f(P) := \begin{cases} f(J), & \text{if } P \doteq \mathrm{T}; \\ J(P), & \text{otherwise.} \end{cases}$$

Admissibility Condition

Ordering

Let f, g be valuations of $\mathfrak{M}$. Then $f \leq g$ iff $f(w, J) \subseteq g(J)$, for all $J \in X$.

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$$\text{if } g \in \Phi(f), \text{ then } f \leq g.$$

- Φ yields the admissible precisifications of an valuations f
- Φ induces an ordering on $\mathrm{Val}_{\mathfrak{M}}^{\mathrm{Adm}}$: $f \leq_{\Phi} g :\leftrightarrow g \in \Phi(f)$.

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Further Assumptions:

- $\leq_\Phi$ is transitive
- if $f \leq g$, then $\Phi(g) \subseteq \Phi(f)$.

Truth Structure

Let $\mathfrak{M} = (D, X, H)$ be a supervaluation structure and $Y \subseteq \mathrm{Val}_{\mathfrak{M}}^{\mathrm{Adm}}$. Then the tupel $(D, X \times Y, H_\Phi)$ is called a **truth structure** iff for all $I, J \in X$ and $f, g \in Y$:

$$(I_f, J_g) \in H_\Phi :\leftrightarrow (I, J) \in H \,\&\, f \leq_\Phi g.$$

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Grounded Truth Structure

Let $\mathfrak{M}_\mathrm{T} = (D, X \times Y, H_\Phi)$ be a truth structure. If there is an $f \in Y$ such that $Y \cap \Phi(f) \neq \emptyset$ and $f \leq g$ for all $g \in Y$, then $\mathfrak{M}_\mathrm{T}$ is called a **grounded truth structure**. A set Y_f with minimal element f is called a grounded truth set.

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Find a grounded truth structure $\mathfrak{M}_{\mathrm{T}}$ with minimal $f \in Y$ such that for all $J \in X$, $w \in W$ and $\varphi \in \mathcal{L}_{\mathrm{T}}$:

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- Transparency is out of reach!

Fixed Points

Definition (Compactness of Φ)

Set $\Phi(X) = \{\Phi(f) \mid f \in X\}$. Φ *is compact on* Y_f *iff for all* $X \subseteq Y_f$*: if* $\Phi(f_1) \cap \ldots \cap \Phi(f_n) \neq \emptyset$ *for all* $n \in \omega$ *and* $f_1, \ldots f_n \in X$*, then* $\bigcap \Phi(X) \neq \emptyset$.

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Proposition

Let $\mathfrak{M} = (D, X, H)$ *be a supervaluation structure and* Φ *compact. Then there exists a grounded truth set* Y_f *and admissible valuation function* f *such that for all* $\varphi \in \mathrm{Sent}_{\mathcal{L}_{\mathrm{T}}}$

$$(D, X \times Y_f, H_\Phi), J_f \Vdash \varphi \text{ iff } (D, X \times Y_f, H_\Phi), J_f \Vdash \mathrm{T}\ulcorner\varphi\urcorner$$

for all $J \in X$.

Some more specifics

Let $\mathrm{Adm}_{\mathfrak{M}}$ be the set of grounded truth sets. Define two operations:

- $\theta^{\Phi}_{\mathfrak{M}} : \mathrm{Val}^{\mathrm{Adm}}_{\mathfrak{M}} \times \mathrm{Adm}_{\mathfrak{M}} \to \mathrm{Val}_{\mathfrak{M}}$ such that for all $f \in Y_f \in \mathrm{Adm}_{\mathfrak{M}}$ and $J \in X$:

$$[\theta^{\Phi}_{\mathfrak{M}}(f, Y_f)](J) := \{\varphi \mid (F, X \times Y_f, H_\Phi), J_f \Vdash \varphi\}$$

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Observation

Let $f \in Y_f \in \mathrm{Adm}_{\mathfrak{M}}$. Then

$$\theta(Y_f, f) = f \text{ iff } \Theta(Y_f) = Y_f.$$

‘Naive’ Fixed Point Property

$$\Phi_{\mathsf{Nve}}(f) := \begin{cases} \emptyset, & \text{if } f \notin \mathsf{Val}^{\mathsf{Adm}}_{\mathfrak{M}}; \\ \{g \in \mathsf{Val}^{\mathsf{Adm}}_{\mathfrak{M}} \mid f \leq g \,\&\, g \text{ is (N3)-naive}\}, & \text{otherwise.} \end{cases}$$

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Proposition (Φ_{Nve}-fixed points)

Let $\mathfrak{M} = (D, X, H)$ be a supervaluation structure. The there exists a grounded truth set Y_f

$$\theta(Y_f, f) = f \text{ and } \Theta(Y_f) = Y_f$$

with admissibility condition Φ_{Nve}.

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- Naive valuation functions and transparency

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  - $\neg \mathrm{T}x \leftrightarrow \mathrm{T} \underset{\cdot}{\neg} x$
  - $\mathrm{T}(x \underset{\cdot}{\wedge} y) \leftrightarrow \mathrm{T}x \wedge Ty$

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  - $\mathrm{T}(x \dot{\wedge} y) \leftrightarrow \mathrm{T}x \wedge Ty$
- truth-iteration axioms:
  - $\mathrm{T}t \leftrightarrow \mathrm{T}\ulcorner \mathrm{T}t \urcorner$

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  - $\mathrm{T}(x \mathbin{\underset{\cdot}{\wedge}} y) \leftrightarrow \mathrm{T}x \wedge Ty$
- truth-iteration axioms:
  - $\mathrm{T}t \leftrightarrow \mathrm{T}\ulcorner \mathrm{T}t \urcorner$
- Truth-principles for $\rightarrow$:
  - $\mathrm{T}x \wedge \mathrm{T}(x \mathbin{\underset{\cdot}{\rightarrow}} y) \rightarrow \mathrm{T}y)$
  - $\mathrm{T}(\underset{\cdot}{\neg}\, x \mathbin{\underset{\cdot}{\vee}} y) \rightarrow \mathrm{T}(x \mathbin{\underset{\cdot}{\rightarrow}} y)$

Validities in Fixed-point models

- N3-logical truths
- Closure under Nec and Conec
- truth commutation axioms for all logical connectives save $\rightarrow$:
  - $\neg \mathrm{T}x \leftrightarrow \mathrm{T} \dot{\neg} x$
  - $\mathrm{T}(x \dot{\wedge} y) \leftrightarrow \mathrm{T}x \wedge Ty$
- truth-iteration axioms:
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Deduction Theorem

Let J_f a fixed-point and $\mathfrak{M}_{J_f}$ the J_f generated substructure of $\mathfrak{M}$. Then

$$\Gamma, \varphi \vDash_{\mathfrak{M}_{J_f}} \psi \text{ iff } \Gamma \vDash_{\mathfrak{M}_{J_f}} \varphi \rightarrow \psi$$

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ω-consistency

There are fixed points for

$$\Phi_{\omega-\mathrm{Nve}}(f) := \begin{cases} \emptyset, & \text{if } f \notin \mathrm{Val}_{\mathfrak{M}}^{\mathrm{Adm}}; \\ \{g \in \mathrm{Val}_{\mathfrak{M}}^{\mathrm{Adm}} \,|\, f \leq g \,\&\, g \text{ is naive a. } \omega \text{ cons.}\}, & \text{else.} \end{cases}$$

Outlook

- Modal strong Kleene supervaluation: modality and natural language conditionals
- First-order approaches
  - External and internal axiomatizations
- Generalized quantifiers