

THE SEQUENCE SPACE

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OUTLINE

- 1 INTRODUCTION
- 2 KRUSSEL-SMITH (1998) ECONOMY
- 3 FAKE NEWS ALGORITHM

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SOLVING MODELS

- Hard when there is rich heterogeneity.
 - Infinitely dimensional state space from the distribution of agent's endogenous state variables—how to handle it?
 - In macro, we are often interested what happens when aggregate shock X hits the economy (monetary, fiscal, etc).
 - State space: must forecast infinite-dimensional distribution to estimate prices / quantities relevant to agents' problem.
 - Sequence space: must find price / quantity sequences that solve market clearing conditions.
- ⇒ Sequence space is of dimension $T \times \text{number of variables}$, versus infinite dimensional state space.
- Cost: linearization / perfect foresight.

SEQUENCE SPACE METHODS

- Sequence Space methods mean we use linear algebra to solve models with rich heterogeneity.
- But how do we get matrices like these:

$$\nabla_{\mathbf{Y}} \mathbf{C} = \begin{pmatrix} \frac{\partial C_0}{\partial Y_0} & \frac{\partial C_0}{\partial Y_1} & \cdots \\ \frac{\partial C_1}{\partial Y_0} & \frac{\partial C_1}{\partial Y_1} & \cdots \\ \frac{\partial C_2}{\partial Y_0} & \frac{\partial C_2}{\partial Y_1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

- Need to solve the household consumption problem: hard.
- Auclert, Bardoczy, Rognlie, Straub (2021): efficient algorithm that exploits model structure and linearity to get the Jacobians of the model really fast!

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HOUSEHOLD

- Preferences:

$$V_t(e, a_-) = \max_{c, k} u(c) + \beta \sum_{e'} V_{t+1}(e', k) P(e, e')$$

- Budget constraint

$$c + k = (1 + r_t)k_- + w_t e n$$

- Borrowing constraint

$$k \geq 0$$

- Optimal policy:

$$c_t^*(e, k_-), k_t^*(e, k_-)$$

- Functions of $\{r_t, w_t\}_{t \geq 0}$.

AGGREGATING CONSUMER PROBLEM

- Distribution of capital and productivity:

$$D_{t+1}(e', K) = \sum_e D_t(e, k_t^{*-1}(e, K)) P(e, e')$$

- Given D_0 , $D_{t+1}(e', K)$ are also functions of $\{r_t, w_t\}_{t \geq 0}$.
- Aggregate capital holdings are therefore also a function of $\{r_t, w_t\}_{t \geq 0}$.

$$\mathbf{A}_t(\{r_s, w_s\}_{s \geq 0}) = \sum_e \int_{k_-} k_t^*(e, k_-) D_t(e, dk_-)$$

- The function \mathbf{A}_t maps an aggregate sequence $\{r_t, w_t\}_{t \geq 0}$ into another aggregate sequence $\{K_t\}_{t \geq 0}$.
- The dimensionality of this problem is the length of the sequence T .

FIRM AND MARKET CLEARING

- Keep this simple:

$$Y_t = Z_t K_{t-1}^\alpha N_t^{1-\alpha}$$

$$r_t = \alpha Z_t K_{t-1}^{\alpha-1} N_t^{1-\alpha} - \delta$$

$$w_t = (1 - \alpha) Z_t K_{t-1}^\alpha N_t^{-\alpha}$$

- Labor supply fixed in this problem so

$$N_t = \sum \pi(e) e n$$

- Market for capital has to clear

$$H_t(K, Z) \equiv \mathbf{A}_t(\mathbf{r}, \mathbf{w}) - K_t = 0$$

DAG REPRESENTATION

IRFs

- You know how to compute $\nabla_{\mathbf{K}}\mathbf{r}, \nabla_{\mathbf{U}}\mathbf{r}, \nabla_{\mathbf{K}}\mathbf{w}, \nabla_{\mathbf{U}}\mathbf{w}$ from firm block.
- How do we compute $\nabla_{\mathbf{r}}\mathbf{A}$ and $\nabla_{\mathbf{w}}\mathbf{A}$?

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FAKE NEWS ALGORITHM

- Goal: efficient computation of Jacobians.
- Framework:

$$\begin{aligned}\mathbf{v}_t &= v(\mathbf{v}_{t+1}, \mathbf{X}_t) \\ \mathbf{D}_{t+1} &= \Lambda(\mathbf{v}_{t+1}, \mathbf{X}_t)' \mathbf{D}_t \\ \mathbf{Y}_t &= y(\mathbf{v}_{t+1}, \mathbf{X}_t)' \mathbf{D}_t\end{aligned}$$

- ▶ \mathbf{X}_t are inputs (e.g., \mathbf{Z}_t).
 - ▶ \mathbf{Y}_t are outputs (e.g., $\mathbf{C}_t, \mathbf{K}_t$).
 - ▶ \mathbf{D}_t is the discretized distribution.
- In sequence space:

$$\mathbf{Y} = h(\mathbf{X})$$

- Want: $\mathcal{J} = \nabla h_{\mathbf{Y}}(\mathbf{X}^{SS})$

BRUTE FORCE

- Column s of \mathcal{J}_X^Y is IRF of outcome (e.g., capital) to a one-time shock (e.g. productivity) at s , $\mathbf{X}^s = \mathbf{X}_{ss} + \mathbf{e}^s dx \equiv \mathbf{X}_{ss} + d\mathbf{X}^s$.
- Solve problem backward given the known sequence of \mathbf{X} to get value functions \mathbf{v}_t , policy functions \mathbf{y}_t^s , and transition matrices $\Lambda(\mathbf{v}_{t+1}, \mathbf{X}_t)'$.
- Given optimal policy functions iterate distribution of agents forward starting from steady state,

$$\mathbf{D}_{t+1}^s = (\Lambda_t^s) \mathbf{D}_t^s$$

- Aggregate policy functions using the distribution to get outcome

$$\mathbf{Y}_t^s = (\mathbf{y}_t^s)' \mathbf{D}_t^s$$

- Repeat for $s = 0, \dots, T$ to get all columns of \mathcal{J} .
- This “brute force” method works but is very slow.

TASKS

- Solve for steady state in consumption problem.
- Get the first column of $\mathcal{J}_r^{\mathcal{H}}$.
- Get the second column of $\mathcal{J}_r^{\mathcal{H}}$.

ALGORITHM: EFFICIENT BACKWARD STEP

- Lemma 1: for any $s \geq 1$, $t \geq 1$:

$$\mathbf{y}_t^s = \begin{cases} \mathbf{y}_{ss} & s < t \\ \mathbf{y}_{T-1-(s-t)}^{T-1} & s \geq t \end{cases}, \quad \Lambda_t^s = \begin{cases} \Lambda_{ss} & s < t \\ \Lambda_{T-1-(s-t)}^{T-1} & s \geq t \end{cases}$$

- Policy functions at t for shock at $t+s$ the same as policy function at 0 for shock at s .
- ⇒ Policy functions need to be solved backwards only once starting with a shock at $T-1$. (vs T times in brute force.)
- For any time after the shock, the policy functions are the same as in steady state.
 - Key: policy function cannot directly depend on distribution of agents

FAKE NEWS

- Denote the sequence $\boldsymbol{\varepsilon}^s$ in which $\varepsilon_s = 1$ and zero otherwise:

$$\varepsilon_t = \begin{cases} 1 & t = s \\ 0 & t \geq 0, t \neq s \end{cases}$$

- Let η_t^s be a fake news shock for ε_t :
 - ▶ At t learn that $\varepsilon_s = 1$ with certainty.
 - ▶ At $t+1$ learn that $\varepsilon_s = 0$.
 - ▶ $\mathbb{E}_k \eta_t^s = 0$ for all $k < t$.
- Lemma: The sequence \boldsymbol{v}^s

$$v_t = \begin{cases} \eta_t^{s-t} & t < s \\ \boldsymbol{\varepsilon}^0 & t \geq s \end{cases}$$

takes on the same expected values and realized values as $\boldsymbol{\varepsilon}^s$.

- Next step: build Jacobian based on \boldsymbol{v}^s rather than $\boldsymbol{\varepsilon}^s$.

ALGORITHM: EFFICIENT FORWARD STEP

- If we return to steady state and the shock is infinitesimal,

$$\begin{aligned} F_{t,s} dx &= dY_t^s - dY_{t-1}^{s-1} \\ &= \mathbf{y}'_{ss} (\Lambda'_{ss})^{t-1} d\mathbf{D}_1^s \end{aligned}$$

- Thought experiment: shock happens at $s-1$ and we are interested in outcomes $t=0, \dots, T-1$. How is this different from a shock that happens at s on outcomes $t=1, \dots, T$?
- From earlier we know the policy functions are the same.
- The only difference is the distribution of agents: \mathbf{D}_s in the first case and \mathbf{D}_1^s in the second.
- To a first order, this only affects outcomes as if all agents followed their steady state policy functions.

FAKE NEWS INTERPRETATION

- For a given s , $F_{t,s}$ can be interpreted as the impulse response to a “date- s fake news shock”: a shock to date s announced at date 0, and retracted at date 1.
- At date 0, agents react to the announcement, which leads to the distribution D_1^s .
- After the announcement is retracted, they revert to steady-state policies, so the effect on output at all dates $t \geq 1$ is $y'_{ss}(\Lambda'_{ss})^{t-1}dD_1^s$.

FAKE NEWS MATRIX

- Define the expectation vector as

$$\mathcal{E}_t = (\Lambda_{ss})^t \mathbf{y}_{ss}$$

- Define the fake news matrix as

$$\mathcal{F}_{t,s} dx \equiv \begin{cases} d\mathcal{Y}_0^s & t = 0 \\ \mathcal{E}'_{t-1} d\mathbf{D}_1^s & t \geq 1 \end{cases}$$

- Then the Jacobian of h is given by

$$\mathcal{J}_{t,s} = \sum_{k=0}^{\min\{s,t\}} \mathcal{F}_{t-k,s-k}$$

- $t = 0$ follows by definition.
- Why does this make sense?

FAKE NEWS MATRIX EXAMPLES

- $t > 0, s = 0$:

$$\mathcal{J}_{t,0} = \mathbf{y}'_{ss}(\Lambda'_{ss})^{t-1} d\mathbf{D}_1^0$$

- ▶ Shock at 0 affected distribution at 1. Now trace out this effect.

- $t = 1, s = 1$:

$$\mathcal{J}_{1,1} = d\mathcal{Y}_0^0 + \mathbf{y}'_{ss} d\mathbf{D}_1^1$$

- ▶ Sum of surprise contemporaneous shock and fake news shock.

- $t = 1, s = 2$:

$$\mathcal{J}_{1,2} = d\mathcal{Y}_0^1 + \mathbf{y}'_{ss} d\mathbf{D}_1^2$$

- ▶ As if we got news today about shock tomorrow, but taking into account change in distribution given that news was known earlier.

$$\mathcal{J}_{3,4} = d\mathcal{Y}_0^1 + \mathbf{y}'_{ss} d\mathbf{D}_1^2 + \mathbf{y}'_{ss}(\Lambda'_{ss}) d\mathbf{D}_1^3 + \mathbf{y}'_{ss}(\Lambda'_{ss})^2 d\mathbf{D}_1^4$$

BOTTOM LINE

- $t = 1, s = 0$:

$$\mathcal{J}_{t,s} = \mathcal{J}_{t-1,s-1} + \mathcal{F}_{t,s}$$

- ▶ A shock at s from time t looks very similar to a shock at $s - 1$ from time $t - 1$.
- ▶ Policy functions are exactly the same.
- ▶ Only difference is that the distribution at t is different from $t - 1$, which is captured by the fake news term.

⇒ Very small number of steps / computations to get all of the Jacobian and therefore the IRFs.

TASKS

- 1 Compute backward step once.
- 2 Compute the first row of the Jacobian / FNM $\{d\mathcal{Y}_0^s\}_{s=0}^T$.
- 3 Compute dD_1^s using steady state distribution and $s = 0, \dots, T$ policy functions.
- 4 Compose the fake news matrix.
- 5 Compose the Jacobian using using the fake news matrix.
- 6 Combine matrices to get H_U and H_K
- 7 Compute IRF to AR(1) productivity shocks with $\rho_a = 0.9$ at $t = 0$, $t = 10$.