Aggregate Risk

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Outline

1. Exorcising misconceptions
2. Before we go to HA
3. Why HA models with aggregate shocks are difficult to solve?
4. Intuition
5. See the code for the Krusell Smith (1998) model

Reference for today

The reference for today is
When Inequality Matters for Macro and Macro for Inequality, by Ahn, Kaplan, Moll, Winberry, Wolf, NBER Macro Annual, 2018.

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2. Before we go to HA
3. Why HA models with aggregate shocks are difficult to solve?
4. Intuition
5. See the code for the Krusell Smith (1998) model

Exorcising misconceptions

Someday someone will tell you...

- Krusell Smith (1998) showed that micro heterogeneity is irrelevant at the aggregate level.
- In heterogeneous agent models there is *approximate aggregation*, meaning that aggregate variables can be summarized by the *average* of the wealth distribution.
- Only need one moment of the distribution (the first one), rather than information on the whole *shape* of the distribution

Exorcising misconceptions

Exorcising misconceptions

- This is not what Krusell Smith (1998) meant or showed
- Approximate aggregation is a result in a particular model: The Aiyagari model with aggregate productivity shocks
- There is no theorem suggesting this is the case for any shock, or any model. There are counterexamples.

Outline

Before we go to HA

- The Reiter method will exploit that HA models in the state space share a structure with RA models. Take the RBC model

$$C_t = \Lambda_t^{-1/\gamma} \text{ Marginal Utility}$$

$$\frac{1}{dt}\mathbb{E}_t \Lambda_t = (\rho - r_t)\Lambda_t \text{ Euler Equation}$$

$$\frac{dK_t}{dt} = w_t + r_t K_t - C_t \text{ Law of motion}$$

$$dZ_t = -\eta Z_t dt + \sigma dW_t \text{ TFP}$$

$$r_t = \alpha e^{Z_t} K_t^{\alpha-1} - \delta \text{ Competitive K market}$$

$$w_t = (1-\alpha) e^{Z_t} K_t^{\alpha} \text{ Competitive L market}$$

Before we go to HA

- If we can show that a HA model shares the same structure

- and we recall that we know how to linearize and solve the RBC model (log-linearization + Blanchard Kahn)

- Then we can solve a HA model

Outline

Source of the difficulty

- Imagine you know a positive TFP shock will realize tomorrow
- You are a firm manager thinking of investing today to take advantage of higher TFP tomorrow
- The marginal benefit of your investment is the NPV of an infinite stream of *MPKs* discounted by $\beta(1-\delta)$
- Your decision depends on the price of capital today versus in the future
- and on your labor choice tomorrow
- which through profit maximization tomorrow depends on the wage tomorrow ($MPL = w$)
- How can you forecast the wage rate tomorrow?

Source of the difficulty

Easy (to write in paper). The wage clears the labor market!

- Individual state variables s, aggregate state variables $\mathscr{S}$ (includes time and agg. shocks)

$$L^d(\mathscr{S}) = \int_0^1 l_j^d(s_j, \mathscr{S}) f(s_j, \mathscr{S}) dj$$

$$L^s(\mathscr{S}) = \int_0^1 l_k^s(s_k, \mathscr{S}) g(s_k, \mathscr{S}) dk$$

- Excess demand Function H

$$H(L^d(\mathscr{S}), L^s(\mathscr{S})) = L^d(\mathscr{S}) - L^s(\mathscr{S})$$

- w is such that $H(L^d(\mathscr{S}), L^s(\mathscr{S})) = 0$
- If I want to form an expectation of the wage tomorrow, I need to forecast policy functions of firms l^d, and households l^s, the distribution of firms in the state space f, and the distribution of households in the state space g.

Source of the difficulty

- In our bmk model $g(s) = g(a, e)$
- Forecasting g means forecasting the mass of households on each point of asset holdings. Infinite dimensional problem
- Even if you discretize a on 100 points, still need to forecast g at 100 points. 100 additional state variables.
- We have learned so far how to solve problems with 2 state variables. Imagine 102. Imagine moving from this simple model. Not feasible.

OUTLINE

State Space Representation

- Today we will work on the state space. That is we need policy functions at "every possible state"

- Advantage of the state space: Once you solve the model, you can wonder how the solution works for every feasible sequence of shocks, and just simulate it from your solution.

- Big problem: Must solve for every combination. If you only care about one sequence of shocks, more efficient to solve in the "sequence space". Johannes will tell you about that next week

Intuition of the Reiter Method

- The Reiter method consists on a couple of steps

  1. Compute a non-linear solution of the model in the absence of aggregate shocks. That is when the aggregate shock $e^{z_t} = e^{\bar{z}} = 1$ for $z_t = \log Z_t$ the log of aggregate TFP let's say.

     ★ Good news. We have already learned how to do that!

  2. First-order Taylor approximation of the state variables x of the model after z changes.

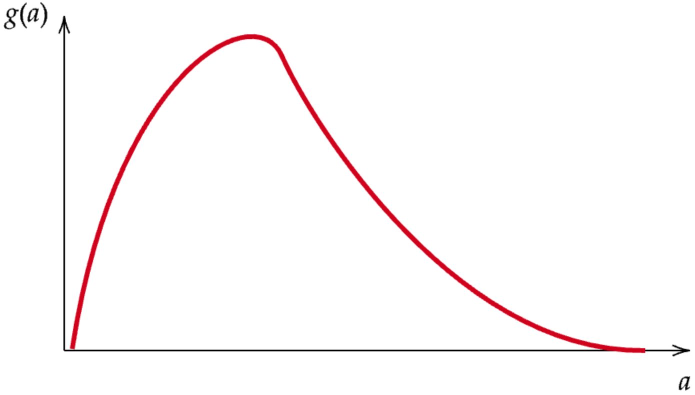
     $$x_t \approx \bar{x} + \left.\frac{\partial x_t}{\partial z_t}\right|_{z_t=\bar{z}} \hat{z}_t$$

     for $\hat{z}_t = z_t - \bar{z}$, and $\bar{x}$ the value of state variable x_t when $z_t = \bar{z}$

     ★ We know how to do that for a RA model. Think the RBC model. It will be almost the same.

Remarks: The Reiter method does not depend on doing things in continuous time. The original papers are in discrete time. But the same arguments that created speed in computing the stationary solution, will help us here.

Illustration of the Reiter Method



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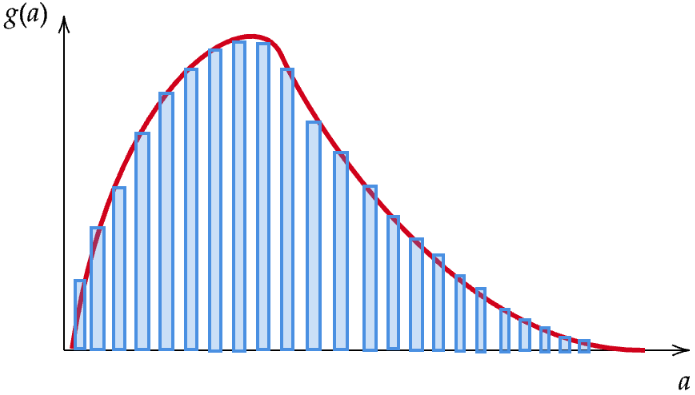

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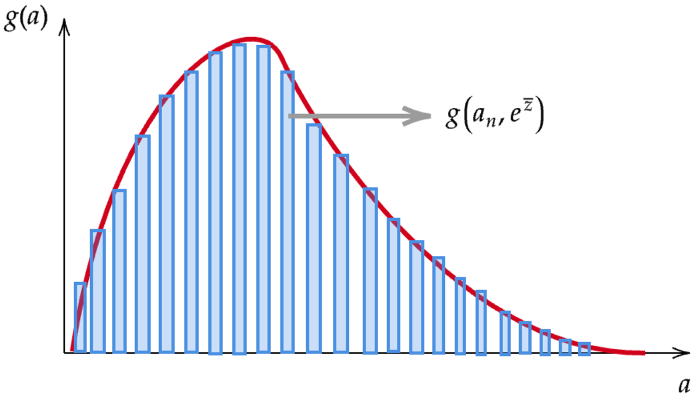

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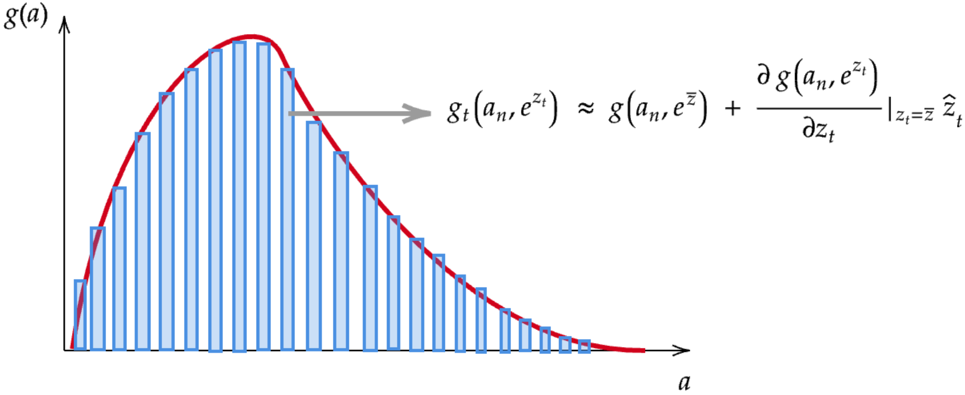

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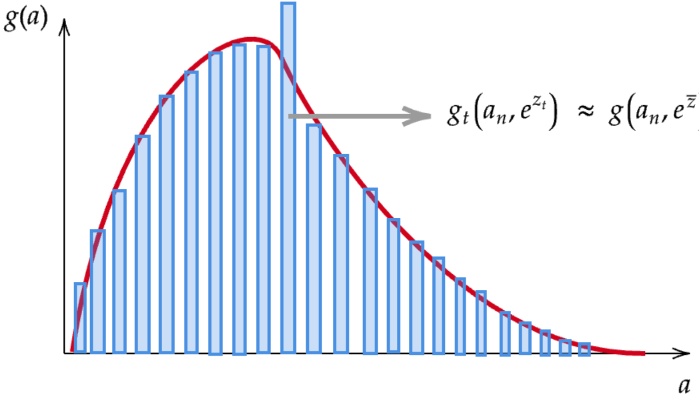
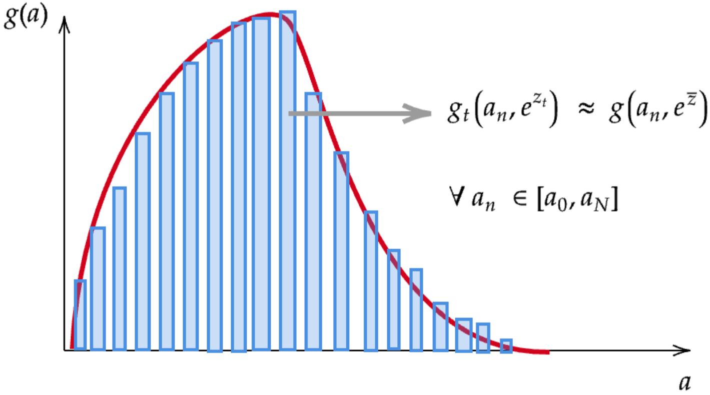


$$g_t\left(a_n, e^{z_t}\right) \approx g\left(a_n, e^{\bar{z}}\right) + \frac{\partial\, g\left(a_n, e^{z_t}\right)}{\partial z_t}\Big|_{z_t=\bar{z}}\, \hat{z}_t$$

Illustration of the Reiter Method



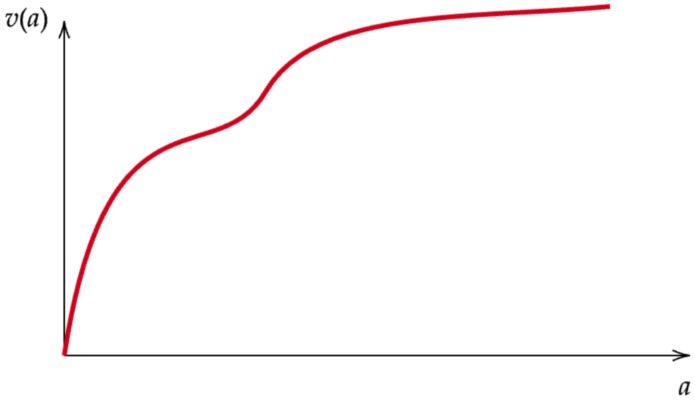
$$g_t\left(a_n, e^{z_t}\right) \approx g\left(a_n, e^{\bar{z}}\right) + \frac{\partial\, g\left(a_n, e^{z_t}\right)}{\partial z_t}\Big|_{z_t=\bar{z}} \hat{z}_t$$

$$\forall\, a_n \in [a_0, a_N]$$

Illustration of the Reiter Method

Can do the same with the value function!

Illustration of the Reiter Method



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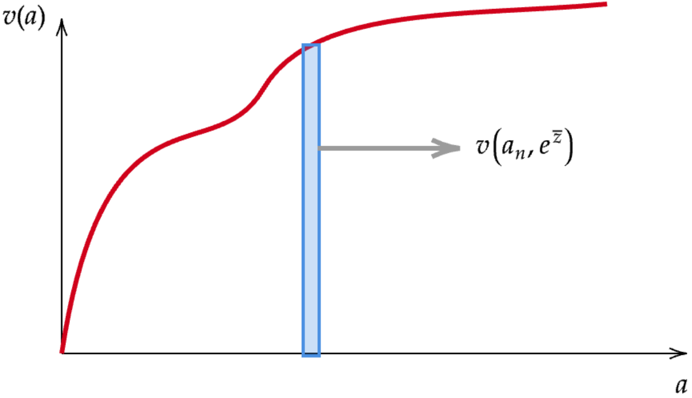
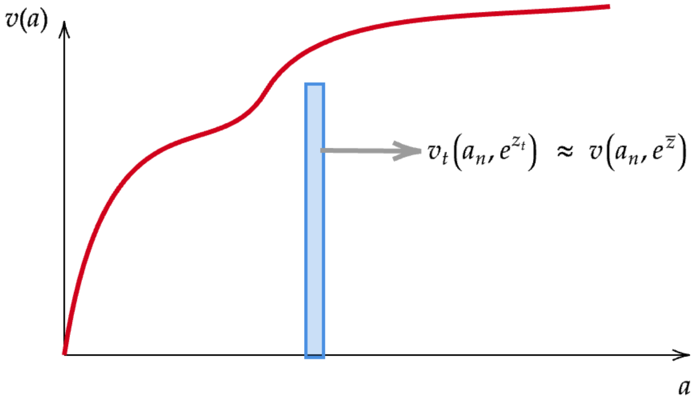
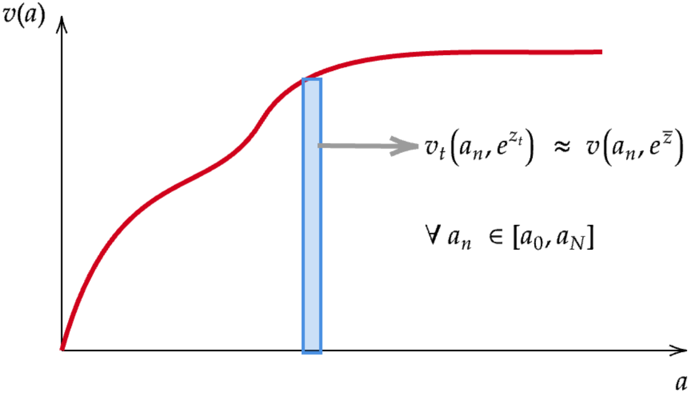


ILLUSTRATION OF THE REITER METHOD



$$v_t\left(a_n, e^{z_t}\right) \approx v\left(a_n, e^{\bar{z}}\right) + \frac{\partial\, v\left(a_n, e^{z_t}\right)}{\partial z_t}\Big|_{z_t=\bar{z}}\, \widehat{z}_t$$

Illustration of the Reiter Method



$$v_t\left(a_n, e^{z_t}\right) \approx v\left(a_n, e^{\bar{z}}\right) + \frac{\partial\, v\left(a_n, e^{z_t}\right)}{\partial z_t}\Big|_{z_t=\bar{z}}\, \hat{z}_t$$

$$\forall\, a_n \in [a_0, a_N]$$

Our old model

$$\rho v(a,e) = \max_c u(c) + \partial_a v s(a,e) + \lambda_e (v(a,e') - v(a,e)) \tag{1}$$

$$s(a,e) = we + ra - c \tag{2}$$

$$a \geq 0 \tag{3}$$

$$0 = -\partial_a s(a,e) g(a,e) - \lambda_z g(a,z) + \lambda_{z'} g(a,z') \tag{4}$$

$$w = (1-\alpha) K^{\alpha} L^{-\alpha} \tag{5}$$

$$r = (\alpha) K^{\alpha-1} L^{1-\alpha} - \delta \tag{6}$$

$$K = \int ag(a,e)dade \tag{7}$$

$$L = \int eg(a,e)dade \tag{8}$$

Our new model

$$\rho v_t(a,e) = \max_c u(c) + \partial_a v_t s_t(a,e) + \lambda_e(v_t(a,e') - v_t(a,e)) + \frac{1}{dt}\mathbb{E}_t dv_t(a,z) \tag{9}$$

$$s_t(a,e) = w_t e + r_t a - c \tag{10}$$

$$a \geq 0 \tag{11}$$

$$\frac{dg_t(a,e)}{dt} = -\partial_a s_t(a,e) g_t(a,e) - \lambda_z g_t(a,z) + \lambda_{z'} g_t(a,z') \tag{12}$$

$$w_t = (1-\alpha)e^{z_t} K_t^{\alpha} L^{-\alpha} \tag{13}$$

$$r_t = (\alpha)e^{z_t} K_t^{\alpha-1} L^{1-\alpha} - \delta \tag{14}$$

$$K_t = \int a g_t(a,e) dade \tag{15}$$

$$L = \int e g_t(a,e) dade \tag{16}$$

$$dz_t = -\eta z_t dt + \sigma dW_t \tag{17}$$

Discretized version

$$\rho \mathbf{v}_t = u(\mathbf{v}) + \mathbf{A}(\mathbf{v}_t, \mathbf{p}_t)\mathbf{v_t} + \frac{1}{dt}\mathbb{E}_t d\mathbf{v}_t \tag{18}$$

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{A}(\mathbf{v}_t, \mathbf{p}_t)^T \mathbf{g}_t \tag{19}$$

$$dz_t = -\eta z_t dt + \sigma dW_t \tag{20}$$

$$\mathbf{p}_t = \mathbf{F}(\mathbf{g}_t; z_t) \tag{21}$$

where $\mathbf{p}$ is a vector that collects the prices of the economy.

- Notice that agents care about the distribution only indirectly, through prices, which depend on distributions.
- Notice that the shock only enters in one equation directly.
- Notice that after an MIT shock (one unexpected dW at time zero), there are no shocks.

Discretized version - Counting Equations

$$\rho \mathbf{v}_t = u(\mathbf{v}) + \mathbf{A}(\mathbf{v}_t, \mathbf{p}_t)\mathbf{v_t} + \frac{1}{dt}\mathbb{E}_t d\mathbf{v}_t \tag{22}$$

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{A}(\mathbf{v}_t, \mathbf{p}_t)^T \mathbf{g}_t \tag{23}$$

$$dz_t = -\eta z_t dt + \sigma dW_t \tag{24}$$

$$\mathbf{p}_t = \mathbf{F}(\mathbf{g}_t; z_t) \tag{25}$$

where $\mathbf{p}$ is a vector that collects the prices of the economy.

- Define N as $2 \times a_n$, the combination of grid points for a, e
- N equations for v
- N equations for g
- 2 equations for w, r
- 1 equation for Z

$2 \times N + 3$ non-linear differential equations

Stack in matrix form even more

$$\mathbb{E}_t \begin{bmatrix} d\mathbf{v}_t \\ d\mathbf{g}_t \\ dZ_t \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{u}(\mathbf{v}_t;\mathbf{p}_t) + \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)\mathbf{v}_t - \rho\mathbf{v}_t \\ A(\mathbf{v}_t;\mathbf{p}_t)^T\mathbf{g}_t \\ -\eta Z_t \\ F(\mathbf{g}_t;Z_t) - \mathbf{p}_t \end{bmatrix} dt. \tag{26}$$

$2\times N+3$ non-linear differential equations

- This structure is actually the same as in a simple RBC model
- The first equation "looks like" the Euler equation: the change in a control variable (C) depends on the value of C and prices
- The second equation looks like the law of motion of K. The change in an endogenous state depends on itself and a function of consumption
- TFP is literally the same
- Prices are determined as in the RBC model

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$$\frac{dK_t}{dt} = w_t + r_t K_t - C_t \text{ Law of motion}$$

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$$r_t = \alpha e^{Z_t} K_t^{\alpha-1} - \delta \text{ Competitive K market}$$

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Analogy to the RBC model

- Because the structure between our model and the RBC model is the same, if we can linearize and solve the RBC model (we can), in principle we can linearize our model.
- But in the RBC model we needed to do Taylor approximations for 5 equations
- Here we need $2 \times N + 3 = 403$ in our parameterization with $e_N = 2, a_N = 100$.
- The cost of the Taylor approximation is to compute derivatives. So the magic of Ahn et al. is to do that fast

Summary of the Intuition

- **If** we knew how to compute the derivatives of g and v to an aggregate shock z, we would be done.
- Notice that:
  1. This is an approximation, accurate to the first order
  2. Solution of the steady state has certainty equivalence w.r.t. the volatility of z. σ does not enter the solution
  3. Notice that we are not imposing that the effect of the shock for v and g must be the same everywhere. It is linear in z at each point of the grid, but the derivative may be (and must be for the case of g) different across a.
  4. g is a distribution, so the net effect of any shock on the overall mass $\int dg(a)da = 0$

Automatic Differentiation

- They will approximate derivatives doing something called automatic differentiation, or algorithmic differentiation.
- Nothing you should understand deeply, but computers represent every function as a sequence of elementary operations (sums, products and exponentials).
- These elementary operations have known derivatives
- So the derivative of a function is the chain rule applied to these elementary operations
- I have **zero** idea how this works under the hood, and have no plans to learn
- But Ahn et al., gave us a toolbox for automatic differentiation

Last stretch before seeing codes

- After you compute the derivatives and have linearized the system, it is as if you lived in the Blanchard Kahn (1980) world

- Which I assume you know from your first year

- But if not, basically what gensys or Dynare are doing under the hood.

Outline

Go to matlab

Further capabilities of the toolbox

- Model-free dimensionality reduction
  - Identify the information in g that is necessary to forecast prices p
  - In the original KS98: the mean of the wealth distribution
  - Idea: $g_t = \gamma_{1t} x_1 + \gamma_{2t} x_2 + \ldots + \gamma_{k_S,t} x_{k_S}$
  - where $x's$ form a basis in $\mathbb{R}^{k_S}$.
  - therefore can keep track of the γ's instead of the whole g_t
  - good idea if $k_S << N$, for N the number of elements in g

Homework

- Write the code for the Aiyagari model with aggregate shocks with the following requirements
  - it computes the steady state and equilibrium conditions using the codes you have created in the past to create the steady state solution and as an input to equilibriumconditions.m, as opposed to the ones provided in phact
  - The ss codes should run as a function of a steady state aggregate TFP value z that is a parameter of the model. You have solved it when $z = 0$.
  - It uses all the dimensionality reduction techniques in phact
  - upload a code that creates the same IRFs we discussed in class and reproduces the results
  - Simulate the same IRF but when the steady state level of z is not 0 but -0.1 (what enters in Y is e^z). Report the results.