Sequence Space Methods

Juan Herreño   Johannes Wieland

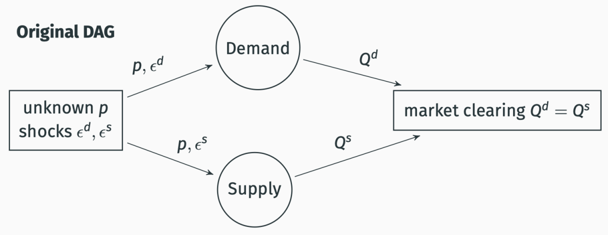
UCSD, Spring 2024

Outline

Outline

Introduction

- We will learn how to solve models in sequence space.
- Basic idea: organize models into "blocks" that represent behavior of (possibly heterogeneous) agents, and interact in GE via a small set of aggregates.
- We will arrange these blocks into Directed Acyclic Graph ("DAG"). Helpful to solve model, think about causality in GE, do decompositions, etc.
- Very useful for solving HA models (not today).



Outline

State Space Approach

- You are likely more familiar with the state space approach.
- Solve for policies and prices as functions of the state.
- State: what is given to you today.
- Policy: what you chose based on FOC.
- Prices: those that implement the equilibrium (market clearing)
- Requires specifying a law of motion for the exogenous states.

General State Space Representation

- State space representation of a linearized macro model (including HANK) is:

$$A\begin{pmatrix} x_{t+1} \\ E_t y_{t+1} \end{pmatrix} + B\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \varepsilon_{t+1} \\ 0 \end{pmatrix}$$

  - x_t is an $n-m \times 1$ vector of predetermined variables.
  - y_t is an $m \times 1$ vector of jump/policy variables.

Solving State Space

- How to solve this?

$$\begin{pmatrix} x_{t+1} \\ E_t y_{t+1} \end{pmatrix} = -A^{-1}B \begin{pmatrix} x_t \\ y_t \end{pmatrix} + A^{-1} \begin{pmatrix} \varepsilon_{t+1} \\ 0 \end{pmatrix}$$

$$= VD^{-1}V^{-1} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + A^{-1} \begin{pmatrix} \varepsilon_{t+1} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{x}_{t+1} \\ E_t \tilde{y}_{t+1} \end{pmatrix} = D \begin{pmatrix} \tilde{x}_t \\ \tilde{y}_t \end{pmatrix} + V^{-1}A^{-1} \begin{pmatrix} \varepsilon_{t+1} \\ 0 \end{pmatrix}$$

- Solve $\tilde{x}$ backward and $\tilde{y}$ forward. Then rotate back using

$$\begin{pmatrix} x_{t+1} \\ E_t y_{t+1} \end{pmatrix} \equiv V^{-1} \begin{pmatrix} \tilde{x}_{t+1} \\ E_t \tilde{y}_{t+1} \end{pmatrix}$$

- Unique stable solution if number of eigenvalues D inside unit circle is $n - m$.
- Dimension of the problem is $n \times n$.

Outline

General Sequence Space Representation

- Equilibrium is a solution to an equation

$$\mathbf{H}(\mathbf{U}, \mathbf{Z}) = 0$$

  where

  - $\mathbf{U}$ represents the time path $U_0, U_1, \ldots,$ of unknown aggregate sequences (e.g., quantities, prices).
  - $\mathbf{Z}$ represents the time path $Z_0, Z_1, \ldots,$ of known exogenous shocks.

- Totally differentiate and evaluate at steady state to get:

$$d\mathbf{U} = -\mathbf{H}_{\mathbf{U}}(\bar{\mathbf{U}}, \bar{\mathbf{Z}})^{-1}\mathbf{H}_{\mathbf{Z}}(\bar{\mathbf{U}}, \bar{\mathbf{Z}})d\mathbf{Z}$$

- Solution requires finding the sequence-space Jacobians $\mathbf{H}_{\mathbf{U}}$ and $\mathbf{H}_{\mathbf{Z}}$.

Sequence Space Approach

- Solve for sequences of policies and prices given a known sequence of exogenous shocks.
- Aligns closely with equilibrium definition.
- Dimension of the problem is $T \times$ number of variables.
- Unlike state space we do not solve for a function of the state.
  - This can be helpful when the state is very high-dimensional
- The sequence space approach assumes perfect foresight.
  - Can get around this to some extent using "MIT-shocks."

Neoclassical Growth Model

- Euler equation:

$$C_t^{-\gamma} = \beta E_t C_{t+1}^{-\gamma}(1-\delta+MPK_{t+1}^k)$$

- Law of motion for capital:

$$K_t = I_t + (1-\delta)K_{t-1}$$

- Production function:

$$Y_t = Z_t K_{t-1}^{\alpha}$$

- Return to capital:

$$MPK_{t+1}^k = \alpha Z_{t+1} K_t^{\alpha-1}$$

- Market clearing:

$$Y_t = C_t + I_t$$

Neoclassical Growth Model Linearized

- Euler equation:

$$-\gamma\hat{c}_t = -\gamma\mathbb{E}_t\hat{c}_{t+1} + (1-\beta(1-\delta))\mathbb{E}_t\hat{mpk}_{t+1}$$

- Law of motion for capital:

$$\hat{k}_t = \delta\hat{\imath}_t + (1-\delta)\hat{k}_{t-1}$$

- Production function:

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_{t-1}$$

- Return to capital:

$$\hat{mpk}_{t+1} = \hat{z}_{t+1} + (\alpha-1)\hat{k}_t$$

- Market clearing:

$$\hat{y}_t = s_c\hat{c}_t + (1-s_c)\hat{\imath}_t$$

Neoclassical Growth Model Linearized

- Euler equation:

$$s_c \hat{c}_t = s_c \hat{c}_{t+1} - \gamma^{-1} s_c (1-\beta(1-\delta))[\hat{z}_{t+1} + (\alpha-1)\hat{k}_t]$$

- Production function:

$$\begin{aligned}
s_c \hat{c}_t &= \hat{z}_t + [\alpha + (1-s_c)\delta^{-1}(1-\delta)]\hat{k}_{t-1} - (1-s_c)\delta^{-1}\hat{k}_t \\
s_c \hat{c}_{t+1} &= \hat{z}_{t+1} + [\alpha + (1-s_c)\delta^{-1}(1-\delta)]\hat{k}_t - (1-s_c)\delta^{-1}\hat{k}_{t+1} \\
s_c \hat{c}_t - s_c \hat{c}_{t+1} &= \hat{z}_t + [\alpha + (1-s_c)\delta^{-1}(1-\delta)]\hat{k}_{t-1} - (1-s_c)\delta^{-1}\hat{k}_t \\
&\quad - \hat{z}_{t+1} - [\alpha + (1-s_c)\delta^{-1}(1-\delta)]\hat{k}_t + (1-s_c)\delta^{-1}\hat{k}_{t+1} \\
&= \hat{z}_t + [\alpha + (1-s_c)\delta^{-1}(1-\delta)]\hat{k}_{t-1} - \hat{z}_{t+1} \\
&\quad - [\alpha + (1-s_c)\delta^{-1}(2-\delta)]\hat{k}_t + (1-s_c)\delta^{-1}\hat{k}_{t+1}
\end{aligned}$$

- Euler equation:

$$\hat{z}_t + [\alpha + (1 - s_c)\delta^{-1}(1 - \delta)]\hat{k}_{t-1} + \left[\gamma^{-1} s_c(1 - \beta(1 - \delta)) - 1\right]\hat{z}_{t+1}$$
$$- \left[\gamma^{-1} s_c(1 - \beta(1 - \delta))(1 - \alpha) + (1 - s_c)\delta^{-1} + \alpha + (1 - s_c)\delta^{-1}(1 - \delta)\right]\hat{k}$$
$$+ (1 - s_c)\delta^{-1}\hat{k}_{t+1}$$

Neoclassical Growth Model Linearized

- Repeated substitution:

$$
\begin{aligned}
0 = H_t(\hat{\mathbf{K}}, \hat{\mathbf{Z}}) \equiv\ & \gamma \left[\alpha + (1 - s_c) \frac{1-\delta}{\delta} \right] \hat{k}_{t-1} \\
& - \gamma \left[\alpha + (1 - s_c) \frac{2-\delta}{\delta} + (1-\alpha) s_c (1 - \beta(1-\delta)) \right] \hat{k}_t \\
& + \gamma (1 - s_c) \frac{1}{\delta} \hat{k}_{t+1} \\
& + \gamma \hat{z}_t \\
& - [\gamma - s_c(1 - \beta(1-\delta))] \hat{z}_{t+1}
\end{aligned}
$$

- The H-matrix is:

$$
\mathbf{0} = \mathbf{H}(\hat{\mathbf{K}}, \hat{\mathbf{Z}}) \equiv \begin{pmatrix} H_0(\hat{\mathbf{K}}, \hat{\mathbf{Z}}) \\ H_1(\hat{\mathbf{K}}, \hat{\mathbf{Z}}) \\ \vdots \\ H_T(\hat{\mathbf{K}}, \hat{\mathbf{Z}}) \end{pmatrix}
$$

Exercise 1: Neoclassical Growth Model

1. Derive $\mathbf{H}_{\hat{\mathbf{K}}} = \begin{pmatrix} \frac{\partial H_0(\hat{\mathbf{K}})}{\partial \hat{k}_0} & \frac{\partial H_0(\hat{\mathbf{K}})}{\partial \hat{k}_1} & \cdots & \frac{\partial H_0(\hat{\mathbf{K}})}{\partial \hat{k}_T} \\ \frac{\partial H_1(\hat{\mathbf{K}})}{\partial \hat{k}_0} & \frac{\partial H_1(\hat{\mathbf{K}})}{\partial \hat{k}_1} & \cdots & \frac{\partial H_1(\hat{\mathbf{K}})}{\partial \hat{k}_T} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial H_T(\hat{\mathbf{K}})}{\partial \hat{k}_0} & \frac{\partial H_T(\hat{\mathbf{K}})}{\partial \hat{k}_1} & \cdots & \frac{\partial H_T(\hat{\mathbf{K}})}{\partial \hat{k}_T} \end{pmatrix}$

2. Derive $\mathbf{H}_{\hat{\mathbf{Z}}} = \begin{pmatrix} \frac{\partial H_0(\hat{\mathbf{K}})}{\partial \hat{z}_0} & \frac{\partial H_0(\hat{\mathbf{K}})}{\partial \hat{z}_1} & \cdots & \frac{\partial H_0(\hat{\mathbf{K}})}{\partial \hat{z}_T} \\ \frac{\partial H_1(\hat{\mathbf{K}})}{\partial \hat{z}_0} & \frac{\partial H_1(\hat{\mathbf{K}})}{\partial \hat{z}_1} & \cdots & \frac{\partial H_1(\hat{\mathbf{K}})}{\partial \hat{z}_T} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial H_T(\hat{\mathbf{K}})}{\partial \hat{z}_0} & \frac{\partial H_T(\hat{\mathbf{K}})}{\partial \hat{z}_1} & \cdots & \frac{\partial H_T(\hat{\mathbf{K}})}{\partial \hat{z}_T} \end{pmatrix}$

3. Assume $\hat{z}_t$ is an AR(1) with persistence $\rho_z = 0.9$. Compute and plot $\hat{\mathbf{k}}$.

Outline

Solving in Sequence Space

- Organize the model in blocks.
  1. Firm block:

$$\hat{y}_t = \hat{z}_t + \alpha \hat{k}_{t-1}$$
$$\hat{mpk}_{t+1} = \hat{z}_{t+1} + (\alpha - 1)\hat{k}_t$$

  2. Household block:

$$\hat{k}_t = \delta \hat{\imath}_{t-1} + (1-\delta)\hat{k}_t$$
$$-\gamma \hat{c}_t = -\gamma \hat{c}_{t+1} + (1-\beta(1-\delta))\hat{mpk}_{t+1}$$

  3. Market clearing block:

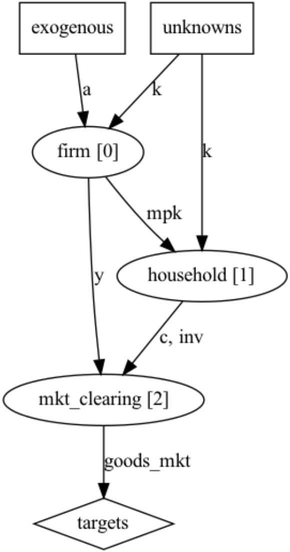
$$0 = s_c \hat{c}_t + (1-s_c)\hat{\imath}_t - \hat{y}_t$$

Sequence Space Algorithm Heuristics

1. Start with an initial guess of the sequences $(\hat{\mathbf{k}}) = \{\hat{k}_t\}_{t=0}^{\infty}$.
   - $\hat{\mathbf{a}} = \{a_t\}_{t=0}^{\infty}$ and $\hat{k}_{t-1}$ are given to us.
2. Solve the firm block for $\hat{\mathbf{y}}$ and $\widehat{\mathbf{mpk}}$.
3. Solve the household block for $\hat{\mathbf{c}}$, $\hat{\imath}$.
4. Check that market clears. If not update guess $\hat{\mathbf{k}}$
   - In turns out we do not need to guess, but can solve the entire system with linear algebra in one step.
   - Approach follows equilibrium definition: find sequences such that everyone optimizes and markets clear.

DAG Representation

- Effectively what we have done is organized our model in a Directed Acyclical Graph (DAG).



DAG Rules

- There are no cycles in a DAG—we travel in one direction only.
- A model has many DAG representations.
    - One representation is to treat every endogenous variable as unknown and have single block.
    - Another representation is to treat each equation as an individual block.
    - These are often not the most useful representation.

DAG Suggestions

- A good DAG minimizes the number of unknowns.
- Generally useful to organize blocks by agent: household, firm, union, government, market clearing.
- Blocks make updating the model easy. Often we change just one problem (e.g., firm for NK model) and leave others untouched.
- Logical check for each block: are the number of variables to solve for equal to the number of equations?
- The computer can organize our model in a DAG and substitute for us.
- Today we will see what the computer does under the hood.

Outline

Solving in Sequence Space

- We want to clear markets at all points in time:

$$\mathbf{H} = \begin{pmatrix} s_c \hat{c}_0 + (1 - s_c)\hat{\imath}_0 - \hat{y}_0 \\ \vdots \\ s_c \hat{c}_T + (1 - s_c)\hat{\imath}_T - \hat{y}_T \end{pmatrix} = \left(\Phi_{gm,c} \hat{\mathbf{c}} + \Phi_{gm,c} \hat{\boldsymbol{\imath}} + \Phi_{gm,y} \hat{\mathbf{y}} \right) = \mathbf{0}$$

  where

$$\begin{aligned} \Phi_{gm,c} &= s_c I_T \\ \Phi_{gm,\imath} &= (1 - s_c) I_T \\ \Phi_{gm,y} &= -I_T \end{aligned}$$

  and I_T is the $T \times T$ identity matrix.

- How does adjusting the sequences $\mathbf{U} = (\hat{\mathbf{k}})$ change the target?

$$\mathbf{H_U} = \left(\Phi_{gm,c} \frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{k}}} + \Phi_{gm,\imath} \frac{\partial \hat{\boldsymbol{\imath}}}{\partial \hat{\mathbf{k}}} + \Phi_{gm,y} \frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{k}}} \right)$$

Solving in Sequence Space

- A simpler and equivalent expression to work with is

$$\mathbf{H_U} = \begin{pmatrix} \Phi_{gm,c} & \Phi_{gm,\imath} & \Phi_{gm,y} & \mathbf{0}_T \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{k}}} \\ \frac{\partial \hat{\imath}}{\partial \hat{\mathbf{k}}} \\ \frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{k}}} \\ \frac{\partial \hat{\mathbf{mpk}}}{\partial \hat{\mathbf{k}}} \end{pmatrix} \equiv \frac{\partial \mathbf{H}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial \mathbf{U}}$$

- Now we move along using the chain rule:

$$\frac{\partial \mathbf{Y}}{\partial \mathbf{U}} = \begin{pmatrix} \frac{\partial (\mathbf{c}, \imath)}{\partial \mathbf{U}} \\ \frac{\partial (\hat{\mathbf{y}}, (\hat{\mathbf{w}} - \hat{\mathbf{p}}))}{\partial \mathbf{U}} \end{pmatrix}$$

where we partitioned based on the two blocks we looked at earlier.

Solving in Sequence Space

- We start with the firm block:

$$\hat{y}_t = \hat{a}_t + \alpha \hat{k}_{t-1}$$

- In matrix notation:

$$\hat{\mathbf{y}} = \hat{\mathbf{z}} + \Phi_{y,k}\hat{\mathbf{k}} + \Phi_{y,k_{-1}}\hat{k}_{-1}$$

- The matrices are:

$$\Phi_{y,k} = \alpha \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \Phi_{y,k_{-1}} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

- Note timing!

Solving in Sequence Space

- The second equation in the firm block:

$$\hat{mpk}_{t+1} = \hat{z}_{t+1} + (\alpha - 1)\hat{k}_t$$

- In matrix notation:

$$\hat{\mathbf{mpk}} = \Phi_{mpk,z}\hat{\mathbf{z}} + \Phi_{mpk,z}\hat{\mathbf{k}}$$

- The matrices are:

$$\Phi_{mpk,z} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\Phi_{mpk,z} = (\alpha - 1) I_T$$

- Note timing!

Solving in Sequence Space

- With $\mathbf{\hat{mpk}}$ and $\mathbf{\hat{k}}$ we solve for $\mathbf{\hat{c}}$ and $\hat{\imath}$ using the household FOC.
- Start with investment:

$$\hat{k}_t = \delta \hat{\imath}_t + (1-\delta)\hat{k}_{t-1}$$

- In matrix notation:

$$\hat{\imath} = \Phi_{\imath,k}\mathbf{\hat{k}} + \Phi_{\imath,k-1}\hat{k}_{-1}$$

- The matrix is:

$$\Phi_{\imath,k} = \delta^{-1}\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ \delta-1 & 1 & \dots & 0 & 0 \\ 0 & \delta-1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \dots & \delta-1 & 1 \end{pmatrix}$$

Solving in Sequence Space

- Next consumption:

$$-\gamma \hat{c}_t = -\gamma \hat{c}_{t+1} + (1-\beta(1-\delta))\hat{mpk}_{t+1}$$

- In matrix notation:

$$\hat{\mathbf{c}} = \Phi_{c,mpk}\hat{\mathbf{mpk}}$$

- The matrices are:

$$\Phi_{c,mpk} = -\gamma^{-1}(1-\beta(1-\delta))\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}^{-1}$$

Solving in Sequence Space

- From the firm block we have:

$$\frac{\partial(\hat{\mathbf{y}}, (\widehat{\mathbf{mpk}}))}{\partial \mathbf{U}} = \begin{pmatrix} \Phi_{y,k} \\ \Phi_{mpk,k} \end{pmatrix}$$

- From the household block we have:

$$\frac{\partial(\hat{\mathbf{c}}, \hat{\imath})}{\partial \mathbf{U}} = \begin{pmatrix} \Phi_{c,mpk} \frac{d\widehat{\mathbf{mpk}}}{\partial \hat{\mathbf{k}}} \\ \Phi_{\imath,k} \end{pmatrix}$$
$$= \begin{pmatrix} \Phi_{c,mpk}\Phi_{mpk,k} \\ \Phi_{\imath,k} \end{pmatrix}$$

- We solved for $\mathbf{H_U}$:

$$\mathbf{H_U} = \begin{pmatrix} \Phi_{gm,c} & \Phi_{gm,\imath} & \Phi_{gm,y} & \mathbf{0}_T \end{pmatrix} \times \begin{pmatrix} \Phi_{c,mpk}\Phi_{mpk,k} \\ \Phi_{\imath,k} \\ \Phi_{y,k} \\ \Phi_{mpk,k} \end{pmatrix}$$
$$= \left(\Phi_{gm,c}\Phi_{c,mpk}\Phi_{mpk,k} + \Phi_{gm,\imath}\Phi_{\imath,k} - \Phi_{y,k}\right)$$

Solving in Sequence Space

- Solving for $\mathbf{H_Z}$ is a bit more straightforward:

$$\mathbf{H_Z} = \frac{\partial \mathbf{H}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}}$$

  We already know the first derivative.
- From the firm block we have:

$$\frac{\partial \hat{\mathbf{y}}}{\partial \mathbf{Z}} = I_T$$

  and

$$\frac{\partial (\hat{\mathbf{mpk}})}{\partial \mathbf{Z}} = \Phi_{mpk,z} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Solving in Sequence Space

- From the household block we have:

$$\frac{\partial \hat{\mathbf{c}}, \hat{\imath}}{\partial \mathbf{Z}} = \begin{pmatrix} \Phi_{c,wp} \frac{\partial \hat{\mathbf{mpk}}}{\partial \hat{\mathbf{z}}} \\ \mathbf{0}_T \end{pmatrix} = \begin{pmatrix} \Phi_{c,mpk} \Phi_{mpk,z} \\ \mathbf{0}_T \end{pmatrix}$$

- We solved for $\mathbf{H_Z}$:

$$\mathbf{H_Z} = \begin{pmatrix} \Phi_{gm,c} & \Phi_{gm,\imath} & \Phi_{gm,y} & \mathbf{0}_T \end{pmatrix} \times \begin{pmatrix} \Phi_{c,mpk} \Phi_{mpk,z} \\ \mathbf{0}_T \\ I_T \\ \Phi_{mpk,z} \end{pmatrix}$$

$$= \left(\Phi_{gm,c} \Phi_{c,mpk} \Phi_{mpk,z} - I_T \right)$$

- We now have the solution to the model:

$$d\mathbf{U} = -\mathbf{H_U}^{-1} \mathbf{H_Z} d\mathbf{Z}$$

and calculate the remaining sequences

$$d\mathbf{Y} = \frac{\partial \mathbf{Y}}{\partial \mathbf{U}} d\mathbf{U} + \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}} d\mathbf{Z} = \left(\frac{\partial \mathbf{Y}}{\partial \mathbf{U}} \mathbf{H_U}^{-1} \mathbf{H_Z} + \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}}\right) d\mathbf{Z}$$

Lessons

- Can solve any linearized dynamic model using linear algebra.
- Extremely fast once matrices are created.
  - If we tell the computer that the matrices are sparse (mostly 0s).
- Replaced substitution with matrix multiplication.
- Strategy follows equilibrium definition: looking for sequence such that everyone optimizes and markets clear.
- Everyone should do this once by hand. Then let the computer do the work for you.

Solving in Sequence Space

- Program this.
- Do you get the same answer as before?

Outline

More Sequence Space

1. Write a DAG for the RBC model without capital.
2. Write a DAG for the RBC model with capital.
3. Write a DAG for the New Keynesian model with capital.
4. For each model, derive the $\mathbf{H_U}$ and $\mathbf{H_Z}$ matrices.
   - You know you will solve three models, so try to set up your code in such a way that you can easily exchange / add blocks.
5. Compute the IRF to a one-time productivity shock of 1% for each model.
6. How much of the IRF in each model is due to the endogenous investment response?