Idiosyncratic Risk

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Outline

1. Where we are and where we are going
2. Idiosyncratic risk
3. Diffusion processes and the KF equation

Outline

Roadmap

- You learned how to solve the problem of one agent living in a deterministic world
- In a number of ways: Iterating v, iterating c, iterating both v and c.
- Today, we will learn to solve the problem of one age living in a stochastic world...
- and then how to compute the stationary distribution of agents given a price
- Lots of things to do today!

Outline

Our problem

$$\max_{\{c_{jt}\}} \int_0^\infty u(c_{jt}) \tag{1}$$

$$u(c_{jt}) = \log(c_{jt}) \tag{2}$$

$$\dot{a}_{jt} = ra_{jt} + y_{jt} - c_{jt} \tag{3}$$

$$y_{jt} = \begin{cases} y_H \\ y_L \end{cases} \tag{4}$$

$$\text{with Poisson intensities } \lambda_L, \lambda_H \tag{5}$$

Very similar to our previous problem! No aggregate shocks.

Our old HJB equation

$$\rho v(a) = \max_c u(c) + v'(a)\dot{a}$$
$$\text{subject to } \dot{a} = ra + y - c.$$

We replaced the constraint into the objective

$$\rho v(a) = \max_c u(c) + v'(a)(ra + y - c) \tag{6}$$

How does the HJB equation changes in the presence of risk?

Our new HJB equation

Instead of

$$\rho v(a) = \max_c u(c) + v'(a)(ra + y - c) \tag{7}$$

we now have

$$\rho v_i(a) = \max_c u(c) + v_i{}'(a)(ra + y_i - c) + \lambda_i [v_j(a) - v_i(a)] \tag{8}$$

- Notice the subscripts first. One HJB equation per income level $i \in L, H$.
- Notice the last term. With some intensity you transition to v_j for $j \neq i$
- v_L and v_H depend on each other

Comparison discrete vs. continuous time

Continuous time

$$\rho v_i(a) = \max_c u(c) + v_i'(a)(ra + y_i - c) + \lambda_i [v_j(a) - v_i(a)] \; \forall i \tag{9}$$

Discrete time

$$v_i(a) = \max_c u(c) + \beta \left[p_{ij} v_j(a') + p_{ii} v_i(a') \right], \tag{10}$$

$$\text{s.t. } a' = a(1+r) + y_i - c \forall i \tag{11}$$

where $i, j \in \{L, H\}, i \neq j$, p is a transition probability

Algorithm

$$\rho v_i(a) = \max_c u(c) + v_i'(a)(ra + y_i - c) + \lambda_i [v_j(a) - v_i(a)] \; \forall i \tag{12}$$

Algorithm:

1. Create a guess for v (now a $a_n \times 2$) matrix
2. Apply the upwind scheme for each v_j
3. Use v_i' from the upwind scheme to back out $c_i \quad \forall i$
4. Iterate using the implicit (preferred) or explicit method

Very similar to what we had before!

Task 1: Solve the consumer problem

1. Open the folder *Task 1*

2. Inspect its contents starting with the file *main.m*. Make sure to inspect every file.

3. The codes solve the deterministic case we worked on last week for two "types". One with income y_H that always has income y_H, one with income y_L that always has income y_L

4. Modify the set of codes so that instead you solve our problem with idiosyncratic risk

5. plot the consumption function c and the saving function $\dot{a}$.

Outline

Diffusion Processes: A crash course

- In their general form they satisfy

$$dx = \mu(x)dt + \sigma(x)dW$$

- As long as you are interested in process that does not jump, you can choose $\mu(x)$ and $\sigma(x)$ functions and get interesting processes as special cases
- Examples:
  1. Browinian motion $\mu(x) = \mu$, $\sigma(x) = \sigma$, the equivalent of the random walk in continuous time

$$dx = \mu dt + \sigma dW$$

  2. Ornstein-Uhlenbeck $\mu(x) = \theta(\bar{x} - x)$, $\sigma(x) = \sigma$, the equivalent of the AR(1) process with autocorrelation $e^{-\theta} \approx 1 - \theta$

$$dx = \theta(\bar{x} - x)dt + \sigma dW$$

  3. Can also get geometric processes where $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$.

Ito's Lemma

- If $x(t)$ is a diffusion (you are world class experts now)

- and you are interested in a function of that diffusion
$$y(t) = f(x(t))$$
that is twice continuously differentiable...

- then Ito's lemma gives you the behavior of $f(x)$.
$$df(x) = \left(\mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)\right)dt + \sigma(x)f'(x)dW$$

- and this extends to multivariate functions. If $\mu(x,t)$, and $\sigma(x,t)$.

Kolmogorov Forward Equation

- I will not bother you with the proof, but the KF equation comes from using Ito's lemma on the density of x, which we call $g(x,t)$ we get the following relation

$$\frac{\partial g(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\mu(x)g(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)g(x,t)\right]$$

- think of x as wealth in our example, and g as the joint distribution of wealth and time in the cross-section
- If a stationary distribution exists, where we define it as $g(x)$ such that if $g(x,t) = g(x)$ then $g(x,\tau) = g(x)\ \forall \tau \geq t$
- Then the Kolmogorov Equation simplifies to

$$0 = -\frac{\partial}{\partial x}\left[\mu(x)g(x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)g(x)\right]$$

- One note before proceeding. We are studying a condition that a stationary distribution needs to satisfy **if it exists**, we are not claiming existence.

When the magic happens

- Cookbook approach and then some words on how it works
- The stationary distribution for Poisson shocks satisfies:

$$0 = -\frac{d}{da}[s_i(a)g_i(a)] - \lambda_i g_i(a) + \lambda_j g_j(a) \tag{13}$$

$$\int_{\underline{a}}^{\infty} (g_1(a) + g_2(a))\, da = 1 \tag{14}$$

$$g_1(a) \geq 0 \forall a \tag{15}$$

$$g_2(a) \geq 0 \forall a \tag{16}$$

- It turns out that discretizing and finding g is extremely easy. Although it looks like a nightmare

$$0 = A(v)^T g$$

- g solves an eigenvalue problem using the transpose of A from the HJB equation
- Notice that to have A you must use the implicit method.

Intuition

- For simplicity assume no shocks
- $g(x)$ is the density of x, $\mu(x)$ the drift of x
- $0 = \mu(x)g'(x)$ from Ito's lemma and no shocks.
- meaning: g is such that $\dot{g} = 0$
- Notice that in the HJB equation we had that the upwind scheme implied $v'(x)\mu(x)$ which we expressed in matrix form as Av (check lecture 3)
- Similarly now we have $\mu(x)g'(x)$ which we will discretize by $A^T g = 0$
- So we only need to find g from

$$0 = A^T g$$

- Transpose because we want to capture how much wealth is flowing to state i in net. Second row of A^T times g:

$$\frac{s_{1F}^{+}}{\Delta k} g_1 + \left(-\frac{s_{3B}^{-}}{\Delta k}\right) g_3 - \left(\frac{s_{2,F}^{+} + (-s_{2,B}^{-})}{\Delta k}\right) g_2$$

Look at some code together to see how this is actually implemented

- I do not have any doubt that you can deduct how to write the code to solve the KF equation
- But it uses a couple of tricks that would require more time than we have today
- So go to exercise1 and inspect the code. Study the following variables
  1. *b*
  2. *Aswitch*
  3. *gg*

Indirect Inference

- Indirect inference refers to the exercise of fixing a parameter value to replicate a moment in the data
- You did one exercise like that when we ran regressions of consumption on wealth
- Today we will perform more exercises that **do not** require simulations

Indirect Inference

- Imagine that you want to target a wealth to income ratio in the economy of 3. Can you find the level of ρ that gets you this number?
- Go back to the original parameterization. Imagine you see in the data that high income earners hold twice as much wealth as low income earners. Fixing λ_H, find the level of λ_L that gets you that number
- Fix the value λ_H. Find the value λ_L that gets you a variance of wealth equal to 0.02.

For each of these exercises write a function that outputs the squared difference between the moment of interest and the target. Then use the function fminsearch to minimize the difference.

Homework

- You observe in the data that the length of an unemployment spell is equal to 3 months. Write a function that simulates the model we developed in class, and sets λ_L such that you match the average length of unemployment.

- Write down a function that gives as an output the aggregate amount of savings in the economy as a function of the interest rate. Fix the parameterization of the model to the value in the prompts for today's lecture. Create a function that plots the aggregate demand for savings as a function of the interest rate in the economy for a range $\vec{r} = [0, 0.04]$ in increments of 0.001