The Sequence Space

Juan Herreño Johannes Wieland

UCSD, Spring 2022

OUTLINE

Outline

Solving Models

- Hard when there is rich heterogeneity.
- Infinitely dimensional state space from the distribution of agent's endogenous state variables—how to handle it?
- In macro, we are often interested what happens when aggregate shock X hits the economy (monetary, fiscal, etc).
- State space: must forecast infinite-dimensional distribution to estimate prices / quantities relevant to agents' problem.
- Sequence space: must find price / quantity sequences that solve market clearing conditions.

⇒ Sequence space is of dimension $T \times$ number of variables, versus infinite dimensional state space.

- Cost: linearization / perfect foresight.

Demand Supply Example

- Demand and Supply

$$q_i^D = -\eta^D p + \mu^D q + v^D$$
$$q_j^S = \eta^S p + \mu^S q + v^S$$

  - Demand/supply elasticities η,
  - "Agglomoration" elasticities μ

- Market clearing:

$$q = \frac{\frac{\eta^S}{1-\mu^S}}{\left(\frac{\eta^D}{1-\mu^D} + \frac{\eta^S}{1-\mu^S}\right)} \frac{1}{1-\mu^D} v^D + \frac{\frac{\eta^D}{1-\mu^D}}{\left(\frac{\eta^D}{1-\mu^D} + \frac{\eta^S}{1-\mu^S}\right)} \frac{1}{1-\mu^S} v^S$$

- Sequence space methods are combining demand / supply elasticities and multiplier effects to solve for equilibrium outcomes.
- New: these are intertemporal objects.

Outline

NK Model

- Simple NK model with perfect forsight:

$$y_t = -\frac{1}{\sigma}(i_t - \pi_{t+1}) + \mathbb{E}_t y_{t+1}$$
$$\pi_t = \kappa(y_t - y_t^{flex}) + \beta \pi_{t+1}$$

  - Exogenous i_t, y_t^{flex} (determinacy?)
  - Perfect foresight.

Writing in Sequence Space

- Equations hold at every point in time starting at $t = 0$.

- First stack Euler equations for $t = 0, 1, ..., T$:

$$\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_T \end{pmatrix} = - \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} i_0 \\ i_1 \\ \vdots \\ i_T \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & 1 \\ 0 & \vdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_T \end{pmatrix}$$

Writing in Sequence Space

- Then stack the NKPC:

$$\begin{pmatrix} 1 & -\beta & 0 & \dots & 0 \\ 0 & 1 & -\beta & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -\beta \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_T \end{pmatrix} = \begin{pmatrix} \kappa & 0 & \dots & 0 \\ 0 & \kappa & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \kappa \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_T \end{pmatrix} - \begin{pmatrix} \kappa & 0 & \dots & 0 \\ 0 & \kappa & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \kappa \end{pmatrix} \begin{pmatrix} y_0^{flex} \\ y_1^{flex} \\ \vdots \\ y_T^{flex} \end{pmatrix}$$

- In matrix form:

$$\boldsymbol{\Phi}_Y \boldsymbol{Y} = -\boldsymbol{\Phi}_i \boldsymbol{i} + \boldsymbol{\Phi}_\pi \boldsymbol{\pi}$$
$$\boldsymbol{\Psi}_\pi \boldsymbol{\pi} = \boldsymbol{\Psi}_Y \boldsymbol{Y} - \boldsymbol{\Psi}_Y \boldsymbol{Y}^{flex}$$

Solution is an Elasticity Formula

- Define $\boldsymbol{\eta}_\pi^D \equiv -\boldsymbol{\Phi}_Y^{-1}\boldsymbol{\Phi}_\pi$, $\boldsymbol{\eta}_\pi^S \equiv \boldsymbol{\Psi}_Y^{-1}\boldsymbol{\Psi}_\pi$, $\eta_i^D = \boldsymbol{\Phi}_Y^{-1}\boldsymbol{\Phi}_i$:

$$\boldsymbol{Y} = -\boldsymbol{\eta}_i^D \boldsymbol{i} - \boldsymbol{\eta}_\pi^D \boldsymbol{\pi}$$
$$\boldsymbol{\eta}_\pi^S \boldsymbol{\pi} = \boldsymbol{Y} - \boldsymbol{Y}^{flex}$$

- Solution:

$$\boldsymbol{Y} = (\boldsymbol{\eta}_\pi^S)[\boldsymbol{\eta}_\pi^S + \boldsymbol{\eta}_\pi^D]^{-1}[-\boldsymbol{\eta}_i^D \boldsymbol{i} + \boldsymbol{\eta}_\pi^D (\boldsymbol{\eta}_\pi^S)^{-1} \boldsymbol{Y}^{flex}]$$

- Interpretation:
  - $-\boldsymbol{\eta}_i^D \boldsymbol{i} + \boldsymbol{\eta}_\pi^D (\boldsymbol{\eta}_\pi^S)^{-1} \boldsymbol{Y}^{flex}$ is the effect holding on output without the endogenous feedback through inflation.
  - $(\boldsymbol{\eta}_\pi^S)[\boldsymbol{\eta}_\pi^S + \boldsymbol{\eta}_\pi^D]^{-1}$ captures the feedback loop of inflation and output through demand / supply elasticities:

Intertemporal Supply and Demand

- Solution:

$$\boldsymbol{Y} = (\boldsymbol{\eta}_\pi^S)[\boldsymbol{\eta}_\pi^S + \boldsymbol{\eta}_\pi^D]^{-1}[-\boldsymbol{\eta}_i^D \boldsymbol{i} + \boldsymbol{\eta}_\pi^D (\boldsymbol{\eta}_\pi^S)^{-1} \boldsymbol{Y}^{flex}]$$

- If supply elasticities very small:

$$\boldsymbol{Y} \approx \boldsymbol{Y}^{flex}$$

- If supply elasticities very large:

$$\boldsymbol{Y} \approx -\boldsymbol{\eta}_i^D \boldsymbol{i}$$

- If demand elasticities very large:

$$\boldsymbol{Y} \approx \boldsymbol{Y}^{flex} - (\boldsymbol{\eta}_\pi^S)(\boldsymbol{\eta}_\pi^D)^{-1} \boldsymbol{\eta}_\pi^D \boldsymbol{i}$$

- If demand elasticities very small:

$$\boldsymbol{Y} \approx 0$$

$\Rightarrow$ All we are doing is intertemporal demand and supply.

Takeaway

- Solving linear models with sequence space is linear algebra.
- Useful to think in terms of demand and supply elasticities in interpreting model output.
- Don't need partial equilibrium or heterogeneity to use sequence space.
- Do need that the solution to the model is a sequence.

Outline

Simplified Consumer Problem

- Preferences:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_{it})$$

- After-tax income:

$$z_{it} \equiv \tau_t \left(\frac{W_t}{P_t} e_{it} n_{it}\right)^{1-\lambda} = \frac{e_{it}^{1-\lambda}}{\int e_{it}^{1-\lambda}} Z_t$$

- Budget constraint:

$$c_{it} + \sum_j a_{it}^j = z_{it} + (1+r_{t-1})\sum_j a_{i,t-1}^j$$

- Nests: RA, TA, HA, HA-IL

Optimal Policy Rules

- Budget constraint with $Z_t = Y_t - T_t$:

$$c_{it} + \sum_j a^j_{it} = \frac{e_{it}^{1-\lambda}}{\int e_{it}^{1-\lambda}}(Y_t - T_t) + (1 + r_{t-1}) \sum_j a^j_{i,t-1}$$

- Idiosyncratic states: e_{it}.
- Assume aggregate variables follow a known sequence $\{Y_t - T_t, r_t\}_{t=0}^{\infty}$.
- Conditional on knowing these sequences, can solve for individual consumption and assets given any initial state $\{a^j_0\}, e_{i0}$:

$$c_t(\{a^j\}, e),\ \{a^j_t(\{a^j\}, e)\}, \qquad t \geq 0$$

- Why? Distribution only enters consumers problem through effect on $\{Y_t - T_t, r_t\}_{t=0}^{\infty}$. No longer have infinite state space.
- Then aggregate all consumption decisions to get:

$$C_t = \int c_t(\{a^j\}, e) d\Psi_t(\{a^j\}, e) = \mathbb{C}(\{Y_{t+s} - T_{t+s}, r_{t+s}\}_{s=0}^{\infty})$$

New Keynesian Model in Sequence Space

- Consumption function:

$$\begin{aligned} C_t &= \mathbb{C}(\{Y_{t+s} - T_{t+s}, r_{t+s}\}_{s=0}^{\infty}) \\ &\equiv \mathbb{C}(\boldsymbol{Y} - \boldsymbol{T}, \boldsymbol{r}) \end{aligned}$$

- NKPC:

$$\pi_t = \mathbb{S}(\boldsymbol{Y} - \boldsymbol{Y}^*)$$

- Interest rate rule:

$$\boldsymbol{r} = \boldsymbol{\Phi}_Y(\boldsymbol{Y} - \boldsymbol{Y}^*) + \boldsymbol{\Phi}_\pi \boldsymbol{\pi} + \boldsymbol{\varepsilon}^r$$

- Excess demand:

$$\boldsymbol{ED} = \mathbb{C}(\boldsymbol{Y} - \boldsymbol{T}, \boldsymbol{r}) + \boldsymbol{G} - \boldsymbol{Y} = 0$$

Linearize to Solve with Linear Algebra

- Linearized Model:

$$\hat{\boldsymbol{C}} = (\nabla_{\boldsymbol{Y}}\mathbb{C})(\hat{\boldsymbol{Y}} - \hat{\boldsymbol{T}}) + (\nabla_{\boldsymbol{r}}\mathbb{C})\hat{\boldsymbol{r}}$$
$$\hat{\boldsymbol{\pi}} = (\nabla_{\boldsymbol{Y}}\mathbb{S})(\hat{\boldsymbol{Y}} - \hat{\boldsymbol{Y}}^*)$$
$$\hat{\boldsymbol{r}} = \boldsymbol{\Phi}_Y(\hat{\boldsymbol{Y}} - \hat{\boldsymbol{Y}}^*) + \boldsymbol{\Phi}_\pi\hat{\boldsymbol{\pi}} + \boldsymbol{\varepsilon}^r$$

where the Jacobians of the model look like:

$$\nabla_{\boldsymbol{Y}}\mathbb{C} = \begin{pmatrix} \frac{\partial C_0}{\partial Y_0} & \frac{\partial C_0}{\partial Y_1} & \cdots \\ \frac{\partial C_1}{\partial Y_0} & \frac{\partial C_1}{\partial Y_1} & \cdots \\ \frac{\partial C_2}{\partial Y_0} & \frac{\partial C_2}{\partial Y_1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

- Equilibrium:

$$\hat{\boldsymbol{Y}} = [\boldsymbol{\Phi}_Y + \boldsymbol{\Phi}_\pi(\nabla_{\boldsymbol{Y}}\mathbb{S})]^{-1}\{[\boldsymbol{\Phi}_Y + \boldsymbol{\Phi}_\pi(\nabla_{\boldsymbol{Y}}\mathbb{S})]^{-1} - [I - \nabla_{\boldsymbol{Y}}\mathbb{C}]^{-1}(\nabla_{\boldsymbol{r}}\mathbb{C})\}^{-1} \times$$
$$\times [I - \nabla_{\boldsymbol{Y}}\mathbb{C}]^{-1}[\hat{\boldsymbol{G}} - (\nabla_{\boldsymbol{Y}}\mathbb{C})\hat{\boldsymbol{T}} + (\nabla_{\boldsymbol{r}}\mathbb{C})\boldsymbol{\varepsilon}^r]$$

Demand and Supply Elasticities Determine GE

- Define:
  - Demand multiplier $\boldsymbol{\mu}^D \equiv [I - \nabla_{\boldsymbol{Y}}\mathbb{C}]^{-1}$
  - Demand elasticity $\boldsymbol{\eta}^{\boldsymbol{D}} \equiv -\nabla_{\boldsymbol{r}}\mathbb{C}$
  - Supply elasticity $\boldsymbol{\eta}^{\boldsymbol{S}} \equiv [\boldsymbol{\Phi}_Y + \boldsymbol{\Phi}_\pi(\boldsymbol{\nabla}_{\boldsymbol{Y}}\mathbb{S})]^{-1}$

$$\hat{\boldsymbol{Y}} = \underbrace{\boldsymbol{\eta}^{\boldsymbol{S}}[\boldsymbol{\eta}^{\boldsymbol{S}} + \boldsymbol{\mu}^D\boldsymbol{\eta}^{\boldsymbol{D}}]^{-1}}_{\text{Demand Incidence}}\boldsymbol{\mu}^D\underbrace{[\hat{\boldsymbol{G}} - (\nabla_{\boldsymbol{Y}}\mathbb{C})\hat{\boldsymbol{T}} + \boldsymbol{\eta}^{\boldsymbol{D}}\boldsymbol{\varepsilon}^r]}_{\text{PE Excess Demand}}$$

$$+\underbrace{\boldsymbol{\mu}^D\boldsymbol{\eta}^{\boldsymbol{D}}[\boldsymbol{\eta}^{\boldsymbol{S}} + \boldsymbol{\mu}^D\boldsymbol{\eta}^{\boldsymbol{D}}]^{-1}}_{\text{Supply Incidence}}\underbrace{\hat{\boldsymbol{Y}}^*}_{\text{PE Excess Supply}}$$

Implications:

- GE effects determined by a standard incidence formula.

$\Rightarrow$ Matrices of micro elasticities are sufficient statistics for GE effects.

Intertemporal Keynesian Cross

- Also assume no real rate change, so $\boldsymbol{\eta}^{\boldsymbol{S}}$ blows up in some meaningful sense.

$$\hat{\boldsymbol{Y}} = [\boldsymbol{I} - \nabla_{\boldsymbol{Y}}\mathbb{C}]^{-1}[\hat{\boldsymbol{G}} - (\nabla_{\boldsymbol{Y}}\mathbb{C})\hat{\boldsymbol{T}}]$$

- This is the Intertemporal Keynesian Cross.
- Balanced budget: $\hat{\boldsymbol{G}} = \hat{\boldsymbol{T}}$

$$\begin{aligned}\hat{\boldsymbol{Y}} &= [\boldsymbol{I} - \nabla_{\boldsymbol{Y}}\mathbb{C}]^{-1}[\boldsymbol{I} - (\nabla_{\boldsymbol{Y}}\mathbb{C})]\hat{\boldsymbol{G}} \\ &= \hat{\boldsymbol{G}}\end{aligned}$$

- The balanced budget multiplier is 1.

Fiscal Policies

- Proposition 4: In the RA model, the G-multiplier is 1.
  - Proof: The G-multiplier is 1 when budget is balanced, and we know for the RA model the timing of T does not matter.
- Proposition 5: in the TA model with μ constrained agents,

$$\hat{\boldsymbol{Y}} = \frac{1}{1-\mu}[\hat{\boldsymbol{G}} - \mu \hat{\boldsymbol{T}}]$$

  - Deficit-financed fiscal expansion ($\hat{T}_0 < \hat{G}_0$) has impact multiplier greater than 1.

MEASURING $\nabla_{\boldsymbol{Y}}\mathbb{C}$

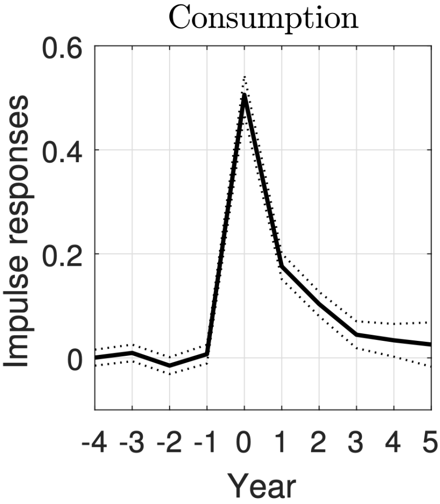
- Demand multiplier: $\boldsymbol{\mu}^D \equiv [I - \nabla_{\boldsymbol{Y}}\mathbb{C}]^{-1}$ where

$$\nabla_{\boldsymbol{Y}}\mathbb{C} = \begin{pmatrix} \frac{\partial C_0}{\partial Y_0} & \frac{\partial C_0}{\partial Y_1} & \cdots \\ \frac{\partial C_1}{\partial Y_0} & \frac{\partial C_1}{\partial Y_1} & \cdots \\ \frac{\partial C_2}{\partial Y_0} & \frac{\partial C_2}{\partial Y_1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

  Columns are PE IRFs of average consumption to income changes at different horizons.

- What evidence do we have?

Lottery Studies

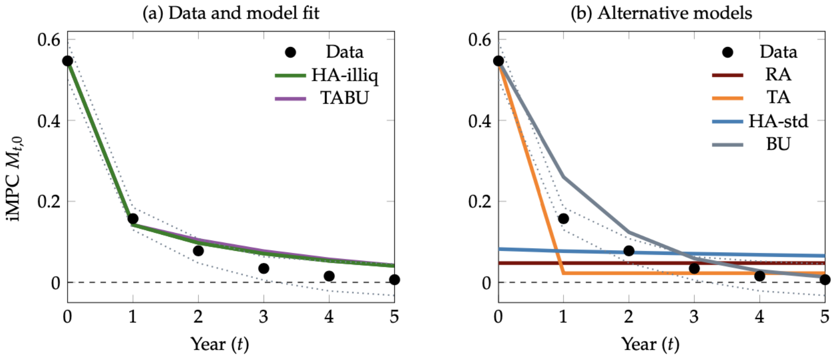


- Provides estimate of first column of $\nabla_{\boldsymbol{Y}}\mathbb{C}$.
- Need a model to extrapolate.

Source: Fagereng, Holm, Natvik (2018)

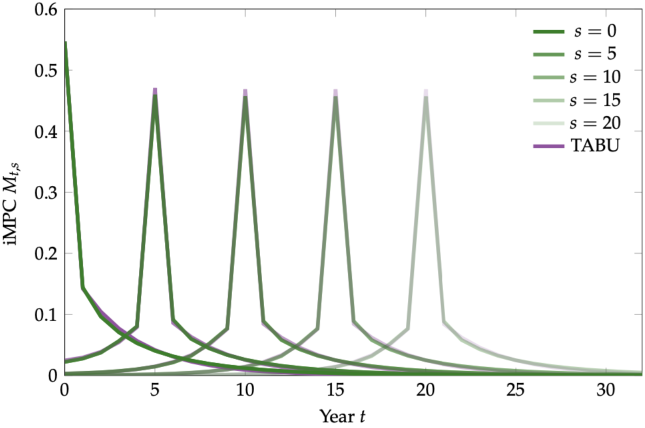
iMPCs in the Models

Figure 2: iMPCs in the Norwegian data and several models.



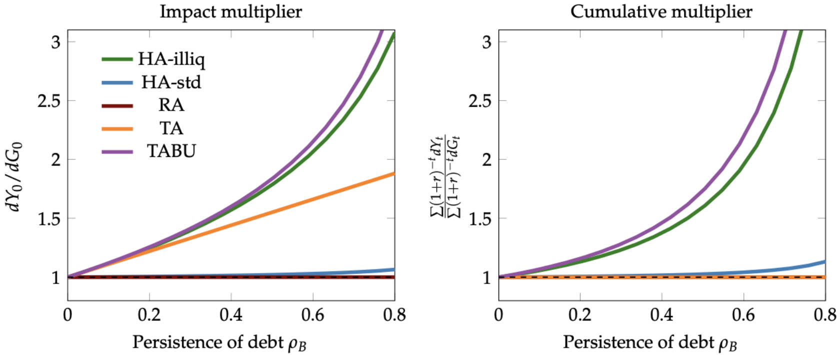
Model Extrapolation

Figure 3: Columns of the iMPC matrix in the HA-illiq model: $M_{\cdot,s}$ for $s = 0, 5, 10, 15, 20$.



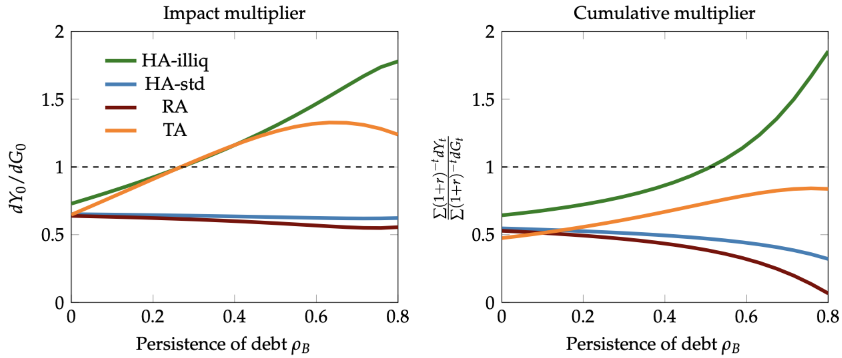
Fiscal Policies: All Models

Figure 4: Multipliers across the benchmark models.



Quantitative Model

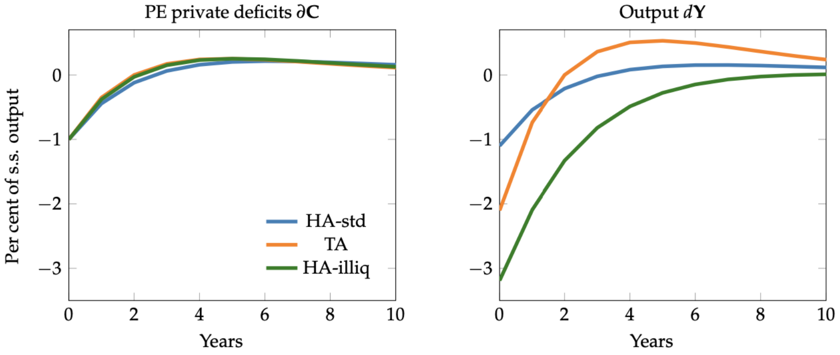
Figure 7: Multipliers in the quantitative models.



- What does this tell us about supply elasticities?

Private Deficits

Figure 8: The effects of deleveraging shocks.



Note. To make responses comparable, ϵ is chosen to equalize the initial direct effect ∂C_0 across models. The persistence parameters are $\rho = 0.7$ and $\rho_{\underline{a}} = 0.7$.

- Why so much amplification?

Outline

Sequence Space Methods

- Sequence Space methods mean we use linear algebra to solve models with rich heterogeneity.
- But how do we get matrices like these:

$$\nabla_{\boldsymbol{Y}}\mathbb{C} = \begin{pmatrix} \frac{\partial C_0}{\partial Y_0} & \frac{\partial C_0}{\partial Y_1} & \cdots \\ \frac{\partial C_1}{\partial Y_0} & \frac{\partial C_1}{\partial Y_1} & \cdots \\ \frac{\partial C_2}{\partial Y_0} & \frac{\partial C_2}{\partial Y_1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

- Need to solve the household consumption problem: hard.
- This paper: efficient algorithm that exploits model structure and linearity to get the Jacobians of the model really fast!

Krussel-Smith (1998) Economy

- Preferences:

$$V_t(e, a_-) = \max_{c,k} u(c) + \beta \sum_{e'} V_{t+1}(e', k) P(e, e')$$

- Budget constraint

$$c + k = (1 + r_t) k_- + w_t e n$$

- Borrowing constraint

$$k \geq 0$$

- Optimal policy:

$$c_t^*(e, k_-),\ k_t^*(e, k_-)$$

- Functions of $\{r_t, w_t\}_{t \geq 0}$.

Aggregating Consumer Problem

- Distribution of capital and productivity:

$$D_{t+1}(e', K) = \sum_e D_t(e, k_t^{*-1}(e, K))P(e, e')$$

- Given D_0, $D_{t+1}(e', K)$ are also functions of $\{r_t, w_t\}_{t\geq 0}$.

- Aggregate capital holdings are therefore also a function of $\{r_t, w_t\}_{t\geq 0}$.:

$$\mathscr{K}_t(\{r_s, w_s\}_{s\geq 0}) = \sum_e \int_{k_-} k_t^*(e, k_-) D_t(e, dk_-)$$

- The function $\mathscr{K}_t$ maps an aggregate sequence $\{r_t, w_t\}_{t\geq 0}$ into another aggregate sequence $\{K_t\}_{t\geq 0}$.

- The dimensionality of this problem is the length of the sequence T.

Firm and Market Clearing

- Keep this simple:

$$\begin{aligned} Y_t &= Z_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\ r_t &= \alpha Z_t K_{t-1}^{\alpha-1} N_t^{1-\alpha} - \delta \\ w_t &= (1-\alpha) Z_t K_{t-1}^{\alpha} N_t^{-\alpha} \end{aligned}$$

- Labor supply fixed in this problem so

$$N_t = \sum \pi(e) e n$$

- Market for capital has to clear

$$\begin{aligned} & H_t(\boldsymbol{K}, \boldsymbol{Z}) \\ & \equiv \mathscr{K}_t \left(\left\{ \alpha Z_s \left(\frac{K_{s-1}}{\sum \pi(e) e n} \right)^{\alpha-1} - \delta, (1-\alpha) Z_s \left(\frac{K_{s-1}}{\sum \pi(e) e n} \right)^{\alpha} \right\}_{s \geq 0} \right) \\ & \qquad - K_t = 0 \end{aligned}$$

IRFs

- Why is this useful? From implicit function theorem,

$$d\boldsymbol{K} = -\boldsymbol{H}_K^{-1}\boldsymbol{H}_{\boldsymbol{Z}}\, d\boldsymbol{Z}$$

$\Rightarrow$ To get IRFs need only the Jacobians of the market clearing condition w.r.t. the sequences $\boldsymbol{H}_K, \boldsymbol{H}_{\boldsymbol{Z}}$.

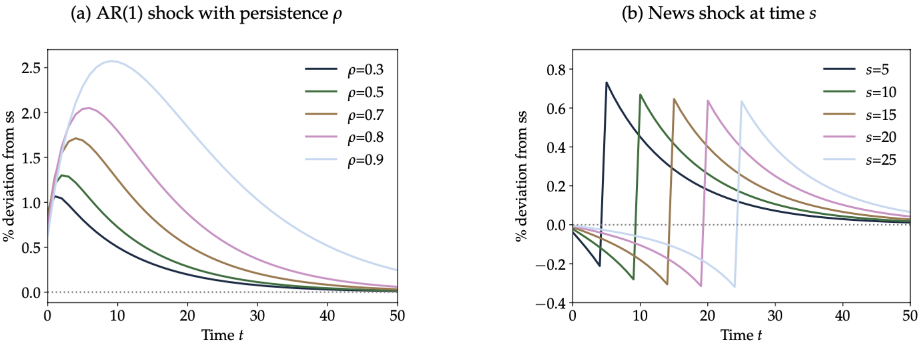
- How do we get these Jacboians? From the chain rule:

$$[\boldsymbol{H}_K]_{t,s} = \frac{\partial \mathscr{K}_t}{\partial r_{s+1}}\frac{\partial r_{s+1}}{\partial K_s} + \frac{\partial \mathscr{K}_t}{\partial w_{s+1}}\frac{\partial w_{s+1}}{\partial K_s} - 1_{\{s=t\}}$$

- Some of these derivatives we can calculate analytically: $\frac{\partial r_{s+1}}{\partial K_s}, \frac{\partial w_{s+1}}{\partial K_s}, \frac{\partial r_s}{\partial Z_s}, \frac{\partial w_s}{\partial Z_s}$.
- Others we have to compute numerically.

General IRFs from One-Time Computation

Figure 1: Impulse responses of capital to 1% TFP shocks in the Krusell-Smith model



Fake News Algorithm

- Goal: efficient computation of Jacobians.
- Framework:

$$\begin{aligned} \boldsymbol{v}_t &= v(\boldsymbol{v}_{t+1}, \boldsymbol{X}_t) \\ \boldsymbol{D}_{t+1} &= \Lambda(\boldsymbol{v}_{t+1}, \boldsymbol{X}_t)' \boldsymbol{D}_t \\ \boldsymbol{Y}_t &= y(\boldsymbol{v}_{t+1}, \boldsymbol{X}_t)' \boldsymbol{D}_t \end{aligned}$$

  - $\boldsymbol{X}_t$ are inputs (e.g., Z_t).
  - $\boldsymbol{Y}_t$ are outputs (e.g., C_t, K_t).
  - $\boldsymbol{D}_t$ is the discretized distribution.
- In sequence space:

$$\boldsymbol{Y} = h(\boldsymbol{X})$$

- Want: $\mathscr{J} = \nabla h_{\boldsymbol{Y}}(\boldsymbol{X}^{SS})$

Brute Force

- Column s of $\mathscr{J}_X^Y$ is IRF of outcome (e.g., capital) to a one-time shock (e.g. productivity) at s, $\boldsymbol{X}^s = \boldsymbol{X}_{ss} + \boldsymbol{e}^s dx \equiv \boldsymbol{X}_{ss} + d\boldsymbol{X}^s$.
- Solve problem backward given the known sequence of $\boldsymbol{X}$ to get value functions $\boldsymbol{v}_t$, policy functions $\boldsymbol{y}_t^s$, and transition matrices $\Lambda(\boldsymbol{v}_{t+1}, \boldsymbol{X}_t)'$.
- Given optimal policy functions iterate distribution of agents forward starting from steady state,

$$\boldsymbol{D}_{t+1}^s = (\Lambda_t^s)\boldsymbol{D}_t^s$$

- Aggregate policy functions using the distribution to get outcome

$$\mathscr{Y}_t^s = (\boldsymbol{y}_t^s)'\boldsymbol{D}_t^s$$

- Repeat for $s = 0, \ldots, T$ to get all columns of $\mathscr{J}$.
- This "brute force" method works but is very slow.

Algorithm: Efficient backward step

- Lemma 1: for any $s \geq 1$, $t \geq 1$:

$$\boldsymbol{y}_t^s = \begin{cases} \boldsymbol{y}_{ss} & s < t \\ \boldsymbol{y}_{T-1-(s-t)}^{T-1} & s \geq t \end{cases}, \qquad \Lambda_t^s = \begin{cases} \Lambda_{ss} & s < t \\ \Lambda_{T-1-(s-t)}^{T-1} & s \geq t \end{cases}$$

- Policy functions at t for shock at $t+s$ the same as policy function at 0 for shock at s.

$\Rightarrow$ Policy functions need to be solved backwards only once starting with a shock at $T-1$.

- For any time after the shock, the policy functions are the same as in steady state.

- Key: policy function cannot directly depend on distribution of agents

Fake News

- Denote the sequence $\boldsymbol{\varepsilon}^s$ in which $\varepsilon_s = 1$ and zero otherwise:

$$\varepsilon_t = \begin{cases} 1 & t = s \\ 0 & t \geq 0, t \neq s \end{cases}$$

- Let η_t^s be a fake news shock for ε_t:
  - At t learn that $\varepsilon_s = 1$ with certainty.
  - At $t+1$ lean that $\varepsilon_s = 0$.
  - $\mathbb{E}_k \eta_t^s = 0$ for all $k < t$.
- Lemma: The sequence $\boldsymbol{v}^s$

$$v_t = \begin{cases} \eta_t^{s-t} & t < s \\ 1 & t = s \\ 0 & t > s \end{cases}$$

takes on the same expected values and realized values as $\boldsymbol{\varepsilon}^s$.

Advantage of Fake News

- Given linearization only care about expected values and realized values.
- Rather than compute IRFs to $\boldsymbol{\varepsilon}^s$ for $s \geq 0$, we need IRFs to fake news shocks η^s for $s \geq 0$ and contemporaneous shock $\boldsymbol{\varepsilon}^0$.
- This is more efficient because η^s and $\boldsymbol{\varepsilon}^0$ share a common feature: the policy function only changes in the period the shock becomes known and then reverts. From then on we only need to solve the distribution forward using steady state policy functions.

Algorithm: Fake News

- Define the difference in outcomes at t for shock at s versus outcomes at $t-1$ for shock at $s-1$.

$$\mathscr{F}_{t,s} dx \equiv d\mathscr{Y}_t^s - d\mathscr{Y}_{t-1}^{s-1}$$

- Not zero: policy function the same, but distribution different.
- Lemma:

$$\mathscr{F}_{t,s} dx = \boldsymbol{y}_{ss}' (\Lambda_{ss}')^{t-1} d\boldsymbol{D}_1^s$$

- Policy functions the same. The only difference is the initial distribution.
- To a first order, the distribution affects outcomes as if all agents followed their steady state policy functions.

Fake News Interpretation

$$\mathscr{F}_{t,s}dx = \boldsymbol{y}_{ss}'(\Lambda_{ss}')^{t-1}d\boldsymbol{D}_1^s$$

- Agents at $t=0$ were told a shock will happen at $t=s$.
- They change their optimal policy rules resulting in a new distribution of outcomes at $t=1$, $d\boldsymbol{D}_1^s$.
- At $t=1$ agents learn (to their surprise) that the shock does not happen.
- So all policy rules revert to steady state.
- But the distribution has not reverted to steady state and will affect economic outcomes.
- $\boldsymbol{y}_{ss}'(\Lambda_{ss}')^{t-1}d\boldsymbol{D}_1^s$ traces out IRF from $t=1$ onwards to a first order.
- Requires only one updating step in solving $d\boldsymbol{D}_1^s$

Fake News Matrix

- Define the fake news matrix as

$$\mathscr{F}_{t,s} dx \equiv \begin{cases} d\mathscr{Y}_0^s & t = 0 \\ \boldsymbol{y}_{ss}'(\Lambda_{ss}')^{t-1} d\boldsymbol{D}_1^s & t \geq 1 \end{cases}$$

- Then the Jacobian of h is given by

$$\mathscr{J}_{t,s} = \sum_{k=0}^{\min\{s,t\}} \mathscr{F}_{t-k,s-k}$$

- $t = 0$ follows by definition.

- Why does this make sense?

Fake News Matrix Examples

- $t > 0, s = 0$:

$$\mathscr{J}_{t,0} = \boldsymbol{y}_{ss}'(\Lambda_{ss}')^{t-1} d\boldsymbol{D}_1^0$$

  - Shock at 0 affected distribution at 1. Now trace out this effect.

- $t = 1, s = 1$:

$$\mathscr{J}_{1,1} = d\mathscr{Y}_0^0 + \boldsymbol{y}_{ss}' d\boldsymbol{D}_1^1$$

  - Sum of surprise contemporaneous shock and fake news shock.

- $t = 1, s = 2$:

$$\mathscr{J}_{1,2} = d\mathscr{Y}_0^1 + \boldsymbol{y}_{ss}' d\boldsymbol{D}_1^2$$

  - As if we got news today about shock tomorrow, but taking into account change in distribution given that news was known earlier.

$$\mathscr{J}_{3,4} = d\mathscr{Y}_0^1 + \boldsymbol{y}_{ss}' d\boldsymbol{D}_1^2 + \boldsymbol{y}_{ss}'(\Lambda_{ss}') d\boldsymbol{D}_1^3 + \boldsymbol{y}_{ss}'(\Lambda_{ss}')^2 d\boldsymbol{D}_1^4$$

Bottom Line

- $t = 1, s = 0$:

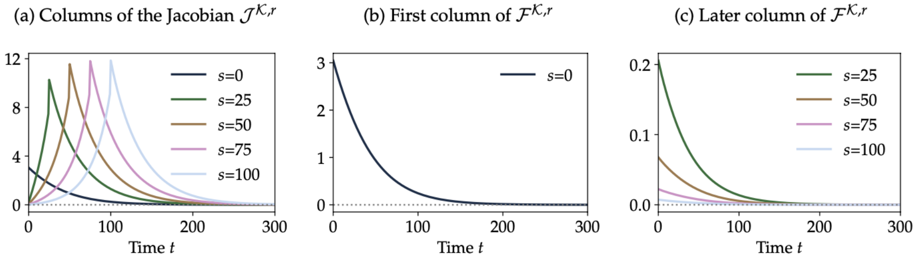
$$\mathscr{J}_{t,s} = \mathscr{J}_{t-1,s-1} + \mathscr{F}_{t,s}$$

  - A shock at s from time t looks very similar to a shock at $s-1$ from time $t-1$.
  - Policy functions are exactly the same.
  - Only difference is that the distribution at t is different from $t-1$, which is captured by the fake news term.
- The objects we need to calculate this are:
  1. Solve consumer problem backwards once for T periods to get policy functions $\{\boldsymbol{y}_{T-1-s}^{T-1}, \boldsymbol{\Lambda}_{T-1-s}^{T-1}\}_{s=0}^{T-1}$.
  2. Get $\{d\mathscr{Y}_0^s\}_{s=0}^{T}$ by combining policy functions for $s = 0, ..., T$ with steady state distribution.
  3. Compute $d\boldsymbol{D}_1^s$ using steady state distribution and $s = 0, ..., T$ policy functions.

$\Rightarrow$ Very small number of steps / computations to get all of the Jacobian and therefore the IRFs.

RESULT

Figure 2: Jacobian $\mathcal{J}^{\mathcal{K},r}$ and fake news matrix $\mathcal{F}^{\mathcal{K},r}$ in the Krusell-Smith model.

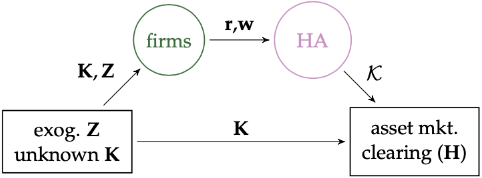


Dimensionality Reduction

- The dimensionality of the problem is $T \times$ number of variables, but with large T and many variables this can get complicated.
- Solution: substitute out for some variables, like we did for r_t, w_t earlier.
- This can be automated by writing the model in separate blocks.
- E.g., firm block $r_t = \alpha Z_t K_{t-1}^{\alpha-1} N_t^{1-\alpha} - \delta$ and $w_t = (1-\alpha) Z_t K_{t-1}^{\alpha} N_t^{-\alpha}$ can be used to solved out for r_t, w_t as a function of Z_t, K_t, N_t.
- Each block takes inputs (e.g., Z_t, K_t, N_t) and computes outputs (r_t, w_t).

Directed Acyclical Graph

Figure 3: DAG representation of Krusell-Smith economy



- The computer can do the substitution if we write the model as a Directed Acyclical Graph (DAG).
- Dimensionality $T \times 2$ even though we have 4 endogenous variables.
- Restriction: cannot have cycle. E.g., if K was not an unknown, the graph would be cylical.
- Always(?) possible to write model as DAG.

More

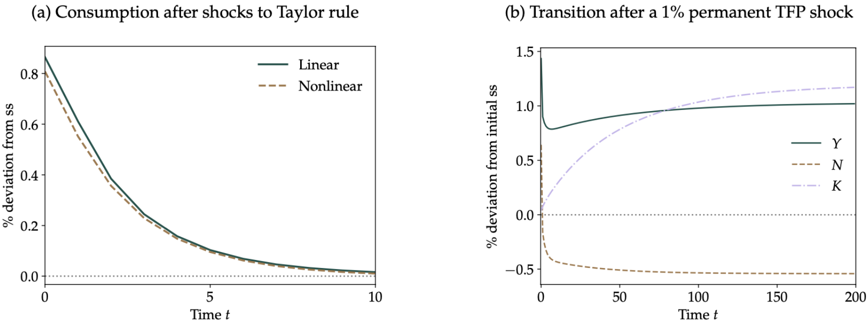
- How to pick T: check if higher T matters for IRFs. Authors recommend $T = 300 - 1000$.

- Estimation: Can compute moments and likelihood quickly from Jacobians. Limit is how quickly you can recompute Jacobians given new parameter values.

- Non-linear perfect foresight dynamics.

More

Figure 7: Nonlinear impulse responses and transitional dynamics for the two-asset HANK model



- See also McKay and Wieland (2022, Econometrica) for how to implement a ZLB.