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Lista Prática

Question 1.

1.1. Funções de Verossimilhança e Estimadores

1) A mistura de normais é caracterizada da seguinte forma:

$$e_i \sim \begin{cases} \mathrm{N}(\mu_1, \sigma_1^2), \text{com probabilidade } 1/2, \\ \mathrm{N}(\mu_2, \sigma_2^2), \text{com probabilidade } 1/2. \end{cases} \tag{1}$$

Para computar a densidade, podemos modelar $f(e_i) \sim \mathrm{Bern}(1/2)$ e calcular $\mathbb{E}_{\mathrm{Bern}(1/2)}\left[f(e_i)\right]$:

$$\mathbb{E}_{\mathrm{Bern}(1/2)}\left[f(e_i)\right] = \frac{1}{2} \times \left[\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left(-\frac{1}{2\sigma_1^2}\left(y_i - \beta_0 - \beta_1 x_i - \mu_1\right)^2\right) + \frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left(-\frac{1}{2\sigma_2^2}\left(y_i - \beta_0 - \beta_1 x_i - \mu_2\right)^2\right)\right] \tag{2}$$

Podemos escrever a função de verossimilhança amostral como

$$\begin{aligned} L(\boldsymbol{\beta}|\mathbf{y}, \mathbf{x}; \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) &= \prod_{i=1}^{n} \frac{1}{2} \times \left[\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left(-\frac{1}{2\sigma_1^2}\left(y_i - \beta_0 - \beta_1 x_i - \mu_1\right)^2\right) + \right. \\ &\left. \frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left(-\frac{1}{2\sigma_2^2}\left(y_i - \beta_0 - \beta_1 x_i - \mu_2\right)^2\right)\right] \\ &= \frac{1}{2^n}\prod_{i=1}^{n} \times \left[\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left(-\frac{1}{2\sigma_1^2}\left(y_i - \beta_0 - \beta_1 x_i - \mu_1\right)^2\right) + \right. \\ &\left. \frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left(-\frac{1}{2\sigma_2^2}\left(y_i - \beta_0 - \beta_1 x_i - \mu_2\right)^2\right)\right]. \end{aligned} \tag{3}$$

Obtendo a log-verossimilhança:

$$\begin{aligned} l(\boldsymbol{\beta}|\mathbf{y}, \mathbf{x}; \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = -n\log(2) + \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left(-\frac{1}{2\sigma_1^2}\left(y_i - \beta_0 - \beta_1 x_i - \mu_1\right)^2\right) + \right. \\ \left. \frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left(-\frac{1}{2\sigma_2^2}\left(y_i - \beta_0 - \beta_1 x_i - \mu_2\right)^2\right)\right). \end{aligned} \tag{4}$$

O Estimador de Máxima Verossimilhança é dado por

$$\hat{\boldsymbol{\beta}} = \underset{\beta_0, \beta_1}{\operatorname{argmax}}\ l(\boldsymbol{\beta}|\mathbf{y}, \mathbf{x}; \mu_1, \mu_2, \sigma_1^2, \sigma_2^2). \tag{5}$$

2) Função de verossimilhança amostral para u, com $u_i \sim \Gamma(a, b)$, definida para $u_i > 0$:

$$\begin{aligned} L(\boldsymbol{\beta}|\mathbf{y}, \mathbf{x}; a, b) &= \prod_{i=1}^{n} \frac{b^a}{\Gamma(a)}\left(y_i - \beta_1 - \beta_2 x_i\right)^{a-1}\exp\left(-b\left(y_i - \beta_1 - \beta_2 x_i\right)\right) \\ &= \left(\frac{b^a}{\Gamma(a)}\right)^n \prod_{i=1}^{n}\left(y_i - \beta_1 - \beta_2 x_i\right)^{a-1}\exp\left(-b\sum_{i=1}^{n}\left(y_i - \beta_1 - \beta_2 x_i\right)\right). \end{aligned} \tag{6}$$

Aplicando tranformação logarítmica

$$l(\boldsymbol{\beta}|\mathbf{y},\mathbf{x};a,b) = n\log\left(\frac{b^a}{\Gamma(a)}\right) + (a-1)\sum_{i=1}^{n}\log\left(y_i-\beta_1-\beta_2x_i\right) - b\sum_{i=1}^{n}\left(y_i-\beta_1-\beta_2x_i\right). \tag{7}$$

O Estimador de Máxima Verossimilhança é dado por

$$\hat{\boldsymbol{\beta}} = \underset{\beta_0,\beta_1}{\operatorname{argmax}}\; l(\boldsymbol{\beta}|\mathbf{y},\mathbf{x};a,b). \tag{8}$$

3) Sabemos que $v_i \sim \text{Cauchy}(0,1) \sim t(\nu=1)$, em que ν representa os graus de liberdade. Portanto, a função de verossimilhança amostral para $\mathbf{v}$, com $v_i \sim \text{Cauchy}(0,1)$ pode ser escrita como:

$$\begin{aligned} L(\boldsymbol{\beta}|\mathbf{y},\mathbf{x};\nu) &= \prod_{i=1}^{n}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{(y_i-\beta_1-\beta_2x_i)^2}{\nu}\right) \\ &= \left[\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\right]^n\prod_{i=1}^{n}\left(1+\frac{(y_i-\beta_1-\beta_2x_i)^2}{\nu}\right). \end{aligned} \tag{9}$$

A log-verossimilhança, por sua vez,

$$l(\boldsymbol{\beta}|\mathbf{y},\mathbf{x};\nu) = n\left[\log\left(\Gamma\left(\frac{\nu+1}{2}\right)\right) - \log\left(\sqrt{\nu\pi}\right) - \log\left(\Gamma\left(\frac{\nu}{2}\right)\right)\right] + \sum_{i=1}^{n}\log\left(1+\frac{(y_i-\beta_1-\beta_2x_i)^2}{\nu}\right). \tag{10}$$

O Estimador de Máxima Verossimilhança é dado por

$$\hat{\boldsymbol{\beta}} = \underset{\beta_0,\beta_1}{\operatorname{argmax}}\; l(\boldsymbol{\beta}|\mathbf{y},\mathbf{x};\nu). \tag{11}$$

1.2. Distribuições Assintóticas

Vide [1], Teorema 14.1.M2, *se são satisfeitas condições de regularidade*,

$$\hat{\boldsymbol{\theta}} \xrightarrow{d} \mathrm{N}\left(\boldsymbol{\theta}_0, \{\mathbf{I}(\boldsymbol{\theta_0})\}^{-1}\right), \tag{12}$$

em que $\hat{\boldsymbol{\theta}}$ é o estimador de Máxima Verossimilhança de $\boldsymbol{\theta}_0$, e $\mathbf{I}(\boldsymbol{\theta_0}) = -\mathbb{E}\left[\frac{\partial^2\log(L)}{\partial\boldsymbol{\theta}_0\partial\boldsymbol{\theta}_0'}\right]$.

Condições de regularidade

blá blá blá

Referências

[1] WH Greene. Econometric analysis, 7th. saddle river, 2011.