7. Equilibrium with Complete Markets

Sequential Trading

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Macroeconomics I, 2023

Introduction

- Today we continue the previous lecture
- We study another structure of complete markets
- We introduce sequential trading, a slightly more realistic setup, and use the celebrated one-period Arrow securities

Lecture Outline

1. Model Setup
2. Financial Wealth as a State Variable
3. Reopening Markets
4. Sequential Trading
5. Equivalence of Allocations
6. Exercises

Main Reference: Ljungqvist and Sargent, 2018, *Recursive Macroeconomic Theory*, Fourth Edition, Chapter 8

1. Model Setup

Model Setup

- The model is the same as in the previous lecture
- The only difference is the market structure
- Consumers no longer have to make all their trades at time 0
- They can trade, more realistically, at every period

Arrow Securities

- Consumers now trade Arrow securities
- Arrow securities are different from Arrow-Debreu securities
- Arrow (1964, *Review of Econ. Studies*) introduced them in a classic paper

Arrow Security

- An Arrow security is a financial instrument or contract that agrees to pay its owner, next period, the following
  - One unit of numeraire, or a fixed amount, if a particular state occurs
  - Zero in all other states
- That is, an Arrow security is a claim to one-period-ahead state-contingent consumption

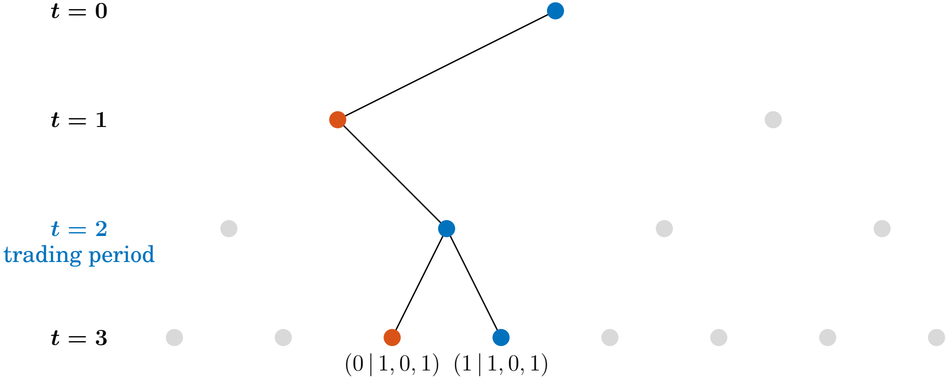
Example with Two States

- Recall the simple example from the previous lecture in which the stochastic process s_t follows a two-state Markov chain, with a good and a bad state

$$s_t \in \boldsymbol{S} = \{0\,,1\} \quad \text{with } s_0 = 1$$

- Let's portray a particular history that the economy has followed up to time 2 together with the two possible one-period continuations in period 3

Example with Two States



- At date 2, agents trade Arrow securities for time-3 goods, ie claims to only those time-3 nodes that can be reached from realized time-2 history $(1, 0, 1)$

Complete Markets

- Markets are complete just like in the previous lecture
- This means there is one Arrow security for every possible state of the world
- These securities are traded at each date $t \geq 0$ after history s^t is realized

2. Financial Wealth as a State Variable

State Variable

- We want to construct a sequential-trading arrangement
- We need to identify a variable that serves as the state in a value function for the consumer at date t and history s^t
- What is this variable?

Implied Wealth

- Remember the time 0 trading budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

- This constraint is like saying that at time 0, consumer i
    - Sells her entire life-time endowment stream $\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$
    - To acquire contingent consumption claims $\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t)$
- Thus this is the implied wealth, or simply wealth, of consumer i

Continuation Wealth

- Suppose we are at time $t \geq 0$ after history s^t has realized
- We discard all claims contingent on time t histories $\tilde{s}^t \neq s^t$ not realized
- The implied continuation wealth of consumer i at time t, after history s^t has realized, and expressed in terms of date t goods, is

$$\sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) c_\tau^i(s^\tau)$$

Two Parts of Wealth

- With sequential trading, consumers do not sell their entire endowment in $t = 0$ but instead retain the ownership to their endowment throughout time
- Hence, at a given point in time t after history s^t, we can decompose the wealth of consumer i into two parts

1. Financial wealth: the consumer's beginning-of-period holdings of Arrow securities contingent on the current state s_t being realized
2. Nonfinancial wealth: the present value of the consumer's current and future endowment (sometimes called human wealth)

Financial Wealth = Total Wealth − Nonfinancial Wealth

- Suppose that the Arrow-Debreu and Arrow trading arrangements yield identical equilibrium allocations: we are going to show this today
- Then it should be the case that

$$\begin{array}{c}\text{Financial wealth with}\\ \text{sequential trading}\end{array} = \begin{array}{c}\text{Total wealth with}\\ \text{time 0 trading}\end{array} - \begin{array}{c}\text{Nonfinancial}\\ \text{wealth}\end{array}$$

Financial Wealth

- Based on the preceding, the financial wealth of consumer i at time t after history s^t, expressed in terms of the date t, history s^t consumption good is

$$\Upsilon_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau)[c_\tau^i(s^\tau) - y_\tau^i(s^\tau)]$$

- $\sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) c_\tau^i(s^\tau)$ is the continuation value of total wealth
- $\sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau)$ is the continuation value of nonfinancial wealth

Initial Financial Wealth

- The time 0 budget constraint holds with equality
- Thus each consumer starts with zero financial wealth at time 0

$$\Upsilon_0^i(s_0) = 0 \quad \text{for all } i$$

- At $t > 0$, financial wealth $\Upsilon_t^i(s^t)$ typically differs from zero for consumer i

Zero Net Supply

- Remember, the feasibility constraint in the Arrow-Debreu equilibrium holds at equality $\sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} y_t^i(s^t)$
- This implies that financial wealth sums to zero across agents

$$\sum_{i=1}^{I} \Upsilon_t^i(s^t) = 0 \quad \text{for all } t,\, s^t$$

- The Arrow securities that make up financial wealth are in zero net supply
- Positive holdings of some consumers (buyers, creditors, lenders) constitute indebtedness of the other consumers (sellers, debtors, issuers)

3. Reopening Markets

Reopening Markets

- Suppose we are in the Arrow-Debreu world
- All trades have occurred at time 0 and the market is closed
- Now what happens if the market reopens at date τ?

Euler Equation

- Remember the formula from the previous lecture

$$q_t^\tau(s^t) \equiv \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = \frac{\beta^t u_i'[c_t^i(s^t)]\pi_t(s^t)}{\beta^\tau u_i'[c_\tau^i(s^\tau)]\pi_\tau(s^\tau)} = \beta^{t-\tau}\frac{u_i'[c_t^i(s^t)]}{u_i'[c_\tau^i(s^\tau)]}\pi_t(s^t|s^\tau)$$

- This is a pricing function for a complete-markets economy with date- and history-contingent commodities
- Whose markets have reopened at date τ, history s^t, starting from a wealth distribution implied by an economy that originally convened at time $t = 0$

No New Trading Occurs

Proposition: Start from the distribution of time τ, history s^τ financial wealth that is implicit in a time 0 Arrow-Debreu equilibrium. If markets reopen at date τ after history s^τ, no trades occur. That is, given the price system

$$q_t^\tau(s^t) = \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = \beta^{t-\tau}\frac{u_i'[c_t^i(s^t)]}{u_i'[c_\tau^i(s^\tau)]}\pi_t(s^t|s^\tau),$$

all consumers choose to continue the tails of their original consumption plans. No new trading occurs.

Intuition

- For a proof of this proposition, see exercise 2 of lecture 6
- The intuition is simple: consumers already insure themselves against all possible future states by trading a complete set of Arrow-Debreu securities
- If markets reopen, agents do not want to make new trades because their time 0 allocation is already optimal

4. Sequential Trading

Sequential Trading

- There is a sequence of markets in one-period-ahead assets
- At each date $t \geq 0$, consumers trade claims to date $t+1$ consumption
- The payment is contingent on the realization of s_{t+1}
- In the time 0 trading arrangement, agents had a unique budget constraint
- Here, consumers have a sequence of budget constraints

Budget Constraint

- Consumer i's time t, history s^t budget constraint is

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t)$$

- $\tilde{c}_t^i(s^t)$ is consumption in t, $y_t^i(s^t)$ is endowment in t
- $\tilde{a}_t^i(s^t)$ is a claim to time t consumption, ie wealth, acquired in $t-1$
- $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$ is a vector of claims on $t+1$ consumption, acquired in t
- $\tilde{Q}_t(s_{t+1}|s^t)$ is a pricing kernel, ie the price of one unit of $t+1$ consumption contingent on the realization of s_{t+1} at $t+1$ when the history at t is s^t

Consumer Choice

- At time t, both endowment $y_t^i(s^t)$ and asset holdings $\tilde{a}_t^i(s^t)$ are given
- Thus at time t, the consumer chooses current consumption $\tilde{c}_t^i(s^t)$ and asset holdings $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$ in the form of a complete set of Arrow securities
- $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$ is a vector, where there is one element of the vector for each value of the time $t+1$ realization of the state s_{t+1}
- In $t+1$ when s_{t+1} realizes, only one of these $\tilde{a}_{t+1}^i(s^{t+1})$ has positive value

Optimal Solution

- Without further constraint, what is the consumer's optimal behavior?
- The agent maximizes utility which depends only on consumption, not assets

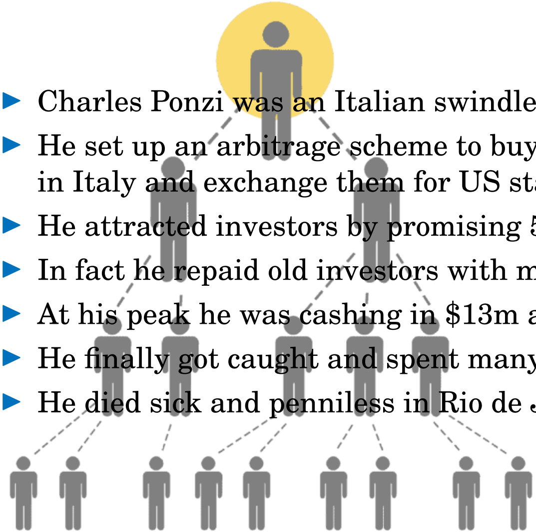
$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t)$$

- So the consumer wants the lowest possible $\sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t)$ to afford the highest possible consumption $\tilde{c}_t^i(s^t)$

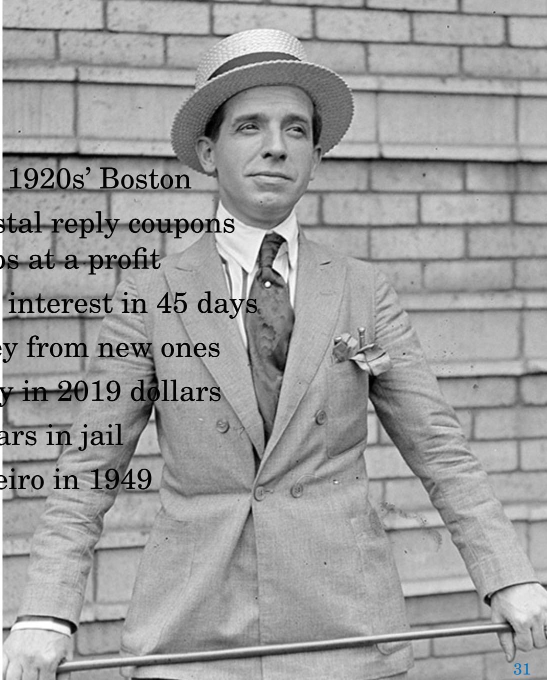
Ponzi Scheme

- The consumer issues (ie sells) as many claims $\{\tilde{a}^i_{t+1}(s_{t+1}, s^t)\}$ as possible
- In the next period she pays off the old claims by issuing new claims
- She keeps rolling over her debt indefinitely, amassing an ever increasing stock of liabilities, ie negative wealth
- In the limit $\{\tilde{a}^i_{t+1}(s_{t+1}, s^t)\} \to -\infty$ and $\tilde{c}^i_t(s^t) \to +\infty$
- This is called a Ponzi scheme

Ponzi



- Charles Ponzi was an Italian swindler in 1920s' Boston
- He set up an arbitrage scheme to buy postal reply coupons in Italy and exchange them for US stamps at a profit
- He attracted investors by promising 50% interest in 45 days
- In fact he repaid old investors with money from new ones
- At his peak he was cashing in \$13m a day in 2019 dollars
- He finally got caught and spent many years in jail
- He died sick and penniless in Rio de Janeiro in 1949



No Ponzi Scheme

- We want to rule out Ponzi schemes
- We impose state-by-state debt limits or borrowing constraints

$$-\tilde{a}^i_{t+1}(s^{t+1}) \leq A^i_{t+1}(s^{t+1})$$

- Debt cannot exceed some value $A^i_{t+1}(s^{t+1})$
- $A^i_{t+1}(s^{t+1})$ is the natural debt limit at $t+1$ and history s^{t+1}

Natural Debt Limit

- The natural debt limit $A_t^i(s^t)$ is the maximum value consumer i can repay assuming her consumption is zero for the rest of her life
- To determine $A_t^i(s^t)$, we start from the equilibrium allocation of the Arrow-Debreu economy
- Let $q_\tau^t(s^\tau)$ be the Arrow-Debreu price expressed in units of date t, history s^t consumption good

Natural Debt Limit

- The value of consumer i's nonfinancial wealth, ie the value of the tail of agent i's endowment sequence at time t, with history s^t, is

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau)$$

- We use this value as the natural debt limit
- The agent cannot credibly promise to pay more than $A_t^i(s^t)$

Problem of the Consumer

- For a given initial wealth $\tilde{a}_0^i(s_0)$, consumer i solves

$$\max_{\{\tilde{c}_t^i(s^t),\{\tilde{a}_{t+1}^i(s_{t+1},s^t)\}_{s_{t+1}}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t}\beta^t u_i[\tilde{c}_t^i(s^t)]\pi_t(s^t)$$

$$\text{subject to} \quad \tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) \leq y_t^i(s^t) + \tilde{a}_t^i(s^t)$$

$$\text{and} \quad -\tilde{a}_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1}) \qquad \text{for all } t,\, s^t,\, s_{t+1}$$

- This is a dynamic optimization problem with two constraints

Lagrangian

- Write a Lagrangian

$$\mathcal{L}^i = \sum_{t=0}^{\infty} \sum_{s^t} \Bigg\{ \beta^t u[\tilde{c}_t^i(s^t)] \pi_t(s^t)$$

$$+ \eta_t^i(s^t) \left[y_t^i(s^t) + \tilde{a}_t^i(s^t) - \tilde{c}_t^i(s^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1}|s^t) \right]$$

$$+ \sum_{s_{t+1}} \nu_t^i(s^t, s_{t+1}) \left[A_{t+1}^i(s^{t+1}) + \tilde{a}_{t+1}^i(s^{t+1}) \right] \Bigg\}$$

- $\eta_t^i(s^t)$ is the multiplier on the budget constraint
- $\nu_t^i(s^t, s_{t+1})$ is the multiplier on the borrowing constraint

Reminder – Intertemporal Optimization

Suppose we want to maximize

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t [y_t - c_t + (1+r_t)b_{t-1} - b_t]$$

λ_t are the multipliers for each period $t = 0, \ldots, T$. The first-order conditions with respect to c_t are

$$\beta^t u'(c_t) - \lambda_t = 0, \quad \text{for } t = 0, \ldots, T.$$

For b_t, it is useful to write out the set of Lagrange terms as

$$\begin{aligned}
&+ \lambda_0[y_0 - c_0 - b_0] \\
&+ \lambda_1[y_1 - c_1 + (1 + r_1)b_0 - b_1] \\
&+ \ldots \\
&+ \lambda_{T-1}[y_{T-1} - c_{T-1} + (1 + r_{T-1})b_{T-2} - b_{T-1}] \\
&+ \lambda_T[y_T - c_T + (1 + r_T)b_{T-1} - b_T],
\end{aligned}$$

to see that b_t appears negatively in the period-t constraint and positively, multiplied by $(1 + r_{t+1})$, in the period-$t + 1$ constraint. Thus, the FOCs with respect to b_t are

$$(1 + r_{t+1})\lambda_{t+1} - \lambda_t = 0, \quad \text{for } t = 0, \ldots, T.$$

First-Order Conditions

- The first-order conditions for $\tilde{c}_t^i(s^t)$ and $\tilde{a}_{t+1}^i(s_{t+1}, s^t)$ are

$$\begin{aligned} \beta^t u_i'[\tilde{c}_t^i(s^t)]\pi_t(s^t) - \eta_t^i(s^t) &= 0 \\ -\eta_t^i(s^t)\tilde{Q}_t(s_{t+1}|s^t) + \nu_t^i(s^t, s_{t+1}) + \eta_{t+1}^i(s_{t+1}, s^t) &= 0 \end{aligned}$$

- There is a pair of these FOCs for each s_{t+1}, t, s^t, and each consumer i

What if the Borrowing Constraint Binds?

- Suppose history s^{t+1} leads to a binding debt limit
- To honor her debt, the agent must consume nothing forever
- But since the utility function satisfies the Inada condition $\lim_{c \to 0} u_i'(c) = \infty$, all future marginal utilities of the agent are infinite
- By postponing earlier consumption to periods after when the constraint begins to bind, the consumer would easily yield higher expected utility

Nonbinding Constraint

- We conclude that the natural debt limit never binds
- Therefore, all Lagrange multipliers $\nu_t^i(s^t; s_{t+1})$ are equal to zero
- To sum up, the no-Ponzi scheme constraint guarantees that an interior solution exists, ie it avoids reaching $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\} \to -\infty$ and $c_t^i(s^t) \to \infty$
- But it does not affect the properties of the solution
- Question: what is the difference between a no-Ponzi scheme condition and a transversality condition?

Optimal Consumption Allocation

- Go back to the FOCs and set $\nu_t^i(s^t, s_{t+1}) = 0$

$$\beta^t u_i'[\tilde{c}_t^i(s^t)]\pi_t(s^t) - \eta_t^i(s^t) = 0$$

$$-\eta_t^i(s^t)\tilde{Q}_t(s_{t+1}|s^t) + \eta_{t+1}^i(s_{t+1}, s^t) = 0$$

- Combine the two FOCs by eliminating $\eta_t^i(s^t)$ and obtain an Euler equation

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u_i'[\tilde{c}_{t+1}^i(s^{t+1})]}{u_i'[\tilde{c}_t^i(s^t)]}\pi_t(s^{t+1}|s^t)$$

- The optimal consumption allocation is such that the pricing kernel equates the marginal rate of substitution between present and future consumption

Wealth Distribution

- A distribution of wealth is a vector $\{\tilde{a}_t^i(s^t)\}_{i=1}^I$ satisfying

$$\sum_i^I \tilde{a}_t^i(s^t) = 0 \quad \text{for all } t \text{ and } s^t$$

- In other words, asset holdings (ie claims) are in zero net supply

Competitive Equilibrium

A competitive equilibrium with sequential trading of Arrow securities is an initial distribution of wealth $\{\tilde{a}_0^i(s_0)\}_{i=1}^I$ satisfying $\sum_i^I \tilde{a}_0^i(s_0) = 0$, a collection of borrowing limits $\{A_t^i(s^t)\}$ satisfying $A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau)$, a feasible allocation $\{\tilde{c}_t^i(s^t)\}_{i=1}^I$, a portfolio $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$, and pricing kernels $\tilde{Q}_t(s_{t+1}|s^t)$ such that

1. Given $\tilde{Q}_t(s_{t+1}|s^t)$, $\tilde{a}_0^i(s_0)$, and $\{A_t^i(s^t)\}$ for all i, the consumption allocation $\tilde{c}_t^i$ and portfolio $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$ solve the consumer's problem for all i
2. For all realizations of $\{s^t\}_{t=0}^\infty$, markets clear, that is aggregate allocation and portfolio satisfy $\sum_i^I \tilde{c}_t^i(s^t) = \sum_i^I y_t^i(s^t)$ and $\sum_i^I \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0$

5. Equivalence of Allocations

Recap

- Equilibrium with time 0 trading of Arrow-Debreu securities

$$\{\{c_t^i\}_{t=0}^{\infty}\}_{i=1}^{I} \quad \text{and} \quad \{q_t^0(s^t)\}_{t=0}^{\infty}$$

- Equilibrium with sequential trading of Arrow securities

$$\{\tilde{a}_0^i(s_0)\}_{i=1}^{I}, \quad \{\{\tilde{c}_t^i\}_{t=0}^{\infty}\}_{i=1}^{I}, \quad \text{and} \quad \{\tilde{Q}_t(s_{t+1}|s^t)\}_{t=0}^{\infty}$$

- We are going to prove that these two allocations coincide

Equivalence of Allocations

Equivalence of allocations. An Arrow-Debreu equilibrium allocation with time 0 trading is also an allocation for a competitive equilibrium with sequential trading of one-period Arrow securities

$$\tilde{c}_t^i(s^t) = c_t^i(s^t) \quad \text{for all periods } t\text{, histories } s^t\text{, agents } i$$

- This holds for a particular initial wealth distribution we need to determine

Three Steps

- We want to prove that $\tilde{c}_t^i = c_t^i$ for a given initial optimal $\{\tilde{a}_0^i(s_0)\}_i^I$
- We start from the result and work backwards in three steps

1. Guess that $\tilde{c}_t^i = c_t^i$ and get a relation for prices
2. Given prices, guess $\{\tilde{a}_0^i(s_0)\}_{i=1}^I$ and $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$, and verify that $\tilde{c}_t^i = c_t^i$
3. Verify that the sequence $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}$ is optimal

Step 1 – Take the Euler Equation in Arrow-Debreu

- Consider the Arrow-Debreu economy
- Take the consumer's FOCs from two consecutive periods

$$\beta^t u_i'[c_t^i(s^t)]\pi_t(s^t) = \mu_i q_t^0(s^t)$$
$$\beta^{t+1} u_i'[c_{t+1}^i(s^{t+1})]\pi_{t+1}(s^{t+1}) = \mu_i q_{t+1}^0(s^{t+1})$$

- Divide one by the other

$$\beta \frac{u_i'[c_{t+1}^i(s^{t+1})]}{u_i[c_t^i(s^t)]}\pi_t(s^{t+1}|s^t) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}$$

Take the Euler Equation in Sequential Trading

- Consider now the sequential trading economy
- Take the consumer's FOC

$$\beta \frac{u_i'[\tilde{c}_{t+1}^i(s^{t+1})]}{u_i'[\tilde{c}_t^i(s^t)]} \pi_t(s^{t+1}|s^t) = \tilde{Q}_t(s_{t+1}|s^t)$$

Equalize Both

- If $\tilde{c}_t^i = c_t^i$, the previous two equations imply

$$\tilde{Q}(s_{t+1}|s^t) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = q_{t+1}^t(s^{t+1})$$

- The one-period-ahead pricing kernel $\tilde{Q}(s_{t+1}|s^t)$ equals the price of a time $t+1$, history s^{t+1} consumption claim expressed in time t, history s^t units

Step 2 – Conjecture an Initial Wealth Distribution

- We conjecture a zero initial wealth vector

$$\{\tilde{a}_0^i(s_0)\}_{i=1}^I = 0$$

- This is reasonable because each consumer must rely on her own endowment stream to finance consumption
- Just like in the Arrow-Debreu world where agents must finance at $t = 0$ their history-contingent purchases for the infinite future

Conjecture a Portfolio Choice

- To prove that the conjecture is correct, we must show that
  - The zero initial wealth vector enables consumer i to finance $\{c_t^i(s^t)\}$
  - The zero initial wealth vector leaves no room to increase c_t^i for any t, s^t
- We guess that at time $t \geq 0$, consumer i chooses a portfolio given by

$$\tilde{a}_{t+1}^i(s_{t+1}, s^t) = \Upsilon_{t+1}^i(s^{t+1}) \quad \text{for all } s_{t+1}$$

- The agent buys Arrow securities until all her financial wealth, equal to total continuation wealth minus the present value of all endowments, is invested

Portfolio Value

- The value of this portfolio in terms of date t, history s^t consumption good is

$$
\begin{aligned}
\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) &= \sum_{s^{t+1}|s^t} \Upsilon^i_{t+1}(s^{t+1}) q^t_{t+1}(s^{t+1}) \\
&= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} q^{t+1}_\tau(s^\tau)[c^i_\tau(s^\tau) - y^i_\tau(s^\tau)] q^t_{t+1}(s^{t+1}) \\
&= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} q^t_\tau(s^\tau)[c^i_\tau(s^\tau) - y^i_\tau(s^\tau)] \qquad (1)
\end{aligned}
$$

where we have used $\Upsilon^i_t(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} q^t_\tau(s^\tau)[c^i_\tau(s^\tau) - y^i_\tau(s^\tau)]$ and the identity $q^{t+1}_\tau(s^\tau) q^t_{t+1}(s^{t+1}) = \frac{q^0_\tau(s^\tau)}{q^0_{t+1}(s^{t+1})} \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = \frac{q^0_\tau(s^\tau)}{q^0_t(s^t)} = q^t_\tau(s^\tau)$

Affordable Strategy

- Can the consumer afford this portfolio strategy?
- To check this, we use her budget constraint to compute the implied consumption plan $\{\tilde{c}^i_\tau(s^\tau)\}_{\tau=0}^{\infty}$
- We start in the initial period $t = 0$ with $\tilde{a}^i_0(s_0) = 0$
- Then we look at all consecutive future periods $t > 0$

Initial Period

- In period $t = 0$ with $\tilde{a}_0^i(s_0) = 0$, the flow budget constraint is

$$\tilde{c}_0^i(s_0) + \sum_{s_1} \tilde{a}_1^i(s_1, s_0)\tilde{Q}_0(s_1|s_0) = y_0^i(s_0) + \underbrace{\tilde{a}_0^i(s_0)}_{=0}$$

- Plug the portfolio choice (1) at time 0 into the budget constraint

$$\tilde{c}_0^i(s_0) + \sum_{t=1}^{\infty} \sum_{s^t} q_t^0(s^t)[c_t^i(s^t) - y_t^i(s^t)] = y_0^i(s_0) + 0 \tag{2}$$

Initial Period

- Remember the Arrow-Debreu time 0 budget constraint at equality $\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$; rewrite this equation as

$$c_0^i(s_0) - y_0^i(s_0) + \sum_{t=1}^{\infty} \sum_{s^t} q_t^0(s^t)[c_t^i(s^t) - y_t^i(s^t)] = 0$$

- This equation together with equation (2) imply that

$$\tilde{c}_0^i(s_0) = c_0^i(s_0)$$

- Thus the proposed portfolio choice is affordable in period 0
- Initial consumption is the same as in the Arrow-Debreu economy

Future Periods – Budget Constraint

- Let's now see for all future periods $t > 0$ and histories s^t
- Take the consumer's flow budget constraint

$$\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) = \tilde{a}^i_t(s^t) + y^i_t(s^t) - \tilde{c}^i_t(s^t)$$

- Use the portfolio guess $\tilde{a}^i_t(s^t) = \Upsilon^i_t(s^t)$

$$\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) = \Upsilon^i_t(s^t) + y^i_t(s^t) - \tilde{c}^i_t(s^t)$$

Future Periods – Portfolio Value

- Now, rewrite the value of the asset portfolio (1) as

$$\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} q^t_\tau(s^\tau)[c^i_\tau(s^\tau) - y^i_\tau(s^\tau)]$$

$$= \underbrace{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} q^t_\tau(s^\tau)[c^i_\tau(s^\tau) - y^i_\tau(s^\tau)] + [c^i_t(s^t) - y^i_t(s^t)]}_{\Upsilon^i_t(s^t)=\sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t} q^t_\tau(s^\tau)[c^i_\tau(s^\tau)-y^i_\tau(s^\tau)]} - [c^i_t(s^t) - y^i_t(s^t)]$$

$$= \Upsilon^i_t(s^t) - [c^i_t(s^t) - y^i_t(s^t)]$$

Step 2 – Equivalence

- The budget constraint we have just computed two slides above is

$$\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) = \Upsilon^i_t(s^t) + y^i_t(s^t) - \tilde{c}^i_t(s^t)$$

- The value of the portfolio we have just computed one slide above is

$$\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) = \Upsilon^i_t(s^t) + y^i_t(s^t) - c^i_t(s^t)$$

- We conclude that $\tilde{c}^i_t(s^t) = c^i_t(s^t)$ for all periods and histories

Step 3 – Optimal Portfolio

- We have just shown that the following proposed portfolio strategy attains the same consumption plan as in the Arrow-Debreu equilibrium

$$\tilde{a}^i_{t+1}(s_{t+1}, s^t) = \Upsilon^i_{t+1}(s^{t+1}) \quad \text{for all } i, t, s^t, s_{t+1}$$

- But is this portfolio choice optimal?
- What precludes consumer i from further increasing current consumption by reducing some component (buying less, selling more) of the asset portfolio?

Actual Debt Limit < Natural Debt Limit

- The answer lies in the debt limit restriction faced by the consumer
- The consumer wants to ensure that the consumption plan $\{c^i_\tau(s^\tau\}$ can be attained starting next period in all possible future states
- Thus to know her actual debt limit, she should subtract the value of this commitment to future consumption from her natural debt limit

$$\underbrace{-\tilde{a}^i_{t+1}(s_{t+1}, s^t)}_{\text{Actual debt limit}} \leq \underbrace{A^i_{t+1}(s^{t+1})}_{\text{Natural debt limit}} - \underbrace{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}_\tau c^i_\tau(s^\tau)}_{\text{Commitment to consumption plan}}$$

- The actual debt limit is more restrictive than the natural debt limit

Actual Debt Limit

- Rewrite the state-by-state borrowing constraint using the definition of the natural debt limit

$$
\begin{aligned}
-\tilde{a}^i_{t+1}(s_{t+1}, s^t) &\leq A^i_{t+1}(s^{t+1}) - \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}_\tau(s^\tau) c^i_\tau(s^\tau) \\
&= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}_\tau(s^\tau) y^i_\tau(s^\tau) - \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}_\tau(s^\tau) c^i_\tau(s^\tau) \\
&= -\Upsilon^i_{t+1}(s^{t+1})
\end{aligned}
$$

Optimal Portfolio

- We obtain
$$\tilde{a}^i_{t+1}(s_{t+1}, s^t) \geq \Upsilon^i_{t+1}(s^{t+1})$$
- Consumer i does not want to increase consumption at time t by reducing next period's financial wealth below $\Upsilon^i_{t+1}(s^{t+1})$
- Doing so would prevent her from attaining the consumption plan satisfying the FOC for all future periods and histories
- We conclude that $\tilde{a}^i_{t+1}(s_{t+1}, s^t) = \Upsilon^i_{t+1}(s^{t+1})$ is optimal

Summary

- For a given zero initial wealth distribution

$$\tilde{a}_0^i(s_0) = 0 \quad \text{for all } i = 1, 2, \ldots, I$$

- The competitive allocation with sequential trading is the same as the competitive allocation with time 0 trading
- The Arrow-Debreu and Arrow worlds are equivalent

Wrap-Up

- How do we transform the Arrow-Debreu price system into an Arrow system?
- We take the time 0 trading economy and account for how individual wealth – the present value of portfolios – evolves as time passes
- In period t, after history s^t, we use the Arrow-Debreu prices to compute the value of claims to current and future goods net of outstanding liabilities
- We then show these wealth levels can be attained in a sequential trading economy with only one-period assets and markets that reopen each period

Conclusion

- Complete markets allow a decentralized economy to reach a competitive equilibrium allocation that is Pareto efficient
- Complete markets insulate agents from idiosyncractic factors and hence underpin every representative-agent model
- Time zero trading of Arrow-Debreu securities and sequential trading of one-period Arrow securities are equivalent

6. Exercises

Exercise 1 – Risk-Averse and Risk-Neutral Agents

Two consumers populate an exchange economy with no production. There is one homogeneous, nondurable good. Consumer 1 has preferences

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t^1(s^t)] \pi_t(s^t), \qquad \beta \in (0,1)$$

where $u' > 0$, $u'' < 0$, $\lim_{c \to 0} u'(c) = \infty$. Consumer 2 has preferences

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t c_t^2(s^t) \pi_t(s^t)$$

The endowment of consumer 1 is $y_t^1(s^t) = s_t$ where $s_t \in (0,1)$ follows a time-invariant Markov chain with transition matrix $\boldsymbol{P}$, initial distribution $\boldsymbol{\pi}_0(s_0 = 0) = 1$ and states $\bar{s}_1 = 0$ and $\bar{s}_2 = 1$. Note that $P_{ij} = \text{Prob}(s_{t+1} = \bar{s}_j | s_t = \bar{s}_i)$. $\boldsymbol{P}$ is a stochastic matrix.

Exercise 1 – Continued

The endowment of consumer 2 is constant over time and histories

$$y_t^2(s^t) = y^2 = (1-\beta)\begin{bmatrix}1 & 0\end{bmatrix}\left[\boldsymbol{I} - \beta\boldsymbol{P}\right]^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$

1. Define the Arrow-Debreu competitive equilibrium, that is with time 0 trading.
2. Compute the Arrow-Debreu competitive equilibrium.
3. Write the Pareto problem that admits the competitive allocation as solution. Find the Pareto weights.
4. Define the sequential trading competitive equilibrium.
5. Compute the sequential trading competitive equilibrium.

Exercise 2 – Incomplete Markets and Money

Two consumers populate an exchange economy with no production and one homogeneous, nondurable good. There is no uncertainty. In even periods $(0, 2, \dots)$ consumer 1 is endowed with one unit of the good. In odd periods $(1, 3, \dots)$ it is consumer 2 who is endowed with one unit of the good

$$\{y_t^1\}_{t=0}^{\infty} = \{1, 0, 1, 0, \dots\}$$
$$\{y_t^2\}_{t=0}^{\infty} = \{0, 1, 0, 1, \dots\}$$

Consumer 1 and 2 have the same preferences

$$\sum_{t=0}^{\infty} \beta^t \ln c_t^i, \qquad i = 1, 2, \qquad \beta \in (0, 1)$$

1. Write and solve the problem of the central planner.
2. Assume complete markets. Define and compute the competitive equilibrium with time 0 trading.

Exercise 2 – Continued

Consider now the following structure of incomplete markets. There are no markets that allow to trade claims to future consumption. However, in $t = 0$ and only at this period, agent 2 only is endowed with M units of a durable good that never depreciates and generates no utility for both agents. In all t there is a market that allows agent $i = 1, 2$ to trade $m_t^i \geq 0$ units of the durable good for units of the consumption good.

The price of the durable good in t is p_t. The budget constraint in t of agent $i = 1, 2$ is

$$p_t m_t^i + c_t^i = y_t^i + p_t m_{t-1}^i$$

Note that $m_t^i \geq 0$, that is the agent cannot short sell the good. The market-clearing condition is $m_t^1 + m_t^2 = M$ for all t.

Exercise 2 – Continued

3. Define a concept of competitive equilibrium for this economy.
4. Compute the competitive equilibrium. *Hint:* conjecture that $m_{2t}^1 = M$ and $m_{2t+1}^1 = 0$.
5. Is the competitive equilibrium different from the Pareto allocation? Explain and give an intuition.

Exercise 3 – Diverse Beliefs

Consider an economy with two consumers, $i = 1, 2$. Each has preferences $\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \ln[c_t^i(s^t)]\pi_t(s^t)$. A feasible allocation satisfies $\sum_i c_t^i(s^t) \leq \sum_i y^i(s_t)$ for all $t \geq 0$ and $s^t \geq 0$. The endowments are functions of the state $s_t \in \{0, 1, 2\}$ and are defined as $y_t^1 = s_t/2$ and $y_t^2 = 1 - y_t^1$. The state variable s_t follows a Markov chain with initial distribution $\boldsymbol{\pi}_0$ and transition matrix $\boldsymbol{P}$

$$\boldsymbol{\pi}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \qquad \boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

where $P_{ij} = \text{Prob}[s_{t+1} = j - 1 | s_t = i - 1]$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. Uncertainty in the economy disappears in $t = 1$.

Exercise 3 – Continued

1. Define the competitive equilibrium with sequential trading of Arrow securities.
2. Compute this equilibrium.
3. Suppose consumer 1 knows $(\boldsymbol{\pi}_0, \boldsymbol{P})$ but consumer 2 knows only $\boldsymbol{\pi}_0$ and thinks the transition matrix is

$$\hat{\boldsymbol{P}} = \begin{bmatrix} 1 & 0 & 0 \\ p & 0 & 1-p \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{with } p \in (0.5, 1)$$

4. Define the competitive equilibrium with time 0 trading of Arrow-Debreu securities.
5. Compute this equilibrium for $p = 0.6$.
6. Ex ante, in $t = 0$, would consumer 2 be better off if his belief were correct? Justify. *Hint:* think what would happen in the limiting case, $p \to 1$.