20. Optimal Control

Discounted Infinite Horizon

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Theory and Practice

- We continue our study of optimal control methods
- We focus on cases where objective functions and laws of motion of state variables are time-invariant and agents discount the future
- We apply these techniques to two classic examples

Lecture Outline

1. Discounted Infinite-Horizon Optimal Control
2. The Growth Model in Continuous Time
3. The Q-Theory of Investment
4. Exercises

Main Reference: Acemoglu, 2009, *Introduction to Modern Economic Growth*, Chapter 7

1. Discounted Infinite-Horizon Optimal Control

Discounting

- In most macroeconomic models, agents' utility function is time-invariant
- Agents discount future utility, and in most case discounting is exponential
- Therefore optimal control problems take a more specific form, where the objective function looks like

$$f(t, x_t, y_t) = e^{-\rho t} f(x_t, y_t)$$

- The payoff function f depends on time only through exponential discouting

The Problem

- We want to solve the following problem

$$\max_{x_t, y_t} \int_0^\infty e^{-\rho t} f(x_t, y_t) dt, \quad \rho > 0$$

$$\text{subject to} \quad \dot{x}_t = g(t, x_t, y_t) \quad \text{for almost all } t$$

$$x_t \in \operatorname{Int} X_t,\ y_t \in \operatorname{Int} Y_t,\ \text{for all } t,\ x_0 \text{ given}$$

- We assume f and g are continuously differentiable in all their arguments

Hamiltonian

- Recall, λ_t is the costate, and we define $\mu_t \equiv \lambda_t / e^{-\rho t}$
- λ_t is the discounted value of the state (price at t in units of t goods)
- μ_t is the current value of the state (price at time t in units of time 0 goods)
- The Hamiltonian writes

$$\begin{aligned} H(t, x_t, y_t, \lambda_t) &= e^{-\rho t} f(x_t, y_t) + \lambda_t g(t, x_t, y_t) \\ &= e^{-\rho t} [f(x_t, y_t) + \mu_t g(t, x_t, y_t)] \end{aligned}$$

Current-Value Hamiltonian

- Rather than using the standard Hamiltonian, we work with the current-value Hamiltonian, defined as

$$\tilde{H}(t, x_t, y_t, \mu_t) \equiv f(x_t, y_t) + \mu_t g(t, x_t, y_t)$$

- μ_t and $\tilde{H}$ are equal to λ_t and H deflated by the discount factor, respectively

$$\mu_t = \frac{\lambda_t}{e^{-\rho t}} \quad \text{and} \quad \tilde{H}(t, x_t, y_t, \mu_t) = \frac{H(t, x_t, y_t, \mu_t)}{e^{-\rho t}}$$

Maximum Principle

Theorem: if f and g are continuously differentiable, and if the problem has a piecewise continuous interior solution, then except at points of discontinuity of y_t, the optimal pair $(\hat{x}_t, \hat{y}_t)$ satisfies the following necessary conditions

$$\tilde{H}_y(t, \hat{x}_t, \hat{y}_t, \mu_t) = 0 \quad \text{FOC control} \tag{1}$$

$$\dot{\mu}_t - \rho\mu_t = -\tilde{H}_x(t, \hat{x}_t, \hat{y}_t, \mu_t) \quad \text{FOC state} \tag{2}$$

$$\dot{\hat{x}}_t = \tilde{H}_\mu(t, \hat{x}_t, \hat{y}_t, \mu_t) = g(t, \hat{x}_t, \hat{y}_t) \quad \text{Law of motion of state} \tag{3}$$

- Since $\dot{\mu}_t - \rho\mu_t = \frac{\dot{\lambda}_t}{e^{-\rho t}}$, FOC (2) is equivalent to $\dot{\lambda}_t = -H_x(t, \hat{x}_t, \hat{y}_t, \lambda_t)$

Transversality Condition

- Let $V(t_0, x_{t_0}) \equiv \max_{x_t \in X, y_t \in Y} \int_{t_0}^{\infty} e^{-\rho t} f(x_t, y_t) dt$ be the optimal value of the problem starting at t_0 with state x_{t_0}

Theorem: if $V(t, \hat{x}_t)$ is differentiable in x and t for t sufficiently large and if $\lim_{t\to\infty} \frac{\partial V(t,\hat{x}_t)}{\partial t} = 0$, then the transversality condition is also a necessary condition

$$\lim_{t\to\infty} e^{-\rho t} \tilde{H}(t, \hat{x}_t, \hat{y}_t, \mu_t) = 0$$

Sufficiency Conditions

- Until now we have presented the necessary conditions that an interior continuous solution must satisfy
- But nothing guarantees that this solution is the global one, the solution may not be interior or continuous
- We need a theorem that gives us the sufficiency conditions for discounted infinite-horizon problems

Concavity

- The key is to assume a concave problem
- Define $M(t, x_t, \mu_t)$ as the maximized current-value Hamiltonian

$$M(t, x_t, \mu_t) \equiv \max_{y_t \in Y_t} \tilde{H}(t, x_t, y_t, \mu_t)$$

- If $\tilde{H}(t, x_t, y_t, \mu_t)$ is jointly concave in (x_t, y_t), then $M(t, x_t, \mu_t)$ is concave in x_t
- Suppose that $\tilde{H}(t, x_t, y_t, \mu_t)$ is concave, so that $M(t, x_t, \mu_t)$ too is concave

Sufficiency Conditions

Theorem: if $V(t, x_t)$ exists and is finite for all t, if for any admissible pair (x_t, y_t), $\lim_{t\to\infty} e^{-\rho t}\mu_t \hat{x}_t \geq 0$, and if X_t is convex and $M(t, x_t, \mu_t)$ is concave in $x \in X_t$ for all t, then the pair $(\hat{x}_t, \hat{y}_t)$ that satisfies the necessary conditions achieves the global maximum of the problem. Moreover, if $M(t, x_t, \mu_t)$ is strictly concave in x, $(\hat{x}_t, \hat{y}_t)$ is the unique solution to the problem

Sufficiency Conditions

- Repeat the sufficiency conditions that guarantee that the pair $(\hat{x}_t, \hat{y}_t)$ is the unique, global solution

1. The value function $V(t, k_t)$ exists and is finite for all t
2. For any admissible pair (x_t, y_t), $\lim_{t\to\infty} e^{-\rho t}\mu_t x_t \geq 0$
3. For all t, X_t is convex and $M(t, x_t, \mu_t)$ is strictly concave in $x_t \in X_t$

Transversality Condition

- With concave problems, the transversality condition is

$$\lim_{t \to \infty} e^{-\rho t} \mu_t \hat{x}_t = 0$$

Recap

- The steps to solve a dynamic optimization problem in continuous time are

1. Set up the current-value Hamiltonian
2. Use Pontryagin's maximum principle to derive the necessary conditions and thus locate a candidate interior solution
3. Verify the sufficiency conditions: if they hold – in particular the maximized Hamiltonian is strictly concave – the solution is unique
4. Work with the first-order conditions to derive the Euler equation(s) and characterize the optimal path

2. The Growth Model in Continuous Time

Neoclassical Growth

- Let's apply the tools of discounted optimal control to a well-known problem: the neoclassical growth model
- The assumptions are the same as in discrete time
- In particular $\beta \in (0, 1)$ implies $\rho > 0$

Planner's Problem

- The central planner solves

$$\max_{k_t \geq 0, c_t \geq 0} \int_0^\infty e^{-\rho t} u(c_t) dt, \quad \rho > 0$$

$$\text{subject to} \quad \dot{k}_t = f(k_t) - \delta k_t - c_t,\ k_0 > 0 \text{ given}$$

Step 1 – Hamiltonian

- The first step consists in writing the Hamiltonian
- Defining $\mu_t \equiv e^{\rho t}\lambda_t$, we write the current-value Hamiltonian as

$$\tilde{H}(k_t, c_t, \mu_t) = u(c_t) + \mu_t[f(k_t) - \delta k_t - c_t]$$

Step 2 – Necessary Conditions

- The first-order conditions are

$$\begin{aligned} \tilde{H}_c(k_t, c_t, \mu_t) = 0 : &\quad u'(c_t) = \mu_t \\ \dot{\mu}_t - \rho\mu_t = -\tilde{H}_k(k_t, c_t, \mu_t) : &\quad \rho\mu_t - \dot{\mu}_t = \mu_t[f'(k_t) - \delta] \\ \dot{k}_t = \tilde{H}_\mu(k_t, c_t, \mu_t) : &\quad \dot{k}_t = f(k_t) - \delta k_t - c_t \end{aligned}$$

  and a transversality condition

Sufficiency Conditions

- Third step: check that the assumptions for the sufficiency conditions hold

1. $V(t, k_t)$ exists and is finite for all t OK
2. For any admissible pair (k_t, c_t), $\lim_{t\to\infty} e^{-\rho t}\mu_t k_t \geq 0$?
   Since $\mu_t = u'(c_t) > 0$ and $0 \leq k_t \leq \bar{k}$ by feasibility, OK
3. For all t, X_t is convex and $M(k_t, \mu_t)$ is strictly concave in $k_t \in X_t$?
   Since $\mu_t > 0$, $\tilde{H}$ is the sum of two strictly concave functions and is thus strictly concave, and so is $M(k_t, \mu_t) \equiv \max_c \tilde{H}$ OK

- We conclude that the candidate solution is the unique global maximum

Transversality Condition

- The tranversality condition is

$$\lim_{t\to\infty} e^{-\rho t}\mu_t k_t = \lim_{t\to\infty} \lambda_t k_t = 0$$

- The transversality condition has the same interpretation as in discrete time
- In the limit, households should not overaccumulate assets: the discounted value of the capital stock they hold should be zero

Step 4 – Combining the FOCs

- Differentiate the first FOC, $\mu_t = u'(c_t)$, with respect to time

$$\dot{\mu}_t = u''(c_t)\dot{c}_t$$

- Combine it with the second FOC, $\mu_t[f'(k_t) - \delta] = \rho\mu_t - \dot{\mu}_t$

$$u'(c_t)[f'(k_t) - \delta - \rho] = -u''(c_t)\dot{c}_t$$

Deriving the Euler Equation

- Define $\varepsilon(c_t)$ as the coefficient of relative risk aversion

$$\varepsilon(c_t) \equiv -\frac{u''(c_t)c_t}{u'(c_t)}$$

- Use $\varepsilon(c_t)$ to rewrite the last equation in the previous slide

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}[f'(k_t) - \delta - \rho]$$

- This is analogous to the Euler equation in discrete time

Alternative Method

- Substitute the constraint into the objective function

$$\max_{k_t} \int_0^\infty e^{-\rho t} u\left[f(k_t) - \delta k_t - \dot{k}_t\right] dt$$

- Define $h(t, k_t, \dot{k}_t) \equiv e^{-\rho t} u[f(k_t) - \delta k_t - \dot{k}_t]$ and derive the necessary conditions

$$k_t : \frac{\partial h}{\partial k} = e^{-\rho t} u'(c_t)[f'(k_t) - \delta]; \qquad \dot{k}_t : \frac{\partial h}{\partial \dot{k}} = -e^{-\rho t} u'(c_t)$$

- Differentiate the second FOC with respect to time

$$\frac{d}{dt}\frac{\partial h}{\partial \dot{k}} = \rho e^{-\rho t} u'(c_t) - e^{-\rho t} u''(c_t)\dot{c}_t$$

- Use the Euler-Lagrange equation to obtain the Euler equation

$$\frac{\partial h(t, k_t, \dot{k}_t)}{\partial k} - \frac{d}{dt}\frac{\partial h(t, k_t, \dot{k}_t)}{\partial \dot{k}} = 0 \quad \Longrightarrow \quad \dot{c}_t = -\frac{u'(c_t)}{u''(c_t)}[f'(k_t) - \delta - \rho]$$

Intertemporal Elasticity and Risk Aversion

- The elasticity of intertemporal substitution between t and s is the elasticity of consumption growth to marginal utility growth

$$\frac{1}{\sigma(c_t, c_s)} \equiv -\frac{d\ln(c_s/c_t)}{d\ln[u'(c_s)/u'(c_t)]}$$

- If $s \to t$

$$\sigma(c_t, c_s) \to \sigma(c_t) = -\frac{u''(c_t)c_t}{u'(c_t)} = \varepsilon(c_t)$$

- The elasticity of intertemporal substitution $1/\sigma(c_t)$ corresponds to the inverse of the coefficient of relative risk aversion $\sigma(c_t)$

Interpreting the Euler Equation

- We can rewrite the Euler equation as

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma(c_t)}[f'(k_t) - \delta - \rho]$$

- *Ceteris paribus*, the growth rate of consumption is higher, the higher the
  - Elasticity of intertemporal substitution $1/\sigma(c_t)$
  - Return on capital net of depreciation $f'(k_t) - \delta$
  - Patience of agents, ie the lower their discount rate ρ

Rewriting the Transversality Condition

- Express the second FOC as $\dot{\mu}_t/\mu_t = -[f'(k_t) - \delta - \rho]$
- Integrate on both sides using $\frac{d \ln \mu_t}{dt} = \frac{\dot{\mu}_t}{\mu_t}$

$$\int_0^t \frac{\dot{\mu}_s}{\mu_s} ds = -\int_0^t [f'(k_s) - \delta - \rho] ds$$

$$\ln \mu_t - \ln \mu_0 = -\int_0^t [f'(k_s) - \delta - \rho] ds$$

- Take the exponential

$$\mu_t = \mu_0 e^{-\int_0^t [f'(k_s) - \delta - \rho] ds} \quad \textbf{with } \mu_0 = u'(c_0) > 0$$

Rewriting the Transversality Condition

- The transversality condition is $\lim_{t\to\infty} e^{-\rho t}\mu_t k_t = 0$
- Plug in the last equation of the previous slide

$$\lim_{t\to\infty} e^{-\rho t}\mu_0 e^{-\int_0^t [f'(k_s)-\delta-\rho]ds} k_t = 0$$

- The transversality condition becomes

$$\lim_{t\to\infty} k_t e^{-\int_0^t [f'(k_s)-\delta]ds} = 0$$

- This is the market value version of the transversality condition

Dynamic System

- The model dynamics are characterized by two difference equations

  1. Euler equation: $$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}[f'(k_t) - \delta - \rho]$$
  2. Law of motion of capital: $$\dot{k}_t = f(k_t) - \delta k_t - c_t$$

- We have an initial condition k_0
- We have a boundary condition, the transversality condition

$$\lim_{t \to \infty} k_t e^{-\int_0^t [f'(k_s) - \delta] ds} = 0$$

Steady State

- In the steady state, $\dot{c}_t = 0$ and $\dot{k}_t = 0$
- The Euler equation becomes

$$f'(k^*) = \delta + \rho$$

- The law of motion of capital becomes

$$c^* = f(k^*) - \delta k^*$$

Golden Rule and Modified Golden Rule

- Remember, k_{gold} is the golden rule, ie the steady-state level of capital that maximizes steady-state consumption

$$k_{\text{gold}} = \arg\max_k [f(k) - \delta k] \quad \Longrightarrow \quad f'(k_{\text{gold}}) = \delta$$

- With discounting we have the modified golden rule

$$k^* = \arg\max_k [f(k) - (\delta + \rho)k] \quad \Longrightarrow \quad f'(k^*) = \delta + \rho$$

- Thus $k^* < k_{\text{gold}}$: agents are impatient and value more early consumption
- k^* is efficient, it is the steady-state level of capital that maximizes agents' consumption given their impatience

Unique Equilibrium

- We have just shown that the steady state (k^*, c^*) is unique
- Next, since the global solution is unique, if we start from an initial $k_0 > 0$, there is only one initial value $c_0 > 0$ that guarantees convergence to (k^*, c^*)
- In other words, in the neoclassical growth model there exists a unique and stable competitive equilibrium path
- We represent this unique equilibrium path by a curve of (k, c), called the stable arm or the saddle path, that converges to the steady state
- Let's see this graphically

Constant Consumption Locus

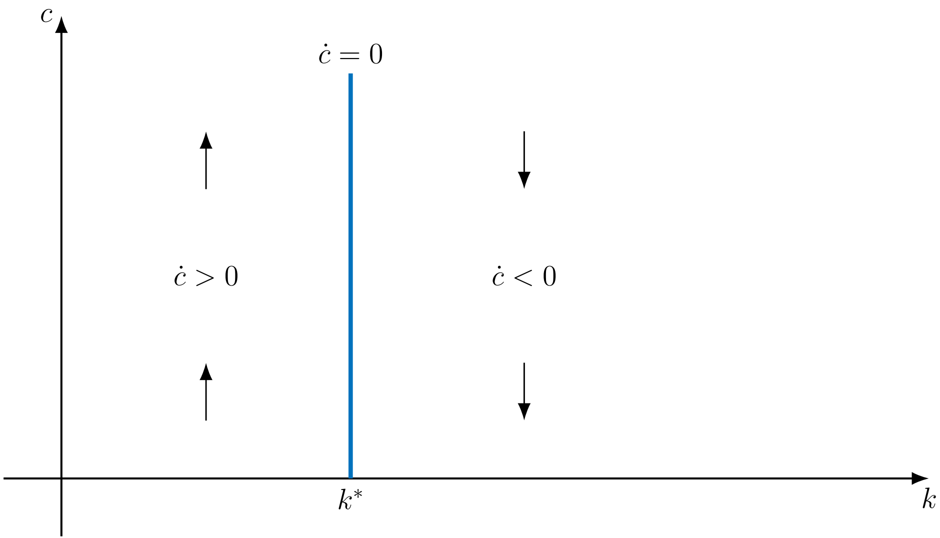
- Repeat the Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}[f'(k_t) - \delta - \rho]$$

- The constant consumption locus, $\dot{c} = 0$, implies $f'(k^*) = \delta + \rho$ and thus is a vertical line at $k_t = k^* = f'^{-1}(\delta + \rho)$; since $f'' < 0$, we deduce that

1. To the right of the $\dot{c} = 0$ line, capital $k > k^*$ is "too high", therefore c decreases, $\dot{c} < 0$, which we represent with a down-pointing arrow
2. To the left of the $\dot{c} = 0$ line, capital $k < k^*$ is "too low", therefore c increases, $\dot{c} > 0$, which we represent with an up-pointing arrow

Constant Consumption Locus



Constant Capital Locus

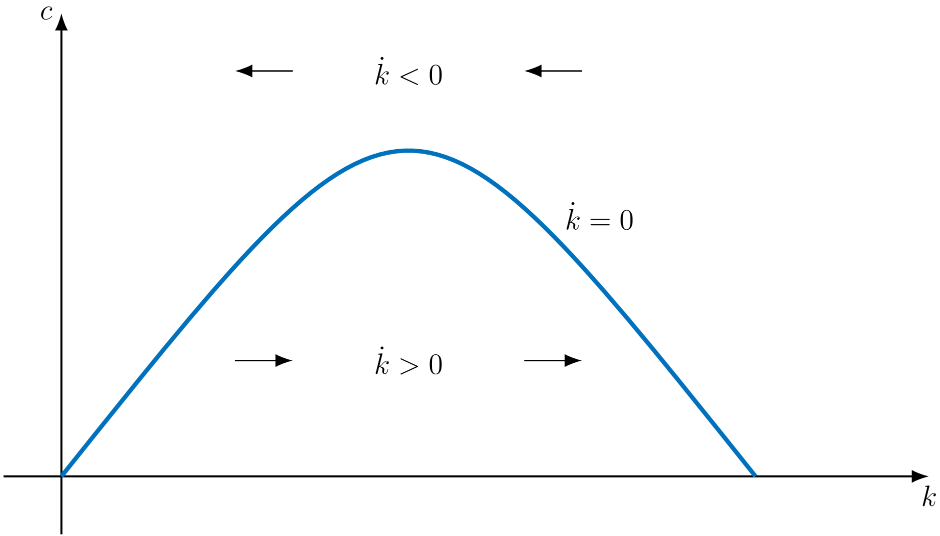
- Repeat the law of motion of capital

$$\dot{k}_t = f(k_t) - \delta k_t - c_t$$

- The constant capital locus, $\dot{k} = 0$, implies $c^* = f(k) - \delta(k)$: since $f' > 0$, $f'' < 0$, c^* is increasing then decreasing in k

1. Above the $\dot{k} = 0$ curve, consumption $c > c^*$ is "too high", therefore k decreases, $\dot{k} < 0$: left-pointing arrow
2. Below the $\dot{k} = 0$ line, consumption $c < c^*$ is "too small", therefore k increases, $\dot{k} > 0$: right-pointing arrow

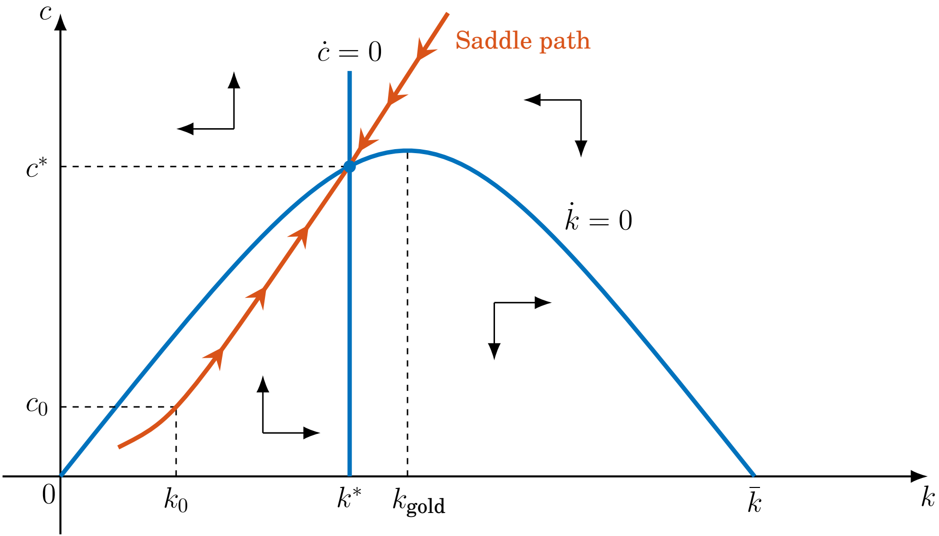
Constant Capital Locus



Complete Phase Diagram

- The intersection of the two loci defines the steady state (k^*, c^*)
- The two loci intersect at the modified golden rule k^*
- For each starting k_0, there is a unique saddle path to the steady state
- Let's see the complete phase diagram

Transitional Dynamics in the Growth Model



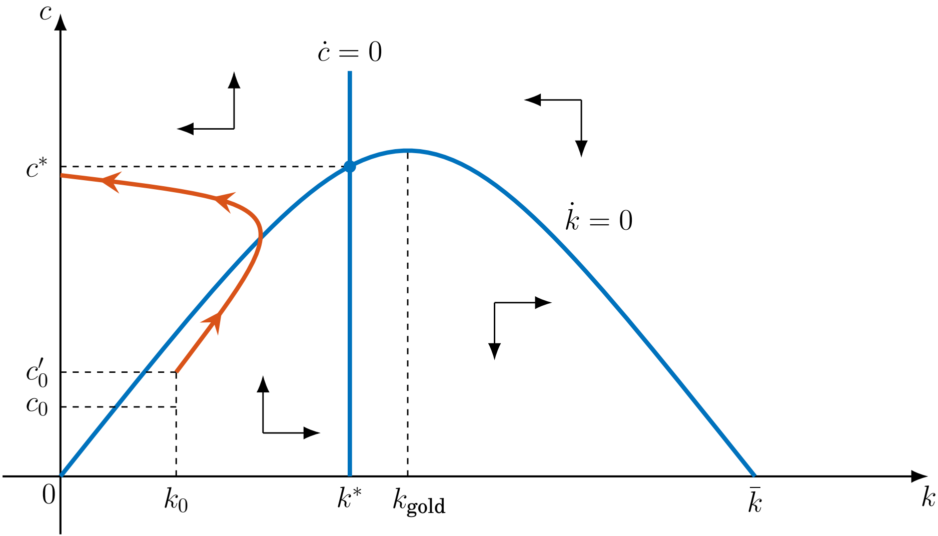
Saddle Path

- Consider the starting point (k_0, c_0)
- For a given initial capital sock k_0, agents consume and save just enough
- Therefore, in the first period $t = 0$, consumption jumps on the saddle path
- As time passes, capital and consumption converge monotonically to their steady-state level (k^*, c^*)

Alternative Trajectories

- What if initial consumption is higher or lower than c_0?
- Let's see the two cases in turn

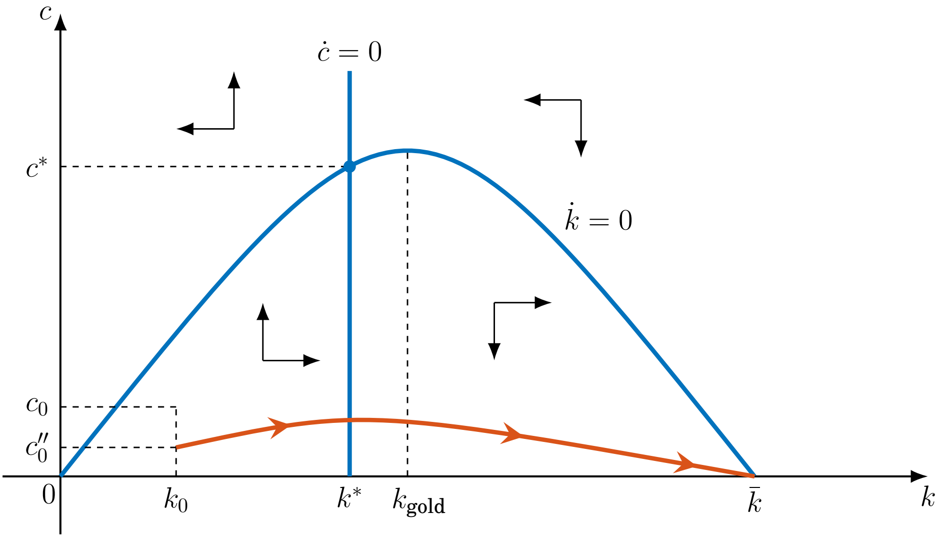
Alternative Trajectory 1



Alternative Trajectory 1

- Consider the point (k_0, c'_0) where $c'_0 > c_0$
- Agents consume too much, don't save enough
- The economy converges to $k = 0$ after a finite number of periods
- With zero production, consumption collapses to $c = 0$
- This is not optimal

Alternative Trajectory 2



Alternative Trajectory 2

- Consider now the point (k_0, c_0'') where $c_0'' < c_0$
- Agents save too much, don't consume enough
- The economy converges to $c = 0$ after a finite period
- This is not optimal

Violated Transversality Condition

- In this last case, using the law of motion of capital, we find that capital accumulates to $\bar{k}$

$$\dot{k} = f(k) - \delta k - c \quad \Longrightarrow \quad f(\bar{k}) - \delta \bar{k} = 0$$

- Note that $\bar{k} > k_{\text{gold}} = \arg\max_k f(k) - \delta k$, thus $f'(\bar{k}) < \delta$ and this violates the transversality condition

$$\lim_{t \to \infty} k_t e^{-\int_0^t [f'(k_s) - \delta] ds} = \bar{k} e^{-\int_0^t [f'(k_s) - \delta] ds} > 0$$

3. The Q-Theory of Investment

A Theory of Investment

- We apply optimal control techniques to the canonical model of investment under adjustment costs: the Q-theory of investment
- It was developed by Tobin (1969, 1971) and Tobin and Brainard (1968, 1977)
- James Tobin received the Nobel Prize in 1981 for his analysis of financial markets and their relations to investment

Model Setup

- A price-taking firm maximizes the discounted value of profits; the only twist is the firm incurs adjustment costs whenever it changes its capital stock
- The capital stock of the firm is $K_t > 0$, this is the only input in production
- The production function is $f(K)$, where $f' > 0$ and $f'' < 0$
- We normalize the price of the firm's output to one at all dates, as in the RBC model; we are dealing with real models

Adjustment Costs

- The firm is subject to investment adjustment costs captured by an increasing and convex function $\phi(I)$ that satisfies

$$\phi' > 0, \quad \phi'' > 0, \quad \text{and} \quad \phi(0) = \phi'(0) = 0$$

- In words, the cost is more than proportionally higher the larger the adjustment of investment, and no new investment means no cost
- Thus, on top of the cost of purchasing investment goods I_t, the firm incurs a cost of adjusting its production structure

Firm's Problem

- The problem of the firm is to solve

$$\max_{K_t \geq 0, I_t \geq 0} \int_0^\infty e^{-rt}[f(K_t) - I_t - \phi(I_t)]dt$$

$$\text{subject to} \quad \dot{K}_t = I_t - \delta K_t,\ K_t \geq 0 \text{ for all } t,\ K_0 > 0 \text{ given}$$

- The firm discount the future by the interest rate r, which is exogenous
- $\phi(I_t)$ does not increase the capital stock, it is just a cost: since it is convex, the firm prefers not to make "large" adjustments, ie ϕ smooths investment

Hamiltonian

- First step, the Hamiltonian
- Define q_t as the costate variable in current value and write the current-value Hamiltonian

$$\tilde{H}(K_t, I_t, q_t) = f(K_t) - I_t - \phi(I_t) + q_t[I_t - \delta K_t]$$

Necessary Conditions

- Second step, the necessary conditions

$$
\begin{aligned}
\tilde{H}_I(K_t, I_t, q_t) &= -1 - \phi'(I_t) + q_t = 0 \\
\tilde{H}_K(K_t, I_t, q_t) &= f'(K_t) - \delta q_t = r q_t - \dot{q}_t \\
\dot{K}_t &= I_t - \delta K_t \\
\lim_{t \to \infty} e^{-rt} q_t K_t &= 0
\end{aligned}
$$

Sufficiency Conditions

- Third step, the sufficiency conditions
- Since $q_t = 1 + \phi'(I_t) > 0$ for all t, $\tilde{H}$ is strictly concave in K_t
- Since $K_t \geq 0$, we have $\lim_{t\to\infty} e^{-\rho t} q_t K_t \geq 0$
- Thus the necessary conditions are sufficient and determine the unique path of investment and capital

Law of Motion of Investment

- Fourth step, the Euler equation
- Differentiate the first FOC with respect to time

$$\dot{q}_t = \phi''(I_t)\dot{I}_t$$

- Substitute this into the second FOC

$$\dot{I}_t = \frac{1}{\phi''(I_t)} \left\{ [r + \delta][1 + \phi'(I_t)] - f'(K_t) \right\}$$

- This Euler equation determines the law of motion of investment

Linear Costs

- Let's analyze the investment Euler equation
- If $\phi''(I) \to 0$, $\dot{I}_t$ diverges, meaning investment I_t can jump to any value
- Note that as $\phi''(I) \to 0$, the adjustment cost $\phi(I)$ becomes linear
- In words, with linear adjustment costs the capital stock immediately reaches its steady-state value
- What is the intuition?

No Smoothing vs Smoothing

- As $\phi''(I) \to 0$, adjustment costs linearly increase the cost of investment and thus do not create any need for smoothing
- For example, to double investment the firm pays the same cost whether it does it all in one period or splits it across two or more periods
- In contrast, with convex costs $\phi''(I_t) > 0$, there is a motive for smoothing: the firm wants to spread its desired investment over a number of periods
- With $\phi''(I_t) > 0$, $\dot{I}_t$ takes a finite value and investment I_t adjusts slowly

Steady State

- In the steady state, $\dot{K} = 0$ and $\dot{I} = 0$
- From the law of motion of capital

$$I^* = \delta K^*$$

- From the law of motion of investment

$$f'(K^*) = (r + \delta)[1 + \phi'(I^*)]$$

Unique Steady State

- The steady state is unique: to see this, combine the previous two equations

$$f'(K^*) = (r + \delta)[1 + \phi'(\delta K^*)]$$

- Since $f'' < 0$, the left side is decreasing in K^*
- Since $\phi'' > 0$, the right side is increasing in K^*
- Thus, the Inada condition ensures there is one unique K^* that satisfies the above equation

Constant Capital Locus

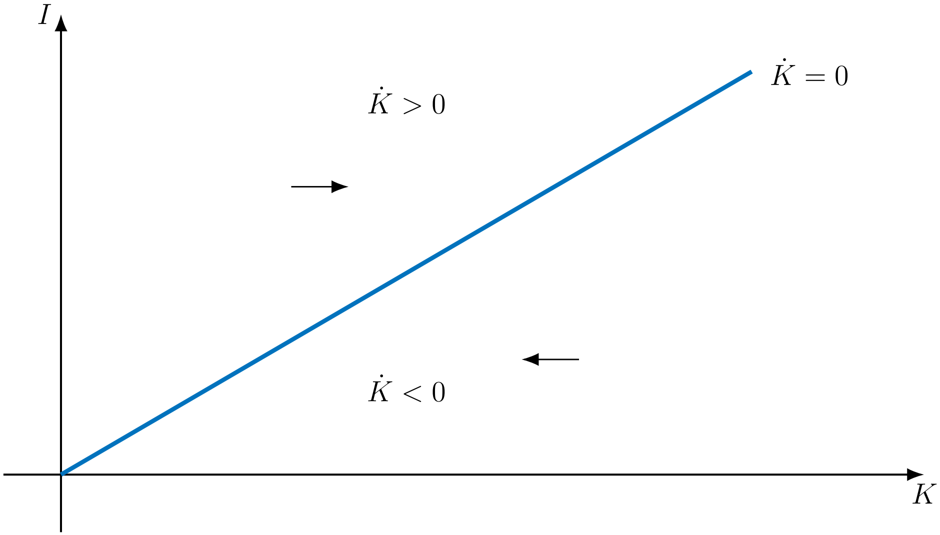
- Repeat the law of motion of capital

$$\dot{K}_t = I_t - \delta K_t$$

- The $\dot{K} = 0$ locus implies $I^* = \delta K^*$: this is an upward-sloping line since more capital requires more investment to replenish the depreciated capital stock

1. Above the $\dot{K} = 0$ line, $I > I^*$, there is more than enough investment for replenishment, therefore K increases, $\dot{K} > 0$: right-pointing arrow
2. Below the $\dot{K} = 0$ line, $I < I^*$, there is not enough investment for replenishment, therefore K decreases, $\dot{K} < 0$: left-pointing arrow

Constant Capital Locus



Constant Investment Locus

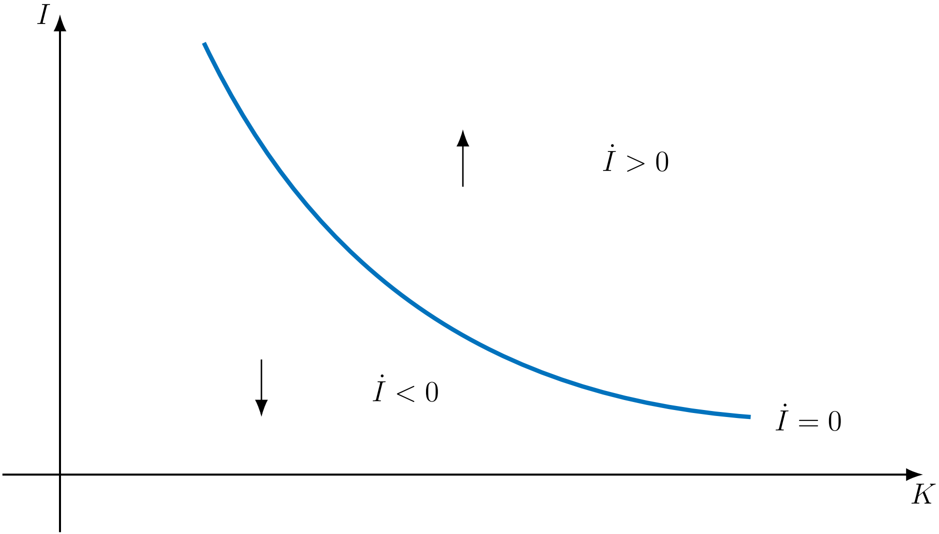
- Repeat the investment Euler equation

$$\dot{I}_t = \frac{1}{\phi''(I_t)} \left\{ [r + \delta][1 + \phi'(I_t)] - f'(K_t) \right\}$$

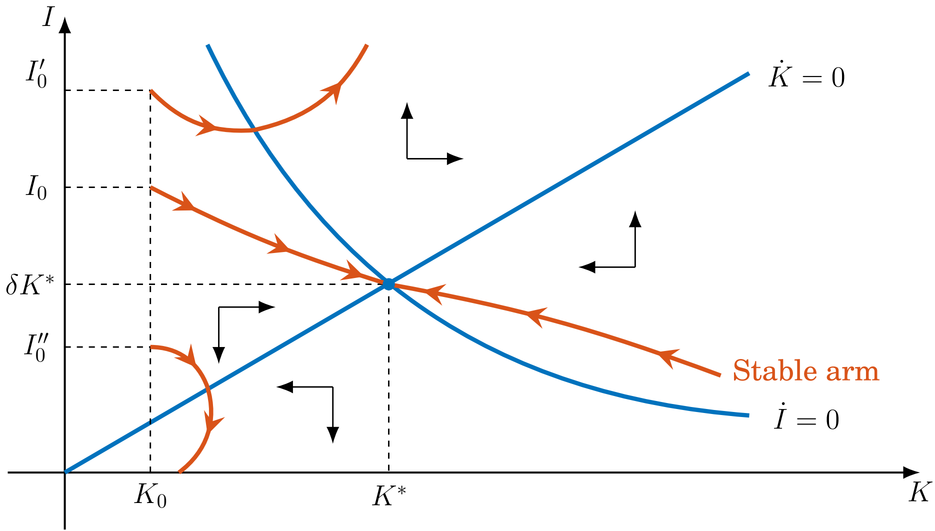
- The $\dot{I} = 0$ locus implies $f'(K^*) = (r + \delta)[1 + \phi'(I^*)]$: since $f'' < 0$ and $\phi'' > 0$, this curve is downward sloping

1. To the right of the $\dot{I} = 0$ curve, $K > K^*$ is too high, $f'(K)$ is too low, therefore I must be increasing, $\dot{I} > 0$: up-pointing arrow
2. To the left of the $\dot{I} = 0$ curve, $K < K^*$ is too low, $f'(K)$ is too high, therefore I must be decreasing, $\dot{I} < 0$: down-pointing arrow

Constant Investment Locus



Phase Diagram



Unique Saddle Path

- We start from an arbitrary K_0
- The unique solution involves a jump to an initial I_0 followed by gradual convergence to the steady-state investment level δK^* along the saddle path
- If $K_0 < K^*$, $I_0 > I^*$ and I_t monotonically falls towards I^*
- As the firm builds up capital, it decreases investment

Q-Theory

- Recall the FOC with respect to investment

$$q_t = 1 + \phi'(I_t)$$

- q_t is all the firm needs to know to decide on new investment I_t
- The larger is q_t the higher is investment in capital
- But what is q_t?

Value of the Firm

- Let the current maximized value of the firm be

$$V(K_t) = \max_{I_s, K_s} \int_t^{\infty} e^{-rs}[f(K_s) - I_s - \phi(I_s)]ds$$
$$\text{subject to} \quad \dot{K}_s = I_s - \delta K_s$$

- Using an envelope theorem-type argument (next slide), we know that the marginal change in V is given by the price of one more unit of capital

$$V'(K_t) = q_t$$

Envelope Theorem

- Take the functional equation where x is the state and y the control

$$V(x) = \max_y \{U(x, y) + \beta V(y)\}$$

- Let y^* be the optimal choice; the total derivative of V is

$$\begin{aligned} V'(x) &= \frac{\partial U(x, y^*)}{\partial x} + \underbrace{\frac{\partial U(x, y^*)}{\partial y} + \beta V'(y^*)}_{=0 \text{ from the Euler equation}} \\ &= \frac{\partial U(x, y^*)}{\partial x} \end{aligned}$$

- Intuitively, V is an optimized value, so its derivative with respect to the control is zero, ie agents have already optimized

Market Value

- Back to our model
$$V'(K_t) = q_t$$
- We conclude that q_t measures exactly by how much a one-dollar increase in capital raises the value of the firm
- Put differently, q_t is the firm's market value of one unit of capital
- The time-varying price of the firm q_t contrasts with the price of output, which is one at all times, meaning the cost of replacing capital is one

Q-Theory

- In steady state, $\phi'(\delta K^*)$ is small, so q is approximately equal to one
- Hence in steady state, the firm's market value of one unit of capital is roughly equal to the cost of replacing capital
- Out of steady state, a high $q_t > 1$ signals a need for further investments, ie it signals the time when the firm's investment demand is high
- On the contrary, a low $q_t < 1$ signals a need to disinvest, ie to sell capital

Tobin's Q

- Tobin (1969) defined what is now known as Tobin's Q
- Tobin's Q is a firm's market value divided by its assets' replacement cost

$$\text{Tobin's Q} = \frac{\text{Total market value of the firm}}{\text{Total asset value of the firm}}$$

- Tobin argued that when Tobin's Q is
  - High, ie more than 1, the firm would like to invest more
  - Low, ie less than 1, the firm would like to sell assets

Tobin's Marginal q

- The model's q_t and Tobin's Q are related; in the model, we have

$$q_t = \frac{V'(K_t)}{1} = 1 + \phi'(I_t)$$

- Thus the model's q_t is actually Tobin's marginal q, which is defined as

$$\text{Tobin's marginal q} = \frac{\text{Market value of one extra unit of capital}}{\text{Replacement cost of one extra unit of capital}}$$

- In the model, Tobin's marginal q is equal to the replacement cost of capital, which is one, plus the marginal adjustment cost: $q_t = 1 + \phi'(I_t)$

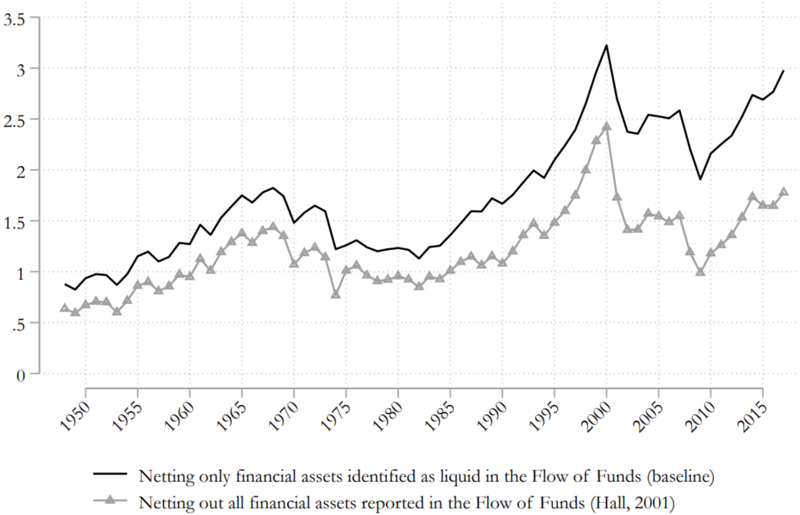
Average Q

- Tobin's Q in the model would be

$$\frac{V(K_t)}{K_t}$$

- Total market value of the firm divided by total asset value
- This is sometimes referred to as Tobin's average Q

Tobin's Average Q in the United States



Source: Crouzet and Eberly (2023)

Model Prediction

- We can check the model's main prediction, that firms invest more when $q_t > 1$, by looking at corporate data
- Tobin's average Q is easy to measure: divide the market value of the firm by its book value, ie the value of all its assets
- But what matters for investment decisions is Tobin's marginal q, as the model makes clear, and this is typically harder to measure

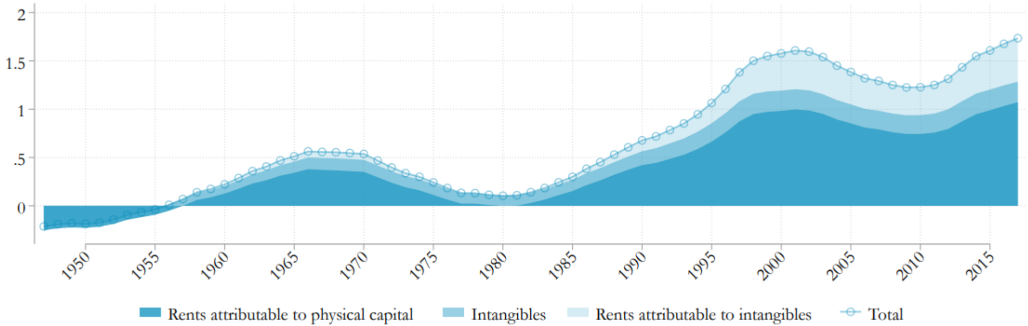
Average Q vs Marginal q in Theory

- Hayashi (1982) shows that if the adjustment cost $\phi(I, K)$ is homogeneous of degree one, then Tobin's average Q equals marginal q
- But homogeneous costs do not seem to be a good description of reality
- In our model, since $V'' < 0$, we have $\frac{d[V(K)-KV'(K)]}{dK} = -KV''(K) > 0$, and thus $V'(K) < \frac{V(K)}{K}$: marginal q is lower than average Q
- Other functional forms create an even higher discrepancy between the two
- Two examples are investment irreversibility, $I_t \geq 0$; and fixed cost: if $I_t > 0$ the firm pays a lump-sum amount (see Caballero 1999 for an old survey)

Valuations vs Investment

- In recent years, valuations have gone up – high average Q – but investment has been lackluster – low marginal q – even with high return to capital
- Thus the Q-theory is increasingly at odds with the data
- One possible explanation is intangible capital: a shift toward intangibles in production causes physical investment to appear low relative to valuations
- Another cause is rising market power: higher rents can account for stable or rising rate of return on assets despite falling user cost of capital

The Investment Gap: Average Q Minus Marginal q



Source: Crouzet and Eberly (2023)

Conclusion

- Optimal control is fun
- We are ready to apply our techniques to any dynamic optimization problem in continuous time
- We will do just that in the next (and last) three lectures by studying models of endogenous growth

4. Exercises

Exercise 1 – A Standard Problem

Characterize, step by step, the optimal path of x_t and c_t that solves the following problem

$$\max \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{subject to} \quad \dot{x}_t = g(x_t) - c_t, \; x_0 \text{ given}$$

where $u' > 0$, $u'' < 0$, $\lim_{c\to\infty} u'(c) = 0$, and $\lim_{c\to 0} u'(c) = \infty$; $g' > 0$, $g'' < 0$, $\lim_{x\to\infty} g'(x) = 0$ and $\lim_{x\to 0} g'(x) = \infty$.

Exercise 2 – Q-Theory

Consider the model with investment adjustment cost of the lecture.

1. Using a phase diagram, show the transitional dynamics after a permanent increase in the interest rate r.

Suppose the adjustment cost is $\phi(\dot{K}/K)K$, where $\phi' > 0$, $\phi'' > 0$, and $\phi(0) = \phi'(0) = 0$.

2. Write the current-value Hamiltonian and derive the necessary conditions.
3. Show that there exists a unique steady state.
4. Define a variable $x = I/K$ and use the phase diagram in the (x, K) plane to characterize the saddle path in the vicinity of the steady state.
5. Check the sufficient conditions. Is the optimal path unique?

Exercise 3 – Fiscal Policy

Consider the standard neoclassical growth model where the representative household has preferences

$$\int_0^\infty e^{-\rho t}\left(\frac{c_t^{1-\theta}-1}{1-\theta}+G_t\right)dt,$$

where G_t is a public good financed by government spending. The production function $Y_t = F(K_t, L_t)$ is standard and the resource constraint is $C_t + I_t \leq Y_t$. Assume that G_t is financed by taxes on investment. In particular, the capital accumulation equation is

$$\dot{K}_t = (1-\tau_t)I_t - \delta K_t,$$

and the fraction τ_t of private investment I_t is used to finance the public good, that is $G_t = \tau_t I_t$. Take the path of tax rates $[\tau_t]_{t=0}^\infty$ as given.

Exercise 3 – Continued

1. Define a competitive equilibrium.
2. Set up the individual maximization problem, and characterize consumption and investment behavior.
3. Assuming that $\lim_{t\to\infty} \tau_t = \tau$, compute the steady state.
4. What value of τ maximizes the steady-state utility of the representative household? Is this value also the tax rate that would maximize the initial utility level when the economy starts away from the steady state? Why or why not?