5. The Neoclassical Growth Model

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Introduction

- Today we study the neoclassical growth model
- Ramsey (1928) first developed the model as a saving problem for a nation's central planner seeking to maximize utility coming from consumption
- Cass (1965) and Koopmans (1965) extended the model to a decentralized economy with a representative agent and neoclassical production
- The neoclassical growth model is also known as the Ramsey growth model or the Ramsey-Cass-Koopmans model

Ramsey vs Solow

- Recall the growth model of Solow (1956) from your undergrad studies? That is a simplified version, or special case, of the neoclassical growth model
- In Solow, the saving rate is an exogenous and constant fraction of income
- In the neoclassical growth model, the choice of consumption and savings is microfounded, resulting in an endogenous and time-varying saving rate

Main Finding of the Solow Growth Model

- The principal conclusion of the Solow growth model is that the accumulation of physical capital cannot account for
  - The vast growth over time in income per person
  - The vast geographic differences in income per person
- This follows directly from the assumption of diminishing returns to capital
- Put differently, a permanent increase in the saving rate, and thus in investment and capital, raises growth only temporarily

Main Finding of the Ramsey Growth Model

- The main conclusion of the neoclassical growth model is that it does not change the conclusion of the Solow growth model
- In other words, relaxing the assumption of constant saving rate does not affect the main result that capital accumulation fails to explain growth

Then Why Bother?

- The neoclassical growth model has one major advantage over Solow's
- It is a fully microfounded macroeconomic model, ie it starts from the behavior of individual consumers and firms and aggregates up
- It is therefore not subject to the Lucas critique
- It enables us to think seriously about welfare and policy issues

The Backbone of Macro

- The neoclassical growth model is the foundational model upon which most modern macroeconomic theory is built

Theory	Foundation
Real business cycle theory	Neoclassical growth with productivity shocks
Endogenous growth theory	Neoclassical growth with endogenous technical change
New Keynesian theory	Neoclassical growth with sticky prices
Dynamic stochastic general equilibrium (DSGE) models	Neoclassical growth with real and nominal frictions
Heterogeneous-agent models	Neoclassical growth with incomplete markets
Behavioral models	Neoclassical growth with non-rational expectations

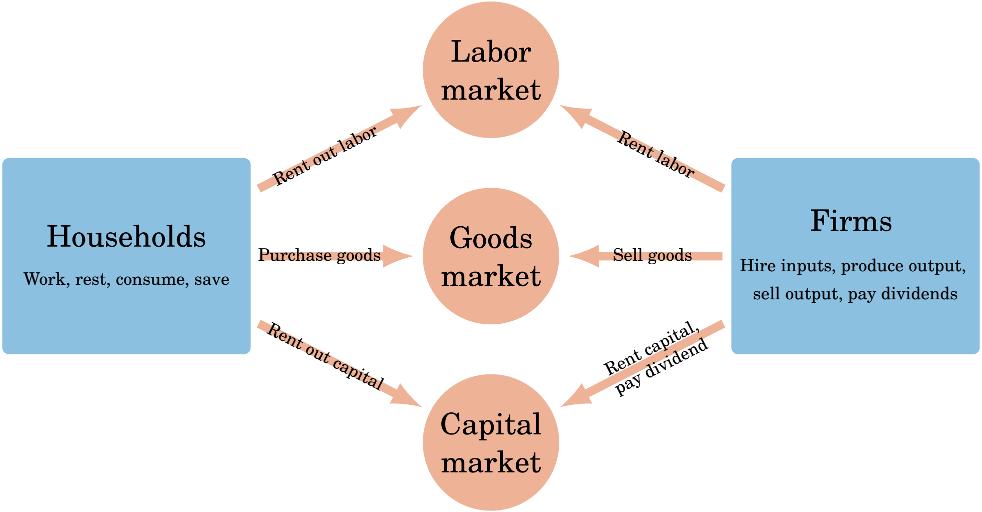
Lecture Outline

1. Model Setup
2. Central Planner
3. Transversality Condition
4. Steady State
5. Dynamics
6. CRRA Utility
7. Adding Growth
8. Competitive Equilibrium
9. Explaining the Data
10. Exercises

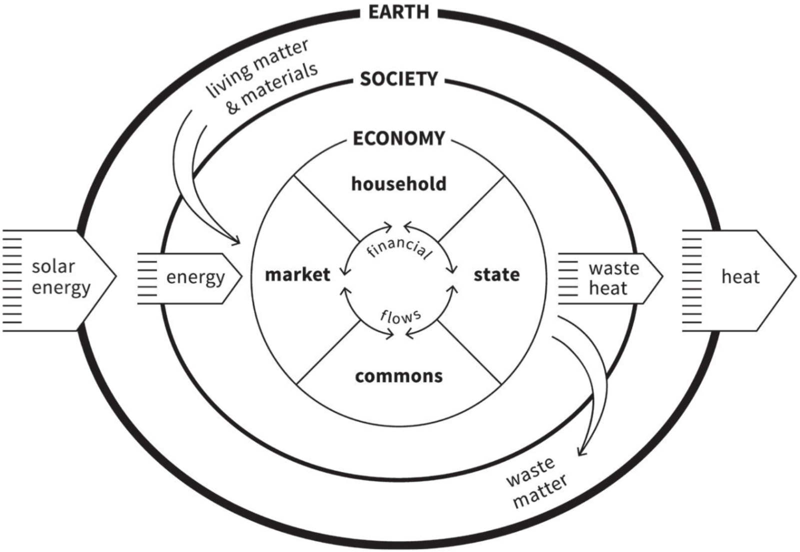
Main Reference: Acemoglu, 2009, *Introduction to Modern Economic Growth*, Chapter 8

1. Model Setup

Overview of the Model



A Better Model of the Economy?



Technology

- We abstract from the firms for now for simplicity
- An aggregate production function combines two inputs to create an output

$$Y_t = F(K_t, L_t)$$

- Y_t is aggregate output
- K_t is aggregate capital
- L_t is aggregate labor
- We abstract from productivity for now ($A_t = 1$)

Assumptions on the Production Function

- The function F is increasing in both inputs, it has positive marginal returns

$$F_K(K,L) > 0 \quad \text{and} \quad F_L(K,L) > 0$$

- Without capital or labor there is no output: $F(0,L) = 0$ and $F(K,0) = 0$
- The production function F has constant returns to scale

$$F(\lambda K, \lambda L) = \lambda F(K,L) \qquad \text{for all } \lambda \geq 0$$

- This implies it has diminishing marginal returns

$$F_{KK}(K,L) < 0 \quad \text{and} \quad F_{LL}(K,L) < 0$$

Discussion

- Constant returns is the critical assumption of the model
- The assumption implies that there are no gains from specialization
- In a very small economy it would be possible to specialize: doubling inputs would more than double output; here the economy is big so it's not possible
- The assumption also implies other inputs like land and natural resources (air, water, soil, minerals, energy, vegetation) are relatively unimportant
- This is of course a drastic oversimplification of the world; to rationalize that think of capital K_t as including land and natural resources

Euler's Theorem

- The function F is homogeneous of degree one

$$\lambda F(K, L) = F(\lambda K, \lambda L) \qquad \text{for all } \lambda \geq 0$$

- Derive this equation with respect to λ

$$F(K, L) = F_K(\lambda K, \lambda L)K + F_L(\lambda K, \lambda L)L$$

- Let $\lambda = 1$

$$F(K, L) = F_K(K, L)K + F_L(K, L)L$$

- This is Euler's homogeneous function theorem

Inada Conditions

- The production function satisfies the Inada conditions, named after Japanese economist Ken-Ichi Inada (1963)

1. No input, no production: $F(0) = 0$
2. Concave production function, ie positive and diminishing marginal returns
3. Infinite marginal product as inputs approach zero

$$\lim_{K \to 0} F_K(K, L) = \infty \quad \text{and} \quad \lim_{L \to 0} F_L(K, L) = \infty$$

4. Zero marginal product as inputs approach infinity

$$\lim_{K \to \infty} F_K(K, L) = 0 \quad \text{and} \quad \lim_{L \to \infty} F_L(K, L) = 0$$

Intensive Form

- Let $k_t \equiv K_t/L_t$ be capital per unit of labor

$$Y_t = F(K_t, L_t) = L_t F\left(\frac{K_t}{L_t}, 1\right) = L_t f(k_t) \quad \text{where } f(k_t) \equiv F(k_t, 1)$$

- We can express output per capita $y_t \equiv Y_t/L_t$ as

$$y_t = f(k_t)$$

- Output per capita is a function only of capital per capita

Inada Conditions, Intensive Form

- We verify that f satisfies the Inada conditions

1. No input, no output: $F(0, L) = Lf(0) = 0 \implies f(0) = 0$
2. Positive and diminishing marginal returns

$$\begin{aligned} F_K(K, L) &= Lf'(K/L)(1/L) = f'(k) > 0 \\ F_L(K, L) &= f(k) + Lf'(k)(-K/L^2) = f(k) - f'(k)k > 0 \\ F_{KK}(K, L) &= f''(k)(1/L) < 0 \implies f''(k) < 0 \end{aligned}$$

3. Infinite MPK as $k \to 0$: $\lim_{k \to 0} f'(k) = \lim_{K \to 0} F_K(K, L) = \infty$
4. Zero MPK as $k \to \infty$: $\lim_{k \to \infty} f'(k) = \lim_{K \to \infty} F_K(K, L) = 0$

Example – Cobb-Douglas Function

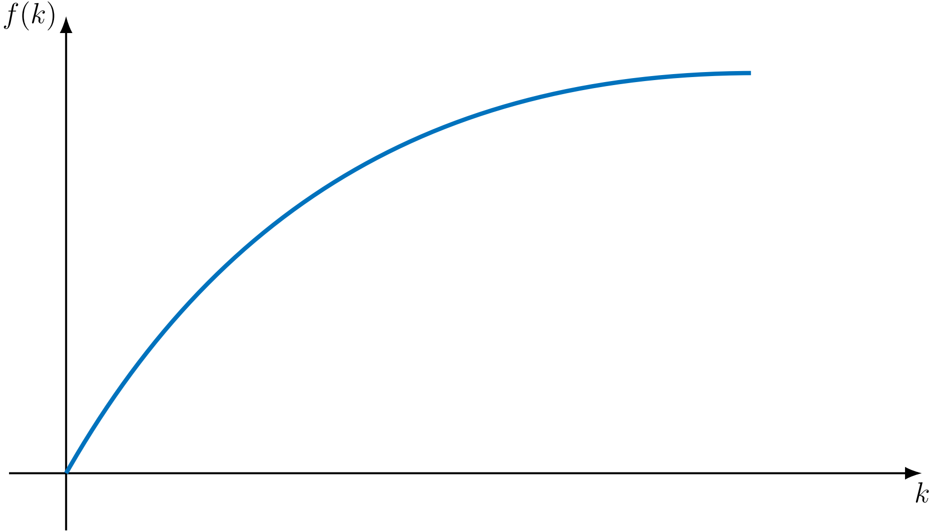
- An example of production function that satisfies the Inada conditions is a Cobb-Douglas production function

$$F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

- Parameter $\alpha \in (0, 1)$ is the share of capital in total output
- The production function in intensive form writes

$$f(k_t) = k_t^{\alpha}$$

An Example of Production Function



Summary

- The production function we use is

$$Y_t = L_t f(k_t)$$

- The function f satisfies

$$f(0) = 0; \qquad f' > 0; \qquad f'' < 0; \qquad \lim_{k \to 0} f'(k) = \infty; \qquad \lim_{k \to \infty} f'(k) = 0$$

Households

- There is a large number of identical households
- All households are equal so we can talk of a representative agent
- Households never die, they are infinitely-lived
- The population is constant, $L_t = L$

Utility Function

- The representative household has the following preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

- $\beta \in (0,1)$ is a discount factor
- u is the utility function, which here depends only on consumption
- c_t is consumption per capita, $c_t = C_t/L$, where C_t is aggregate consumption

Discussion

- These preferences imply the consumer only cares about her consumption
- Earning a lot (high income) or being rich (high wealth, ie high level of capital) does not directly make the consumer happier
- It does so indirectly by allowing the agent to consume more
- One could add more variables in the utility function: hours worked, wealth, money, children, bequest motive, public goods, green or safe environment

Inelastic Labor Supply

- There is no disutility of labor in this simple version of the model
- This means the labor supply is perfectly inelastic: it does not vary with the wage or anything else; all agents work the same amount of time each period
- This is a strong simplifying assumption
- But acceptable since the model is about the long run
- We can normalize $L = 1$ with no loss of generality

Assumptions on the Utility Function

- The utility function u satisfies the Inada conditions

1. No consumption, no utility: $u(0) = 0$
2. Concave utility: $u' > 0$ and $u'' < 0$
3. Infinite marginal utility as consumption approaches zero

$$\lim_{c_t \to 0} u'(c_t) = \infty$$

4. Zero marginal utility as consumption approaches infinity

$$\lim_{c_t \to \infty} u'(c_t) = 0$$

Resource Constraint

- The aggregate resource constraint of the economy is

$$Y_t = C_t + I_t$$

- I_t is aggregate investment
- No government ($G_t = 0$) and closed economy, ie no trade ($X_t = M_t = 0$)
- Everything that is produced in the economy is either consumed or saved (ie invested) privately and domestically

Capital Accumulation

- Investment is used to produce capital goods
- The capital stock evolves according to

$$K_{t+1} = (1-\delta)K_t + I_t$$

- This is the law of motion of capital, ie the capital accumulation equation
- $\delta \in (0,1)$ is a parameter representing the depreciation rate of capital

Reversible Investment

- Any output good Y_t can be transformed one-for-one, at no cost, into either a consumption good C_t or an investment good I_t
- Similarly, any capital good K_t can be freely resold and transformed one-for-one into a consumption good C_t
- In other words we assume capital or investment reversibility

Rewriting the Resource Constraint

- Combine the resource constraint and the capital accumulation equation by substituting out investment

$$Y_t = C_t + K_{t+1} - (1-\delta)K_t$$

- Substitute out output using the production function

$$C_t + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t$$

- Divide by L_t to express in per capita terms (intensive form)

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

Capital Interval

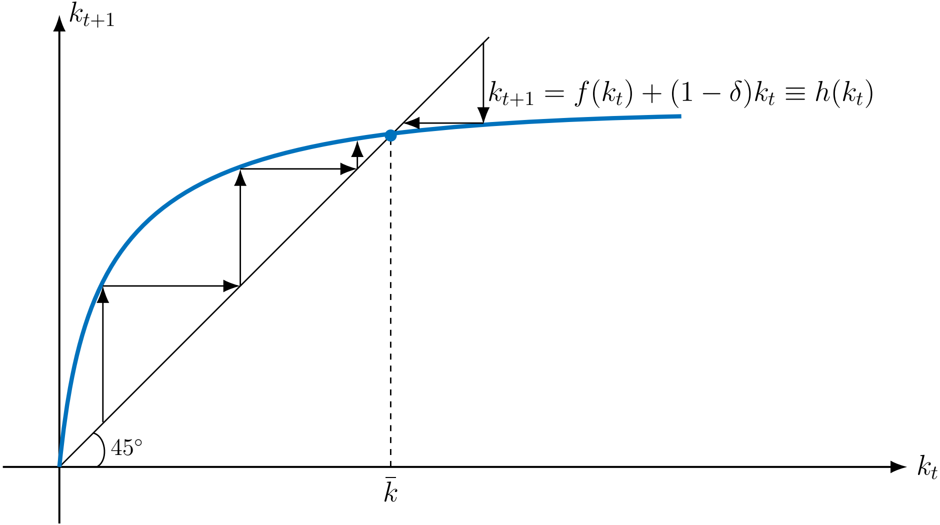
- We can check that capital, the constraint set, is compact: closed + bounded
- Suppose the agent consumes nothing, $c_t = 0$ for all t; capital in $t+1$ is

$$k_{t+1} = f(k_t) + (1-\delta)k_t \equiv h(k_t)$$

- $h'(k_t) = f'(k_t) + 1 - \delta > 0$ and $h''(k_t) = f''(k_t) < 0$: h increasing and concave
- $\lim_{k_t \to 0} h'(k_t) = \infty$ and $\lim_{k \to \infty} h'(k_t) = 1 - \delta < 1$
- The dynamics of capital will thus stabilize in $\bar{k} = h(\bar{k})$ and the possible range of capital is

$$k_t \in [0, \max\{\bar{k}, k_0\}]$$

Capital Dynamics Under Zero Consumption



Centralized vs Decentralized

- There are two ways to solve the model

1. Centralized way, ie from the point of view of a benevolent dictator, also called central planner
2. Decentralized way, from the point of view of households and firms

- Let's start with the central planner

2. Central Planner

Central Planner

- The central planner maximizes the utility of all households in the economy
- Since everyone is identical to one another, this amounts to maximizing the utility of the representative agent
- This will result in a Pareto efficient allocation
- A Pareto efficient allocation means one cannot reallocate to make one agent better off without making at least one other agent worse off

Problem of the Central Planner

- The central planner chooses consumption and capital to solve

$$
\begin{aligned}
\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{subject to} \quad c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \\
c_t \geq 0, \qquad k_{t+1} \geq 0, \qquad t = 0, 1, 2, \ldots \\
k_0 > 0 \quad \text{given}
\end{aligned}
$$

Bellman Equation

- Let $\tilde{k}$ denote the value of capital next period
- Plug the resource constraint into the utility function and write a Bellman equation

$$V(k) = \max_{\tilde{k} \in [0, f(k) + (1-\delta)k]} \left\{ u[f(k) + (1-\delta)k - \tilde{k}] + \beta V(\tilde{k}) \right\}$$

- From the law of motion of capital we know that $\tilde{k} \in [0, f(k) + (1-\delta)k]$

Necessary Conditions

- The first-order condition (FOC) with respect to $\tilde{k}$ is

$$-u'(c) + \beta V'(\tilde{k}) = 0$$

- The envelope theorem (ET) implies

$$V'(k) = [f'(k) + 1 - \delta]u'(c)$$

Euler Equation

- Combine the FOC and ET to obtain the Euler equation

$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{\text{marginal rate of substitution (MRS)}} = \underbrace{f'(k_{t+1}) + 1 - \delta}_{\text{marginal rate of transformation (MRT)}}$$

- The MRS is the rate at which the consumer is willing to give up some amount of consumption today in exchange for more consumption tomorrow
- The MRT is the amount of capital that must be used to produce an extra unit of output; here MRT = marginal product of capital net of depreciation

Intuition

- The Euler equation implies that MRS equal MRT
- At the optimum, the consumer is indifferent between 1) consuming more today and 2) investing in capital and consuming more tomorrow
- Agents smooth consumption over time: the growth rate of consumption depends on future income

Necessary Conditions...

- We have just derived the solution to the model
- Two necessary conditions are

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]$$
$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t$$

- The pair forms a system of two difference equations in two variables (c_t, k_t)

. . . But Not Sufficient

- As such we are not able to solve this system
- We need two boundary conditions

1. An initial condition for the capital stock: $k_0 > 0$
2. A transversality condition, a sort of terminal condition

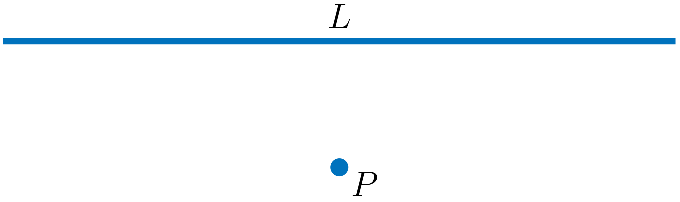
3. Transversality Condition

From Many Paths to One Path

- An Euler equation is a local condition stating that no gain can be achieved by slightly deviating from an optimal path for a short period of time
- If the terminal point is not fixed, there may be many paths satisfying the Euler equation
- The transversality condition enables us to single out the optimal path, or at least to rule out some non-optimal paths

Simple Analogy – Kamihigashi (2006)

- What is the shortest path from point P to line L?



Simple Analogy – Kamihigashi (2006)

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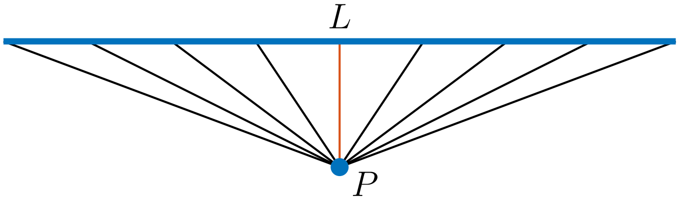


- The answer is the straight line from P to L that is perpendicular to L
- There are two conditions involved here

1. The first condition is that the shortest path be a straight line
2. The second condition is that the shortest path be perpendicular to L

Simple Analogy

- The first condition is analogous to the Euler equation: one cannot make the path shorter by deviating from it and eventually returning to it
- But there are many straight lines from P to L, some of which can be arbitrarily long, so that even very bad choices satisfy the Euler equation
- We thus need the second condition of perpendicularity: one cannot make the path shorter by deviating from it and never returning to it
- In dynamic optimization theory, this condition on end points is called the transversality condition

A Finite Horizon Problem

- Let's see another example that illustrates well the transversality condition
- Consider a finite-horizon problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\text{subject to} \quad c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$k_{t+1} \geq 0, \quad k_0 > 0 \quad \text{given}, \quad t = 0, 1, 2, \ldots, T$$

- Capital can never be negative, ie the agent cannot be in debt
- The world ends in T and the agent knows it

Lagrangian

- Write a Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t [f(k_t) + (1-\delta)k_t - c_t - k_{t+1}] + \sum_{t=0}^{T} \mu_t k_{t+1}$$

- λ_t is the Lagrange multiplier on the resource constraint
- μ_t is the Lagrange multiplier on the nonnegative capital constraint

First-Order Conditions

- Derive the necessary conditions and express them at $t = T$

$$\begin{aligned} \text{FOC for } c_T: &\quad \beta^T u'(c_T) = \lambda_T \\ \text{FOC for } k_{T+1}: &\quad \lambda_T = \mu_T \\ \text{Binding nonnegativity constraint:} &\quad \mu_T k_{T+1} = 0 \end{aligned}$$

- Combine the three FOCs

$$\beta^T u'(c_T) k_{T+1} = 0$$

Terminal Condition

- Repeat the previous FOC, evaluated at the final period $t = T$

$$\beta^T u'(c_T) k_{T+1} = 0$$

- Since $\beta^T u'(c_T) > 0$, we must have $k_{T+1} = 0$
- Intuitively, since the marginal utility of the consumer at time T is positive, the consumer wants to consume all her wealth before she dies
- This is the terminal condition
- It is a first-order condition, so it is necessary for optimality

Transversality Condition

- The transversality condition is like the terminal condition with $T \to \infty$

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0$$

- The transversality condition requires the present discounted value of wealth to converge to zero as the planning horizon reaches infinity
- Intuitively, capital should not grow too fast compared to its marginal value: if the consumer saves too much she is not behaving optimally
- The transversality condition thus rules out over-accumulation of wealth

Proof

- Kamihigashi (2001, *Econometrica*; 2002, *Economic Theory*) proves the necessity of the transversality condition
- See also Acemoglu (2009, Chapter 7)

4. Steady State

Definition

- A steady state is a point in the state space x^* such that

$$x_0 = x^* \quad \text{implies } x_t = x^* \quad \text{for all } t \geq 1$$

- Informally, "if you start there, you stay there"
- In other words, in the steady state all variables are constant

$$c_t = c_{t+1} = c^*; \qquad k_t = k_{t+1} = k^*$$

Steady-State Capital

- From the Euler equation we have $u'(c^*) = \beta u'(c^*)[f'(k^*) + 1 - \delta]$ or

$$f'(k^*) = \frac{1}{\beta} - 1 + \delta$$

- Since $\beta < 1$, we verify that $f'(k^*) > 0$
- Since f' is strictly decreasing ($f'' < 0$), we conclude that k^* is unique
- Therefore, there is a unique level of steady-state capital that depends only on parameters and is independent of the utility function, ie preferences

Steady-State Consumption

- From the capital law of motion we have $k^* = f(k^*) + (1-\delta)k^* - c^*$ or

$$c^* = f(k^*) - \delta k^*$$

- Therefore, there is a unique level of steady-state consumption that depends on the production function and parameters but not on the utility function
- The form of the utility function only affects the dynamics, not the level of endogenous variables in the stationary state

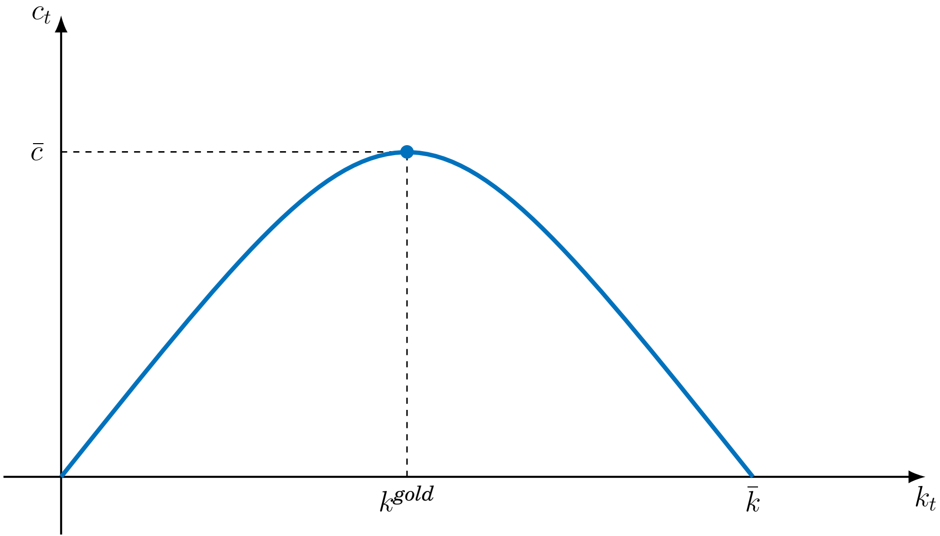
Golden Rule

- The golden rule is the level of capital (or savings rate) that maximizes consumption in the steady state
- In the basic Solow model, the golden rule is

$$f'(k^{gold}) = \delta$$

- Capital accumulates to the point where the marginal product of capital equals depreciation of capital

Golden Rule



Modified Golden Rule

- In the neoclassical growth model, we have

$$f'(k^*) = \delta + \frac{1}{\beta} - 1$$

- Define the discount rate ρ such that $\beta \equiv 1/(1+\rho)$

$$f'(k^*) = \delta + \rho$$

- This is the modified golden rule: the marginal product of capital equals depreciation plus the discount rate, which is the one-period interest rate

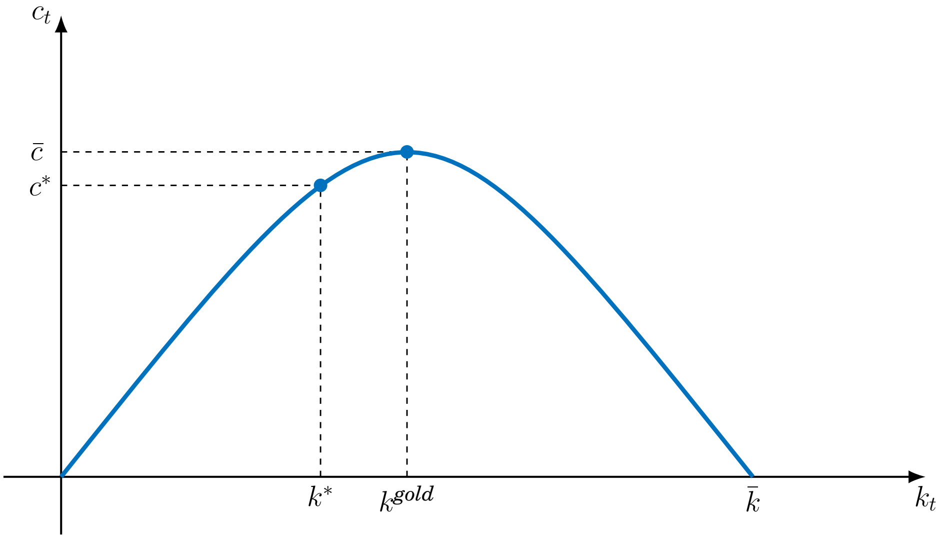
Modified Golden Rule

- The modified golden rule involves a level of capital stock that does not maximize steady-state consumption

$$\delta + \rho > \delta \text{ and } f'' < 0 \text{ imply } k^* < k^{gold}$$

- Agents are impatient and discount the future
- Thus they favor early consumption over late consumption

Modified Golden Rule



Example – Cobb-Douglas Function

- Let's take the previous Cobb-Douglas functional form $f(k_t) = k_t^\alpha$
- In the steady state we have

$$f(k^*) = (k^*)^\alpha \quad \text{and} \quad f'(k^*) = \alpha(k^*)^{\alpha-1}$$

- The level of capital in steady state is

$$k^* = \left(\frac{\alpha}{\beta^{-1} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

- A higher α, a higher β, and a lower δ all increase k^*

Transversality Condition

- Remember the transversality condition is

$$\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0$$

- Suppose $k_t \to k^*$ and $c_t \to c^* = f(k^*) - \delta k^*$
- Since $u' > 0$ and $\beta < 1$, we verify that the transversality condition does indeed hold in steady state

5. Dynamics

Convergence

- We have just shown that there exists a unique steady state
- Now we ask the question: do consumption and capital always converge to their unique steady-state value?
- The answer is yes, given our assumptions

Two-Dimensional System

- Our system has one control (c_t) and one state variable (k_t)

$$
\begin{aligned}
\text{Euler equation:}\quad & u'(c_t) = \beta u'(c_{t+1})[\alpha k_{t+1}^{\alpha-1} + 1 - \delta] \\
\text{Capital accumulation:}\quad & k_{t+1} = k_t^{\alpha} + (1-\delta)k_t - c_t \\
\text{Initial condition:}\quad & k_0 > 0 \text{ given} \\
\text{Transversality condition:}\quad & \lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0
\end{aligned}
$$

- In this two-dimensional setup we can use a phase diagram
- With more dimensions we must use other methods such as policy function iteration or perturbation around the steady state

Phase Diagram

- Phase diagrams are more natural in continuous time
- Here we draw one in discrete time with a slight abuse of notation
- Typically in a phase diagram we put the state state variable on the horizontal axis and the control variable on the vertical axis
- We thus use a (k_t, c_t) space

(k_t, c_t) Space

Isocline

- We want to find the sets of points, or isoclines, where state and control are not changing in the (k_t, c_t) space
- We call these sets of points

1. The constant consumption $c_{t+1}/c_t = 1$ isocline
2. The constant capital $k_{t+1}/k_t = 1$ isocline

- Let's see them in turn

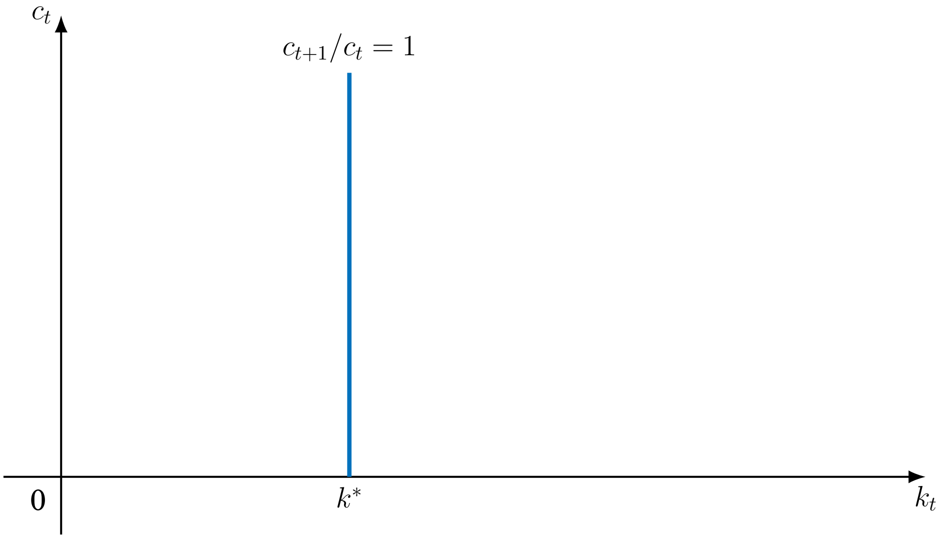
Constant Consumption Isocline

- The only value of k_{t+1} consistent with $c_{t+1}/c_t = 1$ is the steady-state level of capital stock

$$\frac{c_{t+1}}{c_t} = 1 : \qquad k_{t+1} = \left(\frac{\alpha}{\beta^{-1} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

- Thus the $c_{t+1}/c_t = 1$ isocline is a vertical line at $k_{t+1} = k^*$

Constant Consumption Isocline



Constant Capital Isocline

- We find the $k_{t+1}/k_t = 1$ isocline by using the capital accumulation equation

$$\frac{k_{t+1}}{k_t} = 1 : \qquad c_t = k_t^{\alpha} - \delta k_t$$

- Slight complication: k_t shows up in the $\frac{k_{t+1}}{k_t} = 1$ isocline, while k_{t+1} appears in the $\frac{c_{t+1}}{c_t} = 1$ isocline
- This would not be the case in continuous time; we circumvent the issue by assuming $k_{t+1} \approx k_t$ and treat k_{t+1} as k_t in the $\frac{c_{t+1}}{c_t} = 1$ isocline

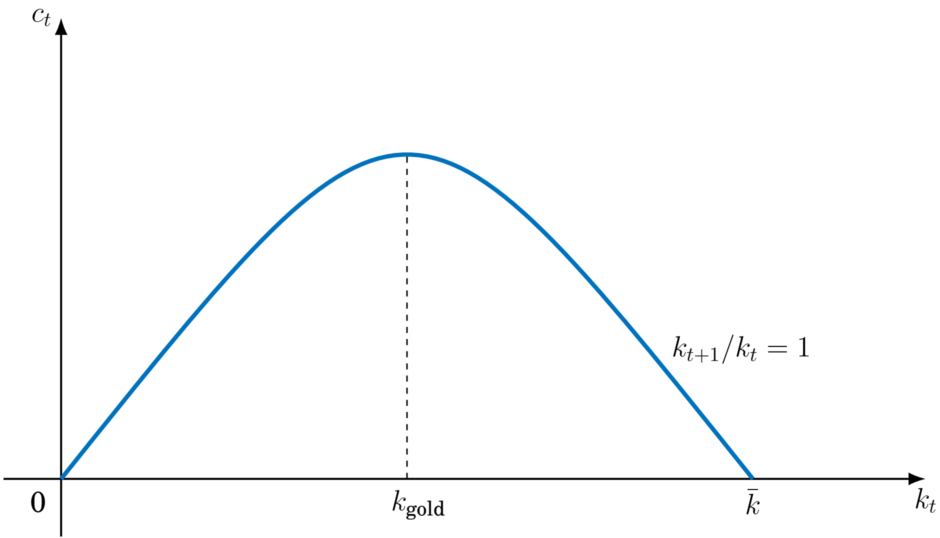
Constant Capital Isocline

- The first derivative of the $k_{t+1}/k_t = 1$ isocline is

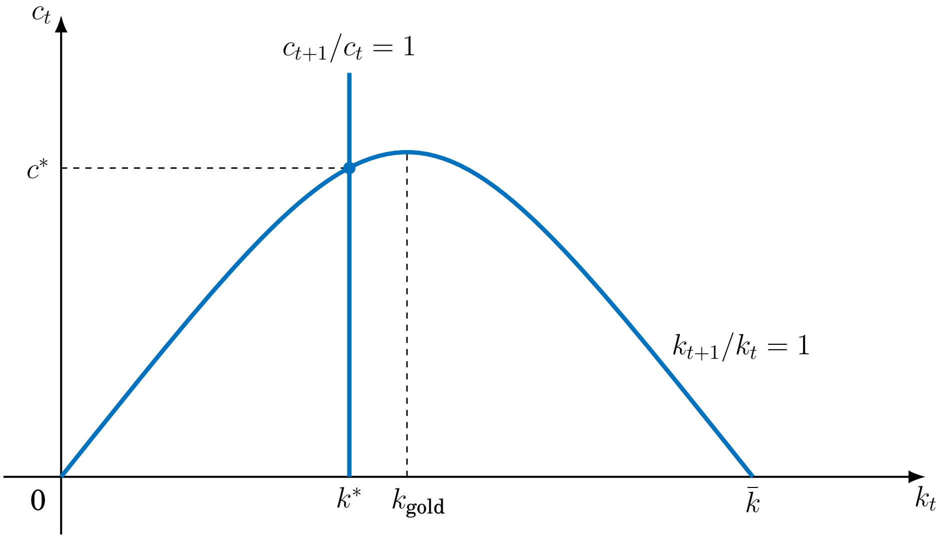
$$\frac{dc_t}{dk_t} = \alpha k_t^{\alpha-1} - \delta$$

- When k_t is small, $\alpha k_t^{\alpha-1}$ is large, the slope is positive
- When k_t is large, the slope is negative and tends to $-\delta$
- The peak occurs at $k_t = (\alpha/\delta)^{\frac{1}{1-\alpha}} > k^*$

Constant Capital Isocline



Combining the Two Isoclines



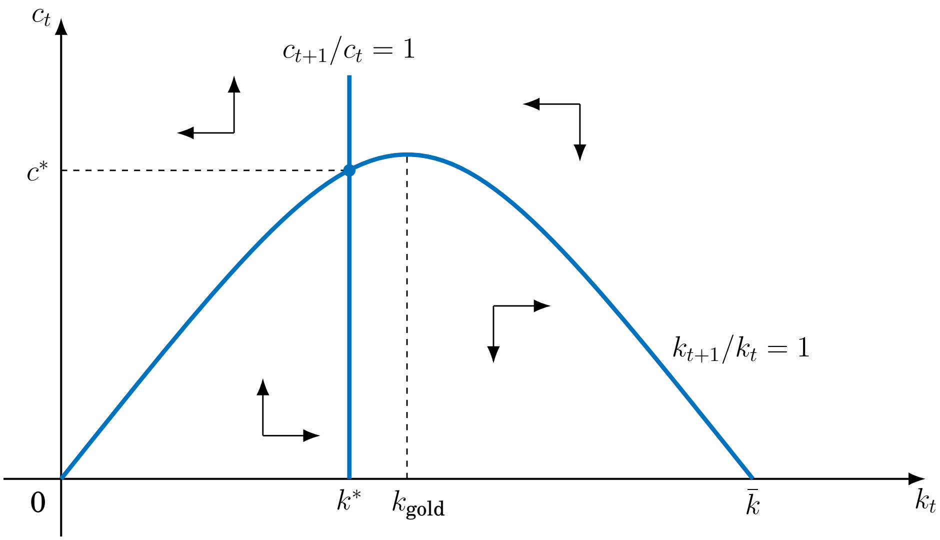
Remarks

- The steady state (k^*, c^*) is where the two isoclines cross
- k^{gold} is the level of capital that maximizes consumption
- $\bar{k}$ is the maximum possible level of capital, attained if $c = 0$
- Now, once we have the two isoclines we want to draw the dynamics

Arrows and Regions

1. Below the $k_{t+1}/k_t = 1$ line, c_t is too low, agents save too much, hence $k_{t+1} > k_t$, ie k_t increases: we draw a right-pointing arrow
2. Above the $k_{t+1}/k_t = 1$ line, c_t is too high, agents save too little, hence $k_{t+1} < k_t$, ie k_t decreases: we draw a left-pointing arrow
3. To the right of the $c_{t+1}/c_t = 1$ line, k_t is too large, the MPK is too low, $c_{t+1} < c_t$, ie c_t decreases: we draw a down-pointing arrow
4. To the left of $c_{t+1}/c_t = 1$ line, k_t is too small, the MPK is too high, $c_{t+1} > c_t$, ie c_t increases: we draw an up-pointing arrow

Arrows and Regions



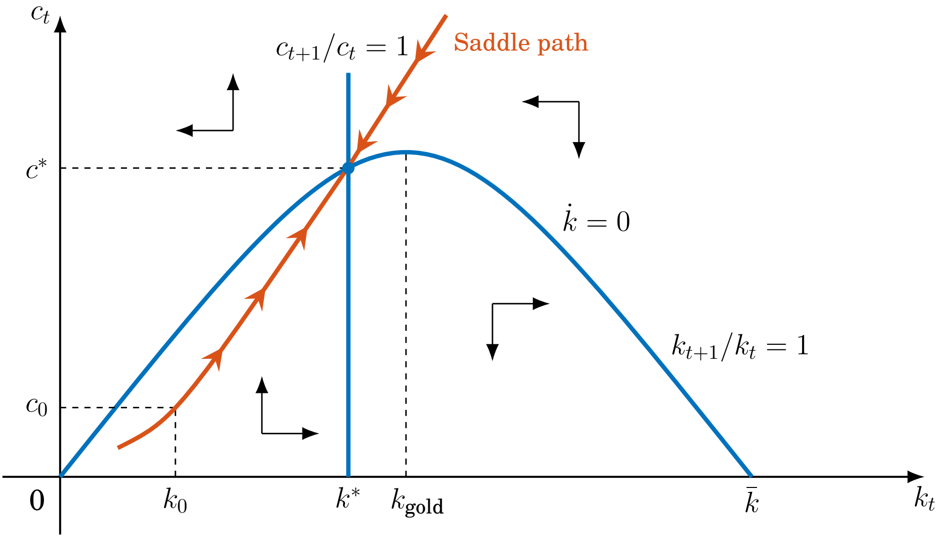
Convergence

- The key assumptions of the neoclassical growth model are

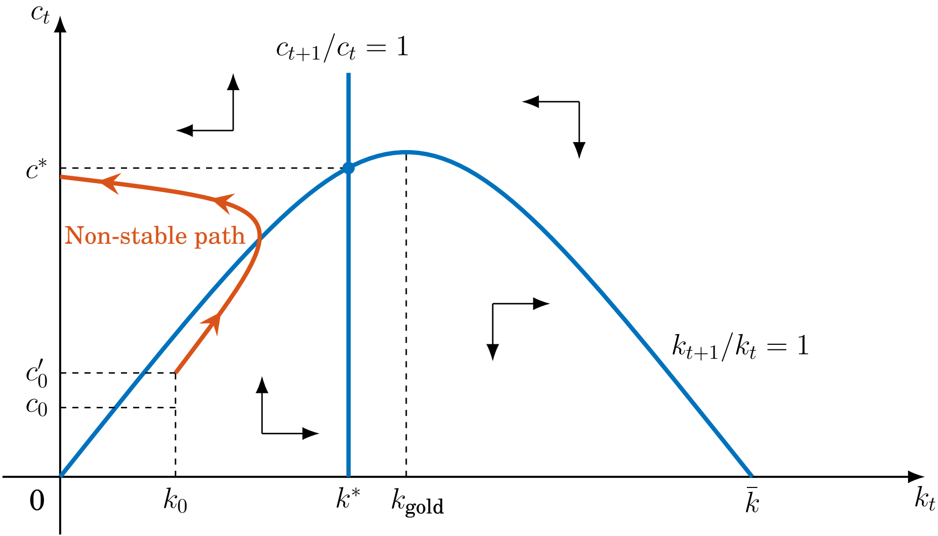
1. Neoclassical production: continuous, differentiable production function with constant returns to scale and diminishing marginal products
2. Standard preferences: increasing, concave, differentiable utility function

- Under these assumptions and starting from any $k_0 > 0$, there exists a unique equilibrium path converging to the unique steady state (k^*, c^*)
- Capital increases up to k^* if $k_0 < k^*$, decreases down to k^* if $k_0 > k^*$

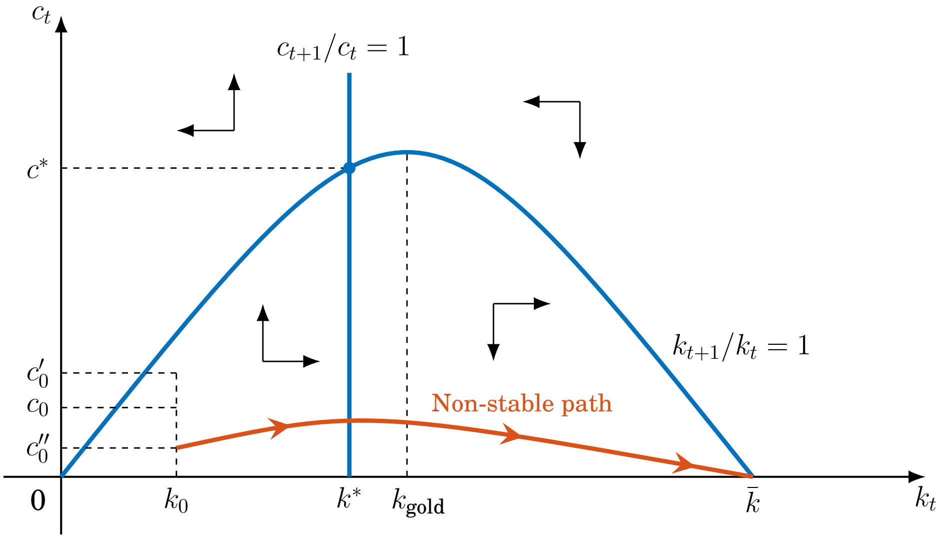
The Phase Diagram and the Unique Saddle Path



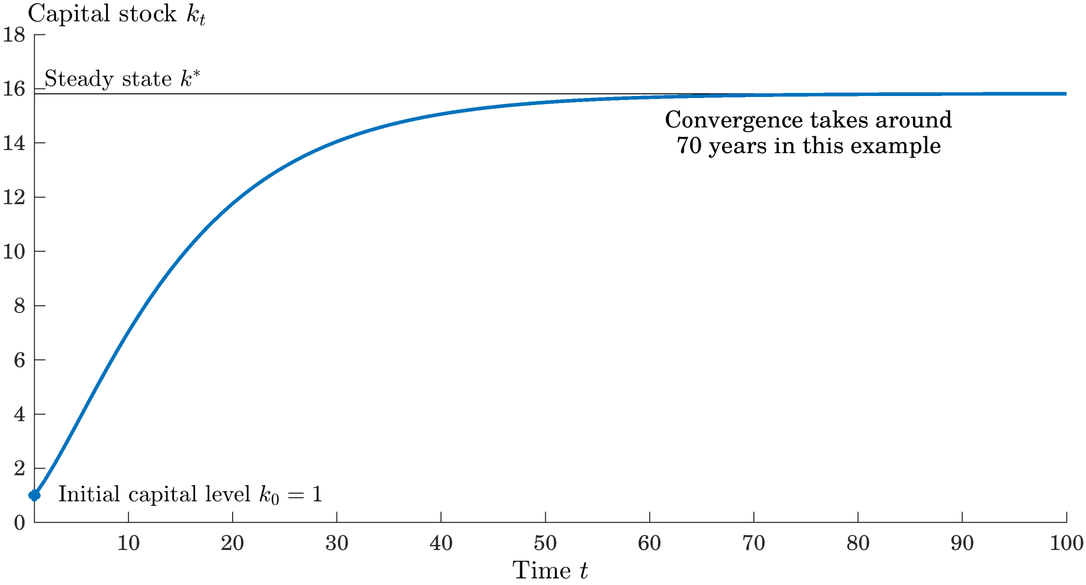
A Non-Stable Arm



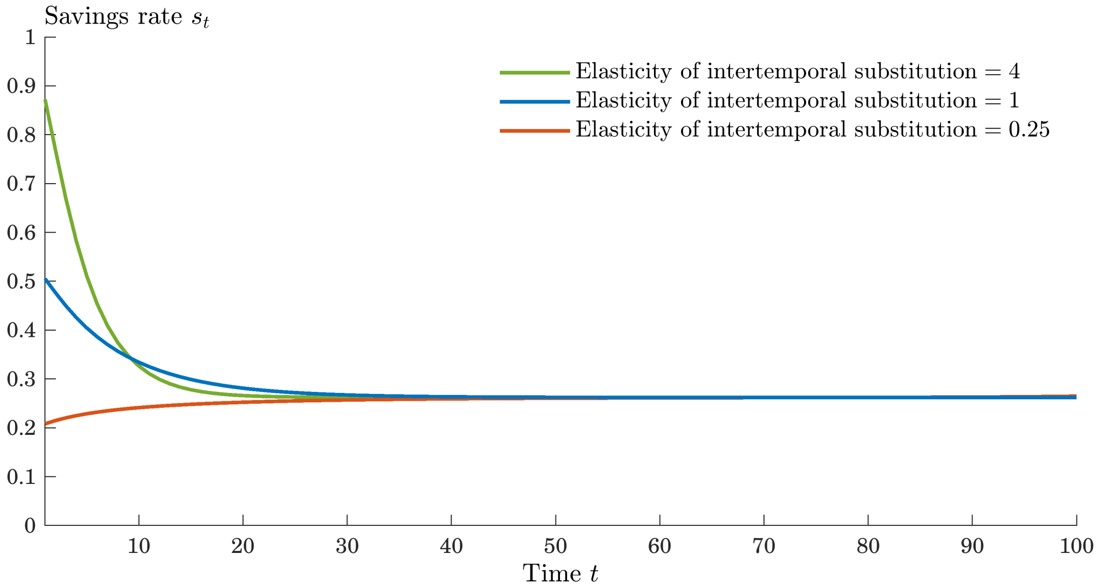
Another Non-Stable Arm



Dynamics of Capital Accumulation



Savings Rate Along the Equilibrium Path



Solow vs Ramsey

- In the Solow growth model, consumption and investment are constant and exogenous fractions of output, $c_t = (1 - \phi)y_t$ and $s_t = \phi y_t$
- In the neoclassical growth model, consumption is endogenous and depends on the return to capital

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]$$

- But the dynamics of the two models are nearly identical
- Capital and consumption converge monotonically to a stationary state, regardless of the initial condition

A Similar Conclusion

- We draw two conclusions

1. As in the Solow growth model, long-term growth in output per capita cannot result from simply accumulating capital
2. Like the Solow growth model, the neoclassical growth model is unable to explain long-term growth in output per capita

- Endogenous consumption-saving decisions do not make a difference

6. CRRA Utility

Constant Relative Risk Aversion

- So far we have not specified any functional form for the utility function
- If we want to study the precise dynamics of the model and compare these to the data we have to assume a particular form
- The form we choose is called constant relative risk aversion (CRRA)
- Also known as isoelastic or power utility function

CRRA Utility

- A CRRA utility function takes the form

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \qquad \sigma > 0$$

- σ is a parameter that has two interpretations (next slides)
- The term -1 in the numerator can be omitted, except if $\sigma = 1$ and $u = \ln c$
- The denominator $1 - \sigma$ serves to simplify the derivative of the function
- If $\sigma = 0$, then $u(c) = c - 1$, utility is linear in consumption, ie the agent is risk neutral, and the function no longer satisfies the Inada conditions

Common in Macro

- The CRRA form is the most popular utility function used in macroeconomic theory thanks to its several advantages
- It is easy to derive and interpret
- It nests the log utility function
- It equalizes risk aversion with inverse intertemporal elasticity
- It is consistent with a balanced growth path

Special Case – Log Preferences

- What if $\sigma = 1$? We apply l'Hôpital's rule: if f and g are differentiable and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

- The utility function writes $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma} = \frac{c^{1-\sigma}}{1-\sigma} - \frac{1}{1-\sigma}$
- Apply l'Hôpital's rule

$$\lim_{\sigma\to 1}\frac{c^{1-\sigma}-1}{1-\sigma} = \lim_{\sigma\to 1}\frac{\frac{d}{d\sigma}e^{(1-\sigma)\ln c}}{\frac{d}{d\sigma}(1-\sigma)} - 0 = -\lim_{\sigma\to 1} e^{(1-\sigma)\ln c}(-\ln c) = \ln c$$

- Thus if $\sigma = 1$ we have log preferences

Relative Risk Aversion

- The coefficient of relative risk aversion is defined by

$$\begin{aligned}\text{Relative risk aversion} &= -c\frac{u''(c)}{u'(c)} \\ &= -\frac{c(-\sigma)c^{-\sigma-1}}{c^{-\sigma}} \\ &= \sigma\end{aligned}$$

- Thus σ is the coefficient of relative risk aversion
- If $\sigma > 0$, consumers are risk averse; the higher σ, the higher the aversion

Elasticity of Intertemporal Substitution

- The elasticity of intertemporal substitution measures the willingness of the consumer to substitute future consumption for present consumption

$$
\begin{aligned}
\text{Elasticity of intertemporal substitution} &= -\frac{d\ln\left(\frac{c_{t+1}}{c_t}\right)}{d\ln\left(\frac{u'(c_{t+1})}{u'(c_t)}\right)} \\
&= -\frac{d\ln\left(\frac{c_{t+1}}{c_t}\right)}{d\left[-\sigma\ln\left(\frac{c_{t+1}}{c_t}\right)\right]} \\
&= \frac{1}{\sigma}
\end{aligned}
$$

- Thus σ is the inverse elasticity of intertemporal substitution

Response to a Change in the Real Interest Rate

- Using the Euler equation, $u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]$, we get

$$-\frac{d\ln\left(c_{t+1}/c_t\right)}{d\ln\left[u'(c_{t+1})/u'(c_t)\right]} = \frac{d\ln\left(c_{t+1}/c_t\right)}{d\ln\left[f'(k_{t+1}) + 1 - \delta\right]} = \frac{1}{\sigma}$$

- Recall, $f'(k_{t+1}) + 1 - \delta$ is the real interest rate, ie the return on investment
- So the elasticity of intertemporal substitution is also a measure of the responsiveness of the consumption growth rate to the real interest rate
- In other words, σ governs how much households change their consumption-saving behavior following a change in the interest rate

Income and Substitution Effects

- The elasticity of intertemporal substitution governs the relative strength of two opposing forces: the income effect and the substitution effect
- The income effect is the change in demand for goods and services resulting from a change in income (labor income, capital income, currency swing)
- The substitution effect is the change in demand for goods and services resulting from a change in relative prices (prices, wages, interest rates)

Which Effect Dominates?

- What happens when the real interest rate $R_t \equiv f'(k_{t+1}) + 1 - \delta$ goes up?
- This means higher capital income and a higher relative return to capital

1. If $\sigma > 1$, EIS < 1, c_{t+1}/c_t increases proportionally less than R_{t+1}: the income effect dominates, household feel richer, save less, consume more
2. If $\sigma < 1$, EIS > 1, c_{t+1}/c_t increases proportionally more than R_{t+1}: the substitution effect dominates, households save a higher fraction of income
3. If $\sigma = 1$, EIS $= 1$, c_{t+1}/c_t increases in the same proportion as R_{t+1}: income and substitution effects cancel out, the saving rate remains unchanged

7. Adding Growth

Adding Growth

- How can the model generate long-term economic growth?
- One way is to assume the existence of exogenous technological change
- Consider the production function

$$Y_t = F(K_t, L_t, A_t)$$

- A_t is productivity or knowledge

Labor-Augmenting Technological Change

- Let's assume technological progress is labor-augmenting or Harrod-neutral

$$Y_t = F(K_t, A_t L_t)$$

- We refer to $A_t L_t$ as effective labor

Alternatives

- Alternatively, technological progress could be capital-augmenting

$$Y_t = F(A_t K_t, L_t)$$

- Or it could be Hicks-neutral

$$Y_t = A_t F(K_t, L_t)$$

- In that last case, A_t would represent total factor productivity (TFP)

Labor-Augmenting Technological Change

- We assume that productivity grows at a constant rate

$$A_{t+1} = (1+\mu)A_t \quad \text{where } \mu > 0 \text{ and } A_0 > 0 \text{ is given}$$

- This implies

$$\begin{aligned} A_1 &= (1+\mu)A_0 \\ A_2 &= (1+\mu)A_1 = (1+\mu)^2 A_0 \\ &\vdots \\ A_t &= (1+\mu)^t A_0 \end{aligned}$$

Units of Effective Labor

- Define capital and consumption per unit of effective labor

$$\tilde{k}_t \equiv \frac{K_t}{A_t L_t} \qquad \tilde{c}_t \equiv \frac{C_t}{A_t L_t}$$

- This implies

$$A_t f(\tilde{k}_t) = A_t F\left(\frac{k_t}{A_t}, 1\right)$$

- The resource constraint becomes

$$\begin{aligned} c_t + A_{t+1}\tilde{k}_{t+1} &= A_t f(\tilde{k}_t) + (1-\delta)A_t\tilde{k}_t \\ \tilde{c}_t + (1+\mu)\tilde{k}_{t+1} &= f(\tilde{k}_t) + (1-\delta)\tilde{k}_t \end{aligned}$$

Problem of the Central Planner

- The problem of the central planner writes

$$
\begin{aligned}
\max_{\{\tilde{c}_t,\tilde{k}_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(A_t \frac{c_t}{A_t}\right) = \max_{\{\tilde{c}_t,\tilde{k}_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[\beta(1+\mu)^{1-\sigma}\right]^t \frac{A_0^{1-\sigma}\tilde{c}_t^{1-\sigma}}{1-\sigma} \\
\text{subject to} \quad \tilde{c}_t + (1+\mu)\tilde{k}_{t+1} = f(\tilde{k}_t) + (1-\delta)\tilde{k}_t \\
\tilde{c}_t \geq 0, \qquad \tilde{k}_{t+1} \geq 0, \qquad \tilde{k}_0 > 0 \quad \text{given}
\end{aligned}
$$

- We need to assume that $\beta(1+\mu)^{1-\sigma} < 1$

Dynamic Programming

- Write a Bellman equation

$$V(\tilde{k}) = \max_{\tilde{s}} \left\{ \frac{A_0^{1-\sigma}}{1-\sigma} [f(\tilde{k}) + (1-\delta)\tilde{k} - (1+\mu)\tilde{s}]^{1-\sigma} + \beta(1+\mu)^{1-\sigma} V(\tilde{s}) \right\}$$

- Derive the first-order and envelope conditions

$$A_0^{1-\sigma}(1+\mu)\tilde{c}^{-\sigma} = \beta(1+\mu)^{1-\sigma} V'(\tilde{s})$$

$$V'(\tilde{k}) = A_0^{1-\sigma}[f'(\tilde{k}) + 1 - \delta]\tilde{c}^{-\sigma}$$

- Combine the two equations and obtain the Euler equation

$$(1+\mu)^{\sigma} \tilde{c}_t^{-\sigma} = \beta[f'(\tilde{k}_{t+1}) + 1 - \delta]\tilde{c}_{t+1}^{-\sigma}$$

Steady State

- In steady state, the Euler equation implies

$$f'(\tilde{k}^*) = \frac{(1+\mu)^\sigma}{\beta} - 1 + \delta$$

- The capital accumulation equation implies

$$\tilde{c}^* = f(\tilde{k}^*) + (1-\delta)\tilde{k}^* - (1+\mu)\tilde{k}^*$$

- There is a unique steady-state level of capital and consumption per unit of effective labor $\tilde{k}^*$ and $\tilde{c}^*$

Convergence

- Under our assumptions (in particular $\beta(1+\mu)^{1-\sigma} < 1$), for any given $\tilde{k}_0 > 0$, there is a unique equilibrium path converging to the steady state $(\tilde{k}^*, \tilde{c}^*)$
- This means that once the economy has converged, capital, consumption, and output per unit of effective labor are constant: $\tilde{k}_t = \tilde{k}^*$, $\tilde{c}_t = \tilde{c}^*$, and $\tilde{y}_t = \tilde{y}^*$
- But remember, $k_t = A_t\tilde{k}$, $c_t = A_t\tilde{c}_t$, and $y_t = A_t\tilde{y}$: thus in steady state, capital, consumption, and output per capita are not constant, they grow

Steady State

- Asymptotic capital is

$$k_t = A_t \tilde{k}^* \quad \Longrightarrow \quad k_{t+1} = A_{t+1}\tilde{k}^* = (1+\mu)A_t\tilde{k}^* = (1+\mu)k_t$$

- Asymptotic output is

$$y_t = A_t f(\tilde{k}^*) \quad \Longrightarrow \quad y_{t+1} = A_{t+1} f(\tilde{k}^*) = (1+\mu)A_t\tilde{y}^* = (1+\mu)y_t$$

- Asymptotic consumption is

$$c_t = A_t f(\tilde{k}^*) + (1-\delta)A_t\tilde{k}^* - (1+\mu)A_t\tilde{k}^*$$

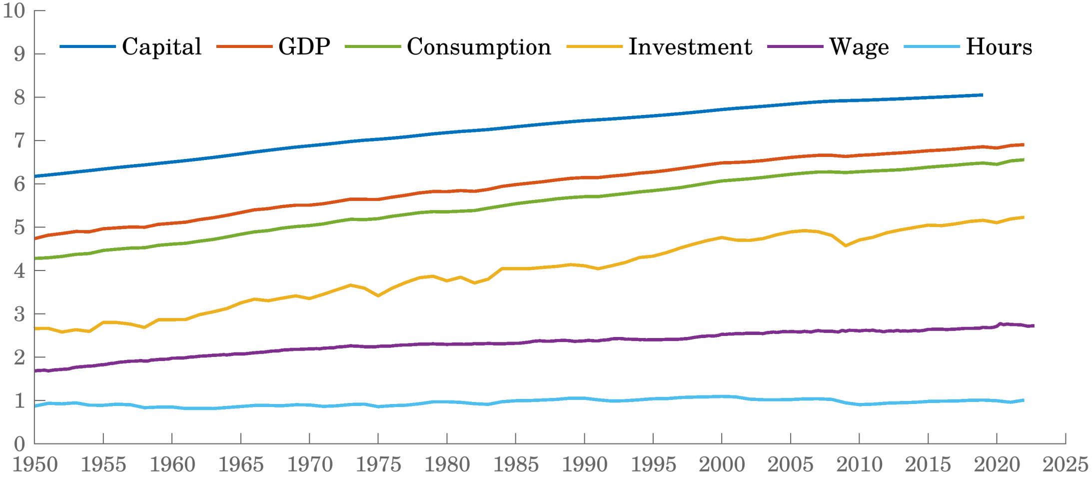
$$c_t = A_t \underbrace{\left[f(\tilde{k}^*) + (1-\delta)\tilde{k}^* - (1+\mu)\tilde{k}^*\right]}_{\tilde{c}^*}$$

$$\Longrightarrow c_{t+1} = A_{t+1}\tilde{c}^* = (1+\mu)A_t\tilde{c}^* = (1+\mu)c_t$$

Balanced Growth Path

- In summary, with labor-augmenting technological change and constant relative risk aversion preferences, we obtain the following result
- Once the economy converges, output, consumption, capital, and investment grow at a the same and constant rate μ, while hours remain constant
- This is what we call a balanced growth path
- Balanced growth means the ratio of capital stock to output does not change
- What about the wage? and the return to capital?

Balanced Growth



Note: All variables are logged; hours are per capita; Sources: Penn World Table 10, US Bureau of Economic Analysis, US Bureau of Labor Statistics

8. Competitive Equilibrium

Competitive Equilibrium

- So far we have solved the model from the point of view of a central planner
- This results in a Pareto efficient allocation
- Now we are going to decentralize the economy
- For this we impose the notion of competitive equilibrium

Firms

- There is a large number of identical firms
- All firms are equal so we can talk of a representative firm
- We assume free entry and exit: this prevents existing firms from making a profit or a loss

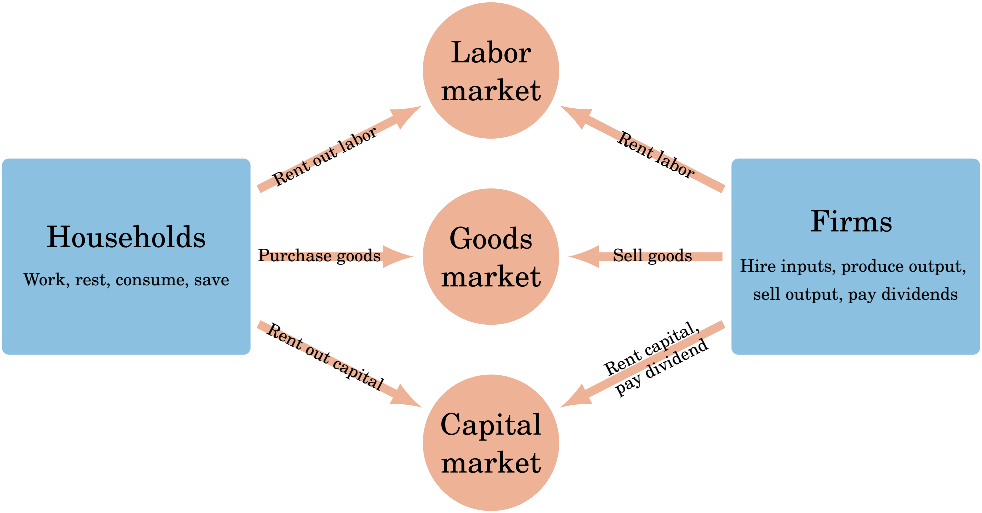
Competitive Markets

- There are three markets in the economy

1. Goods market
2. Capital market
3. Labor market

- All markets are competitive

Model Diagram



Production Function

- The representative firm takes prices as given
- It rents out labor and capital services from households
- It combines the two inputs to produce an output good Y_t, using the same technology as previously

$$F(K_t, L_t, A_t)$$

- We assume that productivity is constant, $A_t = A = 1$

$$F(K_t, L_t)$$

Problem of the Firm

- The representative firm maximizes profit

$$\max_{K_t, L_t} \{F(K_t, L_t) - r_t K_t - w_t L_t\}$$

- r_t is the rental rate of capital
- w_t is the wage rate of labor

First-Order Conditions

- Perfect competition means the firm makes zero profit
- The first-order conditions for the firm's problem are

$$
\begin{aligned}
K_t: \quad & r_t = F_K(K_t, L_t) \\
L_t: \quad & w_t = F_L(K_t, L_t)
\end{aligned}
$$

- The firm demands inputs such that its marginal product of capital equals the capital rental rate and its marginal product of labor equals the wage

Euler's Theorem

- Euler's theorem and the zero profit condition imply

$$\begin{aligned}
F(K_t, L_t) - r_t K_t - w_t L_t &= 0 \\
F(K_t, L_t) - F_K(K_t, L_t) K_t - F_L(K_t, L_t) L_t &= 0 \\
[F(K_t/L_t, 1) - r_t K_t/L_t - w_t] L_t &= 0
\end{aligned}$$

- From the firm's point of view, the capital-labor ratio is the relevant variable
- The firm absorbs any supply of labor L_t as long as K_t/L_t is constant

Example – Cobb-Douglas Production Function

- Assume the production function takes the form

$$Y = F(K, L) = K^{\alpha} L^{1-\alpha}$$

- We have

$$\frac{rK}{Y} = \frac{F_K(K, L)K}{F(K, L)} = \frac{\alpha K^{\alpha-1} L^{1-\alpha} K}{K^{\alpha} L^{1-\alpha}} = \alpha$$

$$\frac{wL}{Y} = \frac{F_L(K, L)L}{F(K, L)} = \frac{(1-\alpha) K^{\alpha} L^{-\alpha} L}{K^{\alpha} L^{1-\alpha}} = 1 - \alpha$$

- α is the share of income that remunerates capital
- $1 - \alpha$ is the share of income that remunerates labor

Households

- The representative household is endowed with one unit of time and $\hat{k}_0 > 0$ units of capital
- The household rents out capital and supplies inelastic labor to the firm
- Given $\hat{k}_0 > 0$ and a sequence of $\{w_t\}_{t=0}^{\infty}$ and $\{r_t\}_{t=0}^{\infty}$, the household solves

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{subject to} \quad \hat{k}_{t+1} = (r_t + 1 - \delta)\hat{k}_t - c_t + w_t$$

$$c_t \geq 0, \qquad \hat{k}_{t+1} \geq 0, \qquad t = 0, 1, 2, \ldots$$

Households

- We solve the household problem in the same way as before
  1. Write a Bellman equation
  2. Derive the necessary first-order and envelope conditions
  3. Combine the two necessary conditions into one Euler equation
- We arrive at the following Euler equation

$$u'(c_t) = \beta(r_{t+1} + 1 - \delta)u'(c_{t+1})$$

Competitive Equilibrium

- We have solved the problem of the firm
- We have solved the problem of the household
- We are now in a position to define a competitive equilibrium

Competitive Equilibrium

- A competitive equilibrium is a sequence of $\{Y_t, c_t, K_t, L_t, w_t, r_t\}_{t=0}^{\infty}$ such that

1. Given prices $\{w_t, r_t\}_{t=0}^{\infty}$ and initial capital stock k_0, the representative household solves its problem

$$u'(c_t) = \beta(r_{t+1} + 1 - \delta)u'(c_{t+1}) \quad \text{and} \quad \lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0$$

2. Given prices $\{w_t, r_t\}_{t=0}^{\infty}$, the representative firm solves its problem

$$r_t = F_K(K_t, L_t) \quad \text{and} \quad w_t = F_L(K_t, L_t)$$

3. All markets clear, ie supply equals demand

$$\begin{aligned} \text{Goods market:} &\quad Y_t = C_t + I_t = C_t + K_{t+1} - (1-\delta)K_t \\ \text{Capital market:} &\quad \hat{k}_t = k_t \\ \text{Labor market:} &\quad 1 = L_t \end{aligned}$$

Equivalence

- Substitute the firm's FOC for capital into the Euler equation

$$u'(c_t) = \beta[f'(k_{t+1}) + 1 - \delta]u'(c_{t+1})$$

- This is the exact same equation as in the central planner's problem
- We conclude that the decentralized, ie competitive, equilibrium and the central planner's equilibrium are identical

Fundamental Theorems of Welfare Economics

First theorem of welfare economics, “the invisible hand”

In an economy with complete markets, zero transaction costs, and perfect information, any competitive equilibrium is Pareto optimal

Second theorem of welfare economics

Assuming price-taking and free entry, any Pareto efficient allocation can be attained by a competitive equilibrium

9. Explaining the Data

Relative Success...

- In the steady state, the neoclassical growth model can explain some characteristics of long-term growth
- Output, consumption, capital and wages grow at a relatively constant rate
- Hours are relatively constant in the long run

. . . And Failure

- But the neoclassical growth model does not explain long-term growth and does not explain the difference in per-capita income across countries
- It cannot explain growth miracles (Japan, China) or growth disasters (Argentina, Africa)
- Many other factors are determinant: institutions, politics, geography
- To learn more about economic growth, read Acemoglu (2009, Chap 1–4) and Romer (2019, Chap 1–4)

10. Exercises

Exercise 1 – Expropriation Risk

Consider the standard neoclassical growth model. For each problem, determine the state variables, write the central planner's Bellman equation, and derive the Euler equation.

1. Risk of expropriation. Consider the production function

$$y_t = \begin{cases} f(k_t) & \text{with probability } 1-p \\ 0 & \text{with probability } p \end{cases}$$

   Assume $f(k_t) = k_t^\alpha$, $k_0 > 0$ given, $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$, $\delta = 1$, and $p = 1/2$. Characterize analytically the optimal path of consumption and capital.

Exercise 1 – Population Growth

2. Population growth. Consider an initial population $L_0 = 1$. Suppose that population grows at a constant rate n, $L_{t+1} = (1+n)L_t$. Given $K_0 > 0$ the central planner solves

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t L_t u(C_t/L_t)$$

$$\text{subject to} \quad C_t + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t$$

$$C_t \geq 0, \qquad K_{t+1} \geq 0, \qquad t = 0, 1, 2, \ldots$$

Does aggregate consumption grow in the steady state? Does consumption per capita grow in the steady state?

Exercise 1 – Habit Formation

3. Habit formation. Consider the instantaneous utility function

$$u(c_t - \gamma c_{t-1})$$

   where $u' > 0$, $u'' < 0$ and $\gamma \geq 0$. Interpret the utility function and give an intuition for the Euler equation.

Exercise 2 – Endogenous Labor Supply

Consider the neoclassical growth model where time can be allocated to labor L or leisure $1 - L$. The agent derives utility from consumption and leisure

$$u(C, 1-L) = \frac{\left[C^{1-\theta}(1-L)^{\theta}\right]^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0, \quad 0 < \theta < 1$$

Suppose technological change $A_{t+1} = (1+\mu)A_t$, with $A_0 = 1$ and $\mu > 0$. The production function is

$$F(A_t, K_t, L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \qquad \alpha \in (0,1)$$

The resource constraint is

$$C_t + K_{t+1} = K_t^{\alpha}(A_t L_t)^{1-\alpha} + (1-\delta)K_t$$

Exercise 2 – Continued

1. Write the problem of the central planner. The planner now chooses a sequence of hours $\{L_t\}_{t=0}^{\infty}$ to maximize the representative agent's utility.
2. Show that in the steady state output, consumption, and capital grow at a constant rate, while hours worked (and leisure) are constant.
3. Using the competitive equilibrium, show that the wage grows at a constant rate in the steady state.

Exercise 3 – Durable Goods

Given k_0 and s_{-1}, consider the problem of the central planner

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(s_t)$$

$$\begin{aligned} \text{subject to} \quad c_t + k_{t+1} &= f(k_t) + (1-\delta)k_t \\ s_t &= (1-\phi)s_{t-1} + c_t \\ c_t \geq 0, \qquad k_{t+1} \geq 0, & \qquad t = 0, 1, 2, \ldots \end{aligned}$$

where s_t is the stock of durable goods, which depreciate at rate $\phi \in (0, 1)$.

1. Write the Bellman equation.
2. Derive the Euler equation.
3. Suppose the planner must pay fixed cost $b > 0$ to produce the durable good

$$\begin{cases} c_t + b & \text{if } c_t > 0 \\ 0 & \text{if } c_t = 0 \end{cases}$$

   Write the Bellman equation of this problem.