6. Equilibrium with Complete Markets

Time Zero Trading

Yvan Becard
PUC-Rio

Macroeconomics I, 2023

From Deterministic to Stochastic

- In the neoclassical growth model of the previous lecture, the environment was deterministic, ie there was no uncertainty in the economy
- This simplifies the analysis but is not realistic: in the real world many, if not most, aspects of our lives are uncertain
- Today we are going to incorporate uncertainty and see how economic agents behave in such a stochastic environment

Endowment Economy

- We study uncertainty in a simple pure exchange infinite horizon economy
- An exchange, or endowment, economy is a world in which agents bring their own endowment and exchange goods among them based on a price system
- Pure exchange means all agents are consumers, there is no production
- Agents cannot save, store, invest, or transform their endowment for the next period; all they can do is eat or exchange their endowment
- Lucas (1978) first used this concept in order to compute asset prices

Stochastic Endowment

- Although there is no production and hence no labor or capital income, agents receive an exogenous endowment each period
- The level of this endowment is uncertain and varies over time
- We are going to characterize the economy's competitive equilibrium
- This model is the foundational framework to study consumption decisions, risk sharing, asset pricing, and later incomplete markets

Complete Markets

- In this lecture we encounter the all-important concept of complete market, or complete system of markets, also known as Arrow-Debreu market
- This is a building block of macroeconomics and finance, underlying all representative-agent models
- Although unrealistic, complete markets are a very useful benchmark

Two Types of Complete Markets

- We are going to study two systems of complete markets

1. An Arrow-Debreu structure with a complete market in dated contingent claims, all traded at time 0 (lecture 6)
2. A sequential-trading structure with a complete market in one-period Arrow securities, traded at each period (lecture 7)

- We will show that these two systems are equivalent

Perfect First, Imperfect Later

“Economies that work smoothly because they are full of perfect markets are neither interesting nor realistic. But because perfect markets provide such a clear benchmark, economists find it much easier to start from them and work out what is going wrong, rather than start from scratch and work out what is going right.”

Tim Harford, 2012, *The Undercover Economist*

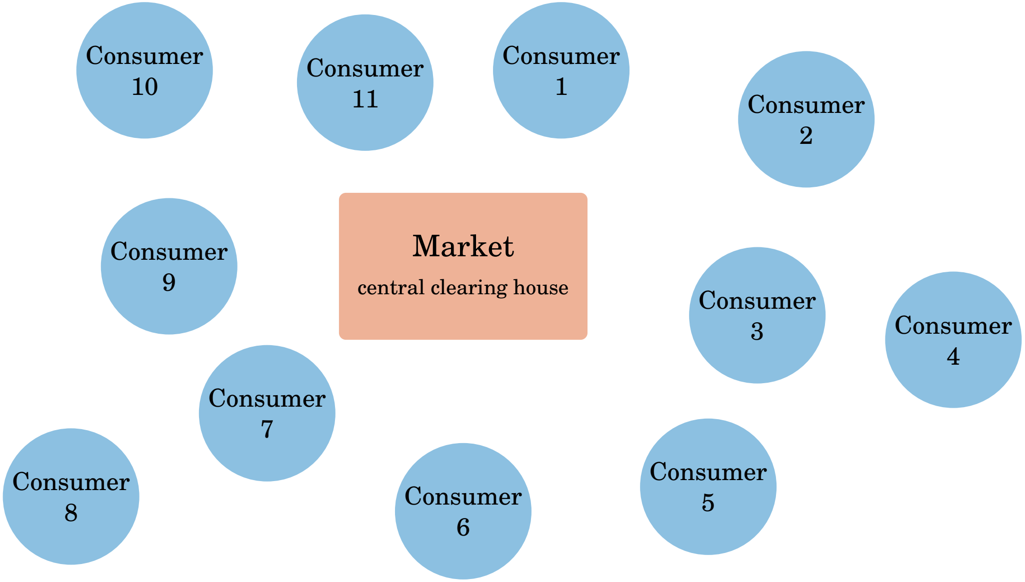
Lecture Outline

1. Model Setup
2. Pareto Problem
3. Time Zero Trading: Arrow-Debreu Securities
4. Risk Sharing and Consumption
5. Asset Pricing
6. Exercises

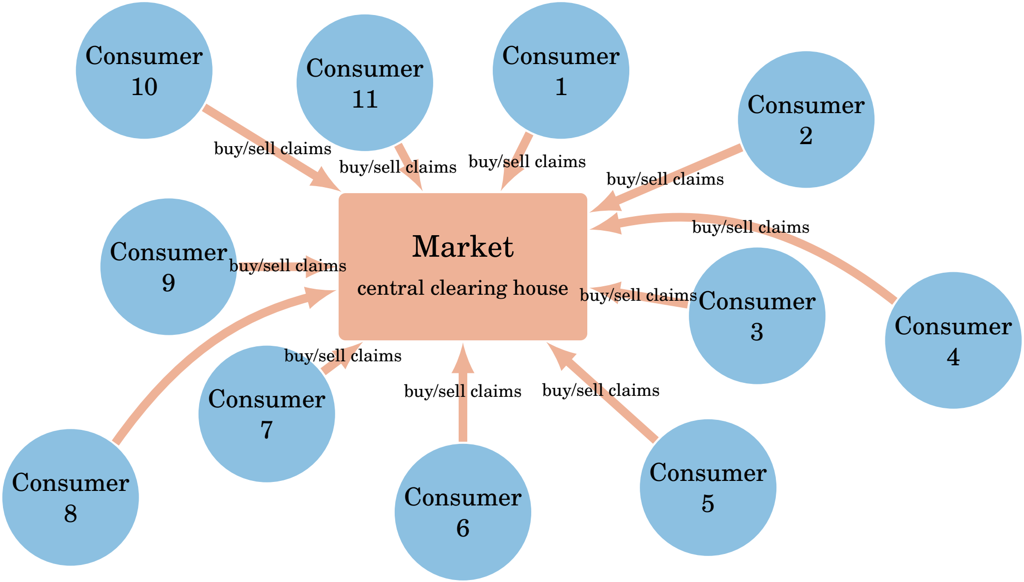
Main Reference: Ljungqvist and Sargent, 2018, *Recursive Macroeconomic Theory*, Fourth Edition, Chapter 8

1. Model Setup

Overview of the Model



Consumers Trade Among Themselves on the Market



Stochastic State

- In each period $t \geq 0$, there is a realization of a stochastic event $s_t \in \boldsymbol{S}$
- s_t is an aggregate state variable affecting all agents in the economy
- Let the history of events up until period t be $s^t = [s_0, s_1, \ldots, s_t]$; note that s_t is a particular event while s^t is the history of past s_t
- The history s^t is publicly observable, ie everybody knows it

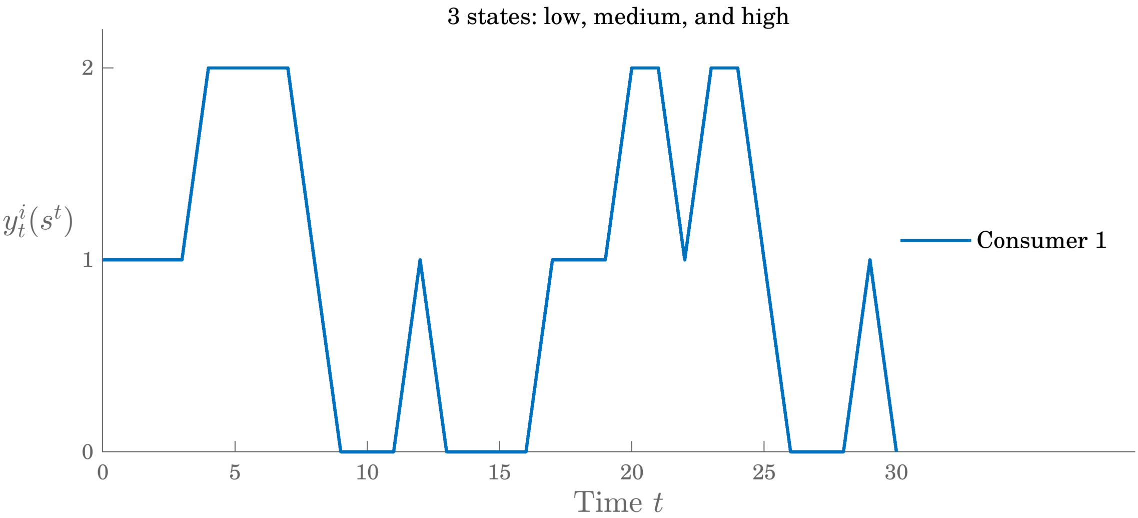
Probabilities

- The unconditional probability of observing a particular sequence of events s^t is given by a probability measure $\pi_t(s^t)$
- For $t > \tau$, we denote as $\pi(s^t|s^\tau)$ the probability of observing s^t conditional on the realization of history s^τ
- We assume that trading always occurs after observing s_0, which we capture by setting $\pi_0(s_0) = 1$ for the initially given value of s_0

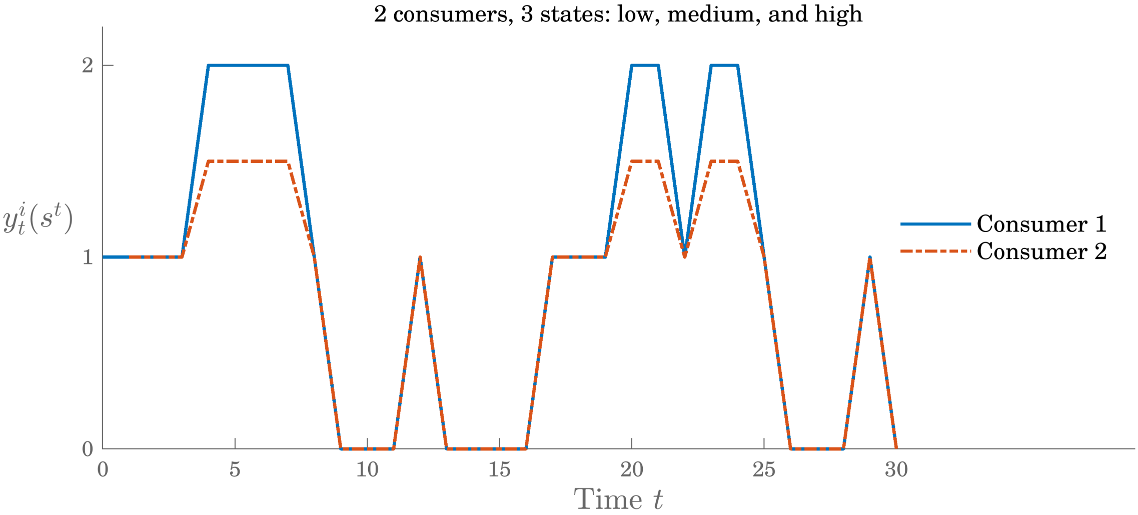
Consumers

- There are I consumers named $i = 1, 2, \ldots, I$
- Consumer i owns a stochastic endowment of one good $y_t^i(s^t)$ that depends on the history s^t; think of some income whose origin is exogenous
- Thus, each consumer receives a different endowment based on the history of common stochastic events s^t, and this endowment changes every period

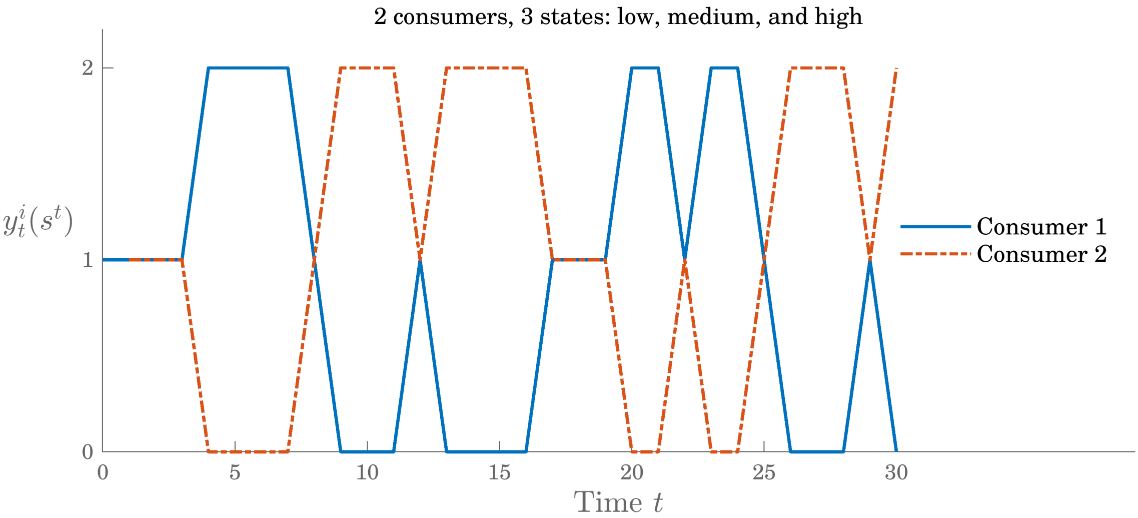
Example of Stochastic Endowment



Economy 1 with Two Consumers



Economy 2 with Two Consumers



Which economy is best? Why?

Preferences

- Consumer i purchases a history-dependent consumption plan $c^i = \{c_t^i\}_{t=0}^{\infty}$ and order these consumption streams by the following utility function

$$
\begin{aligned}
U_i(c^i) &= E_0 \sum_{t=0}^{\infty} \beta^t u_i(c_t^i), \qquad 0 < \beta < 1 \\
&= \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u_i[c_t^i(s^t)] \pi_t(s^t)
\end{aligned}
$$

- E_0 is the mathematical expectation operator conditioned on s_0

Assumptions on the Utility Function

- As usual, the utility function $u_i(c)$ is strictly increasing and concave, twice continuously differentiable, and satisfies the Inada condition

$$\lim_{c \to 0} u_i'(c) = +\infty$$

- The Inada condition implies that the consumer chooses strictly positive consumption at every period, no matter her endowment

Aggregate Resource Constraint

- A feasible allocation satisfies

$$\underbrace{\sum_i^I c_t^i(s^t)}_{C_t(s^t)} \leq \underbrace{\sum_i^I y_t^i(s^t)}_{Y_t(s^t)} \quad \text{for all } t \text{ and all } s^t$$

- At all times, aggregate consumption cannot exceed aggregate endowment
- Implicitly this means there is no borrowing abroad, the economy is closed
- This is the economy's aggregate resource constraint or feasibility constraint

State Variables

- What are the state variables of this economy?
- There is one exogenous aggregate state variable s_t, the stochastic event
- Since consumer i's individual endowment $y_t^i(s^t)$ is a function of the history of s_t only, it effectively acts as a (exogenous) state variable for i

Autarchy

- What happens if no trade is allowed among consumers?
- Since the endowment good $y_t^i(s^t)$ is nondurable and nonstorable, each autarkic consumer i has no choice but eat her entire endowment each period

$$c_t^i(s^t) = y_t^i(s^t) \quad \text{for all } t \text{ and all } s^t$$

- Under autarchy, consumers are not able to smooth their consumption path

2. Pareto Problem

A Benchmark

- We first characterize the Pareto efficient allocation, also known as centralized or social planner problem
- This will set a benchmark to compare to a market economy
- An allocation is efficient if it is Pareto optimal: any reallocation that makes one agent strictly better off also makes at least one other agent worse off

Problem of the Central Planner

- The social planner attaches Pareto weights $\lambda_i \geq 0$, $i = 1, \ldots, I$ to each consumer's utility function
- The planner chooses $c^i = \{c_t^i\}_{t=0}^{\infty}$, $i = 1, \ldots, I$ to maximize the weighted sum of all agents' utility functions

$$W = \sum_{i=1}^{I} \lambda_i U_i(c^i) = \sum_{i=1}^{I} \sum_{t=0}^{\infty} \sum_{s^t} \lambda_i \beta^t u_i[c_t^i(s^t)] \pi_t(s^t)$$

$$\text{subject to} \quad \sum_{i}^{I} c_t^i(s^t) \leq \sum_{i}^{I} y_t^i(s^t) \quad \text{for all } t,\ s^t$$

Lagrangian

- Write a Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_{i=1}^{I} \lambda_i \beta^t u_i[c_t^i(s^t)] \pi_t(s^t) + \theta_t(s^t) \sum_{i=1}^{I} [y_t^i(s^t) - c_t^i(s^t)] \right\}$$

- $\theta_t(s^t)$ is a nonnegative Lagrange multiplier on the feasibility constraint for time t and history s^t, ie there is one multiplier for each t and each s^t
- $\theta_t(s^t)$ is the value, or shadow price, of relaxing the constraint for one unit, ie consuming one more unit

First-Order Condition

- The first-order condition (FOC) for maximizing $\mathcal{L}$ with respect to $c_t^i(s^t)$ is

$$\beta^t u_i'[c_t^i(s^t)]\pi_t(s^t) = \lambda_i^{-1}\theta_t(s^t) \quad \text{for each } i,\ t,\ s^t$$

- For each consumer i, each history of the world s^t, and each period t, there is one FOC

Two Consumers

- Take the FOC for two consumers, say consumers i and 1, at period t and history s^t

$$\beta^t u_i'[c_t^i(s^t)]\pi_t(s^t) = \lambda_i^{-1}\theta_t(s^t)$$
$$\beta^t u_1'[c_t^1(s^t)]\pi_t(s^t) = \lambda_1^{-1}\theta_t(s^t)$$

- Take the ratio, ie divide one by the other

$$\frac{u_i'[c_t^i(s^t)]}{u_1'[c_t^1(s^t)]} = \frac{\lambda_1}{\lambda_i}$$

- Solve for $c_t^i(s^t)$

$$c_t^i(s^t) = u_i'^{-1}\left(\lambda_i^{-1}\lambda_1 u_1'[c_t^1(s^t)]\right)$$

Merge First-Order and Feasibility Conditions

- At the optimum, the resource constraint binds, ie holds with equality, meaning all the endowment is consumed
- Substitute the previous equation for $c_t^i(s^t)$ into the resource constraint at equality

$$\sum_i^I \underbrace{u_i'^{-1}\left(\lambda_i^{-1}\lambda_1 u_1'[c_t^1(s^t)]\right)}_{c_t^i(s^t)} = \sum_i^I y_t^i(s^t)$$

- This is one equation for one unknown, $c_t^1(s^t)$; the other objects are known parameters $\{\lambda_i\}_{i=1}^I$, functional forms, u_i, u_1, and aggregate $\sum_i^I y_t^i(s^t)$

A Function of the Aggregate

- Repeat the previous equation

$$\sum_{i}^{I} u_i'^{-1}\left(\lambda_i^{-1}\lambda_1 u_1'[c_t^1(s^t)]\right) = \sum_{i}^{I} y_t^i(s^t)$$

- The right side is the realized aggregate endowment, observed to all in t
- Thus, given $\{\lambda_i\}_{i=1}^{I}$, u_i, and u_1, we conclude that $c_t^1(s^t)$ depends only on the current realization of the aggregate endowment
- The consumption of an individual consumer does not depend on her individual income

Consumption $\propto$ Aggregate Endowment

- An efficient allocation is a function of the realized aggregate endowment
- An efficient allocation does not depend on
  - The specific history s^t leading up to that aggregate endowment
  - The cross-section distribution of individual endowments realized at t
- Put differently,

$$c_t^i(s^t) = c_\tau^i(\tilde{s}^\tau) \quad \text{for } s^t \text{ and } \tilde{s}^\tau \text{ such that} \quad \sum_i^I y_t^i(s^t) = \sum_i^I y_\tau^i(\tilde{s}^\tau)$$

Intuition

- Each period, the planner observes the aggregate endowment, divides it into shares according to λ_i and u_i, and distributes it to each consumer
- The planner acts as if agents pooled and shared their resources perfectly
- If λ_i and u_i are equal across all agents, then all get the same share of aggregate endowment each period regardless of individual endowment
- Note the consumption of each agent may vary each period if the aggregate endowment fluctuates each period

How to Compute an Optimal Allocation?

- First solve for $c_t^1(s^t)$ using the feasibility constraint

$$\sum_i^I u_i'^{-1}\left(\lambda_i^{-1}\lambda_1 u_1'[c_t^1(s^t)]\right) = \sum_i^I y_t^i(s^t)$$

- Then solve for $c_t^i(s^t)$ using the FOC

$$c_t^i(s^t) = u_i'^{-1}\left(\frac{\lambda_1}{\lambda_i}u_1'[c_t^1(s^t)]\right)$$

- Only the ratios of the Pareto weights matter, so we are free to normalize the weights, for example $\sum_i^I \lambda_i = 1$

3. Time Zero Trading: Arrow-Debreu Securities

Arrow and Debreu

- We now characterize a competitive equilibrium
- We assume consumers can trade a particular financial instrument called an Arrow-Debreu security, also known as state-price security or pure security
- Named after a seminal paper by Kenneth Arrow and Gerard Debreu (1954, *Econometrica*), the foundation of general equilibrium theory
- Both economists won the Nobel Prize (Arrow in 1972, Debreu in 1983) precisely for their pioneering work on the theory of general equilibrium

About the Arrow-Debreu Competitive Equilibrium

“Going back to Leon Walras and Adam Smith, people had some intuition about why [the competitive equilibrium] had to exist, and that it should have some optimal properties, but Kenneth Arrow and Gerard Debreu nailed it. First of all, they closed some open questions like the existence of equilibrium using a theorem. But in doing that, they also took it to a higher and deeper level of analysis by defining what the commodities are, and what the structure of equilibrium has to be.”

Daron Acemoglu, 2010, Interview with Simon Bowmaker

Arrow-Debreu Security

- An Arrow-Debreu security is a financial instrument or contract that agrees to pay its owner the following
  - One unit of numeraire, or a fixed amount, if a particular state occurs at a particular point in time in the future
  - Zero in all other states and periods

Example – Peace and War

- Imagine a world with two possible states tomorrow

State 1: Peace, P State 2: War, W

- Suppose there exist two Arrow-Debreu securities

1. One pays off \$1 if tomorrow's state is P
2. The other pays off \$1 if tomorrow's state is W

- If the agent buys both securities today, she has secured \$1 tomorrow no matter what happens, ie she has purchased a riskless bond

Claims to Consumption

- In the case of our pure exchange economy, an Arrow-Debreu security is a dated history-contingent claim to consumption
- Each security specifies a particular date t and a particular state s_t and entitles the buyer to receive from the seller the following payment
  - $1 \times c$ if that particular date t and that particular state s_t occur
  - 0 in all other states and periods

Almighty Market

- The market for securities opens at time 0 after observing s_0
- All trades occur at time 0
- The market closes forever at the end of time 0
- In other words, after time 0, trades that were agreed at time 0 are executed but no more trades occur

Example with Two States

- Suppose the stochastic process s_t follows a two-state Markov chain with a good state and a bad state

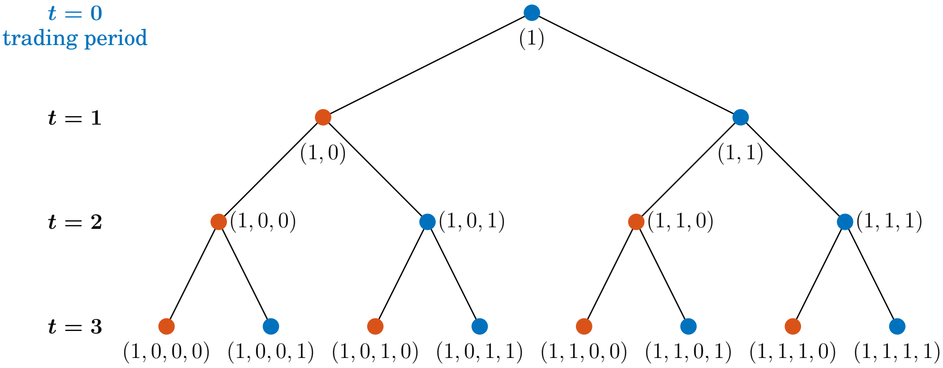
$$s_t \in \boldsymbol{S} = \{0\,,1\} \quad \text{with } s_0 = 1$$

- Now suppose consumer i's endowment is

$$y_t^i(s_t = 1) = 10 \quad \text{and} \quad y_t^i(s_t = 0) = 0$$

- Let's portray all prospective histories possible up to time 3

Example with Two States



- At time 0, consumer i buys and sells one security for each node of each period (14 trades in total): she typically buys when red and sells when blue
- Her sales of securities allow her to finance her purchases of securities

Prices

- Agents exchange claims on time t consumption contingent on history s^t
- Each Arrow-Debreu claim has price $q_t^0(s^t)$, quoted in some unit of account
- In this notation, subscript t denotes the date t agreed in the contract and superscript 0 denotes the date at which trades occur
- For example $q_1^0(s^1)$, $q_{99}^0(\tilde{s}^{99})$

Complete Markets

- Markets are complete
- This means that for all periods t and all possible states of the world s_t, there exists one Arrow-Debreu security

Individual Budget Constraint

- The individual budget constraint of consumer i is

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^i(s^t)$$

- At time 0, the agent sells her total lifetime endowment $\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^i(s^t)$ to finance purchases of claims to lifetime consumption $\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t)$
- The constraint says the consumer cannot buy claims to consumption that together are worth more than the market value of her lifetime endowment

A Lifetime Budget Constraint

- Repeat the budget constraint

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

- This constraint does not mean that at one given period the consumer could not consume more than her income, ie $c_t^i(s^t) > y_t^i(s^t)$ is possible
- This constraint also does not mean that she could not consume more than her lifetime endowment, ie $\sum_{t=0}^{\infty} \sum_{s^t} c_t^i(s^t) > \sum_{t=0}^{\infty} \sum_{s^t} y_t^i(s^t)$ is possible

Numerical Example

- Consider a world with 2 states and 2 periods; consumer i's endowment is $y_1^i = 15$, $y_2^i = 2.5$ but i manages to consume $c_1^i = c_2^i = 10$ thanks to her trades
- The budget constraint of consumer i writes

$$\begin{aligned} q_1^0(s_1)c_1^i(s_1) + q_1^0(s_2)c_2^i(s_2) &\leq q_1^0(s_1)y_1^i(s_1) + q_1^0(s_2)y_2^i(s_2) \\ 1.5 \times 10 + 1 \times 10 &= 1.5 \times 15 + 1 \times 2.5 \end{aligned}$$

- Implicitly, in period 1 when consumer i receives a high endowment, other agents' endowments are low, so i is able to sell her securities at a high price
- We have indeed $c_2^i > y_2^i$ and $c_1^i + c_2^i > y_1^i + y_2^i$

Net Claims

- There are multilateral trades in the market, ie each consumer buys securities from and sells securities to multiple consumers
- Suppose a central clearing house oversees all operations
- The single budget constraint implicitly says that the clearing house keeps track of all net claims
- In other words this is a system with net settlement of trades, not gross

Problem of the Consumer

- In time $t = 0$, given a system of prices $\{q_t^0\}_{t=0}^{\infty}$, consumer i solves the following problem

$$\max_{\{c_t^i(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u_i[c_t^i(s^t)] \pi_t(s^t)$$

$$\text{subject to} \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

Lagrangian

- The Lagrangian writes

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u_i[c_t^i(s^t)] \pi_t(s^t) + \mu_i \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)[y_t^i(s^t) - c_t^i(s^t)]$$

- μ_i is a nonnegative Lagrange multiplier on the budget constraint
- Since each consumer has only one time 0 budget constraint, each consumer has her own unique time-invariant multiplier μ_i

First-Order Condition

- The first-order condition for the consumer's problem is

$$\beta^t u_i'[c_t^i(s^t)]\pi_t(s^t) = \mu_i q_t^0(s^t) \quad \text{for all } i,\, t,\, s^t$$

- For each consumer i, each history of the world s^t, and each period t, there is one FOC

Equilibrium

- A price system is a sequence of functions $\{q_t^0(s^t)\}_{t=0}^{\infty}$
- An allocation is a list of sequences of functions $c_i = \{c_t^i(s^t)\}_{t=0}^{\infty}$, one for each i
- A competitive equilibrium is a feasible allocation and a price system such that, given the price system, the allocation solves each consumer's problem
- Thus a competitive equilibrium is such that the aggregate feasibility constraint, each individual budget constraint, and each FOC are satisfied

Constant Ratio

- Take the FOC for consumers i and 1, say

$$\beta^t u_i'[c_t^i(s^t)]\pi_t(s^t) = \mu_i q_t^0(s^t)$$
$$\beta^t u_1'[c_t^1(s^t)]\pi_t(s^t) = \mu_1 q_t^0(s^t)$$

- Divide one by another

$$\frac{u_i'[c_t^i(s^t)]}{u_1'[c_t^1(s^t)]} = \frac{\mu_i}{\mu_1}$$

- The ratio of marginal utilities between pairs of agents is constant across all histories and dates

Merge First-Order and Feasibility Conditions

- Solve for $c_t^i(s^t)$ from the previous equation

$$c_t^i(s^t) = u_i'^{-1}\left(u_1'[c_t^1(s^t)]\frac{\mu_i}{\mu_1}\right)$$

- Plug this into the aggregate feasibility constraint at equality

$$\sum_i^I u_i'^{-1}\left(u_1'[c_t^1(s^t)]\frac{\mu_i}{\mu_1}\right) = \sum_i^I y_t^i(s^t)$$

- $c_t^1(s^t)$ depends only on today's aggregate endowment and the ratios $\{\frac{\mu_i}{\mu_1}\}_{i=2}^I$

Consumption $\propto$ Aggregate Endowment

- The competitive equilibrium allocation is a function only of the realized aggregate endowment
- The competitive equilibrium allocation does not depend on
  - The specific history s^t leading up to that aggregate endowment
  - The cross-section distribution of individual endowments realized at t
- Put differently,

$$c_t^i(s^t) = c_\tau^i(\tilde{s}^\tau) \quad \text{for } s^t \text{ and } \tilde{s}^\tau \text{ such that} \quad \sum_i^I y_t^i(s^t) = \sum_i^I y_\tau^i(\tilde{s}^\tau)$$

Intuition

- In time 0, each consumer purchases and sells an Arrow-Debreu security for every possible state at all future periods (that's a lot of trades!)
- Complete markets allow agents to pool and share their resources perfectly
- In other words, complete markets allow 1) idiosyncratic risk to be insured away and 2) consumers to smooth consumption over time

The Competitive Allocation is Efficient

- The competitive equilibrium allocation is a particular Pareto optimal allocation, one that sets the Pareto weights $\lambda_i = \mu_i^{-1}$
- These weights are unique (up to a multiplication by a positive scalar)
- This result is the first theorem of welfare economics, the "invisible hand": a competitive equilibrium allocation is efficient

Prices Coincide

- If $\lambda_i = \mu_i^{-1}$ then $\theta_t(s^t) = q_t(s^t)$ for all t and s^t
- The shadow prices of the planning problem are equal to the competitive equilibrium prices for goods delivered at t under history s^t
- This is the second theorem of welfare economics: there exists a price system that supports an efficient allocation as a competitive equilibrium allocation

Perfect Risk Diversification

- Both the centralized economy and competitive equilibrium allocations imply perfect diversification of idiosyncratic risks, ie perfect risk sharing
- Individual consumption changes don't depend on individual income changes
- Only the aggregate risk, ie shocks to aggregate income, matters for individual consumption

Crucial Assumptions

- Complete markets require two implicit assumptions

1. Perfect information: s^t and $\{y_t^i(s^t)\}_{i=1}^I$ are publicly observable, otherwise agents could lie about their endowment
2. Full enforcement: agents always comply with past contracts, ie they always deliver the goods at all t and s^t

Applications

- This simple model can be applied to study several issues
- We are going to see two applications

1. Risk sharing and consumption decisions
2. Asset pricing

4. Risk Sharing and Consumption

Testing the Model

- Is the complete-market model a good description of the world?
- One way to answer this question is to derive empirical implications from the theoretical model and test these implications using actual economic data
- For this application, we assume a power utility function, ie CRRA

$$u(c^i) = \frac{(c^i)^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

A CRRA Utility Function

- Start from the ratio of two consumers' FOC

$$\frac{u_i'[c_t^i(s^t)]}{u_j'[c_t^j(s^t)]} = \frac{\mu_i}{\mu_j} \implies \left(\frac{c_t^i(s^t)}{c_t^j(s^t)}\right)^{-\sigma} = \frac{\mu_i}{\mu_j} \implies \frac{c_t^j(s^t)}{c_t^i(s^t)} = \left(\frac{\mu_j}{\mu_i}\right)^{-\frac{1}{\sigma}}$$

- Sum over all agents j and then solve for $c_t^i(s^t)$

$$\sum_{j=1}^{I} \frac{c_t^j(s^t)}{c_t^i(s^t)} = \sum_{j=1}^{I} \left(\frac{\mu_j}{\mu_i}\right)^{-\frac{1}{\sigma}} \implies c_t^i(s^t) = \underbrace{\frac{\mu_i^{-\frac{1}{\sigma}}}{\sum_{j=1}^{I} \mu_j^{-\frac{1}{\sigma}}}}_{\alpha^i} \underbrace{\sum_{j=1}^{I} c_t^j(s^t)}_{C_t(s^t)}$$

Logarithmic First Difference

- From the previous equation we have

$$c_t^i(s^t) = \alpha^i C_t(s^t)$$

- Express in logs

$$\ln c_t^i(s^t) = \ln \alpha^i + \ln C_t(s^t)$$

- Take the first difference, ie subtract $\ln c_{t-1}^i(s^{t-1}) = \ln \alpha^i + \ln C_{t-1}(s^{t-1})$

$$\Delta \ln c_t^i(s^t) = \Delta \ln C_t(s^t)$$

Individual and Aggregate Consumption

- To sum up, complete markets with CRRA preferences imply

$$\Delta \ln c_t^i(s^t) = \Delta \ln C_t(s^t)$$

- Changes in individual consumption mirror changes in aggregate consumption C_t but are independent of individual income $y_t^i(s^t)$
- Thus, the model predicts that individual consumption responds to aggregate shocks (eg a recession) but not to idiosyncratic shocks (eg loss of a job)

An Empirical Test

- One can test this implication by collecting micro data on individuals' consumption and income and running the following regression

$$\Delta \ln c_t^i = \alpha + \beta_1 \Delta \ln C_t + \beta_2 \Delta \ln y_t^i + \varepsilon_t^i$$

- If markets are complete, we should find $\beta_1 = 1$ and $\beta_2 = 0$
- If agents are in autarchy, we should find $\beta_1 = 0$ and $\beta_2 = 1$
- Let's see the results of a pioneering study by Barbara Mace (1991, *JPE*)

Aren't Complete Markets a Fantasy?

- The idea of financial markets offering to trade a complete set of contingent claims – one for every possible state of nature – is of course not realistic
- But besides explicit contingent assets, a wide variety of formal and informal institutions help provide consumption insurance
  - Unemployment, disability, medical, and disaster insurance
  - Welfare and other government social programs, eg food stamps
  - State-contingent government transfers, eg farm support, drought relief
  - Local, national, and international charities
  - Informal insurance mechanisms, eg gifts and loans from relatives
- Therefore, we should view the test of complete markets as a broader test for the strength of a society's various risk-sharing mechanisms

Results from Mace (1991)

Household Consumption Regressions: Growth Rates

Consumption Measure	Intercept (1)	β_1 ΔC_t^a (2)	β_2 Δy_t^j (3)	F-Ratios: $\beta_1 = 1, \beta_2 = 0$ (4)	F-Ratios: $\beta_1 = 1, \gamma_k = 0\ \forall\ k$ (5)	R^2 (6)
Total consumption	−.04	1.06	.04	14.12*	. . .	.021
	(.01)	(.08)	(.007)			
	−.01	1.05	. . .	. . .	1.85*	.029
	(.01)	(.08)			(A)	
Services	−.02	.93	.04	12.44*	. . .	.011
	(.01)	(.10)	(.01)			
	.02	.94	. . .	. . .	1.34*	.017
	(.02)	(.11)			(B)	
Nondurables	−.02	.97	.04	22.69*	. . .	.027
	(.01)	(.07)	(.006)			
	−.01	.95	. . .	. . .	1.76*	.033
	(.01)	(.07)			(C)	
Durables	−.05	1.00	−.03	.39	. . .	−.046
	(.03)	(.06)	(.03)			
	−.14	1.01	. . .	. . .	1.20	.057
	(.09)	(.06)			(D)	
Food	−.02	.91	.04	18.67*	. . .	.020
	(.01)	(.07)	(.006)			
	.01	.89	. . .	. . .	1.81*	.027
	(.01)	(.07)			(E)	
Housing	−.05	.79	.01	1.77	. . .	.006
	(.01)	(.12)	(.01)			
	−.03	.79	. . .	. . .	1.47*	.018
	(.03)	(.12)			(F)	

Source: Mace (1991, *JPE*)

Complete Markets vs Autarchy

- Mace finds she cannot reject the complete-market hypothesis for 6 out of 12 consumption items; she rejects it for services, nondurables, food, housing
- Mace decisively rejects the autarchy hypothesis: $\beta_1 > 0$ and $\beta_2 < 1$
- Results vary depending on the functional form of the utility function (power vs exponential utility), but the data supports risk sharing to a large extent
- What about other papers? Following Mace, a strand of the literature has tested risk sharing in a number of different countries and setups

Risk and Insurance in Urban United States

"Under full insurance, consumption growth should be cross-sectionally independent of idiosyncratic variables that are exogenous to consumers. This proposition is tested by cross-sectional regressions of consumption growth on a variety of exogenous variables. Full insurance is rejected for long illness and involuntary job loss, but not for spells of unemployment, loss of work due to strike, and an involuntary move."

John Cochrane, 1991, *Journal of Political Economy*

Risk and Insurance in Rural India

"The full insurance model is tested using data from three poor, high risk villages in the semi-arid tropics of southern India ... Although the model is rejected statistically, it does provide a surprisingly good benchmark. Household consumptions comove with village average consumption. More clearly, household consumptions are not much influenced by contemporaneous own income, sickness, unemployment, or other idiosyncratic shocks, controlling for village consumption (ie for village level risk). There is evidence that the landless are less well insured than their village neighbors in one of the three villages."

Robert Townsend, 1994, *Econometrica*

Literature on Consumption Insurance

- Following the classic studies by Mace (1991), Cochrane (1991), and Townsend (1994), the literature has grown large

Townsend 1995; Attanasio and Davis 1996; Hayashi et al. 1996; Dercon 2002; Fafchamps and Lund 2003; Hess 2004; Kaplan and Violante 2010; Schulhofer-Wohl 2011; Mazzocco and Saini 2012; Jack and Suri 2014, Blundell, Pistaferri, and Saporta-Eksten 2018; Santaeulàlia-Llopis and Zheng 2018; Autor, Kostol, Mogstad, and Setzler 2019

Summary of Findings

- Generally, the literature finds that both hypotheses are rejected, implying that markets are incomplete
- In particular, the coefficient on income β_2 is usually positive and significant, meaning that individual income changes affect consumption
- But the data is closer to risk sharing than it is to autarchy

Heterogeneous Preferences

- A few papers argue that heterogeneity in risk preferences bias the result against risk sharing
- Some people are willing to take risks and thus have consumption vary with income, eg entrepreneurs, salespeople, traders, mercenaries
- Accounting for heterogeneous risk tolerance, Schulhofer-Wohl (2011) and Mazzocco and Saini (2012) find even more support for risk sharing

Incomplete Markets

- To describe a world in which individual outcomes matter and inequality is pervasive, researchers have developed incomplete-market models
- With incomplete markets, the number of Arrow-Debreu securities is less than the number of states of nature; financial constraints become crucial
- These models feature partial risk sharing and typically fail to reach the optimal allocation of assets, ie the first welfare theorem no longer holds
- Incomplete markets are one source of market failure

Sources of Incompleteness

- Markets are incomplete for many reasons
- Information asymmetry hinders the well-functioning of financial markets
- Limited enforcement, ie when the economy lacks the institutions to guarantee that the contracts are enforced, is another cause
- In some countries many consumers and firms are cut off from asset markets
- Many claims have yet to be invented; are costly to operate; are so complex that the legal bills would outweigh the benefits; or interest too few people

5. Asset Pricing

Asset Pricing

- Asset pricing is the study of how financial assets – including cash, bank deposits, bonds, equities, derivatives, hybrids – are priced
- A financial asset is simply a right to a stream of future cash flow
- Pricing the asset consists in putting a value to this stream of cash flow

Asset Pricing and Complete Markets

- Many asset-pricing models assume complete markets
- They break the asset into a sequence of history-contingent claims and evaluate each component with the relevant state price deflator $q_t^0(s^t)$
- Then they add up these values to price the asset

Redundant Asset

- In such a setup we say the asset is redundant or synthetic
- The asset offers a bundle of history-contingent claims
- But each of its component is already priced by the market
- Let's see how we price some redundant/synthetic assets
- We normalize $q_0^0(s_0) = 1$

Risk-Free Bond

- A risk-free bond is a bond that repays principal and interest with certainty
- A risk-free bond is therefore a non-state-contingent asset
- The price at time 0 of a one-period risk-free bond (that matures at time 1) is

$$p_0^0(s_0) = \sum_{s^1} \frac{q_1^0(s^1)}{q_0^0(s^0)} = \sum_{s^1} q_1^0(s^1) \equiv \frac{1}{R_{0,1}}$$

- Subscript 0 means the payment for the asset is made in time 0; superscript 0 means the value is measured in time 0 units of consumption
- $R_{0,1}$ is the return on the bond, ie the risk-free interest rate

Stock: Asset Paying a Dividend Forever

- A stock is an asset that pays a dividend as long as the firm operates
- Let $\{d_t(s^t)\}_{t=0}^{\infty}$ be a stream of claims on time t, history s^t consumption
- The price of an asset entitling the owner to this stream is

$$p_0^0(s_0) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t)$$

- The price of the asset is equal to the sum of history-contingent claims to consumption $d_t(s^t)$, ie Arrow-Debreu securities, times their price $q_t^0(s^t)$

Arbitrage

- Why is that so? Suppose that $p_0^0(s_0) > \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t)$
- One could synthesize this asset through purchases of history-contingent dated commodities, then sell the asset for $p_0^0(s_0)$ and make unbounded profit
- Conversely, if $p_0^0(s_0) < \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t)$, one could buy the asset, break it down into a sequence of claims, sell these claims, and make infinite profit
- Thus, by arbitrage the equation must hold at equality

Riskless Consol

- A consolidated annuity, or consol, is a perpetual government bond with no maturity date (eg 4%-rate consol issued in 1927 to pay for UK's war debts)
- Consider a riskless consol, an asset offering to pay one unit of consumption for sure, each period, forever: $d_t(s^t) = 1$ for all t and s^t
- The price of this asset is

$$p_0^0(s_0) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)$$

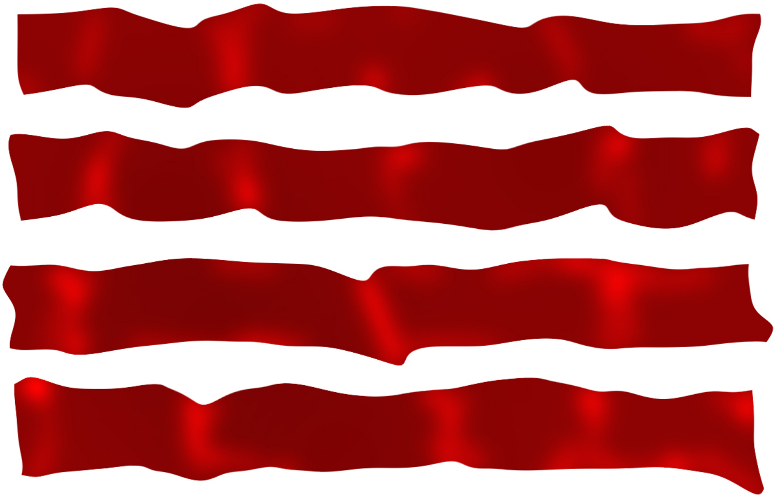
Riskless Strip

- Consider now a strip of payoff on the riskless consol
- The time t strip is just the payoff process

$$\begin{cases} d_\tau = 1 & \text{if} \quad \tau = t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- The owner of the strip is entitled to the time t coupon only, ie the interest payment received by the bondholder at that period only

Strips



Riskless Strip

- The price of the time t strip at time 0 is

$$p_0^0(s_0) = \sum_{s^t} q_t^0(s^t)$$

- Recall that the price of the full consol is $p_0^0(s_0) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)$
- Thus we can think of the t-period riskless strip as a t-period zero-coupon bond, ie a bond that pays no periodic interest but trades at a discount

Tail Assets

- Investors often buy an asset after it was first issued, in $\tau > t = 0$
- Let's go back to the stream of dividends $\{d_t(s^t)\}_{t=0}^{\infty}$ generated by the stock
- For $\tau \geq 1$, we strip off the first $\tau - 1$ periods of the dividend
- That is, we remove $\{d_t(s^t)\}_{t=0}^{\tau-1}$ and keep $\{d_t(s^t)\}_{t\geq\tau}^{\infty}$
- We want the time 0 value of the remaining stream of dividends

Tail Assets

- Let $p_\tau^0(s^t)$ be the time 0 price of an asset that entitles the owner to dividend stream $\{d_t(s^t)\}_{t\geq\tau}$
- The price of that asset is

$$p_\tau^0(s^t) = \sum_{t\geq\tau}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t)$$

A Particular History

- Suppose we want the value of this asset for a particular realization of s^τ
- Let $p_\tau^0(s^\tau)$ be the time 0 price of an asset that entitles the owner to dividend stream $\{d_t(s^t)\}_{t\geq\tau}$ if history s^τ is realized

$$p_\tau^0(s^\tau) = \sum_{t\geq\tau}^{\infty} \sum_{s^t|s^\tau} q_t^0(s^t) d_t(s^t)$$

- The summation over $s^t|s^\tau$ means that we sum over all possible subsequent histories $\tilde{s}^t$ such that realized history in t is $\tilde{s}^\tau = s^\tau$

Deflating

- The price is expressed in units of time 0, history s^τ goods
- To convert the price into units of time τ, history s^τ goods, divide by $q^0_\tau(s^\tau)$

$$p^\tau_\tau(s^\tau) \equiv \frac{p^0_\tau(s^\tau)}{q^0_\tau(s^\tau)} = \sum_{t \geq \tau} \sum_{s^t | s^\tau} \frac{q^0_t(s^t)}{q^0_\tau(s^\tau)} d_t(s^t)$$

- Analogy: suppose the consumer price index is 22 in 1942 and 285 in 2023; to know what \$100 in 1942 is worth in terms of 2023 dollars, compute

$$\$100 \times \frac{285}{22} = \$1{,}295$$

Using the Euler Equation

- Similarly, we have
$$q_t^\tau(s^t) \equiv \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)}$$
- $q_t^\tau(s^t)$ is the price of one unit of consumption good delivered at time t, history s^t in terms of the date τ, history s^τ consumption good
- Using the FOC evaluated at two points in time t and τ, we have
$$q_t^\tau(s^t) = \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = \frac{\beta^t u_i'[c_t^i(s^t)]\pi_t(s^t)}{\beta^\tau u_i'[c_\tau^i(s^\tau)]\pi_\tau(s^\tau)} = \beta^{t-\tau}\frac{u_i'[c_t^i(s^t)]}{u_i'[c_\tau^i(s^\tau)]}\pi_t(s^t|s^\tau)$$
- $\pi_t(s^t|s^\tau)$ is the probability of history s^t conditional on history s^τ at date τ

Tail Assets

- Thus, the price at time τ, history s^τ for the tail asset is

$$p_\tau^\tau(s^\tau) = \sum_{t \geq \tau} \sum_{s^t | s^\tau} q_t^\tau(s^t) d_t(s^t)$$

- This tail asset pricing formula expresses the asset price in terms of prices with time τ, history s^τ good as numeraire
- This formula is useful in finance when we want to create a time series of the price of a stock bought at time τ that yields a dividend from τ onward

One-Period Return

- Remember the Euler equation derived two slides above

$$q_t^\tau(s^t) = \beta^{t-\tau}\frac{u_i'[c_t^i(s^t)]}{u_i'[c_\tau^i(s^\tau)]}\pi_t(s^t|s^\tau)$$

- The one-period version of this equation is

$$q_{t+1}^t(s^{t+1}) = \beta\frac{u_i'[c_{t+1}^i(s^{t+1})]}{u_i'[c_t^i(s^t)]}\pi_{t+1}(s^{t+1}|s^t)$$

- This is the one-period pricing kernel at time t

Pricing Kernel

- The pricing kernel is a key concept in financial economics

$$q_{t+1}^{t}(s^{t+1}) = \beta \frac{u_i'[c_{t+1}^{i}(s^{t+1})]}{u_i'[c_t^{i}(s^t)]} \pi_{t+1}(s^{t+1}|s^t)$$

- The price of any asset is the discounted expected value of its future payoff
- The pricing kernel underscores the law of one price: assets that have the same payoff should have the same price, ie no arbitrage opportunities
- This is the underlying concept behind all asset pricing

An Asset with Random Payoff

- Consider an asset that returns a random payoff $\omega(s_{t+1})$
- The price at time t at history s^t of this claim is

$$
\begin{aligned}
p_t^t(s^t) &= \sum_{s_{t+1}} q_{t+1}^t(s^{t+1})\omega(s_{t+1}) \\
&= E_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\omega(s_{t+1})\right]
\end{aligned}
$$

- We have deleted the i indices because this equation holds for any consumer i; also, c_t depends on s^t as usual

Stochastic Discount Factor

- Let $R_t \equiv \omega(s_t)/p^t_{t-1}(s^{t-1})$ be the one-period gross return on the asset
- Then for any asset, the previous Euler equation implies

$$1 = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} R_{t+1}\right] \equiv E_t[m_{t+1}R_{t+1}] = E[m_{t+1}R_{t+1}]$$

where m_{t+1} is the stochastic discount factor

$$m_{t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

Stochastic Discount Factor

- Repeat the previous equation

$$1 = E[m_{t+1}R_{t+1}], \qquad m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

- The stochastic discount factor is a random variable of s_{t+1}, given s^t
- It describes the relationship between the payoff of an asset in different possible future states of the world and the price of this asset
- It is another name for the pricing kernel
- It is equal to the marginal rate of substitution, ie the rate at which the agent is willing to substitute future consumption for present consumption

Empirical Implication

- The previous equation gives us a testable implication
- Consider annual returns of two major asset classes from 1915 to 2018

1. US government bonds, the world's most liquid risk-free asset
2. S&P500, an index of the 500 largest US stocks

	Average annual nominal return	Average annual real return
Bonds	4.3%	1.1%
Stocks	11.5%	8.3%
Inflation	3.2%	–

Source: Jeremy Siegel

Rejected by the Data

- In the data over the long run, $R_{t+1}^{bond} = 1.01$ and $R_{t+1}^{stock} = 1.08$
- The Euler equations for stocks and bonds are

$$E[m_{t+1} R_{t+1}^{bond}] = 1 \qquad E[m_{t+1} R_{t+1}^{stock}] = 1$$

- Combine the two equations

$$E[m_{t+1}(R_{t+1}^{stock} - R_{t+1}^{bond})] = 0$$

- Unless we assume an extremely high degree of risk aversion σ (contained in m_{t+1}), this equation is rejected by the data, ie $E[m_{t+1}(R_{t+1}^{stock} - R_{t+1}^{bond})] > 0$

Equity Premium Puzzle

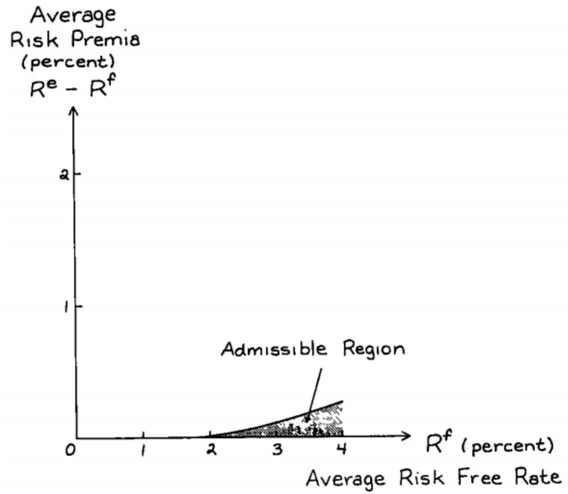


Fig. 4. Set of admissible average equity risk premia and real returns.

Source: Mehra and Prescott (1985)

Equity Premium Puzzle

- Why is the risk premium so large if people are only moderately risk-averse?
- Since stocks yield so much more than bonds, why don't people arbitrate by selling bonds and buying stocks, thus reducing the difference in returns?
- This is the equity premium puzzle, coined by Mehra and Prescott (1985)
- Standard complete-market asset-pricing models fail to explain the average premium of equities over government bonds unless they assume very high σ
- But with high σ, a new puzzle appears, the risk-free rate puzzle: why is the risk-free rate so low if agents are so averse to intertemporal substitution?

Some Solutions to the Equity Premium Puzzle

- Separate risk aversion from EIS, eg with Epstein-Zin preferences
- Habit formation in consumption (Campbell and Cochrane 1999)
- Rare disasters (Barro 2006)
- Behavioral finance (Shiller, Thaler, and others)
- Long-run risk (Bansal and Yaron 2004)
- See Cochrane (2017) for a review

Conclusion

- We have studied the complete market framework, the foundation of general equilibrium theory, risk sharing, and asset pricing
- The framework underpins all representative-agent models and serves as a benchmark for a variety of theories that feature market failures
- These includes models of imperfect information, imperfect competition, financial frictions

On General Equilibrium

"From the time of Adam Smith's Wealth of Nations in 1776, one recurrent theme of economic analysis has been the remarkable degree of coherence among the vast numbers of individual and seemingly separate decisions about the buying and selling of commodities. In everyday, normal experience, there is something of a balance between the amounts of goods and services that some individuals want to supply and the amounts that other, different individuals want to sell. Would-be buyers ordinarily count correctly on being able to carry out their intentions, and would-be sellers do not ordinarily find themselves producing great amounts of goods that they cannot sell. This experience of balance is indeed so widespread that it raises no intellectual disquiet among laymen; they take it so much for granted that they are not disposed to understand the mechanism by which it occur."

Kenneth Arrow, 1973, Nobel Prize Lecture

6. Exercises

Exercise 1 – Risk Sharing and Exponential Utility

Consider the Arrow-Debreu model with time 0 trading. Assume the following exponential utility function

$$u(c^i) = -\frac{1}{\sigma} e^{-\sigma c^i}, \quad \sigma > 0.$$

Derive a testable empirical implication.

Exercise 2 – Reopening Markets

Proposition: Start from the distribution of time τ, history s^τ financial wealth that is implicit in a time 0 Arrow-Debreu equilibrium. If markets are reopened at date τ after history s^τ, no trades occur. That is, given the price system

$$q_t^\tau(s^t) = \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} = \beta^{t-\tau}\frac{u_i'[c_t^i(s^t)]}{u_i'[c_\tau^i(s^\tau)]}\pi_t(s^t|s^\tau),$$

all consumers choose to continue the tails of their original consumption plans.

Prove this proposition.