# 15. Overlapping Generations

## Multiple Equilibria

Yvan Becard
PUC-Rio

Macroeconomics I, 2023

# Recap of the Previous Lecture

- In the last lecture we presented the overlapping generations (OLG) model
- The OLG model features a peculiary type of heterogeneity that distinguishes it from the infinitely-lived agent model in two respects

1. Each individual cares about consumption only at two adjacent dates
2. At any given date in the economy, there are individuals that care about early-age consumption and others that care about late-age consumption

# Efficiency and Uniqueness

- We showed how this special preference and demographic structure affects two key outcomes of the infinitely-lived agent model of lecture 8
- First, the complete-market competitive equilibrium is Pareto suboptimal
- The first generation, born at date $t = 0$, could be made better off without penalizing future generations, by transfering the period $t = 1$ excess goods
- Second, the complete-market competitive equilibrium is not unique
- Today we explore a bit more this property of multiple equilibria

# Lecture Outline

1. Nonstationary Equilibria
2. Money and Nonstationary Equilibria
3. Equivalence of Equilibria
4. Government Deficit Finance
5. Government Deficit: Equivalent Setup
6. Exercises

Main Reference: Ljungqvist and Sargent, 2018, *Recursive Macroeconomic Theory*, Fourth Edition, Chapter 9

# 1. Nonstationary Equilibria

# Model Setup

- The model is the same as in the previous lecture
- Pure exchange economy, no production, only endowments, no uncertainty
- Agents live two periods, young and old, then die

# Example

- Let's consider the example of the previous lecture where the young have a larger endowment than the old (to mimick workers versus retirees)
- Let $\epsilon \in (0, 0.5)$ be a constant, the endowments are

$$
\begin{aligned}
y_i^i &= 1 - \epsilon \quad \text{for all } i \geq 1 \\
y_{i+1}^i &= \epsilon \quad \text{for all } i \geq 0 \\
y_t^i &= 0 \quad \text{otherwise}
\end{aligned}
$$

# Many Equilibria

- We already described the two stationary equilibria, H and L
- This example economy has many more nonstationary equilibria
- To construct more equilibria, we summarize preferences and consumption decisions in terms of an offer curve

# Offer Curve

- The household's offer curve is the locus of early- and late-age consumption $(c_i^i, c_{i+1}^i)$ that solves

$$\max_{\{c_i^i, c_{i+1}^i\}} U(c^i)$$

$$\text{subject to} \quad c_i^i + \alpha_i c_{i+1}^i \leq y_i^i + \alpha_i y_{i+1}^i$$

- $\alpha^i \equiv \frac{q_{i+1}^0}{q_i^0} = R_i^{-1}$ is the reciprocal of the one-period gross rate of return from period $i$ to $i+1$, which we treat as a parameter

# Offer Curve

- For each $\alpha^i > 0$, there is a unique offer curve $(c_i^i, c_{i+1}^i)$ that solves the pair of equations consisting of a budget constraint and a FOC

$$c_i^i + \alpha_i c_{i+1}^i \leq y_i^i + \alpha_i y_{i+1}^i$$
$$\frac{u'(c_{i+1}^i)}{u'(c_i^i)} = \alpha_i$$

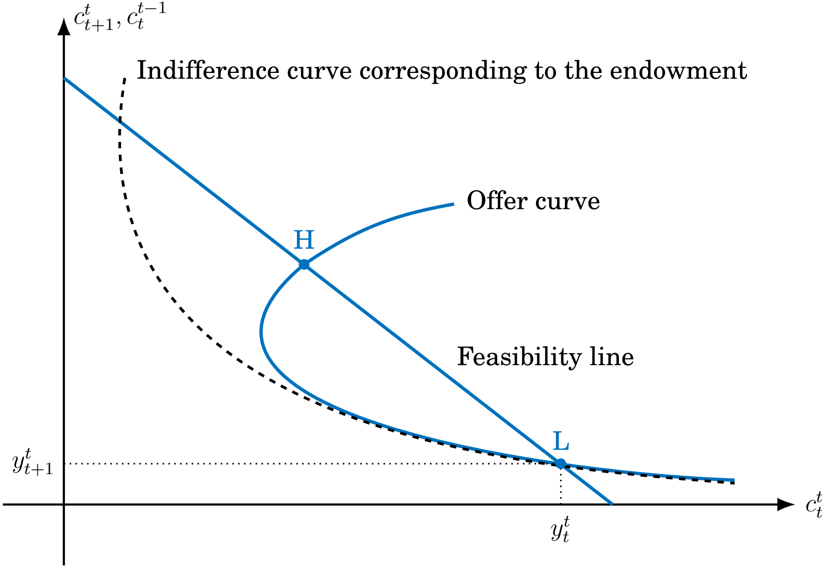
- Let's denote the offer curve by

$$\psi(c_i^i, c_{i+1}^i) = 0$$

# Graphical Representation

- We draw a straight feasibility line, or budget line $c_i^i + c_i^{i-1} = y_i^i + y_i^{i-1}$
- We draw an indifference curve with respect to the two goods $(c_i^i, c_{i+1}^i)$, which depends on the endowment $(y_i^i, y_{i+1}^i)$
- We then trace the offer curve by varying $\alpha_i$ in the household problem and reading tangency points between the indifference curve and the budget line
- The offer curve depends on the endowment vector and lies above the indifference curve through the endowment vector

# Offer Curve and Feasibility Line



# Computing an Equilibrium

- How do we compute equilibrium allocations and prices?
- We use the pair of two difference equations – the offer curve and feasibility constraint – which for $i \geq 1$ are

$$
\begin{aligned}
\psi(c_i^i, c_{i+1}^i) &= 0 \\
c_i^i + c_i^{i-1} &= y_i^i + y_i^{i-1}
\end{aligned}
$$

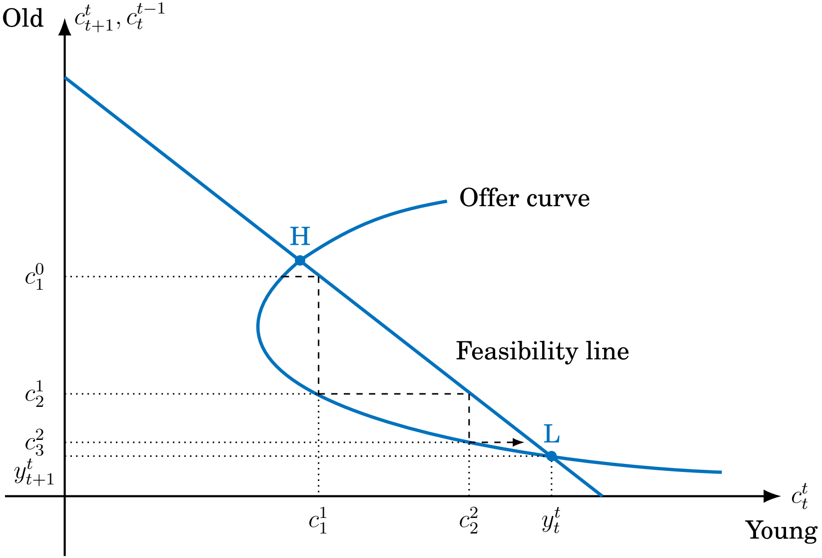
- We take $c_1^1$ as an initial condition and work our way forward
- After computing the allocation, we compute the equilibrium price system

$$q_i^0 = u'(c_i^i) \quad \text{for all } i \geq 1$$

# Computing an Equilibrium Allocation

- Start with a proposed $c_1^1$, a time 1 allocation to the initial young
- Use the budget line to find the maximal feasible value $c_1^0$ for the initial old
- The young choose $c_1^1$ only if $\alpha_1$ is such that $(c_1^1, c_2^1)$ lies on the offer curve, so choose $c_2^1$ from the offer curve point that cuts a vertical line through $c_1^1$
- Then find $c_2^2$ from the intersection of a horizontal line through $c_1^1$ and the feasibility line
- Continue recursively in this way

# Equilibrium Dynamics



# Convergence

- Any consumption $c_1^1$ between the upper (H) and lower (L) intersections of the offer curve and the feasibility line is an equilibrium consumption $c_1^1$
- Each such consumption $c_1^1$ is associated with a distinct allocation and a distinct sequence of interest rates $\alpha_i$
- All but one of these equilibria converge to the low-interest-rate stationary equilibrium allocation L
- The exception is the upper intersection, which corresponds to the high-interest-rate stationary equilibrium H

# Example – Log Utility

- Suppose utility is logarithmic, $u(c) = \ln c$
- The endowments are the same as before, $\epsilon \in (0, 0.5)$

$$
\begin{aligned}
y_i^i &= 1 - \epsilon \quad \text{for all } i \geq 1 \\
y_{i+1}^i &= \epsilon \quad \text{for all } i \geq 0 \\
y_t^i &= 0 \quad \text{otherwise}
\end{aligned}
$$

# Offer Curve

- Given $\alpha_i$, the offer curve solves

$$c_i^i + \alpha_i c_{i+1}^i = 1 - \epsilon + \alpha_i \epsilon$$
$$\frac{c_i^i}{c_{i+1}^i} = \alpha_i$$

- Combining the two equations, we obtain

$$c_i^i = \frac{1}{2}(1 - \epsilon + \alpha_i \epsilon) \quad \textbf{and} \quad c_{i+1}^i = \frac{1}{2\alpha_i}(1 - \epsilon + \alpha_i \epsilon)$$

# Feasibility Constraint

- An equilibrium $\alpha_i$ sequence must satisfy the feasibility constraint and have $\alpha_i > 0$ for all $i$
- The feasibility constraint implies

$$c_i^i + c_i^{i-1} = y_i^i + y_i^{i-1}$$

$$\frac{1}{2}(1 - \epsilon + \alpha_i \epsilon) + \frac{1}{2\alpha_{i-1}}(1 - \epsilon + \alpha_{i-1}\epsilon) = \epsilon + 1 - \epsilon$$

$$\alpha_i = \epsilon^{-1} - \frac{\epsilon^{-1} - 1}{\alpha_{i-1}}$$

- This is a difference equation in $\alpha_i$

# Stationary Equilibria

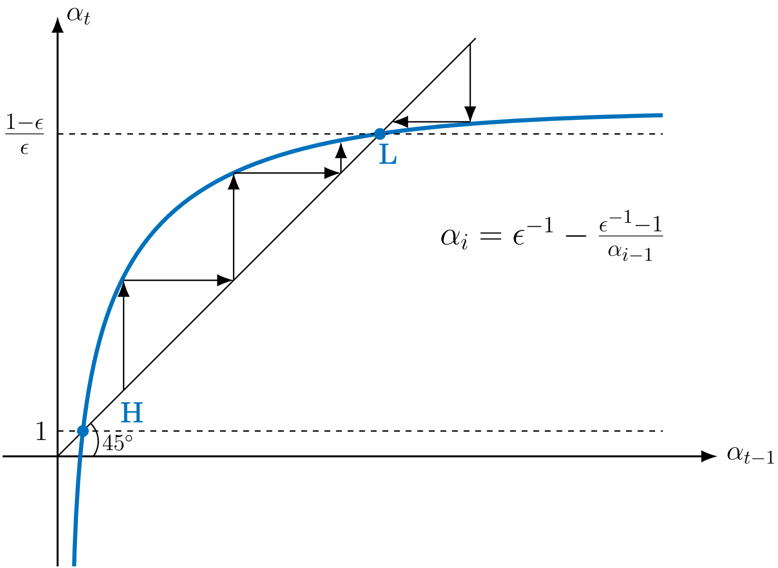
- In a stationary equilibrium, we have

$$\alpha = \epsilon^{-1} - \frac{\epsilon^{-1} - 1}{\alpha} \iff \alpha^2 - \epsilon^{-1}\alpha + \epsilon^{-1} - 1 = 0$$

- There are two solutions to this second-order polynomial

1. $\alpha = 1$: equilibrium H with trade, $R = 1$
2. $\alpha = \frac{1-\epsilon}{\epsilon} = \frac{u'(\epsilon)}{u'(1-\epsilon)} > 1$: equilibrium L without trade, $R < 1$

# L Is Stable, H Is Unstable



# Summary

- There is a continuum of nonstationary equilibria
  - If $c_1^0$ is such that $\alpha_1 > 1$, $\alpha_i$ converges to $\frac{1-\epsilon}{\epsilon}$, ie equilibrium L
  - If $c_1^0$ is such that $\alpha_1 = 1$, $\alpha_i = 1$ for all $i$, ie equilibrium H
  - If $c_1^0$ is such that $\alpha_1 < 1$, $\alpha_i$ converges to $-\infty$, not an equilibrium
- All nonstationary equilibria converge to stationary L
- L is stable, H is unstable

# 2. Money and Nonstationary Equilibria

# Sequential Trading

- We switch to a slightly more realistic framework with sequential trading
- Agents trade a durable asset passed from old to young
- Again, this asset is money

# Budget Constraint

- As in the previous lecture, the initial old generation $i = 0$ is endowed with $M > 0$ units of fiat money
- The budget constraints of a young agent born in $i \geq 1$ are

$$
\begin{aligned}
c_i^i + \frac{m_i^i}{p_i} &\leq y_i^i \\
c_{i+1}^i &\leq \frac{m_i^i}{p_{i+1}} + y_{i+1}^i \\
m_i^i &\geq 0
\end{aligned}
$$

- The young demand money $\frac{m_i^i}{p_i}$ and purchase it from the old

# Saving Function

- Summarize the agent's optimal decisions with a saving function

$$s(\alpha_i; y_i^i, y_{i+1}^i) = y_i^i - c_i^i = \frac{m_i^i}{p_i}$$

- Savings are equal to income minus consumption
- Savings are done by purchasing money holdings

# Equilibrium Conditions

- The equilibrium conditions become

$$
\begin{aligned}
s(\alpha_i; y_i^i, y_{i+1}^i) &= \frac{M}{p_i} \\
\alpha_i &= \frac{p_{i+1}}{p_i} \\
\text{where } c_{i+1}^i &= y_{i+1}^i + \frac{M}{p_{i+1}}
\end{aligned}
$$

- The net savings of generation $i$, $s(\alpha_i; y_i^i, y_{i+1}^i)$, equal the net dissavings of generation $i-1$, $M/p_i$

# Computing an Equilibrium

- To compute an equilibrium, proceed as follows
- First solve the previous difference equations for $\{p_i\}_{i=1}^{\infty}$
- Then get the allocation from the household budget constraints at equality
- Let's see this with an example

# Example

- Suppose the utility function is $u(c) = \ln c$
- The endowment is

$$(y_i^i, y_{i+1}^i) = (w_1, w_2) \quad \text{with } w_1 > w_2$$

- As before, agents have a higher endowment when they are young

# Household Problem

- Household $i$ solves

$$\max_{s(\alpha_i)} \left\{ \ln\underbrace{[w_1 - s(\alpha_i)]}_{c_i^i} + \ln \underbrace{\left( w_2 + \frac{p_i}{p_{i+1}} s(\alpha_i) \right)}_{c_{i+1}^i} \right\}$$

- The first-order condition is

$$\frac{1}{w_1 - s(\alpha_i)} = \frac{p_i/p_{i+1}}{w_2 + (p_i/p_{i+1})s(\alpha_i)}$$

- Solve for $s(\alpha_i)$

$$s(\alpha_i) = \frac{1}{2}\left( w_1 - w_2 \frac{p_{i+1}}{p_i} \right)$$

# Equilibrium

- Market clearing, ie savings equal dissavings, implies

$$\underbrace{\frac{1}{2}\left(w_1 - w_2\frac{p_{t+1}}{p_t}\right)}_{\text{savings}} = \underbrace{\frac{M}{p_t}}_{\text{dissavings}}$$

- Solve for $p_t$

$$p_t = \frac{2M}{w_1} + \frac{w_2}{w_1}p_{t+1}$$

- This is a first-order difference equation in $p_t$

# Theorem

- Consider the following difference equation

$$x_t = a + bx_{t+1}, \qquad -1 < b < 1$$

- The solution of this difference equation has the form

$$x_t = \frac{a}{1-b} + k\left(\frac{1}{b}\right)^t \quad \text{for any scalar } k$$

- There is an infinity of solutions, one for each $k$
- To pin down $k$, we typically need a condition imposed from outside such as an initial condition or a terminal condition

# Solution

- Going back to our example, and applying the theorem of the previous slide, we obtain the following solutions with a positive price level

$$p_t = \frac{2M/w_1}{1 - w_2/w_1} + k\left(\frac{w_1}{w_2}\right)^t, \qquad k \geq 0$$

- The solution $k = 0$ is the unique stationary solution
- Any solution with $k > 0$ leads to ever increasing price levels, $p_t \to \infty$ as $t \to \infty$

# Conclusion

- There exists a continuum of equilibria indexed by $k \geq 0$
- All equilibria except one converge to autarky, where $p_t = \infty$, money has no value, agents don't trade, they simply eat their current endowment
- The exception is $k = 0$, where the price level is finite, money has value, agents engage in intergenerational trade

# 3. Equivalence of Equilibria

# Budget Constraints of All Generations but the Initial One

- Take the sequential problem with money for the initial old and combine the two budget constraints of generation $i \geq 1$

$$c_i^i + c_{i+1}^i \frac{p_{i+1}}{p_i} \leq y_i^i + y_{i+1}^i \frac{p_{i+1}}{p_i}$$

- If $\frac{p_{i+1}}{p_i} = \frac{q_{i+1}^0}{q_i^0} = \alpha_i$, the budget constraint is the same as with time 0 trading
- Now take the budget constraint of the initial old generation $i = 0$

$$c_1^0 = y_1^0 + \frac{M}{p_1}$$

- This is the same as with time zero trading except for the extra term $M/p_1$

# Everything Is Identical

- The budget constraints with time zero and sequential trading are identical
- Also, the optimization problem of agents $i \geq 1$ is identical whether we are in time zero trading or sequential trading
- The feasibility constraint with sequential trading is the same as the feasibility constraint with time zero trading

# Equivalence

Proposition: Suppose the initial old agents in the time zero trading arrangement receive a transfer from the clearinghouse that corresponds to the surplus. If agents have a higher endowment when young, $c_1^1 < y_1^1$, then for any time zero equilibrium there exists a sequential-trading monetary equilibrium with the same allocation. The converse is true.

# Monetary Equilibrium

- In the monetary equilibrium, time $t$ real balances are

$$\text{per capital savings of the young} \;=\; \text{per capital dissavings of the old}$$

- To be a monetary equilibrium, both quantities must be positive for all $t \geq 1$

$$\frac{M}{p_1} = y_1^1 - c_1^1 > 0 \quad \text{requires} \quad c_1^1 < y_1^1$$

# 4. Government Deficit Finance

# Government Spending

- We continue with sequential trading
- We now introduce a government that spends
- The government issues money to finance its deficit
- Agents trade fiat money to smooth consumption

# Young Agents

- The population is constant
- At each date $t \geq 1$, there are $N$ identical young agents
- Each young agent has utility function $u(c_t^t) + u(c_{t+1}^t)$ and is endowed with

$$(y_t^t, y_{t+1}^t) = (w_1, w_2), \qquad w_1 > w_2 > 0$$

- As usual, agents earn more when they are young

# Initial Old Agents

- At time 1, there are $N$ old people
- On top of an endowment $w_2$, each old person receives $M_0 > 0$ units of inconvertible, durable fiat currency
- The initial old have utility function $c_1^0$

# Government

- The government spends a constant stream of expenditures per capita $g$
- At each date $t \geq 1$, the government levies lump-sum taxes of $\tau_1$ on each young agent and $\tau_2$ on each old agent
- At each date $t \geq 1$ the government augments the money supply according to the following government budget constraint

$$M_t - M_{t-1} = p_t(g - \tau_1 - \tau_2)$$

- The budget deficit is financed by issuing new money

# Infinite Price

- $p_t > 0$ is the price level
- If $p_t = +\infty$, the government budget constraint is interpreted as

$$g = \tau_1 + \tau_2$$

- In words, if money has no value, the government finances spending only with taxes, ie it runs a balanced budget

# Household Problem

- Define a saving function $s_t^t = y_t^t - c_t^t$
- For each $t \geq 1$, a young person's behavior is summarized by

$$s_t = f(R_t, \tau_1, \tau_2) = \arg\max_{s \geq 0} [u(w_1 - \tau_1 - s) + u(w_2 - \tau_2 + R_t s)]$$

- The saving function $s_t$ is equal to the demand function for money $f(R_t, \tau_1, \tau_2)$

# Monetary Equilibrium

A monetary equilibrium, ie an equilibrium with valued fiat currency, is a pair of positive sequences $\{M_t, p_t\}$ such that

1. Given the sequence of price levels, money demand equals money supply

$$f(R_t) = \frac{M_t}{p_t}$$

2. $R_t = p_t/p_{t+1}$ and $p_t > 0$ for all $t \geq 1$
3. The government budget constraint is satisfied for all $t \geq 1$

- Optimality conditions and market clearing are implicit in item 1

# Market Clearing

- Note that
$$f(R_t) = \underbrace{\frac{M_t}{p_t}}_{\text{Total}} = \underbrace{\frac{M_{t-1}}{p_t}}_{\text{Old}} + \underbrace{\frac{M_t - M_{t-1}}{p_t}}_{\text{Government}}$$
- $\frac{M_{t-1}}{p_t}$ is the dissavings of the old, ie the real value of currency that the old exchange for time $t$ consumption
- $\frac{M_t - M_{t-1}}{p_t}$ is the dissavings of the government, ie the real value of additional currency that the government prints at time $t$
- The sum of the two, $M_t/p_t$, is used to buy time $t$ goods from the young

# Deficit

- To compute an equilibrium, define deficit $d$

$$d = g - \tau_1 - \tau_2$$

- Write the government budget constraint as

$$\begin{aligned}
\text{for } t \geq 2: \quad & \frac{M_t}{p_t} = \frac{M_{t-1}}{p_{t-1}} \frac{p_{t-1}}{p_t} + d \\
\text{for } t = 1: \quad & \frac{M_1}{p_1} = \frac{M_0}{p_1} + d
\end{aligned}$$

# Difference Equation

- Substitute the equilibrium condition $f(R_t) = M_t/p_t$ into these equations

$$\text{for } t \geq 2: \quad f(R_t) = f(R_{t-1})R_{t-1} + d$$
$$\text{for } t = 1: \quad f(R_1) = \frac{M_0}{p_1} + d$$

- Given $M_0$ and $p_1$, we determine $R_1$ using the $t = 1$ equation
- Next, we have an autonomous difference equation in $R_t$ which we can solve

# Steady State

- Let's find a stationary solution to the previous equations
- $f(R_t)$ is time-invariant because the endowment, tax rates, and government deficit are all time-invariant
- We guess that $R_t = R$ for $t \geq 1$
- The previous equations become

$$\text{for } t \geq 2: \quad f(R) = f(R)R + d$$
$$\text{for } t = 1: \quad f(R) = \frac{M_0}{p_1} + d$$

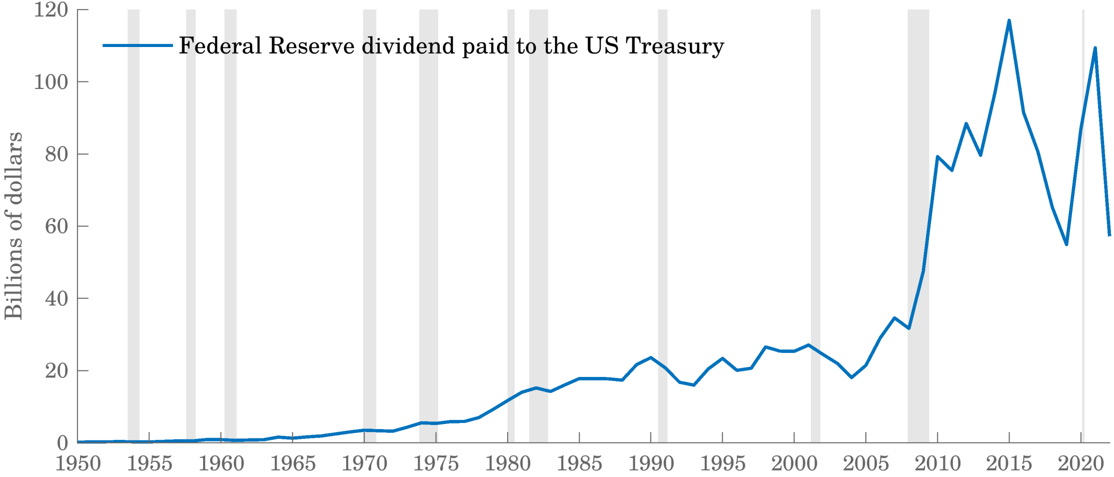
# Inflation Tax

- Since $R = \frac{p}{p_{+1}}$, then $(1 - R) = \frac{p_{+1} - p}{p_{+1}}$, and hence the previous equation becomes
$$d = f(R)(1 - R) = \frac{M}{p} \frac{p_{+1} - p}{p_{+1}}$$
- $(1 - R) = \frac{p_{+1} - p}{p_{+1}}$ is the inflation rate from one period to another
- Inflation acts like a tax on the real value of money balances: higher prices reduce the purchasing power of money, thus "taxing" people's assets
- This is why inflation is sometimes referred to as inflation tax

# Seignorage

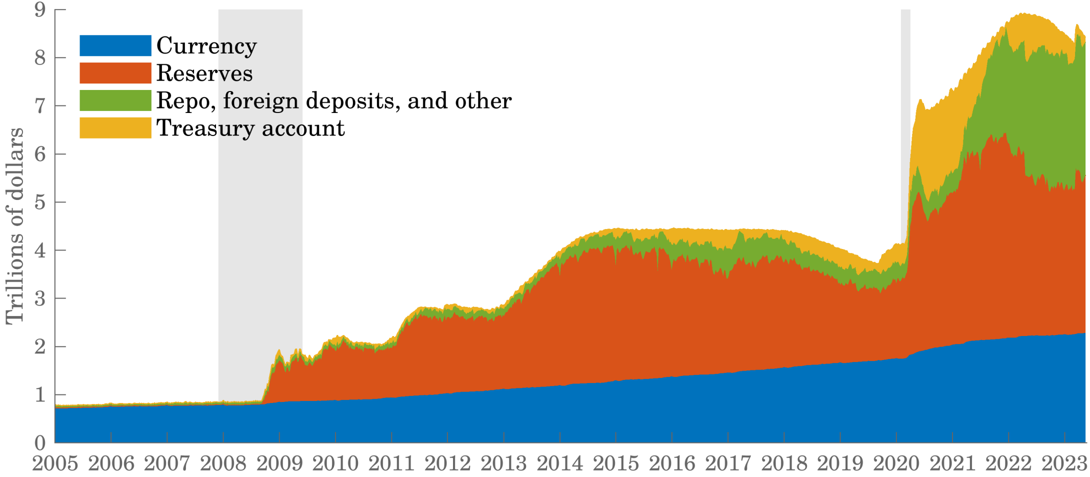
- The government finances its deficit with the inflation tax
- This is sometimes called seignorage but is actually distinct
- Seignorage is the revenue that governments enjoy by having the monopoly to issue the monetary base
- The government prints money for free and invests it in interest-bearing assets such as bonds and foreign exchange

# Seignorage in the United States




Sources: US Bureau of Economic Analysis

# Federal Reserve Liabilities




Source: Federal Reserve Board

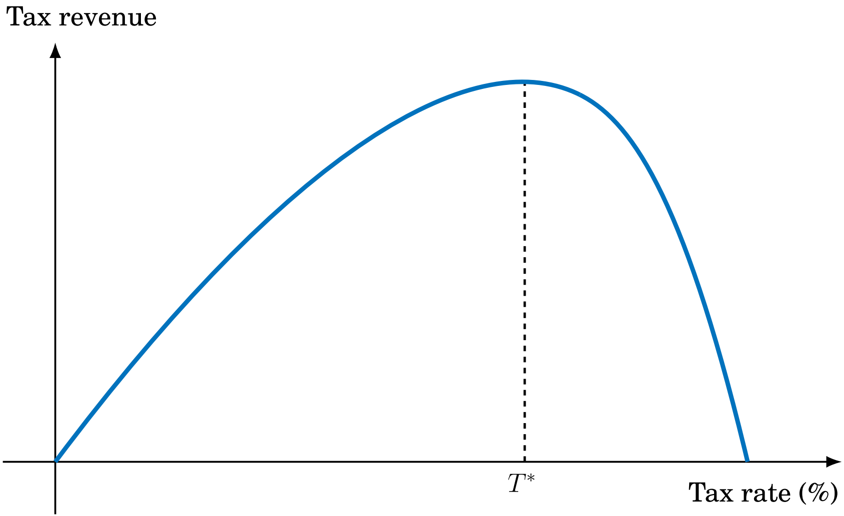
# Laffer Curve

- Repeat the deficit equation

$$d = f(R)(1-R) = \frac{M}{p}\frac{p_{+1}-p}{p_{+1}}$$

- We can interpret this equation as a Laffer curve for the inflation tax rate
- The Laffer curve, named after American economist Arthur Laffer, shows the possible relationship between tax rates and government tax revenues

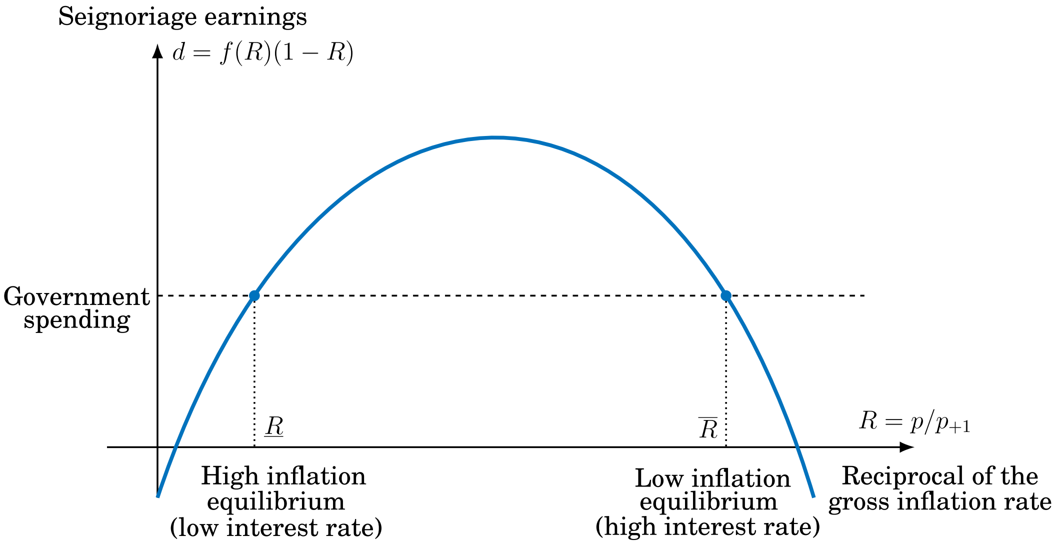
# A Generic Laffer Curve



# Example

- Suppose that $u(c) = \ln c$
- We then have
$$f(R) = \frac{w_1 - \tau_1}{2} - \frac{w_2 - \tau_2}{2R}$$
- Let's plot $f(R)(1-R)$ against $R$

# Laffer Curve



# Stationary Equilibria

- There are two stationary equilibria
- The high-interest-rate, low-tax $R = \overline{R}$ is associated with the good Laffer curve stationary equilibrium
- The low-interest-rate, high-tax $R = \underline{R}$ comes with the bad Laffer curve stationary equilibrium

# Nonstationary Equilibria

- There is a continuum of nonstationary solutions to

$$f(R_t) = f(R_{t-1})R_{t-1} + d$$

- All but one have $R_t \to \underline{R}$ as $t \to \infty$
- Thus the bad Laffer curve equilibrium is stable
- The good Laffer curve equilibrium is unstable

# 5. Government Deficit: Equivalent Setups

# Three Equivalent Setups

- We take a model with a government deficit
- We show that it can be supported with

1. Sequential trading of fiat currency
2. Sequential trading of government-indexed bonds
3. Time zero trading of Arrow-Debreu securities

- All three support the same equilibrium allocations

# Setup 1 – Fiat Money

- Same model as previously
- An initial old person is endowed with $M_0 > 0$ units of unbacked currency and $y_1^0$ units of the consumption good
- The government purchases a stream $\{g_t\}$ per young person but sets all taxes to zero

# Money-Financed Deficit Equilibrium

An equilibrium with money-financed government deficits is a sequence $\{M_t, p_t\}_{t=1}^{\infty}$ with $0 < p_t < \infty$ and $M_t > 0$ that

- Given $\{p_t\}$, satisfies the household problem

$$M_t = \arg\max_{\tilde{M} \geq 0} \left[ u\left( y_t^t - \frac{\tilde{M}}{p_t} \right) + u\left( y_{t+1}^t + \frac{\tilde{M}}{p_{t+1}} \right) \right]$$

- And satisfies the government budget constraint

$$M_t - M_{t-1} = p_t g_t$$

with $M_0 > 0$ given and $c_1^0 = y_1^0 + M_0/p_1$

# Setup 2 – Indexed Government Bonds

- Consider now a version of the same economy in which there is no currency but rather indexed government bonds
- Same demographics and endowments, but now each initial old person is endowed with $B_1$ units of a maturing bond, expressed in units of time 1 good
- In period $t$, the government sells new one-period bonds to the young to 1) finance its spending $g_t$ and 2) pay off the one-period debt falling due at $t$
- Let $R_t > 0$ be the gross real one-period rate of return on government debt between $t$ and $t + 1$

# Bond-Financed Deficit Equilibrium

An equilibrium with bond-financed government deficits is a sequence $\{B_{t+1}, R_t\}_{t=1}^{\infty}$ that

- Given $\{R_t\}$, satisfies the household problem

$$B_{t+1} = \arg\max_{\tilde{B}} \left[ u\left( y_t^t - \frac{\tilde{B}}{R_t} \right) + u\left( y_{t+1}^t + \tilde{B} \right) \right]$$

- And satisfies the government budget constraint

$$\frac{B_{t+1}}{R_t} = B_t + g_t$$

with $B_1 \geq 0$ given and $c_1^0 = y_1^0 + B_1$

# Equivalence

- The two equilibria are isomorphic
- Take an equilibrium with money-financed deficits and set

$$B_t = \frac{M_{t-1}}{p_t} \quad \text{and} \quad R_t = \frac{p_t}{p_{t+1}}$$

- This becomes an equilibrium with bond-financed deficits

# Setup 3 – Equilibrium with Arrow-Debreu Securities

Let $B_1^g$ represent claims to time 1 consumption owed by the government to the old at time 1. An equilibrium with time zero trading is an initial level of government debt $B_1^g$, a price system $\{q_t^0\}_{t=1}^{\infty}$, and a sequence $\{s_t\}_{t=1}^{\infty}$ that

- Given $\{q_t^0\}$, satisfies the household problem

$$s_t = \arg\max_{\tilde{s}} \left[ u\left(y_t^t - \tilde{s}\right) + u\left(y_{t+1}^t + \frac{q_t^0}{q_{t+1}^0}\tilde{s}\right)\right]$$

- And satisfies the government budget constraint

$$q_1^0 B_1^g + \sum_{t=1}^{\infty} q_t^0 g_t = \lim_{T\to\infty} q_T^0 B_T$$

# Equivalence

- Both the money-financed deficit and bond-financed deficit equilibria are isomorphic to the time zero trading equilibrium
- To see this, set

$$s_t = \frac{B_t}{R_t} \quad \text{and} \quad \frac{q_t^0}{q_{t+1}^0} = \frac{p_t}{p_{t+1}}$$

# Equilibrium with Arrow-Debreu Securities

- To obtain the time 0 government budget constraint of the previous slide, start from the recursive form

$$q_{t+1}^0 B_{t+1}^g = q_t^0 B_t^g + q_t^0 g_t$$

- Solve this equation forward

$$B_1^g = \frac{q_2^0}{q_1^0} B_2^g - g_1 = \cdots = \lim_{T \to \infty} \frac{q_T^0}{q_1^0} B_T^g - \sum_{t=1}^{\infty} \frac{q_t^0}{q_1^0} g_t$$

- Obtain the Arrow-Debreu government budget constraint

$$q_1^0 B_1^g + \sum_{t=1}^{\infty} q_t^0 g_t = \lim_{T \to \infty} q_T^0 B_T$$

# Ponzi Scheme

- From the time 0 budget constraint, we see that if $B_1^g > 0$ and $g_t > 0$ at all $t$ then
$$\lim_{T\to\infty} q_T^0 B_T \to +\infty$$
- The government is running a Ponzi scheme with ever-increasing debt
- This ever-decreasing negative net worth corresponds to the unbacked claims that the market nevertheless values
- This is possible only in the OLG model

# Just Need An Asset

- To sum up, what matters is the existence of an asset that can be traded
  - Money
  - One-period government bond
  - Arrow-Debreu security
- In each case, this asset is given by the government to the initial old generation: $M_0$, $B_1$, $B_1^g$

# Adding Lump-Sum Taxes

- What if we add lump-sum taxes $\tau_t^t$?
- The policy sequence is $\{\tau_t^t, \tau_{t+1}^t, g_t\}_{t=1}^{\infty}$ and the equilibrium satisfies

$$s_t = \arg\max_{\tilde{s}} \left[ u\left(y_t^t - \tilde{s} - \tau_t^t\right) + u\left(y_{t+1}^t + \frac{q_t^0}{q_{t+1}^0}\tilde{s} - \tau_{t+1}^t\right)\right]$$

and

$$q_1^0 B_1 + \sum_{t=1}^{\infty} q_t^0 g_t = \sum_{t=1}^{\infty} q_t^0 (\tau_t^{t-1} + \tau_t^t) + \lim_{T\to\infty} q_T^0 B_T$$

# Ricardian Equivalence Fails

- From the government budget constraint, we see that the initial level of debt $B_1$ affects the equilibrium allocation
- Therefore the method of government financing matters
- The tax-debt mix is not irrelevant
- Ricardian equivalence fails in the OLG model

# Intuition

- For an infinitely-lived agent, an increase in government spending means higher taxes either now or later
- Thus the debt-tax mix is irrelevant
- For a mortal person, the timing of taxes matters a lot
- A rise in government spending can be financed through higher taxes now when the person lives or in the future when the person is no longer alive

# Conclusion

- We have presented another feature of OLG models
- There is a continuum of equilibria, both with time zero trading of Arrow-Debreu securities and sequential trading of some durable good
- The stationary equilibrium with trade is unstable
- Provided the initial old receives a transfer, the time zero trading and sequential trading equilibria are equivalent
- Money, debt, and Arrow-Debreu securities are equivalent

# 6. Exercises

# Exercise 1 – Population Growth

Consider an economy with overlapping generations of two-period lived consumers. At each date $t \geq 1$ there are born $N_t$ identical young people each of whom is endowed with $w_1 > 0$ units of single consumption good when young and $w_2 > 0$ units of the consumption good when old. Assume that $w_2 < w_1$. The consumption good is not storable. The population of young people is described by $N_t = nN_{t-1}$, where $n > 0$. Young people born at $t$ rank utility streams according to $\ln c_t^t + \ln c_{t+1}^t$ where $c_t^i$ is the consumption of the time $t$ good of an agent born in $i$. In addition, there are $N_0$ old people at time 1, each of whom is endowed with $w_2$ units of the time 1 good. The old at $t = 1$ are also endowed with one unit of unbacked pieces of infinitely durable but intrinsically worthless pieces of paper called fiat money.

# Exercise 1 – Continued

1. Define an equilibrium without valued fiat currency. Compute such an equilibrium.
2. Define an equilibrium with valued fiat currency.
3. Compute all equilibria with valued fiat currency.
4. Find the limiting rates of return on currency as $t \to \infty$ in each of the equilibria that you found in 3. Compare them with the one-period interest rate in the equilibrium in 1.
5. Are the equilibria in 3 ranked according to the Pareto criterion?

# Exercise 2 – Social Security and the Price Level

Consider an economy (economy I) that consists of overlapping generations of two-period-lived people. At each date $t \geq 1$ there is born a constant number of $N$ of young people, who desire to consume both when they are young, at $t$, and when they are old, at $t+1$. Each young person has the utility function $\ln c_t^t + \ln c_{t+1}^t$, where $c_t^s$ is time $t$ consumption of an agent born at $s$. For all dates $t \geq 1$, young people are endowed with $y > 0$ units of a single nonstorable consumption good when they are young and zero units when they are old. In addition, at time $t = 1$ there are $N$ old people endowed in the aggregate with $H$ units of unbacked fiat currency. Let $p_t$ be the nominal price level at $t$, denominated in dollars per time $t$ good.

# Exercise 2 – Continued

1. Define and compute an equilibrium with valued fiat money for this economy. Argue that it exists and is unique.

Now consider a second economy (economy II) that is identical to the economy I except that it possesses a social security system. In particular, at each date $t \geq 1$ the government taxes $\tau > 0$ units of the time $t$ consumption good away from each young person and at the same time gives $\tau$ units of the time $t$ consumption good to each old person then alive.

2. Does economy II possess an equilibrium with valued fiat currency? Describe the restrictions on $\tau$, if any, that are needed to ensure the existence of such an equilibrium.
3. If an equilibrium with valued fiat currency exists, is it unique?
4. Consider the stationary equilibrium with valued fiat currency. Is it unique? Describe how the value of currency or price level would vary across economies with differences in the size of the social security system, as measured by $\tau$.