21. Endogenous Growth

Neoclassical Setting

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Growth? Yes, but Exogenous

- We return to the central issue of economic growth
- So far we have studied two broad classes of growth theories: 1) the neoclassical growth model and 2) the overlapping generations model
- However, the only way these models generate long-run growth in income per capita is by invoking exogenous technological change
- In other words, these models simply assume growth, they don't explain it

Two Questions

- We would like to understand and model the drivers of economic growth
- This would help us shed light on two major questions

1. Why did economic growth take off only about 300 years ago?
2. Why do some countries grow faster than others over long periods of time?

Endogenous Growth

- We have to endogenize the process of economic growth
- We are going to do this in two steps

1. First-generation models of endogenous growth: we generate sustained growth in (quasi) neoclassical settings
2. Second-generation models of endogenous growth: we endogenize technological progress and we study creative destruction

Lecture Outline

1. Households
2. Growth with Exogenous Technological Change
3. The AK Model
4. The AK Model with Human Capital
5. Exercises

Main Reference: Acemoglu, 2009, *Introduction to Modern Economic Growth*, Chapter 11

1. Households

The Common Household Problem

- The problem of the representative household will be the same in all models of the upcoming three lectures (except one, in this lecture)
- This is because most of these models focus on the supply side, ie firms
- Therefore we derive the household problem here once and for all
- The problem is entirely standard

Population

- A large number of identical households populate the economy
- Within each household, the number of individuals grows at rate $n \geq 0$
- The total population in the economy starts at $L_0 = 1$, therefore the total population at time t is
$$L_t = e^{nt}$$
- Population growth is thus
$$\dot{L}_t = ne^{nt} = nL_t$$

Preferences

- As usual, define per capita consumption $c_t \equiv C_t/L_t$
- The representative household has preferences

$$\int_0^\infty e^{-\rho t} L_t u\left(\frac{C_t}{L_t}\right) dt = \int_0^\infty e^{-(\rho-n)t} u(c_t) dt$$

- We assume $u'(c) > 0$ and $u''(c) < 0$ for all c in the interior of its domain
- We also make the standard discounting assumption $\rho > n$

Risk-Free Asset

- Let $\mathcal{A}_t$ denote total asset holdings of households
- $\mathcal{A}_t$ is a riskless asset that pays market return r_t
- $\mathcal{A}_t$ can be claims to capital stock K_t or treasury bonds B_t
- The law of motion for the asset is

$$\dot{\mathcal{A}}_t = r_t \mathcal{A}_t + w_t L_t - c_t L_t$$

- This is the aggregate budget constraint of households

Individual Budget Constraint

- Define $a_t \equiv \mathcal{A}_t / L_t$ as the per capita demand for assets
- Derive a_t with respect to time and use $\dot{L}_t = nL_t$

$$\dot{a}_t = \frac{\dot{\mathcal{A}}_t L_t - \mathcal{A}_t \dot{L}_t}{L_t^2} = \frac{\dot{\mathcal{A}}_t - \mathcal{A}_t n}{L_t} \quad \Longrightarrow \quad \dot{\mathcal{A}}_t = \dot{a}_t L_t + n a_t L_t$$

- Plug that into the law of motion of $\mathcal{A}_t$ and divide by L_t

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t$$

- This is the individual budget constraint of households

Household Problem

- The problem of the representative household is

$$\max_{c_t, a_t} \int_0^\infty e^{-(\rho - n)t} u(c_t) dt$$

$$\text{subject to} \quad \dot{a}_t = (r_t - n)a_t + w_t - c_t$$

$$\text{and} \quad a_0 \text{ given}$$

Incomplete Problem

- The flow budget constraint is not enough to discipline household behavior, for two reasons

1. Households could borrow an infinitely high amount, $a_t \to -\infty$, to maximize their consumption, $c_t \to +\infty$
2. We want to ensure that the sequential trading formulation coincides with the Arrow-Debreu equilibrium

No-Borrowing Constraint

- A simple solution is to impose a no-borrowing constraint

$$a_t \geq 0 \quad \text{for all } t$$

- In a representative agent model, this is enough since we have $a_t = k_t \geq 0$ in equilibrium without public bonds
- But in a heterogeneous agent model, this constraint breaks the equivalence between time zero and sequential trading
- An initially poor agent with an increasing wage path could want to borrow at time zero, ie choose $a_t < 0$, to smooth consumption

Natural Debt Limit

- Another solution is to use a natural debt limit
- The natural debt limit requires that assets a_t never become so negative that the agent cannot pay its debt even if she consumes nothing
- Take the law of motion of a_t and set $c_t = 0$ from t onward (see next slide)

$$a_t \geq - \int_t^{\infty} w_s e^{-\int_t^s (r_z - n) dz} ds$$

- This is the net present discounted value of labor income
- The problem with this constraint is that in an economy with sustained growth, the debt limit would be $\hat{a} = -\infty$

Computing the Natural Debt Limit

- A first-order homogeneous linear differential equation is of the form $y' + p(t)y = 0$ and solves

$$y' = -p(t)y \implies \int \frac{1}{y} dy = \int -p(t)dt \implies \ln|y| = P(t)+C \implies y \pm e^{P(t)+C} \implies y = Ae^{P(t)}$$

  where $P(t)$ is the antiderivative of $-p(t)$ and $C = \ln A$

- A first-order non-homogeneous linear differential equation is of the form $y' + p(t)y = f(t)$ and, using the *method of variation of parameters*, has solution

$$y(t) = v(t)e^{P(t)} + Ae^{P(t)}$$

  where $v'(t) = e^{-P(t)} f(t)$ and $P(t) = -\int p(t)dt$ is the antiderivative of $-p(t)$

- Consider now our equation $\dot{a}_t = (r_t - n)a_t + w_t$; let $y = a$, $p(t) = (r_t - n)$, and $f(t) = w_t$

$$a_t = \int_t^{\infty} e^{-\int (r_z - n)dz} w_s ds + Ae^{-\int (r_z - n)dz}$$

- Setting $A = 0$, the natural debt limit is therefore $a_t \geq -\int_t^{\infty} w_s e^{-\int_t^s (r_z - n)dz} ds$

No-Ponzi Condition

- The third solution is to use the no-Ponzi scheme condition

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0$$

- The no-Ponzi scheme condition ensures households do not asymptotically tend to negative wealth; it is the best solution because
  - It is consistent with both time zero and sequential trading
  - It can be applied to cases of sustained growth

The Complete Household Problem

- The problem of the household is therefore

$$\max_{c_t, a_t} \int_0^\infty e^{-(\rho-n)t} u(c_t) dt$$

$$\text{subject to} \quad \dot{a}_t = (r_t - n)a_t + w_t - c_t,$$

$$\lim_{t \to \infty} a_t e^{-\int_0^t (r_s - n) ds} \geq 0,$$

$$\text{and} \quad a_0 \text{ given}$$

Hamiltonian

- To solve this problem, we begin by writing the current-value Hamiltonian

$$\tilde{H}(a_t, c_t, \mu_t) = u(c_t) + \mu_t[(r_t - n)a_t + w_t - c_t]$$

Necessary Conditions

- Next, we derive the necessary conditions for a candidate path

$$\begin{aligned}
\tilde{H}_c(a_t, c_t, \mu_t) &= u'(c_t) - \mu_t = 0 \\
\tilde{H}_a(a_t, c_t, \mu_t) &= \mu_t(r_t - n) = -\dot{\mu}_t + (\rho - n)\mu_t \\
\dot{a}_t &= (r_t - n)a_t + w_t - c_t \\
\lim_{t\to\infty} e^{-(\rho-n)t}\mu_t a_t &= 0
\end{aligned}$$

Rewriting the Transversality Condition

- Rearrange the second necessary condition, $\frac{\dot{\mu}_t}{\mu_t} = -(r_t - \rho)$
- Integrate using $\frac{d \ln \mu_t}{dt} = \frac{\dot{\mu}_t}{\mu_t}$, then take the exponential

$$\int_0^t \frac{\dot{\mu}_s}{\mu_s} ds = -\int_0^t (r_s - \rho) ds$$

$$\ln \mu_t - \ln \mu_0 = -\int_0^t (r_s - \rho) ds$$

$$\mu_t = \mu_0 e^{-\int_0^t (r_s - \rho) ds}$$

- Plug in the first necessary condition evaluated at $t = 0$

$$\mu_t = u'(c_0) e^{-\int_0^t (r_s - \rho) ds}$$

Rewriting the Transversality Condition

- Substitute the last equation of the previous slide into the transversality condition

$$\lim_{t\to\infty} e^{-(\rho-n)t} a_t u'(c_0) e^{-\int_0^t (r_s-\rho)ds} = 0$$

- Note that $u'(c_0) > 0$ and combine $e^{-(\rho-n)t}$ with $e^{-\int_0^t (r_s-\rho)ds}$

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s-n)ds} = 0$$

- This is the no-Ponzi condition holding with equality

Transversality Ensures Optimality

- Household maximization plus the transversality condition imply that the no-Ponzi condition holds as equality
- That means the lifetime budget constraint holds as equality: optimizing households never leave any money unspent
- The transversality condition makes sure the household uses its resources to maximize utility even in the far future

Transversality vs No-Ponzi

- The transversality and no-Ponzi conditions are intimately related but they are not the same thing
- The transversality condition is an optimality condition: it must be satisfied in order for the household to maximize its intertemporal utility
- The no-Ponzi condition is an external constraint imposed by the market: it is a lifetime budget constraint (which the agent would like to violate)
- So $\lim_{t \to \infty} a_t e^{-\int_0^t (r_s - n) ds} = 0$ is the lifetime budget constraint holding as equality

Sufficiency Conditions

- Let's check the sufficiency conditions
- $\tilde{H}$ is the sum of a strictly concave function of c and a linear function of (a, c) and thus is strictly concave in (a, c)
- For any admissible pair (a, c), $\lim_{t\to\infty} e^{-(\rho-n)t}\mu_t a_t = 0$
- Any solution satisfying the necessary conditions is therefore the unique solution to the household problem

Euler Equation

- Combining the first-order conditions as usual, we obtain the Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}(r_t - \rho)$$

where $\varepsilon(c_t) \equiv -\frac{u''(c_t)c_t}{u'(c_t)}$ is the inverse elasticity of intertemporal substitution

Balanced Growth Path

- Balanced growth is a pattern of growth consistent with a
  - Constant rate of output growth g_Y
  - Constant consumption-output ratio, $g_C = g_Y$
  - Constant capital-output ratio, $g_K = g_Y$
  - Constant capital share in national income $\alpha = r_t K_t / Y_t$
- Balanced growth implies that the return on capital r_t must be constant
- In per capita terms, we have $g_c = g_y = g_k$

Constant Elasticity of Substitution

- From the Euler equation, if $r_t \to r^*$ then $\dot{c}_t / c_t \to g_c$ is possible only if

$$\varepsilon(c_t) \to \varepsilon$$

- Thus, balanced growth is only consistent with utility functions that have asymptotically constant elasticities of intertemporal substitution
- For simplicity, we assume a CRRA utility function

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma \geq 0$$

- This implies that $\varepsilon(c_t) = \sigma$

Household Problem Solved

- With CRRA utility, the Euler equation becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}(r_t - \rho)$$

- Recall the transversality condition

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0$$

- These two equations solve the household problem

Competitive Equilibrium

- A competitive equilibrium is a path of quantities (k_t, c_t, y_t) and prices (w_t, r_t) such that

1. Given r_t and w_t, k_t solves the firm's problem, to be determined in each case
2. Given r_t and w_t, (a_t, c_t) solves the household's problem
3. Markets clear: the demand for capital k_t equals asset holdings a_t at each t

- By Walras's law, the goods market also clears

2. Growth with Exogenous Technological Change

Exogenous Growth

- Before turning to endogenous growth, we are going to solve the neoclassical growth model with exogenous

1. Population growth
2. Technological progress

Neoclassical Production

- The production technology is

$$Y_t = F(K_t, A_t L_t)$$

- Technological progress is labor-augmenting

Technological Change

- The economy starts with initial technology $A_0 = 1$
- Technology grows at rate $g \geq 0$
- Technology in t is therefore

$$A_t = e^{gt} A_0 = e^{gt}$$

- Technology grows according to

$$\dot{A}_t = g e^{gt} = g A_t$$

Uzawa's Theorem

- Uzawa (1961, *Restud*) proves the following theorem (see proof in Acemoglu 2009 section 2.7.3)

Theorem: In a standard growth model with a constant-returns-to-scale production function and exogenous and constant population growth and technological change, we have

1. $g_Y = g_K = g_C$
2. $Y_t = F(K_t, A_t L_t)$ and $\dfrac{\dot{A}_t}{A_t} = g = g_Y - n$

- In words, Uzawa's theorem says that only labor-augmenting technological progress is consistent with balanced growth

Intuition

- What is the intuition behind Uzawa's theorem?
- Since $r = f'(k) - \delta > n$, capital typically accumulates faster than labor, and thus there is an asymmetry between the two inputs
- But since balanced growth requires $g_K = g_Y$, technological change must make up for this asymmetry
- Thus it should take a labor-augmenting form

Troubling Result

- Uzawa's theorem is rather surprising and troubling
- Any other technological change—capital-augmenting or Hicks-neutral—is not compatible with balanced growth
- But there is no good reasons for why technical progress should be labor-augmenting and not take another form

Normalization

- Define capital per units of effective labor $\tilde{k}_t \equiv K_t/(A_t L_t)$
- Normalize the production function

$$y_t \equiv \frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right) \equiv f(\tilde{k}_t)$$

Firm Problem

- Assume the firm operates the technology transformation, ie builds capital
- The firm's problem writes

$$\max A_t L_t [f(\tilde{k}_t) - w_t - R_t \tilde{k}_t + (1-\delta)\tilde{k}_t]$$

- The FOC with respect to capital is

$$f'(\tilde{k}_t) = R_t - (1-\delta) = r_t + \delta$$

where $R_t \equiv 1 + r_t$

Euler Equation

- Define $\tilde{c}_t \equiv C_t/(A_t L_t) = c_t/A_t$
- Derive $\tilde{c}_t$ with respect to time and divide by $\tilde{c}_t$

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{\dot{c}_t}{c_t} - \frac{\dot{A}_t}{A_t}$$

- Plug this into the Euler equation and recall that $\dot{A}_t/A_t = g$

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1}{\sigma}(r_t - \rho - \sigma g)$$

- This is the Euler equation for consumption per unit of effective labor

Capital Accumulation

- The law of motion of capital is $\dot{K}_t = F(K_t, A_t L_t) - C_t - \delta K_t$
- Derive $\tilde{k}_t = K_t/(A_t L_t)$ with respect to time and rearrange

$$\dot{K}_t = \dot{\tilde{k}}_t A_t L_t + (n+g) K_t$$

- Plug this into the law of motion and divide by $A_t L_t$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (n+g+\delta)\tilde{k}_t$$

- This is the law of motion of capital per unit of effective labor

Steady State

- In steady state, there is growth in per capita variables, ie the steady state corresponds to the balanced growth path
- But variables in units of effective labor are constant

$$\dot{\tilde{k}}_t = 0, \qquad \dot{\tilde{c}}_t = 0, \qquad \text{and} \qquad \frac{\dot{c}_t}{c_t} = g$$

- Our system in steady state reads

$$\begin{aligned} f'(\tilde{k}^*) &= \rho + \delta + \sigma g \\ \tilde{c}^* &= f(\tilde{k}^*) - (n + g + \delta)\tilde{k}^* \end{aligned}$$

Transversality Condition

- Since $a_t = \mathcal{A}_t/L_t$, $\tilde{k}_t = K_t/(A_t L_t)$, and $A_t = e^{gt}$, we have

$$a_t = \tilde{k}_t e^{gt}$$

- The transversality condition $\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0$ becomes

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (r_s - n - g)ds} = 0$$

- Since $f'(\tilde{k}_t) = r_t + \delta$ and $f'(\tilde{k}^*) = \rho + \delta + \sigma g$, we obtain

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (\rho - [1-\sigma]g - n)ds} = 0$$

Discounting Assumption

- The transversality condition only holds if the exponent goes to minus infinity, that is if $\rho - (1-\sigma)g - n > 0$
- We thus modify the discounting assumption to take into account technological change: we assume $\rho - n > (1-\sigma)g$
- Reasonable if $\rho = r \approx .05$, $n \approx .01$, $\sigma > 0$, $g \approx .02$

Summary

- Two differential equations (plus the transversality condition) determine the equilibrium path of $(\tilde{c}_t, \tilde{k}_t)$

$$\text{Euler equation + FOC firm : } \quad \frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1}{\sigma}(r_t - \rho - \sigma g)$$

$$\text{Capital law of motion : } \quad \dot{\tilde{k}}_t = f(k_t) - \tilde{c}_t - (n + g + \delta)k\tilde{k}_t$$

- Given $\tilde{k}_0$, $(\tilde{c}_t, \tilde{k}_t)$ converge monotonically to the unique steady state $(\tilde{c}^*, k^*)$ given by

$$f'(\tilde{k}^*) = \rho + \delta + \sigma g; \qquad \tilde{c}^* = f(\tilde{k}^*) - (n + g + \delta)\tilde{k}^*$$

Balanced Growth Path

- In the long run, ie in the steady state, per capita output, consumption, and capital grow at the rate g
- This means per capita variables grow at the same rate as technical progress
- Aggregate output, consumption, and capital grow at the rate $n + g$, ie technological progress + population growth

3. The AK Model

The Simplest Model

- The AK model is the simplest neoclassical model of sustained growth
- It is not a very realistic model
- The goal here is to introduce endogenous growth theory and get a sense of its key underlying mechanism

Linear Production

- The aggregate production is given by

$$Y_t = AK_t, \quad A > 0$$

- The production function is linear in capital, the only input
- The production function does not depend on labor
- It is as if we imposed $\alpha = 1$ in the standard Cobb-Douglas functional form

No Diminishing Marginal Returns

- Define the usual capital-labor ratio $k_t \equiv K_t/L_t$; per capita output is

$$y_t \equiv \frac{Y_t}{L_t} = Ak_t$$

- In the AK model, there are constant returns to scale but no diminishing marginal returns, $f''(k) = 0$
- The Inada conditions do not hold

$$\lim_{k\to\infty} f'(k) = A > 0$$

- This feature is essential for sustained growth

Constant Return, Zero Wage

- Profit maximization by the firm requires that the marginal product of capital equal the marginal cost of capital
- The marginal product of capital is $f'(k_t) = A$, the marginal cost of capital is $R_t - (1-\delta) = r_t + \delta$, therefore the return to capital is constant

$$r_t = r = A - \delta$$

- Also, since the marginal product of labor is zero, labor earnings are null

$$w_t = 0$$

Equilibrium

- The equilibrium conditions of the AK model are

$$
\begin{aligned}
\text{Euler equation:} \quad & \frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}(r_t - \rho) = \frac{1}{\sigma}(A - \delta - \rho) \\
\text{Law of motion of capital:} \quad & \dot{k}_t = Ak_t - (\delta + n)k_t - c_t = (A - \delta - n)k_t - c_t \\
\text{Transversality condition:} \quad & \lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = \lim_{t\to\infty} k_t e^{-(A-\delta-n)t} = 0
\end{aligned}
$$

Constant Consumption Growth

- Looking at the Euler equation, we see that the growth rate of consumption is constant, ie it is independent of the level of capital
- Let's assume $A - \delta - \rho > 0 \iff A > \rho + \delta$
- This ensures positive growth in the economy

No Transitional Dynamics

- The fact that the growth rate is independent of the capital level implies that there are no transitional dynamics in this model
- Starting from any $k_0 > 0$, consumption per capita immediately starts growing at the constant rate forever
- To see this, integrate the Euler equation from 0 to t using $\frac{d\ln c_s}{ds} = \frac{\dot{c}_s}{c_s}$

$$\int_0^t \frac{\dot{c}_s}{c_s} ds = \int_0^t \frac{1}{\sigma}(A - \delta - \rho) ds$$

$$\ln c_t - \ln c_0 = \frac{1}{\sigma}(A - \delta - \rho)t$$

$$c_t = c_0 e^{\frac{1}{\sigma}(A-\delta-\rho)t}$$

Capital Dynamics

- The law of motion of capital is

$$\begin{aligned}\dot{k}_t &= (A - \delta - n)k_t - c_t \\ &= (A - \delta - n)k_t - c_0 e^{\frac{1}{\sigma}(A-\delta-\rho)t}\end{aligned}$$

- This is a first-order linear differential equation in k_t
- To solve this equation, we apply a theorem shown next slide

Theorem

- A first-order linear differential equation takes the form

$$\dot{x}_t = a_t x_t + b_t$$

- Let κ be a real constant, this equation has solution

$$x_t = \kappa e^{\int_0^t a_s ds} + e^{\int_0^t a_s ds} \int_0^t b_s \left(e^{\int_0^s a_v dv} \right)^{-1} ds$$

- Proof: see Acemoglu (2009), Section B.4

Solution

- Applying the theorem, the solution for k_t is

$$\begin{aligned}
k_t &= \kappa e^{(A-\delta-n)t} + e^{(A-\delta-n)t} \int_0^t \frac{-c_0 e^{\frac{1}{\sigma}(A-\delta-\rho)s}}{e^{(A-\delta-n)s}} ds \\
&= \tilde{\kappa} e^{(A-\delta-n)t} + \left((A-\delta)(\sigma-1)\frac{1}{\sigma} + \rho\frac{1}{\sigma} - n \right)^{-1} c_0 e^{\frac{1}{\sigma}(A-\delta-\rho)t}
\end{aligned}$$

  where $\tilde{\kappa} \neq \kappa$ is a constant to be determined
- The details are next slide

Algebra

$$
\begin{aligned}
k_t &= \kappa e^{\int_0^t (A-\delta-n)ds} - e^{\int_0^t (A-\delta-n)ds} \int_0^t \frac{c_0 e^{\frac{1}{\sigma}(A-\delta-\rho)s}}{e^{(A-\delta-n)s}} ds \\
&= e^{(A-\delta-n)t} \left(\kappa - c_0 \int_0^t e^{[\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta-n)]s} ds \right) \\
&= e^{(A-\delta-n)t} \left(\kappa - c_0 \frac{e^{[\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta-n)]s}\Big|_0^t}{\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta-n)} \right) \\
&= e^{(A-\delta-n)t} \left(\kappa - c_0 \frac{e^{[\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta-n)]t} - 1}{\frac{1}{\sigma}(A-\delta-\rho)-(A-\delta-n)} \right) \\
&= e^{(A-\delta-n)t} \underbrace{\left(\kappa - \frac{c_0}{(A-\delta)(\sigma-1)\frac{1}{\sigma}+\rho\frac{1}{\sigma}-n} \right)}_{\equiv \tilde{\kappa}} + \frac{1}{(A-\delta)(\sigma-1)\frac{1}{\sigma}+\rho\frac{1}{\sigma}-n} c_0 e^{\frac{1}{\sigma}(A-\delta-\rho)t} \\
&\equiv \tilde{\kappa} e^{(A-\delta-n)t} + \left((A-\delta)(\sigma-1)\frac{1}{\sigma} + \rho\frac{1}{\sigma} - n \right)^{-1} c_0 e^{\frac{1}{\sigma}(A-\delta-\rho)t}
\end{aligned}
$$

Using the Transversality Condition

- Substitute the last equation into the transversality condition $\lim_{t\to\infty}\left[k_t e^{-(A-\delta-n)t}\right]=0$

$$\lim_{t\to\infty}\left(\tilde{\kappa}+\left[(A-\delta)(\sigma-1)\frac{1}{\sigma}+\rho\frac{1}{\sigma}-n\right]^{-1}c_0e^{-\left[(A-\delta)(\sigma-1)\frac{1}{\sigma}+\rho\frac{1}{\sigma}-n\right]t}\right)=0$$

- We assume $(A-\delta)(\sigma-1)\frac{1}{\sigma}+\rho\frac{1}{\sigma}-n>0$
- Therefore the second term tends to zero as $t\to\infty$
- We conclude that $\lim_{t\to\infty}\tilde{\kappa}=0 \iff \tilde{\kappa}=0$

Capital Path

- With $\tilde{\kappa} = 0$, the solution for k_t becomes

$$k_t = \left[(A-\delta)(\sigma-1)\frac{1}{\sigma} + \rho\frac{1}{\sigma} - n\right]^{-1} c_0\, e^{\frac{1}{\sigma}(A-\delta-\rho)t}$$

- Since capital is equal to k_0 at $t = 0$, we obtain

$$k_t = k_0\, e^{\frac{1}{\sigma}(A-\delta-\rho)t} \quad \text{where } k_0 = \left[(A-\delta)(\sigma-1)\frac{1}{\sigma} + \rho\frac{1}{\sigma} - n\right]^{-1} c_0$$

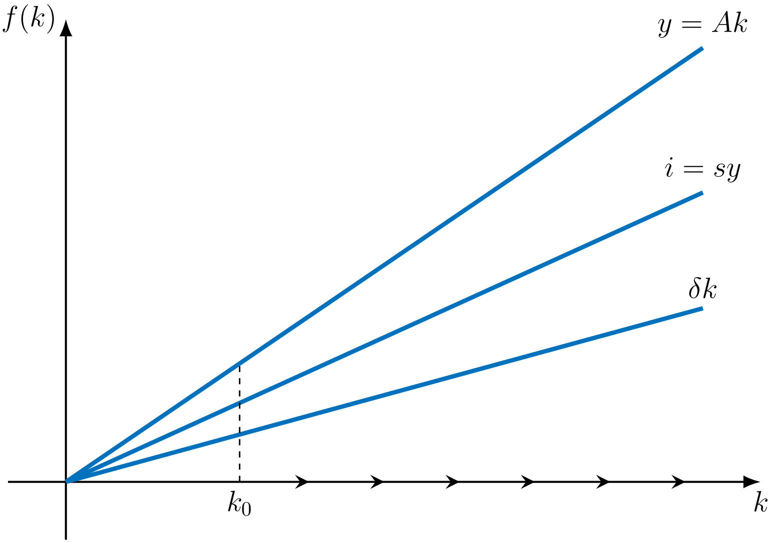
- Finally, with k_0 we obtain c_0

$$c_0 = \left[(A-\delta)(\sigma-1)\frac{1}{\sigma} + \rho\frac{1}{\sigma} - n\right] k_0$$

Constant Growth

- From $k_t = k_0 \, e^{\frac{1}{\sigma}(A-\delta-\rho)t}$, we conclude that capital grows at the same constant rate as consumption
- Output $y_t = Ak_t$ also grows at the same rate as k_t and c_t
- The growth rate of the economy is $g \equiv \frac{1}{\sigma}(A - \delta - \rho)$
- There are no transitional dynamics
- Given an initial capital stock k_0, all variables grow directly at rate g

Sustained Growth



Endogenous Growth

- In this simple AK model, growth is not only sustained but it is also endogenous, ie it is affected by structural parameters
- For example, an increase in the discount rate ρ reduces the economy's growth rate as agents become less patient and accumulate less capital
- In the neoclassical growth model, an increase in ρ reduces only the level of income per capita, not the growth rate
- Here, changes in A, δ, and σ affect both the level and the growth rate of consumption, capital, and output

Pareto Efficiency

- In the AK model, all markets are competitive, there is one representative household, and there are no externalities
- Therefore the competitive equilibrium is Pareto optimal
- The first welfare theorem holds

Nondecreasing Returns

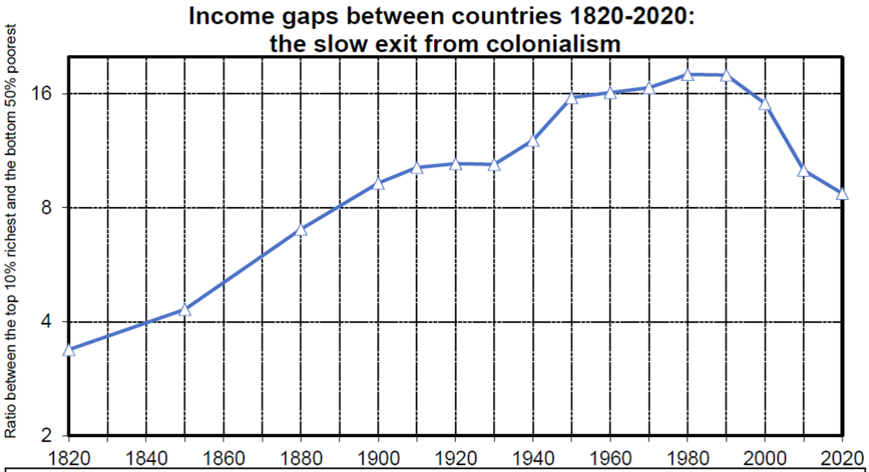
- In the AK model, the engine of growth is capital accumulation
- Technological progress is constant in the model
- The key assumption for sustained growth is the absence of decreasing marginal returns to capital
- This is arguably a strong and disputable assumption

Criticisms

- There are two main shortcomings about the AK model

1. $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$ with $\alpha = 1$ is not a realistic technology
    - There is no labor in the economy
    - The share of national income accruing to capital is one
2. Small variations in technology A and institutions (δ, ρ) generate huge differences in long-run per capita income
    - This is the opposite of the neoclassical growth model
    - This is inconsistent with a relative stable world income distribution, ie inequality does not really diverge worldwide

After Exploding, Global Inequality Is Finally Falling



Lecture. Income gaps between countries, as measured by the ration between the average income of the top 10% of the world population living in the richest countries and the bottom 50% of the population living in the poorest countries, have increased significantly between 1820 and 1960-1980, before beginning a period of reduction. **Note**. For the computation of this ration, the population of overlapping countries has been divided between deciles as if they were multiple countries. **Sources and series**: see piketty.pse.ens.fr/equality (figure 36)

4. The AK Model with Human Capital

Addressing One Criticism

- We address the main criticism of the AK model
- We enrich the model by including human capital on top of physical capital
- Human capital is the stock of skills, education, creativity, competences, and other characteristics embedded in labor
- The term originates from Nobel Prize winner Gary Becker who posited that people invest in their skills in the same way as firms invest in hard capital
- According to this view, human capital represents efficiency units of labor embedded in labor hours supplied by individuals

Production Function

- Suppose the aggregate production function takes the form

$$Y_t = F(K_t, H_t)$$

- H_t denotes human capital, ie efficiency units of labor
- F now again satisfies the usual assumptions: continuous, differentiable, increasing $F' > 0$, concave $F'' < 0$, and the Inada conditions
- We are back with diminishing marginal returns

Physical Capital

- Physical capital is standard and evolves according to

$$\dot{K}_t = I_t^k - \delta^k K_t$$

- I_t^k is the usual investment level in physical capital
- $\delta^k \in (0, 1)$ is the usual depreciation rate of physical capital

Human Capital

- Human capital is similar to physical capital and evolves according to

$$\dot{H}_t = I_t^h - \delta^h H_t$$

- I_t^h is the investment level in human capital
- $\delta_h \in (0, 1)$ is the depreciation rate of the human capital stock

Households

- The problem of the representative household changes (the one exception)
- Besides choosing consumption and asset holdings, the household selects investment in human capital
- Also, there are two state variables now, $\mathcal{A}_t$ and H_t

Constant Population and Normalization

- For simplicity, we assume constant population, $n = 0$
- We normalize population to $L = 1$
- Therefore, aggregate and per capita variables are equal

$$a_t = \mathcal{A}_t \qquad c_t = C_t \qquad h_t = H_t \qquad i_t^h = I_t^h \qquad i_t^k = I_t^k \qquad k_t = K_t$$

Household Problem

- The problem of the representative household is

$$\max_{c_t, a_t, h_t, i_t^h} \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{subject to} \quad \dot{a}_t = r_t a_t + w_t h_t - c_t - i_t^h,$$

$$\dot{h}_t = i_t^h - \delta^h h_t,$$

$$\text{and} \quad \lim_{t \to \infty} a_t e^{-\int_0^t r_s ds} \geq 0$$

Hamiltonian

- Define costate variables μ_t^a and μ_t^h, one for each constraint
- The current-value Hamiltonian writes

$$\begin{aligned}\tilde{H}(a_t, h_t, c_t, i_t^h, \mu_t^a, \mu_t^h) = u(c_t) &+ \mu_t^a[r_t a_t + w_t h_t - c_t - i_t^h] \\ &+ \mu_t^h[i_t^h - \delta^h h_t]\end{aligned}$$

Necessary Conditions

- The necessary conditions are

$$
\begin{aligned}
\tilde{H}_c(a_t, h_t, c_t, i_t^h, \mu_t^a, \mu_t^h) = 0: &\quad u'(c_t) = \mu_t^a \\
\tilde{H}_{i^h}(a_t, h_t, c_t, i_t^h, \mu_t^a, \mu_t^h) = 0: &\quad \mu_t^a = \mu_t^h \\
\rho\mu_t^a - \dot{\mu}_t^a = \tilde{H}_a(a_t, h_t, c_t, i_t^h, \mu_t^a, \mu_t^h): &\quad \rho\mu_t^a - \dot{\mu}_t^a = \mu_t^a r_t \\
\rho\mu_t^h - \dot{\mu}_t^h = \tilde{H}_h(a_t, h_t, c_t, i_t^h, \mu_t^a, \mu_t^h): &\quad \rho\mu_t^h - \dot{\mu}_t^h = \mu_t^a w_t - \mu_t^h \delta^h \\
\dot{a}_t = \tilde{H}_{\mu^a}(a_t, h_t, c_t, i_t^h, \mu_t^a, \mu_t^h): &\quad \dot{a}_t = r_t a_t + w_t h_t - c_t - i_t^h \\
\dot{h}_t = \tilde{H}_{\mu^h}(a_t, h_t, c_t, i_t^h, \mu_t^a, \mu_t^h): &\quad \dot{h}_t = i_t^h - \delta^h h_t \\
\text{Transversality condition 1}: &\quad \lim_{t\to\infty} e^{-\rho t}\mu_t^a a_t = 0 \\
\text{Transversality condition 2}: &\quad \lim_{t\to\infty} e^{-\rho t}\mu_t^h h_t = 0
\end{aligned}
$$

Same Shadow Values

- The first two necessary conditions imply

$$u'(c_t) = \mu_t^a = \mu_t^h = \mu_t$$

- Intuitively, there are no particular constraints on human and physical capital investments, like for example adjustment costs
- Thus at the optimum, the shadow values of these two types of investment must be equal at all points in time

Same Returns

- Combine the third and fourth necessary conditions and use $\mu_t^a = \mu_t^h = \mu_t$

$$r_t = w_t - \delta^h$$

- The rate of return on physical capital is equal to the rate of return on human capital

Euler Equation

- The first necessary condition, $\mu_t = u'(c_t)$, implies

$$\dot{\mu}_t = u''(c_t)\dot{c}_t$$

- Combine it with the third FOC, $\dot{\mu}_t = -\mu_t(r_t - \rho)$

$$u'(c_t)(r_t - \rho) = -u''(c_t)\dot{c}_t$$

- Define $\varepsilon(c_t) \equiv -\frac{u''(c_t)c_t}{u'(c_t)}$ as the inverse elasticity of intertemporal substitution and get the Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\varepsilon(c_t)}(r_t - \rho)$$

CRRA Utility

- Let's assume a CRRA utility function, ie with a constant inverse elasticity of intertemporal substitution $\sigma = \varepsilon(c_t)$

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \quad \sigma \geq 0$$

- With CRRA utility, the Euler equation becomes

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}(r_t - \rho)$$

Firms

- The representative firm chooses physical and human capital as inputs to produce output
- Define the effective capital-labor ratio as

$$\tilde{k}_t \equiv \frac{K_t}{H_t}$$

- The production function rewrites

$$Y_t = F\left(\frac{K_t}{H_t}, 1\right) H_t \equiv f(\tilde{k}_t) H_t$$

Firm's Problem

- The firm solves the problem

$$\max_{K_t, H_t} H_t[f(\tilde{k}_t) - w_t - R_t\tilde{k}_t + (1-\delta^k)\tilde{k}_t]$$

- The first-order conditions are

$$\begin{aligned} K_t: &\quad f'(\tilde{k}_t) = R_t - (1-\delta^k) = r_t + \delta^k \\ H_t: &\quad w_t = f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \end{aligned}$$

where recall $R_t \equiv 1 + r_t$

Equilibrium

- Combine the two FOCs of the firm with the household's necessary condition $r_t = w_t - \delta^h$ to obtain

$$f'(\tilde{k}_t) - \delta^k = f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) - \delta^h$$

- The left-hand side is decreasing in $\tilde{k}_t$
- The right-hand side is increasing in $\tilde{k}_t$
- Thus the effective capital-labor ratio must satisfy

$$\tilde{k}_t = \tilde{k}^* \quad \text{for all } t$$

Human Capital and Physical Capital Track Each Other

- The effective capital-labor ratio $\tilde{k}^*$ is constant at all times
- That means that as physical capital accumulates, human capital accumulates at the same rate
- Since the two inputs are complementary, more physical capital creates demand for human capital and vice versa

Constant Consumption Growth

- Constant $\tilde{k}^*$ implies that the return on capital is also constant

$$r = f'(\tilde{k}^*) - \delta^k$$

- We deduce that consumption grows at a constant rate

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}[f'(\tilde{k}^*) - \delta^k - \rho]$$

- If we assume $f'(\tilde{k}^*) > \delta^k + \rho$, the growth rate is positive

Constant Growth

- We use the same method as in the AK model to compute the growth rate of k_t, h_t, y_t; this is left as an exercise
- It turns out that consumption, physical capital, human capital, and output all grow at the same rate

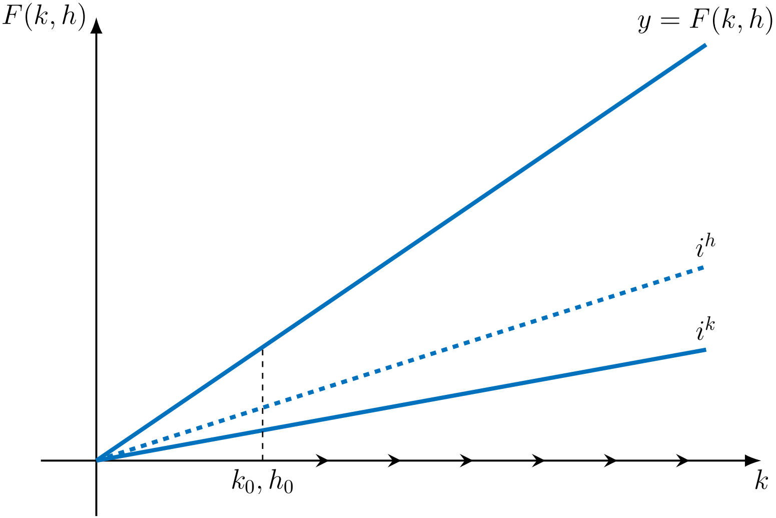
$$g^* \equiv \frac{1}{\sigma}[f'(\tilde{k}^*) - \delta^k - \rho]$$

- This is true whatever the initial condition $\tilde{k}_0 = K_0/H_0$

Sustained Growth

- As in the basic AK model, the AK model with human capital features no transitional dynamics
- For any initial condition (k_0, h_0), the economy grows at the constant growth rate g^* for ever

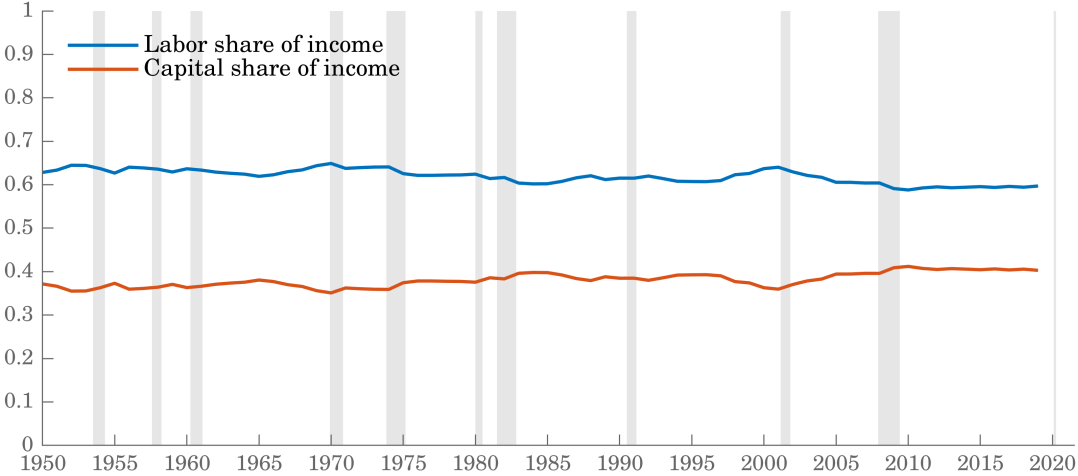
Sustained Growth



Constant and Nonzero Labor Share

- We can also show that in this economy the share of capital income in national income is constant and less than one; also left as an exercise
- Thus this model is more realistic because it generates
  - A stable factor distribution of income
  - A significant fraction of total income accruing to labor

Capital and Labor Share in US GDP



Source: Penn World Table

Just Like an AK Model

- Rewrite the production function as

$$Y_t = F\left(1, \frac{H_t}{K_t}\right) K_t \equiv f\left(\frac{H_t}{K_t}\right) K_t$$

- Since $\tilde{k}^* \equiv K_t/H_t$ is constant, so is H_t/K_t
- Define $A \equiv f\left(H_t/K_t\right)$ and obtain

$$Y_t = AK_t$$

- Thus the model with two types of capital is essentially the same as the basic AK model

Achievement and Limitations

- As in the basic AK model, this model generates long-run differences in growth rates from small policy differences
- It can account for large differences in income per capita across countries
- But it does so partly by generating large human capital differences across countries

To Be Continued

- We took a first look at endogenous growth theory
- The AK model and its variants generate sustained growth by having a linear production technology
- But they don't leave room for agents and governments to innovate, compete, and promote good institutions
- More on endogenous growth in lectures 22 and 23

5. Exercises

Exercise 1 – Neoclassical Growth and AK Model

Consider the neoclassical growth model with Cobb-Douglas technology $y_t = Ak_t^\alpha$, expressed in per capita terms, and log preferences. Characterize the equilibrium path of this economy, and show that as $\alpha \to 1$, the equilibrium path approaches that of the baseline AK economy. Interpret this result.

Exercise 2 – Efficient AK Model

Consider the basic AK model of the lecture and show that its equilibrium path is Pareto efficient.

Exercise 3 – AK Model with Human Capital

Consider the AK model with physical and human capital of the lecture.

1. Show that physical capital, human capital, and output grow at the same, constant rate as consumption.
2. Show that the share of capital income in output is constant and less than one at all times.