19. Optimal Control

Infinite Horizon

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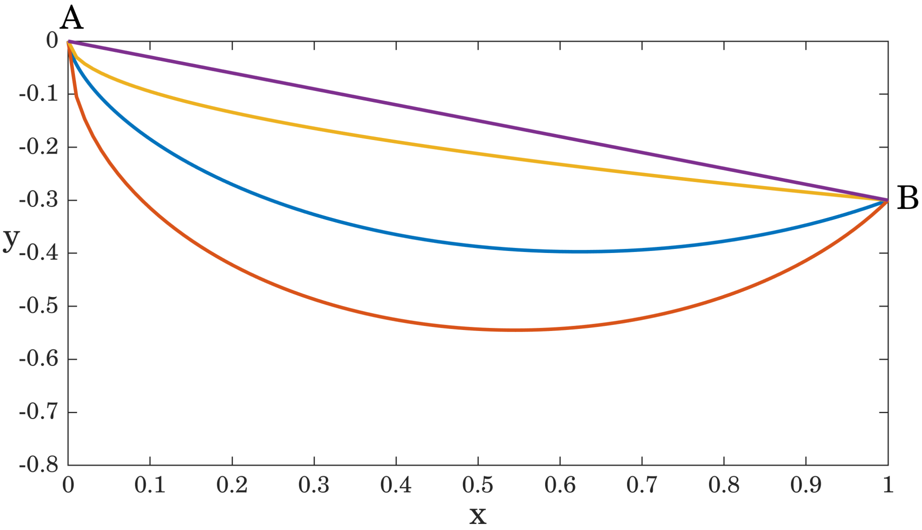
Optimization in Continuous Time

- Optimal control is essentially dynamic programming in continuous time
- Dynamic optimization in discrete and continuous time are two useful tools
- No approach is superior to the other, certain problems are simpler in discrete time while others are naturally cast in continuous time

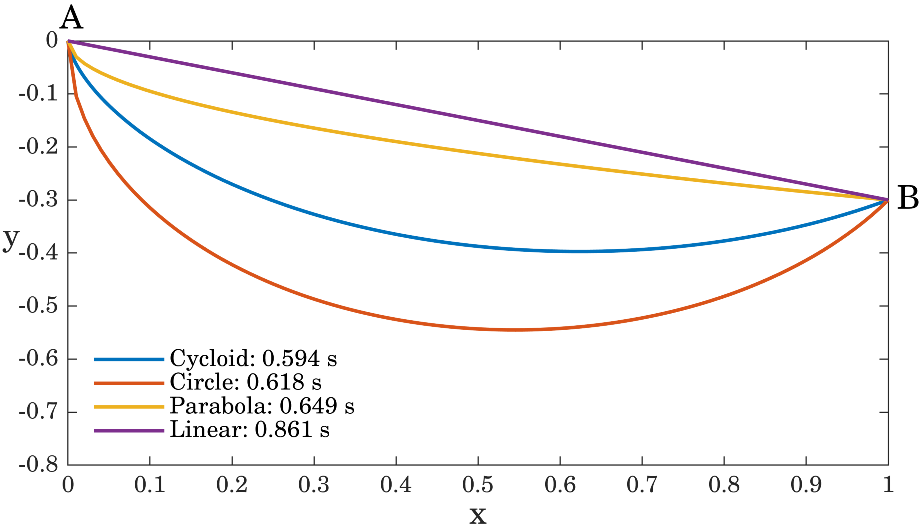
Origin

- Optimal control was born in 1697 when Swiss mathematician Johann Bernoulli posed a challenge to his fellow mathematicians
- The challenge was to solve the brachistochrone problem; in Greek *brachistos* means “the shortest” and *chronos* “time, delay”
- Given two points A and B in a vertical plane, what is the curve of fastest descent from A to B where gravity is the only force?
- Leibniz, L’Hôpital, Newton, and the Bernoulli brothers solved the problem

Which Curve Is Fastest?



The Brachistochrone Curve



Background

- Optimal control draws on the calculus of variations developed by Euler, Lagrange, and Legendre in the 18^{th} century
- Hamilton and Jacobi made contributions in the 19^{th} century that led to the Hamiltonian-Jacobi equation
- Pontryagin et al. (1962) formulated Pontryagin's maximum principle and developed modern optimal control theory
- Around the same time, Bellman was inventing dynamic programming

Lecture Outline

1. Preliminaries
2. Infinite-Horizon Optimal Control
3. The Maximum Principle
4. The Hamilton-Jacobi-Bellman Equation
5. Transversality Condition
6. Consuming a Nonrenewable Resource

Main Reference: Acemoglu, 2009, *Introduction to Modern Economic Growth*, Chapter 7

1. Preliminaries

Discount Factor in Discrete Time

- The usual subjective discount factor in discrete-time models is

$$\beta^t = \left(\frac{1}{1+\rho}\right)^t$$

- β is the discount factor, ρ is the discount rate: $\beta \in (0,1)$, therefore $\rho > 0$
- A high β means agents are very patient and have a low discount rate ρ
- Typically we calibrate β using data on the real return to risk-free bonds r and setting $\rho = r$; eg on an annual basis, $\rho \approx 5\%$ and $\beta \approx 0.95$

Compounding

- Let's uniformly compound the discount rate n times within a same unit of time t

$$\underbrace{\left(\frac{1}{1+\frac{\rho}{n}}\right)^t \times \cdots \times \left(\frac{1}{1+\frac{\rho}{n}}\right)^t}_{n \text{ times}} = \left(\frac{1}{1+\frac{\rho}{n}}\right)^{nt}$$

- For example consider a period t of one year: if the rate is computed each quarter we have $n = 4$, if computed each month we have $n = 12$
- Both $\left(\frac{1}{1+\rho}\right)^t$ and $\left(\frac{1}{1+\frac{\rho}{n}}\right)^{nt}$ denote the present value of one unit of some good in period t under different discounting assumptions

Discount Factor in Continuous Time

- Now if we set $n \to \infty$, we obtain

$$\lim_{n\to\infty} \left(\frac{1}{1+\frac{\rho}{n}} \right)^{nt} = e^{-\rho t}$$

- $e^{-\rho t}$ is the discount factor in continuous time
- As if we computed the discount rate at every point in time
- The proof is next slide

Proof

- Define $s = n/\rho$ and deduce

$$\lim_{n\to\infty}\left(1+\frac{\rho}{n}\right)^{nt} = \lim_{s\to\infty}\left(1+\frac{1}{s}\right)^{s\rho t} = \lim_{s\to\infty}\left[\left(1+\frac{1}{s}\right)^{s}\right]^{\rho t}$$

- We need to show that $\lim_{s\to\infty}\left(1+\frac{1}{s}\right)^{s} = e$
- Define $w = 1/s$ and apply l'Hôpital's rule

$$\lim_{w\to 0}\frac{\ln(1+w)}{w} = \lim_{w\to 0}\frac{1}{w+1} = 1$$

- Since $\frac{\ln(1+w)}{w} = \ln\left[(1+w)^{\frac{1}{w}}\right]$, we have

$$\lim_{s\to\infty}\left(1+\frac{1}{s}\right)^{s} = \lim_{w\to 0}(1+w)^{\frac{1}{w}} = \lim_{w\to 0} e^{\ln\left[(1+w)^{\frac{1}{w}}\right]} = e \quad \blacksquare$$

Law of Motion of Asset in Continuous Time

- Consider the example of the neoclassical growth model; the law of motion of capital between period t and period $t + \Delta$, where $\Delta \in (0, 1)$, is

$$k_{t+\Delta} = \Delta f(k_t) + (1 - \Delta\delta)k_t - \Delta c_t$$

- Subtract k_t and divide by Δ on both sides

$$\frac{k_{t+\Delta} - k_t}{\Delta} = f(k_t) - \delta k_t - c_t$$

- Take the limit as $\Delta \to 0$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t \quad \text{where} \quad \dot{k}_t = \lim_{\Delta \to 0} \frac{k_{t+\Delta} - k_t}{\Delta} = \frac{dk_t}{dt}$$

- This is the law of motion of capital in continuous time

The Neoclassical Growth Model in Continuous Time

- Putting these things together, we can express the problem of the central planner in the continuous-time version of the neoclassical growth model

$$
\begin{aligned}
\max_{k_t, c_t} \int_0^\infty e^{-\rho t} u(c_t) dt \\
\text{subject to} \quad \dot{k}_t = f(k_t) - \delta k_t - c_t, \\
k_0 \text{ given}, \\
\text{and} \quad \rho > 0,\ k > 0,\ c > 0
\end{aligned}
$$

2. Infinite-Horizon Optimal Control

Streamlined Presentation

- Let's see the essential tools to solve a standard optimal control problem
- For a complete treatment read Acemoglu (2009), Section 7.3
- We focus on the simple case with one state variable and one control variable

The Problem

- We want to solve the following problem

$$\max_{x_t, y_t} \int_0^\infty f(t, x_t, y_t)dt$$

$$\text{subject to} \quad \dot{x}_t = g(t, x_t, y_t) \quad \text{for almost all } t$$

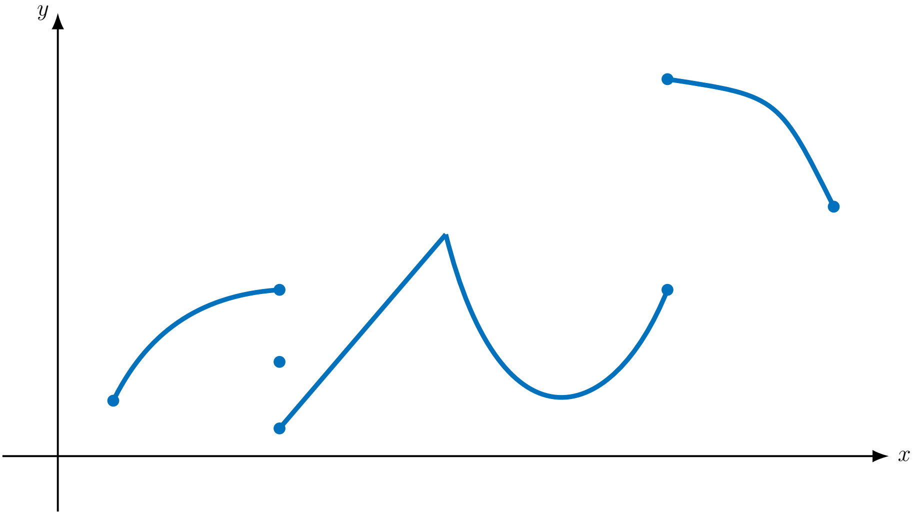
$$x_t \in \text{Int}\, X_t,\ y_t \in \text{Int}\, Y_t,\ \text{for all } t,\ x_0 \text{ given}$$

- x_t is the state, y_t is the control, f is the objective, g is the state law of motion
- $\text{Int}\, X_t$ is the set of all interior points of X_t
- We assume f and g are continuously differentiable in all their arguments

Continuous State and Control

- The state x_t is absolutely continuous
- The control y_t can be continuous, in which case x_t is differentiable
- y_t can be piecewise continuous, ie it can be broken into a finite number of subintervals on which it is continuous
- If y_t is piecewise continuous then x_t is a continuous function differentiable almost everywhere
- By almost everywhere we mean that the set of points in which $\dot{x}_t = g(t, x_t, y_t)$ is not satisfied has measure zero

Example of Piecewise Continuous Function



Value Function

- As in the case with discrete time, we define a value function

$$
\begin{aligned}
V(t_0, x_{t_0}) &= \max_{x_t, y_t} \int_{t_0}^{\infty} f(t, x_t, y_t) dt \\
\text{subject to} \quad \dot{x}_t &= g(t, x_t, y_t) \quad \text{for almost all } t \\
x_t &\in \operatorname{Int} X_t, \; y_t \in \operatorname{Int} Y_t, \; \text{for all } t, \; x_0 \text{ given}
\end{aligned}
$$

- The value function $V(t_0, x_{t_0})$ expresses the optimal value of the problem starting from an arbitrary initial condition x_{t_0}

Principle of Optimality

- The principle of optimality, formulated by Bellman, states that if the pair $(\hat{x}_t, \hat{y}_t)$ solves the above problem, then

$$V(t_0, x_{t_0}) = \int_{t_0}^{t_1} f(t, \hat{x}_t, \hat{y}_t)dt + V(t_1, \hat{x}_{t_1}) \quad \text{for all } t_1 \geq t_0\text{, given } x_{t_0}$$

- The principle of optimality is the same as in discrete time
- Pick whatever initial condition and control; then the controls you choose over the remaining period must be optimal for the remaining problem
- The initial condition could be the state resulting from an early decision

3. The Maximum Principle

Problem

- Recall the problem

$$\begin{aligned} &\max_{x_t, y_t} \int_0^\infty f(t, x_t, y_t) dt \\ \text{subject to} \quad &\dot{x}_t = g(t, x_t, y_t) \quad \text{for almost all } t \\ &x_t \in \text{Int}\, X_t,\ y_t \in \text{Int}\, Y_t,\ \text{ for all } t,\ x_0 \text{ given} \end{aligned}$$

- How do we solve this problem?

Hamiltonian

- We use a very useful device called the Hamiltonian
- The Hamiltonian was developed by Soviet mathematician Lev Pontryagin as part of his maximum principle
- It is the continuous-time equivalent of the Lagrangian

Hamiltonian

- The Hamiltonian is defined as

$$H(t, x_t, y_t, \lambda_t) \equiv f(t, x_t, y_t) + \lambda_t g(t, x_t, y_t)$$

- $f(t, x_t, y_t)$ is the objective, $g(t, x_t, y_t)$ is the law of motion of the state
- λ_t is a costate variable, the equivalent of the Lagrange multiplier
- Since f and g are continuously differentiable, so is H

Maximum Principle

Theorem: if f and g are continuously differentiable, and if the problem has a piecewise continuous interior solution $(\hat{x}_t, \hat{y}_t)$, then

A. The Hamiltonian satisfies Pontryagin's maximum principle: for all $y_t \in Y_t$ and $t > 0$

$$H(t, \hat{x}_t, \hat{y}_t, \lambda_t) \geq H(t, \hat{x}_t, y_t, \lambda_t)$$

B. For all t such that $\hat{y}_t$ is continuous in t, the following necessary conditions are satisfied

$$H_y(t, \hat{x}_t, \hat{y}_t, \lambda_t) = 0 \quad \text{FOC control (to derive)} \tag{1}$$

$$\dot{\lambda}_t = -H_x(t, \hat{x}_t, \hat{y}_t, \lambda_t) \quad \text{FOC state (to derive)} \tag{2}$$

$$\dot{\hat{x}}_t = H_\lambda(t, \hat{x}_t, \hat{y}_t, \lambda_t) = g(t, \hat{x}_t, \hat{y}_t) \quad \text{Law of motion of state (given)} \tag{3}$$

Comments

- The maximum principle says the solution $(\hat{x}_t, \hat{y}_t)$ maximizes the Hamiltonian
- The maximum principle, ie condition A, implies the necessary conditions, ie condition B, but the opposite is not necessarily true
- There exist necessary conditions for discontinuity points but we will ignore them and focus on problems that have a continuous solution
- See Acemoglu (2009), Section 7.3.5, for a proof of the theorem
- Let's now develop the intuition behind the maximum principle

Intuition

- What is the intuition for the first necessary condition (1)?
- Consider the alternative problem of maximizing the Hamiltonian directly

$$\int_0^\infty H(t, \hat{x}_t, y_t, \lambda_t)dt = \int_0^\infty [f(t, \hat{x}_t, y_t) + \lambda_t g(t, \hat{x}_t, y_t)]dt$$

- The first necessary condition, $H_y(t, \hat{x}_t, \hat{y}_t, \lambda_t) = 0$, would also be a necessary condition for this alternative problem
- Thus the maximum principle involves maximizing the sum of the original objective function $\int_0^\infty f(t, \hat{x}_t, y_t)dt$ plus an extra term $\int_0^\infty \lambda_t g(t, \hat{x}_t, y_t)dt$

Costate Variable

- What is the intuition for the costate variable λ_t in this extra term?
- λ_t is like a Lagrange multiplier associated to the constraint
- An envelope theorem-type argument implies that

$$\frac{\partial V(t, \hat{x}_t)}{\partial x} = \lambda_t$$

- λ_t measures the impact of a small increase in the state x on the optimal value of the program
- Thus λ_t is the shadow value of relaxing the constraint by increasing the value of the state x_t at time t

Hamiltonian

- Return to the alternative problem and use $\dot{x}_t = g(t, \hat{x}_t, y_t)$

$$\int_0^\infty H(t, \hat{x}_t, y_t, \lambda_t)dt = \int_0^\infty [f(t, \hat{x}_t, y_t) + \lambda_t \dot{x}_t]dt$$

- Maximizing this equation amounts to maximizing

1. Instantaneous returns given by $f(t, \hat{x}_t, y_t)$
2. Plus the value of stock of x_t, given by λ_t, times the increase in the stock $\dot{x}_t$

Stock–Flow Type Maximization

- The state x_t is a stock, the control y_t is a flow
- The essence of the maximum principle is to maximize the flow return plus the value of the current stock of the state variable
- In other words, when choosing the control y_t the maximizer takes into account the indirect effect of y_t on the value of the stock x_t
- This stock-flow type maximization has a clear economic logic

Costate Equation

- What is the intuition for the second necessary condition (2)?

$$-\dot{\lambda}_t = H_x(t, \hat{x}_t, \hat{y}_t, \lambda_t) = f_x(t, \hat{x}_t, \hat{y}_t) + \lambda_t g_x(t, \hat{x}_t, \hat{y}_t)$$

- λ_t is the shadow value of the stock of the state x_t, so $\dot{\lambda}_t$ is the appreciation, ie the rise in value, of this stock variable
- A small rise in x_t changes the current flow return (less utility now) plus the value of the stock (more capital thus more utility later) by the amount H_x
- But it also affects the value of the stock by the amount $-\dot{\lambda}_t$ (lower value)
- The maximum principle states that this gain H_x should be equal to the depreciation $-\dot{\lambda}_t$, otherwise one could change x_t and increase $\int_0^\infty H(\cdot)dt$

4. The Hamilton–Jacobi–Bellman Equation

Alternative Formulation

- We can express the necessary conditions in Pontryagin's maximum principle theorem in another form
- This other form is analogous to the dynamic programming formulation we saw in lecture 3
- This is the so-called Hamilton-Jacobi-Bellman (HJB) equation

Hamilton–Jacobi–Bellman Equation

- Suppose that f and g are of class $\boldsymbol{C^1}$, ie they are differentiable and their first derivative is continuous
- Let $V(t, x)$ be defined as previously and suppose the problem has a continuous interior solution $(\hat{x}_t, \hat{y}_t)$

Theorem: if $V(t, x)$ is differentiable in t and x, then $(\hat{x}_t, \hat{y}_t)$ satisfies for all $t \geq 0$

$$f(t, \hat{x}_t, \hat{y}_t) + \frac{\partial V(t, \hat{x}_t)}{\partial t} + \frac{\partial V(t, \hat{x}_t)}{\partial x} g(t, \hat{x}_t, \hat{y}_t) = 0$$

Proof

- From the principle of optimality, we have for the optimal pair $(\hat{x}_t, \hat{y}_t)$

$$V(t_0, x_{t_0}) = \int_{t_0}^{t} f(s, \hat{x}_s, \hat{y}_s) ds + V(t, \hat{x}_t)$$

- Differentiate with respect to time and use Leibniz's integral rule

$$\frac{dV(t_0, x_{t_0})}{dt} = f(t, \hat{x}_t, \hat{y}_t) + \underbrace{\int_{t_0}^{t} \frac{\partial}{\partial t} f(s, \hat{x}_s, \hat{y}_s) ds}_{=0} + \frac{\partial V(t, \hat{x}_t)}{\partial t} + \frac{\partial V(t, \hat{x}_t)}{\partial x} \frac{\partial x}{\partial t}$$

$$0 = f(t, \hat{x}_t, \hat{y}_t) + \frac{\partial V(t, \hat{x}_t)}{\partial t} + \frac{\partial V(t, \hat{x}_t)}{\partial x} \dot{x}_t$$

- Set $\dot{x}_t = g(t, \hat{x}_t, \hat{y}_t)$ and obtain the HJB equation ■

Interpreting the HJB Equation

- The HJB equation is used directly in many economic models, eg in endogenous technology models of growth
- The HJB equation is a partial differential equation since it features the derivative of V with respect to both time t and the state variable x
- The HJB equation shares similarity with the discrete-time Euler equation

Euler Equation in Discrete Time

- The Euler equation in discrete time has the form (recall lecture 3 section 5)

$$\frac{\partial U(x, y^*)}{\partial y} + \beta V'(y^*) = 0$$

  where y^* is the optimal policy for the control variable
- This equation requires the current gain from increasing y today to be equal to the discounted loss of value of all future returns $-\beta V'(y^*)$
- Put differently, the sum of the current gain and the discounted gain of all future returns must be equal to zero

A Similar Interpretation

- The HJB equation has a similar interpretation

$$\underbrace{f(t,\hat{x}_t,\hat{y}_t) + \frac{\partial V(t,\hat{x}_t)}{\partial x} g(t,\hat{x}_t,\hat{y}_t)}_{=\text{ maximized Hamiltonian } H(t,\hat{x}_t,\hat{y}_t,\lambda_t)} + \frac{\partial V(t,\hat{x}_t)}{\partial t} = 0$$

- $f(t,\hat{x}_t,\hat{y}_t)$ is the current gain of increasing y_t now, while $\frac{\partial V(t,\hat{x}_t)}{\partial x} g(t,\hat{x}_t,\hat{y}_t)$ is the potential discounted loss of value from increasing y_t now
- The third term $\frac{\partial V(t,\hat{x}_t)}{\partial t}$ results from the fact that the maximized value can also change over time

Stationary HJB Equation

- The HJB equation can give us further intuition to the maximum principle
- Let's derive the stationary version of the HJB equation
- The stationary version applies to exponentially discounted maximization problems with time-independent or time-autonomous constraints
- In these problems, the objective function becomes $f(t, x_t, y_t) = e^{-\rho t} f(x_t, y_t)$ and the law of motion of the state becomes $g(t, x_t, y_t) = g(x_t, y_t)$
- If a pair $(\hat{x}_t, \hat{y}_t)_{t \geq 0}$ is optimal starting at $t = 0$ with initial condition x_0, then $(\hat{x}_t, \hat{y}_t)_{t \geq s > 0}$ is also optimal starting at $s > 0$ with initial condition $x_s = x_0$

Derivative of V with Respect to Time

- In this time-invariant environment, let's define $v(x) \equiv V(0, x)$
- Since $(\hat{x}_t, \hat{y}_t)$ is an optimal plan regardless of the starting date, we have

$$V(t, x_t) = e^{-\rho t} v(x_t) \quad \text{for all } t$$

- Differentiate with respect to time

$$\frac{\partial V(t, x_t)}{\partial t} = -\rho e^{-\rho t} v(x_t)$$

- The value function depends on time only through exponential discounting

Deriving the Stationary HJB Equation

- Let $\dot{v}(x_t)$ be the change in the function v over time

$$\dot{v}(x_t) \equiv \frac{dv(x_t)}{dt} = \frac{\partial v(x_t)}{\partial x}\dot{x}_t$$

- Since v does not directly depend on time t, the change in v only results from the change in the state variable x_t
- Moreover, using $V(t, x_t) = e^{-\rho t} v(x_t)$

$$\frac{\partial V(t, x_t)}{\partial x}\dot{x}_t = e^{-\rho t} \underbrace{\frac{\partial v(x_t)}{\partial x}\dot{x}_t}_{\dot{v}(x_t)} = e^{-\rho t}\dot{v}(x_t)$$

Stationary HJB Equation

- The original HJB equation with discounting is

$$e^{-\rho t} f(\hat{x}_t, \hat{y}_t) + \frac{\partial V(t, \hat{x}_t)}{\partial t} + \frac{\partial V(t, \hat{x}_t)}{\partial x} \dot{x}_t = 0$$

- Plug in the expressions for $\frac{\partial V(t,\hat{x}_t)}{\partial t}$, $\frac{\partial V(t,\hat{x}_t)}{\partial x}\dot{x}_t$, and $\dot{v}(x_t)$ that we just computed to obtain

$$\rho v(\hat{x}_t) = f(\hat{x}_t, \hat{y}_t) + \dot{v}(\hat{x}_t)$$

No Arbitrage

- The stationary HJB equation is like a no-arbitrage asset value equation
- We can think of v as the value of an asset traded in the stock market and ρ as the required return for investors for a large number of investors
- If the asset pays out more than the required rate of return ρ, there is excess demand for it and its value rises until its rate of return becomes equal to ρ
- The rate of return on the asset ρv comes from two sources
  - Dividends or current returns paid to investors $f(\hat{x}_t, \hat{y}_t)$
  - Capital gains or losses $\dot{v}(\hat{x}_t)$, ie appreciation/depreciation of the asset

Maximum Principle and No Arbitrage

- Let d denote dividends, the no-arbitrage condition becomes

$$\rho v(x) = d + \dot{v}(x)$$

- Thus the maximum principle requires that the program's maximized value $v(x)$ and its rate of change $\dot{v}(x)$ be consistent with the no-arbitrage condition
- This is another intuition for the maximum principle

5. Transversality Condition

The Problem

- Recall the problem

$$\max_{x_t, y_t} \int_0^\infty f(t, x_t, y_t)dt$$

$$\text{subject to} \quad \dot{x}_t = g(t, x_t, y_t) \quad \text{for almost all } t$$

$$x_t \in \text{Int}\, X_t,\ y_t \in \text{Int}\, Y_t,\ \text{for all } t,\ x_0 \text{ given}$$

- f and g are continuously differentiable
- The problem has a piecewise continuous solution $(\hat{x}_t, \hat{y}_t)$
- $V(t, \hat{x}_t)$ is the optimal value of the problem as before

Transversality Condition

- We want to avoid possible solutions that satisfy the necessary conditions but imply an explosive path for the value function, $\lim_{t\to\infty} V(t, \hat{x}_t) = \infty$
- For this we assume that in the limit, the value function is time-invariant

$$\lim_{t\to\infty} \frac{\partial V(t, \hat{x}_t)}{\partial t} = 0$$

Theorem: If $V(t, \hat{x}_t)$ is differentiable in x and t for t sufficiently large and if $\lim_{t\to\infty} \frac{\partial V(t,\hat{x}_t)}{\partial t} = 0$, then the transversality condition is also a necessary condition

$$\lim_{t\to\infty} H(t, \hat{x}_t, \hat{y}_t, \lambda_t) = 0$$

Proof

- Recall the Bellman-Jacobi-Hamilton equation

$$\frac{\partial V(t,\hat{x}_t)}{\partial t} + f(t,\hat{x}_t,\hat{y}_t) + \frac{\partial V(t,\hat{x}_t)}{\partial x} g(t,\hat{x}_t,\hat{y}_t) = 0$$

- Since $V(t,\hat{x}_t)$ is differentiable in x and t for t sufficiently large, and since $\frac{\partial V(t,\hat{x}_t)}{\partial x} = \lambda_t$, we have for t sufficiently large

$$\frac{\partial V(t,\hat{x}_t)}{\partial t} + \underbrace{f(t,\hat{x}_t,\hat{y}_t) + \lambda_t g(t,\hat{x}_t,\hat{y}_t)}_{H(t,\hat{x}_t,\hat{y}_t,\lambda_t)} = 0$$

- By assumption, $\lim_{t\to\infty} \frac{\partial V(t,\hat{x}_t)}{\partial t} = 0$, therefore we have

$$\lim_{t\to\infty} H(t,\hat{x}_t,\hat{y}_t,\lambda_t) = 0$$

- This is the tranversality condition ■

The Transversality Condition Ensures a Finite Value Function

- The condition that $\lim_{t\to\infty} V(t, \hat{x}_t)$ is finite is natural, since economic problems with infinite V are not relevant
- If $\lim_{t\to\infty} V(t, \hat{x}_t) = \infty$, there is typically no pair (x_t, y_t) reaching this value
- The assumption here, $\lim_{t\to\infty} \frac{\partial V(t,\hat{x}_t)}{\partial t} = 0$, is only slightly stronger than assuming that $\lim_{t\to\infty} V(t, \hat{x}_t)$ exists and is finite
- It is satisfied in almost all economic problems
- However this transversality condition is not always easy to check so in the next lecture we will present a stronger and more useful version of it

6. Consuming a Nonrenewable Resource

A Simple Problem

- Let's apply our optimal control tools to a concrete example
- We study a simple infinite-horizon optimization problem
- The problem is to choose the optimal path of consuming a nonrenewable or exhaustible natural resource
- Think of fuels (oil, gas, coal) or metals (gold, silver, iron)

Model Setup

- The nonrenewable resource is x_t
- The agent chooses to consume quantity y_t of the resource
- x_t is the state, a stock; y_t is the control, a flow
- There is one initial unit $x_0 = 1$ of the nonrenewable resource
- The instantaneous utility of consuming a flow y_t of the resource is $u(y_t)$, where $u : [0, 1] \to \mathbb{R}$, $u' > 0$, $u'' < 0$

Resource Constraint

- The resource constraint in discrete time is

$$x_{t+1} = x_t - y_t$$

- The resource constraint in continuous time is

$$\begin{aligned} x_{t+\Delta} &= x_t - \Delta y_t \\ \frac{x_{t+\Delta} - x_t}{\Delta} &= -y_t \\ \dot{x}_t &= -y_t \end{aligned}$$

Problem

- The decision-maker solves the problem

$$\max_{y_t} \int_0^\infty e^{-\rho t} u(y_t) dt$$

$$\text{subject to} \quad \dot{x}_t = -y_t, \text{ given } x_0 = 1$$

- How do we solve this problem?

Hamiltonian

- First, we write a Hamiltonian

$$H(t, x_t, y_t, \lambda_t) = e^{-\rho t} u(y_t) - \lambda_t y_t$$

Derive the First-Order Conditions

- Second, we derive the necessary conditions, which by the maximum principle imply that $(\hat{x}_t, \hat{y}_t)$ satisfy

$$
\begin{aligned}
H_y(t, \hat{x}_t, \hat{y}_t, \lambda_t) = 0 : &\quad e^{-\rho t} u'(\hat{y}_t) = \lambda_t \\
\dot{\lambda}_t = -H_x(t, \hat{x}_t, \hat{y}_t, \lambda_t) : &\quad \dot{\lambda}_t = 0 \\
\dot{\hat{x}}_t = H_\lambda(t, \hat{x}_t, \hat{y}_t, \lambda_t) : &\quad \dot{\hat{x}}_t = -\hat{y}_t
\end{aligned}
$$

Define a New Costate and Differentiate the Second FOC

- Third, we define the alternative costate $\mu_t \equiv \lambda_t / e^{-\rho t}$
- λ_t is the discounted marginal value of the state at t in units of time t goods
- μ_t is the current marginal value of the state at t in units of time zero goods
- We then differentiate μ_t with respect to time

$$\dot{\mu}_t = \rho e^{\rho t} \lambda_t + e^{\rho t} \dot{\lambda}_t$$

- Finally we use $\mu_t = e^{\rho t} \lambda_t$ and the second FOC $\dot{\lambda}_t = 0$ to obtain

$$\dot{\mu}_t = \rho \mu_t$$

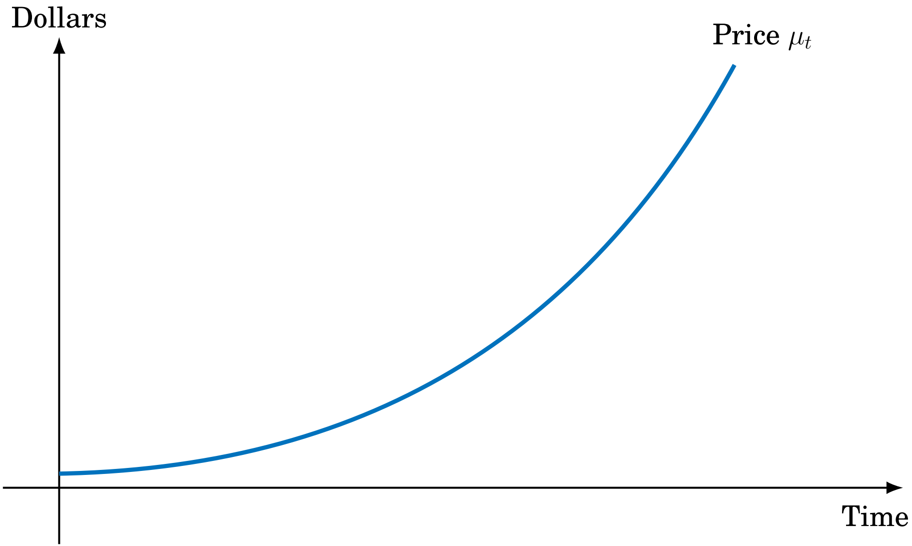
Hotelling's Rule

- The previous expression is known as Hotelling's rule: Hotelling (1931, *JPE*) laid the foundations for the field of natural resource economics

$$\frac{\dot{\mu}_t}{\mu_t} = \rho$$

- μ_t is the shadow price of the resource, ie the marginal net revenue from the sale of the resource, and ρ is the discount rate, ie an interest rate
- Hotelling's rule states that the optimal extraction path of a nonrenewable resource is one along which its price increases at the rate of interest

Price Path



Decreasing Consumption Path

- We can characterize the explicit path of consumption y_t
- Rewrite the first FOC, $e^{-\rho t} u'(\hat{y}_t) = \lambda_t$, as

$$\hat{y}_t = u'^{-1}(e^{\rho t} \lambda_t)$$

- u' is decreasing so u'^{-1} is also decreasing, thus the amount of resource consumed is monotonically decreasing over time
- Agents favor early consumption of the resource because they are impatient, but they do not consume all of it at once because they like a smooth path

Resource Path

- We can characterize the explicit path of the resource $\hat{x}_t$
- The second FOC, $\dot{\lambda}_t = 0$, implies that

$$\lambda_t = \lambda_{t-1} = \cdots = \lambda_0$$

- Combine the two FOCs with the resource constraint, $\dot{\hat{x}}_t = -\hat{y}_t$

$$\dot{\hat{x}}_t = -u'^{-1}(e^{\rho t}\lambda_t) = -u'^{-1}(e^{\rho t}\lambda_0)$$

Resource Path

- Integrate the previous expression, $\dot{\hat{x}}_t = -u'^{-1}(e^{\rho t}\lambda_0)$

$$\int_0^t \dot{\hat{x}}_s ds = -\int_0^t u'^{-1}(e^{\rho s}\lambda_0)ds$$

- Use the initial value of the constraint, $x_0 = 1$

$$\hat{x}_t = 1 - \int_0^t u'^{-1}(e^{\rho s}\lambda_0)ds$$

- This is the optimal path of the resource $\hat{x}_t$

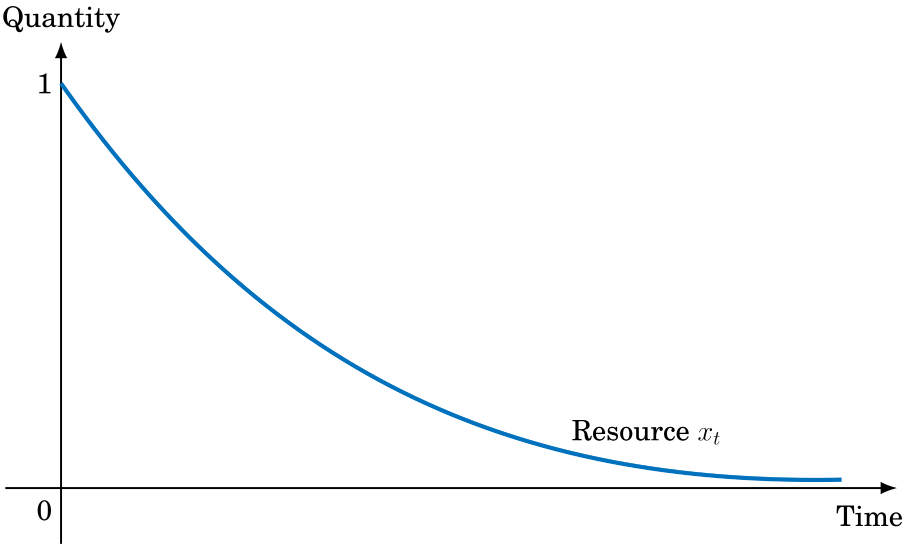
Resource Path

- To get the optimal path, we just need to compute λ_0
- Along any optimal path, $\lim_{t\to\infty} \hat{x}_t = 0$, therefore λ_0 satisfies

$$\int_0^\infty u'^{-1}(e^{\rho s}\lambda_0)ds = 1$$

- The resource is in initial quantity $x_0 = 1$ and is slowly depleted until it reaches zero in the limit
- Exercise: verify that the transversality condition is satisfied

Resource Path



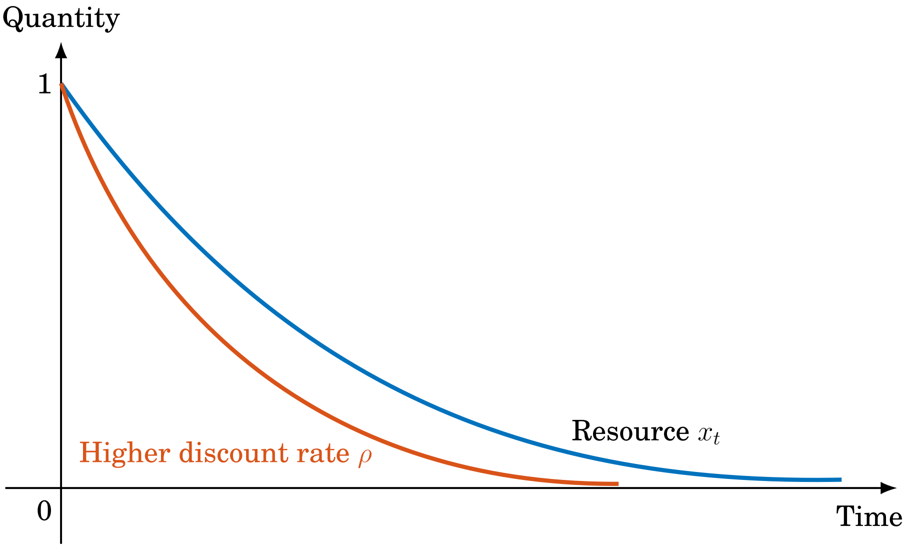
Impatience Shock

- Suppose agents become more impatient, ie the discount rate ρ increases
- From Hotelling's rule $\frac{\dot{\mu}_t}{\mu_t} = \rho$, we know the resource price increases faster
- Let $g(x) \equiv u'^{-1}(x)$; from $\dot{\hat{x}}_t = -u'^{-1}(e^{\rho t}\lambda_0)$ and $\int_0^\infty u'^{-1}(e^{\rho s}\lambda_0)ds = 1$ we get

$$\frac{\partial \dot{\hat{x}}_t}{\partial \rho} = \underbrace{-\,e^{\rho t}}_{<0} \left(\underbrace{t\lambda_0}_{>0} + \underbrace{\frac{\partial \lambda_0}{\partial \rho}}_{<0} \right) \underbrace{g'(e^{\rho t}\lambda_0)}_{<0}$$

- For small t, $|t\lambda_0| < |\frac{\partial \lambda_0}{\partial \rho}|$, so $\frac{\partial \dot{\hat{x}}_t}{\partial \rho} < 0$: the resource depletes faster as ρ rises
- Impatient agents consume the resource faster, so the price rises faster

Impatience Shock



To Be Continued

- More about optimal control in lecture 20