# 9. The Stochastic Growth Model

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# Production Is Back

- In previous lectures, we analyzed optimal behavior under uncertainty in the context of a pure exchange economy with stochastic endowments
- Today, we consider a simple production economy with stochastic technology
- We study the stochastic neoclassical growth model, also referred to as the real business cycle (RBC) model
- This is the last of four lectures on complete markets

# Lecture Outline



Main Reference: Ljungqvist and Sargent, 2018, *Recursive Macroeconomic Theory*, Fourth Edition, Chapter 12

# 1. Model Setup

# The Stochastic Growth Model

- The environment resembles that of the standard neoclassical growth model
- The key difference is that technology is now stochastic
- We also make two minor changes

1. The labor supply is not inelastic anymore, leading to labor supply decisions
2. We introduce a second type of firm that builds capital, so as to induce more trades among agents and price more items, in particular the capital stock

# Stochastic Event

- In each period $t \geq 0$, there is a realization of a stochastic event $s_t \in S$
- The stochastic event $s_t$ is an aggregate, or economy-wide, state variable
- Let the history of events until $t$ be $s^t = [s_0, s_1, \ldots, s_t]$
- The history $s^t$ is publicly observable

# Probabilities

- As usual, the unconditional probability of observing a particular sequence of events $s^t$ is given by probability measure $\pi_t(s^t)$
- For $t > \tau$, the probability of observing $s^t$ conditional on the realization of history $s^\tau$ is $\pi_t(s^t|s^\tau)$
- Again, the initial state $s_0$ in period $0$ is nonstochastic, ie $\pi_0(s_0) = 1$

# Households

- A representative household has preferences over streams of consumption $c_t(s^t)$ and leisure $\ell_t(s^t)$

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t(s^t), \ell_t(s^t)] \pi_t(s^t), \qquad \beta \in (0,1)$$

- $u$ satisfies the usual Inada conditions

$$u(0,\ell) = u(c,0) = 0 \qquad \lim_{c \to 0} u_c(c,\ell) = \lim_{\ell \to 0} u_\ell(c,\ell) = \infty$$

$$u_c, u_\ell > 0, \; u_{cc}, u_{\ell\ell} < 0 \qquad \lim_{c \to \infty} u_c(c,\ell) = \lim_{\ell \to 1} u_\ell(c,\ell) = 0$$

# Work or Chill

- In each period, the household is endowed with one unit of time that can be either devoted to leisure $\ell_t(s^t)$ or labor $n_t(s^t)$

$$1 = \ell_t(s^t) + n_t(s^t)$$

- If the utility function did not depend on leisure, $u[c(s^t)]\pi_t(s^t)$, the household would choose $\ell_t(s^t) = 0$ and $n_t(s^t) = 1$ for all $t$, ie an inelastic labor supply
- Here, because households value leisure, they end up choosing $n_t(s^t) < 1$

# Why Do We Have a Representative Household?

- As soon as we assume complete markets, which is the case in this model, we can rationalize the representative household construct as follows
- Assume there are $I$ consumers named $i = 1, 2, \ldots, I$, and all consumers have the same utility function $u^i[c_t^i(s^t), \ell_t^i(s^t)] = u[c_t^i(s^t), \ell_t^i(s^t)]$
- Each consumer receives idiosyncratic labor productivity shocks $e_t^i(s^t)n_t^i(s^t)$, but optimally trades state-price securities to insure the risk away
- In sum, there is no idiosyncratic risk in this economy, only aggregate risk

# Production Function

- Output is produced according to the production function

$$A_t(s^t)F[k_t(s^{t-1}), n_t(s^t)]$$

- Notice capital in period $t$ depends on the state in period $t-1$
- $A_t(s^t)$ is a stochastic process of Harrod-neutral technology shocks
- $F$ satisfies the standard assumptions

$$F(0,n) = F(k,0) = 0 \qquad \lim_{k\to 0} F_k(k,n) = \lim_{n\to 0} F_n(k,n) = \infty$$

$$F_k, F_n > 0, \;\; F_{kk}, F_{nn} < 0 \qquad \lim_{k\to\infty} F_k(k,n) = \lim_{n\to\infty} F_n(k,n) = 0$$

- Write the function in intensive form: $F(k,n) \equiv nf(\hat{k})$ where $\hat{k} \equiv \frac{k}{n}$

# Constraints

- Output goods are consumption and investment goods

$$c_t(s^t) + i_t(s^t) \leq A_t(s^t)F[k_t(s^{t-1}), n_t(s^t)]$$

- Capital accumulates according to

$$k_{t+1}(s^t) = (1-\delta)k_t(s^{t-1}) + i_t(s^t)$$

- Capital $k_{t+1}(s^t)$, to be used in production in $t+1$, is built in advance in $t$

# Consolidated Constraint

- The investment good $i_t(s^t)$ can take negative values, meaning the capital stock is reversible, ie it can be freely reconverted into consumption
- Consumption, however, cannot be negative
- Plug the law of motion of capital into the resource constraint

$$c_t(s^t) + k_{t+1}(s^t) - (1-\delta)k_t(s^{t-1}) \leq A_t(s^t)F[k_t(s^{t-1}), n_t(s^t)]$$

# 2. Central Planner

# Problem of the Central Planner

- The planner chooses an allocation $\{c_t(s^t), \ell_t(s^t), i_t(s^t), n_t(s^t), k_{t+1}(s^t)\}_{t=0}^{\infty}$ to maximize the utility function, subject to
  - The time constraint
  - The resource constraint
  - The initial capital stock $k_0$
  - The stochastic process for the level of technology $A_t(s^t)$

# Lagrangian

- Write a Lagrangian, in which we directly plug the time constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \Big\{ u[c_t(s^t), 1 - n_t(s^t)]$$
$$+ \mu_t(s^t) \left[ A_t(s^t) F[k_t(s^{t-1}), n_t(s^t)] + (1-\delta) k_t(s^{t-1}) - c_t(s^t) - k_{t+1}(s^t) \right] \Big\}$$

- $\mu_t(s^t)$ is a process of Lagrange multipliers on the resource constraint

# First-Order Conditions

- For each $t$ and $s^t$, the first-order conditions are

$$
\begin{aligned}
c_t &: \; u_c(s^t) = \mu_t(s^t) \\
n_t &: \; u_\ell(s^t) = u_c(s^t) A_t(s^t) F_n(s^t) \\
k_{t+1} &: \; \pi_t(s^t) u_c(s^t) = \beta \sum_{s^{t+1}|s^t} u_c(s^{t+1}) \pi_{t+1}(s^{t+1}) [A_{t+1}(s^{t+1}) F_k(s^{t+1}) + 1 - \delta]
\end{aligned}
$$

# Notation

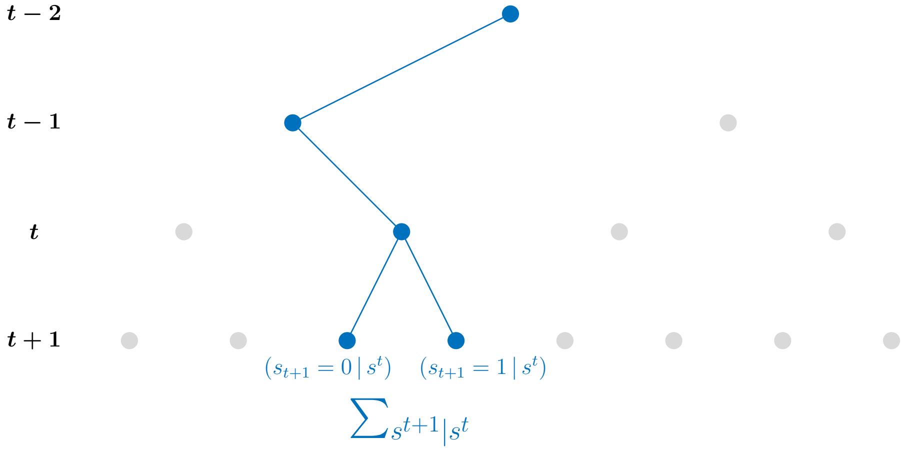
- In the FOC for capital, note that

$$\sum_{s^{t+1}} = \sum_{s^t} \sum_{s^{t+1}|s^t}$$

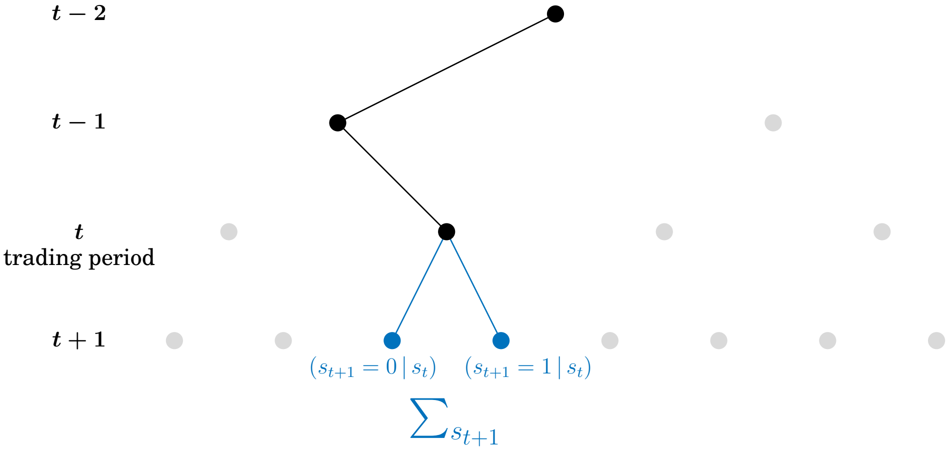
  where the summation over $s^{t+1}|s^t$ means we sum over all possible histories $\tilde{s}^{t+1}$ such that $\tilde{s}^t = s^t$
- Summations over histories and events are different

$$\sum_{s^{t+1}|s^t} \neq \sum_{s_{t+1}}$$

# Summing Over Histories



# Summing Over Events



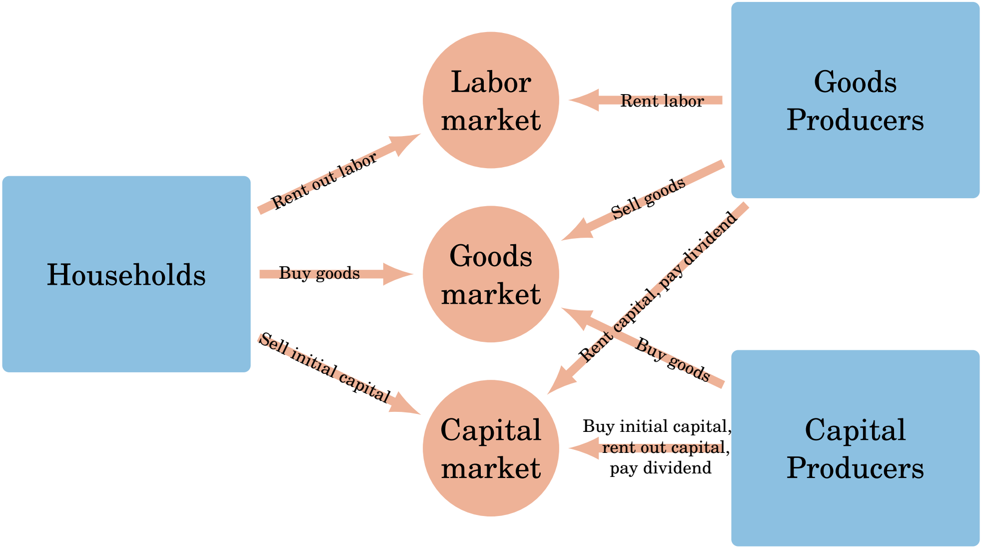
# 3. Time Zero Trading: Arrow-Debreu Securities

# Three Types of Agents

- Let's solve the competitive equilibrium with time 0 trading of a complete set of dated- and history-contingent Arrow-Debreu securities
- Trades occur among three representative agents

1. The representative household
2. A representative goods producer, which we call type I firm
3. A representative capital producer, which we call type II firm

# Model Diagram



# Actions

- Households own the initial capital stock $k_0$, sell it to capital producers at date 0, rent out labor services to and buy goods from goods producers
- Goods producers rent labor from households, rent capital from capital producers, produce, sell output goods to households and capital producers
- Capital producers buy initial $k_0$ from households, buy investment goods from goods producers, produce capital, rent out capital to goods producers

# Problem of the Household

- The household maximizes

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t(s^t), 1 - n_t(s^t)]\pi_t(s^t)$$

$$\text{subject to} \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} w_t^0(s^t) n_t(s^t) + p_{k0} k_0$$

- $q_t^0(s^t)$ is the price of one unit of output/consumption contingent on history $s^t$
- $w_t^0(s^t)$ is the price of one unit of labor contingent on history $s^t$
- $p_{k0}$ is the price of one unit of the initial capital stock

# Interpretation

- We are at time 0, markets open, and all trading takes place
- At time 0, the consumer sells her entire lifetime income stream, made of labor income $\sum_{t=0}^{\infty}\sum_{s^t} w_t^0(s^t)n_t(s^t)$ and a one-off capital sale $p_{k0}k_0$
- The consumer sells her labor to firm I and her capital to firm II
- With the proceeds, the consumer buys an infinite sequence of consumption claims $\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t(s^t)$ from firm I, the goods producer
- At the end of period 0, when trading is complete, markets close forever

# Lagrangian

- Write a Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[c_t(s^t), 1 - n_t(s^t)] \pi_t(s^t)$$
$$+ \eta \left[ \sum_{t=0}^{\infty} \sum_{s^t} w_t^0(s^t) n_t(s^t) + p_{k0} k_0 - \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t(s^t) \right]$$

- $\eta$ is the unique Lagrange multiplier on the time 0 budget constraint

# First-Order Conditions

- The first-order conditions are

$$
\begin{aligned}
c_t(s^t): &\quad \beta^t u_c(s^t)\pi_t(s^t) = \eta q_t^0(s^t) \\
n_t(s^t): &\quad \beta^t u_\ell(s^t)\pi_t(s^t) = \eta w_t^0(s^t)
\end{aligned}
$$

# Goods Producer – Firm of Type I

- The goods producer, or firm I, operates the production technology
- At time 0, the firm enters into state-contingent contracts for each $t$ and each $s^t$ to rent capital $k_t^I(s^t)$ and labor services $n_t(s^t)$ and sell output $y_t(s^t)$
- It trades with households over labor services $n_t(s^t)$ and goods $c_t(s^t)$ and it trades with capital producers over capital services $k_t^I(s^t)$ and goods $i_t(s^t)$

# Problem of the Goods Producer

- The goods producer maximizes

$$\sum_{t=0}^{\infty} \sum_{s^t} \{ q_t^0(s^t)[c_t(s^t) + i_t(s^t)] - r_t^0(s^t) k_t^I(s^t) - w_t^0(s^t) n_t(s^t) \}$$

$$\text{subject to} \qquad c_t(s^t) + i_t(s^t) \leq A_t(s^t) F[k_t^I(s^t), n_t(s^t)]$$

- $r_t^0(s^t)$ is the price for renting capital, ie the rental rate
- Note that all variables in this problem are conditioned on the history $s^t$

# First-Order Conditions

- Plug the constraint into the objective

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \{q_t^0(s^t) A_t(s^t) F[k_t^I(s^t), n_t(s^t)] - r_t^0(s^t) k_t^I(s^t) - w_t^0(s^t) n_t(s^t)\}$$

- The first-order conditions are

$$\begin{aligned} k_t^I(s^t): &\quad q_t^0(s^t) A_t(s^t) F_k(s^t) = r_t^0(s^t) \\ n_t(s^t): &\quad q_t^0(s^t) A_t(s^t) F_n(s^t) = w_t^0(s^t) \end{aligned}$$

# Capital Producer – Firm of Type II

- The capital producer operates a technology to transform output goods (ie investment goods) into capital goods
- At time 0, it enters into state-contingent contracts for each $t$ and each $s^t$
- It purchases initial capital $k_0$ from households and builds new capital $k_{t+1}^{II}(s^t)$ by purchasing investment goods $i_t(s^t)$ from goods producers
- It earns revenues by renting out capital $r_{t+1}^0(s^{t+1})k_{t+1}^{II}(s^t)$ to goods producers

# Problem of the Capital Producer

- The capital producer maximizes profit

$$-p_{k0}k_0^{II} + \sum_{t=0}^{\infty}\sum_{s^t}\left\{r_t^0(s^t)k_t^{II}(s^{t-1}) - q_t^0(s^t)i_t(s^t)\right\}$$

$$\text{subject to} \qquad k_{t+1}^{II}(s^t) = (1-\delta)k_t^{II}(s^{t-1}) + i_t(s^t)$$

- Note that some variables are conditioned on $s^{t-1}$, others on $s^t$

# Who Bears the Risk?

- Capital in period 0 $k_0^{II}$ is bought without any uncertainty about the rental price $r_0^0(s_0)$ in that period
- But for any future period $t$, investment in capital $k_t^{II}(s^{t-1})$, conditioned on $s^{t-1}$, is made without knowing the rental price $r_t^0(s^t)$, conditioned on $s^t$
- So firm II must deal with the risk associated with capital being assembled one period prior to becoming an input for production and yielding a return
- Firm I, on the other hand, can choose how much capital $k_t^I(s^t)$ to rent in period $t$ conditioned on history $s^t$, ie it faces no risk at all

# Problem of the Capital Producer

- Plug the capital accumulation constraint into the objective and rearrange

$$k_0^{II} \left\{ -p_{k0} + r_0^0(s_0) + q_0^0(s_0)(1-\delta) \right\}$$
$$+ \sum_{t=0}^{\infty} \sum_{s^t} k_{t+1}^{II}(s^t) \left\{ -q_t^0(s^t) + \sum_{s^{t+1}|s^t} \left[ r_{t+1}^0(s^{t+1}) + q_{t+1}^0(s^{t+1})(1-\delta) \right] \right\}$$

- Profit is a linear function of investments in capital $k_0^{II}$ and $k_{t+1}^{II}(s^t)$
- What must happen to the two terms in curly brackets $\{\dots\}$?

# Details of the Computation

$$
\begin{aligned}
& -p_{k0}k_0^{II} + \sum_{t=0}^{\infty}\sum_{s^t}\left\{r_t^0(s^t)k_t^{II}(s^{t-1}) - q_t^0(s^t)k_{t+1}^{II}(s^t) + q_t^0(s^t)(1-\delta)k_t^{II}(s^{t-1})\right\} \\
= & -p_{k0}k_0^{II} + r_0^0(s_0)k_0^{II} + q_0^0(s_0)(1-\delta)k_0^{II} + \sum_{t=0}^{\infty}\sum_{s^t} -q_t^0(s^t)k_{t+1}^{II}(s^t) \\
& + \sum_{t=1}^{\infty}\sum_{s^t}\left\{r_t^0(s^t)k_t^{II}(s^{t-1}) + q_t^0(s^t)(1-\delta)k_t^{II}(s^{t-1})\right\} \\
= & \;\; k_0^{II}\left\{-p_{k0} + r_0^0(s_0) + q_0^0(s_0)(1-\delta)\right\} + \sum_{t=0}^{\infty}\sum_{s^t} -q_t^0(s^t)k_{t+1}^{II}(s^t) \\
& + \sum_{t=0}^{\infty}\sum_{s^t}\sum_{s^{t+1}|s^t}\left\{r_{t+1}^0(s^{t+1})k_{t+1}^{II}(s^t) + q_{t+1}^0(s^{t+1})(1-\delta)k_{t+1}^{II}(s^t)\right\} \\
= & \;\; k_0^{II}\left\{-p_{k0} + r_0^0(s_0) + q_0^0(s_0)(1-\delta)\right\} \\
& + \sum_{t=0}^{\infty}\sum_{s^t} k_{t+1}^{II}(s^t)\left\{-q_t^0(s^t) + \sum_{s^{t+1}|s^t}\left[r_{t+1}^0(s^{t+1}) + q_{t+1}^0(s^{t+1})(1-\delta)\right]\right\}
\end{aligned}
$$

# Perfect Competition Means Zero Profit

- If the terms in curly brackets are positive, the firm wants infinite $k_0^{II}, k_{t+1}^{II}$
- If the terms in curly brackets are negative, the firm wants zero $k_0^{II}, k_{t+1}^{II}$
- In equilibrium: 1) perfect competition implies zero profits; 2) supply equals demand, meaning capital cannot be zero or infinite, $0 < k_0^{II}, k_{t+1}^{II} < \infty$
- We conclude that the two terms in curly brackets from the firm's profit equation must be equal to zero: this is the zero-profit condition

# First-Order Conditions

- Based on the preceding, equilibrium prices satisfy

$$
\begin{aligned}
p_{k0} &= r_0^0(s_0) + q_0^0(s_0)(1-\delta) \\
q_t^0(s^t) &= \sum_{s^{t+1}|s^t} \left[ r_{t+1}^0(s^{t+1}) + q_{t+1}^0(s^{t+1})(1-\delta) \right]
\end{aligned}
$$

# Summary of Necessary Conditions

- Households

$$\beta^t u_c(s^t)\pi_t(s^t) = \eta q_t^0(s^t)$$
$$\beta^t u_\ell(s^t)\pi_t(s^t) = \eta w_t^0(s^t)$$

- Goods producer or firm I

$$q_t^0(s^t)A_t(s^t)F_k(s^t) = r_t^0(s^t)$$
$$q_t^0(s^t)A_t(s^t)F_n(s^t) = w_t^0(s^t)$$

- Capital producer or firm II

$$p_{k0} = r_0^0(s_0) + q_0^0(s_0)(1-\delta)$$
$$q_t^0(s^t) = \sum_{s^{t+1}|s^t} [r_{t+1}^0(s^{t+1}) + q_{t+1}^0(s^{t+1})(1-\delta)]$$

# Equilibrium

- In equilibrium, markets clear, ie supply equals demand
- Let's compute the equilibrium price and quantities in the three markets

1. Labor market
2. Capital market
3. Goods market

# Labor Market Equilibrium

- Labor supply is set by the household's FOC for labor
- Labor demand comes from the goods producer's (firm I) FOC for labor
- Combine the two

$$\beta^t u_\ell(s^t)\pi_t(s^t) = \eta q_t^0(s^t) A_t(s^t) F_n(s^t)$$

# Capital Market Equilibrium

- Capital supply comes from the capital producer's (firm II) FOC
- Capital demand comes from the goods producer's (firm I) FOC for capital
- Combine the two

$$q_t^0(s^t) = \sum_{s^{t+1}|s^t} q_{t+1}^0(s^{t+1})[A_{t+1}(s^{t+1})F_k(s^{t+1}) + 1 - \delta]$$

# Goods Market Equilibrium

- Supply of goods comes from the goods producer (firm I)
- Demand for goods comes from households and capital producers (firm II)
- By Walras' law, or by virtue of the resource constraint at equality, the goods market is in equilibrium

$$A_t(s^t)F[k_t(s^{t-1}), n_t(s^t)] = c_t(s^t) + i_t(s^t)$$

# Consumption–Labor Choice

- Plug the household's consumption FOC into the labor market equilibrium equation
$$\frac{u_\ell(s^t)}{u_c(s^t)} = A_t(s^t)F_n(s^t) = w_t(s^t)$$
- This is the intratemporal labor supply–consumption decision
- The marginal rate of substitution (MRS) between leisure and consumption $\frac{u_\ell(s^t)}{u_c(s^t)}$ equals the relative price of leisure, ie the wage $w_t(s^t)$

# Euler Equation

- Plug the household's consumption FOC into the capital market equilibrium equation

$$\pi_t(s^t)u_c(s^t) = \beta \sum_{s^{t+1}|s^t} u_c(s^{t+1})\pi_{t+1}(s^{t+1})[A_{t+1}(s^{t+1})F_k(s^{t+1}) + 1 - \delta]$$

- This is the intertemporal consumption–saving decision, the Euler equation
- The MRS between consumption today and tomorrow $\frac{u_c(s^t)}{\beta E_0 u_c(s^{t+1})}$ equals the relative price of consumption, ie the (expected) interest rate

# Equivalence

- The previous two expressions are identical to the central planner's first-order conditions
- The allocation in the competitive equilibrium with time 0 trading is the same as the Pareto efficient allocation

# 4. Implied Wealth Dynamics

# Change the Numeraire

- In the Arrow-Debreu world, trades are only executed at time 0
- We can still compute how the household's wealth evolves over time
- For this we need to express all prices, wages, and rental rates in terms of time $t$, history $s^t$ consumption goods
- In other words, we change the numeraire

# Deflating

- We obtain

$$q_\tau^t(s^\tau) \equiv \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)} = \beta^{\tau-t}\frac{u_c(s^\tau)}{u_c(s^t)}\pi_\tau(s^\tau|s^t)$$

$$w_\tau^t(s^\tau) \equiv \frac{w_\tau^0(s^\tau)}{q_t^0(s^t)}$$

$$r_\tau^t(s^\tau) \equiv \frac{r_\tau^0(s^\tau)}{q_t^0(s^t)}$$

- Notice that

$$q_t^t(s^t) = \frac{q_t^0(s^t)}{q_t^0(s^t)} = 1$$

# Wealth

- In lecture 7, we computed households' financial wealth as total wealth minus the present value of current and future endowment streams
- Here, we subtract the present value of current and future labor income
- Household wealth, or the value of all current and future net claims, in time $t$, history $s^t$ consumption goods is

$$\Upsilon_t(s^t) \equiv \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \left\{ q_\tau^t(s^\tau) c_\tau(s^\tau) - w_\tau^t(s^\tau) n_\tau(s^\tau) \right\}$$

# Rewriting Wealth

$$
\begin{aligned}
\Upsilon_t(s^t) &\equiv \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \left\{ q_\tau^t(s^\tau) c_\tau(s^\tau) - w_\tau^t(s^\tau) n_\tau(s^\tau) \right\} \\
&= \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \left\{ q_\tau^t(s^\tau) \Big[ A_\tau(s^\tau) F[k_\tau(s^{\tau-1}), n_\tau(s^\tau)] + (1-\delta) k_\tau(s^{\tau-1}) - k_{\tau+1}(s^\tau) \Big] - w_\tau^t(s^\tau) n_\tau(s^\tau) \right\} \\
&= \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \left\{ q_\tau^t(s^\tau) \Big[ A_\tau(s^\tau) \left[ F_k(s^\tau) k_\tau(s^{\tau-1}) + F_n(s^\tau) n_\tau(s^\tau) \right] + (1-\delta) k_\tau(s^{\tau-1}) - k_{\tau+1}(s^\tau) \Big] - w_\tau^t(s^\tau) n_\tau(s^\tau) \right\} \\
&= \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \left\{ r_\tau^t(s^\tau) k_\tau(s^{\tau-1}) + q_\tau^t(s^\tau) \left[ (1-\delta) k_\tau(s^{\tau-1}) - k_{\tau+1}(s^\tau) \right] \right\} \\
&= r_t^t(s^t) k_t(s^{t-1}) + q_t^t(s^t)(1-\delta) k_t(s^{t-1}) + \sum_{\tau=t+1}^{\infty} \sum_{s^{\tau-1} | s^t} \left\{ \sum_{s^\tau | s^{\tau-1}} \Big[ r_\tau^t(s^\tau) + q_\tau^t(s^\tau)(1-\delta) \Big] - q_{\tau-1}^t(s^{\tau-1}) \right\} k_\tau(s^{\tau-1}) \\
&= \left[ r_t^t(s^t) + 1 - \delta \right] k_t(s^{t-1})
\end{aligned}
$$

Second line: resource constraint; Third: Euler's theorem; Fourth: firm's FOCs; Fifth: rearrange; Sixth: $q_t^t(s^t) = \frac{q_t^0(s^t)}{q_t^0(s^t)} = 1$ and zero-profit condition implying curly bracket terms equal zero

# Wealth Is Capital

- We find that the wealth of the representative household, excluding its labor income, is
$$\Upsilon_t(s^t) = \left[r_t^t(s^t) + 1 - \delta\right] k_t(s^{t-1})$$
- Households invest all their wealth in the capital stock
- The entire capital stock is held by households

# 5. Sequential Trading: Arrow Securities

# Sequential Trading

- We now study the same economy with sequential trading
- All markets reopen in each period

1. Goods market
2. Labor market
3. Capital market

# Households

- At each date $t \geq 0$ after history $s^t$, the household brings in assets $\tilde{a}_t(s^t)$, ie claims to time $t$ consumption that it bought in period $t-1$
- In addition, the household earns new labor income $\tilde{w}_t(s^t)n_t(s^t)$ by selling labor services to goods producers
- It uses these revenues to buy consumption goods $\tilde{c}_t(s^t)$ and claims to time $t+1$ consumption whose payment is contingent on the realization of $s_{t+1}$

# Budget Constraint

- The household faces a sequence of budget constraints; the time $t$, history $s^t$ budget constraint is

$$\tilde{c}_t(s^t) + \sum_{s^{t+1}} \tilde{a}_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) \leq \tilde{w}_t(s^t)\tilde{n}_t(s^t) + \tilde{a}_t(s^t)$$

- $\tilde{Q}_t(s_{t+1}|s^t)$ is the pricing kernel: the price of one unit of consumption at time $t+1$ contingent on the realization $s_{t+1}$ at $t+1$ when history at $t$ is $s^t$
- $\{\tilde{a}_{t+1}(s_{t+1}, s^t)\}$ is a vector of claims on time $t+1$ consumption, ie there is one element of the vector for each value of time $t+1$ realization of $s_{t+1}$

# No Ponzi Scheme

- To rule out Ponzi schemes, we must impose borrowing constraints on the household's asset position
- Without these borrowing constraints, the household would find it optimal to borrow as much as possible and roll over debt forever
- What is the maximal amount the household can repay?

# Natural Debt Limit

- Let's compute the state-contingent natural debt limit

$$\sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} \tilde{w}_\tau^t(s^\tau)\tilde{n}_\tau^{\mathbf{max}}(s^\tau)$$

- The maximum the agent could earn and repay is if she promises to work $\tilde{n}_\tau^{\mathbf{max}} = 1$ for all $\tau$ and $s^\tau$ if necessary
- But this is not credible as $\tilde{\ell}_\tau = 0$ would ruin her utility

# Arbitrary Borrowing Constraint

- We can impose that indebtedness in any state next period $-\tilde{a}_{t+1}(s_{t+1}, s^t)$ is bounded by some arbitrary constant
- As long as the budget constraint is bounded, equilibrium forces ensure that the household holds the market portfolio
- Let's impose $\tilde{a}_{t+1}(s_{t+1}, s^t) \geq 0$, ie wealth can never be negative

# Problem of the Household

- The household maximizes

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u[\tilde{c}_t(s^t), 1 - \tilde{n}_t(s^t)] \pi_t(s^t)$$

$$\text{subject to} \quad \tilde{c}_t(s^t) + \sum_{s^{t+1}} \tilde{a}_{t+1}(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1}|s^t) \leq \tilde{w}_t(s^t)\tilde{n}_t(s^t) + \tilde{a}_t(s^t)$$

$$\text{and} \quad \tilde{a}_{t+1}(s_{t+1}, s^t) \geq 0$$

# Lagrangian

- Write a Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \Big\{ \beta^t u[\tilde{c}_t(s^t), 1 - \tilde{n}_t(s^t)] \pi_t(s^t)$$

$$+ \eta_t(s^t) \left[ \tilde{w}_t(s^t) \tilde{n}_t(s^t) + \tilde{a}_t(s^t) - \tilde{c}(s^t) - \sum_{s_{t+1}} \tilde{a}_{t+1}(s_{t+1}, s^t) \tilde{Q}_t(s_{t+1} | s^t) \right]$$

$$+ \nu_t(s^t, s_{t+1}) \tilde{a}_{t+1}(s^{t+1}) \Big\}$$

- $\eta_t(s^t)$ are multipliers on the flow budget constraint
- $\nu_t(s^t, s_{t+1})$ are multipliers on the borrowing constraint

# First-Order Conditions

- The FOCs are

$$\begin{aligned}
\tilde{c}_t(s^t) &: \beta^t u_c[\tilde{c}_t(s^t), 1-\tilde{n}_t(s^t)]\pi_t(s^t) - \eta_t(s^t) = 0 \\
\tilde{n}_t(s^t) &: -\beta^t u_\ell[\tilde{c}_t(s^t), 1-\tilde{n}_t(s^t)]\pi_t(s^t) + \eta_t(s^t)\tilde{w}_t(s^t) = 0 \\
\{\tilde{a}_{t+1}(s_{t+1}, s^t)\}_{s_{t+1}} &: -\eta_t(s^t)\tilde{Q}_t(s_{t+1}|s^t) + \nu_t(s^t, s_{t+1}) + \eta_{t+1}(s_{t+1}, s^t) = 0
\end{aligned}$$

- They hold for all $s_{t+1}, t, s^t$

# Nonbinding Borrowing Constraint

- We conjecture that the arbitrary debt limit is not binding
- As a result, the Lagrange multipliers $\nu_t(s^t, s_{t+1})$ are all equal to zero
- Let's rewrite the FOCs with $\nu_t(s^t, s_{t+1}) = 0$

# Rewriting the First-Order Conditions

- The optimal static consumption–labor choice is

$$\tilde{w}_t(s^t) = \frac{u_\ell[\tilde{c}_t(s^t), 1 - \tilde{n}_t(s^t)]}{u_c[\tilde{c}_t(s^t), 1 - \tilde{n}_t(s^t)]}$$

- The optimal dynamic consumption–saving choice is

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u_c[\tilde{c}_{t+1}(s^{t+1}), 1 - \tilde{n}_{t+1}(s^{t+1})]}{u_c[\tilde{c}_t(s^t), 1 - \tilde{n}_t(s^t)]} \pi_t(s^{t+1}|s^t)$$

# Goods Producer – Firm of Type I

- At each date $t \geq 0$ after history $s^t$, the goods producer solves the usual static problem

$$\max_{\tilde{n}_t(s^t), \tilde{k}_t^I(s^t)} \left\{ A_t(s^t) F[\tilde{k}^I(s^t), \tilde{n}_t(s^t)] - \tilde{r}_t(s^t)\tilde{k}_t^I(s^t) - \tilde{w}_t(s^t)\tilde{n}_t(s^t) \right\}$$

# First-Order Conditions

- The FOCs are

$$
\begin{aligned}
\tilde{k}_t^I(s^t): \quad & \tilde{r}_t(s^t) = A_t(s^t)F_k(s^t) \\
\tilde{n}_t(s^t): \quad & \tilde{w}_t(s^t) = A_t(s^t)F_n(s^t)
\end{aligned}
$$

- The firm makes zero profit and its size is indeterminate
- The firm is willing to produce any quantity of output that the market demands so long as the two FOCs are satisfied

# Capital Producer – Firm of Type II

- The capital producer's problem is a two-period problem

1. At the end of period $t$ after history $s^t$, the firm decides how much capital $\tilde{k}^{II}_{t+1}(s^t)$ to produce and store; the cost of one unit of $\tilde{k}^{II}_{t+1}(s^t)$ is 1
2. In the next period $t+1$, the firm earns a stochastic rental revenue $\tilde{r}_{t+1}(s^{t+1})\tilde{k}^{II}_{t+1}(s^t)$ and a deterministic liquidation value $(1-\delta)\tilde{k}^{II}_{t+1}(s^t)$

- To finance its operations, the firm issues Arrow securities to households
- We use prices $\tilde{Q}_t(s_{t+1}|s^t)$ to express future income streams in today's value

# Problem of the Capital Producer

- At each date $t \geq 0$, the capital producer solves

$$\max_{\tilde{k}^{II}_{t+1}(s^t)} \tilde{k}^{II}_{t+1}(s^t) \left\{ -1 + \sum_{s_{t+1}} \tilde{Q}_t(s_{t+1}|s^t) \left[ \tilde{r}_{t+1}(s^{t+1}) + 1 - \delta \right] \right\}$$

- The price of one unit of capital today in terms of today's output goods is one
- The zero-profit condition is

$$1 = \sum_{s_{t+1}} \tilde{Q}_t(s_{t+1}|s^t) \left[ \tilde{r}_{t+1}(s^{t+1}) + 1 - \delta \right]$$

# 6. Equivalence of Allocations

# Equivalence of Allocations

- Time 0 trading and sequential trading are equivalent if

$$\{c_t(s^t), \ell_t(s^t), n_t(s^t), i_t(s^t), k_{t+1}(s^t)\}_{t=0}^{\infty} = \{\tilde{c}_t(s^t), \tilde{\ell}_t(s^t), \tilde{n}_t(s^t), \tilde{i}_t(s^t), \tilde{k}_{t+1}(s^t)\}_{t=0}^{\infty}$$

- To show the equivalence of allocations, we employ the guess and verify method used in lecture 7; this is left as an exercise

# Guess and Verify

- The trick is to guess that the prices in the sequential equilibrium satisfy

$$\begin{aligned} \tilde{Q}_t(s_{t+1}|s^t) &= q^t_{t+1}(s^{t+1}) \\ \tilde{w}_t(s^t) &= w_t(s^t) \\ \tilde{r}_t(s^t) &= r_t(s^t) \end{aligned}$$

- We also guess that the household chooses the following asset portfolios

$$\tilde{a}_{t+1}(s_{t+1}, s^t) = \Upsilon_{t+1}(s^{t+1}) \quad \text{for all } s_{t+1} \text{ and } t$$

# Initial Capital

- We have to show that the agent can afford these asset portfolios
- In doing that, we will find that the required initial wealth is

$$\tilde{a}_0 = [r_0^0(s_0) + 1 - \delta]k_0 = p_{k0}k_0$$

- The household starts out at time 0 owning the initial capital stock
- This is different from lecture 7 where initial wealth $\tilde{a}_0^i$ was zero for all $i$

# 7. Financing the Firms

# Financing the Goods Producer

- In each period, the goods producer must remunerate workers and capital owners, ie capital producers
- The goods producer finances these expenses by selling output in the very same period to consumers and capital producers
- The firm makes zero profit or loss for all $t$ and $s^t$
- Thus it does not need to issue debt to finance its operations

# Financing the Capital Producer

- By contrast, the capital producer finances its purchases of capital by issuing Arrow securities, ie one-period-ahead state-contingent claims, to households
- To produce $\tilde{k}^{II}_{t+1}(s^t)$ units of capital today in period $t$, firm II issues claims that promise to pay $[\tilde{r}_{t+1}(s^{t+1}) + 1 - \delta]\tilde{k}^{II}_{t+1}(s^t)$ goods tomorrow in state $s_{t+1}$
- Express these payouts in units of today's time $t$ good

$$\sum_{s_{t+1}} \tilde{Q}_t(s_{t+1}|s^t) \left[\tilde{r}_{t+1}(s^{t+1}) + 1 - \delta\right] \tilde{k}^{II}_{t+1}(s^t)$$

# Zero Profit, Zero Net Worth

- The capital producer makes zero profit, implying that it breaks even by issuing these claims and then repaying them next period with interest
- It follows that the capital producer has zero net worth, or zero equity, and is entirely financed by debt; in other words, it has infinite leverage

# Positive Wealth

- The household's wealth is given by

$$\tilde{a}_t(s^t) = \Upsilon_t(s^t) = [\tilde{r}_t(s^t) + 1 - \delta]\tilde{k}_t(s^{t-1})$$

- The wealth of the household is equal to the value of firm II, ie the value of the capital stock
- The capital producer is entirely owned by its unique creditor, the household

# Nonbinding Constraint

- The household willingly holds the capital stock
- Equilibrium prices entice the household to enter each period with a strictly positive net asset level
- We confirm the correctness of our conjecture that the zero debt limit is never binding

# Unique Creditor

- The household is the only creditor of the capital producer

$$\tilde{a}_t(s^t) = [\tilde{r}_t(s^t) + 1 - \delta]\tilde{k}_t(s^{t-1})$$

$$\sum_{s_{t+1}} \tilde{a}_{t+1}(s_{t+1}, s^t)\tilde{Q}(s_{t+1}|s^t) = \underbrace{\sum_{s_{t+1}} [\tilde{r}_{t+1}(s^{t+1}) + (1-\delta)]\tilde{Q}_t(s_{t+1}|s^t)}_{= 1 \text{ by firm II's FOC}} \tilde{k}_{t+1}(s^t)$$

- Thus the household budget constraint can be written as

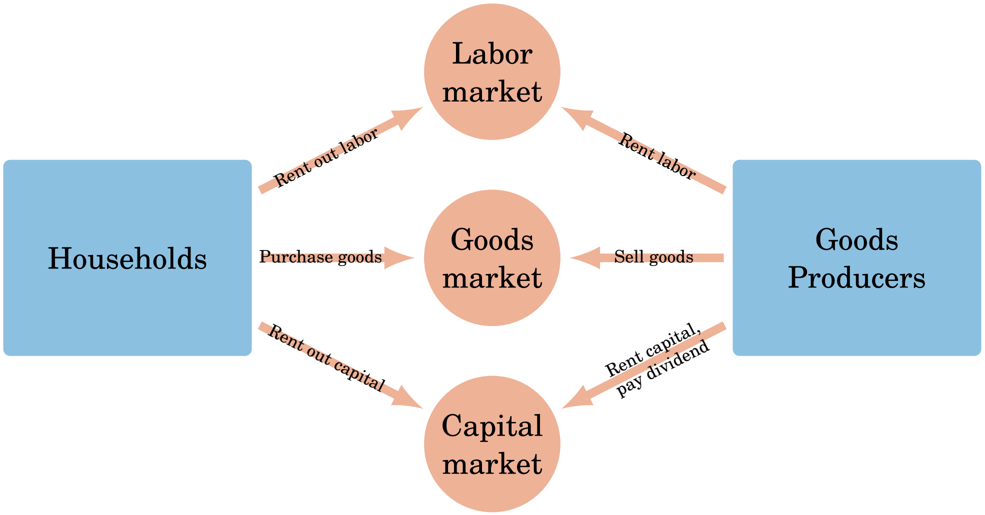
$$\tilde{c}_t(s^t) + \sum_{s^{t+1}} \tilde{a}_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) \leq \tilde{w}_t(s^t)\tilde{n}_t(s^t) + \tilde{a}_t(s^t)$$

$$\tilde{c}_t(s^t) + \tilde{k}_{t+1}(s^t) \leq \tilde{w}_t(s^t)\tilde{n}_t(s^t) + [\tilde{r}_t(s^t) + 1 - \delta]\tilde{k}_t(s^{t-1})$$

# Useless Capital Producer

- The capital producer is entirely owned by the household and there is no financing friction between the two agents
- Therefore, the household can play the role of the capital producer by renting out the capital stock directly to the goods producer (firm I)
- In other words, we can get rid of the capital producer without changing anything to the equilibrium conditions

# Equivalent Model Diagram



# 8. Recursive Formulation

# Equivalence

- We established identical equilibrium allocations in the

1. Complete-market Arrow-Debreu economy with all trading at time 0
2. Complete-market Arrow economy with sequential trading

# Arbitrary Process

- The finding holds for any arbitrary technology process
- $A_t(s^t)$ is a measurable function of the history of events $s^t$
- These events $s^t$, in turn, are governed by some arbitrary probability measure $\pi_t(s^t)$

# Huge State Space

- In this general setup, all prices $\{\tilde{Q}_t(s_{t+1}|s^t), \tilde{w}_t(s^t), \tilde{r}_t(s^t)\}$ and quantities $\{k_{t+1}(s^t), c_t(s^t), \ell_t(s^t)\}$ depend on the entire history of events $s^t$
- They are time-varying functions of all past events $\{s_\tau\}_{\tau=0}^t$
- To obtain a recursive formulation, we need to make further assumptions on the exogenous process for technology

# The Stochastic Event Is Markov

- The first assumption we make is that the stochastic event $s_t$ is governed by a Markov process, $[s \in \boldsymbol{S}, \pi(s'|s), \pi_0(s_0)]$

$$\begin{aligned}
\pi_0(s_0) &= 1 \\
\pi_t(s^t) &= \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\dots\pi(s_1|s_0)\pi_0(s_0) \\
\pi_t(s^t|s^\tau) &= \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\dots\pi(s_{\tau+1}|s_\tau) \quad \text{for } t > \tau
\end{aligned}$$

# Technology Is a Time-Invariant Function

- The second assumption is that aggregate technology is a time-invariant function of its level in the last period and the current stochastic event $s_t$

$$A_t(s^t) = A[A_{t-1}(s^{t-1}), s_t]$$

- Let's consider the multiplicative version

$$A_t(s^t) = s_t A_{t-1}(s^{t-1}) = s_0 s_1 \dots s_t A_{-1}$$

# State Variables

- What are the state variables?

# State Variables

- What are the state variables?
- There are two exogenous aggregate state variables

1. The current value of the stochastic event $s$
2. The current technology level $A$

- There is one endogenous aggregate state variable

3. The beginning-of-period capital stock $K$

# Aggregate State of the Economy

- Thanks to our two assumptions, the state vector $X \equiv \begin{bmatrix} K & A & s \end{bmatrix}$ is a complete summary of the economy's current position
- It is all that is needed for a planner to compute an optimal allocation
- It is all that is needed for the "invisible hand", ie households and firms, to call out prices and implement the first-best allocation

# 9. Recursive Formulation: Central Planner

# Problem of the Central Planner

- Let $C, N, K$ denote objects in the planning problem that correspond to $c, n, k$ in the decentralized economy
- The central planner chooses $C$, $N$, $K'$ to maximize the utility function of the representative household

# Bellman Equation

- The Bellman equation writes

$$v(K,A,s)=\max_{C,N,K'}\left\{u(C,1-N)+\beta\sum_{s'}\pi(s'|s)v(K',A',s')\right\}$$

subject to

$$\begin{aligned} K'+C &\leq AF(K,N)+(1-\delta)K \\ A' &= As' \end{aligned}$$

# Policy Functions

- Using the definition of the state vector $X \equiv \begin{bmatrix} K & A & s \end{bmatrix}$, we denote the optimal policy functions as

$$
\begin{aligned}
C &= \Omega^C(X) \\
N &= \Omega^N(X) \\
K' &= \Omega^K(X)
\end{aligned}
$$

- Equation $A' = As'$ and the Markov transition density $\pi(s'|s)$ induce a transition density $\Pi(X'|X)$ on the state $X$

# First-Order Conditions

- Define for convenience

$$U_c(X) \equiv u_c[\Omega^C(X), 1 - \Omega^N(X)] \qquad F_k(X) \equiv F_k[K, \Omega^N(X)]$$
$$U_\ell(X) \equiv u_\ell[\Omega^C(X), 1 - \Omega^N(X)] \qquad F_n(X) \equiv F_n[K, \Omega^N(X)]$$

- The first-order conditions are

$$U_\ell(X) = U_c(X) A F_n(X)$$
$$1 = \beta \sum_{X'} \Pi(X'|X) \frac{U_c(X')}{U_c(X)} [A' F_K(X') + 1 - \delta]$$

# 10. Recursive Formulation: Sequential Trading

# Endogenous State

- Relative to lecture 8, we now have an endogenous state variable, namely the aggregate capital stock $K_t$
- How do we deal with this in a competitive economy?
- We use a "Big $K$, little $k$" device

# Price Taker vs Price Maker

- So far we have assumed that each individual firm and household is a price taker, ie each acts as if their decisions do not affect current or future prices
- In sequential market setting, prices depend on the state, of which $K_t$ is part
- But of course, in the aggregate, agents choose the motion of capital $K_t$, and so trough their combined actions they determine prices, ie are price makers
- "Big $K$, little $k$" is a device that makes them ignore this fact when they solve their individual decision problem

# Big $K$, Little $k$

- Big $K$ is an endogenous state variable, useful to forecast prices, but which agents regard as beyond their control
- Small $k$ is chosen by firms and consumers
- In the equilibrium, after firms and consumers have optimized, we set

$$K = k$$

# Price System

- We specify price functions (prices are functions of the aggregate state $X$)
  - $r(X)$ is the rental price of capital
  - $w(X)$ is wage rate for labor
  - $Q(X'|X)$ is the price of a claim to one unit of consumption next period when next period's state is $X'$ and this period's state is $X$
- All are measured in units of this period's consumption good

# Perceived Law of Motion

- We take as given an arbitrary perceived law of motion for $K$

$$K' = G(X)$$

- This equation together with $A' = As'$ and a given subjective transition density $\hat{\pi}(s'|s)$ induce a subjective transition density $\hat{\Pi}(X'|X)$ for state $X$
- The perceived law of motion of $K$ and the transition probability $\hat{\Pi}(X'|X)$ describe the beliefs of the household

# Household Problem

- Let $J$ be the value function; the Bellman equation writes

$$J(a,X) = \max_{c,n,\bar{a}(X')} \left\{ u(c, 1-n) + \beta \sum_{X'} J[\bar{a}(X'), X'] \hat{\Pi}(X'|X) \right\}$$

subject to

$$c + \sum_{X'} Q(X'|X)\bar{a}(X') \leq w(X)n + a \quad \textbf{and} \quad \bar{a}(X') \geq 0$$

- $X \equiv \begin{bmatrix} K & A & s \end{bmatrix}$ is the vector of state variables
- $a$ is the household's individual wealth in units of current goods
- $\bar{a}(X')$ is next period's wealth in units of next period's consumption goods

# First-Order Conditions

- The first-order conditions are

$$\bar{u}_\ell(a, X) = \bar{u}_c(a, X) w(X)$$
$$Q(X'|X) = \beta \frac{\bar{u}_c[\sigma^a(a, X; X'), X']}{\bar{u}_c(a, X)} \hat{\Pi}(X'|X)$$

where the household's optimal policy functions are

$$c = \sigma^c(a, X); \qquad n = \sigma^n(a, X); \qquad \bar{a}(X') = \sigma^a(a, X; X')$$

and for convenience

$$\bar{u}_c(a, X) \equiv u_c[\sigma^c(a, X), 1 - \sigma^n(a, X)]$$
$$\bar{u}_\ell(a, X) \equiv u_\ell[\sigma^c(a, X), 1 - \sigma^n(a, X)]$$

# Problem of the Goods Producer

- The static problem of the goods producer writes

$$\max_{k,n}\{AF(k,n) - r(X)k - w(X)n\}$$

- The zero-profit conditions are

$$r(X) = AF_k(k,n)$$
$$w(X) = AF_n(k,n)$$

# Problem of the Capital Producer

- The problem of the capital producer writes

$$\max_{k'} k' \left\{ -1 + \sum_{X'} Q(X'|X)[r(X') + 1 - \delta] \right\}$$

- The zero-profit condition is

$$1 = \sum_{X'} Q(X'|X)[r(X') + 1 - \delta]$$

# 11. Recursive Competitive Equilibrium

# Equilibrium

- So far we have taken the price functions $r(X), w(X), Q(X|X')$, the perceived law of motion $K' = G(X)$, and $\hat{\Pi}(X'|X)$ as given arbitrarily
- We now impose equilibrium conditions on these objects and make them outcomes in the analysis; we impose

$$K = k$$

- Imposing equality afterward makes the household and firms be price takers

# Debt Supply and Debt Demand

- The supply of state-contingent debt issued by the capital producer must be equal to the demand for debt coming from the household

$$\bar{a}(X') = [r(X') + 1 - \delta]K'$$

- Beginning-of-period assets must also satisfy

$$a(X) = [r(X) + 1 - \delta]K$$

# Rewriting the Budget Constraint

- Plug the previous conditions into the household's budget constraint

$$\sum_{X'} Q(X'|X)[r(X') + 1 - \delta]K' = [r(X) + 1 - \delta]K + w(X)n - c$$

- Use the capital producer's FOC $\sum_{X'} Q(X'|X)[r(X') + 1 - \delta] = 1$ and the fact that $K'$ is predetermined when entering next period

$$K' = [r(X) + 1 - \delta]K + w(X)n - c$$

# Rewriting the Budget Constraint

- Plug in the equilibrium prices

$$K' = [AF_k(k,n) + 1 - \delta]K + AF_n(k,n)n - c$$

- Set $K = k$, $N = n = \sigma^n(a,X)$, $C = c = \sigma^c(a,X)$, and use Euler's theorem

$$K' = AF[K, \sigma^n(a,X)] + (1-\delta)K - \sigma^c(a,X)$$

- Use the equilibrium condition $a = [r(X) + 1 - \delta]K$

$$K' = AF\{K, \sigma^n([r(X)+1-\delta]K, X)\} + (1-\delta)K - \sigma^c([r(X)+1-\delta]K, X)$$

# Actual Law of Motion

- We have expressed $K'$ only as a function of the current aggregate state $X = \begin{bmatrix} K & A & s \end{bmatrix}$

$$K' = AF\{K, \sigma^n([r(X) + 1 - \delta]K, X)\} + (1 - \delta)K - \sigma^c([r(X) + 1 - \delta]K, X)$$

- This is the actual law of motion of $K'$ that is implied by the household's and firms' optimal decisions

# Perceived Law of Motion

- Remember the perceived law of motion of capital

$$K' = G(X)$$

- We want $G$ not to be arbitrary but to be an outcome
- We want to find an equilibrium perceived law of motion

# Rational Expectations

- For this we impose rational expectations: we require that the perceived and actual laws of motions be identical, by equating the previous two equations

$$G(X) = AF\{K, \sigma^n([r(X)+1-\delta]K, X)\} + (1-\delta)K - \sigma^c([r(X)+1-\delta]K, X)$$

- The perceived law of motion $G$ affects decisions $\sigma^c$ and $\sigma^n$ via the problem of the household, therefore the right side is itself an implicit function of $G$
- In turn, $G$ and prices imply an actual law of motion of capital
- Mathematically, $G$ is a fixed point: the equation maps a perceived $G$ and a price system into an actual $G$

# Rational Expectations

- Rational expectations mean that the agent's perception is consistent with the equilibrium outcome
- The previous equation requires that the perceived law of motion for the capital stock $G(X)$ equal the actual law of motion
- The actual law is determined jointly by the decisions of the household and the firms in a competitive equilibrium

# Recursive Competitive Equilibrium

A recursive competitive equilibrium with Arrow securities is a price system $r(X)$, $w(X)$, $Q(X'|X)$, a perceived law of motion $K' = G(X)$ and associated induced transition density $\hat{\Pi}(X'|X)$, a borrowing limit $\bar{a}(X')$, a household value function $J(a, X)$, and decision rules $\sigma^c(a, X)$, $\sigma^n(a, x)$, $\sigma^a(a, X; X')$ such that

1. Given $r(X)$, $w(X)$, $Q(X'|X)$, $\hat{\Pi}(X'|X)$, the functions $\sigma^c(a, X)$, $\sigma^n(a, X)$, $\sigma^a(a, X; X')$ and the value function $J(a, X)$ solve the household's problem
2. For all $X$, $r(X)$ and $w(X)$ solve the goods producer's problem

$$r(X) = AF_k \left\{K, \sigma^n([r(X) + (1-\delta)]K, X)\right\}$$
$$w(X) = AF_n \left\{K, \sigma^n([r(X) + (1-\delta)]K, X)\right\}$$

# Recursive Competitive Equilibrium

3. $Q(X'|X)$ and $r(X)$ satisfy the zero-profit condition

$$1 = \sum_{X'} Q(X'|X)[r(X') + 1 - \delta]$$

4. $G(X)$, $r(X)$, $\sigma^c(a, X)$, $\sigma^n(a, X)$ satisfy the law of motion of capital

$$G(X) = AF\{K, \sigma^n([r(X) + 1 - \delta]K, X)\} + (1 - \delta)K - \sigma^c([r(X) + 1 - \delta]K, X)$$

5. The perceived transition density equals the actual one

$$\hat{\pi} = \pi$$

# Remarks

- Item 1 enforces optimization by the household, given the prices it faces and its expectations
- Item 2 requires that the goods producer break even at every capital stock and labor supply chosen by the household
- Item 3 requires that the capital producer break even
- Market clearing is implicit when item 4 requires that the perceived and actual laws of motion of capital be equal
- Item 5 and the equality of perceived and actual $G$ imply that $\hat{\Pi} = \Pi$
- Thus, items 4 and 5 impose rational expectations

# Solving the System

- One could attack directly the fixed point problem at the heart of the equilibrium definition
- Instead we guess a candidate $G$ and a price system
- Then we verify that they form an equilibrium

# Using the Planning Problem

- Which candidates should we pick?
- Remember the welfare theorems: a competitive equilibrium is Pareto efficient
- Thus as our candidates for $G$ and prices we turn to the planning problem

# Using the Planning Problem

- For $G$ we choose the planner decision rule for $K'$

$$K' = \Omega^K(X)$$

- For prices we also choose those of the planner

$$r(X) = AF_k(X)$$
$$w(X) = AF_n(X)$$
$$Q(X'|X) = \beta\Pi(X'|X)\frac{U_c(X')}{U_c(X)}[A'F_K(X') + 1 - \delta]$$

# Equivalence

- In equilibrium the household's decision rules for consumption and labor matches those of the planner

$$\Omega^C(X) = \sigma^c([r(X) + 1 - \delta]K, X)$$
$$\Omega^N(X) = \sigma^n([r(X) + 1 - \delta]K, X)$$

- The key to verifying the guesses is to show that the FOCs for firms and the household are satisfied at these guesses; we leave this as an exercise

# Conclusion

- Economic phenomena are dynamic and uncertain
- We have studied two ways to model these phenomena
- The first way is to use Arrow-Debreu or Arrow general equilibrium structures and search for optimal actions
- These optimal actions are conditional on the sequence of realizations of all past and present random variables

# Conclusion

- The second way is to use recursive methods and search for equilibrium decision or policy rules
- These rules specify current actions as a function of a limited number of state variables that summarize all the necessary information
- Lucas and Prescott (1971) and Mehra and Prescott (1980) introduced the notion of recursive competitive equilibrium
- It is widely used today in macroeconomics and finance

# 12. Exercise

# Exercise – Equivalence of Allocations

1. Prove the equivalence of allocations of Section 6 between the time 0 trading and sequential trading equilibria.
2. Verify that the guesses in Section 11 are correct, ie that the recursive competitive equilibrium with sequential trading matches the recursive equilibrium of the central planner.