8. Recursive Competitive Equilibrium

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Recap of Lecture 6

- We laid out a pure exchange economy
- We determined the Pareto efficient allocation
- We determined the competitive equilibrium allocation in a market structure with time 0 trading of Arrow-Debreu securities

Key Results

- The competitive equilibrium allocation is Pareto efficient
- The allocation does not depend on idiosyncratic history and endowments
- The allocation does not change if markets reopen at a later date

Recap of Lecture 7

- We used the same pure exchange economy
- We determined the competitive equilibrium allocation in a market structure with sequential trading of one-period Arrow securities

Key Results

- The time dimension of the asset space greatly shrinks
- Given a zero initial wealth distribution, the Arrow-Debreu and Arrow equilibrium allocations are equivalent

A Huge State Space

- In both Arrow-Debreu and Arrow environments, the state space is huge
- Endowments $\{y_t^i(s^t)\}_i$, pricing kernels $\tilde{Q}(s_{t+1}|s^t)$, portfolios $\{\tilde{a}_t^i(s^t)\}_i$ all depend on history s^t so are time-varying functions of all past events $\{s_\tau\}_{\tau=0}^t$
- Such a large state space makes it difficult to formulate and solve an economic model that can be used to confront empirical observations

Reducing the State Space

- We want a framework in which economic outcomes are functions of a limited number of state variables
- State variables offer a parsimonious description of the state of the world by summarizing the effects of 1) all past events and 2) current information
- In this lecture, we are going to specialize the exogenous processes in order to facilitate a recursive formulation of the sequential-trading equilibrium

Lecture Outline

1. Markov Endowments
2. Markov Equilibrium Outcomes
3. Recursive Formulation
4. Exercise

Main Reference: Ljungqvist and Sargent, 2018, *Recursive Macroeconomic Theory*, Fourth Edition, Chapter 8, Section 9

1. Markov Endowments

Markov State Space

- We assume the endowments are governed by a Markov process
- Let $\pi(s'|s)$ be a Markov chain with given initial distribution $\pi_0(s)$ and state space $s \in S$

$$\pi(s'|s) = \text{Prob}(s_{t+1} = s'|s_t = s) \quad \textbf{and} \quad \pi_0(s_0) = \text{Prob}(s_0 = s)$$

- As you recall from lecture 2, the chain induces a sequence of probability measures $\pi_t(s^t)$ on histories s^t via the recursions

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \dots \pi(s_1|s_0)\pi_0(s_0)$$

Convenient Property

- As usual, we assume trading occurs after s_0 is observed: $\pi_0(s_0) = 1$
- The Markov property means that the conditional probability $\pi_t(s^t|s^\tau)$ for $t > \tau$ depends only on the state s_τ at time τ and not on the history before τ

$$\pi_t(s^t|s^\tau) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\dots\pi(s_{\tau+1}|s_\tau)$$

- That is, the process is memoryless

Markov Endowment

- Next, we assume that the endowments of consumers in period t are time-invariant measurable functions of s_t

$$y_t^i(s^t) = y^i(s_t) \quad \text{for all } i$$

- In other words, the endowment of each agent follows a Markov chain

Same Results

- In previous lectures, s_t was an arbitrary stochastic process
- Now s_t is a particular stochastic process, a Markov chain
- All our previous results continue to hold
- But the simplifying Markov assumption for s_t gives more structure to the model's equilibrium prices and quantities

2. Markov Equilibrium Outcomes

History Independence

- Remember, each individual's consumption is a function only of the current realization of the aggregate endowment
- It does not depend on the history leading to that outcome
- This holds under any stochastic process for the endowment
- Therefore it holds for a Markov process

Sequential Trading

- Under the assumption that $y_t^i(s^t) = y^i(s_t)$, we have

$$c_t^i(s^t) = \bar{c}^i(s_t) \quad \text{for all } i$$

- Plug that into the FOC of the sequential-trading economy

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u_i'[\bar{c}^i(s_{t+1})]}{u_i'[\bar{c}^i(s_t)]} \pi_t(s_{t+1}|s_t) \equiv Q(s_{t+1}|s_t)$$

- The pricing kernel in the sequential-trading equilibrium is a function only of the current state, ie is independent of history

Time 0 Trading

- Consider now the Arrow-Debreu economy
- If $t-\tau = k-j$ and $[s_\tau, s_{\tau+1}, \dots, s_t] = [\tilde{s}_j, \tilde{s}_{j+1}, \dots, \tilde{s}_k]$ then

$$
\begin{aligned}
q_t^\tau(s^t) &= \beta^{t-\tau} \frac{u_i'[c_t^i(s^t)]}{u_i'[c_\tau^i(s^\tau)]} \pi_t(s^t | s^\tau) \\
&= \beta^{t-\tau} \frac{u_i'[\bar{c}^i(s_t)]}{u_i'[\bar{c}^i(s_\tau)]} \pi_t(s_t | s_{t-1}) \dots \pi(s_{\tau+1} | s_\tau) \\
&= \beta^{k-j} \frac{u_i'[\bar{c}^i(\tilde{s}_k)]}{u_i'[\bar{c}^i(\tilde{s}_j)]} \pi_t(\tilde{s}_k | \tilde{s}_{k-1}) \dots \pi(\tilde{s}_{j+1} | \tilde{s}_j) \\
&= q_k^j(\tilde{s}^k)
\end{aligned}
$$

- The equilibrium Arrow-Debreu price of date $t \geq 0$, history s^t goods expressed in date $\tau \leq t$, history s^τ goods is also not history dependent

Natural Debt Limit

- The natural debt limit inherits the Markov property

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y^i(s_\tau) = \bar{A}^i(s_t)$$

- The natural debt limit depends only on the current state s_t, ie it does not exhibit history dependence

Wealth

- The level of financial wealth inherits the Markov property

$$
\begin{aligned}
\Upsilon_t^i(s^t) &= \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau)[c_\tau^i(s^\tau) - y_\tau^i(s^\tau)] \\
&= \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau)[\bar{c}^i(s_\tau) - y^i(s_\tau)] \\
&= \bar{\Upsilon}^i(s_t)
\end{aligned}
$$

- Wealth depends only on the current state, not on history s^t

No Idiosyncratic Risk

- Each consumer enters every period with a wealth level that is independent of past realizations of her endowment
- In other words, her past trades have fully insured her against the idiosyncratic outcomes of her endowment
- This is how the sequential-trading competitive equilibrium attains the first-best outcome in which consumers bear no idiosyncratic risk

The Current State Is Enough

- The consumer enters the present period t with a wealth level that is a function only of the current state s_t
- She chose that state-contingent wealth in $t-1$, and this wealth is sufficient to continue a trading strategy insuring her against future idiosyncratic risk
- The state s_t determines the current endowment and pricing kernel
- The state s_t contains all information relevant for predicting future realizations of the endowment and future prices

3. Recursive Formulation

Recursive Formulation

- The endowment $y^i(s)$ is a function of a Markov state s
- The pricing kernel $Q(s'|s)$ is a function of a Markov state s
- Therefore we can formulate the consumer's optimization problem in a recursive fashion

State Variables

- Two variables summarize the state of consumer i at time t

1. Her individual wealth a_t^i, an endogenous variable chosen in $t-1$
2. The current aggregate realization s_t, an exogenous variable observed in t

Policy Functions

- We seek a pair of optimal policy functions $h^i(a, s)$ and $g^i(a, s, s')$ such that the consumer's optimal decisions are

$$
\begin{aligned}
c_t^i &= h^i(a_t^i, s_t) \\
a_{t+1}^i(s_{t+1}) &= g^i(a_t^i, s_t, s_{t+1})
\end{aligned}
$$

Value Function

- Let $v^i(a, s)$ be the optimal value of the problem of consumer i, starting from state (a, s)
- $v^i(a, s)$ is the maximum expected discounted utility that consumer i with current wealth a can attain in state s

Bellman Equation

- The Bellman equation for the consumer's problem is

$$v^i(a,s) = \max_{c,\hat{a}(s')} \left\{ u_i(c) + \beta \sum_{s'} v^i[\hat{a}(s'), s']\pi(s'|s) \right\}$$

$$\text{subject to} \qquad c + \sum_{s'} \hat{a}(s')Q(s'|s) \leq y^i(s) + a$$

$$\text{and} \qquad -\hat{a}(s') \leq \bar{A}^i(s'), \quad \text{for all } s'$$

- The two controls are consumption c and next-period wealth $\hat{a}(s')$
- The budget constraint and debt constraint are the same as before

Necessary Conditions

- The first-order condition is

$$u_i'(c)Q(s'|s) = \beta v^{i\prime}[\hat{a}(s'), s']\pi(s'|s)$$

- The envelope condition is

$$v^{i\prime}(a, s) = u_i'(c)$$

Euler Equation

- Combine the two necessary conditions and obtain the Euler equation

$$Q(s_{t+1}|s_t) = \frac{\beta u_i'(c_{t+1}^i)}{u_i'(c_t^i)} \pi(s_{t+1}|s_t)$$

where it is understood that

$$c_t^i = h^i(a_t^i, s_t) \text{ and } c_{t+1}^i = h^i[a_{t+1}^i(s_{t+1}), s_{t+1}] = h^i[g^i(a_t^i, s_t, s_{t+1}), s_{t+1}]$$

Recursive Competitive Equilibrium

A recursive competitive equilibrium is an initial distribution of wealth $\{\tilde{a}_0^i\}_{i=1}^I$, a set of borrowing limits $\{\bar{A}^i(s)\}_{i=1}^I$, a pricing kernel $Q(s'|s)$, sets of value functions $\{v^i(a,s)\}_{i=1}^I$, and decision rules $\{h^i(a,s), g^i(a,s,s')\}_{i=1}^I$ such that

1. The state-by-state borrowing constraints satisfy

$$\bar{A}^i(s) = y^i(s) + \sum_{s'} Q(s'|s)\bar{A}^i(s'|s)$$

2. For all i, given a_0^i, $\bar{A}^i(s)$, and the pricing kernel, the value functions and decision rules solve the consumer's problem
3. For all realizations of $\{s_t\}_{t=0}^\infty$, markets clear

$$\sum_i^I c_t^i = \sum_i^I y^i(s_t) \quad \text{and} \quad \sum_i^I \hat{a}_{t+1}^i(s') = 0 \quad \text{for all } t \text{ and } s'$$

Taking Stock

- We have just showed how to obtain a recursive formulation of the equilibrium with sequential trading
- We've had to assume endowments followed a Markov process
- Under that assumption, we have identified a state vector in terms of which the Arrow securities could be cast
- This aggregate state vector has then become a component of the state vector for each individual problem

Pure Exchange Economy, Exogenous State

- The transformation of price systems in a pure exchange economy is easy
- The only endogenous state variable is wealth, and wealth is a function only of the current exogenous Markov state variable

Production Economy, Endogenous State

- The transformation is more subtle in economies in which part of the aggregate state is endogenous
- This happens when the state emerges from the history of equilibrium interactions of agents' decisions, for example the aggregate capital stock
- In the next lecture, we will study an example of such an economy by adding production into the complete-market framework

4. Exercise

Exercise – Alternating Income

An economy consists of two infinitely lived consumers named $i = 1, 2$. There is one nonstorable consumption good. Consumer i consumes c_t^i at time t. Consumer i ranks consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

where $u(c)$ is increasing, strictly concave, twice continuously differentiable. Consumer 1 is endowed with a stream of the consumption good $y_t^1 = 1, 0, 0, 1, 0, 0, 1, \ldots$. Consumer 2 is endowed with a stream of the consumption good $y_t^2 = 0, 1, 1, 0, 1, 1, 0, \ldots$. Assume that there are complete markets with time 0 trading.

Exercise – Continued

1. Define a competitive equilibrium.
2. Compute a competitive equilibrium.
3. Suppose that one of the consumers markets a derivative asset that promises to pay 0.05 units of consumption each period. What would the price of that asset be?