16. Overlapping Generations

Neoclassical Production

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Adding Production

- So far we have studied pure exchange, endowment OLG economies
- Today we introduce production into the OLG model by means of a standard neoclassical production function
- We thus endogenize output, investment, capital accumulation, factor prices

Lecture Outline

1. The OLG Model with Production
2. A Reminder on the Golden Rule
3. Over-Accumulation of Capital
4. CRRA Utility and Cobb-Douglas Production
5. Evidence on Dynamic Inefficiency
6. Restoring Dynamic Efficiency
7. Social Security
8. Altruism
9. Exercises

Main Reference: Acemoglu, 2009, *Introduction to Modern Economic Growth*, Chapter 9

1. The OLG Model with Production

Population

- The demographic structure is the same as before: individuals live for two periods, young and old
- Time starts in $t = 0$ now, the initial old are born in $t = -1$, the initial young are born in $t = 0$
- There is population growth: in t the young population is

$$L_t = (1 + n)^t L_0, \qquad L_0 \text{ given}$$

Preferences

- Preferences are identical to previous lectures except that agents discount the second period of their life

$$u(c_t^t) + \beta u(c_{t+1}^t), \qquad \beta \in (0,1)$$

- u satisfies the usual assumptions: $u' > 0$, $u'' < 0$, $\lim_{c \to 0} u'(c) = \infty$

Work Then Retire

- There is production and labor in the economy
- We assume that only young individuals can work
- They supply labor inelastically
- Old people retire

Technology

- The production function is as usual

$$Y_t = F(K_t, L_t) = L_t F(K_t/L_t, 1) \equiv L_t f(k_t), \quad k_t \equiv K_t/L_t$$

- For simplicity we assume full depreciation, $\delta = 1$

Accumulating Capital

- Who builds the capital stock?
- In previous lectures we saw two cases

1. A representative firm operates the production technology, the representative household builds capital directly and rents it out to the firm
2. A goods producer operates the production technology, a capital producer builds and transforms capital and rents it out to the goods producer

Three Equivalent Setups

- The two setups are equivalent under two conditions
  - Capital and investment are reversible, convert one-for-one into output
  - There is no friction in the market for capital
- Here we opt for a third setup that is also equivalent
- We assume that the representative firm operates the two technologies: production and capital transformation/accumulation

One Firm, Two Tasks

- We assume that the initial old agent born in $t = -1$ receives an endowment and transforms it into capital for $t = 0$
- The problem of the firm in time 0 is

$$\max_{I_t, L_t} \sum_{t=-1}^{\infty} \{q_t^0 F(K_t, L_t) - w_t^0 L_t - q_t^0 I_t\}$$

$$\text{subject to} \quad K_{t+1} = (1-\delta)K_t + I_t, \qquad K_{-1} = 0 \text{ given}$$

Manipulations

- Plug the capital accumulation equation into the objective and open the summation

$$\sum_{t=-1}^{\infty}\{q_t^0 F(K_t, L_t) - w_t^0 L_t - q_t^0[K_{t+1} - (1-\delta)K_t]\}$$

$$= q_{-1}^0 F(K_{-1}, L_{-1}) - w_{-1}^0 L_{-1} + q_{-1}^0(1-\delta)K_{-1}$$
$$+ \sum_{t=0}^{\infty}\{q_t^0 F(K_t, L_t) - w_t^0 L_t - q_{t-1}^0 K_t + (1-\delta)q_t^0 K_t]\} - \lim_{T\to\infty} q_T^0 k_{T+1}$$

$$= \underbrace{q_{-1}^0 F(0, L_{-1})}_{=0} - \underbrace{w_{-1}^0 L_{-1}}_{=0} + \sum_{t=0}^{\infty} q_t^0\{F(K_t, L_t) - w_t^t L_t - \frac{1}{q_t^{t-1}} K_t + (1-\delta)K_t\}$$
$$- \lim_{T\to\infty} q_T^0 k_{T+1}$$

Intertemporal Borrowing

- In period $t = -1$, $K_{-1} = 0$ by assumption, therefore there is no production and no labor, $F(0, L_{-1}) = 0$ and $L_{-1} = 0$
- The transversality condition imposes

$$\lim_{T \to \infty} q_T^0 k_{T+1} = 0$$

- Define $R_t \equiv 1/q_t^{t-1} = 1/Q_t$ as the gross return to capital, ie the inverse of the pricing kernel

Intertemporal Borrowing

- We conclude that the problem of the representative firm is equivalent to maximizing in each period $t \geq 0$

$$F(K_t, L_t) - w_t L_t - R_t K_t + (1 - \delta) K_t$$

- The firm borrows K_t at the end of $t-1$, hires labor L_t in t, produces and sells output in t, and since $\delta = 1$ repays $R_t K_t$ in t after production
- If $\delta < 1$ the firm retains $(1-\delta)K_t$ after production in t and thus only borrows $K_{t+1} - (1-\delta)K_t = I_t$ at the end of t

Firm Problem

- Based on the preceding, the firm solves

$$\max_{K_t, L_t} \{F(K_t, L_t) - w_t L_t - R_t K_t\} = \max_{k_t, L_t} L_t \{f(k_t) - w_t - R_t k_t\}$$

- The first-order conditions are

$$R_t = f'(k_t)$$
$$w_t = f(k_t) - k_t f'(k_t)$$

- Note that $R_t \equiv 1 + r_t$ where r_t is the net return on capital

Household Problem

- Consumer t, born in t, decides how much to consume in t and $t+1$: to consume when old, she must save s_t when young, and thus solves

$$\max_{c_t^t, c_{t+1}^t, s_t} \quad u(c_t^t) + \beta u(c_{t+1}^t)$$

$$\text{subject to} \quad c_t^t + s_t \leq w_t$$

$$\text{and} \quad c_{t+1}^t \leq R_{t+1} s_t$$

- Young agents rent their savings as capital to the firms at the end of t and receive the return at $t+1$ after production

First-Order Condition

- Plug the two budget constraints into the objective to simplify

$$\max_{s_t} u(w_t - s_t) + \beta u(R_{t+1} s_t)$$

- Derive with respect to s_t

$$u'(\underbrace{w_t - s_t}_{=c_t^t}) = \beta R_{t+1} u'(\underbrace{R_{t+1} s_t}_{=c_{t+1}^t})$$

- This is a standard consumption-saving Euler equation

Saving Function

- The FOC leads to an implicit function that determines savings per person

$$s_t = s(w_t, R_{t+1})$$

- s_t is strictly increasing in the wage $w_t = c_t^t + s_t$
- s_t may be increasing or decreasing in R_{t+1}
  - Increasing if the substitution effect dominates
  - Decreasing if the income effect dominates

Aggregation and Market Clearing

- Aggregate savings in the economy are

$$S_t = \underbrace{L_t}_{\text{number of people}} \times \underbrace{s(w_t, R_{t+1})}_{\text{saving per capita}}$$

- The capital stock is financed by aggregate savings

$$\underbrace{K_{t+1}}_{\text{capital demand}} = \underbrace{L_t s(w_t, R_{t+1})}_{\text{capital supply}}$$

- Divide by L_t

$$(1+n)k_{t+1} = s(w_t, R_{t+1})$$

Competitive Equilibrium

A competitive equilibrium is a sequence of quantities $\{k_t, c_t, s_t\}_{t=0}^{\infty}$ and prices $\{R_t, w_t\}_{t=0}^{\infty}$ such that

- Consumers optimize: $s_t = s(w_t, R_{t+1})$
- Firms optimize: $R_t = f'(k_t)$ and $w_t = f(k_t) - k_t f'(k_t)$
- Markets clear: $s(w_t, R_{t+1}) = (1+n)k_{t+1}$
- The initial old generation has some endowment that determines the initial capital stock K_0

Law of Motion

- Plug the firm's FOCs into the capital market-clearing condition

$$k_{t+1} = \frac{s[f(k_t) - k_t f'(k_t), f'(k_{t+1})]}{1+n}$$

- This difference equation determines the fundamental law of motion of the OLG economy

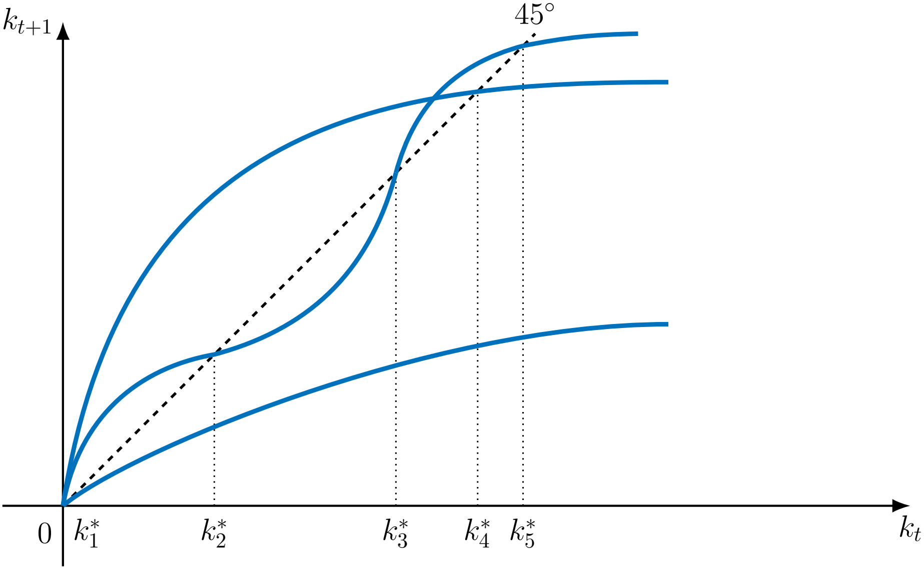
Steady State

- A steady state is such that $k_{t+1} = k_t = k^*$

$$k^* = \frac{s[f(k^*) - k^* f'(k^*), f'(k^*)]}{1+n}$$

- Since we did not specify the form of the utility function, the saving function s can take any form
- The steady state is not necessarily unique

Various Steady-State Equilibria



Multiple Equilibria

- The OLG model can lead to a unique equilibrium
- It can also lead to multiple equilibria
- It can lead to an equilibrium with zero capital stock
- We need more structure on the utility and production functions to make predictions with this model

2. A Reminder on the Golden Rule

Neoclassical Growth Model

- Consider the neoclassical growth model with infinitely-lived agents
- Assume the labor supply is inelastic
- The population grows at a constant rate: $L_{t+1} = (1+n)L_t$, with L_0 given
- The resource constraint, $C_t + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t$, in per capita terms is

$$c_t + (1+n)k_{t+1} = f(k_t) + (1-\delta)k_t$$

Steady State

- This economy converges to a unique steady state
- The Euler equation in steady state is

$$\beta[f'(k_{\text{neo}}) + 1 - \delta] = 1$$

- The steady-state level of capital per worker in the neoclassical growth model is thus

$$f'(k_{\text{neo}}) = \frac{1}{\beta} - 1 + \delta$$

Golden Rule

- What is the steady-state level of capital per worker that maximizes steady-state consumption per worker?
- In steady state the resource constraint is $c = f(k) - (\delta + n)k$
- Thus the optimal level of capital is

$$k_{\text{gold}} = \arg\max_{k}\{f(k) - (\delta + n)k\}$$

- We obtain the golden rule

$$f'(k_{\text{gold}}) = \delta + n$$

Discounting with Constant Population

- Recall the lifetime utility of an agent, $\sum_{t=0}^{\infty} \beta^t u(C_t)$
- With constant population, $n = 0$, we usually assume $\beta < 1$
- This is to ensure discounting of future utility streams
- Why? If $\beta > 1$ lifetime utility typically has infinite value

Discounting with Growing Population

- With population growth, $n > 0$, we impose $\beta(1+n) < 1$
- This is because the household derives utility from the per capita consumption of its additional members in the future
- Using the discount rate, $\beta \equiv 1/(1+\rho)$, we have $\rho > n$
- This is reasonable as annually, $\rho = r \approx 0.04$ (implying $\beta \approx 0.96$) and $n \approx 0.01$

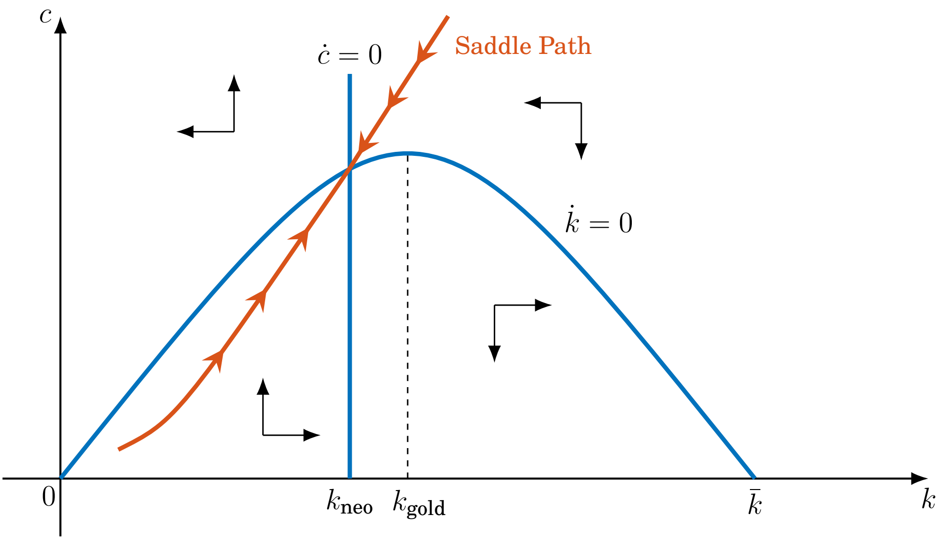
Under-Accumulation

- Using the assumption $\beta(1+n) < 1$, we deduce

$$\frac{1}{\beta} - 1 + \delta > \delta + n \iff f'(k_{\text{neo}}) > f'(k_{\text{gold}}) \iff k_{\text{neo}} < k_{\text{gold}}$$

- In the neoclassical growth model, there is under-accumulation of capital because agents are impatient and value the present more
- Despite $k_{\text{neo}} < k_{\text{gold}}$, the competitive equilibrium is efficient

Under-Accumulation of Capital



3. Over-Accumulation of Capital

Back to Mortal

- Back to our OLG model
- The aggregate resource constraint is

$$\underbrace{C_t^t + C_t^{t-1}}_{\text{consumption}} = \underbrace{L_t[f(k_t) - (1+n)k_{t+1}]}_{\text{output} - \text{investment}}$$

- In steady state this gives us, dividing by L_t

$$c^* \equiv c_y^* + \frac{c_o^*}{1+n} = f(k^*) - (1+n)k^*$$

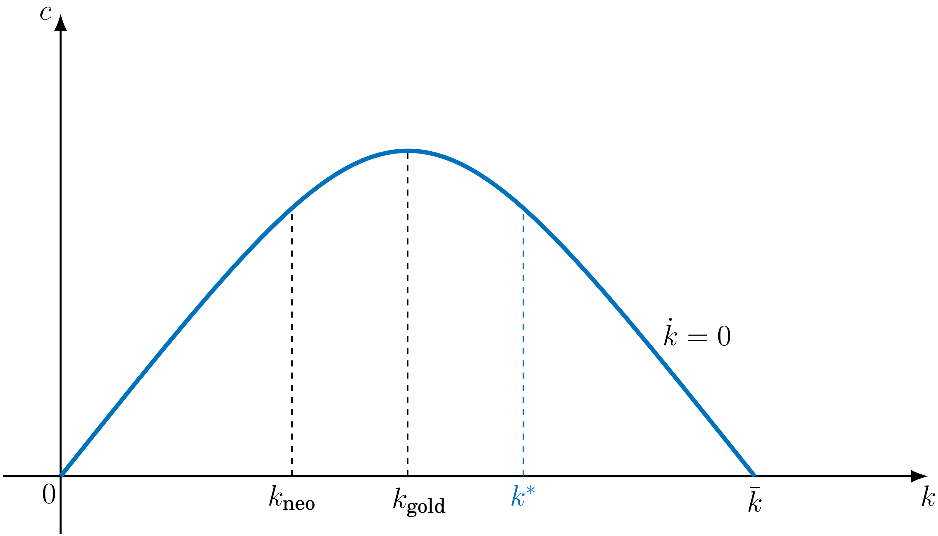
Over-Accumulation

- Remember the golden rule, $f'(k_{\text{gold}}) = \delta + n = 1 + n$
- Note that if we have $\frac{dc^*}{dk^*} = f'(k^*) - (1+n) < 0$ then

$$f'(k^*) < f'(k_{\text{gold}}) \iff k^* > k_{\text{gold}}$$

- The economy is over-accumulating capital
- Intuitively, the young agents are saving too much for retirement

Over-Accumulation



Reducing k^*, Increasing c^*

- Suppose the economy is in steady state at time T
- Now suppose we reduce next period's capital per worker by a small amount $\Delta k \in (0, k^* - k_{\text{gold}})$ and from then on keep the capital-labor ratio at $k^* - \Delta k$
- Since $c_t = f(k_t) - (1+n)k_{t+1}$ we have

$$\Delta c_T = (1+n)\Delta k > 0$$

- Consumption increases in T due to a decrease in savings
- What about other periods $t > T$?

Reducing Capital Increases Consumption

- If Δk is sufficiently small we have for all $t > T$

$$
\begin{aligned}
\Delta c_t &= [f(k^* - \Delta k) - (1+n)(k^* - \Delta k)] - [f(k^*) - (1+n)k^*] \\
&= [f(k^* - \Delta k) - f(k^*) + (1+n)\Delta k] \\
&= \left[\frac{f(k^* - \Delta k) - f(k^*)}{\Delta k} + (1+n)\right]\Delta k \\
&\approx -[f'(k^* - \Delta k) - (1+n)]\Delta k
\end{aligned}
$$

- Since $f'(k^* - \Delta k) - (1+n) < 0$ we have $\Delta c_t > 0$
- Consumption increases in all periods

Pareto Improvement

- The small decrease in capital increases consumption for each generation
- We can allocate that equally during the two periods of life
- Thus the variation increases utility of all generations
- It is Pareto improving

Dynamic Inefficiency

- Dynamic efficiency means one cannot make one generation better off without making any other generation worse off
- If $k^* > k_{\text{gold}}$, the OLG economy is dynamically inefficient
- By saving less, ie reducing the amount of steady-state capital, steady-state consumption would increase
- Too much capital lowers its productivity

Return on Capital vs Population Growth

- Note that $k^* > k_{\text{gold}}$ only if

$$f'(k^*) < 1 + n \iff R^* < 1 + n \iff 1 + r^* < 1 + n \iff r^* < n$$

- Dynamic inefficiency arises when the net return to capital r^* is lower than the growth rate of population n

Infinite Aggregate Income

- Suppose the economy is in steady state at time T
- The value of aggregate income is

$$\sum_{t=T}^{\infty} L_t \frac{w^*}{(1+r^*)^{t-T}} = w^* L_T \sum_{t=T}^{\infty} \frac{(1+n)^{t-T}}{(1+r^*)^{t-T}}$$
$$< \infty \iff \frac{1+n}{1+r^*} < 1 \iff r^* > n$$

- When $r^* < n$ the value of aggregate income is infinite
- Thus the transversality condition is violated when $r^* < n$

Neoclassical Growth

- In the neoclassical growth model the assumption $\beta(1+n) < 1$ ensures that in the steady state $r^* > n$
- Thus dynamic inefficiency never happens in that economy
- The transversality condition is always satisfied

$$\lim_{t\to\infty} \beta^t (1+n)^t u'(c_t) k_{t+1} = 0$$

Limiting Over-Accumulation

- Diamond (1965) first pointed out the inefficiency of the OLG model
- The young generation is saving too much for retirement
- Thus a system that provides consumption to the old when they retire can potentially address the over-accumulation of capital
  - Fiat money (Samuelson 1958)
  - One-period government bonds (Diamond 1965)
  - Social security (Samuelson 1975)

4. CRRA Utility and Cobb-Douglas Production

Functional Forms

- Let's assume the utility function is CRRA

$$U(c_t^t, c_{t+1}^t) = \frac{(c_t^t)^{1-\sigma} - 1}{1-\sigma} + \beta \frac{(c_{t+1}^t)^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

- We also assume the production function is Cobb-Douglas

$$f(k_t) = k_t^\alpha, \quad \alpha \in (0,1)$$

- These functional forms allow us to derive a concrete expression for the saving function

Saving Function

- The first-order condition of the household becomes

$$(w_t - s_t)^{-\sigma} = (R_{t+1} s_t)^{-\sigma} \beta R_{t+1}$$

- Solve for savings s_t

$$s_t = \frac{w_t}{1 + \beta^{-\frac{1}{\sigma}} R_{t+1}^{-\frac{1-\sigma}{\sigma}}}$$

- Since $1 + \beta^{-\frac{1}{\sigma}} R_{t+1}^{-\frac{1-\sigma}{\sigma}} > 1$ we have $s_t < w_t$, savings are always less than earnings

Change in Wages

- How do savings respond to a change in wages?

$$\frac{\partial s_t}{\partial w_t} = \frac{1}{1 + \beta^{-\frac{1}{\sigma}} R_{t+1}^{-\frac{1-\sigma}{\sigma}}} \in (0, 1)$$

- A rise in w_t increases savings but less than one-for-one
- The increase in wages is split into higher savings and higher consumption

Change in the Rate of Return

- How do savings respond to a change in the gross rate?

$$\frac{\partial s_t}{\partial R_{t+1}} = \left(\frac{1-\sigma}{\sigma}\right) \underbrace{(\beta R_{t+1})^{-\frac{1}{\sigma}} \frac{s_t}{1+\beta^{-\frac{1}{\sigma}} R_{t+1}^{-\frac{1-\sigma}{\sigma}}}}_{>0}$$

- The response of savings depends on σ, the inverse elasticity of intertemporal substitution (EIS)

Income and Substitution Effects

1. If $\sigma < 1$, the EIS is “high”, the response is positive
   - The substitution effect dominates
   - Agents save a higher fraction of their earnings as they earn more
2. If $\sigma > 1$, the EIS is “low”, the response is negative
   - The income effect dominates
   - Agents consume more and save proportionally less as they earn more
3. If $\sigma = 1$, the EIS is unity, the saving rate does not respond
   - The income and substitution effects cancel out, the increase in R_{t+1} increases consumption and savings in the same proportion

Capital Dynamics

- Combine the law of motion of capital $k_{t+1}(1+n) = s_t$ and the FOCs $s_t = \frac{w_t}{1+\beta^{-\frac{1}{\sigma}} R_{t+1}^{-\frac{1-\sigma}{\sigma}}}$, $w_t = (1-\alpha)k_t^{\alpha}$, and $R_t = \alpha k_t^{\alpha-1}$

$$k_{t+1} = \frac{(1-\alpha)k_t^{\alpha}}{(1+n)\left(1+\beta^{-\frac{1}{\sigma}}\left[\alpha k_{t+1}^{\alpha-1}\right]^{-\frac{1-\sigma}{\sigma}}\right)}$$

- This is a non-linear first difference equation in k_t

Steady State

- The steady-state level of capital is the solution to

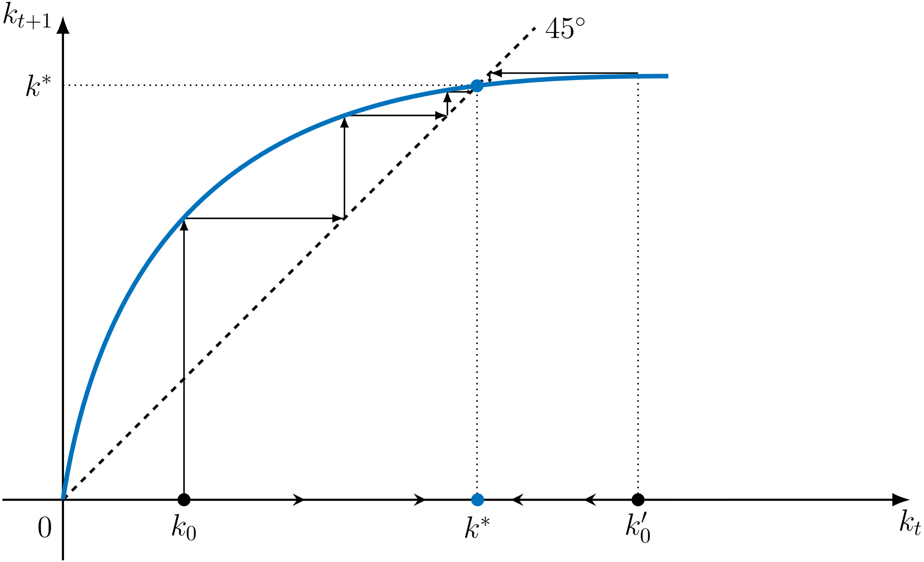
$$(1-\alpha)(k^*)^{\alpha-1} = (1+n)\left(1+\beta^{-\frac{1}{\sigma}}\left[\alpha(k^*)^{\alpha-1}\right]^{-\frac{1-\sigma}{\sigma}}\right)$$

- We can show that k^* is unique

Equilibrium Dynamics

- It is possible to show that the equilibrium dynamics are very similar to those of the neoclassical growth model
- In particular the capital-labor ratio converges monotonically to the unique steady state k^*
- Starting with $k_0 < k^*$ the OLG economy steadily accumulates capital
- Starting with $k_0 > k^*$ the economy de-accumulates capital

Equilibrium Dynamics



Log Preferences

- Let's assume log preferences, ie $\sigma = 1$
- With log utility, income and substitution effects cancel out
- A change in the interest rate, and hence in the capital-labor ratio of the economy, has no effect on the saving rate, which is constant
- The model behaves essentially like the Solow model

Capital Dynamics

- With $\sigma = 1$ the law of motion of capital becomes

$$k_{t+1} = \frac{\beta(1-\alpha)k_t^{\alpha}}{(1+n)(\beta+1)}$$

- In steady state

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+n)(\beta+1)}\right]^{\frac{1}{1-\alpha}}$$

Golden Rule

- Remember the golden rule, $f'(k_{\text{gold}}) = \delta + n$
- With Cobb-Douglas production and full depreciation

$$\alpha k_{\text{gold}}^{\alpha-1} = 1 + n \iff k_{\text{gold}} = \left[\frac{\alpha}{1+n}\right]^{\frac{1}{1-\alpha}}$$

- Hence, $k^* > k_{\text{gold}}$ if

$$\left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}} > \left[\frac{\alpha}{1+n}\right]^{\frac{1}{1-\alpha}} \iff \beta(1-\alpha) > (1+\beta)\alpha$$

Dynamic Inefficiency

- The economy is dynamically inefficient if

$$\beta(1-\alpha) > (1+\beta)\alpha$$

- In the data the share of capital in income α is roughly 0.4

$$0.6 \times \beta > (1+\beta) \times 0.4 \iff \beta > 2$$

- This is implausible as $\beta = \frac{1}{1+r} \approx 0.96$ with annual data

Summary

- Take the OLG model with a Cobb-Douglas production, log preferences, and a reasonable calibration
- This economy does not exhibit dynamic inefficiency

5. Evidence on Dynamic Inefficiency

Empirical Evidence

- What does the data say about capital over-accumulation?
- In general, empirical evidence rejects dynamic inefficiency
- Abel, Mankiw, Summers, and Zeckhauser (1989, *Restud*) test and reject dynamic inefficiency for the US and six rich countries

Testing Dynamic Inefficiency

- Abel, Mankiw, Summers, and Zeckhauser (1989, *Restud*) propose a general criterion to test whether an economy is dynamically efficient
- There is dynamic inefficiency if there exists $\epsilon > 0$ such that for all $t > t_0$

$$r_t^K K_t \leq (1-\epsilon) I_t$$

- The economy is dynamically inefficient if in each period it is investing more than it is getting from capital income, ie if it is sinking resources

A Theoretical Curiosity

"While overaccumulation and dynamic inefficiency have dominated much of the discussion of OLG models in the literature, one should not overemphasize the importance of dynamic inefficiency. . . . The major question of economic growth is why so many countries have so little capital for their workers and why the process of economic growth and capital accumulation started only in the past 200 years. It is highly doubtful that overaccumulation is a major problem for most countries in the world."

Daron Acemoglu, 2008, *Introduction to Modern Economic Growth*

Revisiting Dynamic Inefficiency

- Geerolf (2018) challenges the finding of Abel et al. (1989)
- Consider the problem of a firm with decreasing returns to scale

$$\max_{K_t,N_t,L_t} A_t K_t^{\alpha} N_t^{\beta} L_t^{\gamma} - r_t^K K_t - w_t N_t - r_t^L L_t$$

- N_t is labor and L_t is land, in fixed supply $L_t = \bar{L}$

Measuring Capital Income

- How do we measure $r_t^K K_t$ in the data?
- Using an income approach

$$A_t K_t^{\alpha} N_t^{\beta} L_t^{\gamma} = \underbrace{r_t^K K_t + r_t^L L_t + \textbf{profits}}_{\text{Capital income}} + w_t N_t$$

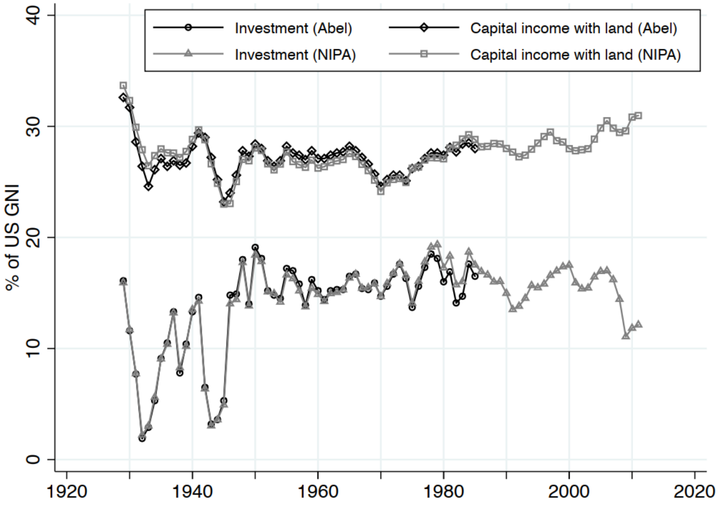
Reassessing Capital Income

- There are at least two issues in measuring $r_t^K K_t$

1. Many unincorporated firms report only mixed income, which includes both labor and capital income
2. Land is not an accumulated factor so its marginal product should not be treated as investment income

- Geerolf uses new data on mixed income and land to update estimates of income from capital and reassess it downwards

Reproducing and Updating Abel et al. (1989)

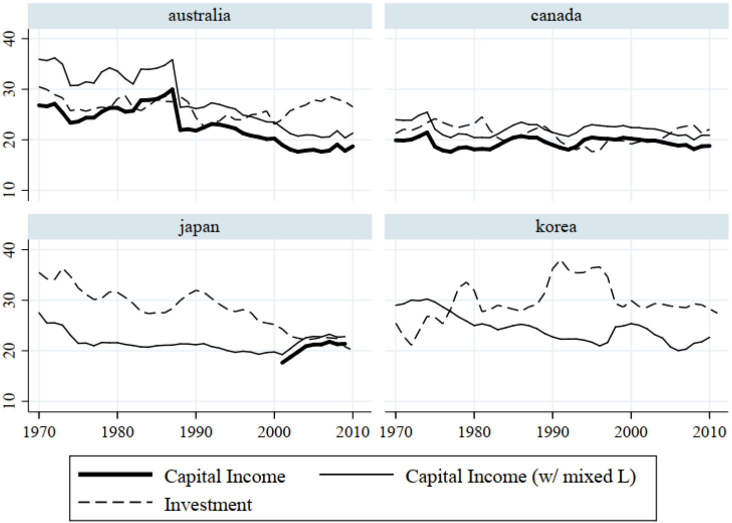


Source: Geerolf (2018)

Including Land and Mixed Income

Source: Geerolf (2018)

Other Countries



Source: Geerolf (2018)

Geerolf on Dynamic Inefficiency

"In a seminal paper, Abel et al. (1989) argue that the United States and six other major advanced economies are dynamically efficient. Updating data on mixed income and land rents, I find in contrast that the criterion for dynamic efficiency is not verified for any advanced economy, and that Japan and South Korea have unambiguously over-accumulated capital."

Geerolf, 2018, "Reassessing Dynamic Efficiency"

Some Countries Over-Accumulate

"This world 'savings glut' can potentially explain otherwise hard-to-understand macroeconomic stylized facts—low interest rates, cash holding by firms, financial bubbles. It is also the macroeconomic counterpart of the microeconomic observation that average firms' return on investment is lower than their measured cost of capital. Subject to some caveats, an increase of public debt, or a generalization of pay-as-you-go systems could therefore be Pareto-improving."

Geerolf, 2018, "Reassessing Dynamic Efficiency"

Summary of the Evidence

- Are economies dynamically efficient or inefficient?
- The debate is not settled for some advanced economies
- But for the vast majority of countries in the world the main challenge is to invest more and accumulate more capital (without destroying our planet)

6. Restoring Dynamic Efficiency

Restoring Efficiency

- The OLG model can generate dynamic inefficiency: young agents save too much and over-accumulate capital
- To understand the intuition about the cause of dynamic inefficiency, we ask the following question: can a benevolent central planner restore efficiency?
- Let's solve the planner's problem

Central Planner

- The planner attributes weight γ^t to generation t and ignores the initial old generation
- The planner solves

$$\max_{\{c_t^t, c_{t+1}^t, k_{t+1}\}} \sum_{t=0}^{\infty} \gamma^t [u(c_t^t) + \beta u(c_{t+1}^t)], \quad \gamma \in (0,1)$$

$$\text{subject to} \quad f(k_t) = (1+n)k_{t+1} + c_t^t + \frac{1}{1+n} c_t^{t-1} \quad \text{for all } t \geq 1$$

Central Planner

- Plug in the resource constraint and open the summation

$$
\begin{aligned}
&\cdots + \gamma^t \left[u\left(f(k_t) - (1+n)k_{t+1} - \frac{1}{1+n} c_t^{t-1} \right) + \beta u(c_{t+1}^t) \right] \\
&+ \gamma^{t+1} \left[u\left(f(k_{t+1}) - (1+n)k_{t+2} - \frac{1}{1+n} c_{t+1}^{t} \right) + \beta u(c_{t+2}^{t+1}) \right] + \ldots
\end{aligned}
$$

- The first-order conditions are

$$
\begin{aligned}
k_{t+1} &: \quad (1+n)u'(c_t^t) = \gamma f'(k_{t+1}) u'(c_{t+1}^{t+1}) \\
c_{t+1}^t &: \quad (1+n)\beta u'(c_{t+1}^t) = \gamma u'(c_{t+1}^{t+1})
\end{aligned}
$$

Same Euler Equation...

- Combine the two FOCs to substitute out $u'(c_{t+1}^{t+1})$

$$u'(c_t^t) = \beta f'(k_{t+1}) u'(c_{t+1}^t)$$

- This is the same cohort-t Euler equation as in the competitive equilibrium
- The planner does what is best for the agent intertemporally

...But Different Capital Accumulation

- However, the first-order condition for capital does not necessarily hold in the competitive equilibrium

$$u'(c_t^t) = \gamma \frac{f'(k_{t+1})}{1+n} u'(c_{t+1}^{t+1})$$

- In the steady state, $c_t^t = c_{t+1}^{t+1} = c^*$

$$1 = \gamma \frac{1+r^*}{1+n} \quad \Longrightarrow \quad r^* > n$$

- The economy is no longer dynamically inefficient

Intuition

- In the competitive equilibrium, agents do not internalize the effect of their behavior on future generations
- When young, they save to guarantee income in old age
- But in some cases the young save too much, capital over accumulates, this inefficiently lowers the return on capital
- This is a pecuniary externality on the next generation

Restoring Efficiency

- Contrary to individual agents, the planner does internalize the effect of savings on future generations
- The planner equalizes the marginal cost and marginal benefit of savings across generations
- This results in an optimal accumulation of capital

7. Social Security

Social Security

- Social security is any government system that provides monetary assistance to people with low or no income
- Social Security in the United States refers to the retirement and disability program of the federal government
- In the OLG model, social security is one way to deal with over-accumulation

Two Systems

- There are two types of social security systems

1. Fully funded pension system (eg United States)
    - The young make compulsory contributions
    - These contributions are paid back in their old age
2. Pay-as-you-go pension system (eg Brazil, France)
    - The young make compulsory contributions
    - These contributions transfer directly to the current old

Fully Funded Social Security

- The government raises some amount d_t from the young by requiring compulsory contributions to their social security accounts
- These funds are invested in the productive asset of the economy, capital
- Workers receive the returns $R_{t+1}d_t$ when they are old

Household Problem

- Consumer t solves

$$\max_{c_t^t, c_{t+1}^t, s_t} u(c_t^t) + \beta u(c_{t+1}^t)$$

$$\text{subject to} \quad c_t^t + s_t + d_t \leq w_t$$

$$\text{and} \quad c_{t+1}^t \leq R_{t+1}(s_t + d_t)$$

- The total amount invested in capital is

$$s_t + d_t = (1+n)k_{t+1}$$

Full Offsetting

- Contributions d_t and savings s_t have the same return R_{t+1}
- Thus the young are indifferent between the two
- An increase in contributions imposed by the government is matched by an equivalent decrease in savings, $\Delta d_t = -\Delta s_t$
- s_t and d_t fully offset each other

Identical Allocation

- Adding contributions has no effect on k_{t+1}, R_t, c_t
- This holds as long as d_t is lower than s_t^* the desired level of savings
- The equilibrium allocation with social security is identical to the equilibrium allocation without social security
- A fully funded social security program does not lead to a Pareto improvement

Pay-As-You-Go Social Security

- The situation is different with unfunded social security
- The government collects d_t from the young and distributes it to the current old with per capita transfer $b_t = (1+n)d_t$
- This takes into account the fact that there are more young than old because of population growth

Household Problem

- Consumer t solves

$$\max_{c_t^t, c_{t+1}^t, s_t} u(c_t^t) + \beta u(c_{t+1}^t)$$

$$\text{subject to} \quad c_t^t + s_t + d_t \leq w_t$$

$$c_{t+1}^t \leq R_{t+1} s_t + (1+n) d_{t+1}$$

- The rate of return on contributions is $(1+n)$, not R_{t+1}

Lower Savings

- Only s_t, rather than $s_t + d_t$ as in the fully funded scheme, goes into capital accumulation
$$s_t = (1+n)k_{t+1}$$
- Unfunded social security reduces capital accumulation

Pareto Improvement

- The contribution d_t reduces the amount of savings invested in the capital stock but increases the interest rate R_{t+1}
- Suppose the economy is dynamically inefficient, $r^* < n$
- Then there exists a funded social security payment d_t that Pareto improves on an economy without social security

From Young to Old

- Unfunded social security reduces over-accumulation
- It transfers resources from future generations to the initial old generation without hurting future generations
- This result holds only with dynamic inefficiency: with dynamic efficiency, $r^* > n$, any transfer of resources makes some future generation worse off
- In practice, if countries do not have enough capital, it might be a good policy to transit from a pay-as-you-go to a fully funded system

Ponzi Scheme

- The government is essentially running a Ponzi game
- Each generation pays d when young and receives $(1+n)d$ from the current young when old
- This is possible only because of population growth
- This is ruled out in the neoclassical growth model because dynamic inefficiency is not allowed by the transversality condition

8. Altruism

Dynastic Altruism

- The general idea of altruism towards offspring is that parents care about the welfare of their children
- Because of that they leave some inheritance when they die
- There are different ways to model altruism
- A common way is Barro's (1974, *JPE*) dynastic altruism: the utility of children enters the utility of parents

Preferences

- People live for two periods as usual
- But now generation t cares about the next generation, $t+1$
- The utility function of generation t is

$$V_t = u(c_t^t) + \beta u(c_{t+1}^t) + \gamma V_{t+1}, \qquad \gamma \in (0,1)$$

- γ is the weight parents give to their offspring's utility, ie the strength of altruism

Introducing Bequest

- Let $b_t \geq 0$ be the bequest received when young and b_{t+1} the bequest left when old to the next (young) generation
- The budget constraints of generation t are

$$
\begin{aligned}
c_t^t + s_t &= w_t + b_t \\
c_{t+1}^t + (1+n)b_{t+1} &= R_{t+1}s_t
\end{aligned}
$$

Household Problem

- Generation t solves

$$\max_{s_t, b_{t+1}} \{u(w_t + b_t - s_t) + \beta u(R_{t+1} s_t - (1+n) b_{t+1})\}$$
$$+ \gamma \max\{u(w_{t+1} + b_{t+1} - s_{t+1}) + \beta u(R_{t+2} s_{t+1} - (1+n) b_{t+2}) + \gamma V_{t+2}\}$$

- Agent t only chooses s_t and b_{t+1}
- s_{t+1} and b_{t+2} are chosen by the agent's offspring, ie agent $t+1$

First-Order Conditions

- Assuming $b_{t+1} > 0$, the first-order conditions are

$$\begin{aligned} s_t : \quad & u'(c_t^t) = \beta R_{t+1} u'(c_{t+1}^t) \\ b_{t+1} : \quad & \beta(1+n) u'(c_{t+1}^t) = \gamma u'(c_{t+1}^{t+1}) \end{aligned}$$

Connected Generations

- The FOCs with altruism are identical to the FOCs of the planner's problem in the model without altruism
- Altruism establishes a connection between generations
- The young generation now cares about future generations and optimally modifies its behavior, which leads to lower savings and higher output
- The economy no longer exhibits dynamic inefficiency

Same Utility Function

- Remember the preferences of the planner in the model without altruism

$$\sum_{t=0}^{\infty} \gamma^t [u(c_t^t) + \beta u(c_{t+1}^t)]$$

- With altruism, the utility function of generation t rewrites

$$V_t = u(c_t^t) + \beta u(c_{t+1}^t) + \gamma V_{t+1} = \sum_{\tau=t}^{\infty} \gamma^{\tau-t} [u(c_\tau^\tau) + \beta u(c_{\tau+1}^\tau)]$$

- The two functions are identical for $t = 0$

Like a Neoclassical Growth Model

- Note that the utility function of generation t with altruism also looks like the utility of a household that lives forever
- Thus dynastic altruism is a way to make the OLG model behave like the neoclassical growth model
- We know there is no dynamic inefficiency in that model

Interpreting the Planner

- We can interpret the objective of the planner in two ways

1. The planner attributes weight γ^t to each generation t, where nobody is altruistic
$$V_t = u(c_t^t) + \beta u(c_{t+1}^t)$$
2. The planner only cares about generation t, but people are altruistic with discount factor γ
$$V_t = u(c_t^t) + \beta u(c_{t+1}^t) + \gamma V_{t+1}$$

Patient Planner

- Independently of γ, the solution to the planner's problem is Pareto efficient
- With altruism the planner places 'double' weight on future generations
- In the words of Caplin and Leahy (2004, *JPE*), "policy makers should be more patient than private citizens"

Useless Social Security

- With altruism and pay-as-you-go social security we have

$$
\begin{aligned}
c_t^t + s_t + d_t &= w_t + b_t \\
c_{t+1}^t + (1+n)(b_{t+1} - d_{t+1}) &= R_{t+1} s_t
\end{aligned}
$$

- Bequest b_t and contribution d_t have the same return
- In equilibrium there is full offsetting, $\Delta d_{t+1} = -\Delta b_{t+1}$
- In other words, social security is no longer useful
- The economy is efficient and social security adds nothing

Ricardian Equivalence

- Ricardian equivalence holds when agents live forever
- In OLG models without altruism Ricardian equivalence fails to hold: the timing of taxes matters
- With altruism Ricardian equivalence holds again because generations are connected and form an immortal dynasty
- This was shown by Barro (1974, *JPE*)

Conclusion

- We have studied the overlapping generations model with production
- We have shown how the model can lead to multiple and inefficient equilibria
- We have shown ways to restore efficiency: money or any durable asset in the pure exchange economy, social security or altruism in the production case
- We can add many features to the OLG model to study real-world issues: debt bubbles, welfare state, labor markets, automation, demographics

9. Exercises

Exercise 1 – Log Utility and Multiple Equilibria

Consider the basic OLG model presented in the lecture. Assume log preferences $\ln c_t^t + \beta \ln c_t^{t+1}$ and the general neoclassical technology $F(K, L)$ satisfying continuity, differentiability, positive and diminishing marginal products, constant returns to scale, and the Inada conditions.

Show that multiple steady-state equilibria are possible in this economy.

Exercise 2 – No Retirement

Consider the basic OLG model presented in the lecture. Assume log preferences $\ln c_t^t + \beta \ln c_t^{t+1}$ and a Cobb-Douglas production function. Assume now that individuals work in both periods of their lives.

1. Define a competitive equilibrium and the steady-state equilibrium.
2. Characterize the steady-state equilibrium and the transitional dynamics in this economy.
3. Can this economy generate over-accumulation?

Exercise 3 – Optimal Taxation

Consider the following problem of optimal taxation with commitment in an OLG model. In each period $t \geq 1$, there are two generations, young and old. Population is constant. The young agent t solves when young

$$\max_{c_t^t, c_{t+1}^t, l_t^t, l_{t+1}^t, k_{t+1}} u(c_t^t, l_t^t) + \beta u(c_{t+1}^t, l_{t+1}^t)$$

$$\text{subject to} \quad c_t^t + k_{t+1} = (1-\tau_t^t) w_t^t l_t^t$$

$$c_{t+1}^t = (1-\tau_{t+1}^t) w_{t+1}^t l_{t+1}^t + [1-\delta+(1-\mathcal{T}_{t+1}) r_{t+1}] k_{t+1}$$

where each young agent saves through capital, which is rented out to the representative firm, and where τ_t^t is the labor income tax and $\mathcal{T}_{t+1}$ is the capital income tax. As usual, $\beta \in (0,1)$, $\delta \in (0,1)$, $u_c > 0$, $u_{cc} < 0$, $u_l < 0$, $u_{ll} < 0$, and u satisfies the Inada conditions to guarantee an interior solution.

Exercise 3 – Continued

Let's ignore the initial old as it won't be relevant to determine the long-run capital tax. Assume it owns k_1 initial capital and $\mathcal{T}_1 = 0$. For all t the government satisfies its budget constraint

$$g_t = \tau_t^t w_t^t l_t^t + \tau_t^{t-1} w_t^{t-1} l_t^{t-1} + \mathcal{T}_t r_t k_t$$

A representative firm solves for all t

$$\max_{l_t^t, l_t^{t-1}, k_t} F(k_t, l_t^t, l_t^{t-1}) - r_t k_t - w_t^t l_t^t - w_t^{t-1} l_t^{t-1}$$

where F satisfies the usual assumptions. The firms operates only the production technology, while households transform and build capital. The resource constraint is

$$c_t^t + c_t^{t-1} + k_{t+1} + g_t = F(k_t, l_t^t, l_t^{t-1}) + (1-\delta)k_t$$

Exercise 3 – Continued

1. Derive the three first-order conditions of the firm.
2. Using Lagrange multipliers, derive the five first-order conditions of agent t's problem.
3. Consolidate the sequential budget constraints of agent $t \geq 1$ and use the first-order conditions to eliminate prices and taxes from the consolidated budget constraint. You should obtain the following implementability condition

$$u_c(c_t^t, l_t^t)c_t^t + u_l(c_t^t, l_t^t)l_t^t + \beta u_c(c_{t+1}^t, l_{t+1}^t)c_{t+1}^t + \beta u_l(c_{t+1}^t, l_{t+1}^t)l_{t+1}^t = 0$$

4. The government assigns weight α^t, $\alpha \in (0,1)$, to generation $t \geq 1$. Use the primal approach to write a Ramsey problem. Ignore restrictions associated to the initial old.
5. Suppose $u(c,l) = \frac{c^{1-\sigma}-1}{1-\sigma} - v(l)$. Show that if the economy converges to a steady state the optimal capital tax rate is zero.