Floating Point Precision

Double-precision (64-bit) floating point values have 53 bits for representation of the significand. (https://en.wikipedia.org/wiki/Double-precision_floating-point_format)

In[ ]:=
```
2^(-53) // N
```

Out[ ]=

1.11022×10^{-16}

So lets assume we're going to need to have 16-17 decimal places of precision in our final formula for calculating $\sin(x)$.

Chebyshev Polynomials

Look at the first few Chebyshev polynomials (of the first kind):

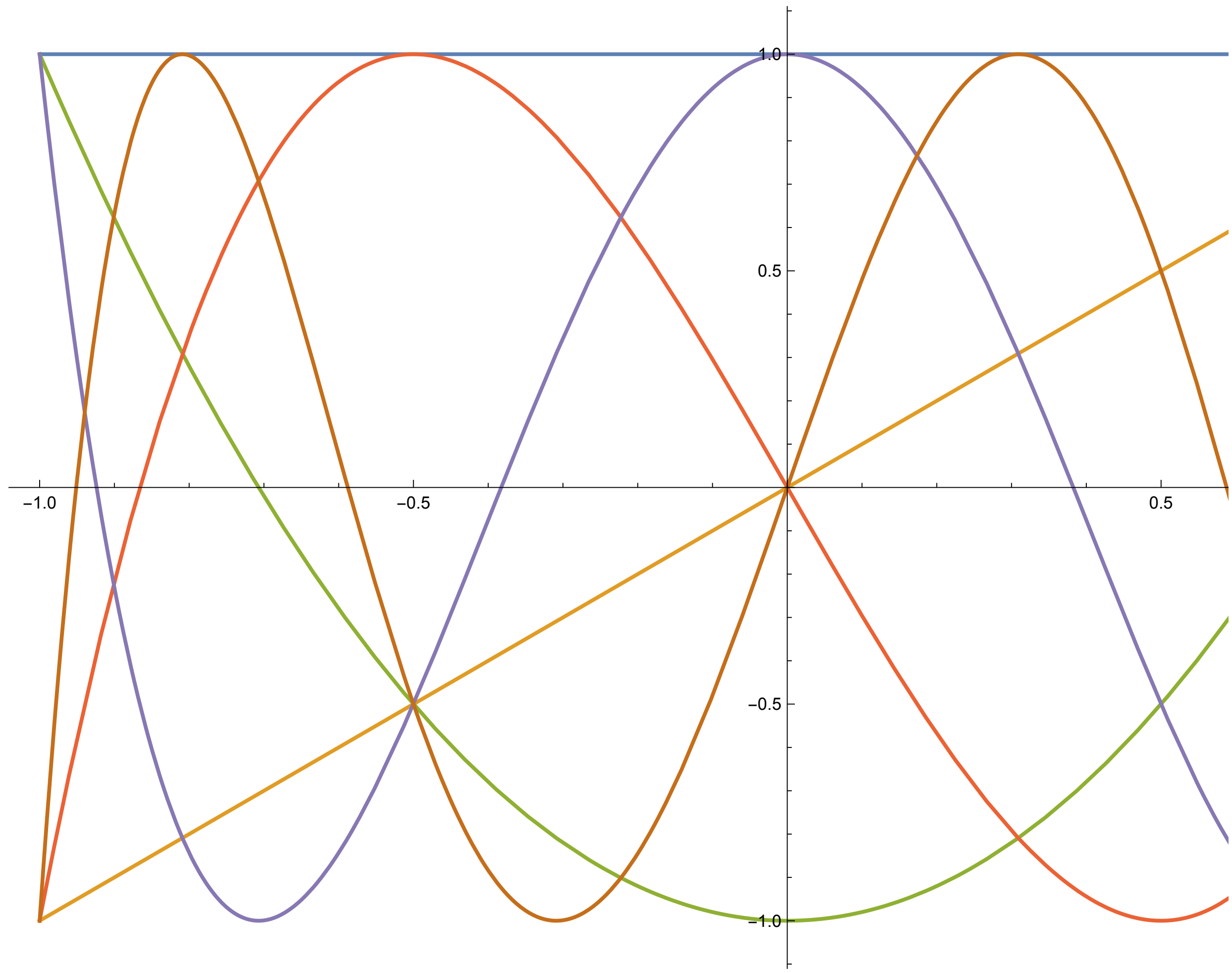
In[ ]:=
```
Table[{T_i, ChebyshevT[i, x]}, {i, 0, 12}] // TableForm
```

Out[ ]//TableForm=

T_0	1
T_1	x
T_2	$-1 + 2 x^2$
T_3	$-3 x + 4 x^3$
T_4	$1 - 8 x^2 + 8 x^4$
T_5	$5 x - 20 x^3 + 16 x^5$
T_6	$-1 + 18 x^2 - 48 x^4 + 32 x^6$
T_7	$-7 x + 56 x^3 - 112 x^5 + 64 x^7$
T_8	$1 - 32 x^2 + 160 x^4 - 256 x^6 + 128 x^8$
T_9	$9 x - 120 x^3 + 432 x^5 - 576 x^7 + 256 x^9$
T_{10}	$-1 + 50 x^2 - 400 x^4 + 1120 x^6 - 1280 x^8 + 512 x^{10}$
T_{11}	$-11 x + 220 x^3 - 1232 x^5 + 2816 x^7 - 2816 x^9 + 1024 x^{11}$
T_{12}	$1 - 72 x^2 + 840 x^4 - 3584 x^6 + 6912 x^8 - 6144 x^{10} + 2048 x^{12}$

In[∘]:=
```
Plot[Evaluate[Table[ChebyshevT[i, x], {i, 0, 5}]], {x, -1, 1},
 PlotLegends → "Expressions"]
```

Out[∘]=



Chebyshev polynomials are orthogonal with the following product:

$$\int_{-1}^{1} \frac{T_i(x)\, T_j(x)}{\sqrt{1-x^2}}\, dx$$

In[∘]:=
```
Integrate[
  ChebyshevT[4, x] * ChebyshevT[1, x] / Sqrt[1 - x^2],
  {x, -1, 1}]
```

Out[∘]=

```
0
```

In[∘]:=
```
innerProd[f_, g_] := Integrate[2 * f[x] * g[x] / (Pi * Sqrt[1 - x^2]), {x, -1, 1}]
```

In[ ]:=
```
innerProd[ChebyshevT[3, #] &, ChebyshevT[3, #] &]
```
Out[ ]=

1

In[ ]:=
```
innerProd[ChebyshevT[1, #] &, ChebyshevT[2, #] &]
```
Out[ ]=

0

In[ ]:=
```
Table[innerProd[ChebyshevT[i, #] &, ChebyshevT[j, #] &],
  {i, 0, 5}, {j, 0, 5}] // TableForm
```
Out[ ]//TableForm=

2	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

In[ ]:=
```
innerProd[ChebyshevT[3, #] &, Sin[Pi * # / 2] &]
```
Out[ ]=

$$-2\ \text{BesselJ}\left[3, \frac{\pi}{2}\right]$$

In[ ]:=
```
coef[n_] := Integrate[
  (2 Sin[x * π / 2] * ChebyshevT[n, x]) / (π Sqrt[1 - x^2]),
  {x, -1, 1}]
```

In[ ]:=
```
sinCoefficients = Table[
   Evaluate[innerProd[Sin[# * Pi / 2] &, ChebyshevT[i, #] &]],
   {i, 0, 20}];
```

In[ ]:=
```
sinCoefficients[[1 ;; 8]] // TableForm
```
Out[ ]//TableForm=

$$0$$
$$2\ \text{BesselJ}\left[1, \frac{\pi}{2}\right]$$
$$0$$
$$-2\ \text{BesselJ}\left[3, \frac{\pi}{2}\right]$$
$$0$$
$$\frac{2\left(\pi\left(-192+\pi^2\right)\text{BesselJ}\left[1,\frac{\pi}{2}\right]-24\left(-64+\pi^2\right)\text{BesselJ}\left[2,\frac{\pi}{2}\right]\right)}{\pi^3}$$
$$0$$
$$\frac{2\left(\pi\left(92160-960\pi^2+\pi^4\right)\text{BesselJ}\left[1,\frac{\pi}{2}\right]-48\left(15360-320\pi^2+\pi^4\right)\text{BesselJ}\left[2,\frac{\pi}{2}\right]\right)}{\pi^5}$$

In[∘]:=
```
N[sinCoefficients, 12] // TableForm
```

Out[∘]//TableForm=

0
1.13364817781
0
-0.138071776587
0
0.00449071424655
0
-0.0000677012758422
0
$5.89129533029 \times 10^{-7}$
0
$-3.33805940892 \times 10^{-9}$
0
$1.32970283845 \times 10^{-11}$
0
$-3.92749958718 \times 10^{-14}$
0
$8.94526011594 \times 10^{-17}$
0
$-1.61896899669 \times 10^{-19}$
0

In[∘]:=
```
sinApproximation[i_] := Simplify[
  N[sinCoefficients[[1 ;; (i + 1)]], 24].Table[ChebyshevT[d, x], {d, 0, i}]
 ]
```

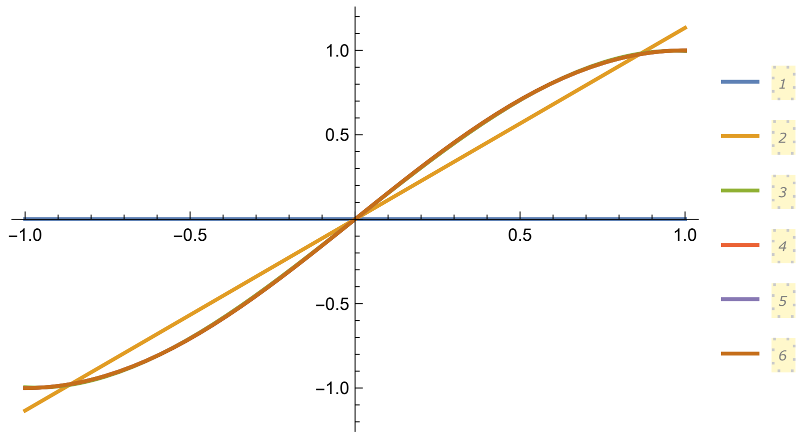
In[∘]:=
```
Table[sinApproximation[i], {i, 0, 12}] // TableForm
```

Out[∘]//TableForm=

0
$1.13364817781174787542249\,x$
$1.13364817781174787542249\,x$
$1.54786350757332418603 13\,x - 0.552287106348768414145 05\,x^3$
$1.5478635075733241860313\,x - 0.55228710634876841414505\,x^3$
$1.570317078806098775594 6\,x - 0.6421013912798667723983 7\,x^3 + 0.0718514279448786866026 52$
$1.5703170788060987755946\,x - 0.6421013912798667723983 7\,x^3 + 0.071851427944878686602652$
$1.5707909877369938430328\,x - 0.6458926627270273119035 6\,x^3 + 0.079433970839199765613030$
$1.5707909877369938430328\,x - 0.6458926627270273119035 6\,x^3 + 0.079433970839199765613030$
$1.5707962899027911034148\,x - 0.6459633582709907836635 8\,x^3 + 0.079688474797468263949114$
$1.5707962899027911034148\,x - 0.6459633582709907836635 8\,x^3 + 0.079688474797468263949114$
$1.5707963266214446015196\,x - 0.6459640926440607457604 9\,x^3 + 0.079692587286660051691813$
$1.5707963266214446015196\,x - 0.6459640926440607457604 9\,x^3 + 0.079692587286660051691813$

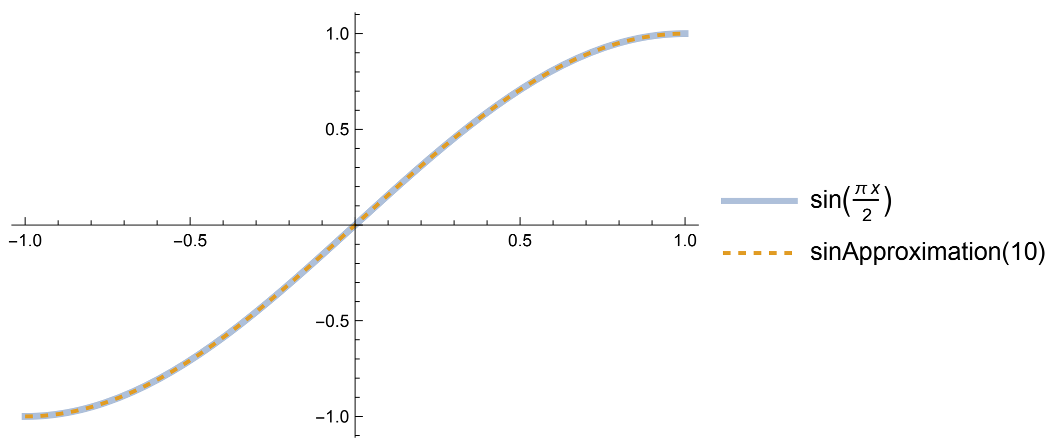
```
In[•]:= Plot[
  Evaluate[Table[sinApproximation[2 * i], {i, 0, 5}]],
  {x, -1, 1}, PlotLegends → Automatic]
```

Out[•]=



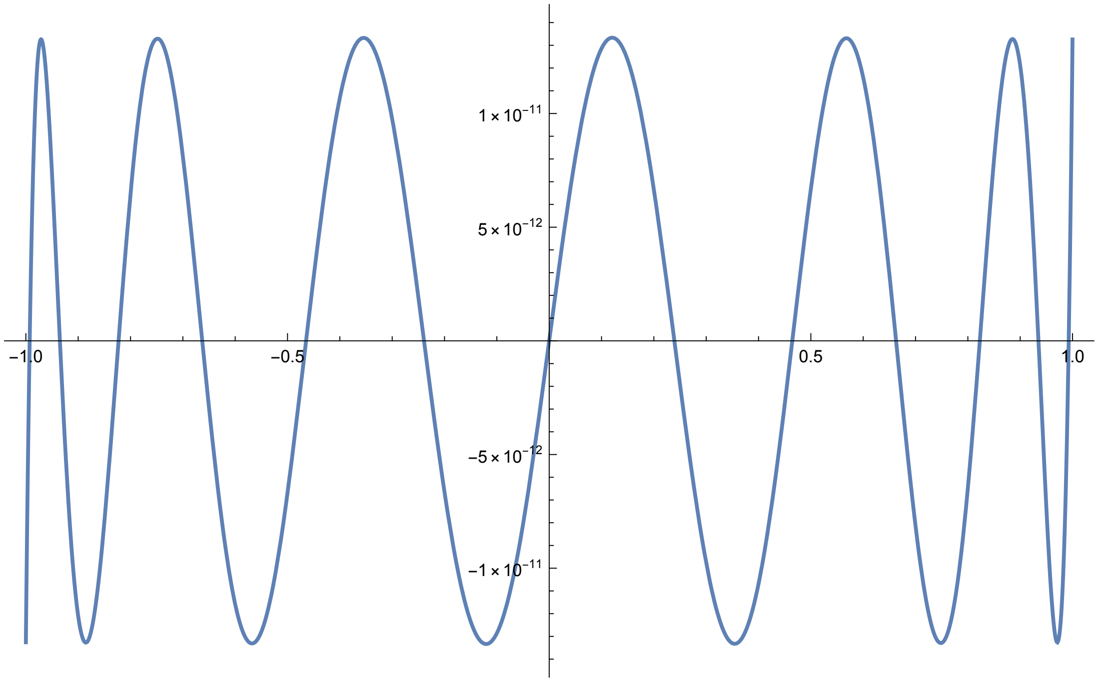

```
In[•]:= Plot[{Sin[Pi * x / 2], sinApproximation[10]}, {x, -1, 1},
  PlotLegends → "Expressions",
  PlotStyle → {{Thickness[0.01], Opacity[0.5]}, Dashed}]
```

Out[•]=



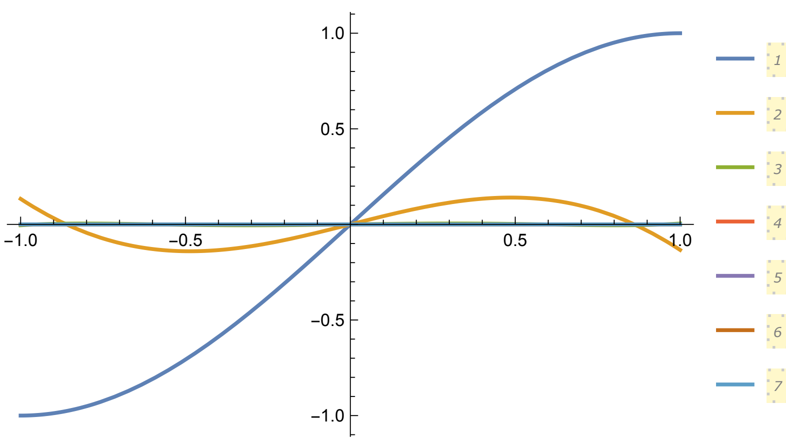
In[ ]:=
```
Plot[Sin[Pi * x / 2] - sinApproximation[12], {x, -1, 1}]
```

Out[ ]=



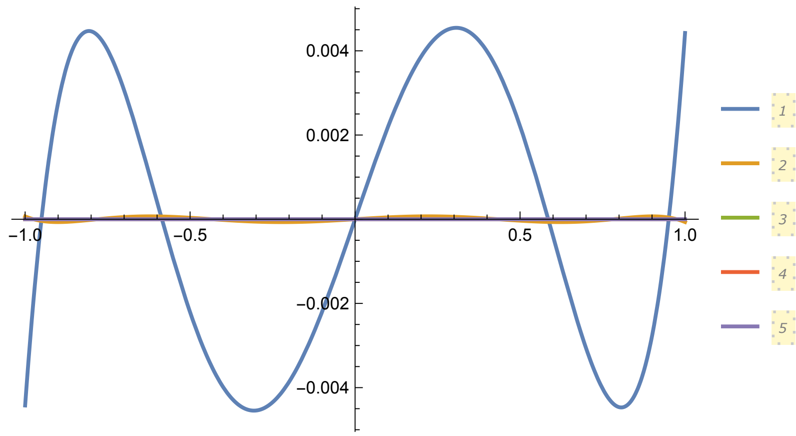
In[ ]:=
```
Plot[
 Evaluate[Table[Sin[Pi * x / 2] - sinApproximation[2 * i], {i, 0, 6}]], {x, -1, 1},
 PlotRange → All, PlotLegends → Automatic]
```

Out[ ]=



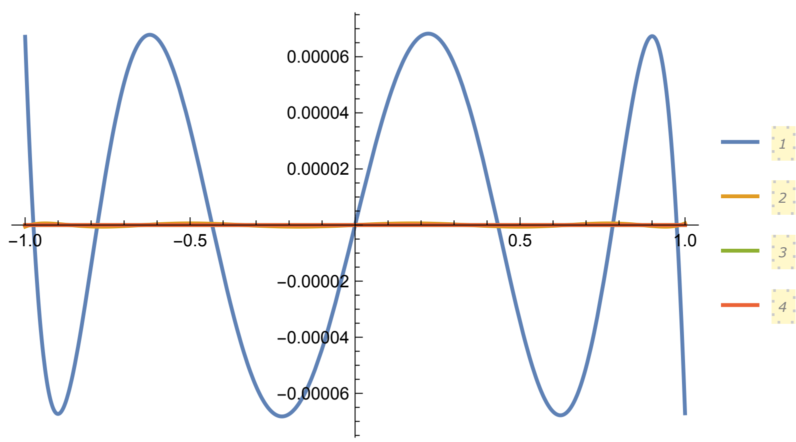
In[ ]:=
```
Plot[
 Evaluate[Table[Sin[Pi * x / 2] - sinApproximation[2 * i], {i, 2, 6}]], {x, -1, 1},
 PlotRange → All, PlotLegends → Automatic]
```

Out[ ]=



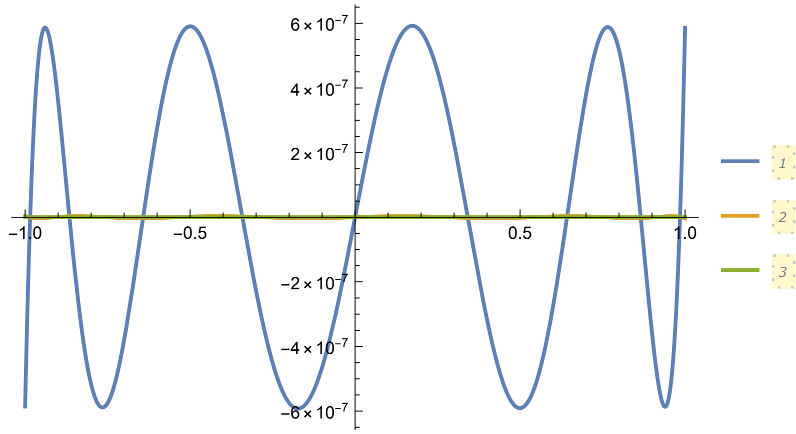
In[ ]:=
```
Plot[
 Evaluate[Table[Sin[Pi * x / 2] - sinApproximation[2 * i], {i, 3, 6}]], {x, -1, 1},
 PlotRange → All, PlotLegends → Automatic]
```

Out[ ]=



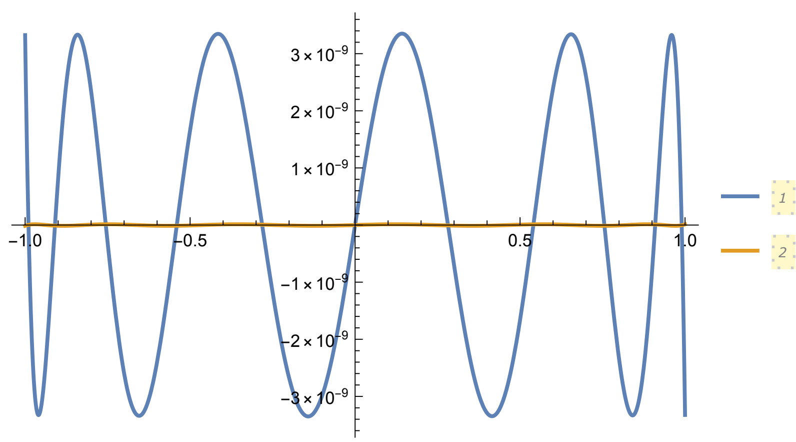
In[ ]:=
```
Plot[
 Evaluate[Table[Sin[Pi * x / 2] - sinApproximation[2 * i], {i, 4, 6}]], {x, -1, 1},
 PlotRange → All, PlotLegends → Automatic]
```

Out[ ]=

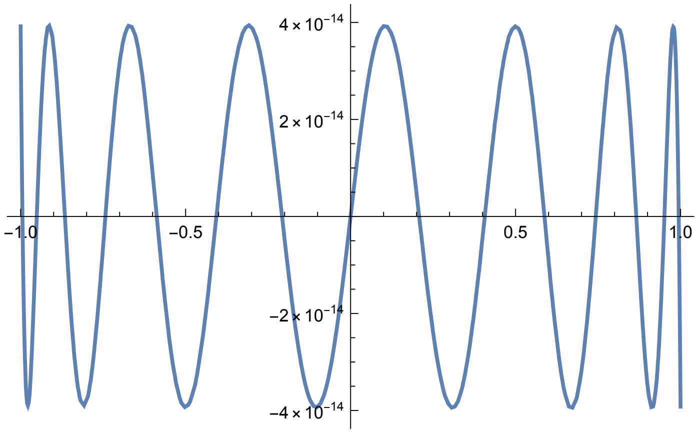


In[ ]:=
```
Plot[
 Evaluate[Table[Sin[Pi * x / 2] - sinApproximation[2 * i], {i, 5, 6}]], {x, -1, 1},
 PlotRange → All, PlotLegends → Automatic]
```

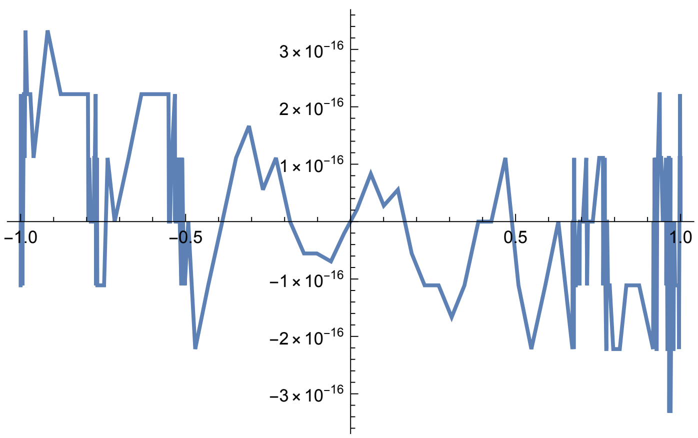
Out[ ]=



In[ ]:=
```
Plot[Sin[Pi * x / 2] - sinApproximation[14], {x, -1, 1}]
```
Out[ ]=

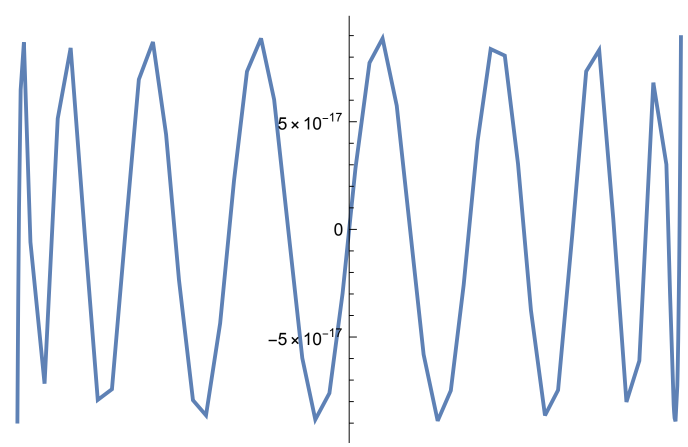


In[ ]:=
```
Plot[Sin[Pi * x / 2] - sinApproximation[16], {x, -1, 1}]
```
Out[ ]=



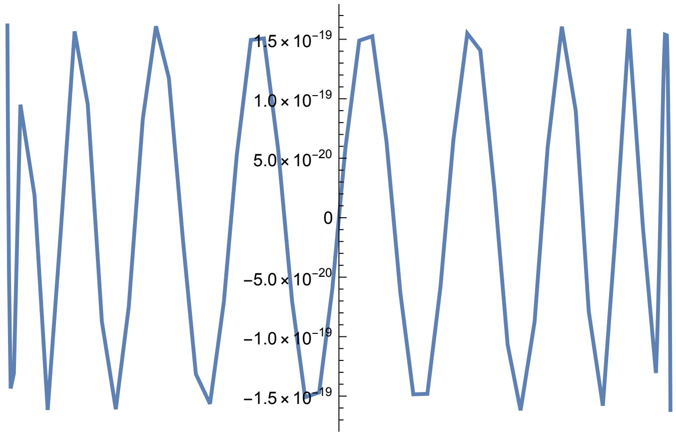
Here we're getting into the realm of machine precision, so we need to increase Mathematica's precision in order to get a reasonable plot:

In[ ]:=
```
Plot[Sin[Pi * x / 2] - sinApproximation[16], {x, -1, 1}, WorkingPrecision → 30]
```
Out[ ]=



In[∘]:=
```
Plot[Sin[Pi * x / 2] - sinApproximation[17], {x, -1, 1}, WorkingPrecision → 30]
```

Out[∘]=



The approximation of order 17 seems to be good enough for our purposes.

In[∘]:=
```
sinApproximation[17]
```

Out[∘]=

$$1.57079632679489661615027\,x - 0.645964097506246068725494\,x^{3} + 0.0796926262461637888741028\,x^{5} - 0.00468175413529262296007412\,x^{7} + 0.000160441184674323158518060\,x^{9} - 3.59884294716944654024508\times 10^{-6}\,x^{11} + 5.69212853539005432284504\times 10^{-8}\,x^{13} - 6.68396586459246317556172\times 10^{-10}\,x^{15} + 5.86236566958322852534316\times 10^{-12}\,x^{17}$$

In[∘]:=
```
BaseForm[
 CoefficientList[sinApproximation[17], x], 16]
```

Out[∘]//BaseForm=

$$\{0_{16},\ 1.921fb54442d18430b3b_{16},\ 0_{16},\ -0.a55de7312df288a2014e_{16},\ 0_{16},\ 0.1466bc6775a9f72f2307_{16},\ 0_{16},\ -0.0132d2cce624822aebf9f_{16},\ 0_{16},\ 0.000a83c1a41fa29a76275f9_{16},\ 0_{16},\ -0.00003c60e9aabd6567a8f55a_{16},\ 0_{16},\ f.4799d7825fec1b5107_{16}\times 16^{-7},\ 0_{16},\ -2.dee8e9e227a0b4f67e8_{16}\times 16^{-8},\ 0_{16},\ 6.721bf72c5d91a3e3651_{16}\times 16^{-10}\}$$

In[∘]:=
```
ExportString[#, "Real64"] & /@ CoefficientList[sinApproximation[17], x]
```

Out[∘]=

{, -DTû!ù?, , Q¾%æ¼«ä¿, , ÷©ug¼f´?, , ♯HbÎ,-s¿, ,
5E?H %?, , ´²^Õt0Î¾, , Ø¿ ¯ 3 n>, , =OG÷ ¾, , Gv±Ü o È=}

In[∘]:=
```
ToCharacterCode[
 ExportString[#, "Real64"] & /@ CoefficientList[sinApproximation[17], x]]
```
Out[∘]=
```
{{0, 0, 0, 0, 0, 0, 0, 0}, {24, 45, 68, 84, 251, 33, 249, 63},
 {0, 0, 0, 0, 0, 0, 0, 0}, {81, 190, 37, 230, 188, 171, 228, 191},
 {0, 0, 0, 0, 0, 0, 0, 0}, {247, 169, 117, 103, 188, 102, 180, 63},
 {0, 0, 0, 0, 0, 0, 0, 0}, {35, 72, 98, 206, 44, 45, 115, 191},
 {0, 0, 0, 0, 0, 0, 0, 0}, {53, 69, 63, 72, 131, 7, 37, 63},
 {0, 0, 0, 0, 0, 0, 0, 0}, {180, 178, 94, 213, 116, 48, 206, 190},
 {0, 0, 0, 0, 0, 0, 0, 0}, {216, 191, 4, 175, 51, 143, 110, 62},
 {0, 0, 0, 0, 0, 0, 0, 0}, {6, 61, 17, 79, 71, 247, 6, 190},
 {0, 0, 0, 0, 0, 0, 0, 0}, {71, 118, 177, 220, 111, 200, 153, 61}}
```

In[∘]:=
```
Map[
 IntegerString[#, 16, 2] &,
 ToCharacterCode[
  ExportString[#, "Real64"] & /@ CoefficientList[sinApproximation[17], x]],
 {2}
]
```
Out[∘]=
```
{{00, 00, 00, 00, 00, 00, 00, 00}, {18, 2d, 44, 54, fb, 21, f9, 3f},
 {00, 00, 00, 00, 00, 00, 00, 00}, {51, be, 25, e6, bc, ab, e4, bf},
 {00, 00, 00, 00, 00, 00, 00, 00}, {f7, a9, 75, 67, bc, 66, b4, 3f},
 {00, 00, 00, 00, 00, 00, 00, 00}, {23, 48, 62, ce, 2c, 2d, 73, bf},
 {00, 00, 00, 00, 00, 00, 00, 00}, {35, 45, 3f, 48, 83, 07, 25, 3f},
 {00, 00, 00, 00, 00, 00, 00, 00}, {b4, b2, 5e, d5, 74, 30, ce, be},
 {00, 00, 00, 00, 00, 00, 00, 00}, {d8, bf, 04, af, 33, 8f, 6e, 3e},
 {00, 00, 00, 00, 00, 00, 00, 00}, {06, 3d, 11, 4f, 47, f7, 06, be},
 {00, 00, 00, 00, 00, 00, 00, 00}, {47, 76, b1, dc, 6f, c8, 99, 3d}}
```

In[∘]:=
```
Map[StringJoin,
 Map[
  IntegerString[#, 16, 2] &,
  ToCharacterCode[
   ExportString[#, "Real64"] & /@ CoefficientList[sinApproximation[17], x]],
  {2}
 ]
]
```
Out[∘]=
```
{0000000000000000, 182d4454fb21f93f, 0000000000000000,
 51be25e6bcabe4bf, 0000000000000000, f7a97567bc66b43f,
 0000000000000000, 234862ce2c2d73bf, 0000000000000000, 35453f488307253f,
 0000000000000000, b4b25ed57430cebe, 0000000000000000, d8bf04af338f6e3e,
 0000000000000000, 063d114f47f706be, 0000000000000000, 4776b1dc6fc8993d}
```

Normalization

In[∘]:=
```
T[0] := (ChebyshevT[0, #] / Sqrt[Pi]) &
T[i_Integer] := (ChebyshevT[i, #] / Sqrt[Pi / 2]) &
```

In[∘]:=
```
T[0]
```

Out[∘]=

$$\frac{\text{ChebyshevT}[0, \#1]}{\sqrt{\pi}} \&$$

In[∘]:=
```
T[0][1.3]
```

Out[∘]=

0.56419

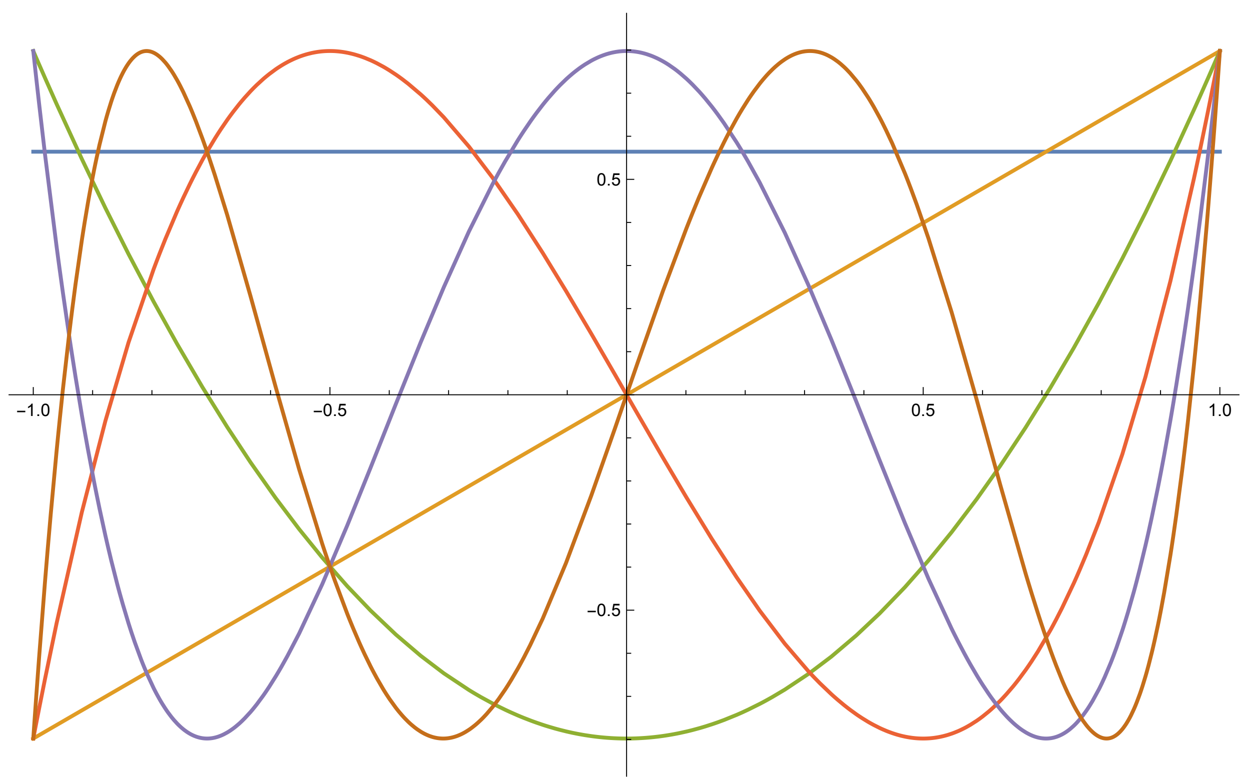
In[∘]:=
```
T[3][x]
```

Out[∘]=

$$\sqrt{\frac{2}{\pi}}\left(-3\,x+4\,x^3\right)$$

In[∘]:=
```
Plot[Evaluate[Table[T[i][x], {i, 0, 5}]], {x, -1, 1},
 PlotLegends → "Expressions"]
```

Out[∘]=



In[∘]:=
```
Table[
  Integrate[(T[i][x] * T[j][x]) / Sqrt[1 - x^2], {x, -1, 1}],
  {i, 0, 5}, {j, 0, 5}] // TableForm
```

Out[∘]//TableForm=

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

In[∘]:=
```
Table[∫_{-1}^{1} (T[i][x] * T[j][x]) / Sqrt[1 - x^2] dx, {i, 0, 5}, {j, 0, 5}] // TableForm
```

Out[∘]//TableForm=

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

In[∘]:=
```
innerProduct[f_, g_] := ∫_{-1}^{1} (f[x] * g[x]) / Sqrt[1 - x^2] dx
```

In[∘]:=
```
innerProduct[T[1], T[2]]
```

Out[∘]=

0

In[∘]:=
```
innerProduct[T[1], Sin]
```

Out[∘]=

$\sqrt{2\pi}$ BesselJ[1, 1]

In[∘]:=
```
innerProduct[T[1], Sin] // N
```

Out[∘]=

1.10304

In[∘]:=
```
NIntegrate[(T[3][x] * T[2][x]) / Sqrt[1 - x^2], {x, -1, 1}]
```

Out[∘]=

-2.77556×10^{-17}

In[∘]:=
```
innerProductN[f_, g_] :=
  NIntegrate[(f[x] * g[x]) / Sqrt[1 - x^2], {x, -1, 1}, PrecisionGoal → 18, WorkingPrecision → 64]
```

In[◦]:=
```
Table[{i, innerProductN[T[i], Sin[Pi * # / 2] &]}, {i, 0, 24}] // TableForm
```

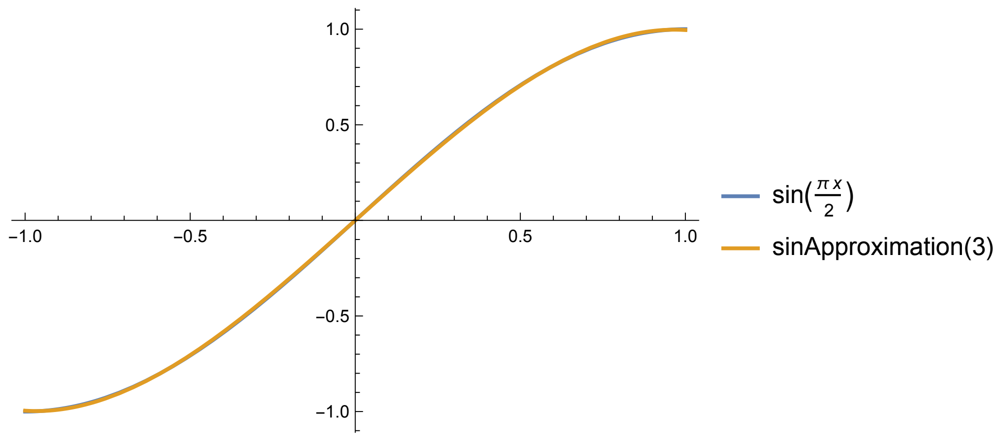
Out[◦]//TableForm=

0	$0. \times 10^{-65}$
1	1.420817287993419621827686337373105253782599598266574841398544928
2	$0. \times 10^{-65}$
3	−0.1730473095609951566947439729462274136537614719927265263274713817
4	$0. \times 10^{-65}$
5	0.005628275651851403639709376069781329291720182568389713314075499712
6	$0. \times 10^{-66}$
7	−0.00008485096612726606722101890597684328358510891098587284634777582229
8	$0. \times 10^{-66}$
9	$7.383643724552385859762704298651691376995297931043241810608200487 \times 10^{-7}$
10	$0. \times 10^{-66}$
11	$-4.183637048396731819784010067708663102793394481473521979964893126 \times 10^{-9}$
12	$0. \times 10^{-67}$
13	$1.666535365856643316497994158603422795969979163016410072617555610 \times 10^{-11}$
14	$0. \times 10^{-67}$
15	$-4.922390756913075022148307035241838965937888555909584092587359933 \times 10^{-14}$
16	$0. \times 10^{-67}$
17	$1.121122096527360389761386748818794703717027047752161616042104844 \times 10^{-16}$
18	$0. \times 10^{-67}$
19	$-2.029076731429751254120007590347292864207052540166977508342749368 \times 10^{-19}$
20	$0. \times 10^{-67}$
21	$2.988471144759487574684163777133537678400087878412817285399409357 \times 10^{-22}$
22	$0. \times 10^{-67}$
23	$-3.651697160026653256973265579881123157208500529026115011922209887 \times 10^{-25}$
24	$0. \times 10^{-67}$

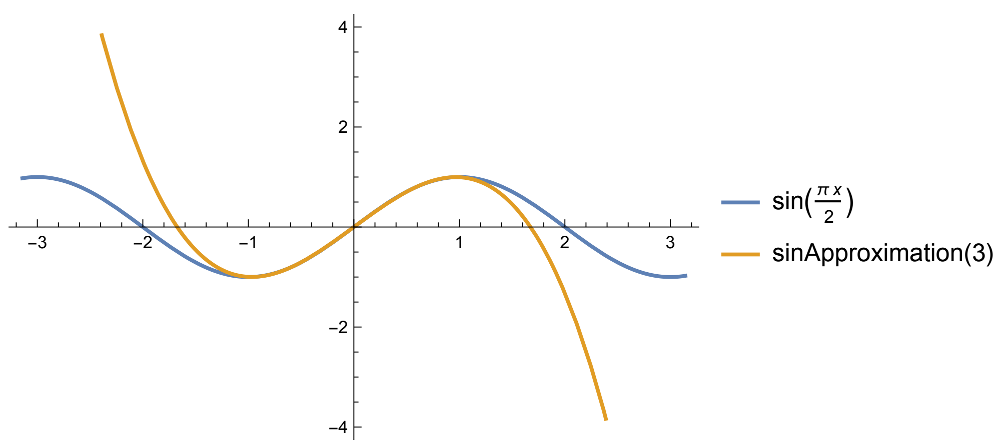
In[◦]:=
```
sinCoefficients = Chop[
  Table[innerProductN[T[i], Sin[Pi * # / 2] &], {i, 0, 24}],
  10^-50]
```

Out[◦]=
```
{0, 1.420817287993419621827686337373105253782599598266574841398544928, 0,
 -0.1730473095609951566947439729462274136537614719927265263274713817, 0,
 0.005628275651851403639709376069781329291720182568389713314075499712, 0,
 -0.00008485096612726606722101890597684328358510891098587284634777582229, 0,
 7.383643724552385859762704298651691376995297931043241810608200487 × 10^-7, 0,
 -4.183637048396731819784010067708663102793394481473521979964893126 × 10^-9, 0,
 1.666535365856643316497994158603422795969979163016410072617555610 × 10^-11, 0,
 -4.922390756913075022148307035241838965937888555909584092587359933 × 10^-14, 0,
 1.121122096527360389761386748818794703717027047752161616042104844 × 10^-16, 0,
 -2.029076731429751254120007590347292864207052540166977508342749368 × 10^-19, 0,
 2.988471144759487574684163777133537678400087878412817285399409357 × 10^-22, 0,
 -3.651697160026653256973265579881123157208500529026115011922209887 × 10^-25, 0}
```

```
In[]:= sinApproximation[i_] :=
        sinCoefficients[[1 ;; i + 1]].Evaluate[Table[T[j][x], {j, 0, i}]] // Simplify

In[]:= sinApproximation[1]

Out[]=
1.13364817781174787542248992693432056708063483108297997787968717 8 x

In[]:= sinApproximation[6]

Out[]=
1.57031707880609877559460779366687186951416003537733929255883751 x -
 0.642101391279866772398366796153653005148670199298768406224455792 x^3 +
 0.0718514279448786866026515686123015222847639122875471839827064203 x^5

In[]:= Plot[{Sin[Pi * x / 2], sinApproximation[3]}, {x, -1, 1}, PlotLegends → "Expressions"]

Out[]=
```




```
In[]:= Plot[{Sin[Pi * x / 2], sinApproximation[3]},
        {x, -Pi, Pi}, PlotLegends → "Expressions"]

Out[]=
```




```
In[]:= $MachinePrecision

Out[]=
15.9546
```

Orthogonality

In[74]:= chebProduct[f_, g_] := $\int_{-1}^{1} \frac{f[x] * g[x]}{\sqrt{1 - x^2}} \, dx$

In[75]:= chebProduct[(1 &), # &]

Out[75]=

0

In[76]:= projection[f_, g_] := $\frac{\text{chebProduct}[f, g]}{\text{chebProduct}[g, g]}$ g

In[77]:= projection[(#^2 &), 1 &]

Out[77]=

$\frac{1 \,\&}{2}$

In[78]:= projection[#^3 &, # &]

Out[78]=

$\frac{3\ (\#1 \,\&)}{4}$

In[79]:= Projection[(#^2 &), 1 &, chebProduct]

Out[79]=

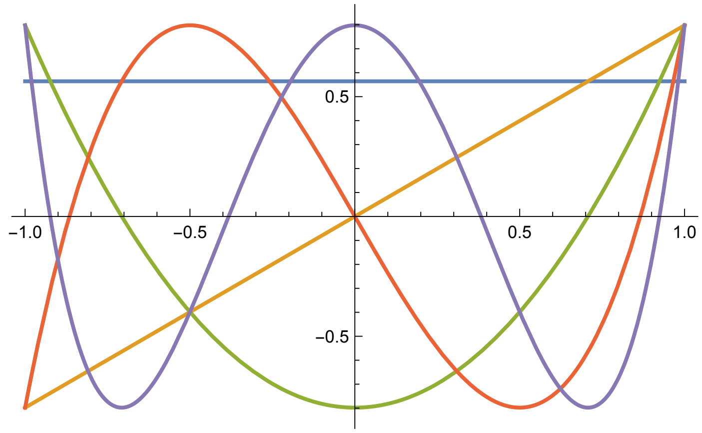
$\frac{1 \,\&}{2}$

In[82]:= Orthogonalize$\left[\{1, x, x^2, x^3, x^4\}, \{f, g\} \mapsto \int_{-1}^{1} \frac{f * g}{\sqrt{1 - x^2}} \, dx\right]$

Out[82]=

$$\left\{\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\, x, 2\sqrt{\frac{2}{\pi}}\left(-\frac{1}{2} + x^2\right), 4\sqrt{\frac{2}{\pi}}\left(-\frac{3x}{4} + x^3\right), 8\sqrt{\frac{2}{\pi}}\left(\frac{1}{8} - x^2 + x^4\right)\right\}$$

In[83]:=
```
Plot[%, {x, -1, 1}]
```
Out[83]=



Nodes

Calculating Chebyshev Nodes

In[4]:=
```
chebNodes[n_Integer, bounds_List : {-1, 1}] := Table[
  (bounds[[1]] + bounds[[2]])/2 + (bounds[[2]] - bounds[[1]])/2 Cos[(2 * k + 1)/(2 * n) π], {k, 0, n - 1}]
```

In[7]:=
```
chebNodes[6] // N
```
Out[7]= {0.965926, 0.707107, 0.258819, -0.258819, -0.707107, -0.965926}

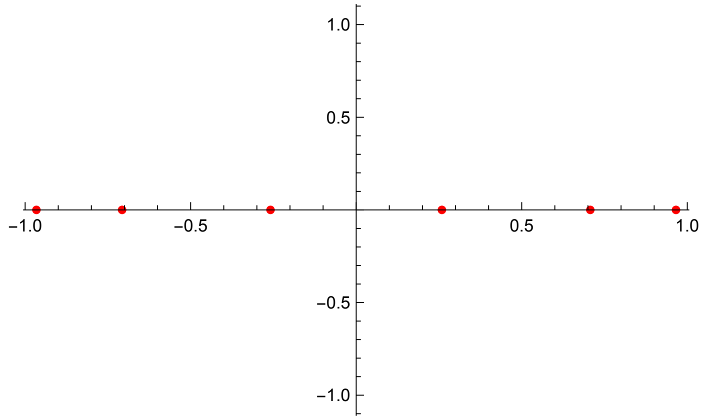
In[9]:=
```
chebNodes[6, {0, Pi / 2}] // N
```
Out[9]= {1.54403, 1.34076, 0.988674, 0.582122, 0.230038, 0.0267618}

In[57]:=
```
Transpose[{chebNodes[6], Table[0, 6]}]
```
Out[57]=

$$\left\{\left\{\frac{1+\sqrt{3}}{2\sqrt{2}}, 0\right\}, \left\{\frac{1}{\sqrt{2}}, 0\right\}, \left\{\frac{-1+\sqrt{3}}{2\sqrt{2}}, 0\right\}, \left\{-\frac{-1+\sqrt{3}}{2\sqrt{2}}, 0\right\}, \left\{-\frac{1}{\sqrt{2}}, 0\right\}, \left\{-\frac{1+\sqrt{3}}{2\sqrt{2}}, 0\right\}\right\}$$

In[58]:=
```
ListPlot[Transpose[{chebNodes[6], Table[0, 6]}], PlotStyle → Red]
```

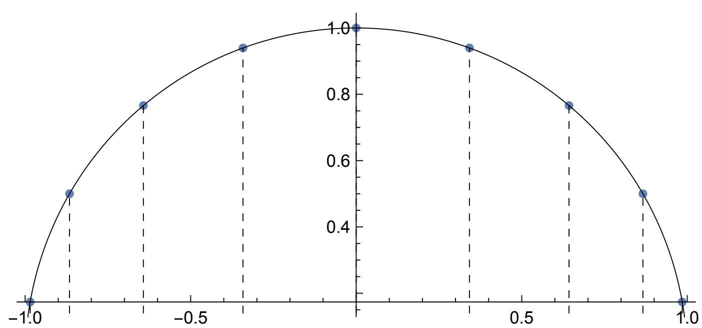
Out[58]=



In[86]:=
```
chebNodePlot[n_Integer] := Show[
  ListPolarPlot[
   Table[{(2*k+1)/(2*n) π, 1}, {k, 0, n-1}]],
  ListPlot[Transpose[{chebNodes[n], Table[0, n]}], PlotStyle → {Red, Large}],
  Graphics[Circle[{0, 0}, 1, {0, Pi}]],
  Graphics[{Dashed, Table[
     Line[{{Cos[(2*k+1)/(2*n) π], 0}, {Cos[(2*k+1)/(2*n) π], Sin[(2*k+1)/(2*n) π]}}], {k, 0, n-1}]}]
 ]
```

In[87]:=
```
chebNodePlot[9]
```

Out[87]=



Approximation Using Nodes

To approximate a function $f(x)$over an interval $[a, b]$ using Chebyshev polynomials up to degree n,

$$g(x) = \sum_{i=0}^{n} \alpha_i T_i\left(2\frac{x-a}{b-a} - 1\right)$$

First, compute the Chebyshev nodes on the interval $[a, b]$:

$$r_k = \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{2k+1}{2n}\right) \text{ for } k = 1 \ldots n$$

Then evaluate the function f at the nodes:

$$y_k = f(r_k) \text{ for } k = 1 \ldots n$$

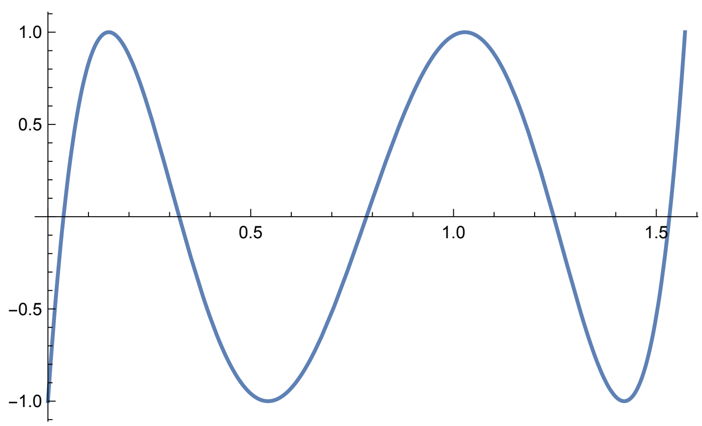
Define the translated Chebyshev functions on $[a, b]$ as:

$$T_i'(x) = T_i\left(2\frac{x-a}{b-a} - 1\right)$$

In[88]:=
```
translatedCheby[n_, x_, bounds_ : {-1, 1}] :=
 ChebyshevT[n, 2 (x - bounds[[1]]) / (bounds[[2]] - bounds[[1]]) - 1]
```

In[89]:=
```
Plot[translatedCheby[5, x, {0, Pi / 2}], {x, 0, Pi / 2}]
```

Out[89]=



In[104]:=
```
Table[T_i(r_k), {i, 0, 5}, {k, 1, 6}] // TableForm
```

Out[104]//TableForm=

$T_0[r_1]$	$T_0[r_2]$	$T_0[r_3]$	$T_0[r_4]$	$T_0[r_5]$	$T_0[r_6]$
$T_1[r_1]$	$T_1[r_2]$	$T_1[r_3]$	$T_1[r_4]$	$T_1[r_5]$	$T_1[r_6]$
$T_2[r_1]$	$T_2[r_2]$	$T_2[r_3]$	$T_2[r_4]$	$T_2[r_5]$	$T_2[r_6]$
$T_3[r_1]$	$T_3[r_2]$	$T_3[r_3]$	$T_3[r_4]$	$T_3[r_5]$	$T_3[r_6]$
$T_4[r_1]$	$T_4[r_2]$	$T_4[r_3]$	$T_4[r_4]$	$T_4[r_5]$	$T_4[r_6]$
$T_5[r_1]$	$T_5[r_2]$	$T_5[r_3]$	$T_5[r_4]$	$T_5[r_5]$	$T_5[r_6]$

In[105]:=
```
nodeMatrix[n_, bounds_ : {-1, 1}] := Table[
  translatedCheby[i, chebNodes[n + 1, bounds][[j]], bounds], {i, 0, n}, {j, 1, n + 1}]
```

In[106]:=

```
nodeMatrix[5, {-1, 1}] // N // MatrixForm
```

Out[106]//MatrixForm=

$$\begin{pmatrix} 1. & 1. & 1. & 1. & 1. & 1. \\ 0.965926 & 0.707107 & 0.258819 & -0.258819 & -0.707107 & -0.965926 \\ 0.866025 & 0. & -0.866025 & -0.866025 & 0. & 0.866025 \\ 0.707107 & -0.707107 & -0.707107 & 0.707107 & 0.707107 & -0.707107 \\ 0.5 & -1. & 0.5 & 0.5 & -1. & 0.5 \\ 0.258819 & -0.707107 & 0.965926 & -0.965926 & 0.707107 & -0.258819 \end{pmatrix}$$

In[110]:=

```
Diagonal[nodeMatrix[5, {-1, 1}].Transpose[nodeMatrix[5, {-1, 1}]]] // N
```

Out[110]=

{6., 3., 3., 3., 3., 3.}

In[112]:=

```
y = Sin[chebNodes[6, {0, Pi / 2}]] // N
```

Out[112]=

{0.999642, 0.973658, 0.835298, 0.549798, 0.228014, 0.0267586}

In[250]:=

```
f = Sin
bounds = {0, Pi / 2}
degree = 5
m = nodeMatrix[degree, bounds];
y = f[chebNodes[degree + 1, bounds]];
coefficients = m.y / Diagonal[m.Transpose[m]];
fApprox =
 coefficients.Table[translatedCheby[i, x, bounds], {i, 0, degree}] // N // Simplify
Plot[{fApprox, f[x]}, {x, bounds[[1]], bounds[[2]]},
 PlotStyle → {Dashed, {Thickness[0.01], Opacity[0.5]}},
 PlotLegends → {"Approximation", "True Function"}]
Plot[fApprox - f[x], {x, bounds[[1]], bounds[[2]]}, PlotLegends → {"Error"}]
```

Out[250]=

Sin

Out[251]=

$\left\{0, \frac{\pi}{2}\right\}$

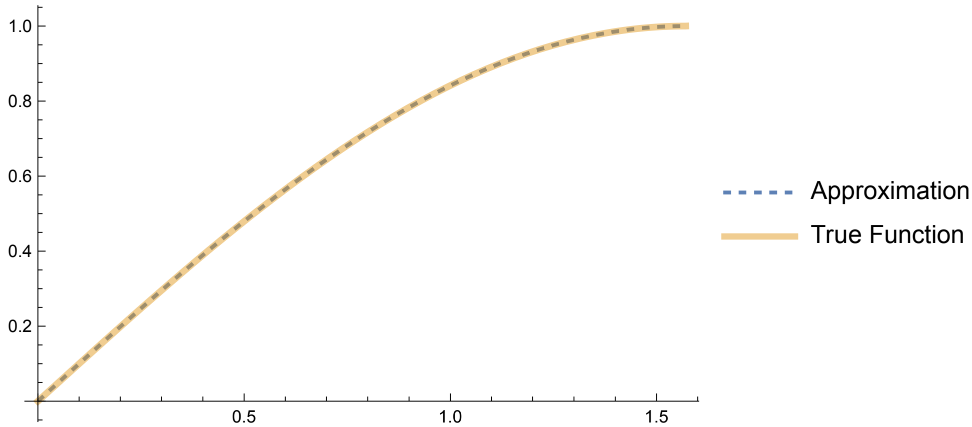
Out[252]=

5

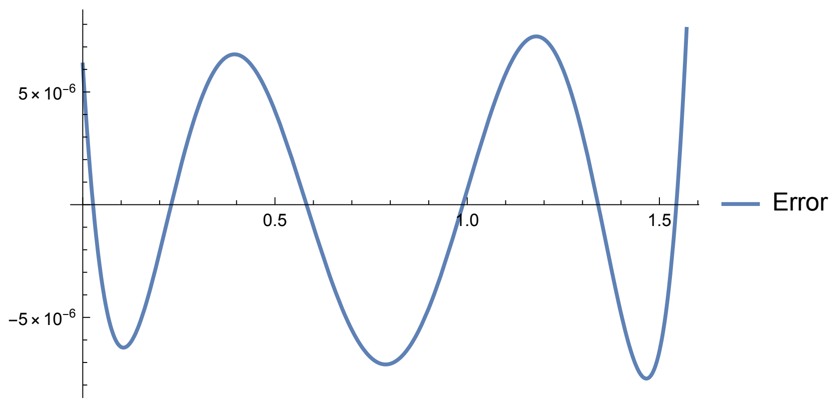
Out[256]=

$6.21629 \times 10^{-6} + 0.999716\, x + 0.00206176\, x^2 - 0.172007\, x^3 + 0.00593014\, x^4 + 0.00576399\, x^5$

Out[257]=



Out[258]=



In[231]:=

```
Diagonal[m]^2 // N
```

Out[231]=

```
{1., 0.5, 0.75, 0.5, 1., 0.0669873}
```

References

https://en.wikipedia.org/wiki/Chebyshev_polynomials
https://www.johndcook.com/blog/2020/03/11/chebyshev-approximation/
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https://mathworld.wolfram.com/ChebyshevApproximationFormula.html
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