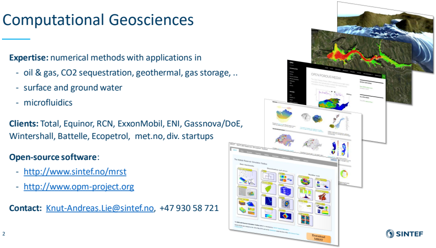
Simple atmospheric models related to global warming

26. October 2020

Presentation and master projects

My name is Xavier Raynaud. I work 80% at SINTEF DIGITAL (applied mathematics and cybernetics) and 20% at NTNU.



see overview of group activity

- We offer Master Projects

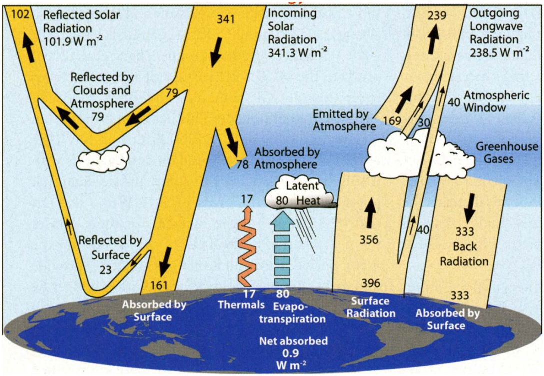
  see link

- To be updated with
  - Water recovery in Somalia
  - Battery modeling
  - H2 underground storage

- Have a look!

Global warming and radiation

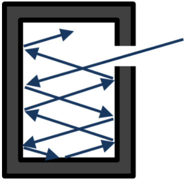
- Main identified cause of global warming : Absorption of radiative heat by greenhouse gas.



Background on radiation

- Warm bodies radiate. This is the way energy travels in vacuum (at the speed of light...)
- A black body is an idealized material, which is a *perfect absorber* and *emitter*.
- A photon brings energy, then energy is stored and does not escape
- Planck's law:

$$E_{b\nu}(T,\nu) = \frac{2\pi h\nu^3}{c_0^2(e^{\frac{h\nu}{kT}} - 1)}$$



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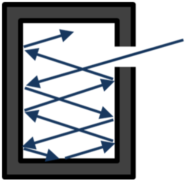
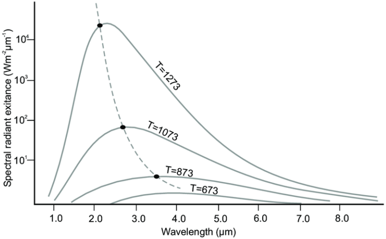




Eigenmodes

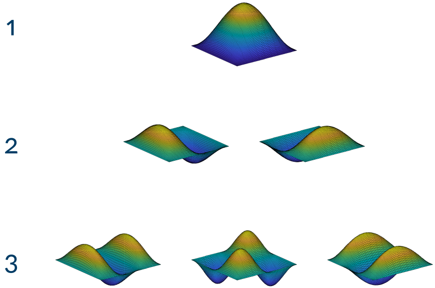
- Solutions of wave equations with vanishing BC,

  $$u = \sin(k_x \pi x)\sin(k_y \pi y)\sin(k_z \pi z)$$

  with $k_x = \frac{l\pi}{L}$, $k_y = \frac{m\pi}{L}$, $k_z = \frac{n\pi}{L}$.
- We have $|k|^2 = (\frac{2\pi\nu}{c})^2$
- Number of modes

  $$dN = C\nu^2\, d\nu$$

- Just too many high frequency modes. The distribution is not integrable.



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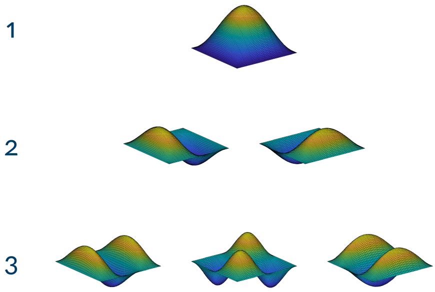

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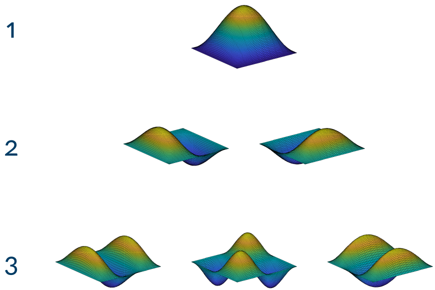

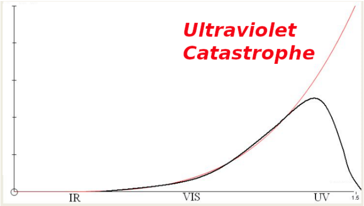

Quantization

- In the classical approach: Every mode has the same energy,
- Planck considered light quantas,

$$E = \hbar\nu.$$

- For a given frequency, the energy that are observed are $\hbar\nu$, $2\hbar\nu$, $3\hbar\nu$, ...
- Statistical mechanics: The distribution in the modes follows the Boltzmann distribution

$$p(E) \quad \alpha \quad e^{-\frac{E}{kT}}$$

- Modes of high frequencies (high energy) are more unlikely to appear
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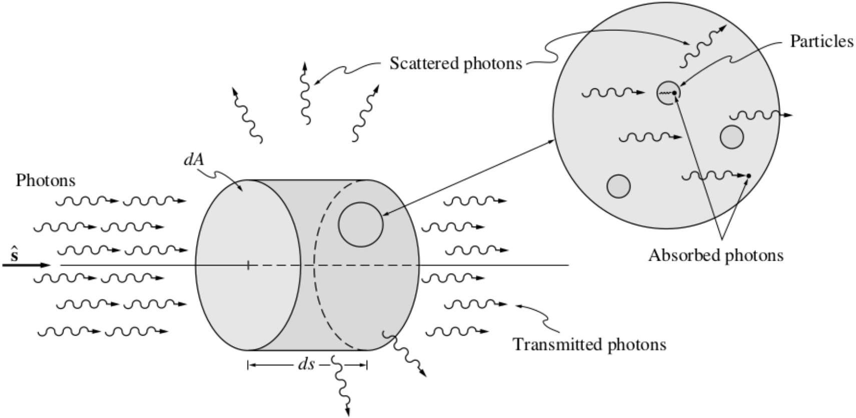
Absorption, transmission, scattering



Spectrography

- Photons are absorbed: The energy of the incoming photon is used to set the atom/molecule in a higher energy level.
- Quantum effects: The levels of energy of the atom/molecule are **discrete**. Hence, only specific frequencies are absorbed : spectrography
- Moreover, there exist
  - **Stimulated** emissions: emission at the same frequency as incoming photon but in random direction,
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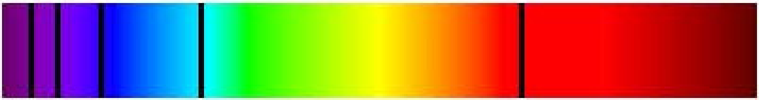

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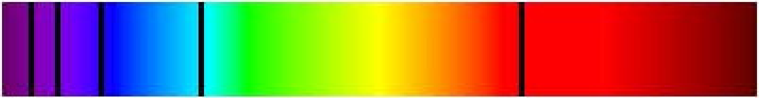

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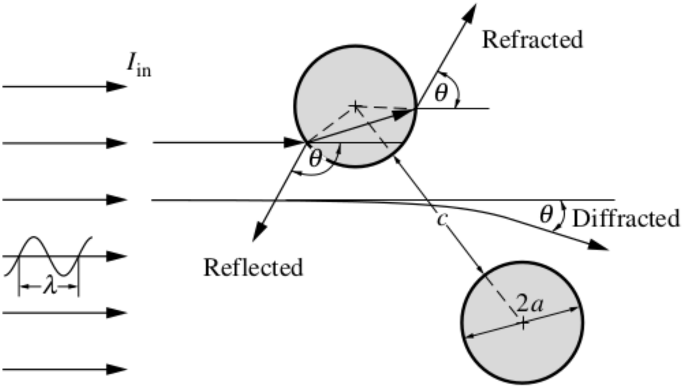
Scattering



I_{in}
Refracted
θ
θ
θ
Diffracted
c
Reflected
λ
$2a$

Scattering

- The scattering equations are obtained by solving **Maxwell equations**
- Different regime according to ratio $x = \frac{2\pi r}{\lambda}$ (r = particle radius)
- For $x = 1$ (most difficult case), **Mie**'s scattering
- For small x: **Rayleigh**'s approximation, large x: **Geometrical optic** approximation

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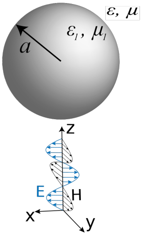
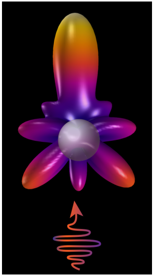
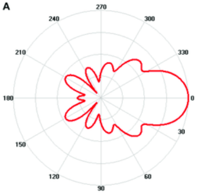
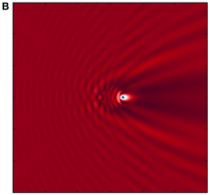






Scattering diagram (MiePlot)



Near-field image (T-matrix software)

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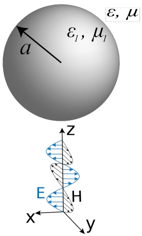

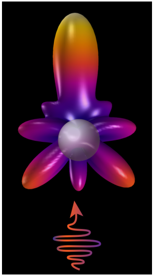

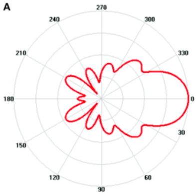

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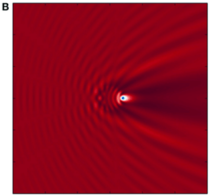

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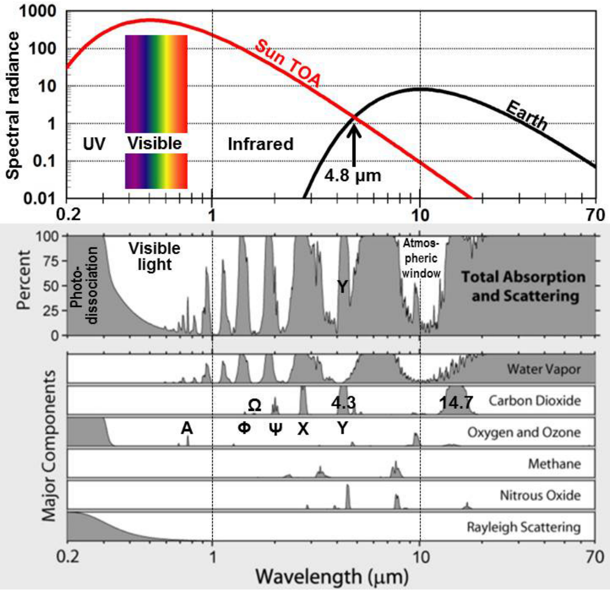
Atmospheric radiation



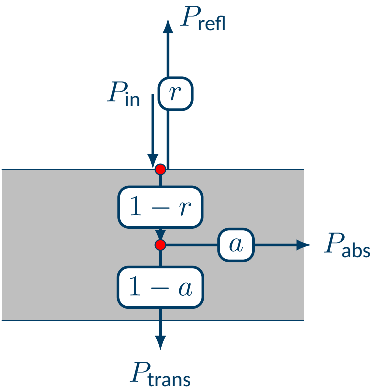
Simple radiation model

- We decompose an incoming radiation using **transfer coefficients**

$$P_{\text{in}} = P_{\text{abs}} + P_{\text{refl}} + P_{\text{trans}}$$

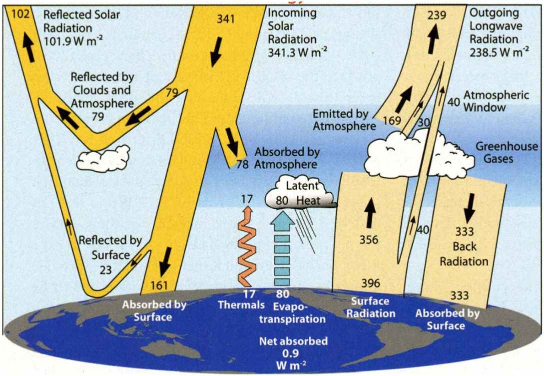
where

$$\begin{aligned} P_{\text{abs}} &= a(1-r)P_{\text{in}}, \\ P_{\text{refl}} &= rP_{\text{in}}, \\ P_{\text{trans}} &= (1-a)(1-r)P_{\text{in}}. \end{aligned}$$



Questions

- We consider two frequency regimes : short wave and long waves



- Compute the radiative heat fluxes using the transfer coefficients given in table (project description)
- Compute the sources (emissivity coefficients)

Questions

- At equilibrium, energy conservation implies

$$\text{sum of fluxes} = \text{sources}$$

- Find expression for atmosphere and Earth's temperature
- Study the sensitivity with respect to the coefficients.
- Note : The model contains many simplifications. The coefficients can be modified.

Polar region

- Albedo : Reflection coefficient
- **Ice** albedo = 0.62 versus average earth albedo = 0.32
- Amplification feedback mechanism

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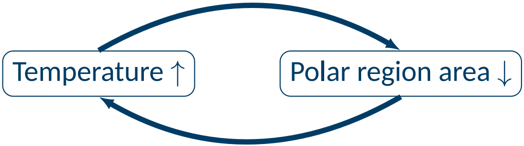
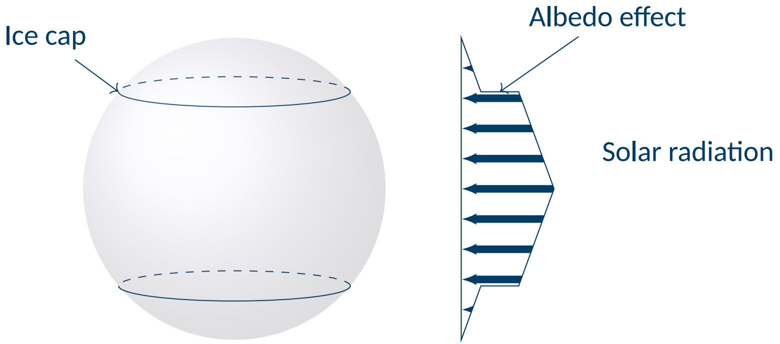


Ice cap location



We want to compute the ice cap location and study its stability.

Energy conservation

- We consider only internal energy (only state variable is temperature)
- Derive the energy conservation equation:

$$\frac{\partial}{\partial t}(c\rho T) = \nabla \cdot (k\nabla T) + Q$$

- Zonal approximation

- We consider that the temperature is constant for the same latitude.
- We consider average radiation for a given latitude
- Rewrite energy conservation equation in form of

$$C_a \frac{\partial T}{\partial t} = D_\phi(K_a \frac{\partial T}{\partial \phi}) + q.$$

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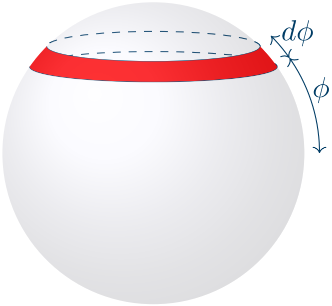


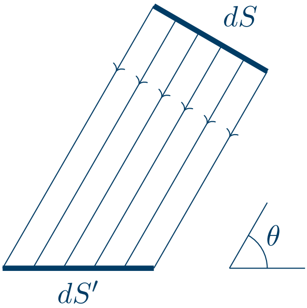
Computation of $S(x)$

- Radiative power I_S and I_E are in $\mathrm{W\,m^{-2}}$
- Flux conservation

$$I_E\, dS' = I_S dS$$

- Hence,

$$I_E = I_S \sin(\theta).$$



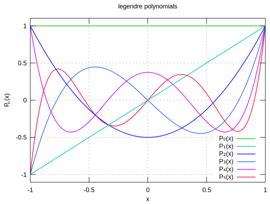
Legendre polynomials - solution technique

- The governing equation at equilibrium:

$$-D\frac{\partial}{\partial x}((1-x^2)\frac{\partial T}{\partial x}) = -I(x) + QS(x)a(x, x_s).$$

- The function a depends on ice cap location x_s.
- We compute Q **given** x_s, that is $Q(x_s)$.
- Legendre polynomials satisfy

$$-((1-x^2)u') = \lambda u$$



- Use Legendre polynomials to compute the solution

Legendre polynomials - solution technique

- More familiar setup for the equation

$$-u'' = f.$$

- The functions $\sin(n\pi x)$ are eigenvectors for the operator $u \to -u''$.
- We have

$$u = \sum u_n \sin(n\pi x), \quad f = \sum f_n \sin(n\pi x)$$

- Hence,

$$u_n = f_n / \left(n\pi\right)^2 .$$

- The **orthogonality** of the Fourrier series is essential to compute the Fourrier expansion. In the same way, the Legendre polynomials are orthogonal with respect to scalar product

$$< u, v >= \int_{-1}^{1} (1 - x^2) u(x) v(x)\, dx$$

Numerical method

- The Legendre polynomials are in fact just a convenient way to **discretize** the equations
- Set up a standard discretization method (finite difference, finite element) to compute the solution.

Equilibrium stability

- Standard setup in finite dimension
$$\dot{X} = F(X)$$
($X \in \mathbb{R}^n$, F is non-linear)
- The equilibrium equation is
$$F(X_0) = 0$$
- We consider a perturbation
$$X(t) = X_0 + \delta X(t)$$
- We obtain a **linear** equation for δX,
$$\delta \dot{X} = DF(X_0)\delta X$$

- We know how to solve linear systems of ODEs.
- The solution takes the form
$$\delta X = \sum e^{\lambda_i t}(U_i + \ldots)$$
- The eigenvalues of $DF(X_0)$ determines the linear stability.
- Proceed in the same way in **infinite dimensional** space (**differential operators** replace matrices).
- We need to discretize to compute the solution...

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$$X(t) = X_0 + \delta X(t)$$
- We obtain a **linear** equation for δX,
$$\delta \dot{X} = DF(X_0)\delta X$$

- We know how to solve linear systems of ODEs.
- The solution takes the form
$$\delta X = \sum e^{\lambda_i t}(U_i + \ldots)$$
- The eigenvalues of $DF(X_0)$ determines the linear stability.
- Proceed in the same way in **infinite dimensional** space (**differential operators** replace matrices).
- We need to discretize to compute the solution...

Equilibrium stability

- Standard setup in finite dimension
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($X \in \mathbb{R}^n$, F is non-linear)
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