Øving 4, Oper. An.

Oppgave 1:

$$\max \quad z = 5x_1 + 3x_2 + x_3 \quad \text{s.t.}$$

$$x_1 + x_2 + 3x_3 \leq 6,$$

$$5x_1 + 3x_2 + 6x_3 \leq 15,$$

$$x_1, x_2, x_3 \geq 0.$$

a) Augmented problem:

$$\max \quad z \quad \text{s.t.}$$

$$z - 5x_1 - 3x_2 - x_3 = 0,$$

$$x_1 + x_2 + 3x_3 + s_1 = 6,$$

$$5x_1 + 3x_2 + 6x_3 + s_2 = 15,$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0.$$

a)

- Begin at initial point (BF solution)

$(0, 0, 0, 6, 15) = (x_1, x_2, x_3, S_1, S_2)$.

Iteration 1:

	x_1	x_2	x_3	S_1	S_2	R.H.S.	
Z	-5	-3	-1	0	0	0	ratio:
S_1	1	1	3	1	0	6	6
S_2	5	3	6	0	1	15	3

+1

$-1/5$

No difference, choose x_2 as entering basic variable.

Iteration 2:

	x_1	x_2	x_3	S_1	S_2	R.H.S.	
Z	0	0	5	0	1	15	ratio:
S_1	0	$2/5$	$9/5$	1	$-1/5$	+3	$+15/2 = 7.5$
x_1	1	$3/5$	$6/5$	0	$1/5$	3	5

$-2/3$

Switch with x_1 as basic variable.

Iteration 3:

	x_1	x_2	x_3	s_1	s_2	R.H.S.	
Z	0	0	$\underline{5}$	0	$\underline{1}$	15	ratio:
s_1	$-2/3$	0	$-3/5$	1	$-1/3$	$+1$	
x_2	$5/3$	1	2	0	$1/3$	5	

Optimality reached twice!

$(x_1^*, x_2^*, x_3^*) = (0, 5, 0)$.

Solution is optimal because all the coeff's in row 0, (z-row) are all ≥ 0.

b) We are in a cycle of optimal solutions if we continue the simplex iterations.

c) I suggest they choose one of the infinitely many optimal solutions along the line

$$\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix}(t) = t \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + (1-t) \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}.$$

d) We find dual values:

$$S_1^* = 1, \quad S_2^* = 0,$$

meaning that we are not constrained by the first functional inequality constraint. Also, we could increase

e) It appears that the sensitivity report is based on the optimal solution $(x_1^*, x_2^*, x_3^*) = (3, 0, 0)$.

As the increase in resource 2 ("orange") is within the allowable increase in the r.h.s. of the second functional constraint, the optimal BF-solution does not change. We can therefore find a new optimal solution at

$$(\hat{x}_1^*, \hat{x}_2^*, \hat{x}_3^*) = (6, 0, 0), \quad \text{with}$$

$$\hat{z} = 30.$$

However, when the availability of resource 2 becomes 37, the linearization about our optimal solution fails, and we have to optimize again.

5) Neither the production plan or the objective function value changes until the price for product 3 increases by more than 5. Until then, the optimal solution we found remains the same.

This we can read off from the sensitivity report.

Ppg. 28 $c = [3, -1, A]^T$,

(P) $\min z = 3x_1 - x_2 - A(-x_3) = c^T x$

s.t. $2x_1 + x_2 - (-x_3) \leq 10$,

$-c\,x_1 + (-x_3) \leq +1$,

$\underline{\underline{A}} \in \mathbb{R}^{3\times 3}$, $(\leq)$?

$x_2 - 2(-x_3)\ B\ 5$,

$x_1, x_2 \geq 0$,

$(-x_3) \geq 0$,

(D) $\max w = 10 y_1 + (-y_2) + 5 y_3$

s.t. $2y_1 + 5(-y_2) \leq 3$,

$y_1 + y_3 \leq -1$,

$-y_1 + (-y_2) - 2y_3 \leq (-D)$,

$y_1 \leq 0$,

$(-y_2) \leq 0$,

$y_3\ E\ 0$ $\cancel{E \geq 0}$,

$E \leq 0$.

Primal problem: | Dual Problem:

$$\max z = c^T x \qquad \min w = b^T y,$$

$$\text{s.t. } Ax \le b, \qquad \text{s.t. } A^T y \le c,$$

$$x \ge 0, \qquad y \le 0,$$

- And: $c^T x^* = b^T y^*$, $x^* = (0, 11, +1)$ $(-x_3)$

Let $B := (\le)$ $\qquad y^* = (-1, 8, 0)$,

$$[3, -1, -A]\begin{bmatrix} 0 \\ 11 \\ +1 \end{bmatrix} = 11 - A = [10, 1, 5]\begin{bmatrix} -1 \\ 8 \\ 0 \end{bmatrix} = -10 + 8 = -2,$$

$$\Rightarrow A = 11 + 2 = 13 \checkmark,$$

$$\underline{A} = \begin{bmatrix} 2 & 1 & -1 \\ -C & 0 & +1 \\ 0 & 1 & -2 \end{bmatrix},$$

$$\underline{A}^T = \begin{bmatrix} 2 & -C & 0 \\ 1 & 0 & 0 \\ -1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & -2 \end{bmatrix}, \Rightarrow C = -5,$$

$$C = \begin{bmatrix} 3 \\ -1 \\ -A \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -D \end{bmatrix},$$

$$\Rightarrow \quad A = D = 13.$$

In summary:

$A = 13$,

$B = (\leq)$,

$C = -5$,

$D = 13$,

$E = (\leq)$.

Oppg. 3

primal problem:

$$\max \quad z = x_1 + 2x_2 + x_3 + x_4$$

$$\begin{aligned}
\text{st.} \quad 2x_1 + x_2 + 5x_3 + x_4 &\leq 8 \\
2x_1 + 2x_2 + 4x_4 &= 12 \\
3x_1 + x_2 + 2x_3 &\geq 18 \\
x_1,\ x_2,\ x_4 &\geq 0 \\
x_3 &\leq 0
\end{aligned}$$

First write primal on standard form:

$$\max \quad z = x_1 + 2x_2 + x_3 + x_4$$

$$\begin{aligned}
\text{st.} \quad 2x_1 + x_2 + 5x_3 + x_4 &\leq 8 \\
2x_1 + 2x_2 + 4x_4 &\leq 12 \\
-2x_1 - 2x_2 - 4x_4 &\leq -12 \\
-3x_1 - x_2 + 2x_3 &\leq -18 \\
x_1,\ x_2,\ x_3,\ x_4 &\geq 0
\end{aligned}$$

This is equivalent to

$$\max \quad z = \underline{c}^T \underline{x}$$
$$\text{st.} \quad \underline{\underline{A}}\,\underline{x} \leq \underline{b}$$
$$\underline{x} \geq \underline{0}$$

with

$$\underline{c} = [1 \quad 2 \quad 1 \quad 1]^T$$

$$\underline{b} = [8 \quad 12 \quad -12 \quad -18]^T$$

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 & 5 & 1 \\ 2 & 2 & 0 & 4 \\ -2 & -2 & 0 & -4 \\ -3 & -1 & 2 & 0 \end{bmatrix}$$

Dual problem

$$\min \quad w = 8y_1 + 12y_2 - 12y_3 - 18y_4$$

$$2y_1 + 2y_2 - 2y_3 - 3y_4 \geq 1$$
$$y_1 + 2y_2 - 2y_3 - y_4 \geq 2$$
$$5y_1 + 2y_4 \geq 1$$
$$y_1 + 4y_2 - 4y_3 \geq 1$$
$$y_1, y_2, y_3, y_4 \geq 0$$

Oppg. 4

$$\min \quad z = 2x_1 + 2x_2 + x_3 - x_4 + x_5$$

$$\begin{aligned} \text{st.} \quad & x_1 + 2x_2 - x_3 + 2x_4 = 6 \\ & 2x_1 + x_3 + x_5 \geq 4 \\ & 3x_2 - 2x_3 + 3x_4 = 7 \end{aligned}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0$$

Assuming the dual solution $\underline{v} = [1 \quad 0 \quad -1]$

a) Determine if the dual solution is feasible.

$$\max \quad w = 6y_1 + 4y_2 + 7y_3$$

$$\begin{aligned} \text{s.t.} \quad & y_1 + 2y_2 \leq 2 \\ & 2y_1 + 3y_3 \leq 2 \\ & -y_1 + y_2 - 2y_3 \leq 1 \\ & 2y_1 + 3y_3 \leq -1 \\ & y_2 \leq 1 \end{aligned}$$

$$y_1 \quad , \quad y_2 \geq 0 \quad , \quad y_3$$

Check if $y = [1 \quad 0 \quad -1]$ is feasible:

LHS	RHS	slack	Satisfied?
1	2	1	✓
-1	2	3	✓
1	1	0	✓
-1	-1	0	✓
0	1	1	✓

The dual solution is feasible.

b) Corresponding primal solution.

Dual w/ SOB method:

$$
\begin{aligned}
\min \quad z = \; & 2x_1 + 2x_2 + x_3 - x_4 + x_5 \\
& x_1 + 2x_2 - x_3 + 2x_4 = 6 \\
& 2x_1 + x_3 + x_5 \geq 4 \\
& 3x_2 - 2x_3 + 3x_4 = 7 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Basisvariable: S_1, S_2, S_5, X_3, X_4

$$-x_3 + 2x_4 = 6$$
$$-2x_3 + 3x_4 = 7$$

$$x_3 = 2x_4 - 6 = 4$$

$$-2(2x_4 - 6) + 3x_4 = 7$$
$$-4x_4 + 12 + 3x_4 = 7$$
$$x_4 = 5$$

Corresponding primal solution:

$$(0, 0, 4, 5, 0)$$

c) Is the dual solution optimal?

Yes, if the primal solution is feasible.

LHS	RHS	"Constraint"	Satisfied?
6	6	=	✓
4	4	$\geq$	✓
-8	7	=	✗

No, the given dual solution is not optimal.