Financial Economics | Problem Set 6

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April 15th, 2024

1 Mean-variance analysis with two risky assets

(a) Mean, Variance and Standard Deviation of Growth and Value

Using the Excel functions =AVERAGE, =VAR, and =STDEV, we have calculated the mean, variance and standard deviation of the two portfolios of value and growth stock with 90% placed in big stocks and 10% in small stocks. The results can be found in table 1

	Growth	Value
Mean	0.1139	0.1505
Variance	0.0449	0.0776
Standard Deviation	0.2118	0.2786

Table 1: Mean, Variance and Standard Deviation of the two portfolios

In table 1, we see that the value portfolio has a higher standard deviation but also delivers a higher average return; thus, investors are being compensated for the additional risk.

(b) Covariance of the two Portfolios

Using the =COVAR function in Excel we have calculated the covariance as $Cov(R_G,R_V) = 0.0483$. It is unsurprising that the covariance of the stocks is positive as we will expect both portfolios to follow the general market and perform well in good times and worse during bad times.

(c) Combining the two portfolios

We use the calculated mean return found in table 1 to calculate the expected return of a combination with

$$\mu_p = w_p^T \cdot \mu, \text{ where } w_p = \begin{pmatrix} w_G \\ w_V \end{pmatrix} \text{ and } \mu = \begin{pmatrix} 0.1139 \\ 0.1505 \end{pmatrix} \tag{1}$$

Next, we calculate the portfolio variance using the variance-covariance matrix using the results from (a) and (b)

$$\sigma_p^2 = w_p^T \cdot \Sigma \cdot w_p, \text{ where } \Sigma = \begin{pmatrix} 0.0499 & 0.0483 \\ 0.0483 & 0.0776 \end{pmatrix} \tag{2}$$

Lastly, we find the portfolio standard deviation as $\sigma_p = \sqrt{\sigma_p^2}$. The results can be found in table 2.

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(d) Plotting the Standard Deviation as a function of Weight in Growth

Using the results in table 2 we can make the following plot

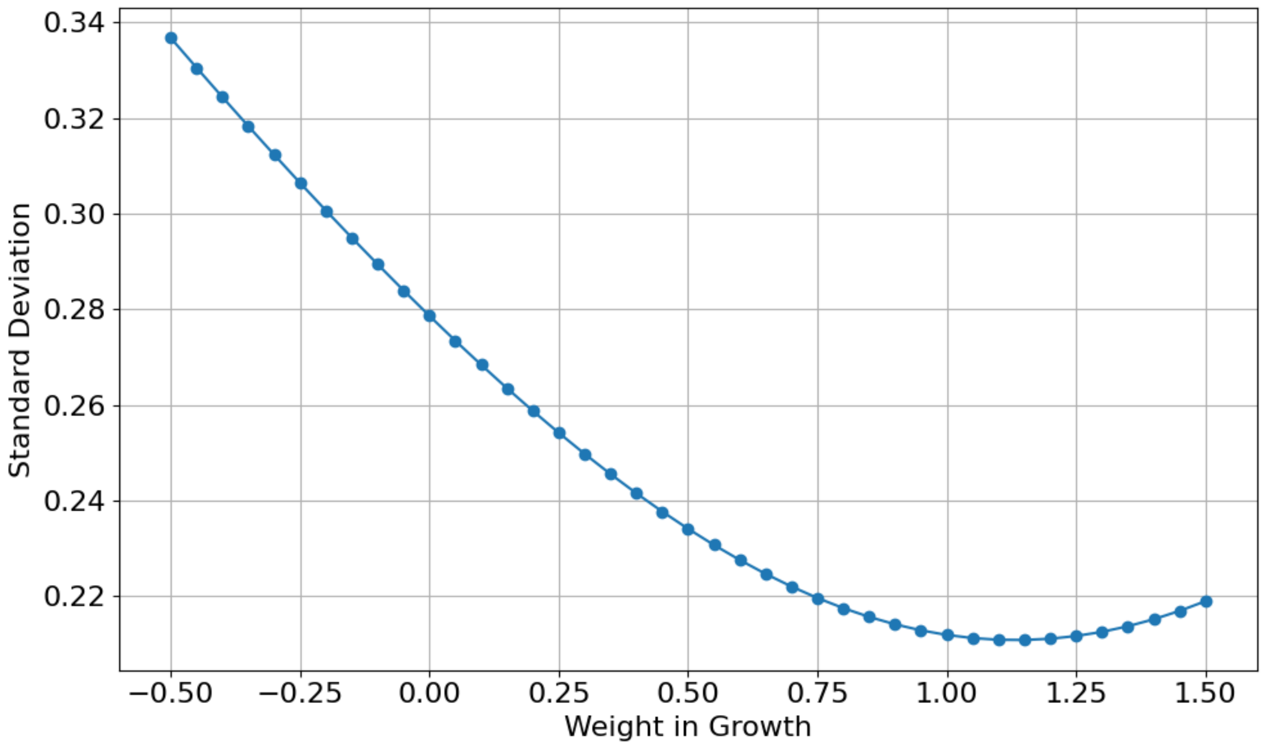

Figure 1: Standard Deviation as a function of the Weight in Growth stock

(e) Plotting the Minimum Variance Frontier

Using the results in table 2 we can plot the minimum variance frontier, efficient frontier and minimum variance portfolio in a world with the two portfolios.

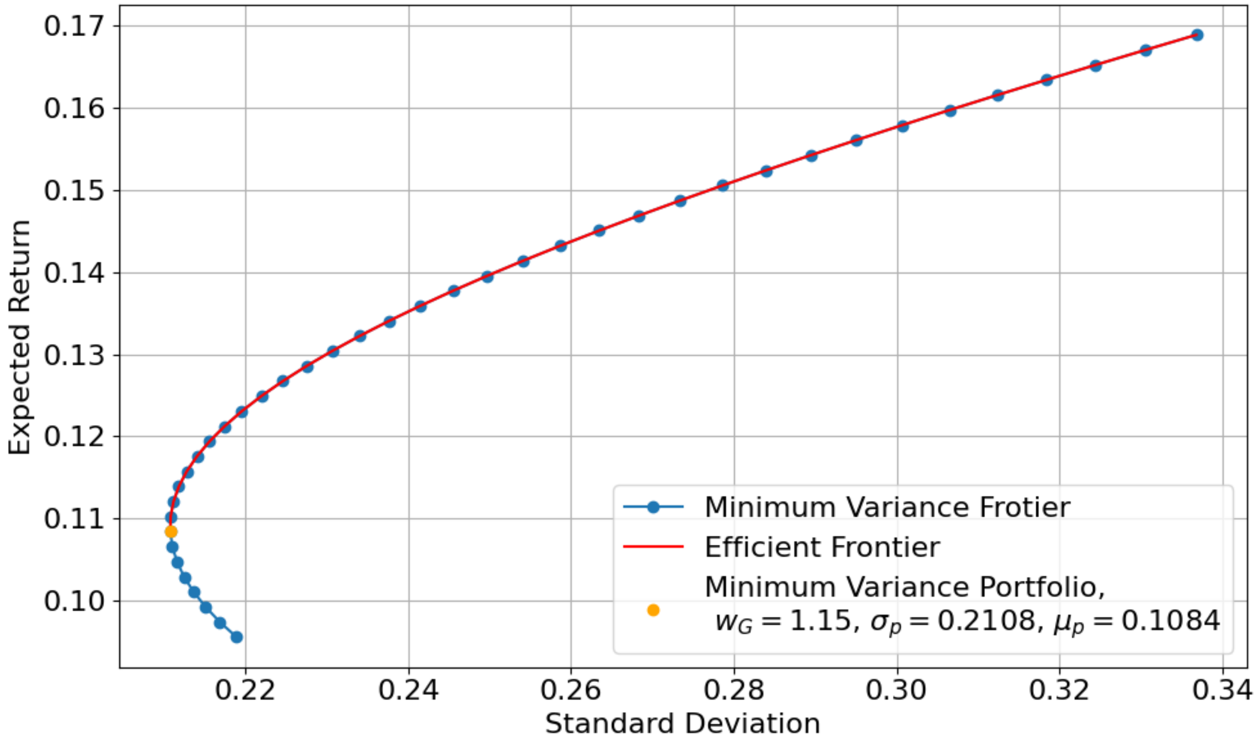

Figure 2: Minimum Variance Frontier, efficient frontier and minimum variance portfolio in a world with just the Growth and Value portfolio

In figure 2, we have plotted the minimum variance frontier in blue, the efficient frontier in red as the part with a return as high or higher than the minimum variance portfolio. Lastly, we have plotted the minimum variance portfolio, based solely on the calculated portfolios in table 2 in orange. The true minimum variance portfolio will be somewhere where w_G is slightly lower, and thus, the true minimum variance portfolio will have $1.10 \leq w_G \leq 1.15$, $\sigma_p \approx 0.211$ and $\mu_p \approx 0.108$. Compared to the pure portfolios from table 1, we find that both the pure portfolios will provide a higher expected return; however, they also come with a higher expected volatility.

2 True or False

(a)

False. As long as stocks' expected excess return is positive, even highly risk-averse investors should still have some risky assets in their portfolios. Only as risk aversion tends toward infinity will the weight of risky assets tend toward 0.

(b)

True. In the CAPM-equilibrium, the market has the highest Sharpe ratio as it lies in the tangency between CML and the efficient frontier

(c)

True. The mutual fund theorem states that every portfolio should be a combination of the tangency portfolio and the risk-free asset, and additionally, all investors should hold risky assets in the same proportions. This example violates the theorem because Sung and Xing have different proportions of risky assets, which breaks the theorem.

(d)

False. CAPM suggests that all assets, including those with expected returns less than the risk-free rate, can be included in the market portfolio because of diversification benefits. If the asset has a negative covariance with the market, an investor would still want to hold such an asset for the optimal trade-off between risk and return

3 CAPM

(a) Expected Return given β

CAPM implies

$$\mathbb{E}(r_i) - r_f = \beta_i(\mathbb{E}(r_m) - r_f) \tag{3}$$

Hence, the expected return premium of an asset with $\beta_i = 1.5$ must be 1.5 times the return premium of the market. In this example, we can use the given information to calculate the expected return of the asset as

$$\mathbb{E}(r_i) = 2\% + 1.5 \cdot (12\% - 2\%) = 17\% \tag{4}$$

Thus, we can see that the asset, i, with $\beta_i = 1.5$, has a 5% higher expected return than the market portfolio, implying an expected return of 17%.

(b) Calculating β

Using (3) we can calculate the the β as

$$\beta_p = \frac{11\% - 2\%}{12\% - 2\%} = 0.9 \tag{5}$$

Thus, we get the β of the portfolio as $\beta_p = 0.9$

(c) Finding β from covariance

The formula for β – the loading of the asset – is

$$\beta_i = \frac{\sigma_{i,m}}{\sigma_m^2} \tag{6}$$

We can combine (6) with (3) to calculate the excess return of asset i

$$\mathbb{E}(r_i) - r_f = \frac{0.1}{0.2^2}(12\% - 2\%) = 25\% \tag{7}$$

Thus, the excess return of asset i is $\mathbb{E}(r_i) - r_f = 25\%$ and the expected return as $\mathbb{E}(r_i) = 27\%$.

4 Stock Pricing

(a) Expected dividend

Using the expected rate of return to find the payment that in perpetuity will equal the price

$$\$20 = \frac{d}{10\%} \Leftrightarrow d = \$20 \cdot 10\% = \$2 \tag{8}$$

We will, therefore, expect the current dividend of the asset to be $2.

(b) β before change in correlation

Using (3) we can calculate the β of the asset before the correlation changes to be

$$\beta = \frac{10\% - 2\%}{6\%} = \frac{4}{3} \approx 1.33 \tag{9}$$

We find the β before the change in correlation, ρ, to be $\beta \approx 1.33$.

(c) Change in β due to changes in ρ

We know the relationship between the covariance and the correlation can be described with the equation

$$\sigma_{i,m} = \rho_{i,m}\sigma_i\sigma_m \tag{10}$$

Combining this with (6) gives

$$\beta_i = \frac{\rho_{i,m}\sigma_i\sigma_m}{\sigma_m^2} = \frac{\rho_{i,m}\sigma_i}{\sigma_m} \tag{11}$$

From here, it is immediately apparent that if the correlation, $\rho_{i,m}$, doubles, then β must double as well. The β after the doubling of the correlation, therefore, is $\beta_i = 8/3 \approx 2.67$.

(d) Expected excess return after a change in β

We now use (3) to recalculate the expected excess return as well as the expected return for the asset after the doubling of the correlation.

$$\mathbb{E}(r_i) - 2\% = \frac{8}{3} \cdot 6\% = 16\% \tag{12}$$

The new expected excess return is 16%, implying that the expected total return has increased from 10% to 18% following the doubling of the correlation.

(e) New market price of after increase in correlation

We can finally calculate the new price of the asset using the new expected return of the asset to price the perpetuity.

$$P = \frac{\$2}{18\%} = \$\frac{10}{9} \approx \$11.11 \tag{13}$$

The new price of the asset thus decreases from \$20 to $\sim$ \$11.11 as the correlation with the market doubles. This makes sense from a CAPM standpoint as the asset now more closely follows the market and is more exposed to systematic risk. The expected return, therefore, must increase, making the price fall as the dividends stay constant.

w_G	w_V	μ_p	σ_p^2	σ_p
-0.50	1,50	0.1688	0.1134	0.3368
-0.45	1,45	0.1670	0.1093	0.3305
-0.40	1,40	0.1652	0.1052	0.3244
-0.35	1,35	0.1633	0.1013	0.3183
-0.30	1,30	0.1615	0.0976	0.3123
-0.25	1,25	0.1597	0.0939	0.3064
-0.20	1,20	0.1578	0.0904	0.3007
-0.15	1,15	0.1560	0.0870	0.2950
-0.10	1,10	0.1542	0.0837	0.2894
-0.05	1,05	0.1524	0.0806	0.2839
0.00	1,00	0.1505	0.0776	0.2786
0.05	0.95	0.1487	0.0748	0.2734
0.10	0.90	0.1469	0.0720	0.2684
0.15	0.85	0.1450	0.0694	0.2635
0.20	0.80	0.1432	0.0669	0.2587
0.25	0.75	0.1414	0.0646	0.2541
0.30	0.70	0.1395	0.0624	0.2497
0.35	0.65	0.1377	0.0603	0.2455
0.40	0.60	0.1359	0.0583	0.2415
0.45	0.55	0.1341	0.0565	0.2376
0.50	0.50	0.1322	0.0548	0.2340
0.55	0.45	0.1304	0.0532	0.2307
0.60	0.40	0.1286	0.0518	0.2275
0.65	0.35	0.1267	0.0504	0.2246
0.70	0.30	0.1249	0.0493	0.2219
0.75	0.25	0.1231	0.0482	0.2196
0.80	0.20	0.1212	0.0473	0.2174
0.85	0.15	0.1194	0.0465	0.2156
0.90	0.10	0.1176	0.0458	0.2140
0.95	0.05	0.1157	0.0453	0.2128
1,00	0.00	0.1139	0.0449	0.2118
1,05	-0.05	0.1121	0.0446	0.2112
1,10	-0.10	0.1103	0.0444	0.2108
1,15	-0.15	0.1084	0.0444	0.2108
1,20	-0.20	0.1066	0.0445	0.2110
1,25	-0.25	0.1048	0.0448	0.2116
1,30	-0.30	0.1029	0.0451	0.2125
1,35	-0.35	0.1011	0.0456	0.2136
1,40	-0.40	0.0993	0.0463	0.2151
1,45	-0.45	0.0974	0.0470	0.2169
1,50	-0.50	0.0956	0.0479	0.2189

Table 2: Expected return, variance, and standard deviation of multiple portfolios