Fixed Income Derivatives - CIR Exam Preparation

In the CIR model, the short rate r_t is assumed to have the following dynamics under the risk-neutral measure $\mathbb{Q}$

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{1}$$

where a, $b > 0$ and σ are model parameters.

a) Solution and distribution of $r(t)$.

   i) Find the closest equivalent to a solution for $r(t)$, $t > 0$, in the CIR model.

   ii) Find the mean $\mathrm{E}[r(t)]$ and the variance $\mathrm{Var}[r(t)]$ of $r(t)$. There is a 'trick' involved when computing the variance, what is that trick called.

   iii) State the distribution of $r(t)$ but do not try to derive it.

   iv) State the distribution of $r(t)$ as $t \nearrow \infty$ but do not try to derive it. That is, find the stationary distribution of $r(\infty)$.

Answers:

   i) To find the closest we get to s solution for the short rate in the CIR model, we apply Ito to $f(t, r) = e^{at}r$ and integrate appropriately to give us that

$$r(t) = r(0)e^{-at} + b\Big[1 - e^{-at}\Big] + \sigma\int_0^t e^{-a(t-u)}\sqrt{r_u}dW_u \tag{2}$$

   ii) The mean and variance can be found by direct computation, however to compute variance Ito asymmetry must be used.

$$\mathrm{E}[r(t)] = r(0)e^{-at} + b\Big[1 - e^{-at}\Big], \qquad \mathrm{Var}[r(t)] = \frac{\sigma^2 r(0)}{a}\Big[e^{-at} - e^{-2at}\Big] + \frac{\sigma^2 b}{2a}\Big[1 - e^{-at}\Big]^2 \tag{3}$$

   iii) The distribution of $r(t)|\mathcal{F}_0$ is that of

$$\frac{\sigma^2}{4b}\Big[1 - e^{-bt}\Big]Y \tag{4}$$

   where Y follows a non-central chi-squared distribution with k degrees of freedom and non-centrality parameter λ

$$k = \frac{4ab}{\sigma^2}, \quad \lambda = \frac{4ae^{-at}}{\sigma^2\big[1 - e^{-at}\big]}r(0) \tag{5}$$

   iv) The stationary distribution is a gamma where

$$r(\infty) \sim \mathrm{Gamma}\big(\alpha, \beta\big), \quad \alpha = \frac{2ab}{\sigma^2}, \quad \beta = \frac{\sigma^2}{2a}, \quad f_{r(\infty)}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta} \tag{6}$$

b) Now assume that the parameters of the CIR model are $r_0 = 0.032$, $a = 1.2$, $b = 0.048$ and $\sigma = 0.1$ Compute ZCB prices, continuously compounded spot rates and instantaneous forward rates in the CIR model.

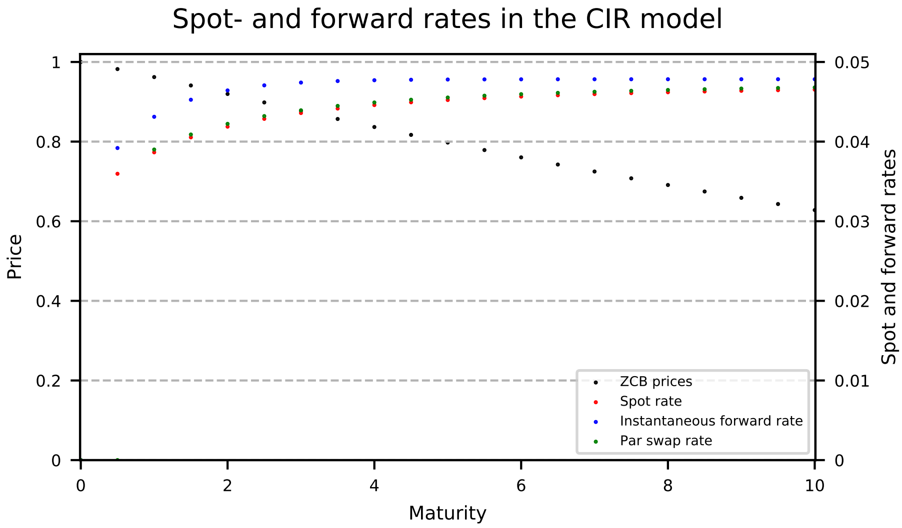
   i) Plot these ZCB prices, spot rates and instantaneous forward rates for values of t in $[0, 10]$.

   ii) Now compute spot rates for maturities $T \in \{0.5, 1, 1.5, ..., 10\}$ using the parameters given above and fit all the parameters in a CIR model to these spot rates. Are you able to recover the parameters of the CIR model as you should be? Are there any difficulties with any of the parameters?

   iii) Now try to fit the parameters this time assuming that you known that $\sigma = 0.1$. Are you able to fit the model parameters and is the fit now better? Explain what this tells you about σ in a CIR model.

Answers:

i) The plot of ZCB prices, spot rates and instantaneous forward rates are shown below.



ii) Fitting the CIR model to spot rates with the objective of recovering all four parameters, we discover that σ is difficult to recover implying that σ is not well-specified in this model. Despite our inability to recover σ, we are nonetheless able to recover the remaining parameters r_0, a and b quite well.

iii) If we fit the CIR model to spot rates assuming that $\sigma = 0.02$ is known the algorithm will converge very fast and the remaining parameters be returned with high accuracy illustrating once again that r_0, a and b are very will identified in the model.

c) We will now study interest rate swaps that involve swapping 6M floating rate payments (EURIBOR say) paid semiannually for fixed payments also paid semi-annually. Assume that present time is $t = 0$ and that the 6M floating rate to be paid at time $T = 0.5$ has just been announced. For simplicity, you can assume the notional of these swaps is just 1.

i) Compute the par swap rate of 10Y interest rate swap.

ii) Compute par swap rates for maturities $T \in \{1, 1.5, ..., 10\}$ and plot the par swap curve along side spot rates and 6M forward rates.

iii) Explain how 6M forward rates and par swap rates are related.

iv) You will notice that par swap rates and zero coupon spot rates are very close. Explain why that is so by appealing to concepts such as the accrual factor of a swap and the duration of all bonds in general.

Answers:

i) The 10Y par swap rate is 0.0468.

ii) The term structure of par swap rates are shown in the plot from i).

iii) Par swap rates are a weighted average of the 6M forward rates with more weight placed on forward rates over the near future.

iv) Par swap rates and zero coupon bond rates are relatively close because an interest rate swap is in nature quite close to a zero coupon bond. Now, a zero coupon bond band pays *no* coupons and has only one cashflow at the very end. The net coupons to an interest rate swap are however are also relatively small as they consist of the difference between the fixed par swap rate and the floating rate in this case 6M EURIBOR. Furthermore, since the fixed par swap rate is a weighted average of 6M forward rates, the average net coupon to an interest rate swap tends to be close to 0.

d) Now we will simulate short rates in the CIR model using the usual first order Euler scheme on a grid of mesh δ that runs from initial time $t_0 = 0$ to terminal time $T = 10$. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$ and the time points will be $[t_0, t_1, ..., t_{M-1}, t_M] = [0, \delta, 2\delta, ..., T-\delta, T]$ and $\delta = \frac{T}{M}$. The scheme you will need to implement is a simple Euler first-order scheme of the form

$$r_m = r_{m-1} + a(b - r_{m-1})\delta + \sigma\sqrt{r_{m-1}}\sqrt{\delta}Z_m, \qquad m = 1, 2, ..., M \tag{7}$$

where $Z_m \sim N(0,1)$, $m = 1, ..., M$ and all the standard normal random variables are independent.

i) Simulate one trajectory of the short rate and plot the trajectory up to time $T = 10$.

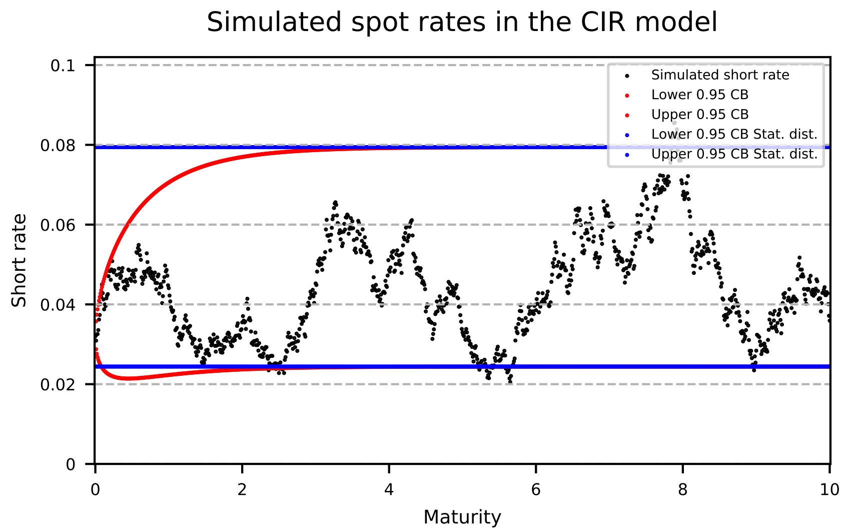
ii) Construct 95 percent two-sided confidence intervals for the short rate and plot these in the same plot.

iii) Construct a 95 percent two-sided confidence interval for the short rate under the stationary distribution and plot this confidence interval in the same plot. Based on the plot, can you say that the distribution of r_3 is roughly the same as that of the stationary distribution? How does this change if you change the parameters?

Answers:

i) A single simulated trajectory of the short rate can be seen in the plot below.

ii) In the same plot, you will also find two-sided 95 percent confidence intervals for the short rate also under the stationary distribution.



iii) The value r_3 in the simulation is the short three years after the initial point, and since the confidence interval of r_3 is close to that of the stationary distribution, we can conclude that after 3 years the short rate has most likely almost settled to its stationary distribution. Convergence to the stationary distribution will be faster if either a is large or σ is small.

e) We will now consider the pricing of an interest rate cap on future EURIBOR fixings that begins immediately and ends in $T = 5$ years. The cap will have a strike of R=0.045. We will price this derivative using simulation.

i) Explain how an interest rate cap is related to caplets and also explain, how a caplet can be seen as a type of European option and on what underlying.

ii) Deduce an expression for the discounted payoff of a caplet at time $t = 0$ on the underlying reference rate $L(T_{i-1}, T_i)$.

iii) Simulate at least $N = 1000$ trajectories for the short rate up to time $T = 5$ and in each simulation, at least $M = 1000$ steps should be taken. For each simulated path, compute the discounted payoffs to all caplets with a maturity less than 5 years. Once you have simulated N trajectories, you can compute the price of the caplets by averaging the discounted payoffs.

iv) Compute the price of the 5Y interest cap with a strike of $F = 0.045$ and discuss how the price of the cap depends on the strike R and σ.

v) Also compute the price of a 2Y3Y payer swaption with a strike of $K = 0.045$ by using a similar approach of computing discounted payoffs for each trajectory and averaging over all simulations.

Answers:

i) An interest rate caplet caps, for a cost, the interest payment on a single floating rate payment. An interest rate cap caps a series of floating rate payments. An interest rate cap can thus be seen as the sum of a series of caplets and the price of a cap can therefore be computed as the sum of the prices of the caplets making up the interest rate cap. A caplet can be seen as a European call option with a specific future LIBOR rate fixing as the underlying asset or equivalently as a European put option on the ZCB price $p(T_{i-1}, T_i)$ realized at T_{i-1}, the time of a the LIBOR rate announcement, for 1 unit of currency delivered at the time T_i of payment of the LIBOR rate.

ii) Assume that we have simulated the short rate from time $t = 0$ to time T_i using M_i steps and denote the realized values of the short rate by $r_0, r_1, ..., r_{M-1}$. The payoff of at caplet with strike R and underlying asset $L(T_{i-1}, T_i)$ discounted back to time $t = 0$, denoted $\tilde{\chi}_i$, is then

$$\tilde{\chi}_i = \exp\Big\{ -\delta \sum_{m=0}^{M_i-1} r_m \Big\} \cdot \big[1 + (T_i - T_{i-1})R\big] \Big(\frac{1}{1 + (T_i - T_{i-1})R} - p(T_{i-1}, T_i) \Big)_+ \tag{8}$$

iii) Simulating the short rate from time $t = 0$ to time $t = 5$ using $M = 5000$ steps, repeating the simulation $N = 20000$ times, computing the discounted payoff to each of the caplets using the above and averaging over the number of simulations, we get caplet prices C_i as given in the table below

Table 1: Caplet prices

T_i	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50	5.00
C_i(bps)	8.8101	16.7477	20.8772	23.0764	23.5194	24.0728	24.0097	23.5051	23.1399

iv) The price of the interest rate cap is simply the sum of the individual caplets and we get that the price of the cap becomes 0.01878 or 187.8 bps. The lower the strike the higher the price of the interest rate cap. Since an interest cap is a sum of caplets each of which are simply European options, the price of the cap is increasing in σ.

v) In this case, the discounted payoff becomes

$$\tilde{\chi} = \exp\Big\{ -\frac{T}{M} \sum_{m=0}^{M-1} r_m \Big\} \cdot S_{swap}\big(R_{swap} - K\big)_+ \tag{9}$$

Here, $M = 2000$ is the number of steps in the simulation up to time $T = 2$, S_{swap} is the accrual factor of the underlying 3Y swap in $T = 2$ years when the swaption expires and R_{swap} is the par swap rate of the then 3Y swap. As mentioned above, the simulation was repeated $N = 20000$ times and we get that the price of the swaption becomes 85.95 bps.