Fixed Income Derivatives - SABR Model Example

In the SABR model, the forward rate denoted F_t, typically a forward par swap rate, follows a stochastic process with stochastic volatility σ_t. The joint dynamics of F_t and σ_t

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta} dW_t^{(1)}, \quad F(0) = F_0 \\ d\sigma_t &= \upsilon \sigma_t dW_t^{(2)}, \quad \sigma(0) = \sigma_0 \\ dW_t^{(1)} dW_t^{(2)} &= \rho \end{aligned} \tag{1}$$

where $\sigma_0 > 0$, $0 < \beta \leq 1$, $\upsilon > 0$ and $-1 < \rho < 1$ are parameters of the model.

Zero coupon bond prices observed from market data are given in the below table.

Table 1: Zero coupon bond prices

T	0.50	1.00	1.50	2.00	2.50	3.00	3.50
$p(0,T)$	0.98429046	0.96633686	0.94690318	0.92655036	0.90568659	0.88460647	0.86352084
T	4.00	4.50	5.00	5.50	6.00	6.50	7.00
$p(0,T)$	0.84257919	0.82188628	0.80151436	0.78151217	0.76191149	0.74273188	0.72398415

Also, we can observe prices of 2Y5Y payer swaptions for a number of strike offsets.

Table 2: $2Y5Y$ Swaption prices

$K_{offset}(bp)$	-300	-250	-200	-150	-100	-50	ATMF
$\Pi_{swaption}$	0.12301549	0.10339456	0.08421278	0.06567338	0.04843543	0.03300976	0.02048677
$K_{offset}(bp)$	50	100	150	200	250	300	-
$\Pi_{swaption}$	0.01173834	0.00648577	0.00361682	0.00215934	0.00137503	0.00093634	-

Problem 1 - The Implied Volatility Smile

Compute the 2Y5Y par swap rate which serves as the underlying asset of the swaption by first computing the corresponding accrual factor S_{swap}

a) Report the $2Y5Y$ forward par swap rate.

b) Compute Black implied volatilities for all strikes and plot these as a function of K_{offset}.

c) Using the implied volatility plot discuss whether the market is pricing swaptions according to Black's model. If not, then what can be said about the distribution of the $2Y5Y$ forward par swap rate implied by the pricing measure chosen by the market? How does that distribution compare to the log normal distribution?

Problem 2 - Fitting the SABR model

We will now fit a SABR model to observed implied volatilities setting initial values $\tilde{\sigma}_0 = 0.055$, $\tilde{\beta} = 0.5$, $\tilde{\upsilon} = 0.48$ and $\tilde{\rho} = -0.25$ and the 'nelder-mead' algorithm.

a) Report the fitted parameter values $\hat{\sigma}_0$, $\hat{\beta}$, $\hat{\upsilon}$ and $\hat{\rho}$.

b) Plot the observed and fitted values of implied volatilities as a function of K_{offset}. Do your fitted values fit the data?

The forward rate, F_t, and the volatility σ_t can be simulated in the SABR model using a simple Euler scheme. Denote by M, the number of steps in the simulation and index the time points in the simulation by m, $m \in \{0, 1, 2, ..., M-1, M\}$ so that the time points will be $\big[t_0, t_1, ..., t_{M-1}, t_M\big] = \big[0, \delta, 2\delta, ..., T-\delta, T = \delta M\big]$ and hence the step in time will be of size $\delta = \frac{T}{M}$. The model can then be simulated using the following equations

$$\begin{aligned} F_m &= F_{m-1} + \sigma_{m-1} F_{m-1}^{\beta} \sqrt{\delta} Z_m^{(1)}, && F(0) = F_0 \\ \sigma_m &= \sigma_{m-1} + \upsilon \sigma_{m-1} \sqrt{\delta} \Big(\rho Z_m^{(1)} + \sqrt{1-\rho^2} Z_m^{(2)} \Big), && \sigma(0) = \sigma_0 \end{aligned} \tag{2}$$

where $Z_m^{(1)}$ and $Z_m^{(2)}$ are independent standard normal random variables.

Problem 3 - Simulating the forward rate in the SABR model

Simulate a single trajectory of the 2Y5Y forward par swap rate and the volatility.

a) Plot the trajectories of the forward par swap rate and the volatility.

b) Repeat the simulation for various values of the 'seed'. Does the volatility behave in strange and perhaps undesired way depend on the choice of the seed.

Problem 4 - Pricing a digital option

Now, we will compute the price of a digital option in the SABR model. The digital option will be one that pays 1 unit of currency if the 5Y spot swap rate in exactly two years exceeds the 2Y5Y forward par swap rate observed right now + 75 bps. So we have a strike $K = F_0 + 0.0075$

a) Which measure is the SABR model defined under? And how does that affect your ability to compute the price of a maturity $T = 2$ derivative such as the digital option considered here. What can you do if you have to price a derivative with a different maturity?

b) Find the payoff function $\chi(T)$ and the discounted payoff function for the digital option.

c) Using at least $M = 2000$ time steps and $N = 10000$ simulations, compute the price of the digital option.

Problem 5 Risk Management in the SABR model

Imagine you have a long position in the $K_{offset} = 150$ 2Y5Y payer swaption and you need to assess how you are exposed to changes in the SABR model parameters. Do this by computing the change in dollar value (DV) in the following cases.

a) If σ_0 falls by 0.001,

b) if the entire spot rate curve drops by 1 bp.