Fixed Income Derivatives E2024 - Problem Set Week 4

Problem 1

Suppose that S_t follows a geometric Brownian motion and has dynamics.

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t \\
S_0 &= s_0
\end{aligned} \tag{1}
$$

where μ and σ are constants. Also assume that there exists a risk-free bank account, B_t, with constant interest rate r and dynamics

$$
\begin{aligned}
dB_t &= r B_t dt \\
B_0 &= b_0
\end{aligned} \tag{2}
$$

a) Show that under the risk neutral measure, the dynamics of S_t must be

$$
\begin{aligned}
dS_t &= r S_t dt + \sigma S_t dW_t \\
S_0 &= s_0
\end{aligned} \tag{3}
$$

by showing that under this measure $\frac{S_t}{B_t}$ is a martingale.

b) Show that the solution $S(T)$ corresponding to the $\mathbb{Q}$-dynamics of S_t is

$$
S(T) = s_0 \exp\Big((r - \frac{1}{2}\sigma^2)T + \sigma W(T) \Big) \tag{4}
$$

Now define $X_t = \ln(S_t)$ and a European call option with contract function $C\big(S(T)\big) = max\big(S(T) - K, 0\big) = \big(S(T) - K, 0\big)_+$

c) Show that $X(T) \sim N\big(x_0 + (r - \frac{1}{2}\sigma^2)T, \sigma^2 T\big)$ under the risk neutral measure and hence that

$$
f_{X(T)}(x) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}} \exp\left(\frac{-\big(x - x_0 - \big(r - \frac{1}{2}\sigma^2\big)T\big)^2}{2\sigma^2 T} \right) \tag{5}
$$

is the probability density function of $X(T)$.

d) Argue that the time 0 price of the call option can be found from

$$
\begin{aligned}
C &= e^{-rT} \mathrm{E}^{\mathbb{Q}}\big[\max\big(S(T) - K, 0\big)\big] = e^{-rT} \int_{-\infty}^{\infty} \big(e^x - K\big)_+ f_{X(T)}(x) dx = e^{-rT} \int_{\ln K}^{\infty} (e^x - K) f_{X(T)}(x) dx \\
&= e^{-rT} \int_{\ln K}^{\infty} e^x f_{X(T)}(x) dx - K e^{-rT} \int_{\ln K}^{\infty} f_{X(T)}(x) dx = I_1 + I_2
\end{aligned} \tag{6}
$$

e) Show that this results in the Black-Scholes formula

$$
C = s_0 \Phi(d_1) - e^{-rT} K \Phi(d_2), \quad d_1 = \frac{\ln\big(\frac{s_0}{K}\big) + \big(r + \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln\big(\frac{s_0}{K}\big) + \big(r - \frac{1}{2}\sigma^2\big)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{7}
$$

where $\Phi()$ is the cumulated distribution function of a standard normal random variable. *Hint: To compute these integrals, you will need to perform a substitution so that you work with the PDF of a standard normal random variable, you will need to complete a square, use that the integral of a density function from $-\infty$ to ∞ equals 1 and you will need to use that for $Z \sim N(0,1)$ we have $P(Z < a) = 1 - P(Z > a)$.*

Problem 2

In this problem, we will consider the Vasicek model for the short rate r_t with dynamics given by

$$\begin{aligned} dr_t &= (b - ar_t)dt + \sigma dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{8}$$

We know that the distribution of r_T is given by

$$r_T \sim N\Big(e^{-aT}r(0) + \frac{b}{a}(1 - e^{-aT}), \frac{\sigma^2}{2a}\big[1 - e^{-2aT}\big]\Big) \tag{9}$$

a) Write a function in Python that takes T, r_0, a, b and σ and a confidence level α as inputs and returns the lower-, upper- or two-sided confidence bounds of r_T.

b) Plot the two-sided confidence bounds for $\alpha = 0.05$ and appropriately many choices of $T < 10$ setting $r_0 = 0.04$, $a = 2$, $b = 0.1$, $\sigma = 0.02$. Also include the two-sided confidence bounds under the stationary distribution in your plot.

c) For combinations of $a = [1, 2, 4, 8]$ and b such that $\frac{b}{a} = 0.05$ (That is, as you are changing a also change b to keep $\frac{b}{a} = 0.05$), and $\sigma = [0.01, 0.02, 0.03, 0.04]$, redo the plot from b) for a sufficiently large T. For each of the combinations of parameters, assess how large T must be for r_T to have settled to it's stationary distribution. How does the rate at which r_T settles to it's stationary distribution depend on a and σ?

In the following, you will simulate the short rate on a grid of mesh δ that runs from initial time $t_0 = 0$ to some terminal time T. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$, the time points will be $\big[t_0, t_1, ..., t_{M-1}, t_M\big] = \big[0, \delta, 2\delta, ..., T-\delta, T\big]$ and $\delta = \frac{T}{M}$. In the following, please consider the following three schemes

i) An Euler scheme

ii) A Milstein scheme

iii) An exact scheme

d) Derive the difference equation for the short rate for each of the three schemes in terms of a standard normal variable denoted Z_m drawn in each of the steps. Are some of the schemes equivalent? Which of the three schemes do you expect to be more accurate?

e) Write a python function that take as inputs T, M, r_0, a, b, σ, and "*scheme*", and returns a simulated trajectory of the short rate. Plot a single trajectory for each of the three schemes setting $r_0 = 0.04$, $a = 2$, $b = 0.1$, $\sigma = 0.02$ up to time $T = 10$ and for $M = 10,000$. Include the lower, upper and two-sided confidence bounds in your plot for a choice of $\alpha = 0.1$.

f) Now set $T = 3$, repeat the simulations N times and denote the value of the short rate at $T = 3$ in the n'th simulation by r_{3n}, $n = 1, 2, ..., N$. Construct at least 50 but ideally more equally spaced bins to cover the range of r_{3n} from the smallest to the largest value. Sort your simulated values into these bins and use the proportion in each bin to construct an empirical probability mass function. Plot the empirical mass function and the theoretical mass function for $N = 1,000$ and $M = 1,000$.

g) Now, we will investigate how the difference between the empirical and theoretical PMF's depend on M and N. For a choice of 100 bins and combinations of values of M in [4000,6000,8000,10000] and N in [4000,6000,8000,10000], compute the total square difference between the theoretical probabilities and empirical frequencies across the 100 bins. Report the total squared differences for all combinations of M and N, and for all three schemes. Compare the accuracy of the three schemes and try to assess how large M and N need to be in each of the three cases to arrive at a reasonable accuracy.

Problem 3

In this problem, we will consider the CIR model for the short rate r_t with dynamics given by

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{10}$$

We know that $r_T|r_0$ is equal in distribution to

$$\frac{\sigma^2}{4b}\left[1 - e^{-bT}\right]Y \tag{11}$$

where Y follows a non-central chi-squared distribution with k degrees of freedom and non-centrality parameter λ

$$k = \frac{4ab}{\sigma^2}, \quad \lambda = \frac{4ae^{-aT}}{\sigma^2\left[1 - e^{-aT}\right]}r_0 \tag{12}$$

The stationary distribution of the short rate is a gamma where

$$r_\infty \sim \text{Gamma}(\alpha, \beta), \quad \alpha = \frac{2ab}{\sigma^2}, \quad \beta = \frac{\sigma^2}{2a}, \quad f_{r_\infty}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}, \; x > 0$$

a) Write a function in Python that takes T, r_0, a, b and σ and a confidence level α as inputs and returns the lower-, upper- or two-sided confidence bounds of r_T.

b) Plot the two-sided confidence bounds for $\alpha = 0.05$ and appropriately many choices of $T < 10$ setting $r_0 = 0.04$, $a = 2$, $b = 0.05$, $\sigma = 0.1$. Also include the two-sided confidence bounds under the stationary distribution in your plot.

c) For combinations of $a = [1, 2, 4, 8]$, b fixed at 0.05, $\sigma = [0.05, 0.1, 0.15, 0.2]$, redo the plot from b) for a sufficiently large T. For each of the combinations of parameters, assess how large T must be for r_T to have settled to it's stationary distribution. How does the rate at which r_T settles to it's stationary distribution depend on a and σ?

In the following, you will simulate the short rate on a grid of mesh δ that runs from initial time $t_0 = 0$ to some terminal time T. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M - 1, M$, the time points will be $\left[t_0, t_1, ..., t_{M-1}, t_M\right] = \left[0, \delta, 2\delta, ..., T - \delta, T\right]$ and $\delta = \frac{T}{M}$. In the following, please consider the following three schemes

i) An Euler scheme

ii) A Milstein scheme

iii) An exact scheme

d) Derive the difference equation for the short rate for each of the three schemes (if it is possible!) in terms of a standard normal variable denoted Z_m drawn in each of the steps. Are some of the schemes equivalent? Which of the three schemes do you expect to be more accurate?

e) Write a python function that take as inputs T, M, r_0, a, b, σ, and "*scheme*", and returns a simulated trajectory of the short rate. Plot a single trajectory setting $r_0 = 0.04$, $a = 2$, $b = 0.05$, $\sigma = 0.1$ for each of the three schemes and include the lower, upper and two-sided confidence bounds in your plot for a choice of $\alpha = 0.1$.

f) Now set $T = 3$, repeat the simulations N times and denote the value of the short rate at $T = 3$ in the n'th simulation by r_{3n}, $n = 1, 2, ..., N$. Construct at least 50 but ideally more equally spaced bins to cover the range of r_{3n} from the smallest to the largest value. Sort your simulated values into these bins and use the proportion in each bin to construct an empirical probability mass function. Plot the empirical mass function and the theoretical mass function for $N = 10,000$ and $M = 10,000$.

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