Fixed Income Derivatives Final Exam Fall 2024(18.01.2025)

Imagine that you are working for a major financial institution and that you have been approached by a client who has a 10 year floating rate loan on which he pays 6M Euribor semi-annually plus a spread. The client is worried that future interest rates will be high and that his future loan payments will rise beyond what is acceptable to him. For simplicity and without loss of generality, you will work as if the notional of the clients loan obligation is one Euro and all your answers will be per 1 Euro of total debt. Your task will be to present to the client three different solutions to manage his interest rate risk and weigh the pros and cons of each of the three solutions. The three different solutions you will offer are:

1) To enter into a 10Y payer interest rate swap in which the client receives 6M floating Euribor and pays a fixed rate instead. That way, the client will turn his future unknown floating rate payments into a known fixed rate.

2) To construct a 10Y interest rate cap with a strike of $K = 0.06$ to prevent that the clients future floating rate payments will exceed the strike of $K = 0.06$.

3) To buy a 3Y7Y payer swaption with a strike as close to $K = 0.06$ as possible so that if interest rates have risen at the time of exercise in 3 years, the client can enter into the underlying 7Y payer interest rate swap and convert his floating rate payments into fixed rate payments of size equal to the strike of the swaption.

In Problem 1, we will examine the first of these three options, in Problem 2 the second, in Problem 3 the third. In Problem 4, you will compare the three different ways of managing interest rate risk and assess the pros and cons of each of the three. Problem 5 is independent of the first four problems.

For problems 1, 2 and 3, we will use market data that includes the 6M Euribor fixing recently announced, Euribor Forward Rate Agreements and Euribor denominated interest rate swaps constructed at the exact same time as the Euribor rate announcement. The swap rates represent the par swap rates for a range of swap agreements with different maturities in which the floating leg pays 6M Euribor and the fixed leg pays an annual fixed rate. All interest rate swaps throughout this exam pay 6M floating Euribor against a fixed rate paid annually. The Euribor rate, the FRA rates and par swap rates are all, as is usually the case, to be understood as "simple" interest rates. The data for these rates is shown in the table below.

Table 1: Euribor, FRA and Swap Market Data

EURIBOR	Fixing	FRA	Midquote	IRS	Midquote
6M	0.03772	1X7	0.04026	2Y	0.05228
		2X8	0.04261	3Y	0.05602
		3X9	0.04477	4Y	0.05755
		4X10	0.04677	5Y	0.05791
		5X11	0.04860	7Y	0.05718
		6X12	0.05029	10Y	0.05539
		7X13	0.05183	15Y	0.05324
		8X14	0.05324	20Y	0.05205
		9X15	0.05452	30Y	0.05087

In addition, we also observe market prices for caplets on 6M forward Euribor rates for a strike of $K = 0.055$ and a notional of 1 Euro. The caplet price on the future Euribor rate $L(T_{i-1}, T_i)$ announced at time T_{i-1} is denoted by π_i and the payment on the caplet contract thus occurs at time T_i. Caplet prices are given in basispoints, so to transfer to monetary units, simply divide by 10,000.

Table 2: Caplet Market Prices

T_i	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
π_i(bps.)	-	3.5920	19.2679	32.1887	37.2136	36.4750	32.2678	26.9031	21.2176	16.2022
T_i	5.5	6	6.5	7	7.5	8	8.5	9	9.5	10
π_i(bps.)	12.0628	8.8952	6.5191	4.8435	3.6485	2.8098	2.2067	1.7814	1.4707	1.2443

Finally, we also have data for the Black implied volatility of 3Y7Y payer swaptions for a range of different strikes. These swaptions give the owner the right to enter into a 7Y payer swap at a strike K exactly 3 years from now. The ATMF is simply the 3Y7Y par swap rate observed right now and the offsets, denoted K_{offset} and measured in basispoints, are relative to the 3Y7Y forward par swap rate.

Table 3: $3Y7Y$ Swaption Market Prices

$K_{offset}(bp)$	-300	-250	-200	-150	-100	-50	ATMF
$\Pi_{swaption}$	0.220675	0.183310	0.155103	0.129001	0.108120	0.084411	0.071866
$K_{offset}(bp)$	50	100	150	200	250	300	-
$\Pi_{swaption}$	0.066535	0.073942	0.082751	0.093605	0.098971	0.108909	-

At the end of this document, a Python script is presented containing the data from the tables above. You do not have to use this piece of code but it might save you some time.

Problem 1

We will begin by fitting a term structure of continuously compounded zero coupon bond rates to the market data presented in Table 1 above. Please perform the fit and answer the following questions:

a) Plot the term structures of spot- and forward rates from your fit for all maturities up to 30 years and also report the 6M, 1Y, 2Y, 5Y, 10Y, 15Y, 20Y and 30Y continuously compounded spot rates from your fit. Does your fit match the data? If yes, provide evidence that this is so for example by reporting the SSE(Sum-of-Squared-Errors) of the fit.

b) Discuss which properties the term structures of spot- and forward rates should have and also discuss if your fit has those properties. Also explain which type of curve you have used when fitting the market data and why you chose that type of fit.

c) Compute 6M forward Euribor rates $L(0, T_{i-1}, T_i)$ up to $T_i = 10$ and the 10Y par swap rate. Plot the term structure of 6M forward Libor rates and the 10Y par swap rate as a horizontal line in a separate plot. Report the 10Y par swap rate and explain how it is related to 6M forward Euribor rates?

Problem 2

Now, we will focus on pricing the strike $K = 0.06$ 10Y interest rate cap that the investor considers as a tool to protect himself against interest rate increases. We will do so using two different models. First, we will use the Vasicek model in which the dynamics of the short rate r_t are given by

$$dr_t = (b - ar_t)dt + \sigma dW_t, \qquad r(0) = r_0 \tag{1}$$

where $a > 0$, $b > 0$ and σ are all constant parameters and W_t is a Brownian motion. Second, we will use the Hull-White Extended Vasicek(HWEV) model in which the dynamics of the short rate are

$$dr_t = \big(\Theta(t) - ar_t\big)dt + \sigma dW_t, \qquad r(0) = r_0 \tag{2}$$

where $a > 0$ and σ are constant parameters, $\Theta(t)$ is a deterministic function and W_t is a Brownian motion.

a) Set $\sigma = 0.02$ and use the initial values $r_0 = 0.035$, $a = 6$ and $b = 0.25$ to fit a Vasicek model to the ZCB spot rates you found in Problem 1 and answer the following questions:

   i) Report the parameters of the Vasicek model from your fit and plot the term structure of spot rates implied by the fitted parameters along with the term structure of spot rates you found in Problem 1a. Is the Vasicek model able to fit the term structure of market spot rates?

   ii) Could it be expected that a Vasicek model is able to fit this particular term structure of observed spot rates? Please explain your reasoning.

   iii) Suggest another model that might produce a better fit and explain why you expect that method to work better.

b) Now we will fit the Hull-White Extended Vasicek (HWEV) model to the prices of caplets for a strike $K = 0.055$ given in Table 2. That is, please perform the following steps:

   i) Write a function that takes as its first argument the parameters a and σ and returns the SSE between observed caplet prices from the table above and computed caplet prices as a function of a and σ.

   ii) Using an initial values for a of 2.5 and an initial value for σ of 0.018, fit the HWEV model to caplet prices using ”nelder-mead” and report the fitted values $\hat{a}$ and $\hat{\sigma}$ you obtain.

   iii) Plot the term structure of caplet Black implied volatility corresponding to both market- and fitted caplet prices in the same plot and use this plot to assess if the HWEV model is able to fit observed caplet prices well.

   iv) Now that we have fitted the HWEV model to caplet prices, how do we ensure that the model also fits ZCB spot rates and do we even have to? Please explain your reasoning.

c) Next, we will simulate trajectories of the short rate in both the Vasicek model and in the HWEV model with the parameters you obtained when fitting the models. If for some reason you have been unable to fit one or both of the models, simply proceed assuming the values of the parameters that were given as the initial parameter values.

   i) Simulate the short rate in both the Vasicek and HWEV model by taking $M = 500$ steps for values of t from $t = 0$ to $t = 10$ so that the step-size is $\delta = 0.02$. Plot the two trajectories in separate plots in which you also include the mean and a 95% two-sided confidence interval.

   ii) Interpret the behavior of the short rate in these two models and relate your findings to the parameters of the models as well as the initial fit of the ZCB spot rates you obtained in 1a.

   iii) Briefly explain which of the two models you think fits the market data best and which is more realistic. Please also explain which of the two models is more likely to produce a reliable price of a 10Y caplet with a strike that is different from the one for which we have caplet price data.

d) Finally, we will compute the price of the 10Y interest rate cap with a strike of $K = 0.06$ on a series of 6M Euribor payments using the model you found to be the best in problem 2c. Please explain how you compute first caplet prices and then the cap price. Also, explain which parameter values you will use. Note that the caplet with a maturity of T_i, pays the amount $\big(L(T_{i-1}, T_i) - K\big)_+$ at time T_i where $L(T_{i-1}, T_i)$ is the Euribor fixing announced 6 months prior at time T_{i-1}.

i) Compute prices of 6M Euribor caplets for maturities ranging from $T_i = 1$ to $T_i = 10$ and report these for $T_i \in [1, 2, 4, 6, 8, 10]$

ii) Find the price of a 10Y interest rate cap with a strike of $K = 0.06$ on 6M Euribor that begins as early as possible and ends in exactly $T = 10$ years. Report the price in basispoints, both as an amount paid upfront as well as spread on top of regular semi-annual interest rate payments.

Problem 3

Next, we will focus on the 3Y7Y payer swaption that the investor also considers as a possible tool to protect himself against interest rate increases. The underlying asset of the 3Y7Y payer swaption is the 3Y7Y forward payer swap and we will model the forward 3Y7Y par swap rate F_t and its volatility σ_t using the SABR model. That is, we assume that F_t and σ_t have the following joint dynamics

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta} dW_t^{(1)}, \qquad & F(0) = F_0 \\ d\sigma_t &= \upsilon \sigma_t dW_t^{(2)}, \qquad & \sigma(0) = \sigma_0 \\ dW_t^{(1)} dW_t^{(2)} &= \rho dt \end{aligned} \tag{3}$$

where $0 \leq \beta \leq 1$, $0 < \upsilon$ and $-1 < \rho < 1$ are constants and $W_t^{(1)}$ and $W_t^{(2)}$ are correlated Brownian motions with correlation coefficient ρ. Also recall that we have data for market implied volatilities from market prices of 3Y7Y payer swaptions in Table 3.

a) First, we examine the market implied volatilities of swaption prices and find the price of the swaption that the client considers as a tool to manage interest rate risk.

   i) Plot the market implied volatilities of the payer swaptions as a function of their strike. Interpret the plot and in particular assess if the shape of the graph is common for market implied volatilities. What does the implied volatility curve tell us about the distribution of the underlying par swap rate implied by swaption prices.

   ii) Find the ATMF forward rate corresponding to the 3Y7Y payer swaption. That is, simply find the 3Y7Y forward par swap rate.

   iii) Find the price of a the 3Y7Y payer swaption that has a strike closest to $K = 0.06$. Report the price of this swaption in basispoints as well as its exact strike. This particular swaption will be the one that the client will consider as a tool to hedge against rising interest rates.

b) Assuming we know that $\beta = 0.55$, fit the SABR model to the 3Y7Y payer swaption market implied volatilities given in Table 3 and solve the following questions:

   i) Report the fitted parameter values of σ_0, υ and ρ and plot fitted implied volatilities in the same plot as implied volatilities from observed market prices.

   ii) Do fitted implied volatilities match market implied volatilities? Report the SSE of your fit.

c) Now we will consider a so called strangle consisting of a long position in one ATMF + 100 basispoints payer swaption and a long position in one ATMF - 100 basispoints receiver swaption. We will use the parameter values from the fit of the SABR model. If you were not able to get sensible parameter values when fitting the SABR model, simply use the initial values suggested in Problem 3b.

   i) Argue that a payer swaption can be seen as a call option on the par swap rate of the underlying interest rate swap and that a receiver swaption can be seen as a put option on the par swap rate of the underlying interest rate swap.

   ii) Find the value of the strangle in the SABR model using the fitted parameter values. Bump the initial value F_0 of underlying swap rate by one basispoint up and down and report the loss/gain to the value of the strangle in both cases. Also bump the initial volatility σ_0 up and down by 0.001 and report the loss/gain in both cases.

   iii) Based on your answer to ii) assess the nature of the exposure you get from a strangle. Is the strangle very sensitive to changes in the forward par swap rate and what is the direction of the exposure? Also, is the strangle very sensitive to changes in volatility and what is the direction of the exposure? Please provide some intuition for your conclusions.

Problem 4

Finally, in this problem we will compare the three options the client is offered to manage the risk of his 10Y floating rate obligation on which he must pay 6M Euribor semi-annually.

a) We will begin by reiterating the results we have for the three different options offered to the client.

   i) Option 1 was to enter into a 10Y payer swap immediately. Report the 10Y par swap rate.

   ii) Option 2 was to buy a 10Y interest rate cap with a strike of $K = 0.06$. Report the price of the interest rate cap both in terms of an upfront payment and in terms of a spread paid on top of the clients regular semi-annual payments.

   iii) Option 3 was to buy a 3Y7Y payer swaption with a strike as close to $K = 0.06$ as possible. Report the price and strike of the 3Y7Y payer swaption.

b) Imagine that the client will choose exactly one of the three options 1, 2 or 3 and consider the total payments implied by each of the three options. Please describe which future evolution of interest rates that would *minimize* the total payments of the client and how low these payments would they be in this scenario. Also, please describe which future evolution of interest rates that would *maximize* the total payments of the client and how high these payments would they be in this scenario. Please do so for

   i) The 10Y interest rate swap

   ii) The 10Y interest rate cap

   iii) The 3Y7Y payer swaption

c) Explain which of the three options you deem the most risky and which do you deem the safest. In particular, explain which of the three options that gives the client exposure to the worst outcome and outline this outcome.

d) The 10Y interest rate swap costs nothing upfront and eliminates all future interest rate uncertainty. Options 2 and 3, on the other hand, involve an upfront payment, possibly spread out over the future. Yet, these options do not eliminate all uncertainty about future interest rate payments. Explain why, nonetheless, options 2 and 3 are valid alternatives and that it is fair that these options come at a cost.

Problem 5

We will now consider a model in which the short rate at time t denoted r_t is driven by two factors X_t and Y_t. The dynamics of the two factors are given by

$$\begin{aligned} dX_t &= -\gamma X_t dt + \phi dW_t^{(1)}, & X_0 &= x_0, \\ dY_t &= \big(b - aY_t\big)dt + \sigma dW_t^{(2)}, & Y_0 &= y_0, \end{aligned} \tag{4}$$

where $\gamma > 0$, $\phi \in \mathbb{R}$, $a > 0$, $b > 0$ and $\sigma \in \mathbb{R}$ are known constants, the initial values x_0 and y_0 are known, and $W_t^{(1)}$ and $W_t^{(2)}$ are independent Brownian motions. The short rate for all $t \geq 0$ is the sum of the two factors.

$$r_t = X_t + Y_t \text{ for } t > 0 \quad \text{and} \quad r_0 = x_0 + y_0 \text{ for } t = 0 \tag{5}$$

a) Consider the function

$$f(t, X, Y) = e^{-\gamma(T-t)}X + e^{-a(T-t)}Y \tag{6}$$

for $T > 0$ fixed where X and Y have dynamics as given in (4). Use the function $f(t, X, Y)$ to find a solution for $r_T|x_0, y_0$. Alternatively, carefully solve the SDE's for X_t and Y_t individually to find expressions for $X_T|x_0$ and $Y_T|y_0$ and argue why you can use these to find the solution for $r_T|x_0, y_0$.

b) Show that the distribution of r_T given x_0 and y_0 is Gaussian, $r_T|x_0, y_0 \sim N\big(M, V\big)$, with mean $M = M(T; x_0, y_0)$ and variance $V = V(T)$ where

$$\begin{aligned} M(T; x_0, y_0) &= x_0 e^{-\gamma T} + y_0 e^{-aT} + \frac{b}{a}\Big[1 - e^{-aT}\Big] \\ V(T) &= \frac{\phi^2}{2\gamma}\Big[1 - e^{-2\gamma T}\Big] + \frac{\sigma^2}{2a}\Big[1 - e^{-2aT}\Big] \end{aligned} \tag{7}$$

and also give the stationary distribution of r_∞.

c) Now assume that $x_0 = 0$, $\gamma = 32$, $\phi = 0.03$, $y_0 = 0.03$, $a = 0.5$, $b = 0.025$, $\sigma = 0.015$ and simulate one trajectory of each of X_t, Y_t and r_t using $N = 100$ steps from time $t = 0$ to $t = 1$ such that the step-size is $\delta = 0.01$ and plot the three trajectories along with the mean of r_t and a 95 per cent two-sided confidence interval for r_t. Also, simulate one trajectory of each of X_t, Y_t and r_t using $N = 1000$ steps from time $t = 0$ to $t = 10$ such that the step-size is still $\delta = 0.01$. In a separate plot show these three trajectories of X_t, Y_t and r_t along with the mean of r_t and a 95 per cent two-sided confidence interval for r_t. Report the two-sided upper- and lower confidence bounds for both $T = 1$ and $T = 10$.

d) Please briefly explain the role that X_t and Y_t play in the overall dynamics of the short rate r_t based on the parameter-values of the processes of X_t and Y_t as well as the plots of their simulated trajectories. In doing so, please consider the following questions:

   i) Which of the two factors mean-revert the fastest and how is your answer related to the relevant parameter(s) from the dynamics of X_t and Y_t in (4)

   ii) Explain how the fluctuations in X_t and Y_t influence the fluctuations in r_t differently and relate your answer to i) above.

   iii) Determine roughly when r_T settles to its stationary distribution and relate your answer to your answer to i) and ii) above.

A Python Code

```
import numpy as np
from scipy.stats import norm
from scipy.optimize import minimize
import fixed_income_derivatives_E2024 as fid
import matplotlib.pyplot as plt

alpha_caplet = 0.5
N_caplet = 21
T_caplet = np.array([i*alpha_caplet for i in range(0,N_caplet)])
strike_caplet_market = 0.055
price_caplet_market = np.array([0, 0, 3.592, 19.2679, 32.1887, 37.2136, 36.475, 32.2678, 26.9031, 21.2176, 16.2022, 12.0628,
8.8952, 6.5191, 4.8435, 3.6485, 2.8098, 2.2067, 1.7814, 1.4707, 1.2443])
price_caplet_market = price_caplet_market/10000

K_swaption_offset = np.array([-300,-250,-200,-150,-100,-50,0,50,100,150,200,250,300])
iv_swaption_market = np.array([0.220675, 0.18331, 0.155103, 0.129001, 0.10812, 0.084411, 0.071866, 0.066535, 0.073942, 0.082751, 0.093605, 0.098971, 0.108909])

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fra_market = [{"id": 1,"instrument": "fra","exercise": 1/12,"maturity": 7/12, "rate": 0.04026},
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swap_market = [{"id": 10,"instrument": "swap","maturity": 2, "rate": 0.05228, "float_freq": "semiannual", "fixed_freq": "annual","indices": []},
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{"id": 18,"instrument": "swap","maturity": 30, "rate": 0.05087, "float_freq": "semiannual", "fixed_freq": "annual","indices": []}]
data_zcb = EURIBOR_fixing + fra_market + swap_market
```