Fixed Income Derivatives - Vasicek Exam Preparation

In the Vasicek model, the short rate r_t is assumed to have the following dynamics under the risk-neutral measure $\mathbb{Q}$

$$dr_t = (b - ar_t)dt + \sigma dW_t \tag{1}$$

where a, $b > 0$ and σ are model parameters.

a) Solution and distribution of $r(t)$.

   i) Find the solution for $r(t)$, $t > 0$, in the Vasicek model.

   ii) Find the mean $\mathrm{E}[r(t)]$ and the variance $\mathrm{Var}[r(t)]$ of $r(t)$. There is a 'trick' involved when computing the variance, what is that trick called?

   iii) Find the distribution of $r(t)$.

   iv) Find the distribution of $r(t)$ as $t \nearrow \infty$. That is, find the stationary distribution of $r(\infty)$.

b) Now assume that the parameters of the Vasicek model are $r_0 = 0.02$, $a = 0.6$, $b = 0.03$ and $\sigma = 0.02$. Compute ZCB prices, continuously compounded spot rates and instantaneous forward rates in the Vasicek model.

   i) Plot the ZCB prices, spot rates and instantaneous forward rates for values of t in $[0, 10]$.

   ii) Now compute spot rates for maturities $T \in \{0.5, 1, 1.5, ..., 10\}$ using the parameters given above and fit all the parameters in a Vasicek model to these spot rates. Are you able to recover the parameters of the Vasicek model as you should be? Are there any difficulties with any of the parameters?

   iii) Now try to fit the parameters again, this time assuming that you known that $\sigma = 0.02$. Are you able to fit the model parameters and is the fit now better? Explain what this tells you about σ in a Vasicek model.

c) We will now study interest rate swaps that involve swapping 6M floating rate payments (EURIBOR say) paid semiannually for fixed payments also paid semi-annually. Assume that present time is $t = 0$ and that the 6M floating rate to be paid at time $T = 0.5$ has just been announced. For simplicity, you can assume the notional of these swaps is just 1.

   i) Compute the par swap rate of 10Y interest rate swap.

   ii) Compute par swap rates for maturities $T \in \{1, 1.5, ..., 10\}$ and plot the par swap curve along side spot rates and 6M forward rates.

   iii) Explain how 6M forward rates and par swap rates are related.

   iv) You will notice that par swap rates and zero coupon spot rates are very close. Explain why that is so by appealing to concepts such as the accrual factor of a swap and the duration of all bonds in general.

d) We will now consider the pricing of an interest rate cap with a strike of $R = 0.05$.

   i) Explain how an interest rate cap is related to caplets and also explain, how a caplet can be seen as a type of European option and on what underlying.

   ii) Compute the prices of a cap that begins as early as possible and ends in exactly 10 years.

   iii) Investigate how the price of the cap depends on σ and compute the DV01 of changing σ by 0.001 both up and down. Is the $DV01$ you have computed a first-order approximation or is it exact? Can both exact and approximated values be computed?

e) Now we will simulate short rates in the Vasicek model using the usual first order Euler scheme on a grid of mesh δ that runs from initial time $t_0 = 0$ to terminal time $T = 10$. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$, the time points will be $[t_0, t_1, ..., t_{M-1}, t_M] = [0, \delta, 2\delta, ..., T-\delta, T]$ and $\delta = \frac{T}{M}$. The scheme you will need to implement is a simple Euler first-order scheme of the form

$$r_m = r_{m-1} + (b - a r_{m-1})\delta + \sigma\sqrt{\delta} Z_m, \quad m = 1, 2, ..., M \tag{2}$$

where $Z_m \sim N(0,1)$, $m = 1, ..., M$ and all the standard normal random variables are independent.

i) Simulate one trajectory of the short rate and plot the trajectory up to time $T = 10$.

ii) Construct 95 percent two-sided confidence intervals for the short rate and plot these in the same plot.

iii) Construct a 95 percent two-sided confidence interval for the short rate under the stationary distribution and plot this confidence interval in the same plot. Based on the plot, can you say that the distribution of r_3 is roughly the same as that of the stationary distribution? How does this change if you change the parameters of the Vasicek model?

f) We will now introduce a 2Y8Y payer swaption with a strike of $K = 0.045$. That is, we will introduce a swaption that gives the owner the right but not obligation to enter into a 8Y swap at exercise in $T_n = 2$ years so that we have $T_N = 10$. To compute the price of this swaption, you will need to use simulation.

i) Argue that the payoff function $\chi(T_n)$ and the discounted payoff function $\tilde{\chi}(T_n)$ of the payer swaption are.

$$\begin{aligned}
\chi(T_n) &= S_n^N(T_n)\big(R_n^N(T_n) - K\big)_+ \\
\tilde{\chi}(T_n) &= \exp\Big\{ -\int_0^{T_n} r_t dt \Big\} S_n^N(T_n)\big(R_n^N(T_n) - K\big)_+
\end{aligned} \tag{3}$$

ii) Find a method to compute the price at $t = 0$ of the swaption by simulating at least $L = 1000$ trajectories and having at least $M = 1000$ steps in your simulation.

iii) Investigate if the price you have computed is accurate by plotting the value of the derivative for various choices of L.

iv) Explain how the price of the swaption depends on σ, T_n, T_N and of course K.