Fixed Income Derivatives - Problem Set Week 6

Problem 1

Consider the Hull-White model where the short rate r_t has dynamics

$$dr_t = \big[\Theta(t) - ar_t\big]dt + \sigma dW_t \tag{1}$$

a) Argue that ZCB prices are of the form

$$p(t,T) = e^{A(t,T)-B(t,T)r_t} \tag{2}$$

where

$$\begin{aligned} A(t,T) &= \int_t^T \Big[\frac{1}{2}\sigma^2 B^2(s,T) - \Theta(s)B(s,T)\Big]ds \\ B(t,T) &= \frac{1}{a}\Big[1 - e^{-a(T-t)}\Big] \end{aligned} \tag{3}$$

b) Show that forward rates $f(t,T)$ are of the form

$$f(t,T) = -\frac{\partial}{\partial T}A(t,T) + r_t\frac{\partial}{\partial T}B(t,T) \tag{4}$$

c) Argue that the forward rate dynamics can be found from

$$df(t,T) = -\frac{\partial}{\partial T}\Big(A_t(t,T)dt - B_t(t,T)r_t dt - B(t,T)dr_t\Big) \tag{5}$$

where $A_t(t,T) = \frac{\partial}{\partial t}A(t,T)$

d) Show that the forward rate dynamics are

$$df(t,T) = \frac{\sigma^2}{a}e^{-a(T-t)}\Big[1 - e^{-a(T-t)}\Big]dt + \sigma e^{-a(T-t)}dW_t \tag{6}$$

Now, we will find the forward rate dynamics in a different way. Let us recall that in the Hull-White model, zero coupon bond prices become

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\Big\{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)\big(1 - e^{-2at}\big) - B(t,T)r_t\Big\} \tag{7}$$

e) Use the above expression to find an expression for forward rates and treat this expression as a function $g = g(t,T,r)$.

f) Show from $g(t,T,r)$ that the forward rate dynamics are of the form

$$df(t,T) = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW_t \tag{8}$$

where $\alpha(t,T)$ is yet to be determined.

g) Use the HJM drift condition to find $\alpha(t,T)$ and thus show that it is of the same form as in d).

Problem 2

Take as given an HJM model under the risk neutral measure $\mathbb{Q}$ where

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t \tag{9}$$

a) Show that all forward rates and also the short rate are normally distributed.

b) Show that zero coupon bond prices are log-normally distributed.

Problem 3

Consider the Ho-Lee model where the short rate has dynamics

$$dr_t = \Theta(t)dt + \sigma dW_t. \tag{10}$$

Recall that ZCB prices in the Ho-Lee model are given by

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\Big\{(T-t)f^*(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r\Big\}. \tag{11}$$

a) Use the procedure outlined in Problem 1e-1g to find the forward rate dynamics in this model.

b) Find the dynamics of zero coupon bond prices under $\mathbb{Q}$.

c) Find the distribution of forward rates in this model and argue that they are Gaussian.

d) Use a result from the chapter 'Change of Numeraire' in Bjork to directly compute the time t price of a European call option with exercise date $T_1 > t$ on a maturity $T_2 > T_1$ zero coupon bond.

Problem 4

Take as given an HJM model under the risk neutral measure $\mathbb{Q}$ of the form

$$df(t,T) = \alpha(t,T)dt + \sigma_1(T-t)dW_{1t} + \sigma_2 e^{-a(T-t)}dW_{2t} \tag{12}$$

a) Use the HJM drift condition to find $\alpha(t,T)$.

b) Find the dynamics of zero coupon bond prices under $\mathbb{Q}$.

c) Find the distribution of forward rates in this model and argue that they are Gaussian.

d) Use a result from the chapter 'Change of Numeraire' in Bjork to directly compute the time $t = 0$ price of a European call option with exercise date $T_1 > t$ on a maturity $T_2 > T_1$ zero coupon bond.