Fixed Income Derivatives - Vasicek Exam Preparation

In the Vasicek model, the short rate r_t is assumed to have the following dynamics under the risk-neutral measure $\mathbb{Q}$

$$dr_t = (b - ar_t)dt + \sigma dW_t \tag{1}$$

where a, $b > 0$ and σ are model parameters.

a) Solution and distribution of $r(t)$.

   i) Find the solution for $r(t)$, $t > 0$, in the Vasicek model.

   ii) Find the mean $\mathrm{E}[r(t)]$ and the variance $\mathrm{Var}[r(t)]$ of $r(t)$. There is a 'trick' involved when computing the variance, what is that trick called?

   iii) Find the distribution of $r(t)$.

   iv) Find the distribution of $r(t)$ as $t \nearrow \infty$. That is, find the stationary distribution of $r(\infty)$.

Answers:

   i) The solution for the short rate in the Vasicek model can be found by applying Ito to $f(t,r) = e^{at}r$ and integrating appropriately giving us that

$$r(t) = r(0)e^{-at} + \frac{b}{a}\Big[1 - e^{-at}\Big] + \sigma \int_0^t e^{-a(t-u)} dW_u \tag{2}$$

   ii) The mean and variance can be found by direct computation, however to compute variance Ito asymmetry must be used along to way.

$$\mathrm{E}[r(t)] = r(0)e^{-at} + \frac{b}{a}\Big[1 - e^{-at}\Big], \qquad \mathrm{Var}[r(t)] = \frac{\sigma^2}{2a}\Big[1 - e^{-2at}\Big] \tag{3}$$

   iii) The distribution of $r(t)$ is given by

$$r(t)|\mathcal{F}_0 \sim N\bigg(r(0)e^{-at} + \frac{b}{a}\Big[1 - e^{-at}\Big], \frac{\sigma^2}{2a}\Big[1 - e^{-2at}\Big]\bigg) \tag{4}$$

   iv) The stationary distribution can be found be sending $t \nearrow \infty$

$$r(\infty) \sim N\bigg(\frac{b}{a}, \frac{\sigma^2}{2a}\bigg) \tag{5}$$

b) Now assume that the parameters of the Vasicek model are $r_0 = 0.02$, $a = 0.6$, $b = 0.03$ and $\sigma = 0.02$. Compute ZCB prices, continuously compounded spot rates and instantaneous forward rates in the Vasicek model.

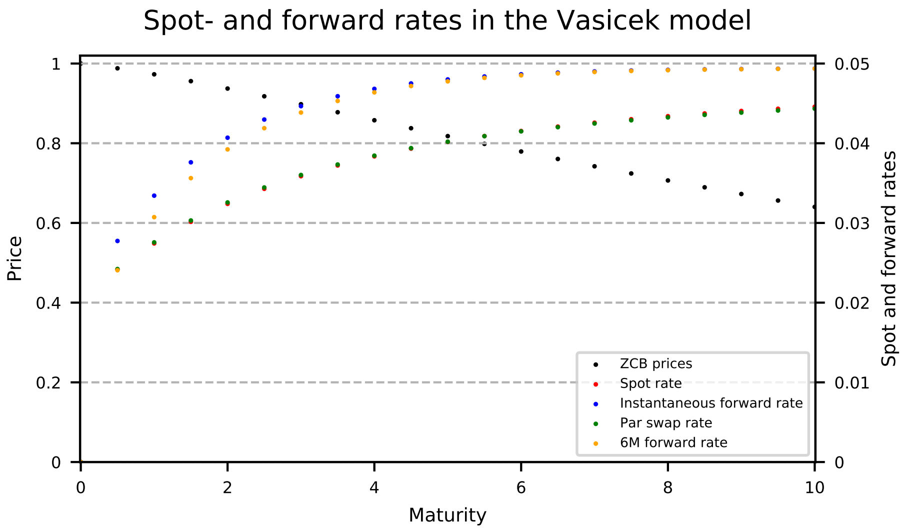
   i) Plot the ZCB prices, spot rates and instantaneous forward rates for values of t in $[0, 10]$.

   ii) Now compute spot rates for maturities $T \in \{0.5, 1, 1.5, ..., 10\}$ using the parameters given above and fit all the parameters in a Vasicek model to these spot rates. Are you able to recover the parameters of the Vasicek model as you should be? Are there any difficulties with any of the parameters?

   iii) Now try to fit the parameters again, this time assuming that you known that $\sigma = 0.02$. Are you able to fit the model parameters and is the fit now better? Explain what this tells you about σ in a Vasicek model.

Answers:

i) The plot of ZCB prices, spot rates, instantaneous forward rates and par swap rates becomes



ii) Fitting the Vasicek model to spot rates with the objective of recovering all four parameters, we discover that σ is difficult to recover implying that σ is not well-specified in this model. Despite our inability to recover σ, we are nonetheless able to recover the remaining parameters r_0, a and b quite well.

iii) If we fit the Vasicek model to spot rates assuming that $\sigma = 0.02$ is known the algorithm will converge very fast and the remaining parameters be returned with high accuracy illustrating once again that r_0, a and b are very will identified in the model.

c) We will now study interest rate swaps that involve swapping 6M floating rate payments (EURIBOR say) paid semiannually for fixed payments also paid semi-annually. Assume that present time is $t = 0$ and that the 6M floating rate to be paid at time $T = 0.5$ has just been announced. For simplicity, you can assume the notional of these swaps is just 1.

i) Compute the par swap rate of 10Y interest rate swap.

ii) Compute par swap rates for maturities $T \in \{1, 1.5, ..., 10\}$ and plot the par swap curve along side spot rates and 6M forward rates.

iii) Explain how 6M forward rates and par swap rates are related.

iv) You will notice that par swap rates and zero coupon spot rates are very close. Explain why that is so by appealing to concepts such as the accrual factor of a swap and the duration of all bonds in general.

Answers:

i) The 10Y par swap rate becomes 0.044335.

ii) Par swap rates and 6M forward rates are included in the plot from b).

iii) Par swap rates are a weighted average of the 6M forward rates with more weight placed on forward rates over the near future.

iv) Par swap rates and zero coupon bond rates are relatively close because an interest rate swap is in nature quite close to a zero coupon bond. Now, a zero coupon bond band pays *no* coupons and has only one cashflow at the very end. The net coupons to an interest rate swap are however are also relatively small as they consist of the difference between the fixed par swap rate and the floating rate in this case 6M EURIBOR. Furthermore, since the fixed par swap rate is a weighted average of 6M forward rates, the average net coupon to an interest rate swap tends to be close to 0.

d) We will now consider the pricing of an interest rate cap with a strike of $R = 0.05$.

i) Explain how an interest rate cap is related to caplets and also explain, how a caplet can be seen as a type of European option and on what underlying.

ii) Compute the prices of a cap that begins as early as possible and ends in exactly 10 years.

iii) Investigate how the price of the cap depends on σ and compute the DV01 of changing σ by 0.001 both up and down. Is the $DV01$ you have computed a first-order approximation or is it exact? Can both exact and approximated values be computed?

Answers:

i) A caplet is an instrument which, for a price of course, caps the interest payment on some given future floating rate payment of in this case 6M EURIBOR. An interest rate cap consists of multiple caplet and is thus a tool to cap a series of floating rate payments at some a priori chosen level. A caplet can be seen as a European call option on a future floating rate and using a simple replication argument, this European call option on the floating rate can equivalently be priced as a put option on the forward ZCB price corresponding to the forward floating rate.

ii) The interest rate cap can be priced by first computing the price of the individual caplets. The caplet prices can be found using Black-Scholes formula and the caplet prices are shown in the table below

Table 1: Caplet prices

T_i	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50	5.00
C_i(bps)	0	0.82727	4.92962	9.78459	14.0030	17.2847	19.6804	21.3383	22.4156	23.0522

T_i	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00	9.50	10.0
C_i(bps)	23.3596	23.4261	23.3187	23.0873	22.7691	22.3912	21.9735	21.5303	21.0722	20.6066

The price of the 10Y interest rate cap with a strike of $R = 0.05$ can be found as the sum of the prices of the individual caplets and the price of the cap becomes 0.035685 or 356.85 basispoints. The price of the cap corresponds to a semi-annual premium of 21.9867 basispoints.

iii) To investigate how the price of the interest rate cap changes if we change σ, both an approximate and an exact method is available. To compute the change in price exactly, simply change σ and recompute the price of the cap. This is of course very easy since we have an explicit formula for the price of a caplet in the Vasicek model and is the preferred method. However, a first-order approximation can also be employed using the *vega* of the caplet price as implied by the Black-Scholes formula. There is however no reason to settle for the approximation when the exact value is so readily available and we get that if σ is increased by 0.001, the price of the cap increases to 0.037699, and if σ is reduced by 0.0001, the price of the cap falls to 0.033677.

e) Now we will simulate short rates in the Vasicek model using the usual first order Euler scheme on a grid of mesh δ that runs from initial time $t_0 = 0$ to terminal time $T = 10$. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$, the time points will be $[t_0, t_1, ..., t_{M-1}, t_M] = [0, \delta, 2\delta, ..., T-\delta, T]$ and $\delta = \frac{T}{M}$. The scheme you will need to implement is a simple Euler first-order scheme of the form

$$r_m = r_{m-1} + (b - ar_{m-1})\delta + \sigma\sqrt{\delta}Z_m, \quad m = 1, 2, ..., M \tag{6}$$

where $Z_m \sim N(0, 1)$, $m = 1, ..., M$ and all the standard normal random variables are independent.

i) Simulate one trajectory of the short rate and plot the trajectory up to time $T = 10$.

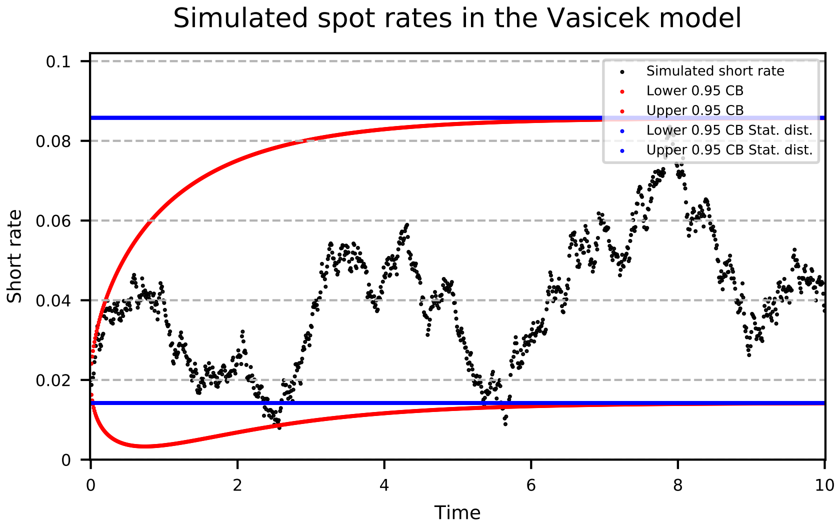
ii) Construct 95 percent two-sided confidence intervals for the short rate and plot these in the same plot.

iii) Construct a 95 percent two-sided confidence interval for the short rate under the stationary distribution and plot this confidence interval in the same plot. Based on the plot, can you say that the distribution of r_3 is roughly the same as that of the stationary distribution? How does this change if you change the parameters of the Vasicek model?

Answers:

i) A single simulated trajectory of the short rate can be seen in the plot below.

ii) In the same plot, you will also find two-sided 95 percent confidence intervals for the short rate also under the stationary distribution.



iii) The value r_3 in the simulation is of course the short three years after the initial point, and since the confidence interval of r_3 is different from that of the stationary distribution, we can conclude that 3 years is not enough time for the short rate to settle to its stationary distribution. Convergence to the stationary distribution will be faster if either a is large or σ is small.

f) We will now introduce a 2Y8Y payer swaption with a strike of $K = 0.045$. That is, we will introduce a swaption that gives the owner the right but not obligation to enter into a 8Y swap at exercise in $T_n = 2$ years so that we have $T_N = 10$. To compute the price of this swaption, you will need to use simulation.

i) Argue that the payoff function $\chi(T_n)$ and the discounted payoff function $\tilde{\chi}(T_n)$ of the payer swaption are.

$$
\begin{aligned}
\chi(T_n) &= S_n^N(T_n)\big(R_n^N(T_n) - K\big)_+ \\
\tilde{\chi}(T_n) &= \exp\Big\{-\int_0^{T_n} r_t dt\Big\} S_n^N(T_n)\big(R_n^N(T_n) - K\big)_+
\end{aligned}
\tag{7}
$$

ii) Find a method to compute the price at $t = 0$ of the swaption by simulating at least $L = 1000$ trajectories and having at least $M = 1000$ steps in your simulation.

iii) Investigate if the price you have computed is accurate by plotting the value of the derivative for various choices of L.

iv) Explain how the price of the swaption depends on σ, T_n, T_N and of course K.

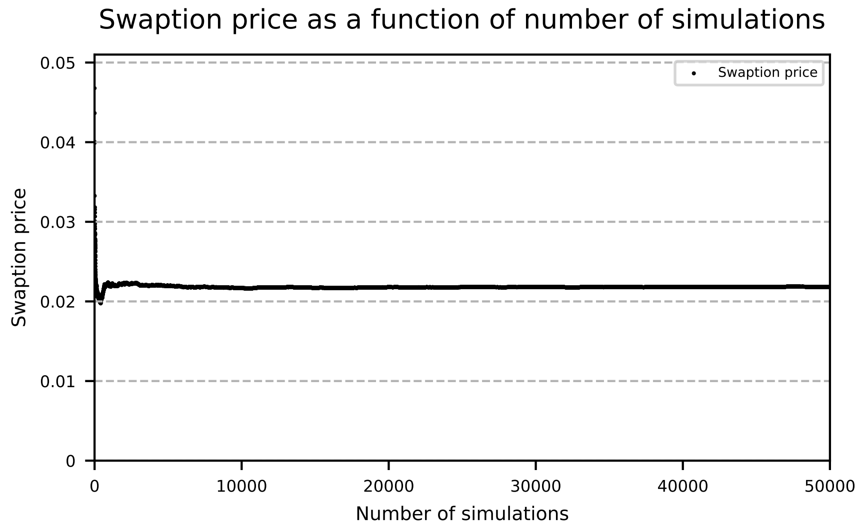
Answers:

i) We know that a 2Y8Y payer swaption is a European option which gives the holder the right but not obligation to enter into a payer swap paying a fixed rate $K = 0.045$ at maturity $T_n = 2$.

The value of the underlying 8Y swap at time $T_n = 2$ is $S_n^N(T_n)\big(R_n^N(T_n) - K\big)$ and if this value is positive, it will be beneficial to exercise the swaption giveing us the first equation in (7). The second equation in (7) simply comes from discounting.

ii) To compute the price of the swaption in the Vasicek model, we will simulate the short rate up until maturity at $T_n = 2$. Given the realization of the short rate and the parameters of the Vasicek model, we can then calculate ZCB prices and both S_{swap} and the par swap rate for an 8Y interest rate swap. Knowing these quantities, and the trajectory of the short rate, we can compute the discounted value of the payer swaption from (7) for this specific realization of the short rate. We can then repeat this procedure for L trajectories of the short rate and for each trajectory store the discounted value of the swaption. Once we have repeated this procedure L times, we simply average over the L different outcomes of the realized value of the 2Y8Y payer swaption. The price of the swaption for $M = 4000$ and $L = 50000$ simulations is 0.021829.

iii) A plot of the swaption price as a function of the number of simulations can be found below



iv) The price of the swaption will be increasing in σ and T_N and decreasing in T_n and K.