Fixed Income Derivatives - SABR Exam Preparation

In the SABR model, the forward rate denoted F_t, typically a forward par swap rate, follows a stochastic process with stochastic volatility σ_t. The joint dynamics of F_t and σ_t

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta} dW_t^{(1)}, \quad F(0) = F_0 \\ d\sigma_t &= \upsilon \sigma_t dW_t^{(2)}, \quad \sigma(0) = \sigma_0 \\ dW_t^{(1)} dW_t^{(2)} &= \rho \end{aligned} \tag{1}$$

where $\sigma_0 > 0$, $0 < \beta \leq 1$, $\upsilon > 0$ and $-1 < \rho < 1$ are parameters of the model. Zero coupon bond prices observed from market data are given in the below table.

Table 1: Zero coupon bond prices

T	0.50	1.00	1.50	2.00	2.50	3.00	3.50
$p(0,T)$	0.98429046	0.96633686	0.94690318	0.92655036	0.90568659	0.88460647	0.86352084
T	4.00	4.50	5.00	5.50	6.00	6.50	7.00
$p(0,T)$	0.84257919	0.82188628	0.80151436	0.78151217	0.76191149	0.74273188	0.72398415

Also, we can observe prices of 2Y5Y payer swaptions for a number of strike offsets.

Table 2: $2Y5Y$ Swaption prices

$K_{offset}(bp)$	-300	-250	-200	-150	-100	-50	ATMF
$\Pi_{swaption}$	0.12301549	0.10339456	0.08421278	0.06567338	0.04843543	0.03300976	0.02048677
$K_{offset}(bp)$	50	100	150	200	250	300	-
$\Pi_{swaption}$	0.01173834	0.00648577	0.00361682	0.00215934	0.00137503	0.00093634	-

a) Compute the 2Y5Y par swap rate which serves as the underlying asset of the swaption by first computing the corresponding accrual factor S_{swap}

  i) Report the $2Y5Y$ forward par swap rate.

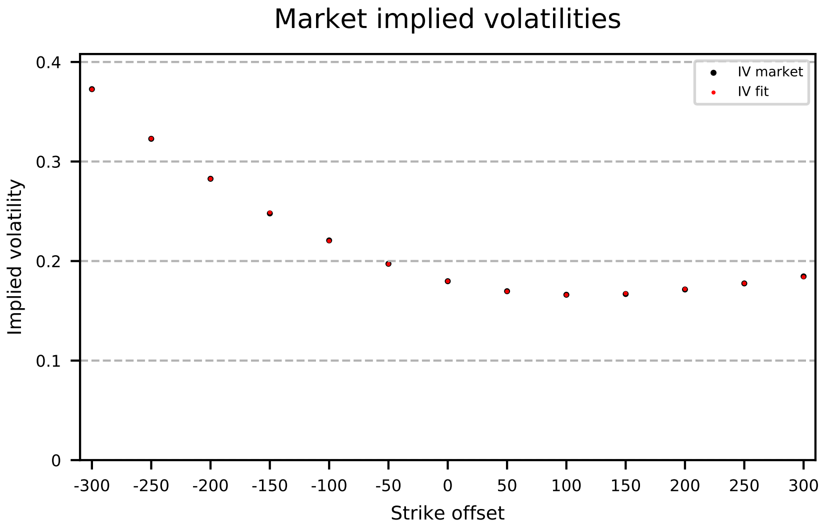
  ii) Compute Black implied volatilities for all strikes and plot these as a function of K_{offset}.

  iii) Using the implied volatility plot discuss whether the market is pricing swaptions according to Black's model. If not, then what can be said about the distribution of the $2Y5Y$ forward par swap rate implied by the pricing measure chosen by the market? How does that distribution compare to the log normal distribution?

Answers:

  i) The 2Y5Y par swap rate becomes 0.04983 and the corresponding accrual factor becomes 4.0650.

  ii) Inverting Black's formula for the price of a swaption, we can compute implied volatilities corresponding to the observed swaption prices. Implied volatilities for the different strikes are shown in the plot below.



iii) We can see from the plot of market implied volatilities and the fact that implied volatility is *not* constant for all strikes that the market is not pricing 2Y5Y payer swaptions according to Black's formula. The 'fat' tails both on the left and the right indicate that the distribution of the underlying asset implied by market swaption prices have tails that are fatter than that of the log-normal distribution. Also, the fact that the slope of the smile is negative ATM and that the left tail is more pronounced than the right suggests that the distribution of the underlying implied by market prices is asymmetric with more weight placed in the left tail.

b) You are now to fit a SABR model to observed implied volatilities setting initial values $\tilde{\sigma}_0 = 0.055$, $\tilde{\beta} = 0.5$, $\tilde{\upsilon} = 0.48$ and $\tilde{\rho} = -0.25$ and the 'nelder-mead' algorithm.

i) Report the fitted parameter values $\hat{\sigma}_0$, $\hat{\beta}$, $\hat{\upsilon}$ and $\hat{\rho}$.

ii) Plot the observed and fitted values of implied volatilities as a function of K_{offset}. Do your fitted values fit the data?

Answers:

i) Fitting the SABR model to swaption prices using 'nelder-mead' is not difficult and the result should be something like $\hat{\sigma_0} = 0.03794$, $\hat{\beta} = 0.4941$, $\hat{\upsilon} = 0.5736$ and $\hat{\rho} = -0.3218$

ii) The fitted values are indeed very close to the observed values which can be seen from the plot in a) above.

The forward rate, F_t, and the volatility σ_t can be simulated in the SABR model using a simple Euler scheme. Denote by M, the number of steps in the simulation and index the time points in the simulation by m, $m \in \{0, 1, 2, ..., M-1, M\}$ so that the time points will be $\big[t_0, t_1, ..., t_{M-1}, t_M\big] = \big[0, \delta, 2\delta, ..., T-\delta, T = \delta M\big]$ and hence the step in time will be of size $\delta = \frac{T}{M}$. The model can then be simulated using the following equations

$$
\begin{aligned}
F_m &= F_{m-1} + \sigma_{m-1} F_{m-1}^{\beta} \sqrt{\delta} Z_m^{(1)}, && F(0) = F_0 \\
\sigma_m &= \sigma_{m-1} + \upsilon \sigma_{m-1} \sqrt{\delta} \Big(\rho Z_m^{(1)} + \sqrt{1-\rho^2} Z_m^{(2)} \Big), && \sigma(0) = \sigma_0
\end{aligned} \tag{2}
$$

where $Z_m^{(1)}$ and $Z_m^{(2)}$ are independent standard normal random variables.

c) Simulate a single trajectory of the 2Y5Y forward par swap rate and the volatility.

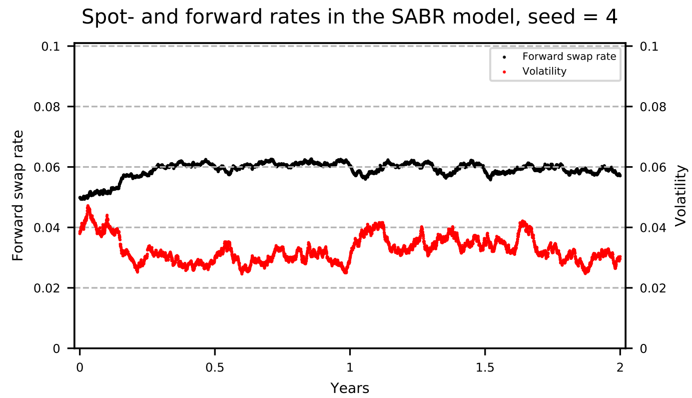
i) Plot the trajectories of the forward par swap rate and the volatility.

ii) Repeat the simulation for various values of the 'seed'. Does the volatility behave in strange and perhaps undesired way depend on the choice of the seed.

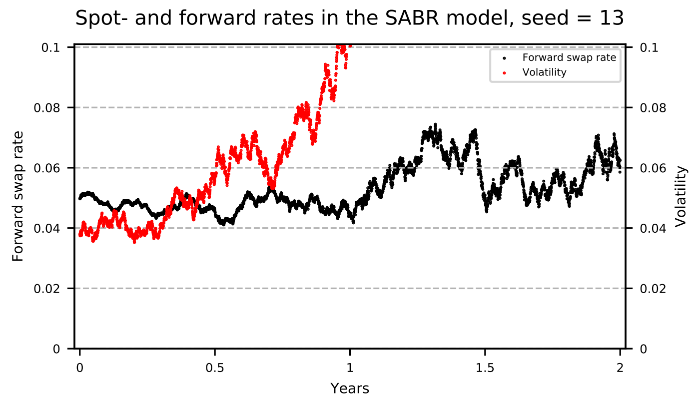
Answers:

i) See the plots below

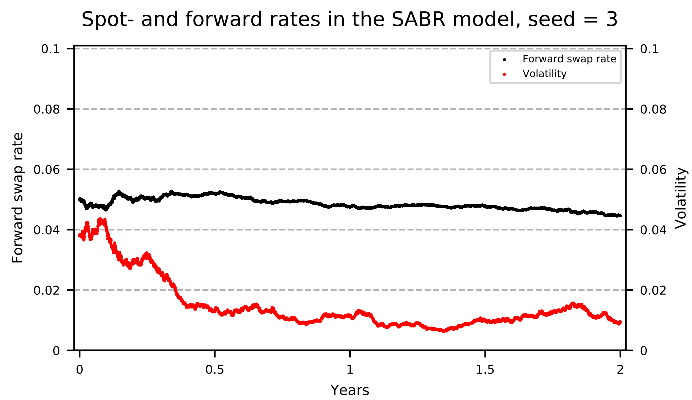
ii) The first plot for a seed chosen to be 4, the plot in this case becomes as shown below. In this case, the underlying parswap rate and its volatility both behave relatively realistically.



Next, we look at the same plot for a seed of 13. In this case, the volatility 'blows up'.



Finally, we look at the same plot for a seed of 13. In this case, volatility 'freezes' at 0 in that at becomes relatively small and can't seem to escape simply because the diffusion coefficient of volatility depends on volatility itself.



d) Now, we will compute the price of a digital option in the SABR model. The digital option will be one that pays 1 unit of currency if the 5Y spot swap rate in exactly two years exceeds the 2Y5Y forward par swap rate observed right now + 75 bps. So we have a strike $K = F_0 + 0.0075$

i) Which measure is the SABR model defined under? And how does that affect your ability to compute the price of a maturity $T = 2$ derivative such as the digital option considered here. What can you do if you have to price a derivative with a different maturity?

ii) Find the payoff function $\chi(T)$ and the discounted payoff function for the digital option.

iii) Using at least $M = 2000$ time steps and $N = 10000$ simulations, compute the price of the digital option.

Answers:

i) The SABR model is defined under the forward measure, specifically the forward measure corresponding to the maturity of the underlying swaption. If we denote the maturity of the swaption by T, the SABR model is in other words defined under the measure $\mathbb{Q}^T$, the measure under which all assets scaled by the price $p(t,T)$ of the maturity T ZCB are martingales. This is fact is important because it implies that we can discount future cashflows back to time 0 using the price $p(t,T)$ that we can observe from market prices. If we were to price other a derivative with a different maturity, we would need to either switch to the forward measure corresponding to that new maturity or use an entirely different model.

ii) The payoff function $\chi(T)$ and the discounted payoff function $\tilde{\chi}(T)$ for the digital option are

$$\begin{aligned}\chi(T) &= \mathbb{1}_{R_2^5(2)>K} \\ \tilde{\chi}(T) &= p(0,2)\mathbb{1}_{R_2^5(2)>K}\end{aligned} \tag{3}$$

where $R_2^5(2)$ is the 5Y par swap rate in exactly 2 years when the digital option matures.

iii) The digital option requires many simulation for price pricing, but choosing $M = 4,000$ timesteps and $N = 100,000$ simulations, the price in this case became 0.2364.

e) Imagine you have a long position in the $K_{offset} = 150$ 2Y5Y payer swaption and you need to assess how you are exposed to changes in the SABR model parameters. Do this by computing the change in dollar value (DV) in the following cases.

i) If σ_0 falls by 0.001,

ii) if the entire spot rate curve drops by 1 bp.

Answers:

i) Computing the change in the value of the long position in a payer swaption if σ_0 changes, simply recompute the implied volatility for the new value of σ_0 and then use Black's formula for the value of a payer swaption to get that the value of the payer swaption falls by roughly 2.25 if σ_0 falls by 0.001.

ii) To compute the change in the value of the payer swaption if the entire spot rate curve drops by one bps, we need to first bump the entire spot rate curve, then recompute the 2Y5Y forward par swap rate and then recompute the price of the payer swaption using Black's formula. We get that the value of the payer swaption drops by 0.083033 bps if the entire spot curve drops by one bp. The change in value corresponds to a drop of roughly 0.2 percent.