Fixed Income Derivatives - Problem Set Week 6

Problem 1

Consider the Hull-White model where the short rate r_t has dynamics

$$dr_t = \big[\Theta(t) - ar_t\big]dt + \sigma dW_t \tag{1}$$

a) Argue that ZCB prices are of the form

$$p(t,T) = e^{A(t,T)-B(t,T)r_t} \tag{2}$$

where

$$\begin{aligned} A(t,T) &= \int_t^T \Big[\frac{1}{2}\sigma^2 B^2(s,T) - \Theta(s)B(s,T)\Big]ds \\ B(t,T) &= \frac{1}{a}\Big[1 - e^{-a(T-t)}\Big] \end{aligned} \tag{3}$$

b) Show that forward rates $f(t,T)$ are of the form

$$f(t,T) = -\frac{\partial}{\partial T}A(t,T) + r_t\frac{\partial}{\partial T}B(t,T) \tag{4}$$

c) Argue that the forward rate dynamics can be found from

$$df(t,T) = -\frac{\partial}{\partial T}\Big(A_t(t,T)dt - B_t(t,T)r_t dt - B(t,T)dr_t\Big) \tag{5}$$

where $A_t(t,T) = \frac{\partial}{\partial t}A(t,T)$

d) Show that the forward rate dynamics are

$$df(t,T) = \frac{\sigma^2}{a}e^{-a(T-t)}\Big[1 - e^{-a(T-t)}\Big]dt + \sigma e^{-a(T-t)}dW_t \tag{6}$$

Now, we will find the forward rate dynamics in a different way. Let us recall that in the Hull-White model, zero coupon bond prices become

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\Big\{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)\big(1-e^{-2at}\big) - B(t,T)r_t\Big\} \tag{7}$$

e) Use the above expression to find an expression for forward rates and treat this expression as a function $g = g(t,T,r)$.

f) Show from $g(t,T,r)$ that the forward rate dynamics are of the form

$$df(t,T) = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW_t \tag{8}$$

where $\alpha(t,T)$ is yet to be determined.

g) Use the HJM drift condition to find $\alpha(t,T)$ and thus show that it is of the same form as in d).

Solution

a) Since the drift and the squared diffusion coefficients of the short rate are both affine, the Hull-White model admits an affine term structure with $p(t,T) = e^{A(t,T)-B(t,T)}r_t$ where $A(t,T)$ and $B(t,T)$ satisfy the system of ODE's

$$\begin{aligned} A_t(t,T) &= \Theta(t)B(t,T) - \frac{1}{2}\sigma^2B^2(t,T), \qquad && A(T,T) = 0 \\ B_t(t,T) &= aB(t,T) - 1, && B(T,T) = 0 \end{aligned} \tag{9}$$

The equation for $B(t,T)$ is the same as in the Vasicek model and $A(t,T)$ can then be found directly by integration and we have

$$\begin{aligned} A(t,T) &= \int_t^T \Big[\frac{1}{2}\sigma^2 B^2(s,T) - \Theta(s)B(s,T)\Big]ds \\ B(t,T) &= \frac{1}{a}\Big[1 - e^{-a(T-t)}\Big] \end{aligned} \tag{10}$$

b) To use the Hull-White model in practice, we would need to fit the model to observed forward rates $f^*(0,t)$ and thereby find $\Theta(t)$. This involves computing the integral in (10) to find $A(t,T)$ but as we will see next, the forward rate dynamics are independent of $\Theta(t)$ and can be found before fitting the model to data. From the definition of forward rates, we immediately have

$$f(t,T) = -\frac{\partial}{\partial T}p(t,T) = -\frac{\partial}{\partial T}A(t,T) + r_t\frac{\partial}{\partial T}B(t,T) \tag{11}$$

c) The forward rate dynamics can be found directly the product rule of differentiation and Ito's formula

$$df(t,T) = -\frac{\partial}{\partial T}d\Big(A(t,T) - B(t,T)r_t\Big) = -\frac{\partial}{\partial T}\Big(A_t(t,T)dt - B_t(t,T)r_tdt - B(t,T)dr_t\Big) \tag{12}$$

d) To find the forward rate dynamics, we need to use $A_t(t,T)$ and $B_t(t,T)$ but they are given from the system of ODE's in (10)

$$\begin{aligned}
df(t,T) =& -\frac{\partial}{\partial T}d\Big(\Theta(t)B(t,T)dt - \frac{1}{2}\sigma^2B^2(t,T)dt - aB(t,T)r_tdt + r_tdt - B(t,T)\big[\Theta(t) - ar_t\big]dt - B(t,T)\sigma dW_t\Big)\\
=& \frac{1}{2}\sigma^2\frac{\partial}{\partial T}B^2(t,T)dt + \sigma\frac{\partial}{\partial T}B(t,T)dW_t = \sigma^2B(t,T)\frac{\partial}{\partial T}B(t,T)dt + \sigma\frac{\partial}{\partial T}B(t,T)dW_t\\
=& \frac{\sigma^2}{a}e^{-a(T-t)}\Big[1 - e^{-a(T-t)}\Big]dt + \sigma e^{-a(T-t)}dW_t
\end{aligned} \tag{13}$$

e) Using the expression for ZCB prices and the definition of forward rates gives us that

$$f(t,T) = g(t,T,r) = -\frac{\partial}{\partial T}\Big(\log p^*(0,T) - \log p^*(0,t) + B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)\big(1 - e^{-2at}\big) - B(t,T)r\Big) \tag{14}$$

f) Applying Ito's formula and simply denoting the drift term by $\alpha(t,T)$ gives us that

$$\begin{aligned}
df(t,T) = dg(t,T,r) =& \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial r}\big[\Theta(t) - ar_t\big]dt + \frac{\partial g}{\partial r}\sigma dW_t + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}\sigma^2dt = \alpha(t,T)dt + \frac{\partial g}{\partial r}\sigma dW_t\\
=& \alpha(t,T)dt + \frac{\partial}{\partial T}\frac{1}{a}\Big[1 - e^{-a(T-t)}\Big]\sigma dW_t = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW_t
\end{aligned} \tag{15}$$

g) Now, we know that the drift of the forward rates must satisfy the HJM drift condition and that can be used to find $\alpha(t,T)$. The HJM drift condition tells us that the drift coefficient of forward rates depends on the diffusion coefficient $\sigma(t,T)$ as follows

$$\begin{aligned}
\alpha(t,T) =& \sigma(t,T)\int_t^T\sigma(t,s)ds = \sigma e^{-a(T-t)}\int_t^T\sigma e^{-a(T-s)}ds = \frac{\sigma^2}{a}e^{-a(T-t)}e^{-aT}\Big[e^{as}\Big]_t^T\\
=& \frac{\sigma^2}{a}e^{-a(T-t)}\Big[1 - e^{-a(T-t)}\Big]
\end{aligned} \tag{16}$$

And indeed, we rediscover the dynamics of forward rates in the Hull-White model just as in d).

Problem 2

Take as given an HJM model under the risk neutral measure $\mathbb{Q}$ where

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t \tag{17}$$

a) Show that all forward rates and also the short rate are normally distributed.

b) Show that zero coupon bond prices are log-normally distributed.

Solution

a) Since the drift- and diffusion coefficients of the SDE satisfied by $f(t,T)$ has deterministic coefficients, the solution for $f(t,T)|\mathcal{F}_0$ for $0 < t \leq T$ can be found directly by integration

$$\begin{aligned}
f(t,T) - f(0,T) =& \int_0^t df(s,T) = \int_0^t\alpha(s,T)ds + \int_0^t\sigma(s,T)dW_s \quad \Rightarrow\\
f(t,T) =& f(0,T) + \int_0^t\alpha(s,T)ds + \int_0^t\sigma(s,T)dW_s
\end{aligned} \tag{18}$$

The first of the two integrals in (18) is a regular $\mathcal{F}_0$-adapted Riemann-Stieltjes time integral that, in principle at least, can be computed. The second integral of (18) is an Ito integral with a deterministic coefficient and hence, this integral is Gaussian with mean zero and a variance that can be computed using Ito isometry. We thus have that

$$f(t,T)\big|\mathcal{F}_0 \;\sim\; N\Big(f(0,T) + \int_0^t \alpha(s,T)ds, \int_0^t \sigma^2(s,T)ds\Big) \tag{19}$$

b) To show that zero coupon bond prices, $p(t,T)$, follow a log-normal distribution, we first recall the link between $p(t,T)$ and $f(t,T)$

$$\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T}\log p(t,T) \quad \Leftrightarrow \quad p(t,T) = \exp\Big\{-\int_t^T f(t,s)ds\Big\} \quad \Rightarrow \\
\log p(t,T) &= -\int_t^T \Big[f(t,T) + \int_t^s \alpha(u,T)du + \int_t^s \sigma(u,T)dW_u\Big]ds \\
&= -\big(T-t\big)f(t,T) - \int_t^T\int_t^s \alpha(u,T)duds - \int_t^T\int_t^s \sigma(u,T)dW_u ds
\end{aligned} \tag{20}$$

In order to proceed, we will need to change the order of integration and for that we will need a version of Fubini's theorem that we strictly speaking have not proven. In particular, have not proven such a theorem when integration with respect to Brownian motion is involved. However, under suitable conditions and $\alpha(\cdot,T)$ and $\sigma(\cdot,T)$, we proceed to change the order of integration to get that

$$\log p(t,T) = -\big(T-t\big)f(t,T) - \int_t^T\int_t^u \alpha(s,T)dsdu - \int_t^T\int_t^u \sigma(s,T)dsdW_u = -\big(T-t\big)f(t,T) + I_1 + I_2 \tag{21}$$

Now, $\sigma(s,T)$ is a deterministic function and so

$$g(u) = \int_t^u \sigma(s,T)ds \tag{22}$$

is also a deterministic function of u and we can find the distribution of I_2 using what we know about the distribution of an Ito integral with a deterministic integrand

$$I_2 = -\int_t^T g(u)dW_u \quad \Rightarrow \quad I_2 \;\sim\; N\Big(0, \int_t^T\Big(\int_t^u \sigma(s,T)ds\Big)^2 du\Big). \tag{23}$$

We can therefore conclude that

$$\log p(t,T) \;\sim\; N\Big(-\big(T-t\big)f(t,T) - \int_t^T\int_t^u \alpha(s,T)dsdu, \int_t^T\Big(\int_t^u \sigma(s,T)ds\Big)^2 du\Big). \tag{24}$$

Problem 3

Consider the Ho-Lee model where the short rate has dynamics

$$dr_t = \Theta(t)dt + \sigma dW_t. \tag{25}$$

Recall that ZCB prices in the Ho-Lee model are given by

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\Big\{\big(T-t\big)f^*(0,t) - \frac{\sigma^2}{2}t\big(T-t\big)^2 - \big(T-t\big)r\Big\}. \tag{26}$$

a) Use the procedure outlined in Problem 1e-1g to find the forward rate dynamics in this model.

b) Find the dynamics of zero coupon bond prices under $\mathbb{Q}$.

c) Find the distribution of forward rates in this model and argue that they are Gaussian.

d) Use a result from the chapter 'Change of Numeraire' in Bjork to directly compute the time t price of a European call option with exercise date $T_1 > t$ on a maturity $T_2 > T_1$ zero coupon bond.

Solution

a) Forward rates in the Ho-Lee model can be found from

$$f(t,T) = -\frac{\partial}{\partial T}\Big(\log p^*(0,T) - \log p^*(0,t) + \big(T-t\big) f^*(0,t) - \frac{\sigma^2}{2} t \big(T-t\big)^2 - \big(T-t\big) r \Big) \tag{27}$$

Applying Ito's formula to $g(t,T,r) = f(t,T)$ and denoting the drift of the result by $\alpha(t,T)$ gives us

$$\begin{aligned} df(t,T) &= dg(t,T,r) = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial r}\Theta(t)dt + \frac{\partial g}{\partial r}\sigma dW_t + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}\sigma^2 dt = \alpha(t,T)dt + \frac{\partial g}{\partial r}\sigma dW_t \\ &= \alpha(t,T)dt + \sigma dW_t = \alpha(t,T)dt + \sigma dW_t \end{aligned} \tag{28}$$

Recall that the dynamics of forward rates in the HJM model are of the form

$$df_(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t \tag{29}$$

From the HJM drift condition, we can now find the drift coefficient $\alpha(t,T)$ from the diffusion coefficient of the forward rates.

$$\alpha(t,T) = \sigma \int_t^T \sigma ds = \sigma^2 \big(T-t\big) \tag{30}$$

The dynamics of the forward rates become

$$df(t,T) = \sigma^2 \big(T-t\big)dt + \sigma dW_t \tag{31}$$

b) Finding the dynamics of ZCB prices from the dynamics of forward prices can be done in many ways, but we will use a result from the chapter 'Bonds and Interest Rates' in Bjørk. This results allows us to directly find the dynamics of ZCB prices from the dynamics of forward rates in the general case and thus also in this case. The result states that

$$dp(t,T) = \big(r_t + A(t,T) + \|\mathbf{S}(t,T)\|^2\big)p(t,T)dt + p(t,T)\mathbf{S}'(t,T)d\mathbf{W}_t$$

where $\|\cdot\|$ denotes the Euclidean norm and

$$\begin{aligned} A(t,T) &= -\int_t^T \alpha(t,s)ds \\ \mathbf{S}(t,T) &= -\int_t^T \boldsymbol{\sigma}(t,s)ds \end{aligned} \tag{32}$$

and we get that

$$\begin{aligned} A(t,T) &= -\frac{1}{2}\sigma^2\big(T-t\big)^2 \\ S(t,T) &= -\sigma\big(T-t\big) \\ dp(t,T) &= \big(r_t + \sigma(T-t) - \frac{1}{2}\sigma^2(T-t)^2\big)p(t,T)dt - \sigma(T-t)p(t,T)dW_t \end{aligned} \tag{33}$$

c) Forward rates in this model satisfy an SDE with deterministic coefficients and we can therefore easily find the solution for $f(S,T)\big|\mathcal{F}_t$ where $t < S \leq T$.

$$\begin{aligned} f(S,T) &= \int_t^S \alpha(u,T)du + \int_t^S \sigma(u,T)dW_u = \int_t^S \sigma^2\big(T-u\big)du + \int_t^S \sigma dW_u \\ &= \sigma^2\big[S(T-t) - \frac{1}{2}\big(S^2 - t^2\big)\big] + \int_t^S \sigma dW_u \end{aligned} \tag{34}$$

The distribution of $f(S,T)\big|\mathcal{F}_t$ can then be found to be a Gaussian and the variance can be found using Ito isometry.

$$f(S,T)\big|\mathcal{F}_t \sim N\Big(\sigma^2\big[S(T-t) - \frac{1}{2}\big(S^2 - t^2\big)\big], \sigma^2\big[S-t\big]\Big) \tag{35}$$

d) The price $\Pi = \Pi(0; K, T_1, T_2)$ of a European call option with maturity T_1 on a T_2 zero coupon bond can be found using a result from chapter 22 in Bjørk on the pricing of such an option when forward rates are Gaussian. To use this formula, we compute

$$\sigma_{T_1,T_2}(t) = -\int_{T_1}^{T_2} \sigma(t,s)ds = -\int_{T_1}^{T_2} \sigma ds = -\sigma\big(T_2 - T_1\big)$$
$$\Sigma^2_{T_1,T_2}(t) = \int_t^{T_1} \|\sigma_{T_1,T_2}(s)\|^2 ds = \int_t^{T_1} \sigma^2\big(T_2 - T_1\big)ds = \sigma^2\big(T_2 - T_1\big)\big(T_1 - t\big) \tag{36}$$

We get that

$$\Pi = p(0,T_2)\Phi(d_1) - Kp(0,T_1)\Phi(d_2) \tag{37}$$

where

$$d_1 = \frac{\ln\left(\frac{p(t,T_2)}{p(t,T_1)K}\right) + \frac{1}{2}\Sigma^2_{T_1,T_2}(t)}{\sqrt{\Sigma^2_{T_1,T_2}(t)}} = \frac{\ln\left(\frac{p(t,T_2)}{p(t,T_1)K}\right) + \frac{1}{2}\sigma^2\big(T_2 - T_1\big)\big(T_1 - t\big)}{\sigma\sqrt{\big(T_2 - T_1\big)\big(T_1 - t\big)}}$$
$$d_2 = d_1 - \sqrt{\Sigma^2_{T_1,T_2}(t)} = d_1 - \sigma\big(T_2 - T_1\big)\big(T_1 - t\big) \tag{38}$$

Problem 4

Take as given an HJM model under the risk neutral measure $\mathbb{Q}$ of the form

$$df(t,T) = \alpha(t,T)dt + \sigma_1(T-t)dW_{1t} + \sigma_2 e^{-a(T-t)}dW_{2t} \tag{39}$$

a) Use the HJM drift condition to find $\alpha(t,T)$.

b) Find the dynamics of zero coupon bond prices under $\mathbb{Q}$.

c) Find the distribution of forward rates in this model and argue that they are Gaussian.

d) Use a result from the chapter 'Change of Numeraire' in Bjork to directly compute the time $t = 0$ price of a European call option with exercise date $T_1 > t$ on a maturity $T_2 > T_1$ zero coupon bond.