Fixed Income Derivatives - Problem Set Week 7

Problem 1

In this problem, we will study a version of the Nelson-Siegel function often used when modeling the term structure of spot or forward rates. In particular, we will assume that instantaneous forward rates are given by

$$f(t,T) = f_\infty + a_0 e^{-b_0(T-t)} + a_1(T-t)e^{-b_1(T-t)} + a_2(T-t)e^{-b_2(T-t)} + ... = f_\infty + \sum_{k=0}^{K} a_k (T-t)^k e^{-b_k(T-t)} \quad (1)$$

where $b_k > 0$ and K is a strictly positive integer. As usual, t denotes present time and T is some time in the future

a) Show that we have the following relationship between forward rates $f(t,T)$ and zero coupon bond prices $p(t,T)$

$$p(t,T) = \exp\Big(- \int_t^T f(t,s)ds \Big) \quad (2)$$

b) Show that when forward rates are defined as in (1), ZCB prices are given by

$$\begin{aligned} p(t,T) &= \exp\Big(- \int_t^T f(t,s)ds \Big) = \exp\bigg(- f_\infty(T-t) - \sum_{k=0}^{K} a_k \int_t^T (T-s)^k e^{-b_k(T-s)} ds \bigg) \\ &= \exp\bigg(- f_\infty(T-t) - \sum_{k=0}^{K} a_k I_k \bigg), \quad I_k = \int_t^T (T-s)^k e^{-b_k(T-s)} ds \end{aligned} \quad (3)$$

c) Show that I_k can be written as

$$I_k = b_k^{-k-1} \int_0^{b_k(T-t)} u^k e^{-u} du \quad (4)$$

An integral of the form

$$\Gamma\big(a,b\big) = \int_0^b x^{a-1} e^{-x} dx \quad (5)$$

for $a > 0$ and $b > 0$ is called an *Incomplete Gamma Function.* The incomplete gamma function is not defined for a a non-positive integer and has to be evaluated numerically for a general a. However, when a is a positive integer, $a \in \mathbb{Z}^+$, the situation is much simpler and we will now explore this case.

d) Show that

$$\Gamma\big(1,b\big) = 1 - e^{-b} \quad (6)$$

e) Use integration by parts to show the following recursive relationship

$$\Gamma\big(a+1,b\big) = a\Gamma\big(a,b\big) - b^a e^{-b} \quad (7)$$

f) Using the recursive relationship between incomplete gamma functions, show that

$$\Gamma\big(a+1,b\big) = a! + e^-b \sum_{k=0}^{a} b^{a-k} \frac{a!}{(a-k)!} \quad (8)$$

for $a \in \mathbb{Z}^+$ and $b > 0$.

g) Finally, show that if forward rates are given by (1) then ZCB prices are given by

$$p(t,T) = \exp\bigg(- f_\infty(T-t) - \sum_{k=0}^{K} a_k I_k \bigg) = \exp\bigg(- f_\infty(T-t) - \sum_{k=0}^{K} a_k b_k^{-k-1} \Gamma\big(k+1, b_k(T-t)\big) \bigg) \quad (9)$$

Solution

a) The result follows immediately from the definition of forward rates.

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T) \quad \Rightarrow \quad -\int_t^T f(t,s)ds = \int_t^T \frac{\partial}{\partial s}\log p(t,s)ds = \log p(t,T) - \log p(t,t) \quad \Rightarrow$$
$$P(t,T) = \exp\Big(-\int_t^T f(t,s)ds\Big) \tag{10}$$

b) Using the result from a) and (1) gives us that

$$p(t,T) = \exp\Big(-\int_t^T f(t,s)ds\Big) = \exp\Big(-\int_t^T f_\infty ds - \int_t^T \sum_{k=0}^K a_k(T-s)^k e^{-b_k(T-s)}ds\Big)$$
$$= \exp\Big(-f_\infty(T-t) - \sum_{k=0}^K a_k \int_t^T (T-s)^k e^{-b_k(T-s)}ds\Big) \tag{11}$$

c) By substituting $u = b_k(T-s)$ which implies that $du = -b_k ds$ and reversing the order of integration, we get that

$$I_k = \int_t^T (T-s)^k e^{-b_k(T-s)}ds = -b_k^{-k-1}\int_{b_k(T-t)}^0 u^k e^{-u}ds = b_k^{-k-1}\int_0^{b_k(T-t)} u^k e^{-u}ds \tag{12}$$

d) $\Gamma(1,b)$ can be found directly by integration

$$\Gamma(1,b) = \int_0^b e^{-x}dx = -\big[e^{-x}\big]_0^b = 1 - e^{-b} \tag{13}$$

e) To use integration by parts, we notice that

$$\Gamma(a,b) = \int_0^b x^{a-1}e^{-x}dx = \frac{1}{a}\int_0^b \Big(\frac{\partial}{\partial a}x^a\Big)e^{-x}dx = \frac{1}{a}\int_0^b e^{-x}\Big(\frac{d}{dx}x^a\Big)dx = \frac{1}{a}\int_0^b udv = \frac{1}{a}\big[uv\big]_0^b - \frac{1}{a}\int_0^b vdu \tag{14}$$

Setting $u = e^{-x}$ so that $du = -e^{-x}dx$ and $dv = \Big(\frac{d}{dx}x^a\Big)dx = d(x^a)$ so that $v = x^a$ gives us that

$$\Gamma(a,b) = \frac{1}{a}\int_0^b udv = \frac{1}{a}\big[e^{-x}x^a\big]_0^b - \frac{1}{a}\int_0^b x^a e^{-x}dx = \frac{1}{a}e^{-b}b^a - \frac{1}{a}\Gamma(a+1,b) \tag{15}$$

and hence

$$\Gamma(a+1,b) = a\Gamma(a,b) - b^a e^{-b} \tag{16}$$

f) The recursive relationship satisfied by the incomplete gamma function can be shown by recursive substitution as follows

$$\Gamma(a+1,b) = a\Gamma(a,b) - b^a e^{-b} = a\big[(a-1)\Gamma(a-1,b) - b^{a-1}e^{-b}\big] - b^a e^{-b} = a(a-1)\Gamma(a-1,b) - e^{-b}\big[ab^{a-1} + b^a\big]$$
$$= a(a-1)\big[(a-2)\Gamma(a-2,b) - b^{a-2}e^{-b}\big] - e^{-b}\big[ab^{a-1} + b^a\big]$$
$$= a(a-1)(a-2)\Gamma(a-2,b) - e^{-b}\big[a(a-1)b^{a-2} + ab^{a-1} + b^a\big] = \ldots$$
$$= a! - e^{-b}\sum_{k=0}^a b^{a-k}\frac{a!}{(a-k)!} \tag{17}$$

g)

$$p(t,T) = \exp\Big(-f_\infty(T-t) - \sum_{k=0}^K a_k I_k\Big) = \exp\Big(-f_\infty(T-t) - \sum_{k=0}^K a_k b_k^{-k-1}\Gamma\big(k+1, b_k(T-t)\big)\Big) \tag{18}$$

Problem 2

In this problem, we will consider the Ho-Lee model in which the short rate under the risk neutral measure $\mathbb{Q}$ has dynamics

$$dr_t = \Theta(t)dt + \sigma dW_t. \tag{19}$$

Our objective will be to fit the Ho-Lee model to observed forward rates extracted from the market. So, assume that we observe the forward rates given in the vector *f_star* in the file *homework_7.py* for the maturities in the vector T from that same file and denote these observed forward rates by f^*. Also assume that $\sigma = 0.03$. To estimate $\Theta(t)$ in the Ho-Lee model, we will fit a Nelson-Siegel type function $f(t,T)$ to the observed prices f^*

$$f(t,T) = f_\infty + a_0 e^{-b_0(T-t)} + a_1(T-t)e^{-b_1(T-t)} + a_2(T-t)e^{-b_2(T-t)} + ... = f_\infty + \sum_{k=0}^{K} a_k (T-t)^k e^{-b_k(T-t)} \tag{20}$$

where $b_k > 0$ and K governs the number of terms included in the fit.

a) Set present time to $t = 0$ and plot the forward rates, spot rates and zero coupon bond prices generated by the function $f(T) = f(t = 0, T)$ for all maturities in the $T = \big[0, 0.1, ..., 9.9, 10\big]$ and parameters $\big[f_\infty, a_0, a_1, b_0, b_1\big] = \big[0.05, -0.02, 0.01, 0.5, 0.4\big]$. Explain the role each parameter plays in the shape of the spot and forward rate curves.

b) Plot the observed forward rates $f^*(T)$ in a separate plot and try to guess how many terms K will at least need to be included in the fit. Also, based on the plot come up with a set of plausible parameter values so that $f(T)$ with your choice of parameters is likely to fit $f^*(T)$ for $K = 1$.

c) Fit the function $f(T)$ to the observed values in $f^*(T)$ using 'scipy.optimize' and the 'nelder-mead' method. Your objective function should compute the total squared error between the fitted and observed values and hence, you should solve the following minimization problem

$$\min \sum_{m=0}^{M} \Big(f^*(T_m) - f(f_\infty, \mathbf{a}, \mathbf{b}; T_m)\Big)^2 \quad \text{wrt. } f_\infty, \mathbf{a}, \mathbf{b} \tag{21}$$

Do your fit recursively for increasing values of K starting with $K = 1$ and try to go up to no more than $K = 4$. Plot the fitted values $\hat{f}(T)$ versus the observed values $f^*(T)$ for the best fit you achieve.

d) Given your choice of K and preferred parameter estimates, find the function $\Theta(t)$ in the drift of the Ho-Lee model using that

$$\Theta(t) = \frac{\partial f^*(0,t)}{\partial T} + \sigma^2 t \tag{22}$$

where $\frac{\partial f^*(0,t)}{\partial T}$ denotes derivative in the second argument of f^* evaluated at $(0,t)$.

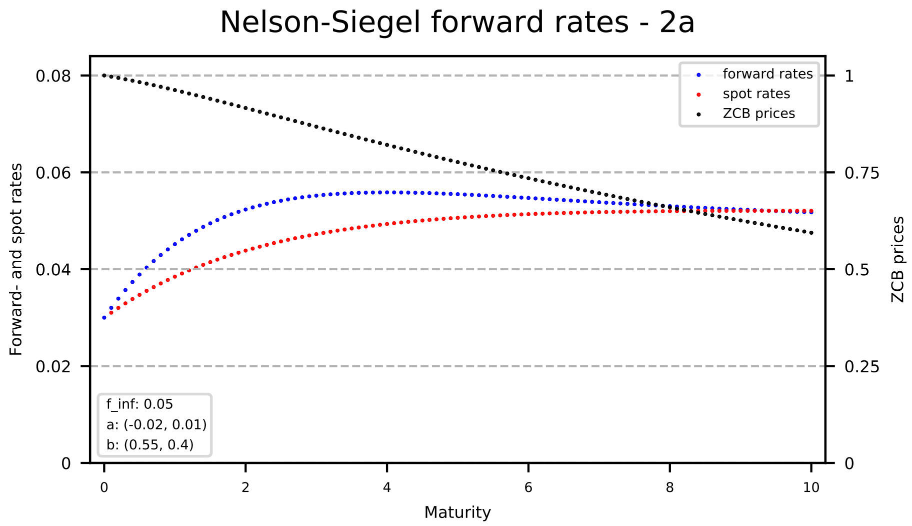
e) Now try to fit the function $f(T)$ using the method 'Newton-CG'. To do this, you will need to supply the algorithm with a function that returns the Jacobian (a vector of first-order derivatives of the objective function wrt. the parameters) and the Hessian (a matrix of second-order derivatives of the objective function wrt. the parameters). For example, the derivative of your objective function with respect to a_0 will be

$$\sum_{m=0}^{M} 2 \cdot \Big(f^*(T_m) - f(f_\infty, \mathbf{a}, \mathbf{b}; T_m)\Big) \cdot \frac{\partial f}{\partial a_0} \tag{23}$$

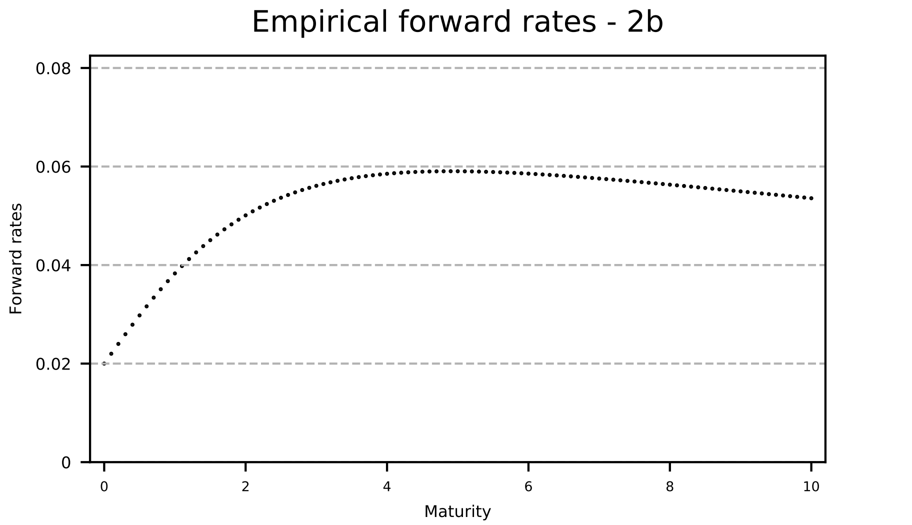
Report the parameter estimates you find using this method and plot the both the empirical and fitted values.

Solution

a) A plot of the term structures of spot- forward and ZCB prices for the parameters $[f_\infty, a_0, a_1, b_0, b_1] = [0.05, -0.02, 0.01, 0.5, 0.4]$ looks as follows

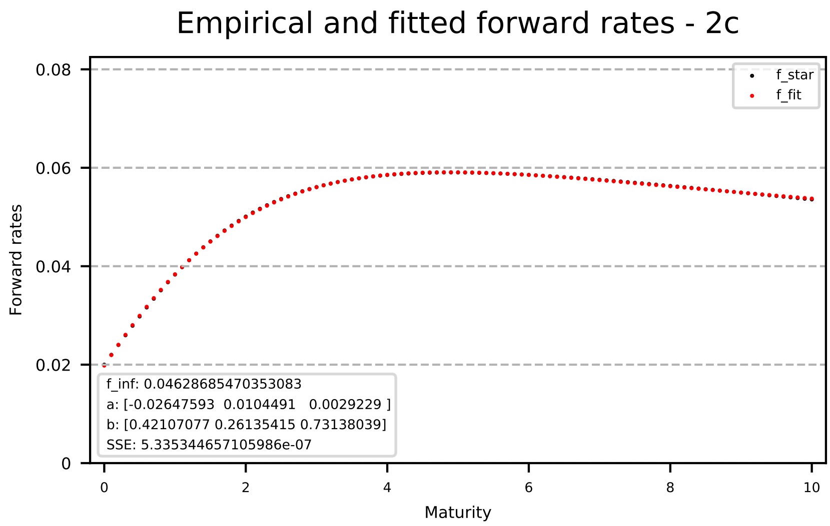


b) A plot of the market forward rates we have observed looks as follows

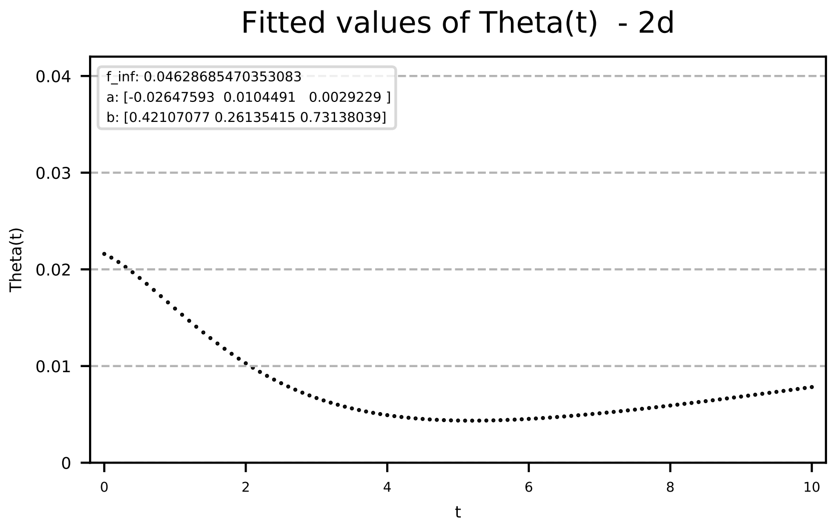


If we choose $K = 0$ there will be no hump in the fitted term structure of forward rates $f^*(T)$, if we choose $K = 1$ and choose a_1 sufficiently large, we will get one hump. If we choose $K = 2$ and choose a_1 and a_2 sufficiently large, we will get two humps and so on. There is one 'hump' in the term structure of market forward rates and hence, we have to choose K to be at least 1 but we could choose K higher than that provided the corresponding values of a_k are sufficiently small to not create a second hump.

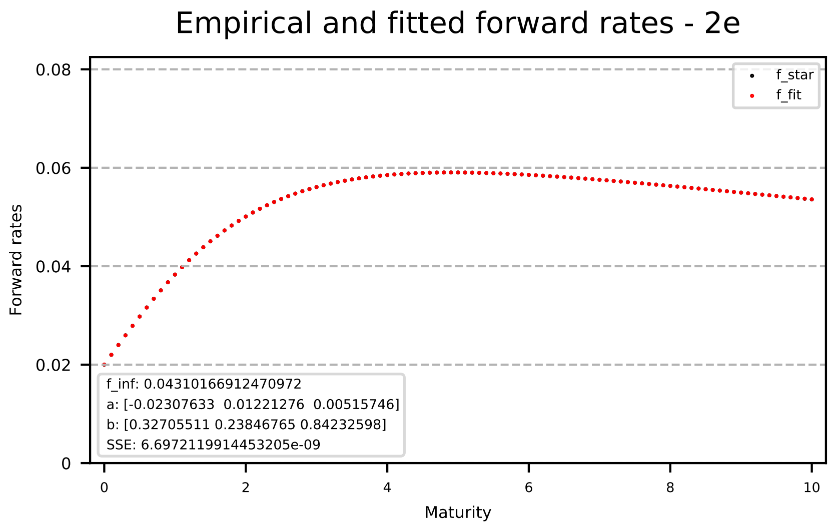
c) A plot of the fitted values of f^* for $K = 3$ is shown below and we see that the fitted values are very close to observed market forward rates.



d) The function $\Theta(t)$ corresponding to the fit performed in 2c looks as follows.



e) Fitting market forward rates using the 'Newton-CG' should and does also result in a very good fit that is plotted below. Not surprisingly, this algorithm results in parameter estimates that are very close to the ones we found in 2c.



Problem 3

Now that we have fitted the Ho-lee model to observed zero coupon bond prices and found the corresponding $\Theta(t)$, we can proceed to use the fitted values to price a complicated derivatives. We will consider two different derivatives and in both cases find a fair value of the derivative at initial time $t = 0$ by simulating the short rate. For each trajectory of simulated values of the short rate, compute the discounted value of the derivative at maturity and repeat the simulation sufficiently many times so that the value of the derivative has converged. When simulating the short rate use the Nelson-Siegel function and the estimated parameters from the best fit you obtained in Problem 1 with $\sigma = 0.03$. Denote by M, the number of steps in your simulation and index the time points in your simulation by $m = 0, 1, 2, ..., M-1, M$, the time points will then be denoted $[t_0, t_1, ..., t_{M-1}, t_M] = [0, \delta, 2\delta, ..., (m-1)\delta, T = M\delta]$ and $\delta = \frac{T}{M}$. The scheme you will need to implement is a simple Euler first-order scheme of the form

$$r_m = r_{m-1} + \Theta(t_{m-1})\delta + \sigma\sqrt{\delta}Z_m, \qquad m = 1, 2, ..., M \tag{24}$$

where $Z_m \sim N(0, 1)$, $m = 1, ..., M$ and all the standard normal random variables are independent.

a) Using what we know about the relationship between the short rate and forward rates, find a good starting value r_0 for your simulation in terms of the parameters f_∞, $\mathbf{a}$ and $\mathbf{b}$.

First we consider an Asian-style derivative which at maturity $T = 2$ pays the average short rate over the period from $t = 0$ to $T = 2$ provided the average short rate is positive. The contract function for this option is in other words

$$\chi(T) = \frac{1}{T}\max\Big(\int_0^T r_u du, 0\Big) = \frac{1}{T}\Big(\int_0^T r_u du\Big)_+ \tag{25}$$

and the time $t = 0$ value Π of the Asian-style derivative can be expressed as

$$\Pi = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^T r_u du\Big)\chi(T)\Big|\mathcal{F}_0\Big] \tag{26}$$

b) Run the simulation N times where you can start by setting N to something small, say $N = 10$. For each trajectory denoted by n, collect the discounted value of $\chi_n(T)$ in a vector of length N and use these values to find an estimate for Π as a function of the number of simulations.

c) Now run the above scheme for large values of M and N and plot the value of the derivative as a function of the number of simulations. Assess how many simulations are for the scheme to have converged as a function of N. Does the fact that the simulation has converged implies that the price of the derivative is accurate?

Second, we consider a 1Y4Y payer swaption which at time of exercise T gives the owner of the swaption the right to enter into a 4Y payer swap at a strike of $K = 0.04$.

d) Find an expression for the payoff at time of exercise and an expression for the time $t = 0$ price of the 1Y4Y payer swaption. These expressions will depend on ZCB prices $p(t, T)$ in the Ho-Lee model when the model has been fitted to forward rates at time $t = 0$ as was done in problem 2.

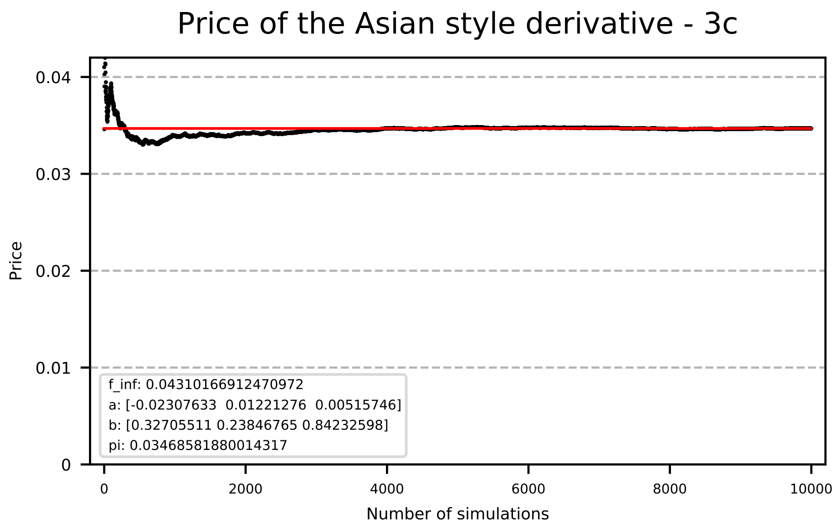
e) Run the simulation for a reasonably large M and N and plot the price of the 1Y4Y swaption as a function of the number of simulations and be sure to choose a sufficiently large N to insure that the scheme has converged as a function of N. Assess if the value of your swaption is accurate. Could you have computed the value of the swaption using the explicit formula for option prices that are available for the Ho-Lee model?

Solution

a) We know that $r_0 = f(0,0)$ and from 1, we get that $r_0 = f_\infty + a_0$, So, we choose the starting value from the Newton-CG fit and get $r_0 = \hat{f}_\infty + \hat{a}_0 = 0.04310166912470972 - 0.02307633 = 0.020025334538834955$.

b) The estimate of the price of the Asian derivative will require many simulations to converge, so only using $N = 10$ will very likely result in an estimate that is far from the true value. As an example, the value for a seed of 2024 gives us an estimate of $\hat{\Pi} = 0.045457$.

c) Running the simulation for $N = 10,000$ gives us a price estimate of $\hat{\Pi} = 0.0346858$ and clearly, the algorithm has converged well for such a large N.



d) Assume that we take M steps to simulate the trajectory of the short rate up to exercise at $T = 1$. The payoff χ_n at exercise for the n'th simulation, the discounted payoff $\tilde{\chi_n}$ and the time 0 price estimate $\hat{\Pi}$ of the 1Y4Y payer swaption can be found from

$$\begin{aligned} \chi_n &= S_1^5 \big(R_{1n}^5 - K \big)_+ \\ \tilde{\chi}_n &= \exp\Big(-\frac{1}{M+1} \sum_{m=0}^{M} r_{m/M} \Big) S_{1n}^5 \big(R_{1n}^5 - K \big)_+ \\ \hat{\Pi} &= \frac{1}{N} \sum_{n=1}^{N} \tilde{\chi}_n \end{aligned} \tag{27}$$

e) A plot of the swaption price estimate as a function of the number of simulations can be seen below, and we see that we get a price estimate of roughly $\hat{\Pi} = 0.0593$. It seems that the algorithm has converged reasonably well and that the price estimate is accurate. We have an explicit expression for the price of a caplet in the Ho-Lee model and thus also for an interest rate cap. However, we do not have an explicit expression for the price of a swaption.