Fixed Income Derivatives - CIR Exam Preparation

In the CIR model, the short rate r_t is assumed to have the following dynamics under the risk-neutral measure $\mathbb{Q}$

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{1}$$

where a, $b > 0$ and σ are model parameters.

a) Solution and distribution of $r(t)$.

  i) Find the closest equivalent to a solution for $r(t)$, $t > 0$, in the CIR model.

  ii) Find the mean $\mathrm{E}[r(t)]$ and the variance $\mathrm{Var}[r(t)]$ of $r(t)$. There is a 'trick' involved when computing the variance, what is that trick called.

  iii) State the distribution of $r(t)$ but do not try to derive it.

  iv) State the distribution of $r(t)$ as $t \nearrow \infty$ but do not try to derive it. That is, find the stationary distribution of $r(\infty)$.

b) Now assume that the parameters of the CIR model are $r_0 = 0.032$, $a = 1.2$, $b = 0.048$ and $\sigma = 0.1$ Compute ZCB prices, continuously compounded spot rates and instantaneous forward rates in the CIR model.

  i) Plot these ZCB prices, spot rates and instantaneous forward rates for values of t in $[0, 10]$.

  ii) Now compute spot rates for maturities $T \in \{0.5, 1, 1.5, ..., 10\}$ using the parameters given above and fit all the parameters in a CIR model to these spot rates. Are you able to recover the parameters of the CIR model as you should be? Are there any difficulties with any of the parameters?

  iii) Now try to fit the parameters this time assuming that you known that $\sigma = 0.1$. Are you able to fit the model parameters and is the fit now better? Explain what this tells you about σ in a CIR model.

c) We will now study interest rate swaps that involve swapping 6M floating rate payments (EURIBOR say) paid semiannually for fixed payments also paid semi-annually. Assume that present time is $t = 0$ and that the 6M floating rate to be paid at time $T = 0.5$ has just been announced. For simplicity, you can assume the notional of these swaps is just 1.

  i) Compute the par swap rate of 10Y interest rate swap.

  ii) Compute par swap rates for maturities $T \in \{1, 1.5, ..., 10\}$ and plot the par swap curve along side spot rates and 6M forward rates.

  iii) Explain how 6M forward rates and par swap rates are related.

  iv) You will notice that par swap rates and zero coupon spot rates are very close. Explain why that is so by appealing to concepts such as the accrual factor of a swap and the duration of all bonds in general.

d) Now we will simulate short rates in the CIR model using the usual first order Euler scheme on a grid of mesh δ that runs from initial time $t_0 = 0$ to terminal time $T = 10$. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$ and the time points will be $\big[t_0, t_1, ..., t_{M-1}, t_M\big] = \big[0, \delta, 2\delta, ..., T - \delta, T\big]$ and $\delta = \frac{T}{M}$. The scheme you will need to implement is a simple Euler first-order scheme of the form

$$r_m = r_{m-1} + a(b - r_{m-1})\delta + \sigma\sqrt{r_{m-1}}\sqrt{\delta}Z_m, \quad m = 1, 2, ..., M \tag{2}$$

where $Z_m \sim N(0, 1)$, $m = 1, ..., M$ and all the standard normal random variables are independent.

  i) Simulate one trajectory of the short rate and plot the trajectory up to time $T = 10$.

ii) Construct 95 percent two-sided confidence intervals for the short rate and plot these in the same plot.

iii) Construct a 95 percent two-sided confidence interval for the short rate under the stationary distribution and plot this confidence interval in the same plot. Based on the plot, can you say that the distribution of r_3 is roughly the same as that of the stationary distribution? How does this change if you change the parameters?

e) We will now consider the pricing of an interest rate cap on future EURIBOR fixings that begins immediately and ends in $T = 5$ years. The cap will have a strike of R=0.045. We will price this derivative using simulation.

i) Explain how an interest rate cap is related to caplets and also explain, how a caplet can be seen as a type of European option and on what underlying.

ii) Deduce an expression for the discounted payoff of a caplet at time $t = 0$ on the underlying reference rate $L(T_{i-1}, T_i)$.

iii) Simulate at least $N = 1000$ trajectories for the short rate up to time $T = 5$ and in each simulation, at least $M = 1000$ steps should be taken. For each simulated path, compute the discounted payoffs to all caplets with a maturity less than 5 years. Once you have simulated N trajectories, you can compute the price of the caplets by averaging the discounted payoffs.

iv) Compute the price of the 5Y interest cap with a strike of $F = 0.045$ and discuss how the price of the cap depends on the strike R and σ.

v) Also compute the price of a 2Y3Y payer swaption with a strike of $K = 0.045$ by using a similar approach of computing discounted payoffs for each trajectory and averaging over all simulations.