Linear Programming: Introduction

Optimization problems are present in our daily lives, stemming from our need to make the right decisions. For example, which route to take to Deakin, or how many hours to allocate to studies.

The decisions in an optimization problem are often represented as the symbols $x_1,\ x_2, \dots, x_n$.

In many optimization problems, we are faced with restrictions or constraints.

The generally ways to express constraint optimization are:

- Less or equal to a constraint: $f(X_1,\ X_2, \dots, X_n) \leq b$
- More or equal to a constraint: $f(X_1,\ X_2, \dots, X_n) \geq b$
- Equal to a constraint: $f(X_1,\ X_2, \dots, X_n) = b$

There is a goal or objective that the decision maker considers when deciding which action is best.

The mathematical formulation can be written as:

Max or min $f(X_1,\ X_2, \dots, X_n)$

Subject to $f_1(X_1,\ X_2, \dots, X_n) \leq b_1$

$$f_2(X_1,\ X_2, \dots, X_n) \geq b_2$$

$$\dots$$

$$f_m(X_1,\ X_2, \dots, X_n) = b_m$$

Linear programming (LP) involves creating and solving optimization problems with linear objective functions and linear constraints only.

We can rewrite the formula as:

Max or min $c_1X_1 + c_2X_2 + \cdots + c_nX_n$

Subject to $a_{11}X_1 + a_{12}X_2 + \cdots + c_{1n}X_n \leq b_1$

$$a_{21}X_1 + a_{22}X_2 + \cdots + c_{2n}X_n \geq b_2$$

$$\dots$$

$$a_{m1}X_1 + a_{m2}X_2 + \cdots + c_{mn}X_n = b_m$$

Linear Programming: A Graphical Approach

We look at a certain case:

Blue Ridge Hot Tubs manufactures and sells two models of hot tubs: the Aqua-Spa and the Hydro-Lux. Howie Jones, the owner and manager of the company, needs to decide how many of each type of hot tub to produce during his next production cycle. Howie buys prefabricated fiberglass hot tub shells from a local supplier and adds the pump and tubing to the shells to create his hot tubs. (This supplier has the capacity to deliver as many hot tub shells as Howie needs.) Howie installs the same type of pump into both hot tubs. He will have only 200 pumps available during his next production cycle. From a manufacturing standpoint, the main difference between the two models of hot tubs is the amount of tubing and labor required. Each Aqua-Spa requires 9 hours of labor and 12 feet of tubing. Each Hydro-Lux requires 6 hours of labor and 16 feet of tubing. Howie expects to have 1,566 production labor hours and 2,880 feet of tubing available during the next production cycle. Howie earns a profit of $350 on each Aqua-Spa he sells and $300 on each Hydro-Lux he sells. He is confident that he can sell all the hot tubs he produces. The question is, how many Aqua-Spas and Hydro-Luxes should Howie produce if he wants to maximize his profits during the next production cycle?

Max $350X_1 + 300X_2$

Subject to $X_1 + X_2 \leq 200$

$9X_1 + 6X_2 \geq 1556$

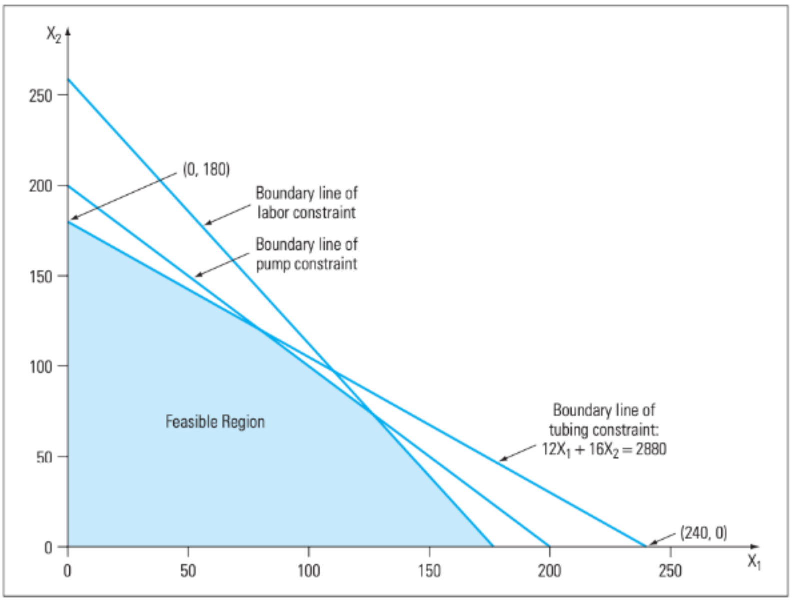
$12X_1 + 16X_2 \leq 2880$

$X_1 \geq 0$

$X_2 \geq 0$

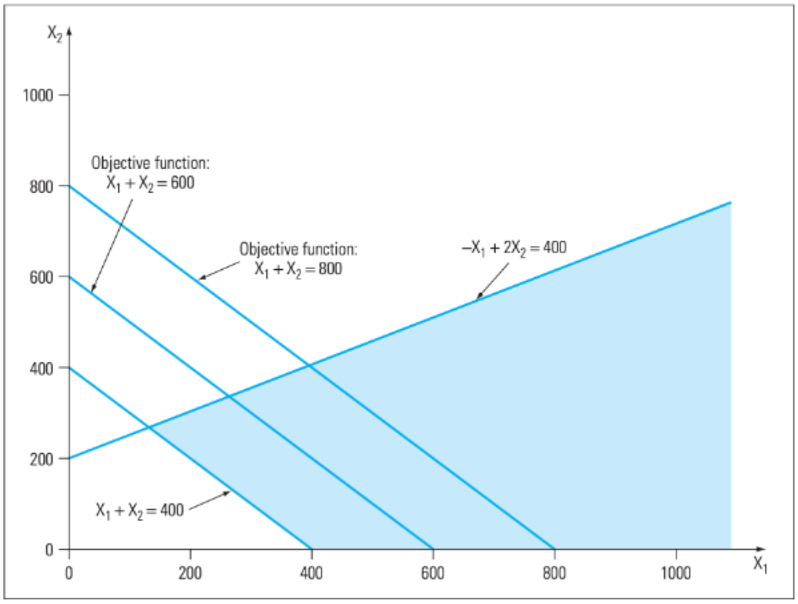
We are interested in finding the values X_1 and X_2.

By graphing all the formulas, we get:



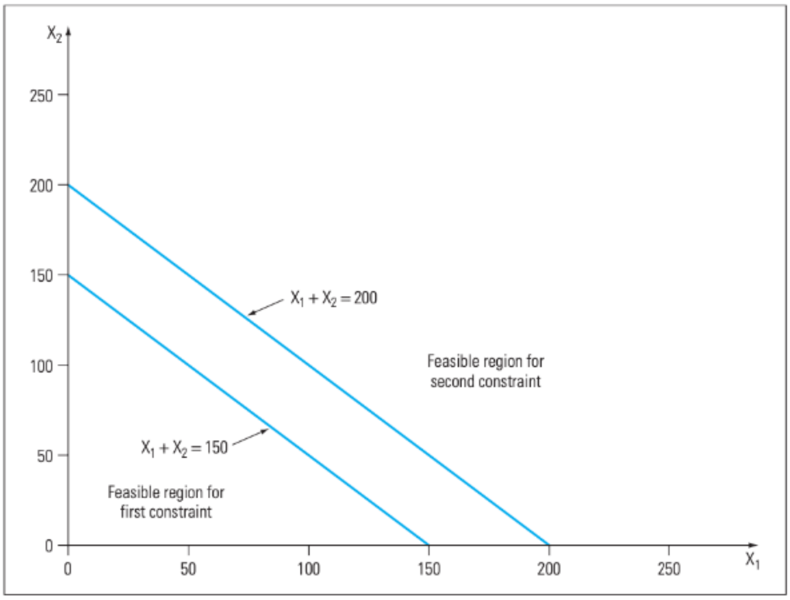
As we keep adding formulas, the feasible region (where optimal values of X_1 and X_2 lie) will decrease.

Issues in linear programming include: alternate optimal solutions, redundant constraints, unbounded solution and infeasibility. Unbounded solutions and infeasibility prevent us from solving an LP model whereas the rest are just anomalies.



As you can see in this graph, as the level curves shifts farther away from the origin, the objective function increases. The feasible region is not bounded to the origin so you can make the objective function infinitely large. This is called an unbounded solution.

An LP problem is infeasible if there is no way to satisfy all the constraints simultaneously. For example:



Linear Programming: Standard and Slack Forms

Various algorithms for linear problems needs objective function and constraints in a certain form.

In the standard form, all the constraints are in inequalities.

In the slack form all constraints must be in equalities.

An optimization problem in standard form looks like:

Maximize $\sum_{j=1}^{n} c_i \, x_j$

Subject to $\sum_{j=1}^{n} a_{ij} \, x_j \leq b_i$ for i=1,2,...,m

$x_j \geq 0$ for j=1,2,...,n

The formula for the slack form is:

$x_{n+1} = b_i - \sum_{j=1}^{n} a_{ij} \, x_j$, where $x_{n+i} \geq 0$

Linear Programming: Gaussian Elimination

We define a system of linear equations as:

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

...

$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$

We can write this in matrix-vector notation:

Ax = b

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

If A is non-singular, it has an inverse A^{-1}, which can be computed by $x = A^{-1}b$.

LU-Decomposition is recommended to calculate inverse A^{-1}

In LU decomposition, we find two nxn matrices L and U such that A=LU.

L is a lower-triangular matrix

U is an upper-triangular matrix

We can rewrite Ax=b as LUx=b

Forward substitution solves the lower triangular system first for unknown y is calculated by Ly=b, where y=Ux

Backward substitution solves the upper triangular system for unknown x is calculated by Ux=y.

Linear Programming: Simplex Algorithm

To execute the simplex method, we need to:

- Check if the linear problem is a standard maximization problem in standard form
- Create slack variables
- Create a system of equations using the variables
- Place the equations into a matrix, with the objective equation in the bottom row
- Select a pivot column by finding the most negative indicator
- Select a pivot row
- Find pivot
- If we don't get all non-negative indicators, repeat step 5 and 7

Example. Maximize $P = 3x_1 + x_2$ Subject to: $\begin{array}{l} 2x_1 + x_2 \le 8 \\ 2x_1 + 3x_2 \le 12 \\ x_1, x_2 \ge 0 \end{array}$

<u>Solution</u>

Step 1. This is of course a standard maximization problem in standard form.

Step 2. Rewrite the two problem constraints as equations by using slack variables:

$$\begin{array}{l} 2x_1 + x_2 + s_1 \qquad = 8 \\ 2x_1 + 3x_2 \qquad + s_2 = 12 \end{array}$$

Step 3. Rewrite the objective function in the form $-3x_1 - x_2 + P = 0$. Put it together with the problem constraints: $\begin{array}{l} 2x_1 + x_2 + s_1 \qquad = 8 \\ 2x_1 + 3x_2 \qquad + s_2 \qquad = 12 \\ -3x_1 - x_2 \qquad\qquad + \mathrm{P} = 12 \end{array}$, we get a linear system with 5 variables and 3 equations, which is called **initial system**.

Step 4. Write the initial system in matrix form (**initial simplex tableau**). See below.

Step 5 to step 9:

$$\begin{array}{c} \\ s_1 \\ s_2 \\ P \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 2 \text{ pivot} & 1 & 1 & 0 & 0 & 8 \\ 2 & 3 & 0 & 1 & 0 & 12 \\ \hdashline -3 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \frac{8}{2} = 4 \text{ smallest positive pivot row} \\ \frac{12}{2} = 6 \\ \\ \end{array}$$

most negative indicator
pivot column

$$\underset{\sim}{\tfrac{1}{2}R_1 \to R_1} \quad \begin{array}{c} x_1 \\ s_2 \\ P \end{array} \left[\begin{array}{ccccc|c} 1 & 1/2 & 1/2 & 0 & 0 & 4 \\ 2 & 3 & 0 & 1 & 0 & 12 \\ \hdashline -3 & -1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} -2R_1+R_2 \to R_2 \\ \sim \\ 3R_1+R_3 \to R_3 \end{array} \quad \begin{array}{c} x_1 \\ s_2 \\ P \end{array} \left[\begin{array}{ccccc|c} 1 & 1/2 & 1/2 & 0 & 0 & 4 \\ 0 & 2 & -1 & 1 & 0 & 4 \\ \hdashline \underbrace{0 \quad 1/2 \quad 3/2 \quad 0}_{\text{no negative indicators, stop!}} & & & & 1 & 12 \end{array}\right]$$

max. value of P

@ $x_1 = 4$, $x_2 = 0$

Check: compare to the method we did in 5-3, we got same answer!

Maximize $18X_1 + 12.5X_2$

subject to $X_1 + X_2 \le 20$

$X_1 \le 12$

$X_2 \le 16$

$X_1, X_2 \ge 0$

$\therefore$ introduce slack numbers

$\therefore \cancel{X_1 + X_2 \le 20}$

$X_1 + X_2 + S_1 = 20$

$X_1 + S_2 = 12$

$X_2 + S_3 = 16$

X_1	X_2	S_1	S_2	S_3	C
1	1	1	0	0	20
1	0	0	1	0	12
0	1	0	0	1	16
-18	-12.5	0	0	0	0

X_1	X_2	S_1	S_2	S_3	C
0	1	1	-1	0	8
1	0	0	1	0	12
0	1	0	0	1	16
0	-12.5	0	18	0	216

~~X_1 X_2 X_3 X_4 X~~

X_1	X_2	S_1	S_2	S_3	C
0	1	1	-1	0	8
1	0	0	1	0	12
0	0	-1	1	1	8
0	0	12.5	5.5	0	316

$X_1 = 12$

$X_2 = 8$

$C = 316$