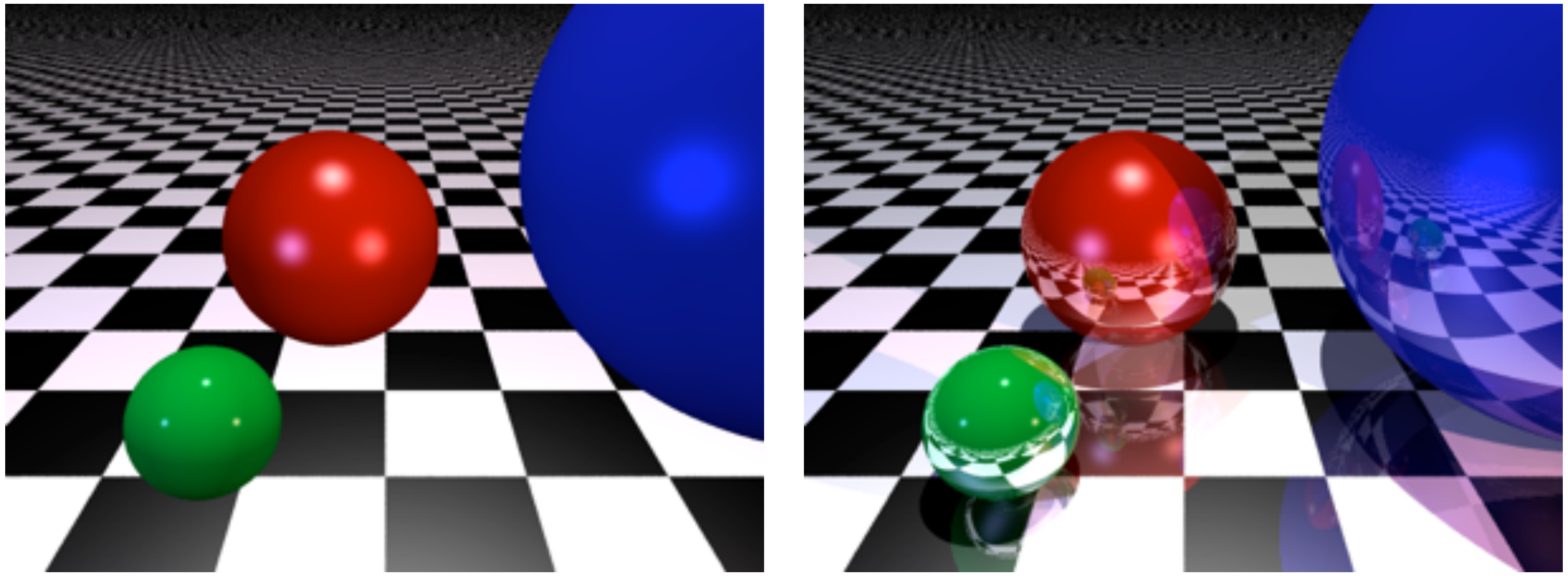
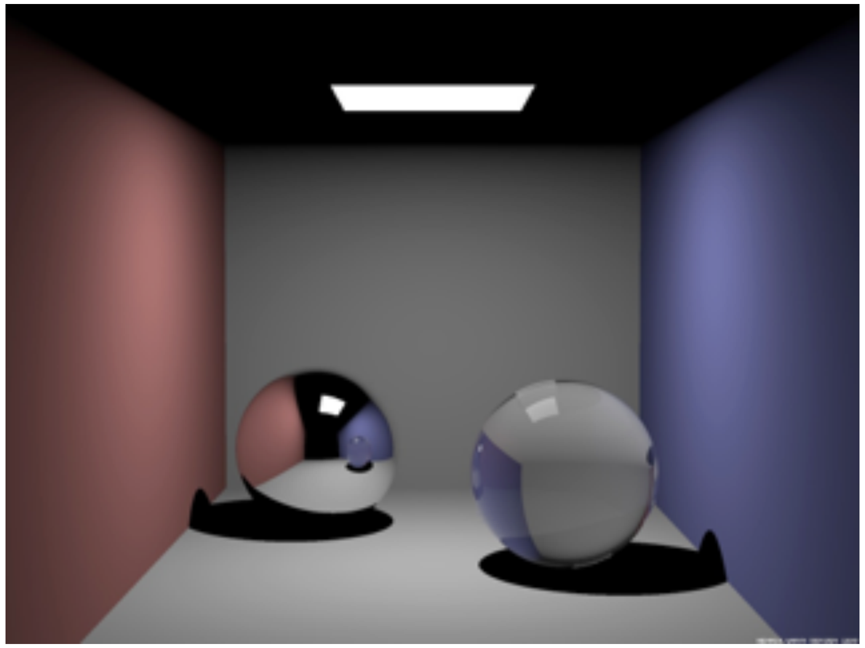
Introduction to Computer Graphics
The Rendering Equation



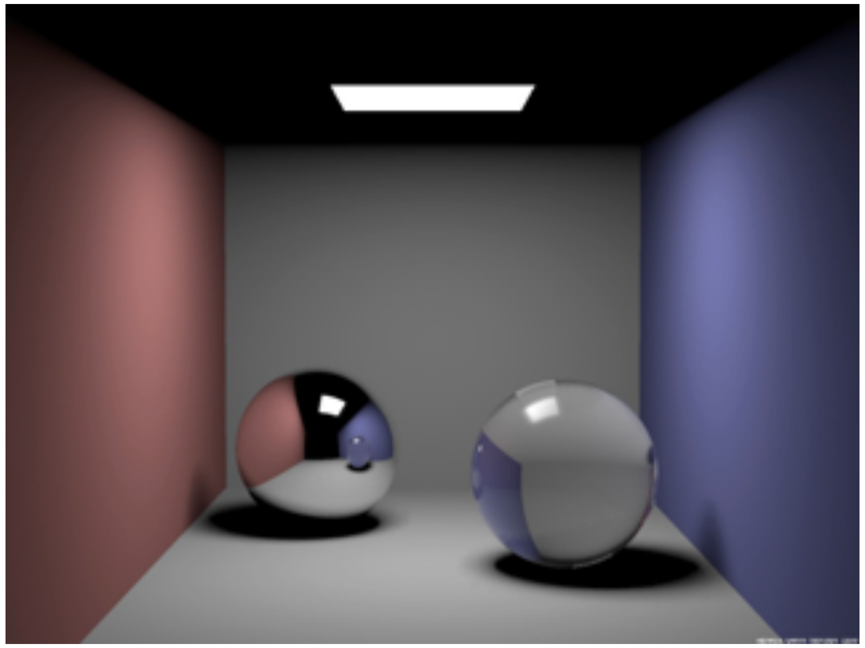
Prof. Dr. Mario Botsch

Computer Graphics & Geometry Processing

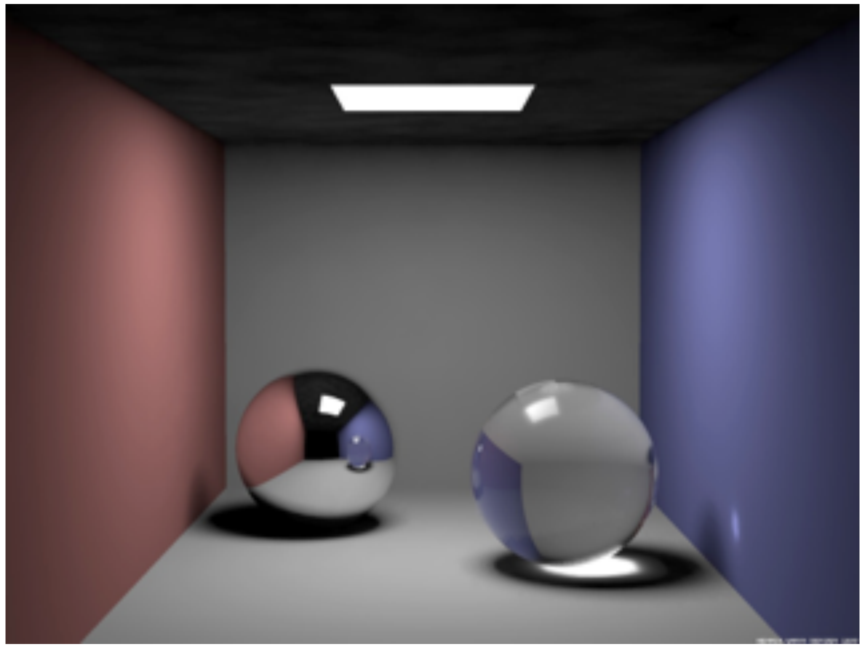
The Quest for Realism



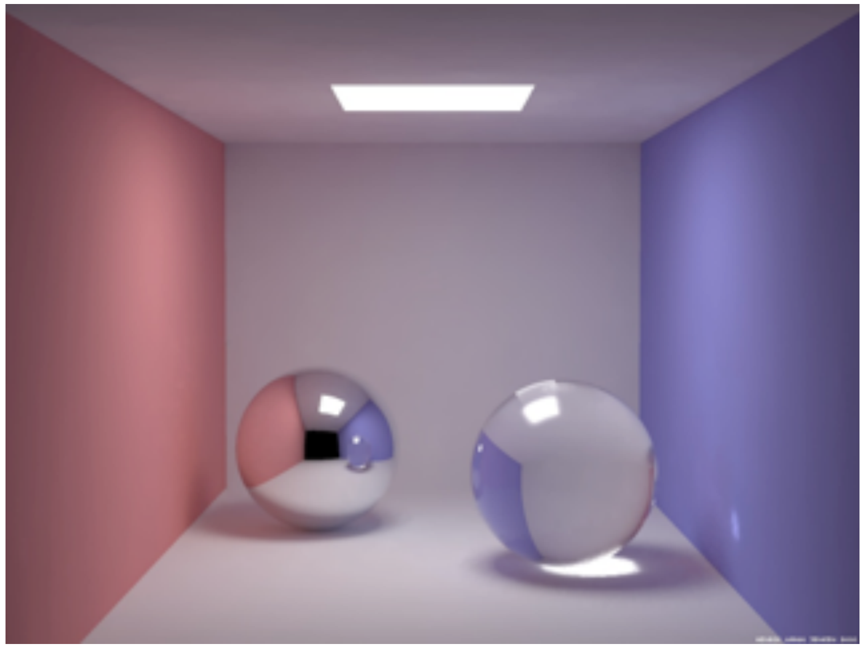
Ray Tracing



Soft Shadows



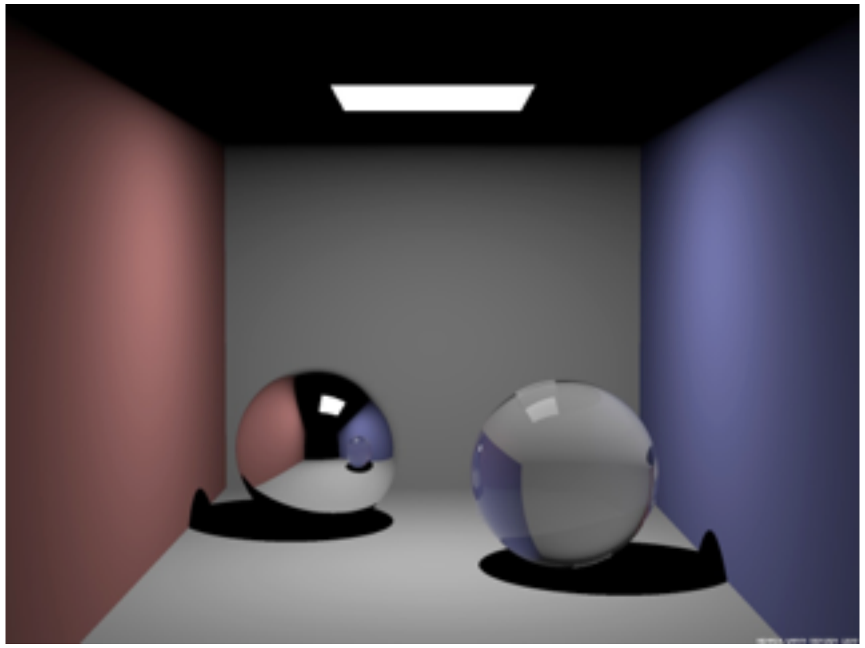
Caustics

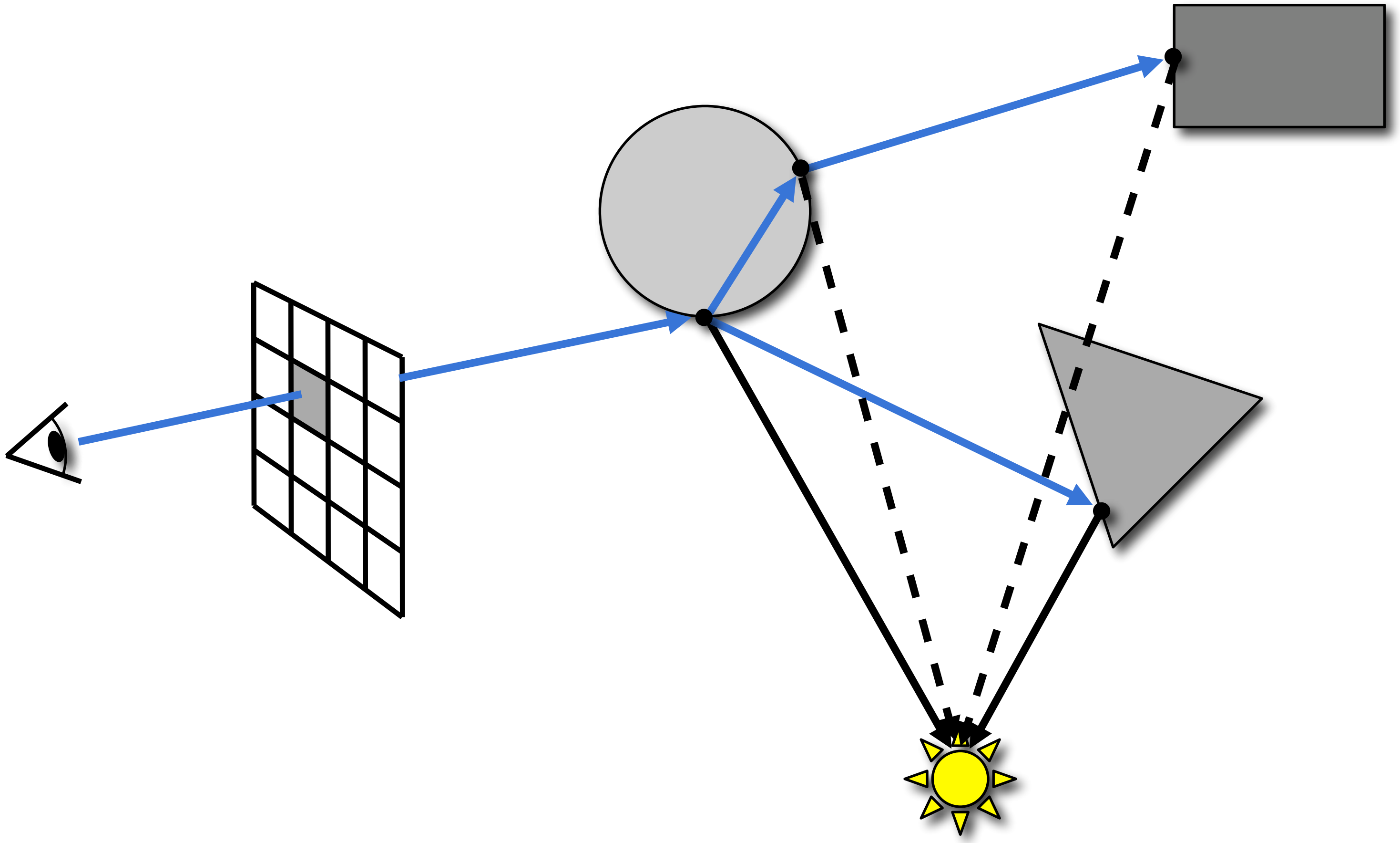


Indirect Illumination

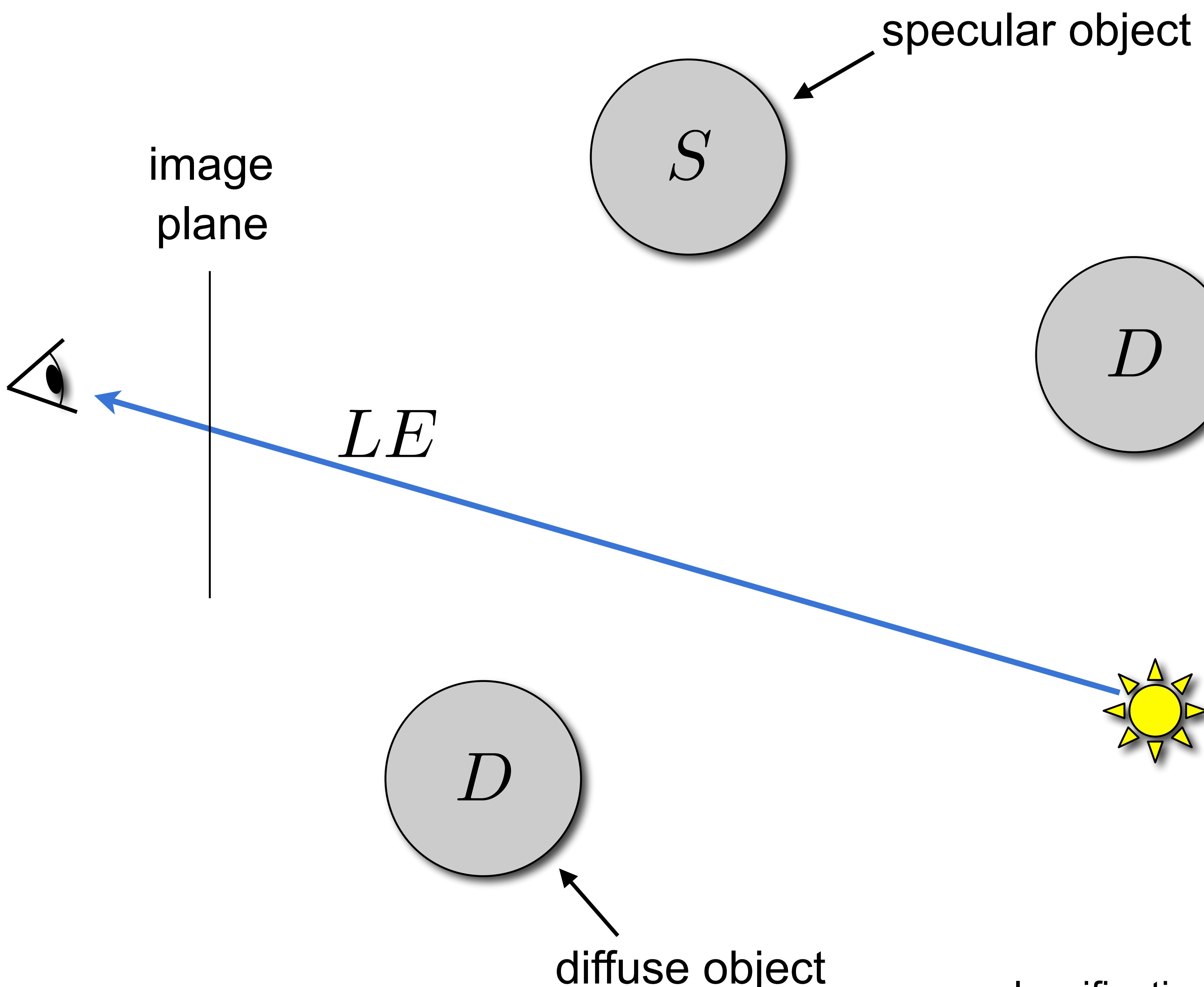
[Henrik van Jensen]

Recursive Raytracing



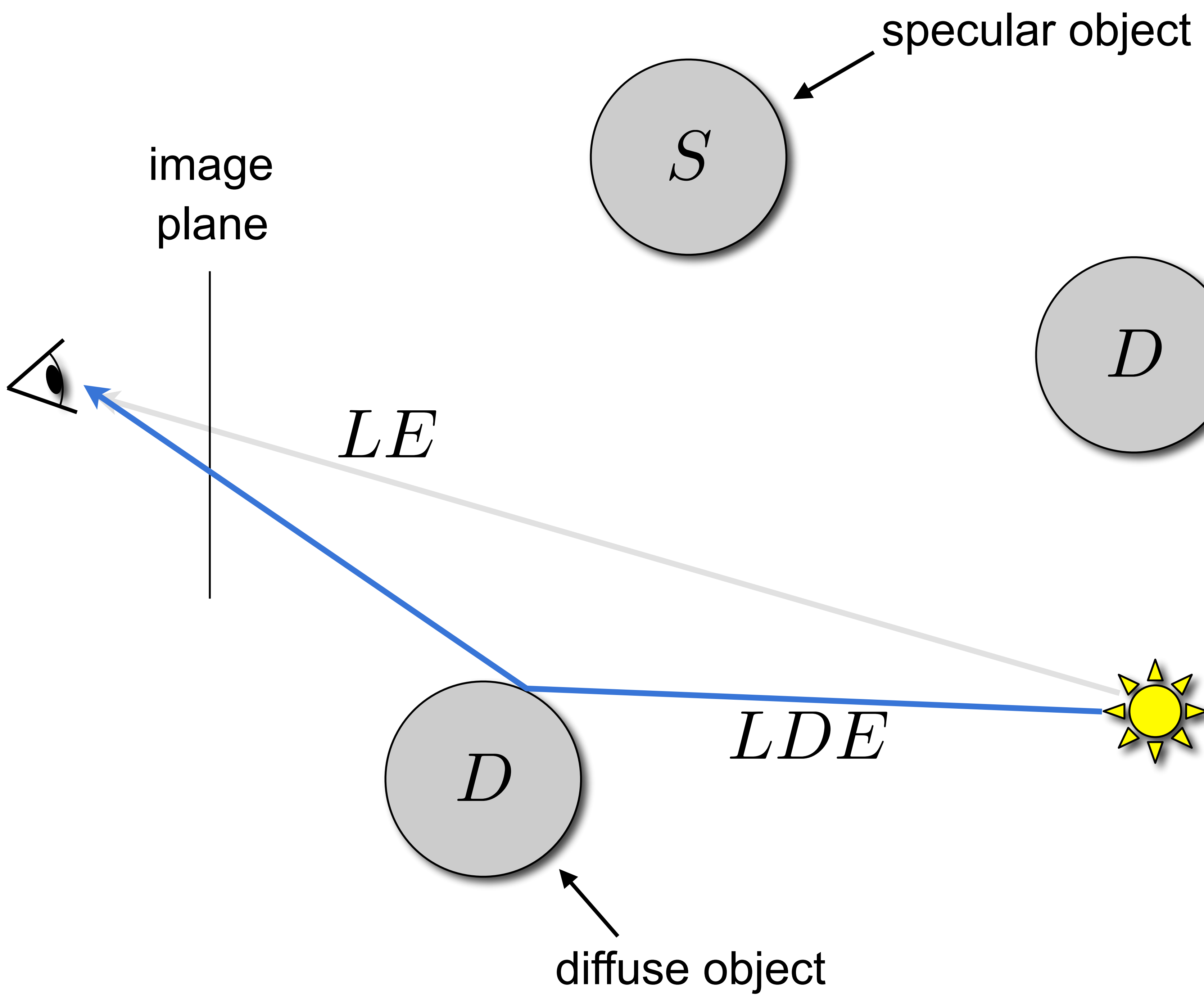


Light Paths

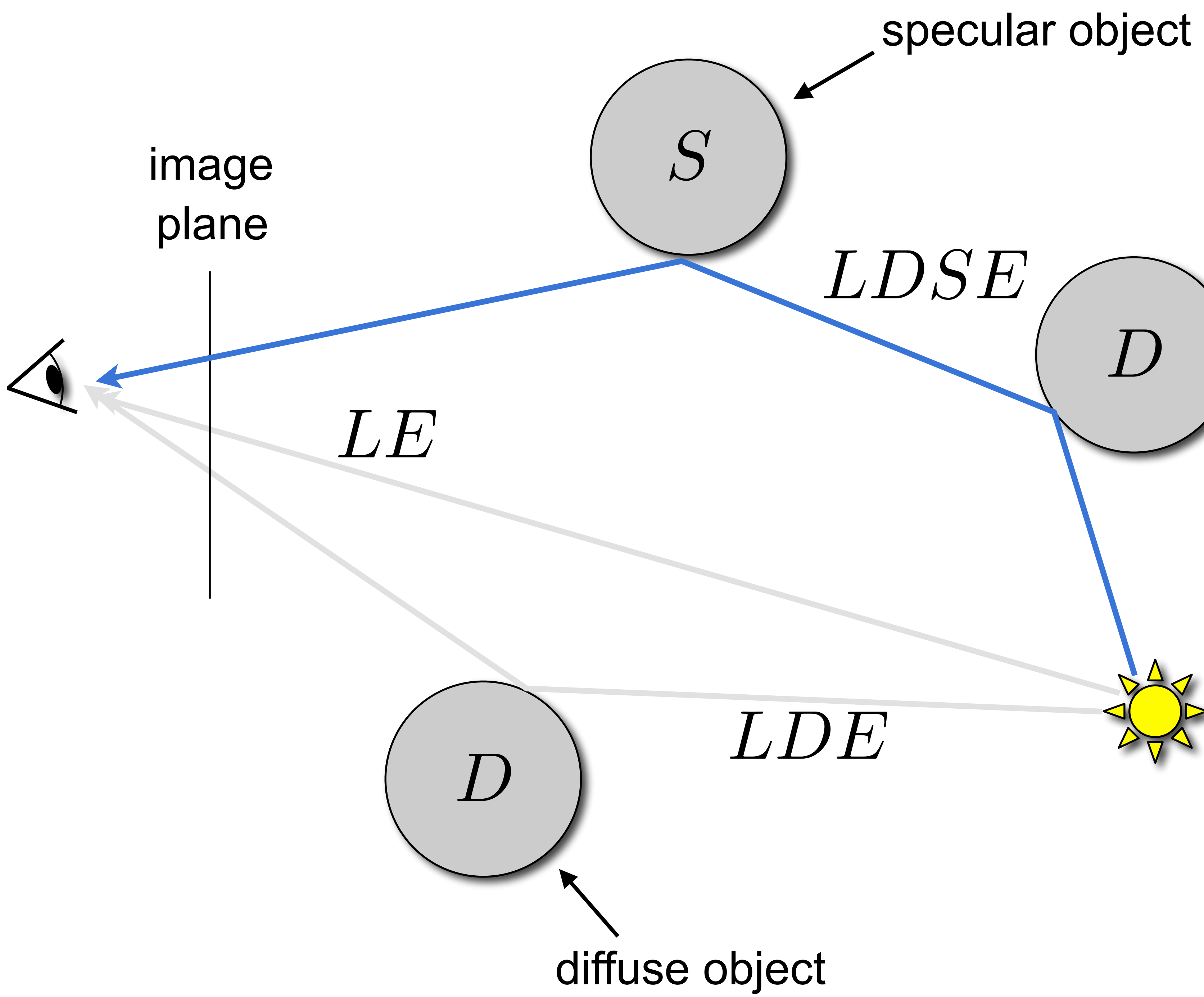


classification by Paul Heckbert

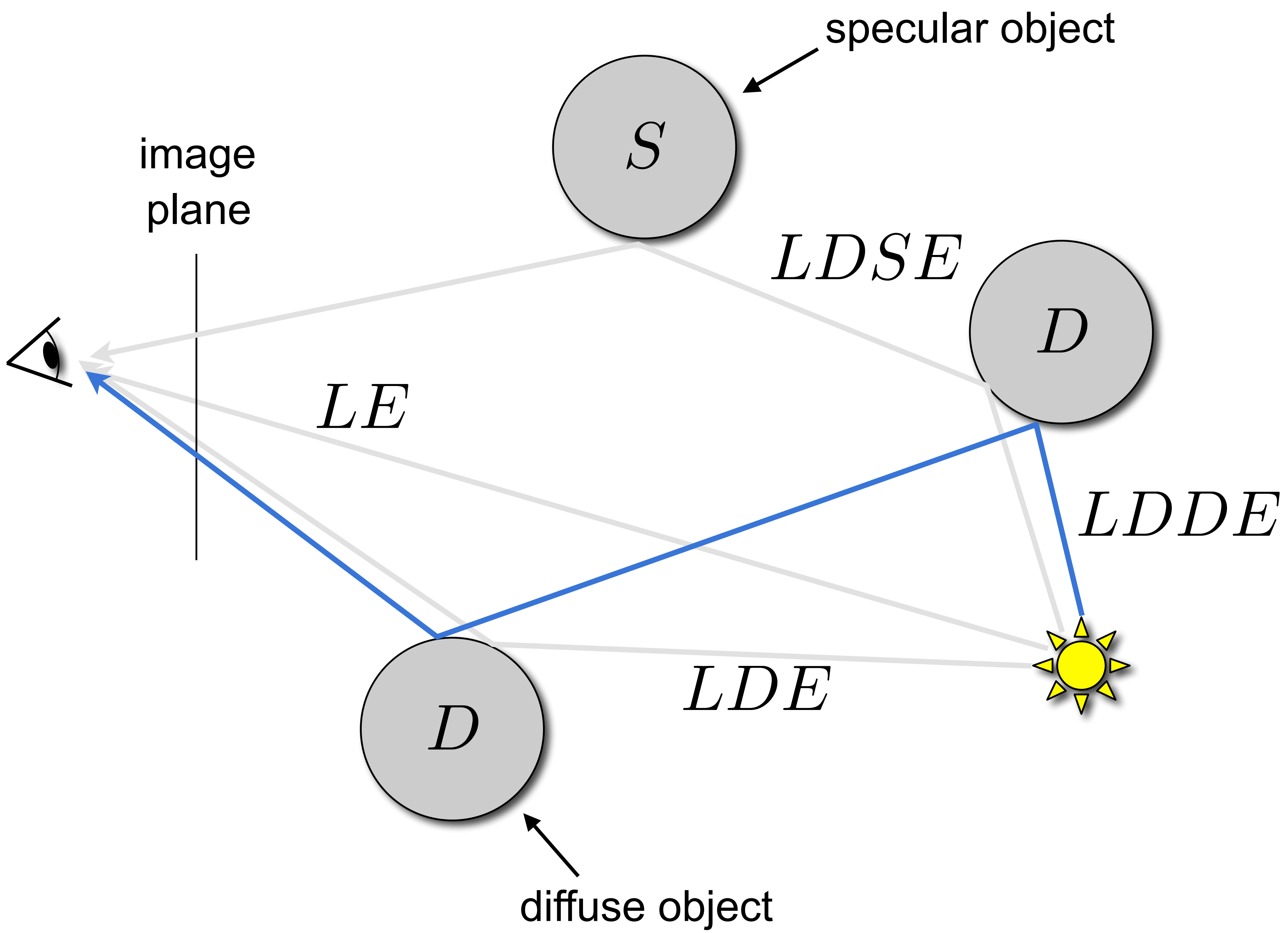
Light Paths



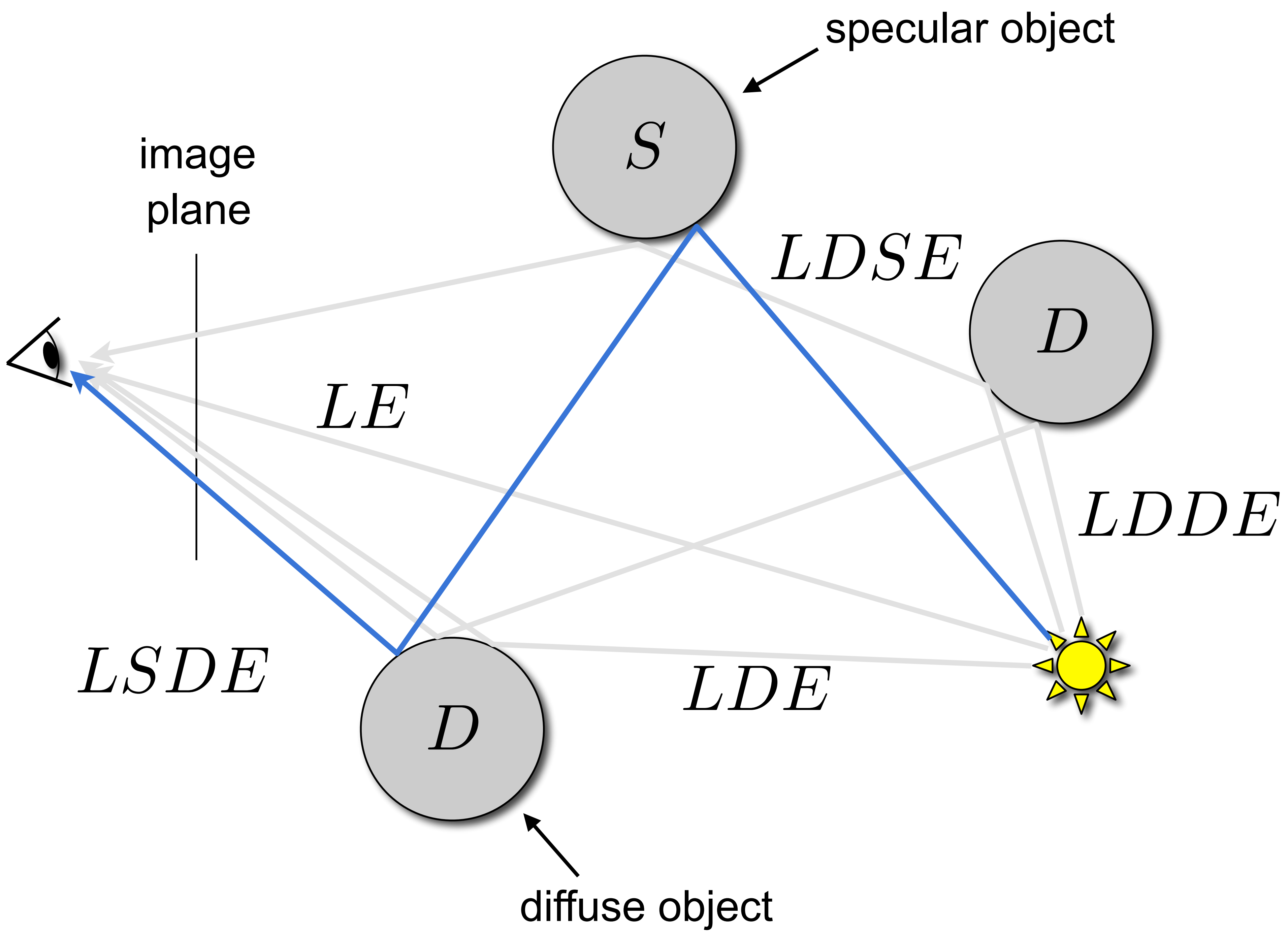
Light Paths



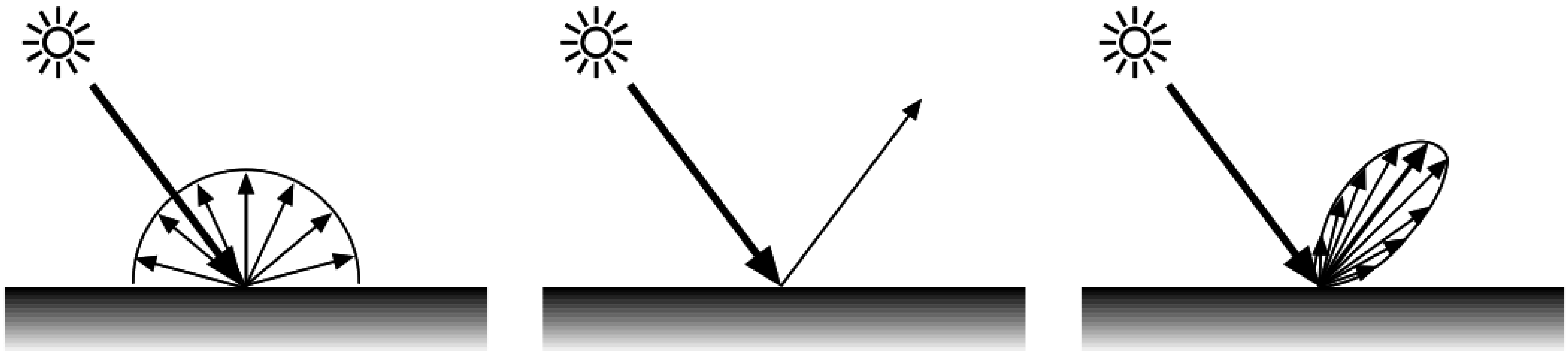
Light Paths


specular object
image
plane
S
$LDSE$
D
LE
$LDDE$
LDE
D
diffuse object

Light Paths


specular object
image
plane
S
$LDSE$
D
LE
$LDDE$
$LSDE$
D
LDE
diffuse object

Types of Reflection

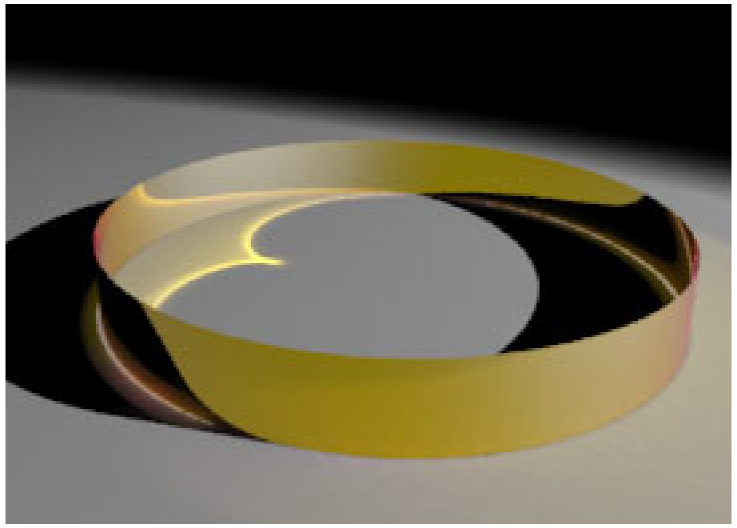


D: diffuse S: specular G: glossy

Hendrik Lensch, *"Efficient Image-Based Appearance Acquisition of Real-World Objects"*, Ph.D. thesis, 2004

Recursive Raytracing

- Classical raytracing (from the eye) can only handle paths of the form L(D|G)[S*]E
  - No multiple diffuse inter-reflections
  - No caustics

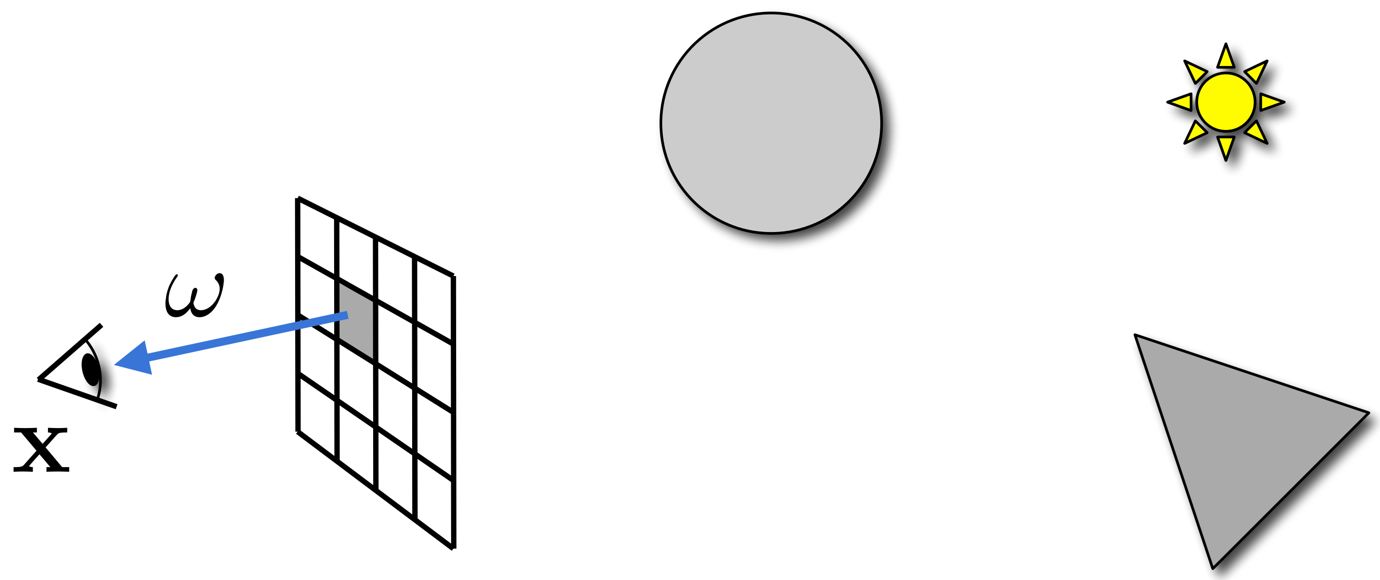


The Rendering Equation

James Kajiya, *"The Rendering Equation"*, SIGGRAPH 1986

See also Pat Hanrahan's course:
https://graphics.stanford.edu/wikis/cs348b-08

Image Synthesis



pixel color

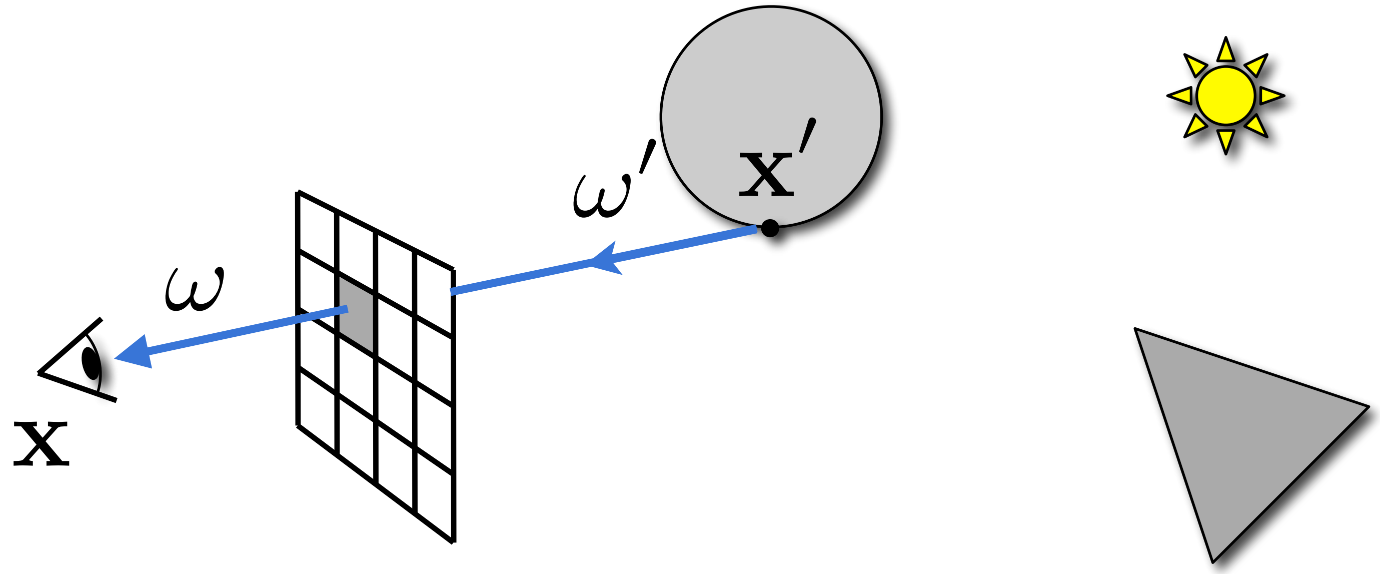
incoming ‘light’

$$c \sim L_i(\mathbf{x}, \omega)$$

position

direction

Image Synthesis



pixel color

incoming ‘light’

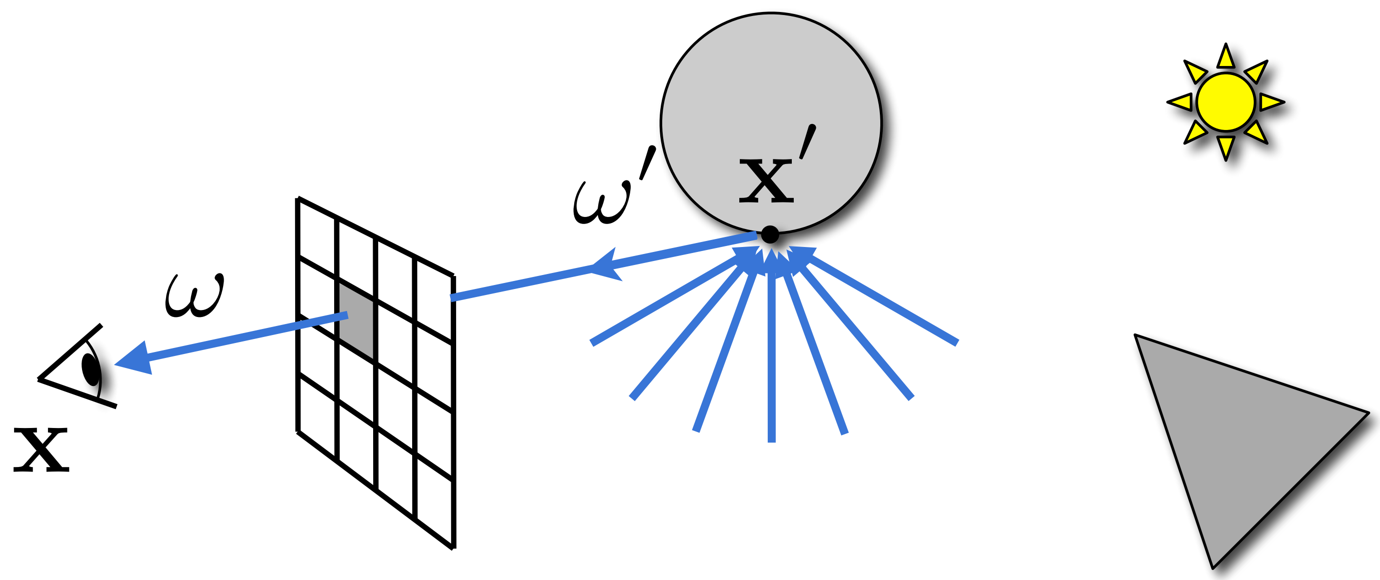
$$c \sim L_i(\mathbf{x}, \omega) = L_o(\mathbf{x}', \omega')$$

position

direction

outgoing ‘light’

Image Synthesis



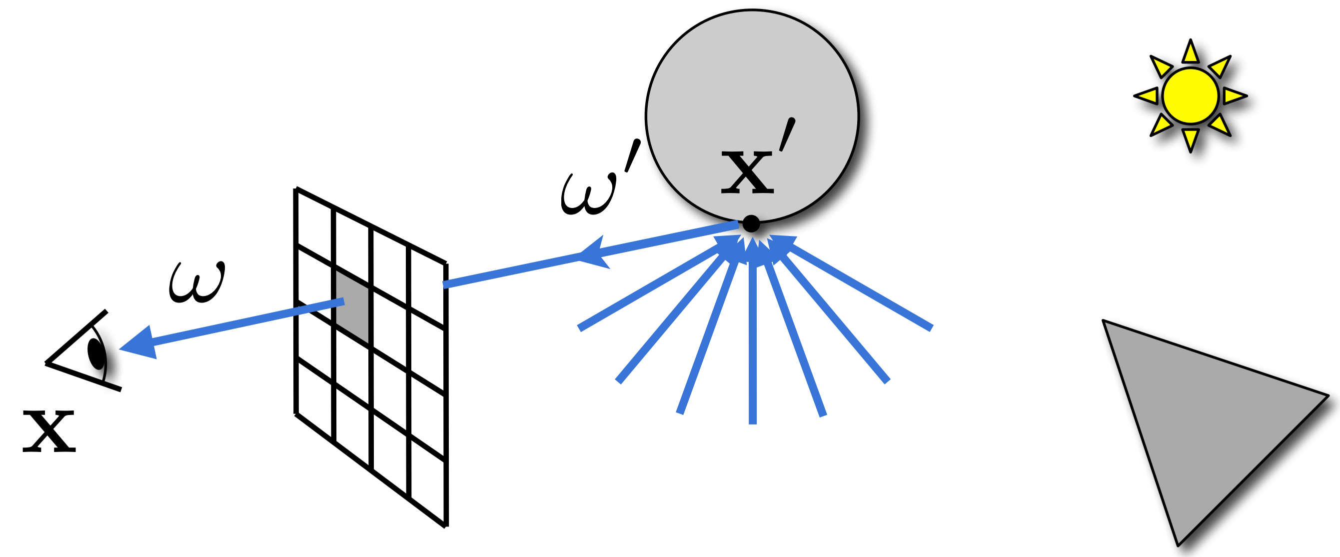
pixel color — incoming 'light' — emitted 'light'

$$c \sim L_i(\mathbf{x}, \omega) = L_o(\mathbf{x}', \omega') = L_e(\mathbf{x}', \omega') + L_r(\mathbf{x}', \omega')$$

position — direction — outgoing 'light' — reflected 'light'

Ingredients

- **Radiometry**
  - What is ‘light’?



- Reflection models
  - Relation between incoming and reflected ‘light’
- Light transport
  - Relation between

    $(\mathbf{x}, \omega) \leftrightarrow (\mathbf{x}', \omega')$

Radiative Transfer

- Study of transfer of radiative energy
  - – Based on geometric optics
- Simplifying assumptions
  - Linearity
  - Energy conservation
  - No polarization, fluorescence, phosphorescence
  - Steady state (equilibrium)

Radiometry

- Basic quantities (depend on wavelength)
  - flux Φ
  - irradiance E
  - radiance L

will be the most important quantity for us

Radiometry

- *Flux* (radiant flux, power)
  - Total amount of energy passing through surface or space per unit time

$$\Phi \qquad \left[\frac{J}{s} = W\right]$$

Radiometry

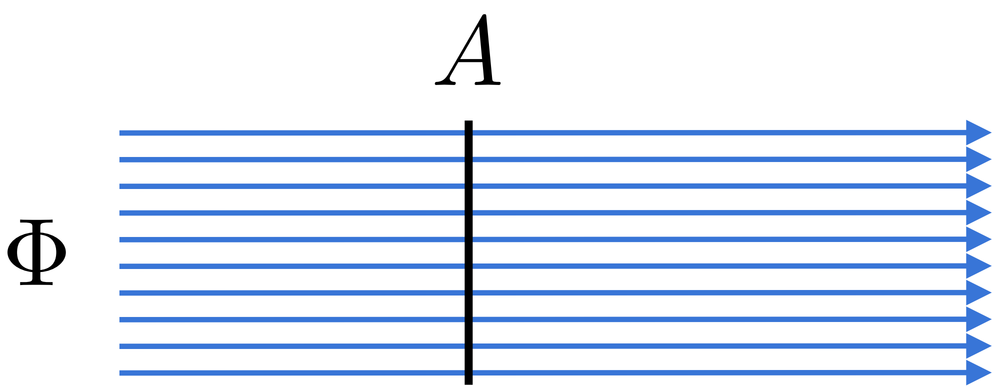
- *Irradiance*
  – Power per unit area incident on a surface

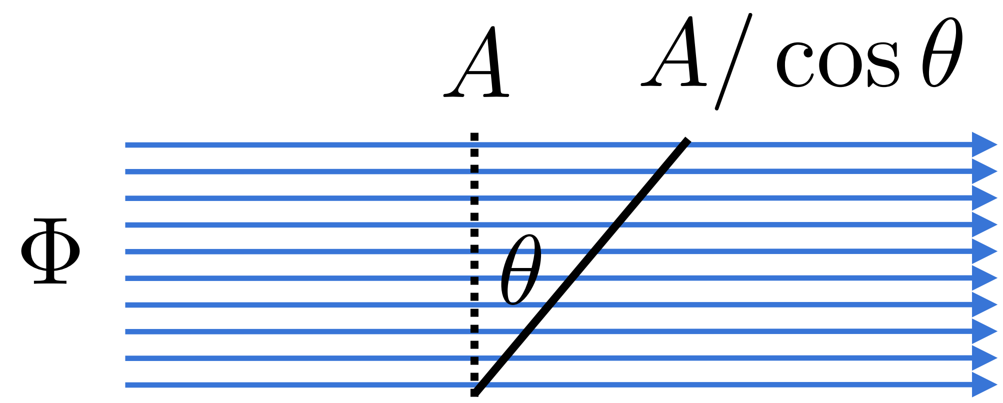
    = area density of flux

$$E(\mathbf{x}) = \frac{d\Phi}{dA} \qquad \left[\frac{W}{m^2}\right]$$

Radiometry

- *Irradiance*
  - Lambert's cosine law





$$E = \frac{\Phi}{A}$$

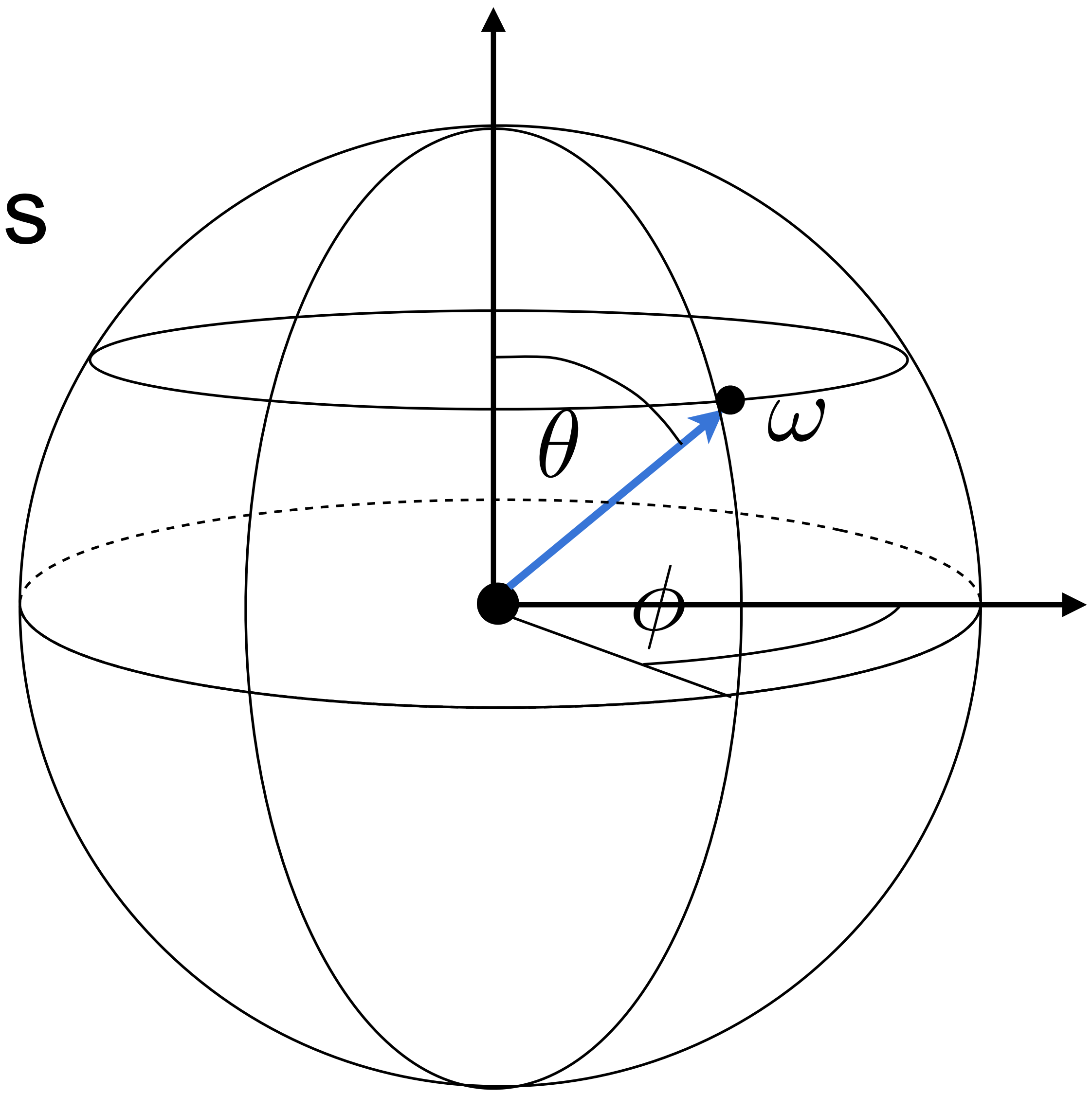
$$E = \frac{\Phi}{A/\cos\theta} = \frac{\Phi}{A}\cos\theta$$

Directions & Solid Angle

- Direction
  - point on the unit sphere
  - parameterized by two angles

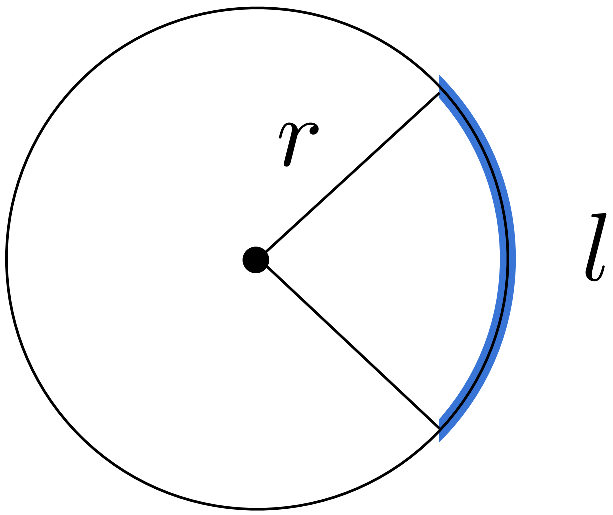
$$\omega = (\theta, \phi)$$

zenith azimuth

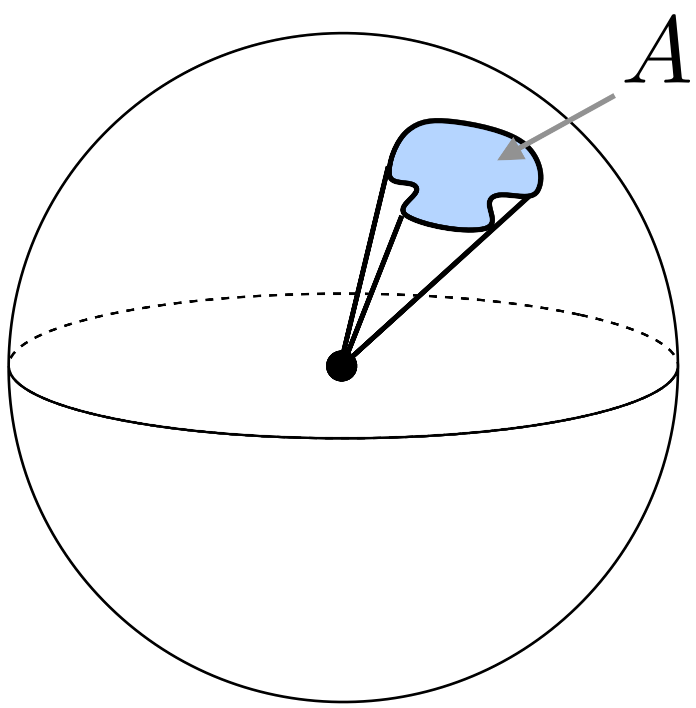


Directions & Solid Angle

- Angle $\theta = \frac{l}{r}$
  - circle: 2π radians



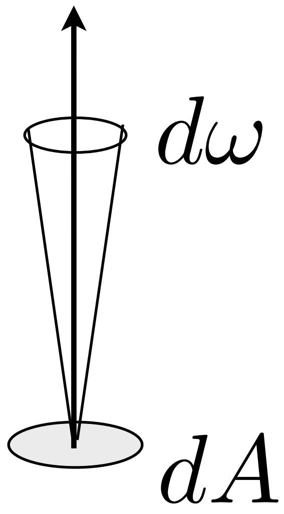
- *Solid angle* $\omega = \frac{A}{r^2}$
  - sphere: 4π steradians



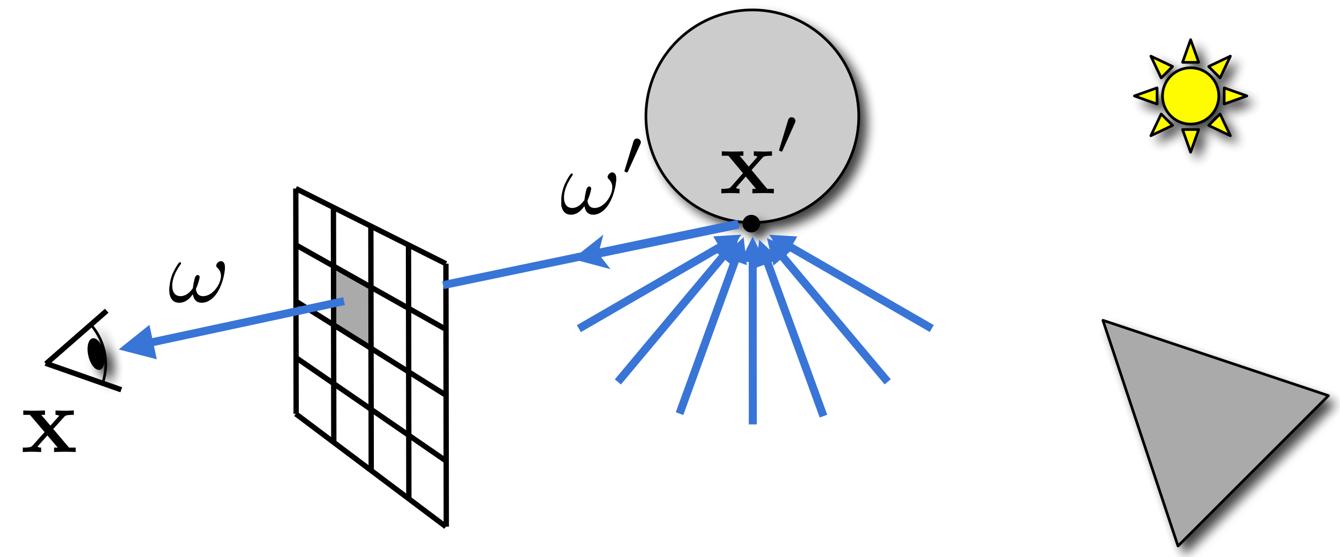
Radiometry

- *Radiance*
  - Flux density per unit area, per unit solid angle
  - remains constant along ray

$$L = \frac{d^2\Phi}{d\omega dA\cos\theta} \qquad \left[\frac{W}{m^2 sr}\right]$$



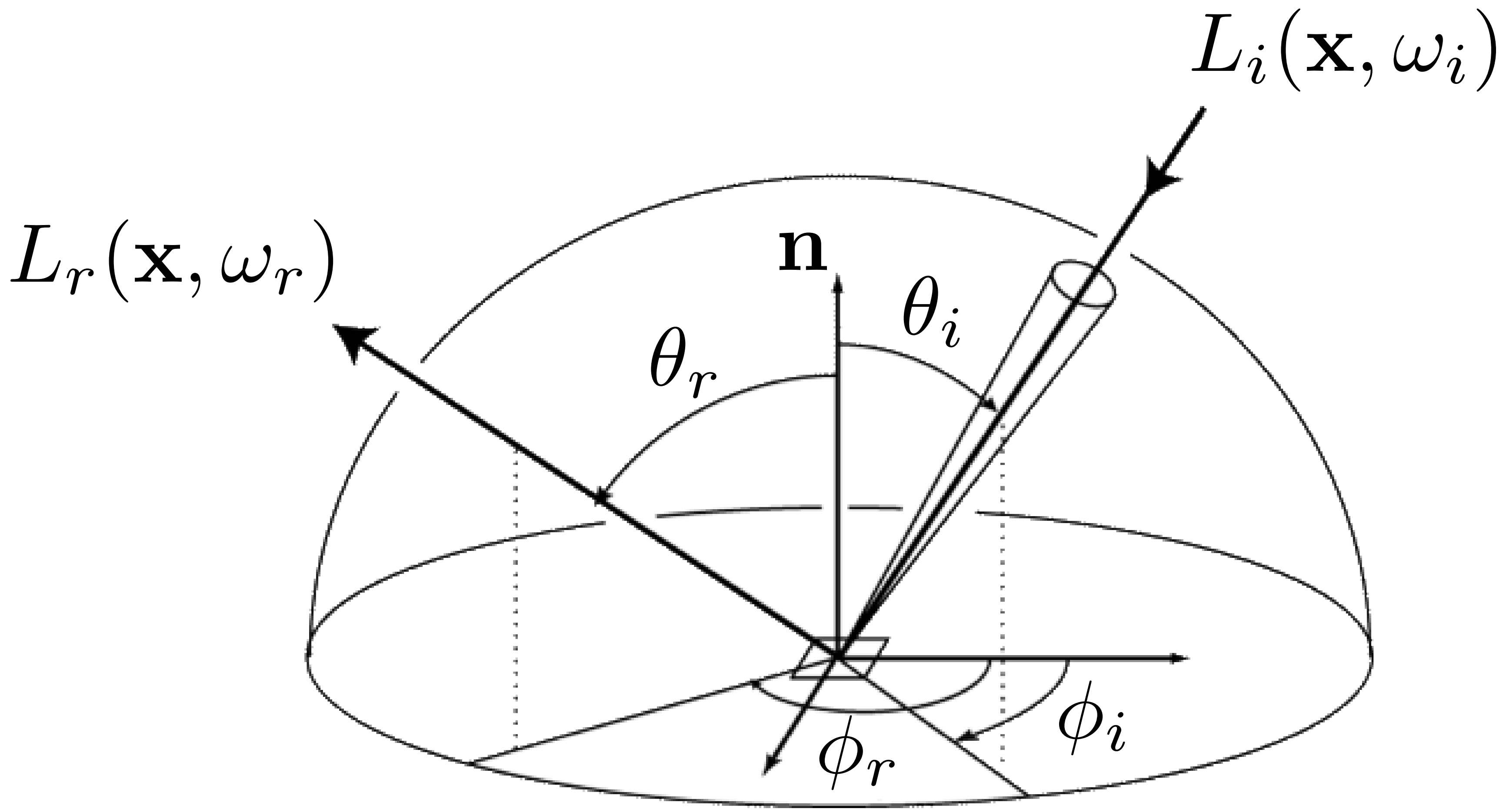
Ingredients



- Radiometry
  - What is 'light'?
- **Reflection models**
  - Relation between incoming and reflected 'light'
- Light transport
  - Relation between

    $(\mathbf{x}, \omega) \leftrightarrow (\mathbf{x}', \omega')$

Light Reflection Equation

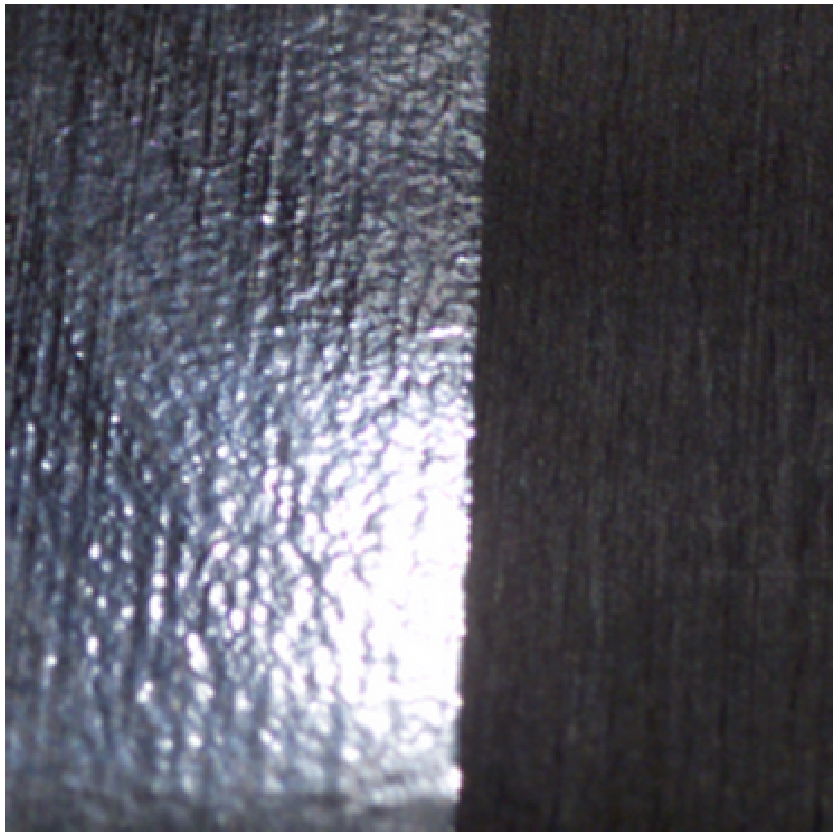


Simple Phong lighting might not be sufficient!

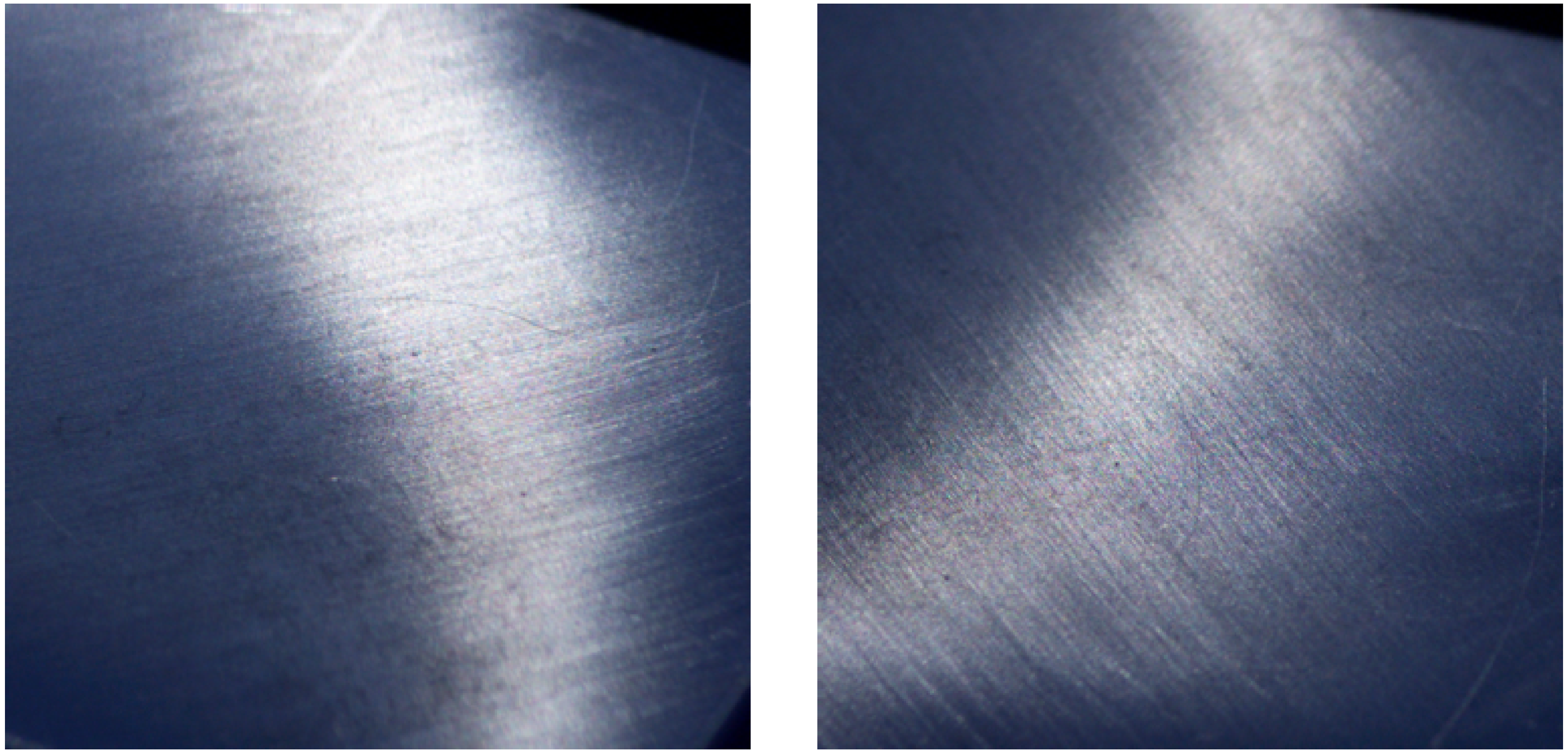
Complex Materials?

© Henrik Wann Jensen

Complex Materials?

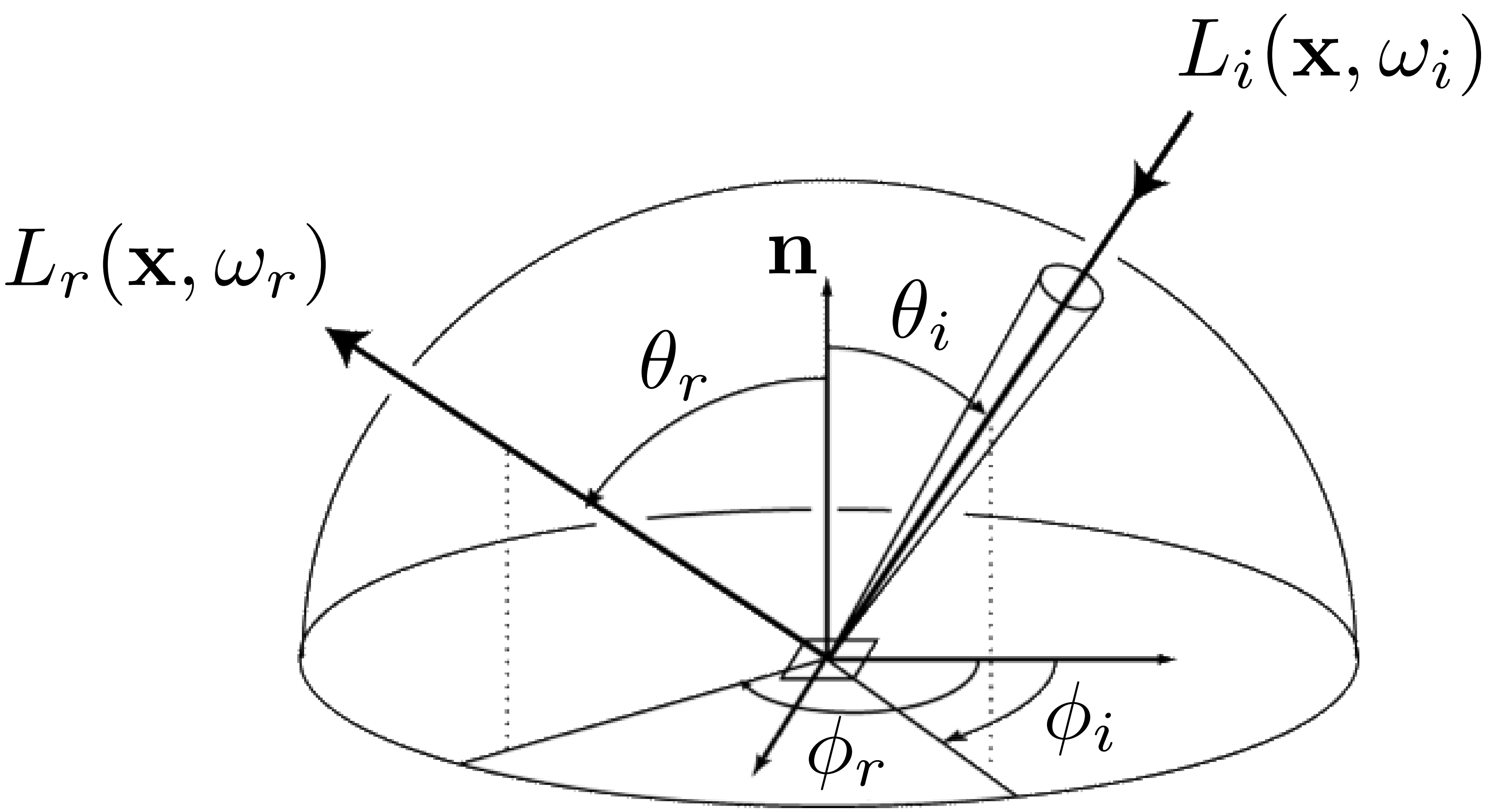


layered materials



anisotropic reflections

Light Reflection Equation



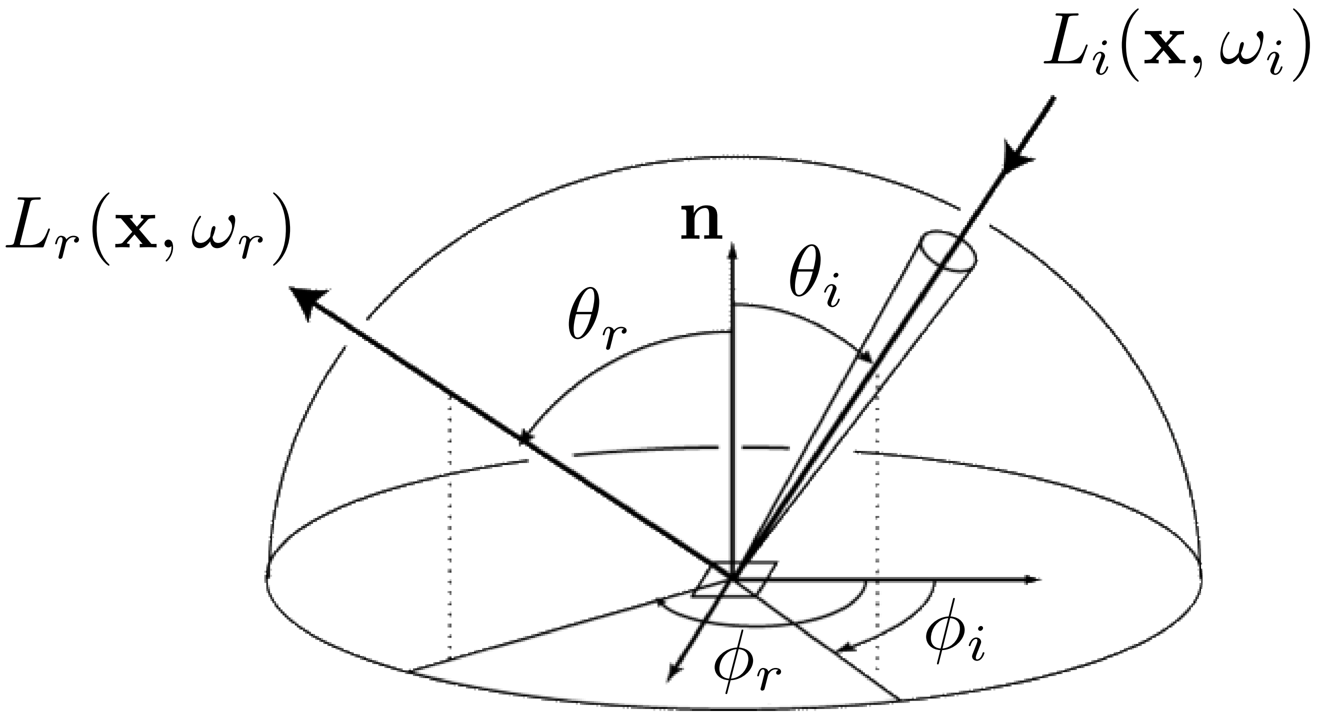
$$L_r(\mathbf{x}, \omega_r) = \int_{H^2} f_r(\mathbf{x}, \omega_i, \omega_r)\, L_i(\mathbf{x}, \omega_i) \cos\theta_i \, \mathrm{d}\omega_i$$

hemisphere

BRDF

BRDF

- **B**idirectional **R**eflectance **D**istribution **F**unction
  - "How much of the light coming from direction ω_i is reflected into direction ω_r ?"
  - Directions ω correspond to two angles (θ, ϕ)



BRDF

- **B**idirectional **R**eflectance **D**istribution **F**unction
  - two directions 4 DoFs
  - spatially varying +2 DoFs

BSSRDF

- **B**idirectional **S**urface **S**cattering **R**eflectance **D**istribution **F**unction
  – two directions 4 DoFs
  – spatially varying +2 DoFs
  – subsurface scattering +2 DoFs

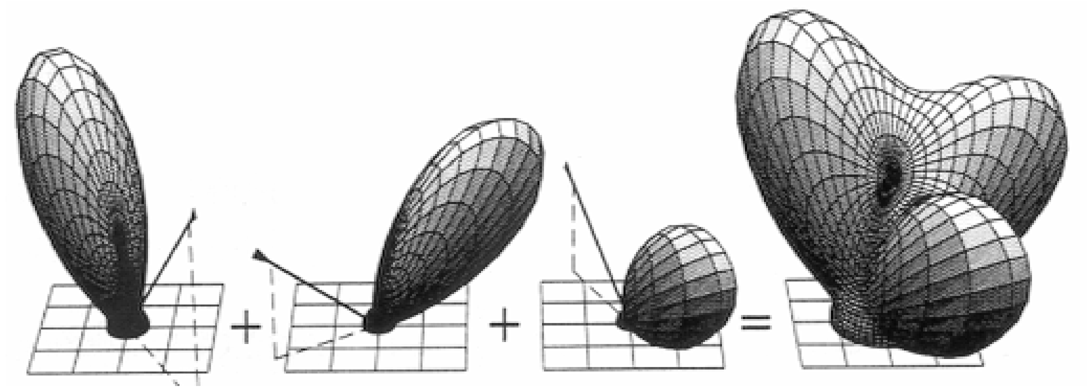
Subsurface Scattering

© Henrik Wann Jensen

Subsurface Scattering

BRDF

- Properties
  - Linearity



  - Reciprocity

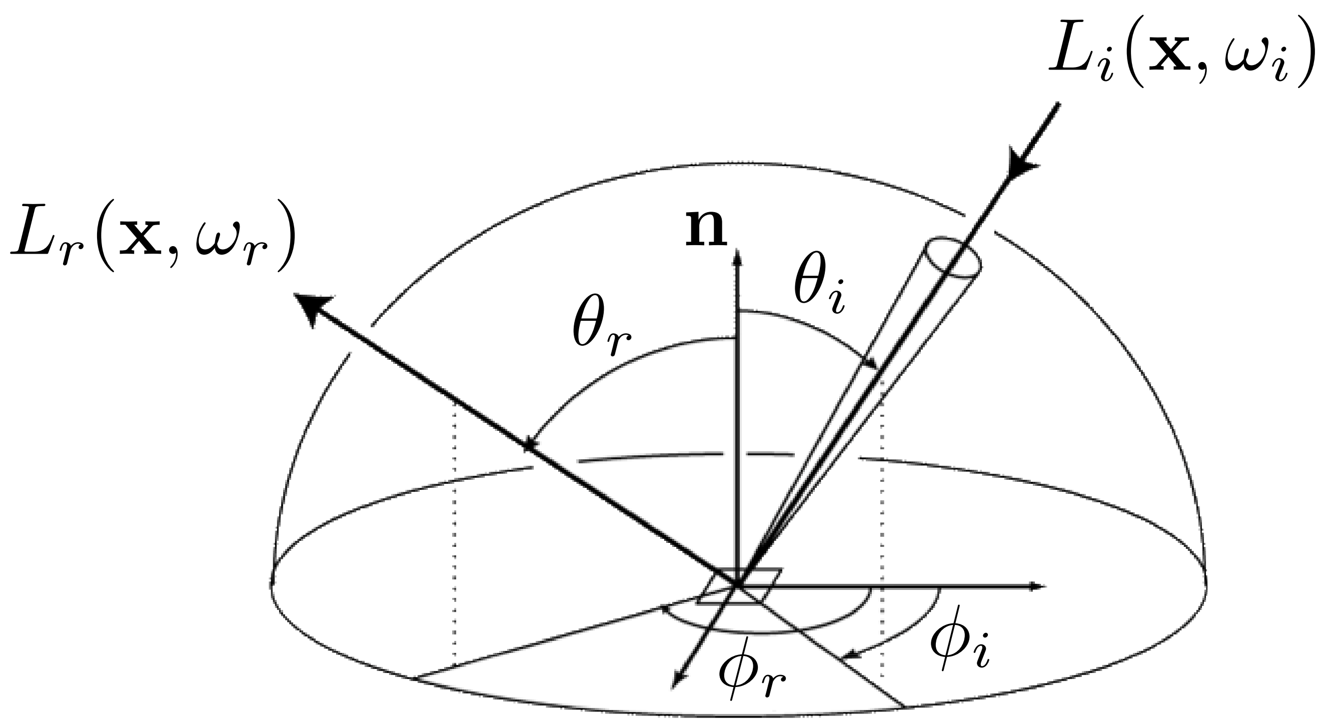
$$f_r(\omega_i \rightarrow \omega_r) = f_r(\omega_r \rightarrow \omega_i)$$

  - Energy conservation

$$\int_{H^2} f_r(\omega_i \rightarrow \omega_r) \cos\theta d\omega_r \leq 1$$

Reflection

- Reflected radiance depends on incident radiance
- Incident radiance depends on reflected radiance



$$L_r(\mathbf{x}, \omega_r) = \int_{H^2} f_r(\mathbf{x}, \omega_i \to \omega_r) L_i(\mathbf{x}, \omega_i) \cos\theta_i d\omega_i$$

Energy Balance

- Equilibrium

incoming absorbed

$$\Phi_o - \Phi_i = \Phi_e - \Phi_a$$

outgoing emitted

Energy Balance

- Surface Balance Equation
  - (incoming - absorbed = reflected + transmitted)

$$\underbrace{L_o(\mathbf{x},\omega_o)}_{\text{outgoing}} = \underbrace{L_e(\mathbf{x},\omega_o)}_{\text{emitted}} + \underbrace{L_r(\mathbf{x},\omega_o)}_{\text{reflected}} + \underbrace{L_t(\mathbf{x},\omega)}_{\text{transmitted}}$$

$$L_o(\mathbf{x},\omega_o) = L_e(\mathbf{x},\omega_o) + \int_{H^2} f_r(\mathbf{x},\omega_i \rightarrow \omega_o) L_i(\mathbf{x},\omega_i) \cos\theta_i d\omega_i$$

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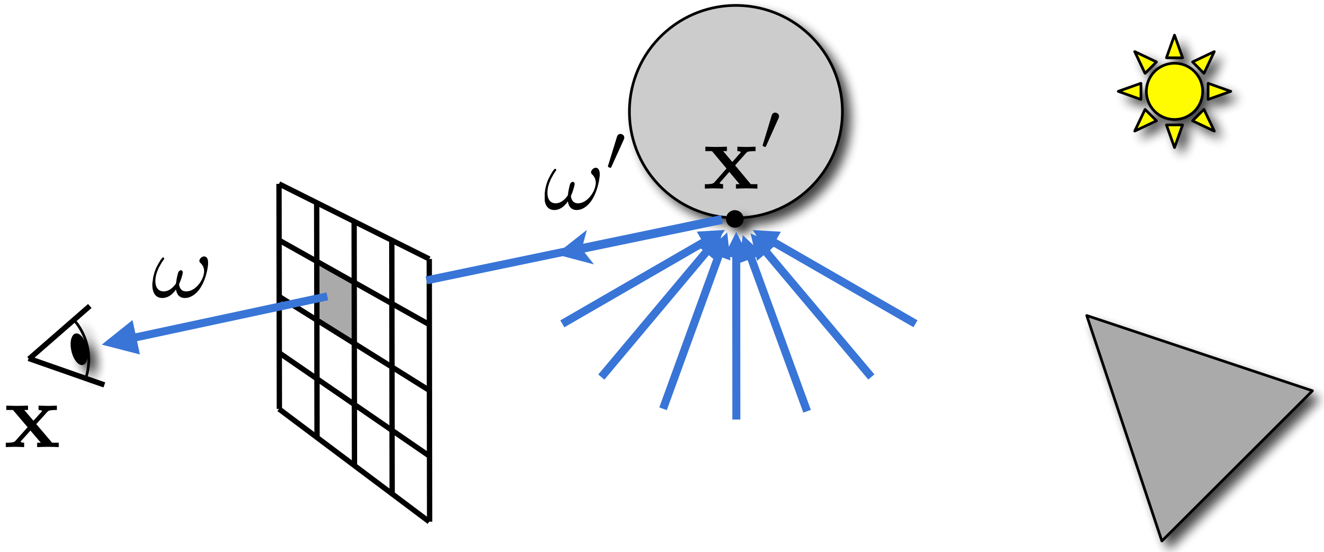
- **Light transport**
  - Relation between

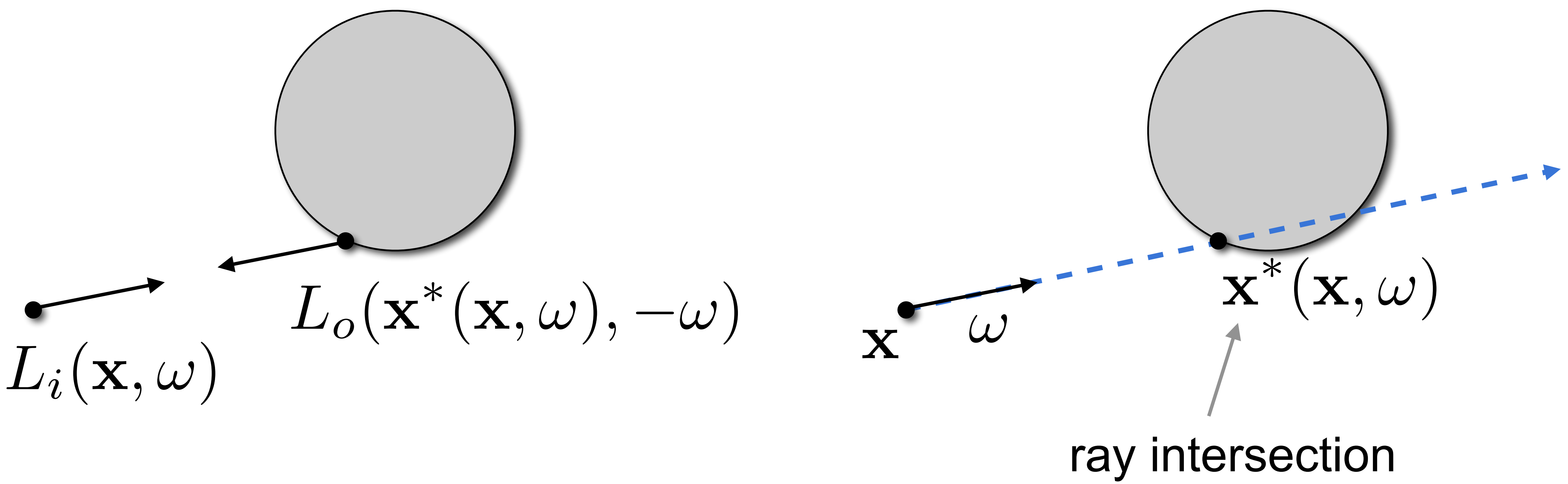
    $(\mathbf{x}, \omega) \leftrightarrow (\mathbf{x}', \omega')$



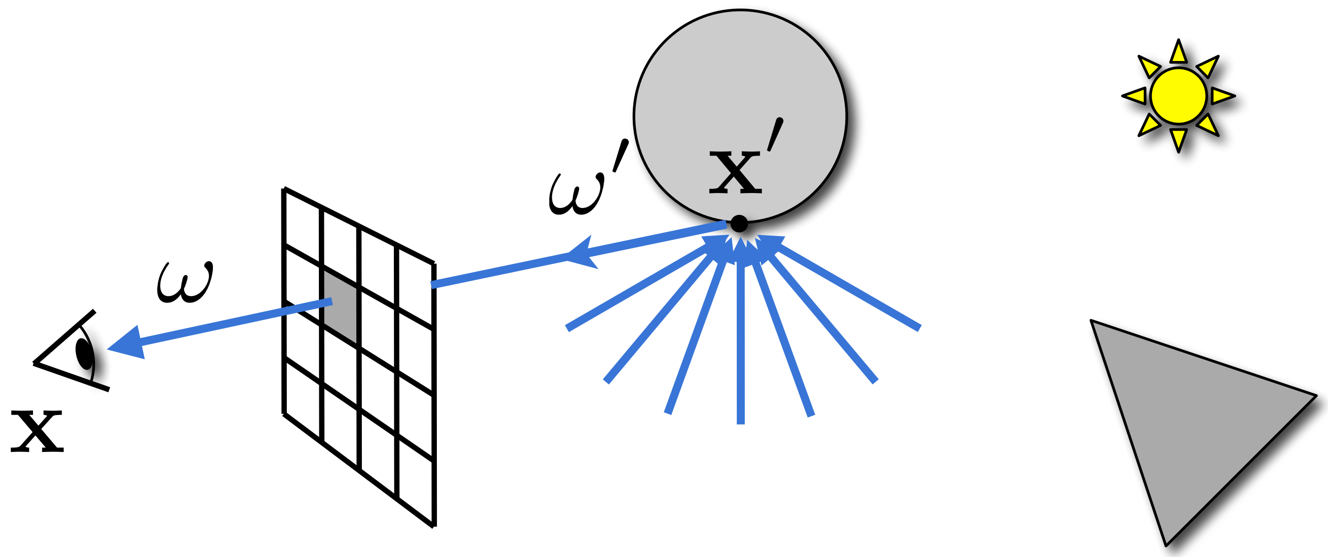
Light Transport

- No participating media → radiance is constant along a ray
- Relate incoming radiance to outgoing radiance

$$L_i(\mathbf{x}, \omega) = L_o(\mathbf{x}^*(\mathbf{x}, \omega), -\omega)$$



Ingredients



- Radiometry
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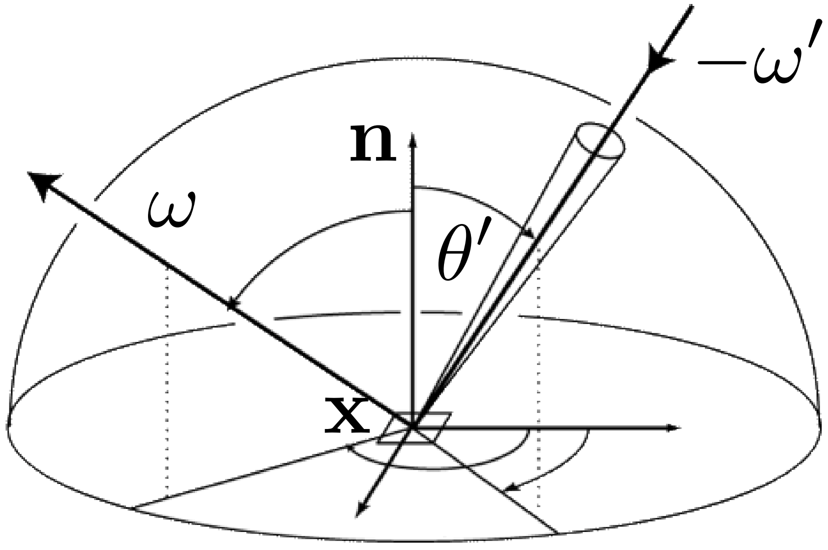
The Rendering Equation

$$L(\mathbf{x},\omega) = L_e(\mathbf{x},\omega) + \int_{H^2} f_r(\mathbf{x},\omega' \to \omega) L(\mathbf{x}^*(\mathbf{x},\omega')), -\omega') \cos\theta' d\omega'$$

integrate over hemisphere of directions

transport operator, i.e., ray tracing



Integral Equations

- Integral equations of the 1st kind

$$f(x) = \int k(x, x')\, g(x')\, \mathrm{d}x'$$

- Integral equations of the 2nd kind

$$f(x) = g(x') + \int k(x, x')\, f(x')\, \mathrm{d}x'$$

Linear Operators

- Linear operators act on functions like matrices act on vectors

$$g(x) \;=\; (K \circ f)(x')$$

- Linearity means

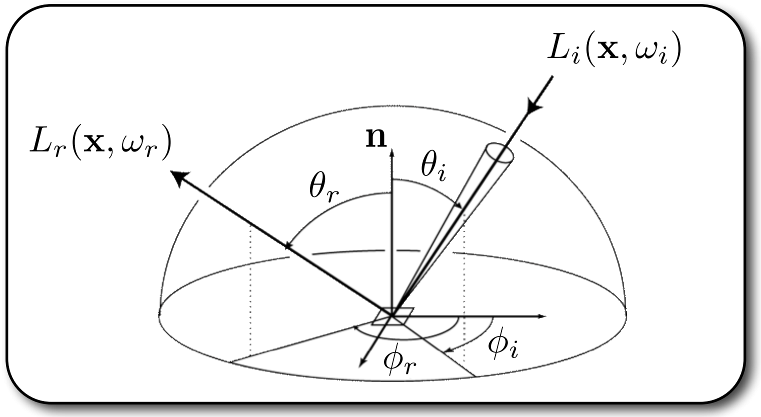
$$K \circ (a \cdot f + b \cdot g) \;=\; a \cdot (K \circ f) + b \cdot (K \circ g)$$

- Our operator is of the form

$$(K \circ f)(x) \;=\; \int k(x, x')\, f(x')\, \mathrm{d}x'$$

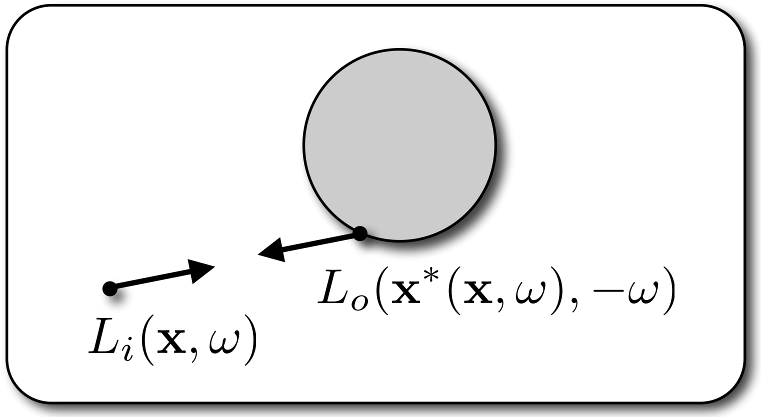
The Rendering Equation

- Scattering operator



$$L_o(\mathbf{x}, \omega_o) = \int_{H^2} f_r(\mathbf{x}, \omega_i \to \omega_o) L_i(\mathbf{x}, \omega_i) \cos\theta_i d\omega_i \equiv S \circ L_i$$

- Transport operator



$$L_i(\mathbf{x}, \omega_i) = L_o(\mathbf{x}^*(\mathbf{x}, \omega_i)), -\omega_i) \equiv T \circ L_o$$

The Rendering Equation

- Operator notation

$$L = L_e + K \circ L \qquad\qquad K \equiv S \circ T$$

- Solution

$$(I - K) \circ L = L_e \longrightarrow L = (I - K)^{-1} \circ L_e$$

The Rendering Equation

- Neumann series

$$(I - K)^{-1} = \frac{1}{I - K} = I + K + K^2 + \ldots$$

- Verify

$$\begin{aligned}(I - K) \circ (I - K)^{-1} &= (I - K) \circ (I + K + K^2 + \ldots) \\ &= (I + K + \ldots) - (K + K^2 + \ldots) \\ &= I\end{aligned}$$

The Rendering Equation

- Successive approximations

$$L^1 = L_e$$

$$L^2 = L_e + K \circ L^1$$

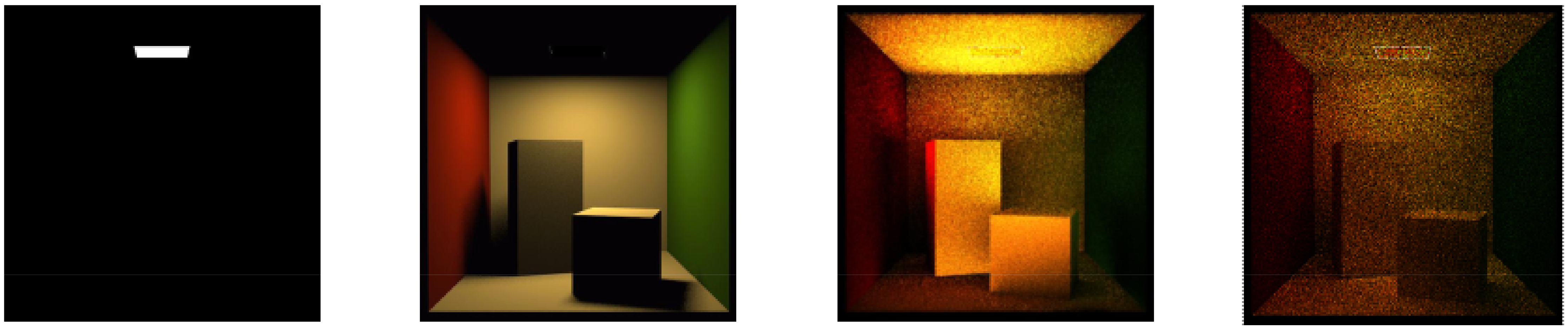
$$\dots$$

$$L^n = L_e + K \circ L^{n-1}$$

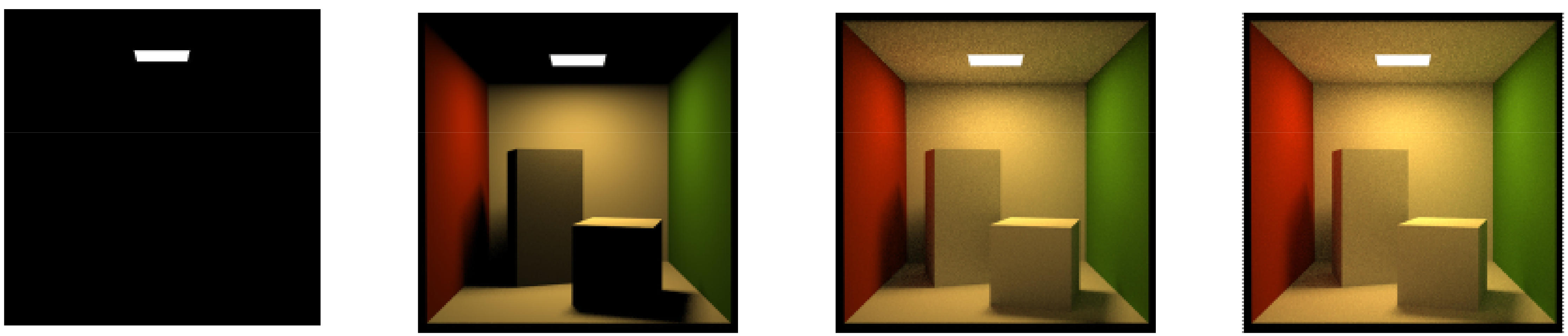
- Convergence

$$L^n = L^{n-1} \rightarrow L^n = L_e + K \circ L^n$$

Successive Approximation



L_e | $K \circ L_e$ | $K^2 \circ L_e$ | $K^3 \circ L_e$



L_e | $+K \circ L_e$ | $+K^2 \circ L_e$ | $+K^3 \circ L_e$

Rendering Equation

- Rendering Equation

$$L(\mathbf{x},\omega) = L_e(\mathbf{x},\omega) + \int_{H^2} f_r(\mathbf{x},\omega' \rightarrow \omega) L(\mathbf{x}^*(\mathbf{x},\omega')), -\omega') \cos\theta' d\omega'$$

- Solution with Neumann series
  – from an integral equation to an infinite sum of integrals

$$L = \sum_{i=0}^{\infty} K^i \circ L_e$$

Rendering Equation

- How to evaluate these integrals?
- Closed-form solution almost never exists
  - Multidimensional, complex BRDF
  - Discontinuous integrand (visibility)
  - Complex integration domains
- Numerical Methods
  - Gaussian quadrature → inefficient
  - Monte Carlo methods (next time)