1. (Kernel Trick for polynomial SVM) Consider the polynomial Kernel $K(u,v) = (1 + u * v)^2$ where $u, v \in [-1,1]^d$.

   (a) show that for $d = 2$ that $K(u,v)$ can be written as a scalar product $\Phi(u) * \Phi(v)$ in a six-dimensional state space and provide explicit expressions for the feature transform $\Phi(u)$.

   Solution:

$$
\begin{aligned}
K(u,v) = (1 + u * v)^2 &= (1 + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} * \begin{pmatrix} v_1 \\ v_2 \end{pmatrix})^2 && (1) \\
&= (1 + u_1 v_1 + u_2 v_2)^2 && (2) \\
&= 1 + 2u_1 v_1 + 2u_2 v_2 + 2u_1 v_1 u_2 v_2 + (u_1 v_1)^2 + (u_2 v_2)^2 && (3) \\
&= \begin{pmatrix} 1 \\ \sqrt{2}u_1 \\ \sqrt{2}u_2 \\ \sqrt{2}u_1 u_2 \\ u_1^2 \\ u_2^2 \end{pmatrix} * \begin{pmatrix} 1, \\ \sqrt{2}v_1, \\ \sqrt{2}v_2, \\ \sqrt{2}v_1 v_2, \\ v_1^2, \\ v_2^2, \end{pmatrix} && (4) \\
&= \Phi(u) * \Phi(v) && (5)
\end{aligned}
$$

   (b) generalize your result for general (integer) $d > 1$.

Solution:

$$
\begin{aligned}
K(u,v) = (1+u*v)^2 = (1 + \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} * \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix})^2 && (6)\\
= 1 + 2u_1v_1 + \dots + 2u_dv_d + && (7)\\
2u_1v_1u_2v_2 + \dots + 2u_1v_1u_dv_d + && (8)\\
2u_2v_2u_3v_3 + \dots + 2u_2v_2u_dv_d + && (9)\\
\vdots && (10)\\
2u_{d-1}v_{d-1}u_dv_d + && (11)\\
(u_1v_1)^2 + \dots + (u_dv_d)^2 && (12)\\
= \begin{pmatrix} 1 \\ \sqrt{2}u_1 \\ \vdots \\ \sqrt{2}u_d \\ \sqrt{2}u_1u_2 \\ \vdots \\ \sqrt{2}u_1u_d \\ \sqrt{2}u_2u_3 \\ \vdots \\ \sqrt{2}u_2u_d \\ \vdots \\ \sqrt{2}u_{d-1}u_d \\ u_1^2 \\ \vdots \\ u_d^2 \end{pmatrix} * \begin{pmatrix} 1 \\ \sqrt{2}v_1 \\ \vdots \\ \sqrt{2}v_d \\ \sqrt{2}v_1v_2 \\ \vdots \\ \sqrt{2}v_1v_d \\ \sqrt{2}v_2v_3 \\ \vdots \\ \sqrt{2}v_2v_d \\ \vdots \\ \sqrt{2}v_{d-1}v_d \\ v_1^2 \\ \vdots \\ v_d^2 \end{pmatrix} && (13)\\
= \Phi(u) * \Phi(v) && (14)
\end{aligned}
$$

(c) demonstrate for this case explicitly the validity of Mercer's condition, if the integration range is chosen as the hypercube $[-1,1]^d$.

Solution: