Lecture September 3-2020

double ** A $\quad (A \in \mathbb{R}^{4\times 4})$

$$\begin{array}{l} A[0] \longrightarrow \\ A[1] \longrightarrow \\ A[2] \longrightarrow \\ A[3] \longrightarrow \end{array} \begin{bmatrix} A[0][0] & A[0][1] & A[0][2] & A[0][3] \\ A[1][0] & A[1][1] & A[1][2] & \dots \\ A[2][0] & - & - & - \\ - & - & - & - \end{bmatrix}$$

A = new double[n]

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & - & - & - \\ a_{41} & - & - & - \end{bmatrix}$$

- $\underline{A \cdot x = b}$ $\quad A \in \mathbb{R}^{n\times n}$ known.
  $x \in \mathbb{R}^n$ (unknown)
  $b \in \mathbb{R}^n$ known

- why LU decomposition?
  and what is LU-decomp?

$\underline{A \cdot y = c}$

$\underline{A \cdot z = d}$

$$\underline{\frac{d^2x}{dt^2} = \underline{b}(t)}$$

Gaussian elimination on dense matrix A $\rightarrow$

$\frac{2}{3} n^3$ FLOPs

For the inverse of a matrix this would lead to n^4 FLOPs

with LU decomp, # FLOPs $\sim n^3$

LU-decomp: if A is non-singular, then A can be written as (Ex $A \in \mathbb{R}^{4 \times 4}$)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & & & 0 \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$\sim \frac{2}{3} n^3 + O(n^2)$

Lower Triangular

upper triangular

normal $\boxed{l_{ii} = 1}$

immediate benefit:

$$\det(A) = \det(LU) = \underbrace{\det L}_{= 1} \cdot \det U =$$

$$\boxed{u_{11}\, u_{22}\, u_{33}\, u_{44}}$$

$$A \cdot x = L \underbrace{U \cdot x}_{w} = b$$

$$= \boxed{L \cdot \boxed{w} = b}$$

$$U \cdot x = \underline{w}$$

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$l_{11} w_1 = b_1 \Rightarrow w_1 = b_1 / l_{11}$$

$$l_{21} w_1 + l_{22} w_2 = b_2 \Rightarrow$$

$$w_2 = (b_2 - l_{21} / l_{11} \cdot b_1) / l_{22}$$

$\sim n^2$ FLOPs

$\vdots$

w_4

$$U \cdot x = w$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & \underline{u_{44}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$u_{44} x_4 = w_4 \Rightarrow x_4 = \underline{w_4 / u_{44}}$$

$$u_{33} x_3 + u_{34} x_4 = w_3 \Rightarrow$$

$$x_3 = \left(w_3 - \frac{u_{34} \cdot w_4}{u_{44}} \right) / u_{33}$$

$\vdots$

x_1. $\sim n^2$ FLOPs

2nd immediate benefit:

Inverse of matrix:

$$A^{-1} A = I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \\ & & & 1 \end{bmatrix} = A A^{-1}$$

$$A = L \cdot U$$

$$A^{-1} = \begin{bmatrix} a_1^{-1} & a_2^{-1} & \cdots & a_n^{-1} \end{bmatrix}$$

$$a_1^{-1} = \begin{bmatrix} a_{11}^{-1} \\ a_{21}^{-1} \\ \vdots \\ a_{n1}^{-1} \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$(\underline{LU}) \begin{bmatrix} \underline{a_1^{-1}} & a_2^{-1} & \cdots & a_n^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & \\ \vdots & 0 & \vdots & & \\ \vdots & \vdots & 0 & \ddots & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

$$(LU)a_1^{-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\overbrace{(LU)}^{n^3} a_2^{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{etc}$$

Have to solve n-equations with an LU-decomposed matrix this leads

$$n \cdot n^2 \text{ FLOPs} \sim n^3 \text{ FLOPs}$$

$n = 100000$

Need for $\underbrace{A \in \mathbb{R}^{n \times n}}_{8 \text{bytes} \times 10^{10}}$

Proj 1 $\quad n \sim 10^4$