Lecture october 2

ORDINARY DIFFERENTIAL EQS
(ODEs)

- $\forall$ Euler-family
- $\forall$ Verlet-family
- $\forall$ Project 3

] Thursday october 1

- Friday:
  - Project 3 and ODEs
  - Runge-Kutta family
  - object orientation

Simple-Euler method:

$$m\frac{d^2x}{dt^2} = F(x, v, t)$$

$$\frac{dx}{dt} = v(t) \qquad \frac{dv}{dt} = \frac{F(x,v,t)}{m} = a(x,v,t)$$

$$h = \frac{t_n - t_0}{n}$$

$$x_{i+1} = x_i + h v_i$$
$$v_{i+1} = v_i + h a_i$$

Euler-Cromer

$$v_{i+1} = v_i + h a_i$$
$$x_{i+1} = x_i + h v_{i+1}$$

Velocity-Verlet

First $x_{i+1} = x_i + h v_i + \frac{h^2}{2} a_i \quad (O(h^3))$

$$v_{i+1} = v_i + \frac{h}{2}\left[a_{i+1} + \boxed{a_i}\right]$$

$a_i = a_i(t_i)$. Here a depends only on x

$a_i = a(x_i) \qquad a_{i+1} = a(x_{i+1})$

Earth-Sun case

Scaling: $v_0 = 2\pi\, AU/yr$

$$\boxed{G M_\odot = 4\pi^2 (AU)^3 / yr^2}$$

$$\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y$$

$$\frac{dv_x}{dt} = \frac{-4\pi^2 x}{\left(\sqrt{x^2+y^2}\right)^3} \qquad \frac{dv_y}{dt} = \frac{-4\pi^2 y}{\left(\sqrt{x^2+y^2}\right)^3}$$

Adding Jupiter

$M_J = 1.9 \cdot 10^{27}\,kg$

$$F_x^{EJ} = -\frac{G M_J M_E (x_E - x_J)}{\left\{\left(\sqrt{(x_E-x_J)^2 + (y_E-y_J)^2}\right)^3\right\}}$$

$\dots^E$

$$\frac{dv_x}{dt} = -\frac{GM_\odot x_E}{r^3} - \frac{GM_J(x_E - x_J)}{r_{EJ}^3}$$

Scaled version

$$= -\frac{4\pi^2 x_E}{r^3} - \frac{4\pi^2\left(\frac{M_J}{M_\odot}\right)(x_E - x_J)}{r_{EJ}^3}$$

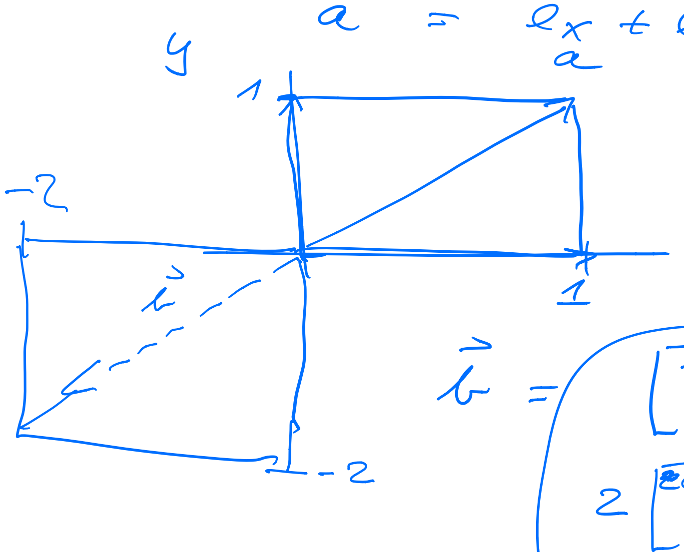
$+ \ \frac{dv_y^E}{dt}$ and $\frac{dx_E}{dt}$, $\frac{dy_E}{dt}$

+ similar equations for Jupiter.

—— Symplectic ODEs & Energy conservation

Area conservation:

$$\vec{a} = \vec{e}_x + \vec{e}_y$$



$A = 1$

$$\vec{b} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$2\begin{bmatrix} \cos\theta & -\cancel{\sin\theta} \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\cos\theta = -1$

$\sin\theta = 0$

$A = +4$

$$A = \det\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

if A is conserved, we say the ODE is symplectic and this leads to energy conservation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Euler's method

$$x_{i+1} = x_i + h\cdot v_i$$

$$v_{i+1} = v_i - \underbrace{\omega^2 h\cdot x_i}_{a_i\cdot h}$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & h \\ -\omega^2 h & 1 \end{bmatrix}}_{\det = (1+\omega^2h^2)^m} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

Euler-Cromer

$$x_{i+1} = x_i + h\, v_{i+1}$$

$$v_{i+1} = v_i - \omega^2 h\, x_i$$

$$x_{i+1} = x_i(1-\omega^2h^2) + h v_i$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1-\omega^2h^2 & h \\ -\omega^2 h & 1 \end{bmatrix}}_{\det = 1} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

Area is conserved

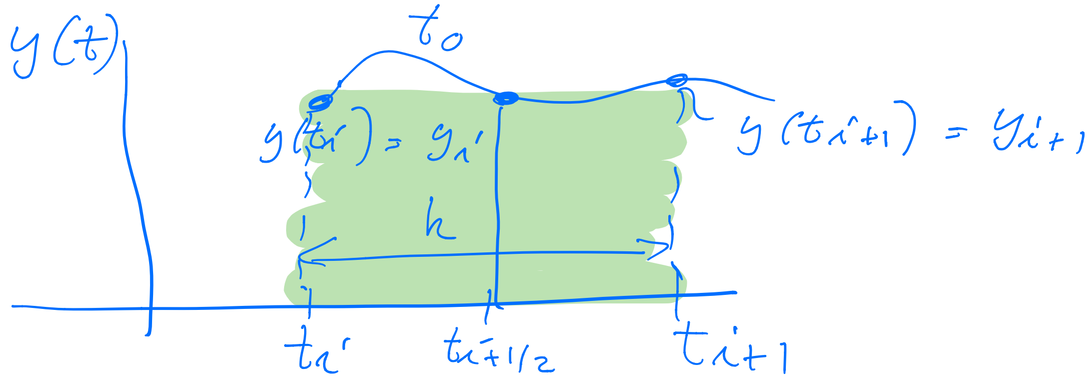
$\Rightarrow$ energy conservation

Runge-Kutta family:

$$\frac{dy}{dt} = f(t, y)$$

$$t \in [t_0, t_n]$$

$$y(t) = \int_{t_0}^{t_n} f(t', y)\, dt'$$



$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t, y)\, dt$$

$$\int_{t_i}^{t_{i+1}} f(t, y)\, dt = h\, f(t_{i+1/2}, y_{i+1/2}) + O(h^3)$$

(Midpoint/rectangular rule)

$$y_{i+1} = y_i + h \cdot f(t_{i+1/2}, y_{i+1/2}) + O(h^3)$$

Runge-Kutta 2 = RK2

$$y_{i+1/2} = ?$$

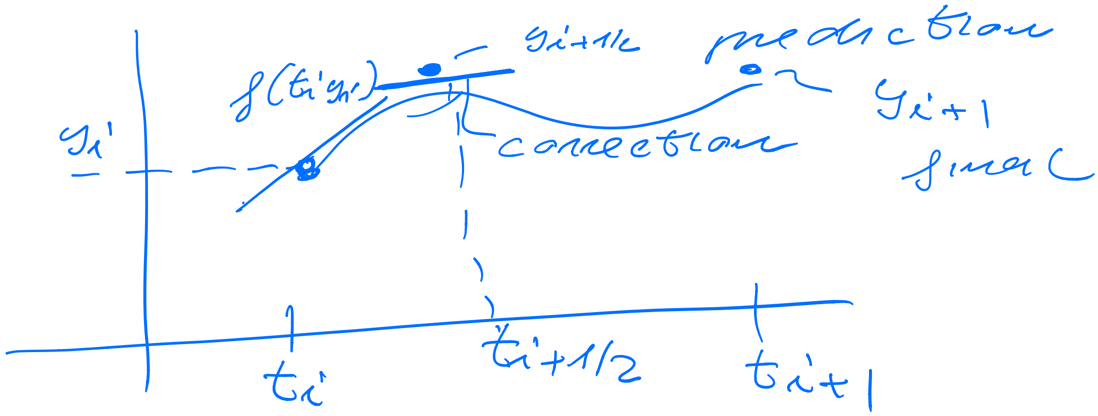
$$y_{i+1/2} = y_i + \frac{h}{2} f(t_i, y_i)$$

$$k_1 = h f(t_i, y_i)$$

$$k_2 = h f\left(t_{i+1/2}, y_i + \frac{k_1}{2}\right)$$

$$y_{i+1} = y_i + k_2 + O(h^3)$$

$$\simeq y_i + k_2$$

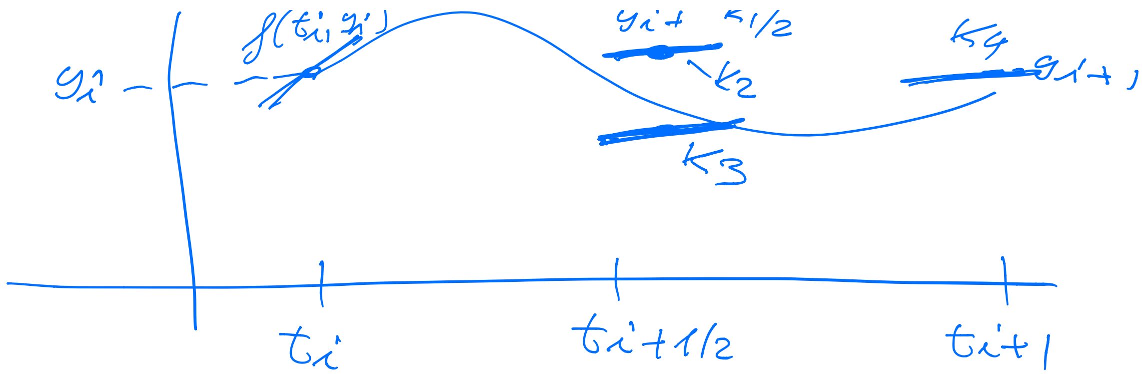


Runge-Kutta 4 = RK4

$$\int_{t_i}^{t_{i+1}} f(t,y)\,dt = \frac{h}{6}\left[y_i + y_{i+1} + 4\cdot y_{i+1/2}\right] + O(h^5)$$

Simpson's rule

$$= \frac{h}{6}\left[y_i + 2y_{i+1/2} + 2y_{i+1/2} + y_{i+1}\right] + O(h^5)$$



$$k_1 = hf(t_i, y_i)$$

$$k_2 = hf(t_{i+1/2}, y_{i+1/2})$$

$$= hf(t_{i+1/2}, y_i + k_1/2)$$

$$k_3 = hf\left(t_{i+1/2}, y_i + \frac{k_2}{2}\right)$$

$$k_4 = hf(t_{i+1}, y_i + k_3)$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + O(h^5)$$

<u>RK4</u>

$$\boxed{\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)}$$