Lecture September 24

$$T = \begin{bmatrix} a_1 & b_1 & & & 0 \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ 0 & & & b_{n-1} & a_n \end{bmatrix}$$

$$P(T - \lambda) = \det(T - \lambda \mathbb{I}) = 0$$

$$P_0(x) = \underline{1} \qquad \lambda \Rightarrow x$$

$$\begin{vmatrix} \boxed{a_1 - x} & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 - x & b_2 & & \\ & b_2 & \ddots & & \\ 0 & & & \ddots & \\ & & & & a_n - x \end{vmatrix} = 0$$

$$P_1(x) = a_1 - x = 0 \Rightarrow a_1 = x$$

$$P_2(x) = \underline{(a_1 - x)}(a_2 - x) - b_1^2 = 0$$

$$= (a_2 - x) P_1(x) - b_1^2 P_0(x)$$

$$= 0$$

$$P_n(x) = (a_n - x) P_{n-1}(x) - b_{n-1}^2 \times P_{n-2}(x)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} = 0$$

$$\det(A - \lambda I) = 0$$

$A \in \mathbb{R}^{2\times 2}$

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

$\#$ FLOPS : 3

$A \in \mathbb{R}^{3\times 3}$

$$\det A = \underline{a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \underline{- a_{12}\begin{vmatrix} x & x \\ x & x \end{vmatrix}} \underline{+ a_{13}\begin{vmatrix} x & x \\ x & x \end{vmatrix}}$$

$\#$ FLOPS $= 14$

$A \in \mathbb{R}^{4\times 4}$ $\quad$ $\#$ FLOPS $= 63$

$A \in \mathbb{R}^{5\times 5}$ $\quad$ $\#$ — " — $= 324$

$A \in \mathbb{R}^{6\times 6}$ $\quad$ $\#$ — " — $= \underline{1955}$

Householder + efficient Tridiagonal

Solver : # Flops $O(n^3)$

$$T = \begin{bmatrix} \boxed{1} & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

$x \in [0,4]$

$a_1 \pm 2$

$x \in [0,2]$

Polynomial degree

$P_1(x) = 1 - x \Rightarrow x = 1 = \lambda_1^{(1)}$

$P_2(x) = (2-x)(1-x) - 1 = 0$

$\lambda_1^{(2)} = 0.382 \qquad \lambda_2^{(2)} = 2.618$

$P_3(x) = (3-x)\,P_2(x) - P_1(x) = 0$

$\lambda_1^{(3)} = 0.268 \qquad \lambda_2^{(3)} = 2.00$

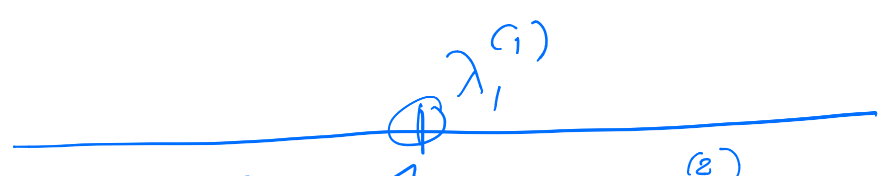
$\lambda_3^{(3)} = 3.732$

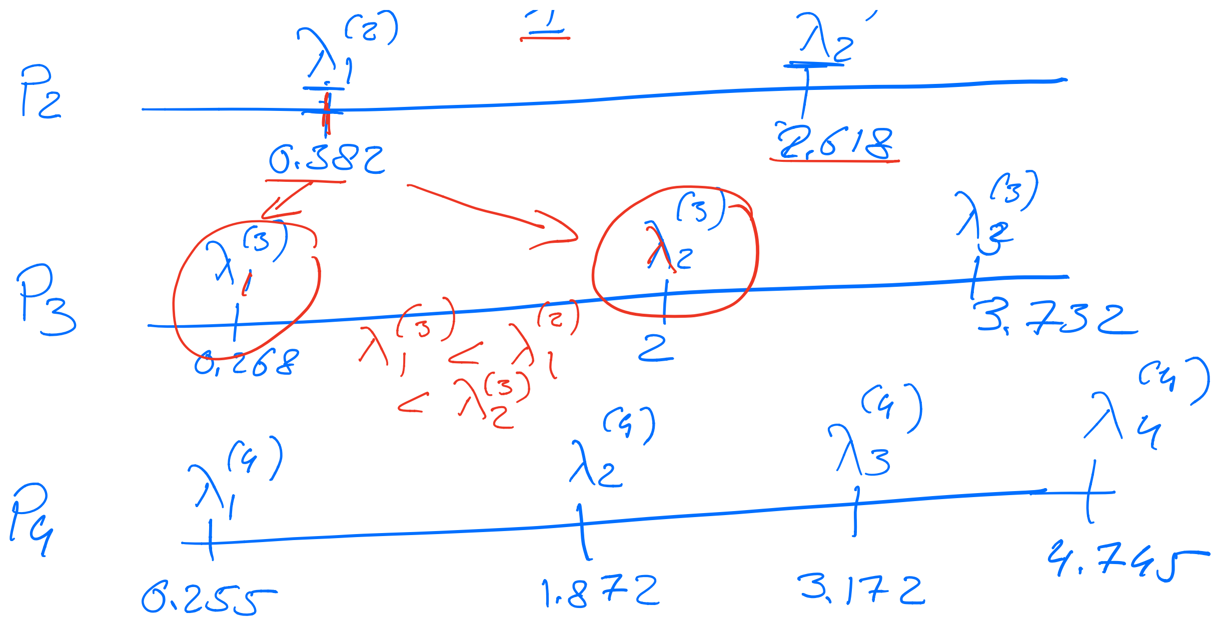
$P_4(x) = (4-x)\,P_3(x) - P_2(x) = 0$

$\lambda_1^{(4)} = 0.256 \qquad \lambda_2^{(4)} = 1.872$

$\lambda_3^{(4)} = 3.177 \qquad \lambda_4^{(4)} = 4.745$

P_1





$$\boxed{\lambda_j^{(k+1)} < \lambda_j^{(k)} < \lambda_{j+1}^{(k+1)}} \Leftarrow$$

$$\lambda_1^{(k)} < \lambda_2^{(k)} < \lambda_3^{(k)} \dots < \lambda_k^{(k)}$$

$$\boxed{|a_1| \le |a_2| \le |a_3| \dots \le |a_n|}$$

$$\lambda_k(T) \in [y, z]$$

$$y = \min_{1 \le i \le n} \; a_i - \underbrace{|b_i|}_{1/h^2} - \underbrace{|b_{i-1}|}_{1/h^2}$$

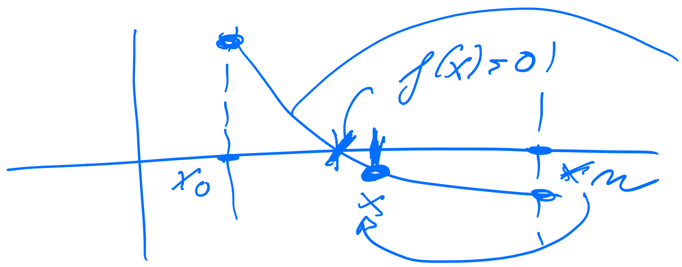
$$= a_i - 2/h^2$$

$$z = \max_{1 \le i \le n} \; a_i + |b_i| + |b_{i+1}|$$

$$= a_i + 2/h^2$$

with Domains $\lambda \in \boxed{[x_0, x_n]}$
we can use bisection to
find roots (one root at the
time).

$f(x) = 0$



$P_n(x) P_n(x_0) < 0$

$f(x) = P_n(x)$

$\underline{P_n(x_0) P_n(x_n)} < 0$ and
$x_0 < x_n$

if $P_n(x_0) P_n(x_n) < 0$ and
$x_0 < x_n$

$\approx 10^{-10}$

while $|\underline{x_n - x_0}| > \varepsilon(|x_0| + \underline{|x_n|})$

$$x = \frac{x_0 + x_n}{2}$$

if $P_n(x) P_n(x_0) < 0$

$x_n = x$

else

$x_0 = x$

end while,

Householder transformation

<u>Project 2</u>

We have

$$T = \begin{bmatrix} \times & \times & & & 0 \\ \times & \times & \times & & \\ & \times & \times & \times & \\ & & \times & \times & \times \\ 0 & & & \times & \times \end{bmatrix} \longrightarrow$$

$$D = \begin{bmatrix} \times & & & 0 \\ & \times & & \\ & & \ddots & \\ 0 & & & \times \end{bmatrix}$$

$A \in \mathbb{R}^{n \times n}$, symmetric.

$$A = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \vdots & & & & \\ \times & - & - & \cdots & \times \end{bmatrix} \xrightarrow{SAS^T} T = \begin{bmatrix} \times & \times & & & 0 \\ \times & \times & \times & & \\ & \times & \times & \times & \\ & & \times & \times & \times \\ 0 & & & \times & \times \end{bmatrix}$$

$$\longrightarrow D = \begin{bmatrix} \times & & 0 \\ & \times & \\ 0 & \times & \\ & & \times \end{bmatrix}$$

Householder's approaching view

$$S_1^T A S_1 = \begin{bmatrix} a_{11} & \underline{e_1} & \overbrace{0 & \cdots & 0}^{n-2} \\ \boxed{e_1} & \tilde{a}_{22} & \tilde{a}_{23} & \cdots & \\ 0 & \vdots & & & \\ \vdots & \vdots & & & \\ 0 & \tilde{a}_{n2} & & \cdots & \tilde{a}_{nn} \end{bmatrix}$$

$$S_1^T S_1 = \mathbb{1}$$

$$= \begin{bmatrix} \underline{a_{11}} & e_1 & 0 & \cdots & 0 \\ e_1 & x & x & \cdots & x \\ 0 & x & & & \\ \vdots & & & & \\ 0 & x & x & \cdots & x \end{bmatrix}$$

$$S_2^T S_1^T A S_1 S_2$$

$$= \begin{bmatrix} a_{11} & e_1 & 0 & \cdots & 0 \\ e_1 & \tilde{a}_{22} & e_2 & \cdots & 0 \\ 0 & e_2 & x & x & x \; x \\ \vdots & \vdots & x & x & x \; x \\ 0 & 0 & x \; x & x & x \end{bmatrix}$$

$$S_{n-1}^T S_{n-2}^T \cdots S_1^T A S_1 \cdots S_{n-1}$$

$$= \begin{bmatrix} a_{11} & e_1 & & & 0 \\ e_1 & \tilde{a}_{22} & e_2 & & \\ & e_2 & \tilde{a}_{33} & & \\ 0 & & e_3 & \ddots & e_{n-1} \\ & & & e_{n-1} & \tilde{a}_{nn} \end{bmatrix}$$

$\boxed{S_i = ?}$

$$\underline{S_1} = \begin{bmatrix} 1 & 0^T \\ 0 & P \end{bmatrix} \quad P \in \mathbb{R}^{n-1 \times n-1}$$

$$S_1 \in \mathbb{R}^{n \times n}$$

$$0^T = [0, \ldots 0] \in \mathbb{R}^{n-1}$$

Householder:

unknown

$$P = \mathbb{1} - 2 \underbrace{u u^T}_{\text{outer product}}$$

$$\boxed{P^T = P} \qquad P^2 = \mathbb{1} =$$

$$\mathbb{1} - 4 (u u^T) u u^T \qquad u^T u = 1$$

$$+ 4 (u u^T) u u^T = \mathbb{1}$$

$$\underline{P_{ij} = \delta_{ij} - 2 u_i u_j}$$

$$\underline{S_1^T A S_1} = \begin{bmatrix} a_{11} & (P_1 v)^T \\ P_1 v & \tilde{A} \end{bmatrix}$$

$$\left(v^T = \underbrace{[\underline{a_{12}} \; a_{13} \; \dots \; a_{1n}]}_{\text{known}} \right)$$

$$= \begin{bmatrix} a_{11} & k & 0 & \dots & 0 \\ k & x & x & x & x \\ 0 & x & & & \\ \vdots & x & & & \\ 0 & x & \dots & & x \end{bmatrix}$$

$$(P_1 v)^T = [\underline{k}, 0, 0, \dots 0] \in \mathbb{R}^{n-1}$$

Define $e^T = [1, 0, 0, \dots 0] \in \mathbb{R}^{n-1}$

$$P_1 v = \mathbb{1} \cdot v - 2 \underline{u (u^T v)} = \underline{\underline{k e}}$$

$$(P_1 v)^T (P_1 v) = v^T \underbrace{P^T P}_{\mathbb{1}} v = v^T v$$

$$= \sum_{i=2}^{n} a_{1i}^2 = k^2 \Rightarrow$$

$$\boxed{k = \pm v} \leftarrow \text{absolute value of } v$$

$$\boxed{v - 2u(u^T v) = ke}$$

$$v - ke = 2u(u^T \cdot v) \quad \Big/ \text{square both sides}$$

$$v^2 - 2ke^T v + k^2 = 4(u^T u)(u^T v)^2$$

$k = \pm|v|$ $\quad v^2$ $\quad 1$

$$\cancel{2}(v^2 - k a_{12}) = \cancel{4}^{2}(u^T v)^2$$

$$\boxed{u^T \cdot v} = \sqrt{\frac{(v^2 \mp a_{12}|v|)}{2}}$$

$$u = \frac{v - ke}{2u^T v}$$

repeat for $S_2, S_3 \ldots S_{n-1}$

$$A \rightarrow T = \begin{bmatrix} a_{11} & e_1 & & & 0 \\ e_1 & \tilde{a}_{22} & e_2 & & \\ & e_2 & \tilde{a}_{33} & \ddots & \\ & & \ddots & \ddots & e_{n-1} \\ 0 & & & e_{n-1} & \tilde{a}_{nn} \end{bmatrix}$$