Lecture November 27

Diffusion eq in 1+2-dim

$$\nabla^2 u(x,y,t) = \partial u/\partial t$$

boundary
$$\begin{cases} u(x,0,t) = g(x,t) \\ u(x,L,t) = h(x,t) \\ u(0,y,t) = f(y,t) \\ u(L,y,t) = e(y,t) \end{cases}$$

$$x \in [0,L] \wedge y \in [0,L]$$

initial $\quad u(x,y,0) = i(x,y)$

Discretize:

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{u^{l}_{i+1,j} + u^{l}_{i-1,j} - 2u^{l}_{ij}}{(\Delta x)^2}$$

$x \to x_i \qquad y \to y_j \qquad t \to t_l$

$$\frac{\partial u}{\partial t} \simeq \begin{cases} \dfrac{u^{l+1}_{ij} - u^{l}_{ij}}{\Delta t} \quad \text{Explicit scheme} \\[2ex] \dfrac{u^{l}_{ij} - u^{l-1}_{ij}}{\Delta t} \leftarrow \end{cases}$$

Implicit scheme

wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{ij}^{l+1} + u_{ij}^{l-1} - 2u_{ij}^{l}}{(\Delta t)^2}$$

$$\boxed{\frac{\partial u}{\partial t} = \frac{u_{ij}^{l+1} - u_{ij}^{l-1}}{2\Delta t}}$$

Explicit scheme for Diffusion eq in 2-Dim

$$\alpha = \frac{\Delta t}{h^2} \qquad h = \Delta x, \Delta y$$

$$h = \frac{L-0}{n} = L/n$$

$$\boxed{u_{ij}^{l+1} = u_{ij}^{l} + \alpha\left[u_{i+1j}^{l} + u_{i-1j}^{l} + u_{ij+1}^{l} + u_{ij-1}^{l} - \underline{4}\, u_{ij}^{l}\right]}$$

Implicit scheme for Diff in 2 dim

intermediate step

- Laplace's and Poisson's equation:

$$\nabla^2 u(x,y) = \frac{\cdot}{\epsilon}\rho(x,y)$$

$$u_{ij} = \frac{1}{4}\left[u_{ij+1} + u_{ij-1} + u_{i+1j} + u_{i-1j}\right] + \frac{h^2}{4}\rho_{ij}$$

Solved iteratively using for example Jacobi or Gauss-Seidel methods.

Back to Diffusion in 2 dims

$$u_{ij}^{l} = \frac{1}{1+4\alpha}\left[\alpha\Delta_{ij}^{l} + \underset{\text{known}}{u_{ij}^{l-1}}\right]$$

$$\Delta_{ij}^{l} = u_{i+1j}^{l} + u_{i-1j}^{l} + u_{ij+1}^{l} + u_{ij-1}^{l}$$

Wave Equation in 2 dim

$$\Rightarrow \frac{\partial^2 u(x,y,t)}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$x \in [0,1]$ $y \in [0,1]$

$(L = 1) \qquad t \in [0, t_{final}]$

initial condition:

$$u(x,y,0) = g(x,y)$$

$$\rightarrow \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

Boundary condition:

$$u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$$

Discretized equation:

$$\frac{u_{ij}^{l+1} + u_{ij}^{l-1} - 2u_{ij}^{l}}{(\Delta t)^2} = \frac{u_{i+1,j}^{l} + u_{i-1,j}^{l} - 2u_{ij}^{l}}{h^2} + \frac{u_{i,j+1}^{l} + u_{i,j-1}^{l} - 2u_{ij}^{l}}{h^2}$$

Explicit scheme

$$u_{ij}^{l+1} = 2u_{ij}^{l} - u_{ij}^{l-1} + \frac{\Delta t^2}{h^2}\left[\Delta_{ij}^{l} - 4u_{ij}^{l}\right]$$

$l = 0$

$$\underbrace{u^{1}_{i,j}}_{\text{unknown}} = \underbrace{2u^{0}_{i,j}}_{\text{known}} - \underbrace{u^{-1}_{i,j}}_{\text{unknown}} + \frac{\Delta t^2}{h^2}\left[\overbrace{\Delta^{0}_{i,j}}^{\text{known}} - \underbrace{4u^{0}_{i,j}}_{\text{known}}\right]$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} \simeq \frac{u^{1}_{i,j} - u^{-1}_{i,j}}{2\Delta t} = 0$$

$$u^{-1}_{i,j} = u^{1}_{i,j}$$

$$\underline{u^{1}_{i,j}} = u^{0}_{i,j} + \frac{\Delta t^2}{2h^2}\left[\Delta^{0}_{i,j} - 4u^{0}_{i,j}\right]$$

Self-starting also