September 4

FLOPs for Gaussian Elimination:

$$
\begin{array}{c}
\frac{a_{21}}{a_{11}} \\ \\ \frac{a_{31}}{a_{11}} \\ \frac{a_{41}}{a_{11}}
\end{array}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
0 & \left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) & \left(a_{23} - \frac{a_{21}}{a_{11}} a_{13}\right) & \left(a_{24} - \frac{a_{21}}{a_{11}} a_{14}\right) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

precalculate $\frac{a_{21}}{a_{11}}$ $\frac{a_{31}}{a_{11}}$ $\frac{a_{41}}{a_{11}}$

$(n-1)$ FLOPs

$2(n-1)(\cancel{2}n-1)$

$$
\begin{bmatrix}
x & x & x & x \\
0 & x & x & x \\
0 & x & x & x \\
0 & x & x & x
\end{bmatrix}
$$

$n-2$

$2(n-2)(n-2)$

$$
\begin{bmatrix}
x & x & x & x \\
0 & x & x & x \\
0 & 0 & x & x
\end{bmatrix}
$$

$$\begin{bmatrix} 0 & 0 & \times & \times \end{bmatrix}$$

RHS : $2(\underline{n-1})$, $2(n-2)$

Dominating term ;

$$2\sum_{i=n}^{2}(i-\underline{1})(i-\underline{1})$$

$$= 2\sum_{i=2}^{n}(i-\underline{1})(i-\underline{1})$$

$j = i-\underline{1}$ $\quad$ $i=2$ $\quad$ $j=\underline{1}$

$i=n$ $\quad$ $j=n-1=m$

$$= \boxed{2\sum_{j=1}^{m} j^2} = \frac{2}{6}\left(2m^3+\underline{2m^2}+m\right)$$

$$\sim \frac{2}{3}m^3$$

$$\sim O\left(2/3\, m^3\right)$$

LU-decomp Algorithm

$$\begin{bmatrix} \boxed{a_{11}} & \boxed{a_{12}} & a_{13} & a_{14} \\ \boxed{a_{21}} & a_{22} & a_{23} & a_{24} \\ a_{31} & & & \end{bmatrix} =$$

$$\begin{bmatrix} a_{41} & - & - & - & - \end{bmatrix}$$

$A \in \mathbb{R}^{4\times 4}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

First column $\quad a_{ij} \quad j=1$

$a_{11} = \boxed{u_{11}} \leftarrow$ Fixed

$a_{21} = l_{21}\, u_{11} \qquad l_{21} = a_{21}/u_{11}$

$a_{31} = l_{31}\, u_{11} \qquad l_{31} = a_{31}/u_{11}$

$\underline{a_{41}} = \underline{l_{41}}\, u_{11}$

Second column $\quad a_{ij} \quad j=2$

$a_{12} = \boxed{u_{12}}$

$a_{22} = l_{21} u_{12} + u_{22}$

$\underline{u_{22}} = a_{22} - l_{21} u_{12}$

$a_{32} = l_{31} u_{12} + \underline{l_{32}} \cdot u_{22}$

$\underline{l_{32}} = (a_{32} - l_{31} \cdot u_{12}) / \underline{u_{22}}$

$$a_{42} = l_{41} u_{12} + \boxed{l_{42}}\, u_{22}$$

Third column $\quad a_{ij} \quad j = 3$

$$\boxed{a_{13} = u_{13}}$$

$$a_{23} = l_{21} u_{13} + u_{23}$$

$$a_{33} = l_{31} u_{13} + l_{32} u_{23} + u_{33}$$

general algo

$\boxed{i > j}$

$$l_{ij} = \Big(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \Big) / u_{jj}$$

$\boxed{i = j}$

$$u_{ii} = a_{ii} - \sum_{k=1}^{i-1} l_{ik} u_{ki}$$

$\boxed{i < j}$

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$$

Small Exercise

Find the LU decomp of

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 & 2 \end{bmatrix}$$

Pivoting

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \simeq \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix}$$

$$\underline{a_{11} = 0 = u_{11}}$$

$$l_{21} = a_{21}/u_{11} = \infty$$

Cheat matrix

$$u_{11} = 10^{-20} \qquad u_{12} = a_{12} = 1$$

$$l_{21} = a_{21}/u_{11} = 10^{20}$$

$$u_{22} = 1 - 10^{20} \simeq -10^{20}$$

$$L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

$$L \cdot U = \begin{bmatrix} 10^{-20} & 1 \\ 1 & \ldots \end{bmatrix} \neq A$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

Can LU-decompose, but decomp is wrong!

Pivoting

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_1 \end{bmatrix}$$

$$u_{11} = \underline{1} \qquad u_{12} = \underline{1}$$

$$l_{21} = 0 \qquad u_{22} = a_{22} - l_{21} u_{12} = \underline{1}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A!$$

Iterative methods

- Gauss-Seidel
- Jacobi
- Relaxation algos

$Ax = b \qquad A \in \mathbb{R}^{4\times 4}$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$$

$$- \quad - \quad - \quad - \quad - \quad -$$

$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4$$

initial guess $x_i^{(0)}$

$$x_1^{(1)} = \left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - a_{14}x_4^{(0)}\right) / a_{11}$$

i

$$x_i^{(k+1)} = \left(b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}x_j^{(k)}\right) \Big/ a_{ii}$$

Theorem if A is semi-positive definite (all eigenvalues ≥ 0) then $Ax = b$ converges iteratively to the exact x irrespective of initial $x^{(0)}$

Algorithm

- give initial guess $x^{(0)}$
- compute
  $x_i^{(k+1)}$
- iterate till

$$|x^{(k+1)} - x^{(k)}| \le \varepsilon$$

$\sim 10^{-10}$

Jacobi's iterative algo.