Lecture November 26

PS and diffusion eq,
analytical solution.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad u = u(x,t)$$

no function that depends on x and t

$x \in [0,1]$ $\qquad u(x,0) = g(x)$

$0 < x < 1$

Boundary $\qquad u(0,t) = 0$

$u(1,t) = 0$

$t \geq 0$

ansatz: separation of variables

$$u(x,t) = F(x)\,G(t)$$

$$G(t)\frac{\partial^2 F}{\partial x^2} = F\frac{\partial G}{\partial t} \quad \Rightarrow$$

$$\frac{F''}{F} = G'/G$$

$$\frac{F''}{F} = -\lambda^2 \quad \Rightarrow$$

$$F'' + \lambda^2 F = 0 \quad \wedge \quad G' = -\lambda^2 G$$

$$G = \underline{C}\, e^{-\lambda^2 t}$$

$F(x) = A\sin(\lambda x) + B\cos(\lambda x)$

using boundary conditions

$x=0 \quad F(x)=0 \quad \Rightarrow B=0$

$x=\underline{1} \quad F(1) = \underline{A}\sin(\lambda) = 0$

$\lambda = \underline{n}\cdot\pi \qquad n = \pm1, \pm2, \ldots$

$\Big(x \in [0,1] \;, x\in[0,L] \Rightarrow$

$\lambda = n\pi/L \Big)$

$$u_n(x,t) = \underset{A\cdot C}{\underline{A_n}}\sin(\underline{n}\pi x/L)\, e^{-n^2\pi^2/L^2\, t}$$

$$\| \; u(x,t) = \sum_{n=1}^{\infty} \underline{A_n}\sin(n\pi x/L)\, e^{-n^2\pi^2/L^2\, t}$$

used boundary conditions

initial conditions

$u(x,0) = g(x)$

$$= \sum_{n=1}^{\infty} A_n \sin(n\pi x/L)$$

$$\int_a^b dx \,\sin(mx)\sin(nx) \propto \delta_{mn}$$

$x\in[a,b]$

$$\int_0^L g(x)\sin(mx)\,dx =$$

$$\sum \int^L A_m \sin(m\pi x/L)\sin(nx)\,dx$$

$n=1$ ... 0

$$\Rightarrow \boxed{A_n = \frac{2}{L}\int_0^L g(x)\sin(n\pi x/L)\,dx}$$

1-DIM

Crank-Nicolson: $O(\Delta x^2)$, $O(\Delta t^2)$
stable for all Δx & Δt

Implicit scheme: $O(\Delta x^2)$, $O(\Delta t)$
stable for all Δx & Δt

Explicit scheme: $O(\Delta x^2)$, $O(\Delta t)$

$$\frac{\Delta t}{(\Delta x)^2} \le 1/2$$

2-Dim: Explicit scheme

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u(x,y,t) = \underbrace{F(x)G(y)}_{X(x,y)}H(t)$$

Discretize;

$$x \to x_i = x_0 + i\Delta x \qquad h = \Delta x = \Delta y$$

$$y \to y_j = y_0 + j\Delta y$$

$$t \to t_\ell = t_0 + \ell \Delta t$$

$$\Delta x = \Delta y = h = \frac{1}{n}$$

$$x \in [0, 1]$$

$$y \in [0, 1]$$

$$i, j = 0, 1, 2, \ldots, n$$

$$u(x, y, t) \to u(x_i, y_j, t_\ell) = u_{ij}^{\ell}$$

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{u_{i+1,j}^{\ell} + u_{i-1,j}^{\ell} - 2u_{ij}^{\ell}}{h^2}$$

$$\frac{\partial u}{\partial t} \simeq \frac{u_{ij}^{\ell+1} - u_{ij}^{\ell}}{\Delta t}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{ij+1}^{\ell} + u_{ij-1}^{\ell} - 2u_{ij}^{\ell}}{h^2}$$

$$u_{ij}^{\ell+1} = u_{ij}^{\ell} + \alpha\Big[u_{i+1j}^{\ell} + u_{i-1j}^{\ell}$$

$$- 4\alpha u_{ij}^{\ell} \qquad + u_{ij+1}^{\ell} + u_{ij-1}^{\ell}\Big]$$

$u_{ij}^{*}(1-4\alpha)$ $\alpha = \frac{\Delta t}{h^2}$

Explicit scheme using boundary & initial condition we can find

$$u_{ij}^{1} = u_{ij}^{0} + \alpha\left[u_{i+1j}^{0} + u_{i-1j}^{0} + u_{ij+1}^{0} + u_{ij-1}^{0}\right]$$

boundary + initial condition

$$\frac{\Delta t}{(\Delta x)^2}, \frac{\Delta t}{(\Delta y)^2} \leq 1/2$$

Implicit scheme

- intermediate step

Laplace/Poisson, needed for the spatial part of the diffusion eq in 2(3)-dim.

Laplace's equation

$$\nabla^2 u = 0$$

2-dim

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\rho(x,y)$$

Discretized version

$$\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}}{h^2}$$

$$\left(\begin{array}{l} x \to x_i = x_0 + i\,\Delta x = x_0 + i \cdot h \\ y \to y_j = y_0 + j\,\Delta y = y_0 + j \cdot h \end{array}\right)$$

$$= -\rho_{ij} \qquad \left(\rho(x,y) = \rho(x_i,y_j) = \rho_{ij}\right)$$

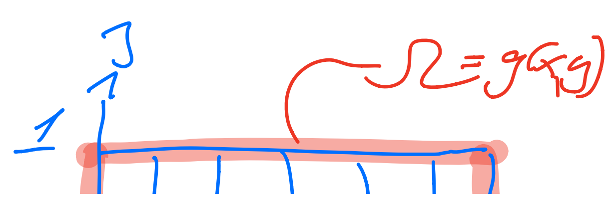
$$\boxed{u_{ij} = \frac{1}{4}\left[u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j}\right] + \frac{h^2}{4}\rho_{ij}}$$

$$x \in [0,1] \wedge y \in [0,1]$$

Boundary condition Γ:

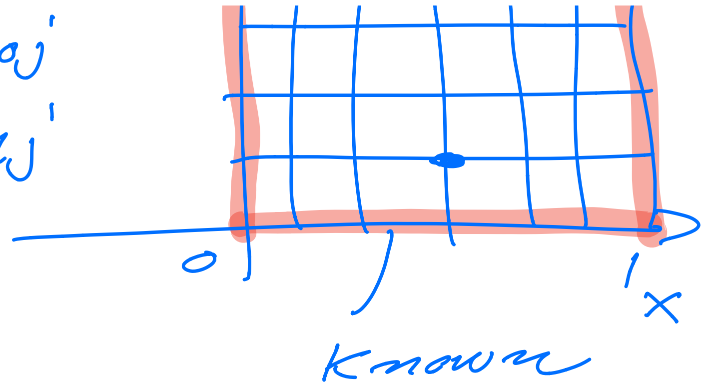
$$u_{i0} = g_{i0}$$

$$u_{i1} = g_{i1}$$



$u_{0j} = g_{0j}$

$u_{1j} = g_{1j}$



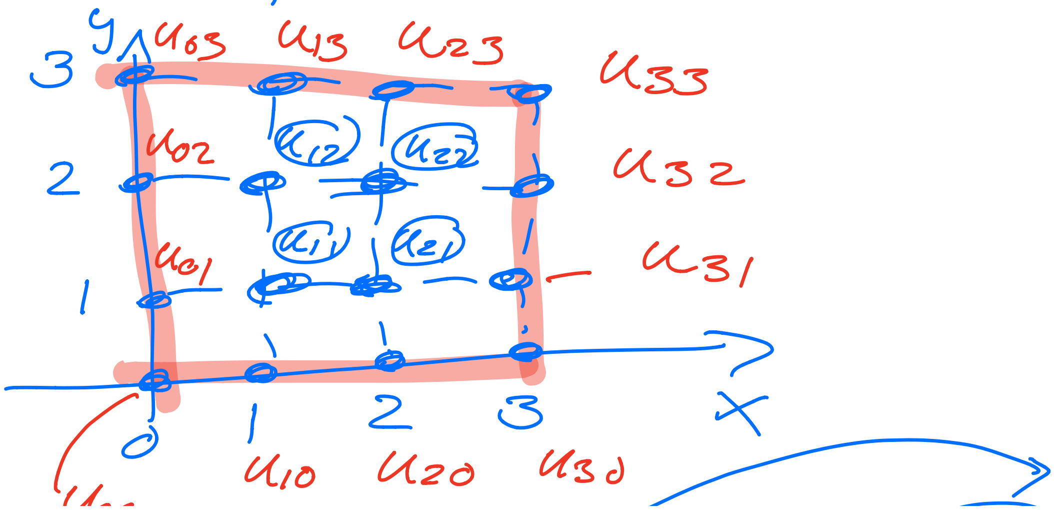
$$u_{ij} = \frac{1}{4}\left[u_{i+1j} + u_{i-1j} + u_{ij+1} + u_{ij-1}\right] + \frac{h^2}{4} f_{ij}$$

Boundary is not enough?

$$4u_{ij} = \overbrace{\left[u_{i+1j} + u_{i-1j} + u_{ij+1} + u_{ij-1}\right]}^{\text{Laplacian } \Delta_{ij}} + \tilde{f}_{ij} \qquad \tilde{f}_{ij} = h^2 f_{ij}$$

$$\underline{4u_{ij}} = \Delta_{ij} + \tilde{f}_{ij}$$

$n = 4$



$$4u_{11} - u_{21} - u_{01} - u_{12} - u_{10} = f_{11}$$

$$4u_{12} - u_{02} - u_{22} - u_{13} - u_{11} = f_{12}$$

$$4u_{21} - u_{11} - u_{31} - u_{22} - u_{20} = f_{21}$$

$$4u_{22} - u_{12} - u_{32} - u_{23} - u_{21} = f_{22}$$

unknown

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{22} \\ u_{22} \end{bmatrix} = \begin{bmatrix} f_{11} + u_{01} + u_{10} \\ f_{12} + u_{13} + u_{02} \\ f_{21} + u_{31} + u_{20} \\ f_{22} + u_{32} + u_{23} \end{bmatrix}$$

Known

$$\boxed{A \cdot x = b}$$

A is positive definite matrix. $\lambda_i > 0$

Family of iterative

Solvers: Jacobi-method

Gauss-Seidel

⋮

$A \in \mathbb{R}^{n \times n}$

Gauss-elim

$O(n^3)$