Lecture September 18

Standard Eigenvalue finding approach:

$$A = \begin{bmatrix} x & x & x & \cdots & x \\ x & & & & \\ \vdots & & & & \\ x & \cdots & \cdots & \cdots & x \end{bmatrix} \qquad A \in \mathbb{R}^{n \times n} \text{ symmetric}$$

$$\longrightarrow \quad SAS^T = T$$

$$\begin{bmatrix} x & x & & & & 0 \\ x & x & x & & & \\ & x & \ddots & \ddots & & \\ & & \ddots & \ddots & x & \\ 0 & & & x & x & x \\ & & & & x & x \end{bmatrix}$$

Householder transformation.

QR algo.

Find $D = [\lambda_1 \lambda_2 \cdots \lambda_n]$

$$\longrightarrow \quad \begin{bmatrix} x & & & 0 \\ & x & & \\ & & \ddots & \\ 0 & & & x \end{bmatrix}$$

How do we go from $T \to D$?

* Bisection and Sturm's method.

$$T = \begin{bmatrix} a_1 & b_1 & & & 0 \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & 0 & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}$$

Def polynomial of degree $-r-$

$$P_r(x) = \det(T_r - x\mathbb{I}) = 0$$

$r = 1:n$

$$P_r(x) = (a_r - x)\,P_{r-1}(x) - b_{r-1}^2\,P_{r-2}(x)$$

for $r = 2:n$

$$\begin{vmatrix} \boxed{a_1 - x} & b_1 & & \\ b_1 & a_2 - x & & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n - x \end{vmatrix} = 0$$

$$P_0(x) = 1 \qquad P_1(x) = a_1 - x$$

$$P_2(x) = \begin{bmatrix} a_1 - x & b_1 \\ b_1 & a_2 - x \end{bmatrix}$$

$$= (a_2 - x)(a_1 - x) - b_1^2$$

$$= (a_2 - x) P_1(x) - b_1^2 P_0(x)$$

$$P_1(x) = 0 \Rightarrow x = a_1$$

$$P_3(x) = \begin{bmatrix} a_1 - x & b_1 & 0 \\ b_1 & a_2 - x & b_2 \\ 0 & b_2 & a_3 - x \end{bmatrix} \cdots 0 \cdots 0$$

$$(a_1 - x) \begin{vmatrix} a_2 - x & b_2 \\ b_2 & a_3 - x \end{vmatrix}$$

$$- b_1 \begin{vmatrix} b_1 & b_2 \\ 0 & a_3 - x \end{vmatrix}$$

$$= (a_3 - x) P_2(x) - P_1(x) b_2^2$$

$$P_n(x) = (a_n - x) P_{n-1}(x) - b_{n-1}^2 P_{n-2}(x)$$

$P_n(x)$ can be evaluated in $O(n)$ FLOPS by bisection.

$$|a_1| \le |a_2| \le |a_3| \ldots \le |a_n|$$

$$\lambda_k(T) \in \underline{[y, z]}$$

$$z = \max a_i + |b_i| + |b_{i-1}|$$

$$y = \min a_i - |b_i| - |b_{i-1}|,$$

for all $i = 1:n$