# Lecture November 19

Covariance

$\mathrm{cov}(x_i, x_j) =$

$\int dx_1 \cdots dx_n \, (x_i - \mu_{x_i})(x_j - \mu_{x_j})$
$\times P(x_1, x_2 \cdots x_n)$

if $P(x_1, x_2 \cdots x_n) = p(x_1)\, p(x_2) \cdots p(x_n)$

$X = \{x_1, x_2 \cdots x_n\}$ are <u>i.i.d</u> $\quad \mu_{x_i} = \mu_{x_j}$

$\mathrm{cov}(x_i, x_j) = 0$

$\mu = \mu_{x_i} = \int dx \, p(x)\, x$

$\sigma^2 = \int dx \, p(x) (x - \mu)^2$

$$z = \frac{x_1 + x_2 + \cdots x_n}{n}$$

$\tilde{p}(z) = ?$

$n \to \infty$

$$\tilde{p}(z) = \frac{1}{\left(\sqrt{2\pi}\, \sigma / \sqrt{n}\right)^2} e^{-\frac{(z-\mu)^2}{2\sigma^2/n}}$$

$$\Rightarrow \boxed{\tilde{\sigma}^2 = \frac{\sigma^2}{n}} \Leftarrow$$

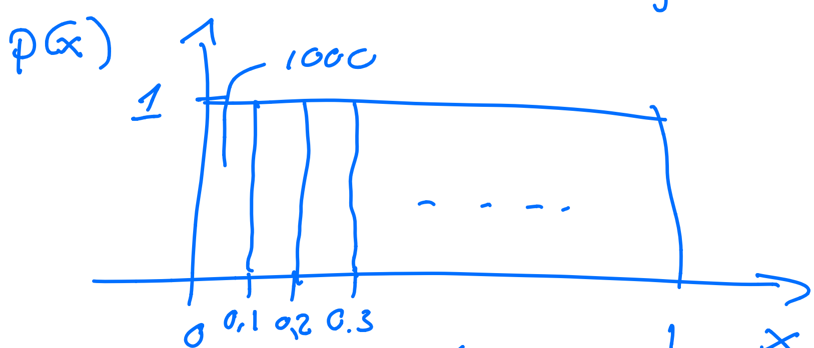
if not iid:

$$\tilde{\sigma}^2 = \frac{\sigma^2}{n} + \underline{cov}$$

in general $\tilde{\sigma}^2 \geq \frac{\sigma^2}{n}$

$$cov = \frac{1}{n} \sum_{\substack{i,j \\ i \neq j}} (x_i - \mu_{x_i})(x_j - \mu_{x_j})$$

$$= \frac{2}{n} \sum_{i<j} (x_i - \mu_{x_i})(x_j - \mu_{x_j})$$




$x \in [0, 1]$

$$\mu = \int_0^1 x\, p(x)\, dx = 1/2$$

$$\sigma^2 = \int_0^1 (x-\mu)^2 p(x)\, dx = \frac{1}{12}$$

RNG

Basic algo - linear

congruential algo

$N_i$ = RN

$$N_i = (\underline{a} N_{i-1} + \underline{c})\, \underline{MOD}(\underline{M})$$

$a = \underline{1}$ $\quad c = 0 \quad N_{i-1} = 13$

$M = 2$

$N_i = (13)\, MOD\,(2) = \underline{1}$

in fortran $\quad MOD(a, b)$

in C++ $\quad a \% b$

$N_0$ = seed

$x_i = N_i / M \qquad x_i \in [0, 1]$

M = period

$M = 5 \qquad c = 7 \qquad a = 6$

$N_0 = 2$

$N_1 = (19)\, MOD(5) = 4$

$$\underbrace{4, 1, 3, 0, 2}, \underbrace{4, 1, 3, 0, 2}, \ldots$$

period of 5

$N_0 = 2 \qquad M = 54 \qquad c = 11$

$a = 27$

$N_1 = (54 + 11)\ \text{MOD}(54) = 11$

$N_2 = 38, \quad N_3 = 11, \quad N_4 = 38 \ldots$

period $= 2$ !

– Famous RNG = <u>RAND</u>, <u>RAN0</u>

$$\boxed{N_i = (a N_{i-1})\ \text{MOD}(M)}$$

$M \sim 10^9$

Warning; always check period $M$ of a given RNG.

Can you use any seed?

Can't use $N_0 = 0$.

$N_i$ is an integer, $M$ same

$a$ also.

standard int variable

has 32 bits $\sim \underline{2^{31}}$

$\boxed{a N_{i-1}}$ ~ integer overflow

Trick after Schrage (1968)

Define $M = a \cdot q + r$

$r < q$

$q =$ integer division

$= \left[ M/a \right]$

$r = (M) \text{ MOD } (a)$

$M = 13 \qquad a = 3$

$q = \left[ 13/3 \right] = 4$

$r = (M) \text{ MOD } (a) = \underline{1}$

$M = 13 = 3 \cdot 4 + 1 = 13$

$$\Big( \underbrace{\left[ N_{i-1}/q \right]}_{\text{integer}} M \Big) \text{ MOD } (M) = 0$$

$(a N_{i-1}) \text{ MOD } (M) =$

$$\left( \underline{a N_{i-1}} - \left[ N_{i-1}/q \right] (\underline{a \cdot q} + r) \right)$$
$$\text{MOD } (M)$$

$a \left( N_{i-1} - \left[ N_{i-1}/q \right] q \right)$

$= a N_{i-1} \text{ MOD } (q)$

$(aN_{i-1}) \text{ MOD } (M) =$

$$\left[ (aN_{i-1}) \text{ MOD} (q) - \underbrace{[N_{i-1}/q]}\, r \right] \text{ MOD } (M)$$

$$[N_{i-1}/q]\, r \le N_{i-1}\, r/q$$

$r < q$, we get a number which is smaller than $N_{i-1}$ and is smaller than $M \Rightarrow aq < M$

## PDEs

- Diffusion in one dim.
  - scaled and dimless variables;

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$$u = u(x, t)$$

initial conditions $\left[ u(x, t=0) = g(x) \right.$

$$x \in [0, 1]$$

Boundary conditions
Dirichlet conditions

$$\begin{cases} u(0,t) = a \ (=0) \\ u(1,t) = b \ (=0) \end{cases}$$

$$x \to x_i = x_0 + i \cdot \Delta x \quad i = 0,1,2,\dots n$$

$$\Delta x = \frac{b-a}{n}$$

$$t \to t_j = t_0 + j \cdot \Delta t \quad j = 0,1,\dots m$$

define $t_0$ and final $t = t_m$

Discretize $u(x,t) = u(x_i, t_j) = u_{ij}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1\,j} + u_{i-1\,j} - 2u_{ij}}{\Delta x^2} + O(\Delta x^2)$$

$$\frac{\partial u}{\partial t} = \frac{u_{i\,j+1} - u_{ij}}{\Delta t} + O(\Delta t)$$

Euler's forward

$$\frac{\partial u}{\partial t} = \frac{u_{ij} - u_{i\,j-1}}{\Delta t} + O(\Delta t)$$

Euler's backward