Lecture November 20

Diffusion eq. (scaled)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad u = u(x,t)$$

initial condition:

$$u(x,0) = g(x)$$

Boundary conditions

$$x \in [0,1]$$

$$u(0,t) = a = 0$$

$$u(1,t) = b = 1$$

$$x \to x_i = x_0 + i\Delta x \qquad i = 0,1,\dots n$$

$$\Delta x = \frac{b-a}{n}$$

$$x_0 = 0 \qquad x_n = 1$$

$$t \to t_j = t_0 + j \cdot \Delta t \qquad j = 0,1,\dots m$$

$$\Delta t = \frac{t_{final} - t_{initial}}{m}$$

$$t_0 = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1j} + u_{i-1j} - 2u_{ij}}{(\Delta x)^2} + O(\Delta x^2)$$

$$\frac{\partial u}{\partial t} = \begin{cases} \dfrac{u_{i,j+1} - u_{i,j}}{\Delta t} + O(\Delta t) \\ \text{Forward Euler} \\ \dfrac{u_{i,j} - u_{i,j-1}}{\Delta t} + O(\Delta t) \\ \text{Backward Euler} \end{cases}$$

Forward Euler: Explicit scheme

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

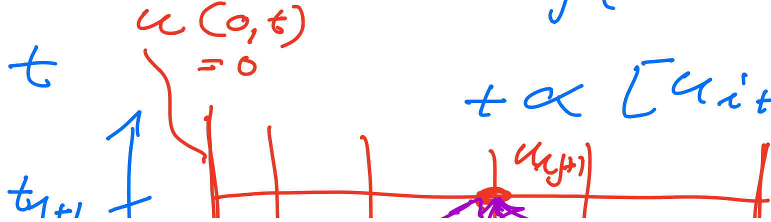
$$u(x,t) = u(x_i, t_j) = u_{i,j}$$

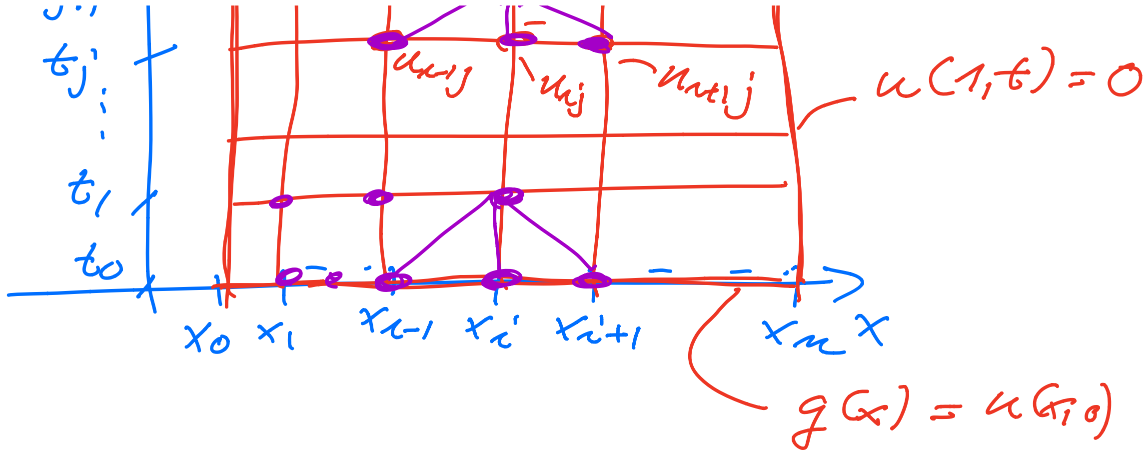
$$\frac{u_{i+1} + u_{i-1} - 2u_{i,j}}{(\Delta x)^2} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

$$\frac{\Delta t}{(\Delta x)^2} = \alpha$$

$$u_{i,j+1} = u_{i,j} + \alpha\left[u_{i+1,j} + u_{i-1,j} - 2u_{i,j}\right]$$

$$= u_{i,j}(1 - 2\alpha) + \alpha\left[u_{i+1,j} + u_{i-1,j}\right]$$





$$u_{ij+1} = u_{ij}(1-2\alpha) + \alpha\left[u_{i+1j} + u_{i-1j}\right]$$

$$V_j = \begin{bmatrix} u_{0j} \\ u_{1j} \\ \vdots \\ u_{nj} \end{bmatrix}$$

known

known

$$V_{j+1} = A\,V_j$$

$$A = \mathbb{1} - \alpha B$$

$$B = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & & \\ \vdots & & & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Töplitz Tridiag matrix.

$$u_{1,j+1} = \boxed{\underline{u_{1j}}(1-2\alpha) + \alpha\left[u_{2j} + u_{0j}\right]}$$

$$u_{2,j+1} = u_{2j}(1-2\alpha) + \alpha\left[u_{3j} + u_{1j}\right]$$

$$\vdots$$

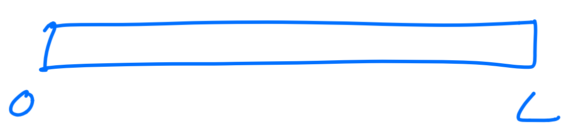
$$u_{m-1,j+1} = u_{m-1,j}(1-2\alpha) + \alpha\left[u_{mj} + u_{m-2,j}\right]$$

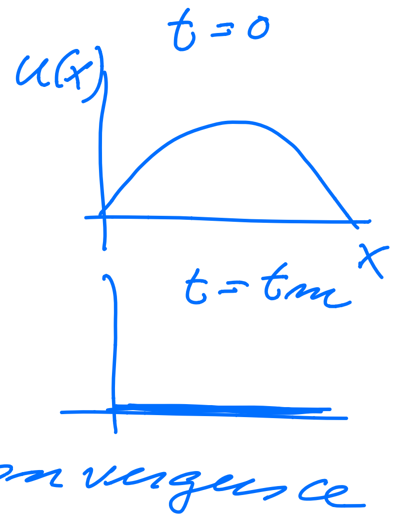
$V_1 = A V_0$ $\quad$ V_0 is known

$V_2 = A V_1 = A^2 V_0$

$$\vdots$$

$$\boxed{\underline{V_m} = A^m V_0}$$





The criterion for convergence

$$\alpha = \frac{\Delta t}{(\Delta x)^2} \le 1/2$$

$(\Delta x)^2$

$\Delta x = 0{,}1 \Rightarrow \Delta t \le 1/2 \cdot 10^{-2}$

$x \in [0,1]$

$\Delta x = 0{,}01 \Rightarrow \Delta t \le 1/2 \cdot 10^{-4}$

Backward Euler: Implicit scheme

$$\frac{u_{ij} - u_{ij-1}}{\Delta t} = \frac{u_{i+1j} + u_{i-1j} - 2u_{ij}}{(\Delta x)^2}$$

$$u_{ij} + 2\alpha u_{ij} - \alpha u_{i+1j} - \alpha u_{i-1j} = u_{ij-1}$$

$$u_{ij}(1+2\alpha) - \alpha(u_{i+1j} + u_{i-1j}) = u_{ij-1}$$

$j = 1$

unknown

$$u_{i1}(1+2\alpha) - \alpha(u_{i+1\,1} + u_{i-1\,1}) = u_{i0} \text{ - known}$$

$$= g(x_i) = g_i$$

$$V_j = \begin{bmatrix} u_{0i} \\ u_{1i} \\ \vdots \end{bmatrix}$$

$$\lfloor u_{mi} \rfloor$$

$$A V_j = V_{j-1}$$ - known

$$V_j = A^{-1} V_{j-1}$$ (inefficient)

$$u_{1j}(1+2\alpha) - \alpha(u_{2j} + u_{0j}) = u_{1j-1}$$ (known)

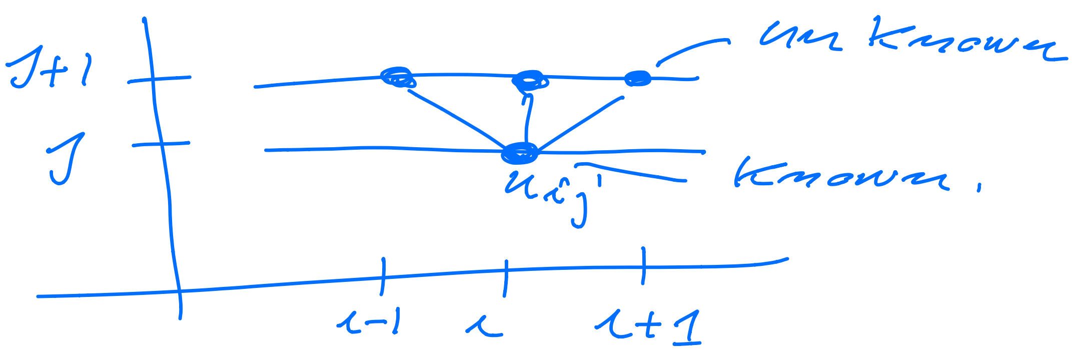
$$u_{2j}(1+2\alpha) - \alpha(u_{3j} + u_{1j}) = u_{2j-1}$$

$$\vdots$$

$$u_{m-1j}(1+2\alpha) - \alpha(u_{mj} + u_{m-2j}) = u_{m-1j-1}$$

$$A = \mathbb{1} + \alpha B$$

$$B = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}$$



$$A = \begin{bmatrix} 1+2\alpha & -\alpha & & 0 \\ -\alpha & 1+2\alpha & & \\ & & \ddots & \\ 0 & & 1+2\alpha & -\alpha \\ & & -\alpha & 1+2\alpha \end{bmatrix}$$

use the tridiagonal solver from project 1.

Implicit scheme is stable for all Δt and Δx

CN-scheme

$$-\alpha u_{i-1,j} + (2+2\alpha) u_{i,j} - \alpha u_{i+1,j}$$

$$= \alpha u_{i-1,j-1} + (2-2\alpha) u_{i,j-1}$$

$$+ \alpha u_{i+1,j-1}$$

$$\Rightarrow \underbrace{(2I + \alpha B)}_{\substack{\text{modified} \\ \text{implicit} \\ \text{part}}} V_j = \underbrace{(2I - \alpha B)}_{\substack{\text{modified} \\ \text{explicit} \\ \text{part.}}} V_{j-1}$$

- precalculate (explicit scheme) $(2I - \alpha B) V_{j-1} =$

$$w_{j-1}$$

$$-\quad \underbrace{(2I+\alpha B)}_{\substack{\text{Tridiagonal} \\ \text{Töplitz} \\ \text{matrix}}} v_j = w_{j-1}$$

$2I+\alpha B$ has a spectral radius $\rho(2I+\alpha B) > 1$

$$D v_j = w_{j-1}$$

$$D = 2I+\alpha B$$

$$v_j = D^{-1} w_{j-1}$$

$$\rho(D^{-1}) < 1 \quad \Rightarrow$$

CN converges for all values of Δt and Δx.