Project 1 (Ex 3.1) → Lecture notes ch 3.1

Elements to study:

- Discretization of a continuous equation

$$\frac{d^2u}{dx^2} = f(x) \qquad x \in [0, 1]$$

Dimensionless eq.
Two-point boundary value problem
$u(0) = u(1) = 0$

$f(x)$ is known, $u(x)$ is unknown.

- Mathematical approx to a 2nd derivative.

- This approximation leads to potential errors
  - Mathematical ones
  - Numerical round-off → numerical loss of precision

- Representation of arrays (vectors and matrices)

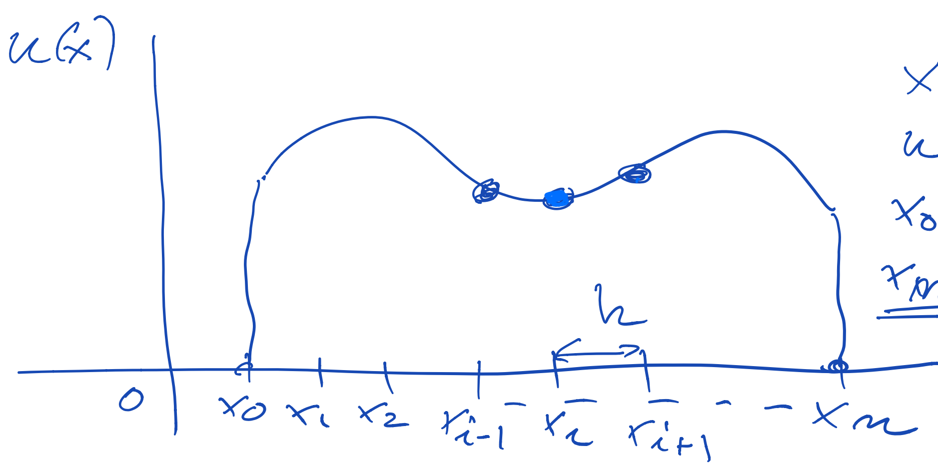
Numpy, Armadillo and Eigen
$\longrightarrow$ segmentation fault?
Memory handling.

- Read and write to file
  Timing your program.
  FLOPs = Floating point operations.

Mathematical Discretization and numerical calculation of derivatives:

$$\frac{d^2 u(x)}{dx^2} = \underbrace{f(x)}_{\text{known}}$$

$$x \rightarrow x_i = x_0 + i \cdot h$$
$$i = 0, 1, 2, \ldots n$$



$x \in [0, 1]$
$u(0) = u(1) = 0$
$x_0 = 0$
$\underline{x_n} = 1$

$$h = \text{step size} = \frac{x_n - x_0}{n}$$
(Δx)

$\underline{u(x)} \longrightarrow v(x_i) = \underline{u(x_i)}$

$u(x_i) = \underline{u_i}$

$u(x_i \pm h) = \underline{u_{i\pm 1}}$

$$\left.\frac{d^2u}{dx^2}\right|_{x_i} = \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2} \underline{+ O(h^2)}$$

can be studied.

$$\left.\frac{du}{dx}\right|_{x_i} = \begin{cases} 2p \quad \dfrac{u_{i+1} - u_i}{h} + O(h) \\ 2p \quad \dfrac{u_i - u_{i-1}}{h} + O(h) \\ 3p \quad \underline{\dfrac{u_{i+1} - u_{i-1}}{2h} + O(h^2)} \end{cases}$$

Taylor expansion

$$u(x_i \pm h) = \underline{u(x_i)} \pm h\, u'(x_i) + \frac{h^2}{2!} u''(x_i) \pm \frac{h^3}{3!} u'''(x_i) + O(h^4)$$

Euler's forward :

$$u(x_i+h)-u(x_i) = h\,u'(x_i)+o(h^2)$$

$$u'\big|_{x_i} = \frac{u_{i+1}-u_i}{h} + o(h)$$

$$\frac{u_{i+1}-u_{i-1}}{2h} = \ ?$$

$$u(x_i+h) - u(x_i-h) = u_{i+1}-u_{i-1}$$

$$\frac{u_{i+1}+u_{i-1}-u_i}{h^2} = \frac{u_{i+1}+u_{i-1}}{\cdots\cdots}$$

Chapter 3.1 :

3p formulaes $\quad u', u'' \quad \underline{u''\big|_{x_i}} = u''_i$

$$\frac{u_{i+1}-u_{i-1}}{2h} = u'_i + \sum_{j=1}^{\infty} \frac{u_i^{(2j+1)}}{(2j+1)!} h^{2j}$$

$$\frac{u_{i+1}+u_{i-1}-2u_i}{h^2} = u''_i + \sum_{j=1}^{\infty} \frac{u_i^{(2j+2)}}{(2j+2)!} h^{2j}$$

Project 1 (Ex 3.1) we have

an analytical solution,
Relative error

$$\varepsilon = \left| \frac{u_{computed} - u_{exact}}{u_{exact}} \right|$$

<u>Example</u>

$$u(x) = e^x$$
$$u''(x) = e^x$$

ε as function of h

$$h = \frac{x_n - x_0}{n} \qquad x \in [x_0, x_n]$$

plot/analyze in terms of $\log_{10}$

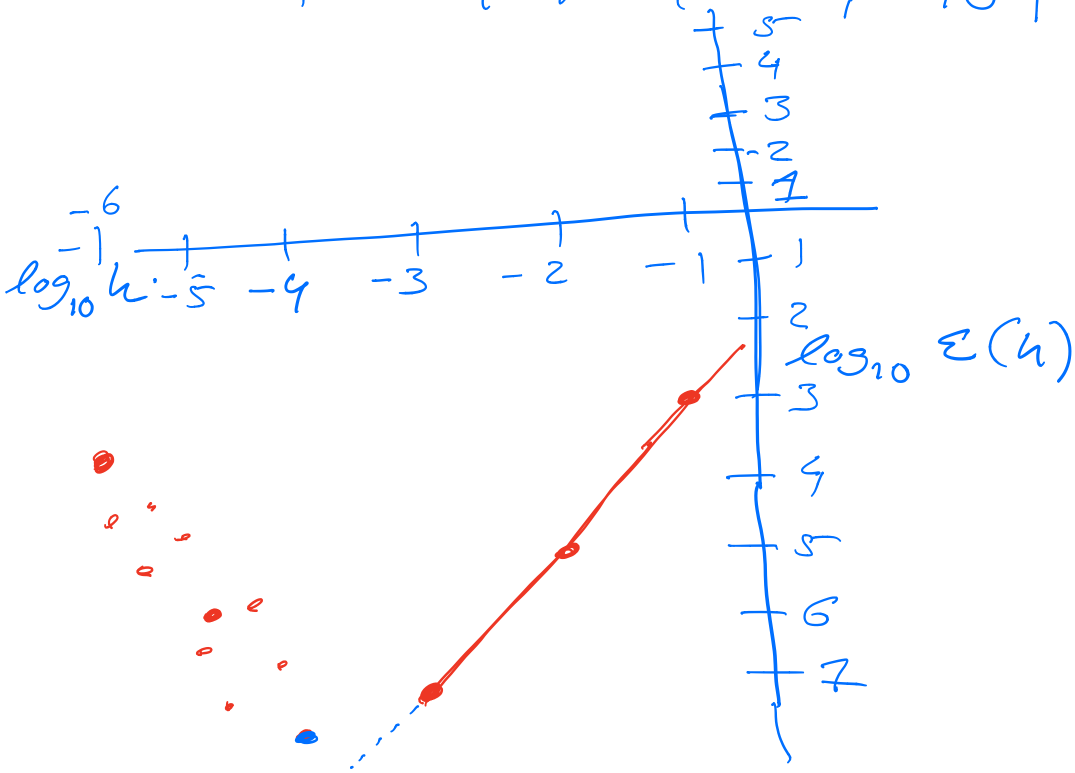
$$\boxed{u_i'' = \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2}} + \underline{O(h^2)}$$

$$\varepsilon(h) = \left| \frac{u''_{exact} - u_i''}{u''_{exact}} \right| \sim O(h^2)$$

$$u_i'' \simeq \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2}$$

$\log_{10} \varepsilon(h) \sim$ slope $= 2$

$h = 0.1,\ 0.01,\ 0.001,\ 10^{-4},\ 10^{-5},\ \ldots$



Make a model for the error

$$\varepsilon_{Model} = \varepsilon_{Math} + \varepsilon_{RO}$$

$$u_i'' = \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2} + \sum_{j=1} \frac{u_i^{(2j+2)}}{(2j+2)!} \times h^{2j}$$

assume $\varepsilon_{Math} = \dfrac{u_i^{(4)}}{12} h^2$

$\varepsilon_{RO} = \ ?$

Floating point number

$$fl(a) = a(1 \pm \varepsilon_M)$$

ε_M = machine precision

double precision $\varepsilon_M \sim 10^{-15}$

(chapter 2)

$$u_{i+1} + u_{i-1} - 2u_i = (u_{i+1} - u_i) + (u_{i-1} - u_i) \simeq 2\varepsilon_M$$

$$\varepsilon_{RO} = \frac{\frac{u_{i+1} + u_{i-1} - 2u_i}{h^2}}{} = \frac{2\varepsilon_M}{h^2}$$

$$\varepsilon_{RO} \leq \frac{2\varepsilon_M}{h^2}$$

$$\varepsilon_{Model} = \frac{2\varepsilon_M}{h^2} + \frac{u_i^{(4)} h^2}{12}$$

where does ε_{model} have its min as function of h?

$$\frac{d\varepsilon_{Model}}{dh} = 0 \implies h = \left(\frac{24\varepsilon_M}{u_1^{(4)}}\right)^{1/4}$$

$$x = 2 \implies h \simeq 10^{-4}$$