Lecture November 13

- RNG, central limit theorem and the covariance.

$p(x)dx$ (continuous)

$$\mu_f = \mathbb{E}[f(x)] = \int_a^b f(x)\,p(x)\,dx$$

$x \in [a, b]$

$$\mathrm{var}[f] = \mathbb{E}[f^2(x)] - \mu_f^2$$

multivariate probability

$$X \in \{x_1, x_2, \dots x_n\}$$

$$p(x_1, x_2 \dots x_n) = ?$$

i.i.d = independent and identically distributed

$$p(x_1, x_2, \dots x_n) = p(x_1)\,p(x_2) \dots p(x_n)$$

$$\int p(x_1, x_2 \dots, x_n)\,dx_1\,dx_2 \dots dx_n$$

$$= \int p(x_1)\,dx_1 \int p(x_2)\,dx_2 \dots$$

$$\int p(x_n)\,dx_n = \underline{1}$$

$$\mathbb{E}[x_1] = \mu_{x_1} = \int p(x_1)\, x_1\, dx_1 \underbrace{\int p(x_2)\, dx}_{=1} \cdots \int p(x_n)\, dx_n$$

$$\left(\int p(x_i)\, dx_i = \underline{1} \right)$$

$$= \mu_{x_1} = \int p(x_1)\, x_1\, dx_1$$

$$= \int p(x)\, x\, dx$$

not i.i.d :

$$\mu_{x_1} = \int \cdots \int p(x_1 x_2 \ldots x_n)\, x_1 \times dx_1 \ldots dx_n$$

covariance :

$$\mathbb{E}[(x_i, x_j)] = \int \cdots \int dx_1 \ldots dx_n$$

$$\boxed{\times}\; p(x_1 x_2 \ldots x_n)(x_i - \mu_{x_i})(x_j - \mu_{x_j})$$

if i.i.d :

$$\mathbb{E}[(x_i x_j)] = \mathrm{cov}(x_i, x_j)$$

$$= \int \cdots \int dx_1 \ldots dx_n\, p(x_1)\, p(x_2) \ldots p(x_n)$$

$$\boxed{\times}\; (x_i - \mu_{x_i})(x_j - \mu_{x_j})$$

$$\mu_{x_i'} = \mu_{x_j'} = \mu$$

$$= \iint dx_i' \, dx_j' \, p(x_i') \, p(x_j') \, (x_i' - \mu)(x_j' - \mu)$$

$$= \mathbb{E}[x_i' x_j'] - \mu^2$$

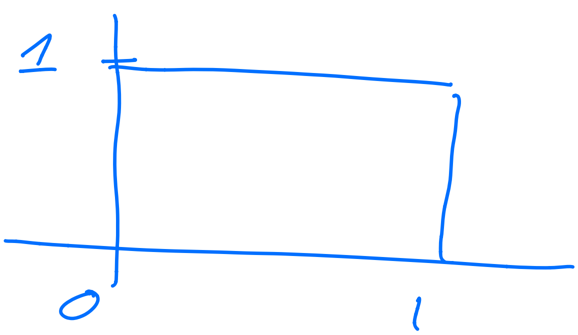
$$\iint dx_i' \, dx_j' \, x_i' x_j' \, p(x_i') \, p(x_j')$$

$$= \mu^2 \Rightarrow \text{if i.i.d}$$

then $\mathrm{cov}(x_i', x_j') \equiv 0$

our <u>RNG</u> is based on the uniform distribution and should produce RNs that are i.i.d

if $i' = j'$, cov $= 0$



$$p(x)\,dx = \begin{cases} dx & \text{if } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$x \in [0, 1]$

Assessing the <u>RNG</u>

3 - simplest tests

(i) $\mu = \int_0^1 p(x)\,dx\,x = \int_0^1 dx\, x$

$= 1/2$

(ii) $var(x) = \int_0^1 (x-\mu)^2 p(x)\,dx$

$STD = \sqrt{var(x)} = \frac{1}{\sqrt{12}}$

(iii) $cov(x_i, x_j) = 0$

normally not the case.

central limit theorem

$\underline{p}(x_1)\,dx_1$, $\underline{p}(x_2)\,dx_2$ ... $\underline{p}(x_n)\,dx_n$

what is the PDF where the mean value is

$z = \frac{x_1 + x_2 + \dots x_n}{n}$

x_i's are i.i.d.

$\tilde{p}(z) = \int dx_1\, p(x_1) \int dx_2\, p(x_2)$

$\dots \int dx_n\, p(x_n)\, \delta\left(z - \frac{x_1 + x_2 \dots x_n}{n}\right)$

$$\delta\left(z - \frac{x_1 + x_2 + \dots x_n}{n}\right)\Bigg)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \exp\left(iq\left(z - \frac{x_1 + x_2 + \dots x_n}{n}\right)\right)$$

insert $e^{i\mu q - i\mu q}$

$$\mu = \int x\, p(x)\, dx$$

$$\tilde{p}(z) = \frac{1}{2\pi}\int dq\, e^{iq(z-\mu)}$$

$$\times \left[\int_{-\infty}^{\infty} dx\, p(x)\, e^{iq(\mu - x)/n}\right]^n$$

$$\int_{-\infty}^{\infty} dx\, p(x)\, e^{iq(\mu-x)/n}$$

$\int p(x)\,dx = 1$

$\int p(x)\,dx\, x = \mu$

$$= \int_{-\infty}^{\infty} dx\, p(x)\left[1 + \frac{iq(\mu - x)}{n} - \frac{q^2(\mu - x)^2}{2n^2} + \dots\right]$$

$$= 1 + 0 - \frac{q^2}{2n^2}\sigma^2$$

$\int p(x)\,dx\,(\mu - x)^2 = \sigma^2$

$$\left[\int^{\infty} \dots e^{iq(\mu-x)/\dots}\right]^n$$

$$\left[\int_{-\infty} dx\, p(x)\, e \right]$$

$$\simeq \left[1 - \frac{q^2 \sigma^2}{2n^2} \right]^n$$

$\Rightarrow$

$$\tilde{p}(z) = \frac{1}{\sqrt{2\pi (\sigma/\sqrt{n})}} e^{-\frac{(z-\mu)^2}{2(\sigma/\sqrt{n})^2}}$$

normal distribution with mean value $\boxed{\mu}$ and variance

$$\sigma^2/n \Rightarrow$$

$$\text{STD} = \sigma/\sqrt{n}$$