CSCI 430: Homework 5

John E. Buckley III

October 8, 2017

1 3.1-1

We need to prove the following inequations:

1. c1(f(n)+g(n)) $\geq$ 0
2. c1(f(n)+g(n)) $\leq$ max(f(n), g(n))
3. max(f(n), g(n)) $\leq$ c2(f(n)+g(n))

The **first** inequation holds since f(n) and g(n) are asymptotically non-negative functions which was stated in the problem.

The **second** inequation holds because max(f(n),g(n)) $\geq$ f(n) and max(f(n),g(n)) $\geq$g(n). This gives us max(f(n),g(n)) $\geq$ $\frac{(f(n)+g(n))}{2}$

The **third** inequation holds because max(f(n),g(n)) $\leq$ f(n)+g(n). c2 $\geq$ 1.

2 3.1-2

Proof: There exists n_0 ϵ $\mathbb{N}$ and there exists c1,c2 ϵ $\mathbb{R}$ s.t. for all n $\geq$ n_0: c1n^b $\leq$ $(n+a)^b$ $\leq$ c2n^b. n+a>n because a>0 and n+a $\leq$ 2n for all n $\geq$ a. The inequalities are preserved by raising both sides by the power of b: (n+a)> n^b and $(n+a)^b \leq 2^b n^b$.
Thus, this equation holds when n_0=a, c1=1, c2=2^b

3 3.1-3

Big O notation is usually used to indicate an upper bound of a function but the way the problem is worded,"is at least" which implies $\geq$, it actually gives us no information about the upper bound. The best running time could be anything slower than O(n^2).

4 3.1-4

The first statement is true because:
$2^{n+1} = 2 \times 2^n \leq c2^n$ where c $\geq$ 2.

The second statement is false because:
$2^{2n} \leq c \times 2^n$
ln2 $\times$ 2n $\leq$ lnc+ln2 $\times$ n
2n $\leq$ lnc+n
n $\leq$ lnc

5 3.1-6

If T(n)=Θ(g(n)) then: 0 $\leq$ c1g(n) $\leq$ T(n) $\leq$ c2g(n) for n $\geq$ n_0
Since T(n) > 0 and T(n) $\leq$ c2g(n): T(n)=O(g(n)) which is the upper bound or worst case running time.
Since T(n) $\geq$ c1g(n) and c1g(n) $\geq$ 0: T(n)=Ω(g(n)), which is the lower bound or best case running time.

6 3.1-7

From our notes, if we were to think calculus:
f(n) is o(g(n)):

$$\lim_{x \to \infty} \frac{f(n)}{g(n)} = 0$$

f(n) is ω(g(n)):

$$\lim_{x \to \infty} \frac{f(n)}{g(n)} = \infty$$

Both of these cannot be simultaneously true, thus no f(n) exists.

7 3.2-1

From our notes, f(n) and g(n) are monotonically increasing iff n $\geq$ m $\rightarrow$ f(n),g(n) $\geq$ f(m),g(m).

Therefore: f(n)+g(n) $\geq$ f(m)+g(m): f(n)+g(n) is monotonically increasing
Therefore: f(g(n)) $\geq$ f(g(m)): f(g(n)) is monotonically increasing
Therefore: f(n) $\times$ g(n) $\geq$ f(m) $\times$ g(m): f(n) $\times$ g(n) is monotonically increasing

8 3.2-2

Using the identities from our notes and some arithmetic:

$a^{log_b c} = a^{log_a^{\frac{c}{log_a b}}}$
$= (a^{log_a c})^{\frac{1}{log_a b}}$
$= c^{frac1log_a b}$
$= c^{log_b a}$

9 3.2-3

Using Stirling's approximation:

lg(n!) $\approx$ lg($\sqrt{2\pi n}$)($\frac{n}{e}$)^n)
= lg($\sqrt{2\pi n}$) + lg($\frac{n}{e}$)^n
= lg$\sqrt{2\pi}$ + lg$\sqrt{n}$ + nlg($\frac{n}{e}$)
= lg$\sqrt{2\pi}$ + $\frac{1}{2}lgn$ + nlgn - nlge
= $\Theta(1)$ + $\Theta(lgn)$ + $\Theta(nlgn)$ - $\Theta(n)$
= $\Theta(nlgn)$

n! = n $\times$ (n - 1) $\times$ (n - 2) $\times$ .... < n $\times$ n $\times$ n .... n number of times, for n $\geq$ 2.
= o(n^n)

n! = n $\times$ (n - 1) $\times$ (n - 2) $\times$ .... > 2 $\times$ 2 $\times$ 2 .... n number of times, for n $\geq$ 4.
=$\omega(2^n)$

10 3.2-6

From our notes: we substitute ϕ and $\hat{\phi}^2$ and use some arithmetic.

$\phi^2 = (\frac{1+\sqrt{5}}{2})^2$
$= \frac{1+2\sqrt{5}+5}{4}$
$= \frac{3+\sqrt{5}}{2}$
$= \frac{1+\sqrt{5}}{2} + 1$
$= \phi + 1$

$\hat{\phi}^2 = (\frac{1-\sqrt{5}}{2})^2$
$= \frac{1-2\sqrt{5}+5}{4}$
$= \frac{3-\sqrt{5}}{2}$
$= \frac{1-\sqrt{5}}{2} + 1$
$= \hat{\phi} + 1$

11 3-2

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes	no	no	no
b.	n^k	c^n	yes	yes	no	no	no
c.	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d.	2^n	$2^{n/2}$	no	no	yes	yes	no
e.	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f.	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

a Polynomial grows faster than logarithm.

b Exponential grows faster than polynomial.

c No relations among them in terms of growth.

d Decreasing the base of an exponential function makes it grow slower.

e Both functions are same (see 3.2-2).

f Round the sum for O and Ω and Stirling's Approximation for Θ.

12 3-4

a False: Let f(n)=n and g(n)=n^2. n=$O(n^2)$ but $n^2 \neq O(n)$.

b False: Using the previous example: $n^2 + n \neq \Theta(min(n^2, n))$.

c True:

d False: Let f(n)=2n and g(n)=n. f(n)=$O(g(n))$ but $n^{2n} = 4^n \neq O(2^n)$.

- e True: $0 \leq f(n) \leq cg(n)$ We need to prove that: $0 \leq df(n) \leq g(n)$ Which is straightforward with $d = 1/c$.

- f False: Let f(n)=4^n. $4^n \neq \Theta(4(n/2)) = \Theta(2^n)$.

- g True: From our notes: $0 \leq o(f(n)) \leq f(n)$. Then $f(n) \leq f(n) + o(f(n)) \leq 2f(n)$. Thus, $f(n) + o(f(n)) = \Theta(f(n))$.