

Effective Homology of the Pushout of Simplicial Sets

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Motivation

- Kenzo:

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 - Symbolic Computation system devoted to Algebraic Topology

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 - Compute homology groups of spaces of not finite type: loop spaces, classifying spaces,...

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 - Particular cases: wedges, joins, ...

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Concrete goal

New Kenzo module for constructing the Pushout of simplicial sets

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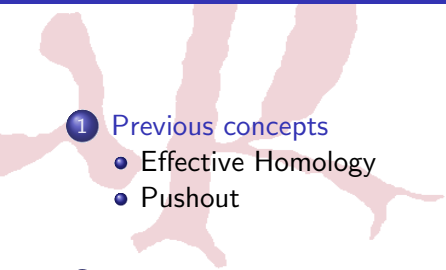
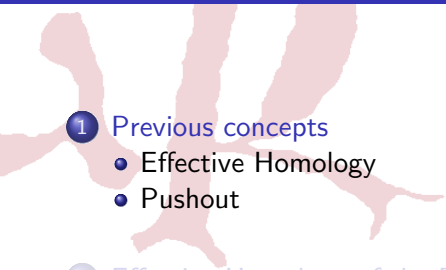
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 - Effective Homology
 - Pushout
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 - 3 Examples
 - 4 Conclusions and Further Work

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Effective Homology

- Finite nature objects:

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 - Adjacency matrix is an integer matrix

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Effective vs Locally Effective Chain Complexes

Definition

An effective chain complex is a free chain complex of \mathbb{Z} -modules, $C_* = (C_n, d_n)_{n \in \mathbb{N}}$, where each group C_n is finitely generated and

- an algorithm returns a \mathbb{Z} -base in each grade n
- an algorithm provides the differentials d_n

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- an algorithm provides the differentials d_n
- differentials $d_n : C_n \rightarrow C_{n-1}$ can be expressed as integer matrices
- possible to compute $\text{Ker } d_n$ y $\text{Im } d_{n+1}$

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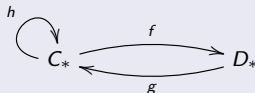
A locally effective chain complex is a free chain complex of \mathbb{Z} -modules, $C_* = (C_n, d_n)_{n \in \mathbb{N}}$, where each group C_n is formed by a infinite number of generators

- impossible to compute $\text{Ker } d_n$ y $\text{Im } d_{n+1}$
- possible to perform local computations, differential of a generator

Effective Homology

Definition

A reduction ρ between two chain complexes C_* y D_* (denoted by $\rho : C_* \rightrightarrows D_*$) is a triple $\rho = (f, g, h)$



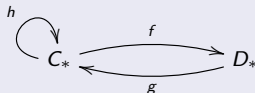
satisfying the following relations:

- 1) $fg = \text{Id}_{D_*}$;
- 2) $d_C h + h d_C = \text{Id}_{C_*} - g f$;
- 3) $fh = 0$; $hg = 0$; $hh = 0$.

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Theorem

If $C_* \Rightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n .

Effective Homology

Definition

A (strong chain) equivalence ε between C_* and D_* , $\varepsilon : C_* \Leftrightarrow D_*$, is a triple $\varepsilon = (B_*, \rho, \rho')$ where B_* is a chain complex, $\rho : B_* \Rightarrow C_*$ and $\rho' : B_* \Rightarrow D_*$.

$$\begin{array}{ccc}
 & B_* & \\
 \swarrow & & \searrow \\
 C_* & & D_*
 \end{array}$$

$$\begin{array}{ccc}
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Definition

An object with effective homology is a quadruple $(X, C_*(X), HC_*, \varepsilon)$ where:

- X is a locally effective object
- $C_*(X)$ is a (locally effective) chain complex canonically associated with X , which allows the study of the homological nature of X
- HC_* is an effective chain complex
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Theorem

Let an object with effective homology $(X, C_*(X), HC_*, \varepsilon)$ then $H_n(X) \cong H_n(HC_*)$ for all n .

Pushout

Definition

Let f, g morphisms, the pushout of f, g

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \\ Z & & \end{array}$$

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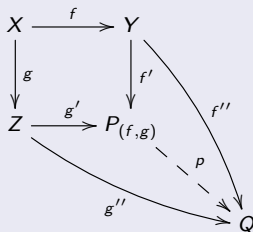
is an object P for which the diagram:

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Pushout

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is an object P for which the diagram:

- commutes
- respects the universal property


Pushout

Standard Construction

$P_{(f,g)} \cong (Y \amalg (X \times I) \amalg Z) / \sim$ where:

- I is the unit interval
- for every $x \in X$, \sim :
 - $(x, 0) \sim f(x) \in Y$
 - $(x, 1) \sim g(x) \in Z$

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Construction of the Effective Homology of the Pushout: Step 1

Given $f : X \rightarrow Y$ and $g : X \rightarrow Z$ simplicial morphisms where X , Y and Z are simplicial sets:

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Theorem (Algorithm: Standard Construction, Implementation: J. Heras)

Input: two simplicial morphisms $f : X \rightarrow Y$ and $g : X \rightarrow Z$ where X , Y and Z are simplicial sets.

Output: the pushout $P_{(f,g)}$.

Construction of the Effective Homology of the Pushout: Step 2

Given $f : X \rightarrow Y$ and $g : X \rightarrow Z$ simplicial morphisms where X , Y and Z are simplicial sets with effective homology:

$$\begin{array}{ccc}
 (X, C_*(X), HX_*, \varepsilon_X) & \xrightarrow{f} & (Y, C_*(Y), HY_*, \varepsilon_Y) \\
 \downarrow g & & \downarrow \\
 (Z, C_*(Z), HZ_*, \varepsilon_Z) & \longrightarrow & (P_{(f,g)}, C_*(P_{(f,g)}), -, -)
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Theorem (Algorithm: F. Sergeraert, Implementation: J. Heras)

Input: two simplicial morphisms $f : X \rightarrow Y$ and $g : X \rightarrow Z$ where X , Y and Z are simplicial sets with effective homology.

Output: the effective homology version of $P_{(f,g)}$, that is, an equivalence $C_*(P_{(f,g)}) \Leftrightarrow HP_*$, where HP_* is an effective chain complex.

Effective Homology of the Pushout

Theorem

Input:

$C_*(B)$ a chain complex;

$(C_*(A), HA_*, \varepsilon_A);$

$(C_*(C), HC_*, \varepsilon_C);$

$$0 \xleftarrow{0} C_*(A)_* \xrightleftharpoons[j]{\sigma} C_*(B) \xrightleftharpoons[i]{\rho} C_*(C) \xleftarrow{0} 0$$

Output: $(C_*(B), HB_*, \varepsilon_B)$

$$0 \xleftarrow{\quad} M \xrightleftharpoons{\quad} CP_{(f,g)} \xrightleftharpoons{\quad} CY \oplus CZ \xleftarrow{\quad} 0$$

where $M = X \times I \setminus ((X \times \{0\}) \cup (X \times \{1\}))$

Effective Homology of the Pushout

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Theorem

Input: two simplicial sets X and Y with effective homology

Output: an equivalence $C_(X \oplus Y) \Leftarrow DD_* \Rightarrow HD_*$, where HD_* is effective.*

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Consider the short exact sequence:

$$0 \longleftarrow M \rightleftarrows C(X \times I) \rightleftarrows C(X \times \{0\}) \oplus C(X \times \{1\}) \longleftarrow 0$$

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Theorem (Eilenberg-Zilber Theorem)

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Output: an equivalence $C_(X \times Y) \Leftarrow DC_* \Rightarrow C_*(X) \otimes C_*(Y)$,
where $C_*(X) \otimes C_*(Y)$ are effective.*

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- New module allows the computation of homology groups of spaces

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- Demo

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Conclusions

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 - New Kenzo module (1600 lines) allows the computation of homology groups of spaces defined as the pushout of simplicial sets

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- Further Work:
 - Implementation of new constructions

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