Noname manuscript No.
(will be inserted by the editor)

A Hierarchy of Mathematical Structures in ACL2

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Received: date / Accepted: date

Abstract In this paper, we present a methodology which allows one to deal with *mathematical structures* in the ACL2 theorem prover. Namely, we cope with the representation of mathematical structures, the certification that an object fulfills the axioms characterizing an algebraic structure and the generation of generic theories about concrete structures. As a by-product, an *ACL2 algebraic hierarchy* has been obtained. Our framework has been tested with the definition of *homology groups*, an example coming from Homological Algebra which involves several notions related to Universal Algebra. The method presented here means a great profit, with respect to a from scratch approach, when we work with complex mathematical structures; for instance, the ones coming from Algebraic Topology.

Keywords Mathematical structures, ACL2, Algebraic Hierarchy, Homological Algebra, Proof Engineering

Partially supported by Ministerio de Educación y Ciencia, project MTM2009-13842-C02-01, and by the European Union's 7th Framework Programme under grant agreement nr. 243847 (ForMath).

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1 Introduction

The implementation of algebraic structures in *theorem proving environments* is a well-known problem; and most of the theorem prover systems offer a set of tools to deal with it. As a result, several implementations of *algebraic hierarchies* have been produced for different systems.

These algebraic hierarchies are, in some cases, the foundation for large proof developments. There are several proposals for the Coq system: the CCorn hierarchy [25] based on dependent records and used in the proof of the Fundamental Theorem of Algebra, the SSReflect hierarchy [24] based on packed classes and currently used in the development of the proof of the Feit-Thompson Theorem, and also an approach based on the Coq's type class mechanism [52]; other examples can be found in systems such as Isabelle [23,35], Mizar [50], Nuprl [32] and Lego [9].

Nevertheless, up to the best of our knowledge, a similar development had not been undertaken for the ACL2 theorem prover until now. In the work presented in this paper we have tackled the challenge of devising a methodology to deal with mathematical structures in ACL2. As a result, we have obtained an algebraic hierarchy which allows one to build theories about usual mathematical structures taking advantage of the ACL2 capabilities. The hierarchy ranges from setoids (a setoid is a set together with an equivalence relation on it) to R-modules including structures such as groups or rings. In addition, this hierarchy is flexible enough to be extended with new structures following our methodology.

As we will show throughout this paper, our hierarchy of algebraic structures can be used to formalize non-trivial mathematical concepts. However, we are not trying to compete with systems such as Coq or Mizar where an impressive amount of results about algebra have been already formalized; on the contrary, our final aim is oriented towards ACL2's forte: software and hardware verification, see [36]. In particular, we are mainly focused on the verification of *Computer Algebra systems*, a kind of software where algebraic structures are instrumental. In addition, our work can be also useful for the verification of non mathematical software which needs, from time to time, some results from algebra.

The rest of this paper is organized as follows. In the next section, we present a brief introduction to the ACL2 system and the tools employed in our development. Section 3 is devoted to introduce our methodology to deal with mathematical structures and the resultant algebraic hierarchy. As a benchmark to test our approach, the formalization of homology groups is explained in Section 4. The suitability of our method to cope with formalizations related to complex mathematical structures is presented in Section 5, showing the advantages of using our approach instead of working from scratch. In Section 6 we discuss related work associated with both the formalization of algebraic structures using Theorem Provers, and its use to verify the correctness of Computer Algebra systems. The paper ends with a section of conclusions and further work, and the bibliography.

2 A brief introduction to ACL2

ACL2 [37,39] is a programming language, a logic, and a theorem prover supporting reasoning in the logic. The ACL2 programming language is an extension of an

applicative subset of Common Lisp. The ACL2 logic is a first-order logic with equality, used for specifying properties and reasoning about the functions defined in the programming language. All the variables in the formulas allowed by the ACL2 system are implicitly universally quantified. The syntax of its terms and formulas is that of Common Lisp and it includes axioms for propositional logic, equality and for a number of predefined Common Lisp functions and data types. Rules of inference of the logic include those for propositional calculus, equality and instantiation.

One important rule of inference is the *principle of induction*, that allows proofs by well-founded induction on the ordinal ε_0. The logic has a constructive definition of the ordinals up to ε_0, in terms of lists and natural numbers. The system also includes the usual well-founded order relation defined on this set of ordinals. Although this is the only built-in well-founded relation, the user may define new well-founded relations, provided that an ordinal mapping is explicitly given and proved to be monotone.

By the *principle of definition*, new function definitions are admitted as axioms only if there exists a measure and a well-founded relation with respect to which the arguments of each recursive call decrease, thus ensuring that the function terminates. In this way, no inconsistencies are introduced by new function definitions. Usually, the system can prove automatically termination properties using a predefined ordinal measure and the built-in well-founded relation on ordinals. Nevertheless, if the termination proof is not trivial, the user has to explicitly provide a measure on the arguments and a well-founded relation with respect to which this measure decreases. In addition, new function definitions must be total on the language of terms, so when functions are naturally defined only working on a subset of terms, some behavior must also be defined on arguments outside of that subset.

An additional way to introduce new function symbols in the logic is by means of the `encapsulate` mechanism [38]. Instead of giving their definitional body, only certain properties are assumed about them; to ensure consistency, witness local functions (which are witness functions local to an encapsulate block) having the same properties have to be exhibited. Inside an `encapsulate`, the properties stated need to be proved for the local witnesses, and outside, they work as assumed axioms.

A derived rule of inference, called *functional instantiation*, see [37], provides a limited higher-order-like reasoning mechanism instantiating the function symbols of a previously proved theorem. This rule replaces function symbols with other ones, provided it can be proved that the new functions satisfy the constraints or the definitional axioms of the replaced functions (depending on whether they were introduced by an `encapsulate` or by the principle of definition, respectively).

The ACL2 theorem prover mechanizes the ACL2 logic, and is particularly successful obtaining mechanical proofs mainly based on simplification and induction. The role of the user in this mechanization is important: usually a non-trivial result is not proved in a first attempt, and the user has to lead the prover to a successful proof providing a set of lemmas that the prover uses mainly as rewrite rules.

In addition to the built-in inference rules and tools provided by the ACL2 system we extensively use two tools developed by external authors: the defstructure tool [16], and the generic instantiation tool [44].

The defstructure tool provides the ACL2 `defstructure` macro that creates and characterizes general purpose record structures. Thus, this tool provides a convenient way to group and access data, rendering specifications more readable and modular, and it also creates an extensive theory about the creation, access, and update of the defined structures, providing many of the definitions and lemmas that a user will normally need to be able to effectively and automatically reason about ACL2 specifications using structures.

The generic instantiation tool generalizes the functional instantiation rule in ACL2 in such a way that given a "generic theory", consisting of a set of "independent" functions and assumed properties about them, and a sequence of events (functions definitions and properties) built from the independent functions; then an instance of the whole generic theory can be obtained by means of an instantiation process, provided a concrete version of the independent functions satisfying the assumed properties.

We will skip many details and some of the function definitions will be omitted. We urge the interested reader to consult the original and complete source code at `http://www.unirioja.es/cu/joheras/ahomsia/`. In addition a detailed explanation of the implementation of the tools presented in this paper can be read in [31].

3 Modeling a hierarchy of algebraic structures in ACL2

In this section a framework to deal with both mathematical structures and morphisms between them is presented. Namely, we have developed a methodology to model the hierarchy depicted in Figure 1, where boxes represent types of structures. Moreover, our methodology is flexible enough to extend this hierarchy without any special hindrance.

Let us present in a nutshell some remarks about our hierarchy, the concrete details will be provided throughout this section.

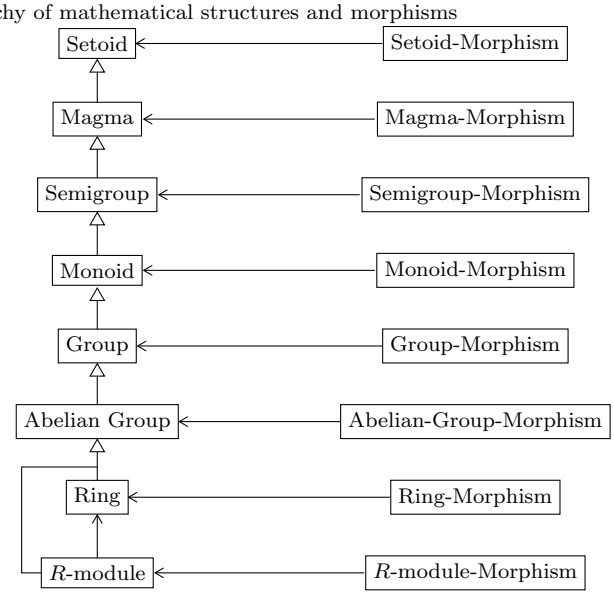
We have depicted the mathematical structures of our hierarchy ranging from *setoids* to *R-modules* in the left side of Figure 1. A mathematical structure can be encoded by means of a structure with several components of functional nature, satisfying the definitional axioms of the intended mathematical structure.

A continuous arrow with an open triangle as tip represents an *inheritance* relationship modeling that the source mathematical structure *is-a* target mathematical structure, e.g. an abelian group is-a group with some additional properties. Whereas a continuous arrow with a normal tip describes a *use* relationship in the sense that the target mathematical structure *is used* to define the source one.

The morphisms included in our hierarchy are presented in the right side of Figure 1. It is worth noting that a morphism always consists of a source structure A of type T, a target structure B of type T, and a map $f : A \to B$. The previous fact is depicted in the diagram using a continue arrow between a T-morphism and the corresponding T-structure.

Now, let us present how we implement the hierarchy in ACL2. We start by introducing in detail the implementation of the simplest algebraic structure: *setoid* [14]. Subsequently, we extrapolate the methodology to the rest of mathematical structures and morphisms.

Fig. 1 Hierarchy of mathematical structures and morphisms



3.1 Modeling setoids in ACL2

A *setoid* $\mathcal{X} = (X, \sim_X)$ is a set X together with an equivalence relation $\sim_X$ on it. Setoids are commonly used in the development of algebraic structures since we can form the quotient of a setoid by changing its equivalence relation, we will see an example of this in Section 4.

A setoid can be represented by means of two functions: the characteristic function of the underlying set (the *invariant*) and a binary function encoding the *equivalence relation*. Therefore, if we are interested in modeling, for instance, the setoid whose underlying set is the integers and the equivalence relation is that which makes integers with same absolute equivalent, we could use the ACL2 `integerp` function as invariant (`integerp` is a recognizer for integer numbers, it returns `true` if its argument is an integer, and `nil` otherwise), and the `eq-abs` function as equivalence relation.

```
(defun eq-abs (a b)
  (equal (abs a) (abs b)))
```

Moreover, it is necessary to prove the events ensuring that `eq-abs` is an equivalence relation on the set characterized by `integerp`.

```
(implies (integerp x) ;; Reflexive
         (eq-abs x x))

(implies (and (integerp x) (integerp y) (eq-abs x y)) ;; Symmetry
         (eq-abs y x))

(implies (and (integerp x) (integerp y) (integerp z) ;; Transitive
```

```
            (eq-abs x y) (eq-abs y z))
       (eq-abs x z))
```

ACL2 provides a way to define equivalence rules [27], but those equivalence rules must be total; so, we cannot use them since we are restricting the domain of our equivalence rules to a concrete set.

In this way, concrete setoids can be modeled in ACL2. However, this is not enough to work with setoids as in standard mathematical textbooks where, for example, *universal properties* about setoids are proved. In order to tackle this problem, we should use the `encapsulate` mechanism [37]. This tool allows us to define a *generic setoid*, namely we can define two generic functions `X-inv` (the invariant function) and `X-eq` (the equivalence relation) where the properties of setois are assumed about them.

Now, using the functions which define the generic setoid, we could prove universal properties which, afterwards, could be instantiated for concrete setoids using the *functional instantiation* mechanism [37]. For example, we could prove the following property:

```
(defthm symmetry-transitive
   (implies (and (X-inv x) (X-inv y) (X-inv z)
                 (X-eq y x) (X-eq y z))
            (X-eq x z)))
```

and subsequently instantiate it for the concrete setoid defined previously.

Then, this from scratch approach can be applied to deal with setoids in ACL2; nevertheless, from our point of view, several enhancements can be introduced to improve the use of this mathematical structure in ACL2.

First of all, it is worth noting that the use of a structure which gathers the functions encoding the invariant and the equivalence relation of a setoid would model the setoid more accurately than having the functions separately. This can be carried out in ACL2 by means of a record, implemented with the `defstructure` macro [16], with two fields (`inv` and `eq`) which store respectively the names of the invariant function and the intended equivalence relation (in addition, those functions must have been introduced previously). The following `defstructure` construction is used to define the setoid structure.

```
(defstructure setoid
  inv eq)
```

It is worth remarking that as ACL2 is an untyped system, then there is no need to attach types to the field names of the structures. Now, using this representation, the setoid whose underlying set is the set of integer numbers having the same absolute value can be encoded as an instance of the `setoid` record, where the values of the `inv` and `eq` slots are respectively the names `integerp` and `eq-abs`. Moreover this instance can be assigned to an ACL2 constant, called `*Zabs*`[1], for latter use in our development.

```
(defconst *Zabs* (make-setoid :inv 'integerp :eq 'eq-abs))
```

[1] A constant in ACL2 is a symbol beginning and ending with the character `*`

Let us note that, since ACL2 is not higher-order and does not use some set theoretic encoding of functions, the *only* way of treating functions as data is to refer them by name.

In order to facilitate the statement of the event which ensures the definitional setoid axioms for `setoid` instances, the `setoid-algebraic-structure` function has been defined. This function takes as argument a `setoid` instance and produces a quoted conjunction of formulas with the definitional axioms of setoids for the functions of the `setoid` (that is, the `eq` component of the `setoid` instance is Boolean, reflexive, symmetric and transitive on the set characterized by the `inv` component of the `setoid` instance). This function is internally invoked by a macro called `check-setoid-p` which can be used to certify that a concrete `setoid` instance is really a setoid. For example, if `*S*` is a constant storing a `setoid` instance, in the invocation:

```
(check-setoid-p *S*)
```

the macro expands into a call of `defthm` whose name is `*S*-is-a-setoid`. The term generated for the `defthm` is provided by the `setoid-algebraic-structure` function, and, as we have said, such term states that the functions of the setoid instance `*S*` satisfy the setoid definitional axioms. In this way, we have automated the creation of the event (a function which extends the ACL2 logic) associated with the definitional properties of setoids. The `setoid-algebraic-structure` function together with the `check-setoid-p` can be seen as a characteristic function for the type of setoids.

The last enhancement is related to the definition of generic setoids without providing, at least explicitly, an `encapsulate`. This has been achieved by means of a macro called `defgeneric-setoid` which makes the creation of generic setoids easier. This macro takes as argument a symbol, for instance `X`, and expands into an `encapsulate` which produces the constant `*X*`, that stores a generic setoid, and the theorem `*X*-is-a-setoid`, which ensures that `*X*` satisfies the setoid axioms.

3.2 A hierarchy of algebraic structures

Following the same ideas presented for setoids, we have defined a number of algebraic structures in ACL2. In the left side of Figure 1 we have depicted the mathematical structures of our hierarchy with the relations among them. A detailed description of each one of these structures can be seen in [18].

The implementation of the *inheritance* relationship between structures is translated into a definition of the source mathematical structure in terms of the target one; this fact will allow us to reuse several code fragments.

We say that a B structure is defined in terms of an A structure if B is an A structure together with (in some cases) additional operations, $op_1, \ldots, op_n$, and satisfying further properties, $P_1, \ldots, P_m$. Then, the ACL2 representation of a B structure which is an A structure together with operations $op_1, \ldots, op_n$ is:

```
(defstructure B
  A op1 ... opn)
```

where the value of the `A` field will be an `A` instance, and the values of `op1`, ..., `opn` slots will be function symbols. For instance, a *magma* is a setoid with a binary operation; so, a magma is represented as:

```
(defstructure magma
  setoid binary-op)
```

This idea is extrapolated to represent all the structures of our hierarchy. Just a remark, the representation of R-modules follows the same strategy but with a small nuance, the *use* relationship between R-modules and rings is handled by means of a field in the definition of `R-module` whose value will be a `Ring` instance.

```
(defstructure R-module
  Abelian-group ring external_operation)
```

Now, we have to deal with the statement of the event which ensures the definitional axioms for an instance of a structure. Let us retake the general case of a B structure which is an A structure together with operations $op_1, \ldots, op_n$ and satisfying properties $P_1, \ldots, P_m$. First, we define the function `B-algebraic-structure` which invokes the functions `A-algebraic-structure` (that is a previously defined function which generates a term with the definitional axioms for an `A` structure) and `P1`, ..., `Pm` (the functions which produce, respectively, the statement of properties $P_1, \ldots, P_m$).

Therefore, the `B-algebraic-structure` function takes a `B` instance as argument and produces a list with the B definitional axioms for that instance. In addition, this function is invoked by a macro called `check-B-p` which has an analogous behavior to the one presented for `check-setoid-p` but for B structures.

As an example, we can consider the magma structure, which is a setoid with a binary operation that is *closed* on the set characterized by the invariant function of the setoid and *compatible* with the equivalence relation of the underlying setoid. It is worth remarking that these closure and compatibility requirements, which usually in algebra are implicit, are needed here explicitly because of the untyped nature of ACL2 logic. In this case, the `magma-algebraic-structure` function invokes the functions `setoid-algebraic-structure` (that produces the definitional axioms for the underlying setoid of the magma instance), `closed-op` (which generates the closure property for the binary operation) and `compatible-op` (which produces the compatibility requirement).

The `check-magma-p` macro, which given a constant `*M*`, that stores a `magma` instance, as argument, expands into a call of `defthm` whose name is `*M*-is-a-magma`. The term of this event states the magma definitional axioms for the components of `*M*`. Analogously, this method can be applied in the rest of the structures of our hierarchy.

The last set of tools that we have defined for the mathematical structures of Figure 1 allows us to work with generic instances of them. Namely, we have defined a number of macros called `defgeneric-<structure>` (where `<structure>` is the name of the structure). These macros take as argument a symbol, `X`, and produce a constant, `*X*`, which stores a generic instance of the structure `<structure>` and a theorem, `*X*-is-a-<structure>`, which states the definitional axioms of `<structure>` for the generic instance.

It is worth noting that the names of the components of the `<structure>` instance stored in the constant produced by `defgeneric-<structure>` always follow the same convention: `<symbol>-<slot>` where `<symbol>` is the symbol given in the call to `defgeneric-<structure>` and `<slot>` is the name of each one of the slots

of `<structure>` (these components are introduced by means of the `encapsulate` principle).

Using these tools we can prove *universal properties* about the structures of our hierarchy. For instance, the result which says that given $\mathcal{M} = (M, \sim_M, \circ_M)$ a magma and N a subset of M closed with respect to $\circ_M$; then $\mathcal{N} = (N, \sim_M, \circ_M)$ is a magma, will be proved as follows. First, we define a generic magma using the `defgeneric-magma` macro taking the symbol `M` as argument. Afterwards, a *generic subset* of M closed with respect to $\circ_M$ is defined using the `encapsulate` principle, where `N-inv` is the invariant of that generic subset.

Now, we can construct a `magma` instance where `N-inv` is the invariant function, `M-eq` is the equivalence relation and `M-binary-op` is the binary operation; and store it in the constant `*N*`.

```
(defconst *N* (make-magma :setoid (make-setoid :inv 'N-inv :eq 'M-eq)
                          :binary-op 'M-binary-op))
```

Finally, we can certify that `*N*` is really a magma using `check-magma-p`.

```
(check-magma-p *N*)
```

It is worth noting that ACL2 is able to find the proof of this event without any external help and from now on, we could instantiate this result for concrete magmas. This property about magmas is a particular case of a well known result of Universal Algebra called *Subalgebra criterion* [18]. This criterion says that given $\mathcal{X} = (X, op_1, \ldots, op_n)$ a T-structure where X is the underlying set of $\mathcal{X}$, and Y is a subset of X closed with respect to $op_1, \ldots, op_n$; then $\mathcal{Y} = (Y, op_1, \ldots, op_n)$ is also a T-structure. This result has been proved for all the structures of our hierarchy following the same schema presented for the magma case.

3.3 Morphisms between algebraic structures

Until now, we have presented how to model algebraic structures in ACL2; now we are going to tackle the task of representing *morphisms* over those mathematical structures as the ones presented in the right side of Figure 1.

In order to deal with morphisms we follow the same strategy presented for mathematical structures. First, we define a record to represent the morphism. It is worth remarking that a morphism consists of: the source structure, S, the target structure, T, and a map $f : S \to T$ (where S and T have the same mathematical structure type). Therefore, all the morphisms of our hierarchy, unlike what happened in the case of algebraic structures, can be encoded with the pattern:

```
(defstructure <structure>-morphism
  source target map)
```

where `<structure>` is the name of one of the mathematical structures of our hierarchy, the value of both `source` and `target` slots will be `<structure>` instances, and the value of `map` will be a function symbol. So, for instance, the identity setoid morphism on `*Zabs*` is defined as:

```
(defconst *id-Zabs*
  (make-setoid-morphism :source *Zabs* :target *Zabs* :map 'id))
```

where `id` is the identity function.

Following the same schema as in the case of structures, a function in charge of creating the event which ensures the axioms of a morphism between `<structure>`s; called `<structure>-morphism-structure`, is defined. Furthermore, we also have defined the macro `check-<structure>-morphism-p`, which internally invokes to `<structure>-algebraic-structure`, whose behavior is analogous to the one presented for the macro `check-<structure>-p` presented in the previous subsection.

We also have a macro, called `defgeneric-<structure>-morphism`, to define generic morphism instances between `<structure>`s.

To summarize this section, we have defined several tools which facilitate the use of mathematical structures and morphisms in ACL2. First of all, the representation of both structures and morphisms is improved thanks to the records which allow us to define the hierarchy of Figure 1 accurately. In addition, the verification that an object fulfills the definitional axioms of a structure or morphism has been enhanced. The reason is twofold: the generation of the necessary events is automated with a set of macros (which can be seen as the characteristic functions for each one of the types of structure), and even if ACL2 is not able to find the proof of the event generated by these macros in the first attempt, the user only has to focus on the trickiest properties since the trivial ones are automatically proved. Finally, the definition of generic instances of structures and morphisms is reduced to a macro call, making the development of generic theories about those structures and morphisms easier.

4 Application: Homological Algebra

In this section we present an example of the application of our tools to the context of Homological Algebra, an introduction to this mathematical subject can be seen in [53]. In particular, we are going to work with *homology groups*, an important concept in the Homological Algebra setting.

Definition 1 Let $f : G_1 \to G_2$ and $g : G_2 \to G_3$ be abelian group morphisms such that $\forall x \in G_1, gf(x) \sim_{G_3} 0_{G_3}$ (where 0_{G_3} is the neutral element of G_3), then the *homology group* of (f, g), denoted by $H_{(f,g)}$, is the abelian group $H_{(f,g)} = ker(g)/im(f)$.

The condition $\forall x \in G1, gf(x) \sim_{G_3} 0_{G_3}$, known as *nilpotency condition*, makes the above definition meaningful, since $im(f) \subseteq ker(g)$. It is worth noting that this definition involves several constructions of Universal Algebra such as subalgebras, morphisms or quotients; for an introduction to Universal Algebra see [18].

Given $f : G_1 \to G_2$ and $g : G_2 \to G_3$ abelian group morphisms such that $\forall x \in G1, gf(x) \sim_{G_3} 0_{G_3}$; let us present how we can use our framework to define $H_{(f,g)}$ and prove that it is an abelian group.

First of all, we define three generic abelian groups (`*G1*`, `*G2*` and `*G3*`) using the `defgeneric-abelian-group` macro.

```
(defgeneric-abelian-group G1)
(defgeneric-abelian-group G2)
(defgeneric-abelian-group G3)
```

The components of these generic abelian groups are: `G<i>-inv`, the invariant function of the underlying setoid of the group, `G<i>-eq`, the equivalence relation of the group, `G<i>-binary-op`, the binary operation, `G<i>-id-elem`, the identity element, and `G<i>-inverse`, the inverse function, with `<i>=1,2,3`.

Now, using the `encapsulate` principle we define two generic abelian group morphisms $f : G_1 \to G_2$ and $g : G_2 \to G_3$ such that the nilpotency condition is satisfied, in this case the abelian group morphism definitional axioms are stated with the `check-abelian-group-morphism-p` macro.

```
(encapsulate
   ; Signatures
   (((f *) => *)
    ((g *) => *))

   ; Generic Abelian Group Morphisms Definition
   (defconst *f*
     (make-abelian-group-morphism :source *G1* :target *G2* :map 'f))
   (defconst *g*
     (make-abelian-group-morphism :source *G2* :target *G3* :map 'g))

   ; Abelian Group Morphism Axioms
   (check-abelian-group-morphism-p *f*)
   (check-abelian-group-morphism-p *g*)

   ; Nilpotency condition
   (defthm nilpotency-condition
      (implies (G1-inv x)
               (G3-eq (g (f x)) (G3-id-elem))))
)
```

The above encapsulate must be read as follows. First of all, we provide the signatures of the functions `f` and `g`, the notation `(f *) => *` means that the function `f` has an argument, which belongs to the universe of ACL2 terms, as input and returns a term as output. Subsequently, we define the constants `*f*` and `*g*` which store the two generic abelian group morphisms. Afterwards, using the `check-abelian-group-morphism-p` macros we impose the axioms of abelian group morphisms to `*f*` and `*g*`. Finally, we additionally impose the nilpotency condition. It is worth noting that the macro `defgeneric-abelian-group-morphism` cannot be used here since we do not only want to build generic abelian group morphisms but also impose the nilpotency condition.

Now, let us note that the set $ker(g) = \{x \in G_2 : g(x) \sim_{G_3} 0_{G_3}\}$ (encoded in ACL2 by means of a function called `ker-g-inv`) is both a subset of the underlying set of G_2 and closed with respect to the group operations of G_2; in addition, we have proved the *Subalgebra criterion* for abelian groups (in a similar way to the one presented for the magma case at the end of Subsection 3.2). Therefore, we can instantiate this criterion for our concrete case and define the abelian group $ker(g)$.

```
(defconst *ker-g*
   (make-abelian-group :group
        (make-group :monoid
```

```
          (make-monoid :semigroup
               (make-semigroup :magma
                    (make-magma :setoid
                         (make-setoid :inv 'ker-g-inv :eq 'G2-eq)
                                :binary-op 'G2-binary-op))
                       :id-elem 'G2-id-elem)
                 :inverse 'G2-inverse))
```

Now, we define $im(f) = \{x \in G_2 : \exists y \in G_1, f(y) \sim_{G_2} x\}$ as a subgroup of G_2 using the same idea presented for $ker(g)$. The existential quantifier in the definition of the invariant function of $im(f)$ is introduced using `defun-sk`, which is the way ACL2 provides support for first-order quantification.

```
(defun-sk im-f-inv (x)
  (exists (y)
          (and (G2-inv x) (G1-inv y) (G2-eq (f y) x))))
```

Then, using `im-f-inv` and the operations of `*G2*`, we construct an `abelian-group` instance which encodes the abelian group $im(f)$.

Afterwards, we can tackle the task of defining the homology group $H_{(f,g)}$ as the quotient $ker(g)/im(f)$. Quotienting a structure of our hierarchy is achieved by changing the equivalence relation of the underlying setoid of the structure with another equivalence relation compatible with the operations of the structure. This result has been proved for each one of the structures of our hierarchy following a similar process to the one presented to prove the Subalgebra criterion. In addition, we have proved that if $im(f)$ is a subgroup of $ker(g)$, then $im(f)$ induces an equivalence relation on $ker(g)$ which is given in ACL2 by the following definition.

Equivalence Relation: $\forall x, y \in ker(g), x \sim_{im(f)} y \Leftrightarrow xy^{-1} \in im(f)$

```
  (defun im-f-eq (x y)
    (im-f-inv (G2-binary-op x (G2-inverse y))))
```

Therefore, $H_{(f,g)}$ is defined using `ker-g-inv` as invariant, `im-f-eq` as equivalence relation, `G2-binary-op` as binary operation, `G2-id-elem` as the identity element, and `G2-inverse` as the inverse operation.

```
(defconst *homology-fg*
  (make-abelian-group :group
       (make-group :monoid
            (make-monoid :semigroup
                 (make-semigroup :magma
                      (make-magma :setoid
                           (make-setoid :inv 'ker-g-inv :eq 'im-f-eq)
                                  :binary-op 'G2-binary-op))
                         :id-elem 'G2-id-elem)
                   :inverse 'G2-inverse))
```

The last step consists in certifying that `*homology-fg*` satisfies the definitional axioms of an abelian group, to this aim the `check-abelian-group-p` macro, taking as argument `*homology-fg*`, is invoked.

```
(check-abelian-group-p *homology-fg*)
```

ACL2 is not able to find the proof of the event generated by the call to this macro in the first attempt and some previous lemmas, suggested by the failed proof, are necessary. The way of facing those lemmas is the usual when trying to prove a result with ACL2: inspect the failed proof attempt and provide the necessary lemmas and hints, this is known in ACL2 as "*the Method*" [37].

In this way, we have defined the homology group $H_{(f,g)}$ and proved that it is an abelian group.

5 Dealing with complex mathematical structures

The framework that we have presented in the previous sections can be enriched with more complex mathematical structures than the ones introduced up to now. It is in those cases when the profit of using our tools is most remarkable with respect to an ad-hoc approach. Let us illustrate this fact with an example coming from Algebraic Topology [46]; to that end, it is necessary to introduce how we deal with *indexed families of structures* in our context.

5.1 Indexed families of structures

In Algebraic Topology, we do not usually work with just an object of a structure but with a *family* of objects of that structure *indexed* on a set, called the *index set*. This can be seen, for instance, in the definition of two instrumental notions in Algebraic Topology.

Definition 2 A *chain complex* is a pair $(C_n, d_n)_{n\in\mathbb{Z}}$ where $(C_n)_{n\in\mathbb{Z}}$ is a graded R-module indexed on the integers and $(d_n)_{n\in\mathbb{Z}}$ (the *differential* map) is a graded R-module endomorphism of degree -1 ($d_n : C_n \to C_{n-1}$) such that $d_{n-1}d_n = 0$ (this property is known as *nilpotency* condition).

Let $(C_n, d_n)_{n\in\mathbb{Z}}$ and $(D_n, \widehat{d}_n)_{n\in\mathbb{Z}}$ be two chain complexes, a *chain complex morphism* between them is a family of R-module morphism $(f_n)_{n\in\mathbb{Z}}$ such that $\widehat{d}_n f_n = f_{n-1}d_n$ for each $n \in \mathbb{Z}$.

The approach that we have followed to represent indexed families of structures in our framework is based on the one presented in [43]. Roughly speaking, the representation of a graded structure indexed on a set is achieved thanks to the introduction of an additional parameter, which ranges the elements of the index set, in each one of the operations of the structure.

Then, in order to deal with families of structures in our context, we have created a hierarchy of graded structures which is paralleled to the one presented in the left side of Figure 1. The basic object of this hierarchy is *graded setoid* which is defined using the `defstructure` construction with three fields: `inv`, `eq` and `index-sets`. When, we are working with a graded setoid, the value of `inv` and `eq` will be respectively a function symbol, whose arity is 2, representing the underlying graded set of the setoid and a function symbol, whose arity is 3, encoding the intended equivalence relation; and, the value of `index-sets` will be a list with a sole element which is a function name that represents the characteristic function of the index set of the graded setoid.

It is worth noting that we can deal with n-graded setoids (that is to say, a family of setoids indexed on n sets) using the same record structure. In the general case, the arities of `inv` and `eq` functions will be $n+1$ and $n+2$ respectively; and, the value of the `index-sets` slot will be a list whose elements are n function names encoding the n characteristic functions of the n sets. Let us note that if the value of `index-sets` is an empty list, we have an object *"equivalent"* to a `setoid` instance as we have presented it in Subsection 3.1.

The ideas presented in Subsection 3.2 to define structures in terms of others can also be applied in the case of graded structures; that is to say, if B is a graded structure defined in terms of an A graded structure, B is an A graded structure together with additional operations and satisfying further properties. Then, the `index-sets` slot will be inherited from the `graded-setoid` structure to the rest of the graded structures of the hierarchy.

In addition to the tools in charge of representing graded structures, we have also defined the functions which generate the events that provide the definitional axioms of the graded structures. Namely, those functions have been defined in order to produce a term depending on the size of the list stored in `index-sets`. Therefore, there are available `check-graded-<structure>-p` macros which behave as `check-<structure>-p` macros but for graded structures.

Furthermore, we also have `defgeneric-graded-<structure>` macros in order to define generic indexed families of `structure`s. The definition of these macros includes a *keyword* parameter called `index-sets` whose value will be a list of function names encoding the characteristic functions of the underlying index sets of the generic indexed family of `structure`s.

Now, we can tackle the task of working with chain complexes in our environment. To this aim, the instrumental notion is the one of graded R-module; this graded structure *is-a* graded abelian group that *uses* a ring as part of its definition; so both the graded hierarchy and the one depicted in Figure 1 are necessary to define graded R-modules. From this graded structure, we can represent chain complexes using the following `defstructure` construction.

```
(defstructure chain-complex
  graded-R-module diff)
```

The value of `graded-R-module` will be a `graded-R-module` instance, let us call it $\mathcal{C}$, indexed on the set of integer numbers, and `diff` will be a function symbol whose arity is 2 encoding the differential map, `diff`: $\mathbb{Z} \times \mathcal{C} \to \mathcal{C}$; that is, the differential map is *uncurried* (the subscript of the differential map is now one of the inputs). The macro `check-chain-complex-p`, which has been defined following the ideas presented in Section 3, is in charge of verifying that given a `chain-complex` instance, its `graded-R-module` component is really a graded R-module, and `diff` is a family of R-module endomorphism of degree -1 satisfying the nilpotency condition. We have defined the macro `defgeneric-chain-complex` which produces generic chain complex instances.

Finally, we can focus on the second notion included in Definition 2, *chain complex morphisms*. The record structure to represent chain complex morphisms is

```
(defstructure chain-complex-morphism
  source target map)
```

where the value of both `source` and `target` slots will be a `chain-complex` instance, and the value of `map` will be a function symbol whose arity is 2 (as in the case of the differential map in chain complexes, the map of the chain complex morphism is uncurried).

As in the rest of structures and morphisms presented in this paper, we have introduced a macro to certify that `chain-complex-morphism` instances are really chain complex morphisms, `check-chain-complex-morphism-p`, and another one, called `defgeneric-chain-complex-morphism`, to generate generic chain complex morphisms. It is worth noting that the notion of chain complex morphism can be generalized to *chain complex morphism of degree q, with* $q \in \mathbb{Z}$ (a family of R-module morphism $(f_n)_{n\in\mathbb{Z}} : C_n \to C_{n+p}$ such that $\widehat{d}_{n+q} f_n = f_{n-1} d_n$ for each $n \in \mathbb{Z}$); in the general case, the representation of chain complex morphism of degree q is the same presented for chain complex morphisms; on the contrary, the macros in charge of verifying that an object is really a chain complex morphism of degree q and generating generic chain complex morphism of degree q take as input an additional argument which is the degree q.

In the next subsection, we are going to present several examples of the usage of the tools related to chain complexes and chain complexes morphisms and the great profit of using them.

5.2 Benefits of our approach

Once that we have introduced how we model chain complexes and chain complex morphisms; let us present the advantages of using our tools in the context of Algebraic Topology by means of the *Cone construction*.

Definition 3 Let $C_* = (C_n, dC_n)_{n\in\mathbb{Z}}$ and $D_* = (D_n, dD_n)_{n\in\mathbb{Z}}$ be two chain complexes and $\phi : D_* \to C_*$ be a chain complex morphism. Then the cone of ϕ, denoted by $Cone(\phi) = (A_n, dA_n)_{n\in\mathbb{Z}}$, is defined as: $A_n := C_{n+1} \oplus D_n$ (an element $x \in A_n$ is a pair such that its first component belongs to C_{n+1} and the second one to D_n); and

$$\begin{array}{rl} dA_n : C_{n+1} \oplus D_n \to & C_n \oplus D_{n-1} \\ (c_{n+1}, d_n) \mapsto & (dC_{n+1}(c_{n+1}) + \phi(d_n), dD_n(d_n)). \end{array}$$

A more detailed description of this construction can be seen in [49]. The task that we are going to tackle consists in verifying that given a chain complex morphism ϕ, then $Cone(\phi)$ is a chain complex.

Using the tools that have been presented previously, we can define a generic chain complex morphism ϕ.

```
(defgeneric-chain-complex-morphism PHI)
```

As the rest of this kind of macros, the above macro call produces the constant `*PHI*` which stores a generic chain complex morphism; and the theorem which ensures that the components of `*PHI*` satisfy the chain complex morphism axioms.

Afterwards, we introduce the chain complex operations (9 operations are necessary to define a chain complex) associated with the cone construction from the components of `*PHI*`. From them, we create a `chain-complex` instance which is assigned to a new constant, called `*Cone-PHI*`, for latter use.

Subsequently, we use the `check-chain-complex-p` macro with `*Cone-PHI*` as argument to prove an event which ensures the chain complex axioms for the functions of this `chain-complex` instance.

```
(check-chain-complex-p *Cone-PHI*)
```

The system is not able to find the proof of the event generated by this macro in the first attempt. Namely, the event consists of the 49 chain complex definitional axioms, 40 of which are automatically proved by ACL2, and the rest, the trickiest ones, need some auxiliary lemmas suggested by the failed proof.

Finally, in order to make the instantiation of the cone construction for concrete chain complex morphisms easier, we have used the *generic instantiation tool* [44].

In the above development we have taken advantage of the tools presented throughout this paper. However, the same formalization could be performed from scratch, but not without difficulty, as we will see as follows.

First of all, to define the generic chain complex morphism ϕ, the `encapsulate` mechanism should be employed. The definition of this generic object involves 19 function symbols, which define the operations of the chain complex morphism, the corresponding 19 witnesses and 84 axioms which ensures that the 19 function symbols fulfill the properties which characterize the chain complex morphism operations.

Afterwards, from the function symbols of the generic chain complex morphism, we introduce the operations which define the chain complex associated with the cone construction; as we said previously, this means 9 new definitions.

Finally, we could state the 49 events which claim that the 9 operations introduced in the previous step satisfy the axioms which characterize the chain complex operations. The non trivial events are the same as before, so; in 9 of them some auxiliary lemmas are necessary.

In this way, we can prove that given a chain complex morphism the cone construction produces a chain complex; however, in spite of being able to carry out this task, this way of working is, from our point of view, worse than the one presented using our tools. First of all, in the from scratch approach, there is a considerable amount of definitions and theorems, both in the definition of the generic chain complex morphism and in the certification of the correctness of the cone construction; so, it is likely that some of them can be forgotten causing unexpected problems. In addition, the functions which represent the operations of the structure are not gathered in any way, then, the structure is not explicitly given as is done when working with pencil and paper or using other theorem provers such as Coq or Isabelle.

Table 1 shows a comparison between the two approaches (the one which uses our tools will be called hierarchical) considering the cone construction. In addition, we also consider a half-way method presented in [30] where the macros in charge of certifying that an object satisfies the axioms which characterize an algebraic structure were defined, but not the functionality to generate generic instances of concrete structures.

As can be noticed, there are some figures which are the same in all the cases (the number of definitions of the cone construction and the auxiliary lemmas), this is due to the fact that the result that we are proving is always the same; so, ACL2 always needs help in the same places. However, the use of the tools presented in this paper means a great improvement with respect to the other approaches.

Table 1 Comparison between the different approaches

	Definition of generic chain complex morphism	Definition of cone construction	Proof of the correctness of the construction
from scratch	19 function symbols 19 witnesses 84 axioms	9 definitions	49 theorems 34 auxiliary lemmas
half-way	19 function symbols 19 witnesses 84 axioms	9 definitions 1 `chain-complex`	1 macro call 34 auxiliary lemmas
hierarchical	1 macro call	9 definitions 1 `chain-complex`	1 macro call 34 auxiliary lemmas

First, the amount of definitions and theorems is considerably reduced; then, both the number of lines of our development and the chance of forgetting some results decrease. Related to the previous fact, it is also worth noting that the developments are more readable thanks to the use of macros, an important issue when we are documenting our work. Besides, the use of the hierarchical methodology presented in this paper makes the work of the user easier because he only has to focus on the difficult parts of the proofs.

As a final remark, we can say that the certification of the cone construction is interesting not only from the point of view of the formalization of a theoretical result but also in the context of program verification. Kenzo [21] is a Common Lisp system devoted to Algebraic Topology, which was developed by Francis Sergeraert. The Kenzo system has obtained some results not confirmed nor refuted by neither theoretical or computational means. Therefore, a wide project was launched several years ago to prove the correctness of Kenzo using different theorem proving tools, see [7, 20, 40]. In the next section, we comment the previous efforts to formalize the Kenzo system, followed by why we think that the work presented in this paper is beneficial to that task.

6 Related work

In this section we are going to discuss previous work considering three different points of view: the formalization of algebraic structures, its use to verify the correctness of Computer Algebra systems, and the verification of the Kenzo system using different theorem provers.

6.1 Algebraic Hierarchies and Theorem Provers

As we have commented in the introduction of this paper, the formalization of algebraic structures within formal proof systems has been broadly studied in the literature.

Coq is probably the most prolific system in this sense. Up to the best of our knowledge, 4 different approaches have been considered in this system to formalize algebraic structures. An algebraic hierarchy which tries to imitate the one of the Axiom Computer Algebra system [33] was implemented in [47]. The formalization

of the Fundamental Theorem of Algebra, see [26], employed the hierarchy presented in [25]. The SSReflect hierarchy, introduced in [24], plays a key role in the project to formalize the Feit Thompson theorem [2]. Moreover, a new hierarchy was developed in [52] having as a final goal the formalization of practical exact real arithmetic.

Two Ph.D. theses have been devoted, at least partially, to this topic. A hierarchy for the Nuprl system appeared in the thesis of P. Jackson, see [32], and was the basis for proving some results about abstract algebra. A. Bailey implemented in his Ph.D. thesis [9] an algebraic hierarchy in Lego which was used to formalize, part of, Galois theory.

There are also different approaches in the family of HOL theorem provers. A basic theory of groups was developed in HOL using the hierarchy presented in [28]. As can be seen in [11], the Isabelle/HOL systems provides a library to formalize abstract algebra which has been successfully used to prove, for instance, Sylow theorems. In addition, there is also a hierarchy for relation and Kleene algebras in Isabelle, see [22].

We can find a classical set-theoretic treatment of algebra in Mizar. The different structures, like group, ring and field are defined in several articles by various authors [1]. A report about some formalization issues faced during these developments can be seen in [50]. Abstract algebra has been also formalized in constructive set theory using the MetaPRL system [54].

Our ACL2 algebraic hierarchy shares some features with the hierarchies developed in all these systems. First of all, all the hierarchies have a top structure which is embedded in every lower level one. We can classify the hierarchies depending on this top level structure. The hierarchies presented in [50, 54] are set based, the ones introduced in [28, 47, 9, 25, 52, 11] are setoid based, and the SSReflect hierarchy [24] uses a choice structure as top level object. The hierarchy presented in this paper is setoid based, this is quite common in the rest of hierarchies since, using this approach, we just need to change the equivalence relation of the underlying setoid to construct the quotient of a structure. In addition, the algebraic structures which have been modeled in our hierarchy are the same that appears in the rest of the formalizations. The only important structure which is missed in our hierarchy is the one of field, mainly because we did not need it; however it can be included without any special hindrance. Finally, in all the cases algebraic structures are defined using a record (called `locale` in Isabelle, `Class` or `Record` in Coq, `struct` in Mizar and so on) where the operations of the structure are packed. In some cases, as in Isabelle or Coq, such records also include the axioms about the operations; the situation in ACL2 is similar to the one of Mizar where the axioms about the operations are external to the record.

It is worth noting that our approach has some limitations when we compare it with the other ones. The first drawback is the ACL2 inheritance mechanism which produces nested record structures; this makes tiresome the task of accessing some components of the structure. This issue is solved in other systems using subtyping mechanisms and automatic inference of coercions, see [25, 24]. Another pitfall is the lack of existential quantification over algebraic structures, however we can universally quantify over them using the `defgeneric-<structure>` macros. Moreover, the definition of families of structures can seem a bit strange since we cannot define a function which returns a structure as output, this is the common method in the rest of theorem provers. Nevertheless, the introduction of the index set as a parameter in the operations of a structure is a well-known technique to

represent families using first order logic, see [42]. The last disadvantage is that we always need to state and prove that the operations of the algebraic structures are closed on the underlying set characterized by the invariant function, nevertheless most of these properties are automatically proved by ACL2 without any external guidance. This last issue is not relevant in other systems because the operations can be defined in the desired domain, but this is not possible in ACL2 since it is an untyped system.

On the other hand, there are some advantages if we compare our approach with the other ones, specially if we focus on the final aim of our hierarchy: the verification of Computer Algebra systems.

6.2 Formal Algebraic Hierarchies and Computer Algebra systems

Algebraic structures are instrumental in Computer Algebra systems since they are the basis for several constructions. Therefore, it makes sense to use Theorem Prover tools in order to increase the consistency of those algebraic structures.

Axiom is a general purpose Computer Algebra system whose hierarchy of algebraic structures has been formalized both in Coq [47] and Isabelle [10]. The Nuprl hierarchy presented in [32] had as final aim the connection with the Weyl system [55]. The correctness of Maple has been studied with the systems PVS [5], Isabelle [12] and HOL [29]. There is also a development to create a Computer Algebra system on top of the HOL system, see [34].

A different approach is the one of the FOC system [15] where an environment to develop certified programs for symbolic computation has been constructed. The FOC environment contain a classical algebraic hierarchy. A similar project is the one of Analytica [13], a theorem prover which runs on top of Mathematica and can be used to verify programs of this system.

However, we cannot consider any of these approaches fully satisfactory. The drawback of the approaches which formalize Computer Algebra systems using proof assistant is that the code which is verified is far from the actual code. Building the Computer Algebra system on top of the theorem prover seems a solution to this problem, but it has as disadvantage one of the weak points of proof assistants: inefficiency. The problem of systems like FOC or Analytica is that they are ad hoc systems; so, both the community of users and the libraries of results are small.

These problems does not appear with our ACL2 hierarchy of algebraic structures. First of all, Common Lisp is the language of several Computer Algebra systems such as Axiom [33], Maxima [4], Reduce [3], Weyl [55] and Kenzo; then, we can verify actual code of these systems using ACL2. In addition, the ACL2 community is one of the biggest in the context of Theorem Proving tools; so, the great amount of libraries previously developed by other users can be used in our work.

Let us present now the benefits of our approach in the particular case of Kenzo; a system whose correctness have been studied with three theorem provers: Isabelle, Coq and ACL2.

6.3 Verification of the Kenzo system

The Kenzo system has been able to compute some unknown results [51], and also have been used to refute some computations obtained by theoretical means, see [48]. This implies that increasing user's trust in the system is relevant.

One feature of Kenzo is its use of higher order functional programming to handle spaces of infinite dimension. Thus, the first attempts to apply theorem proving assistants in the analysis of Kenzo were oriented towards higher order logic tools. Concretely, the Isabelle/HOL proof assistant was used to verify in [7] a very important algorithm in Homological Algebra: the Basic Perturbation Lemma [49]; the algebraic hierarchy used in this development was the one presented in [11]. In the same line, we can find the work of [20] where the Effective Homology of the Bicomplexes (another important result in Homological Algebra) was formalized in Coq extending the algebraic hierarchy of [25]. Let us note that these formalizations were related to algorithms and not to the real programs implemented in Kenzo. The problem of extracting programs from the Isabelle/HOL proofs has been dealt with in [8], but even there the programs are generated in ML, far from Kenzo.

Since both Kenzo and ACL2 are Common Lisp programs, we can undertake the task of verifying real Kenzo code in ACL2. For instance, ACL2 has been successfully used to study some critical fragments of Kenzo in [45,6], but algebraic structures were not involved in any of these two formalizations. However, ACL2 can also be used to formalize results about the algebraic structures implemented in the Kenzo system. This can be seen in [41], where the Eilenberg-Zilberg theorem [49] (an instrumental result in the Kenzo system involving a construction about chain complexes) has been formalized. The formalization of the Eilenberg-Zilberg theorem was undertaken from scratch; so, the problems presented in Subsection 5.2 appeared during that development. Hence, the use of our methodology can mean a great benefit when dealing with this kind of formalizations.

As a final point, we can compare the formalization in ACL2 and Coq of the cone construction, the Coq proof of this result was presented in [19]. In addition to the fact that the gap between the ACL2 formalization and the Kenzo code is much smaller than the one between Coq and Kenzo, there is another advantage of the ACL2 approach and is that several parts of the proof are automated by ACL2 and the user only has to focus on the difficult parts; whereas in the Coq formalization all the steps must be given by the user.

7 Conclusions and Further work

In this paper, we have presented an ACL2 infrastructure to deal with algebraic structures and morphisms between them. Namely, we have provided several tools which make the handling of this kind of objects easier. As a result, an ACL2 hierarchy of the most common algebraic structures has been provided; a task, that as far as we are aware, had not been undertaken up to now for this system.

The feasibility of using our framework has been illustrated with the example of the homology group of two abelian group morphisms satisfying the nilpotency condition. This example involves several common constructions in Universal Algebra such as subalgebras, morphisms or quotients.

In addition, we have presented the benefits of using our approach when dealing with complex mathematical structures instead of working from scratch. In this kind of problems the development effort is considerably reduced using our tools. This is important when we are facing the final aim of our work: the formalization of Computer Algebra systems.

With the acquired experience, the method presented in this paper could be extrapolated to other algebraic structures, for instance to the case of Tarski Kleene Algebras which was previously studied in Isabelle [23]. It is also appealing the idea of formalizing the generic theory of Universal Algebra as was previously done in Coq, see [17,52].

In addition, as we have seen in Section 3, the definition of morphism between structures always follows the same pattern; so, it would be desirable to have a tool able to automatize, at least part of, the process to generate the tools related to morphisms between structures.

Nevertheless, our main research line for the future is the application of the tools that we have presented here to verify actual Computer Algebra systems. We are mainly interested in the Kenzo system, where an environment like the one presented in this paper is desirable when dealing with its mathematical structures. Moreover, we are also keen on the formalization of other Lisp-based systems such as Axiom or Maxima.

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