Effective Homology of the Pushout of Simplicial Sets

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Motivation

- Kenzo:

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  - Particular cases: wedges, joins, ...

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Concrete goal

New Kenzo module for constructing the Pushout of simplicial sets

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Effective Homology

- Finite nature objects:

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Effective vs Locally Effective Chain Complexes

Definition

An effective chain complex is a free chain complex of $\mathbb{Z}$-modules, $C_* = (C_n, d_n)_{n \in \mathbb{N}}$, where each group C_n is finitely generated and

- *an algorithm returns a $\mathbb{Z}$-base in each grade n*
- *an algorithm provides the differentials d_n*

Definition

A locally effective chain complex is a free chain complex of $\mathbb{Z}$-modules, $C_* = (C_n, d_n)_{n \in \mathbb{N}}$, where each group C_n is formed by a infinite number of generators

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- differentials $d_n : C_n \to C_{n-1}$ can be expressed as integer matrices
- possible to compute $Ker\ d_n$ y $Im\ d_{n+1}$

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- possible to compute *Ker* d_n y *Im* d_{n+1}
- possible to compute the homology groups

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A locally effective chain complex is a free chain complex of $\mathbb{Z}$-modules, $C_* = (C_n, d_n)_{n\in\mathbb{N}}$, where each group C_n is formed by a infinite number of generators

- impossible to compute *Ker* d_n y *Im* d_{n+1}
- possible to perform local computations, differential of a generator

Effective Homology

Definition

A reduction ρ between two chain complexes C_* y D_* (denoted by $\rho : C_* \Rightarrow D_*$) is a triple $\rho = (f, g, h)$

$$
\begin{tikzcd}
C_* \arrow[loop left, "h"] \arrow[r, bend left, "f"] & D_* \arrow[l, bend left, "g"]
\end{tikzcd}
$$

satisfying the following relations:

1) $fg = \mathrm{Id}_{D_*}$;
2) $d_C h + h d_C = \mathrm{Id}_{C_*} - gf$;
3) $fh = 0; \quad hg = 0; \quad hh = 0.$

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Theorem

If $C_* \Rightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n.

Effective Homology

Definition

A (strong chain) equivalence ε *between* C_* *and* D_*, $\varepsilon : C_* \Leftrightarrow D_*$, *is a triple* $\varepsilon = (B_*, \rho, \rho')$ *where* B_* *is a chain complex,* $\rho : B_* \Rightarrow C_*$ *and* $\rho' : B_* \Rightarrow D_*$.

$$\begin{tikzcd} & B_* \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ C_* & & D_* \end{tikzcd} \qquad \begin{tikzcd} & \frac{42}{30} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \frac{14}{10} & & \frac{21}{15} \end{tikzcd}$$

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Definition

An object with effective homology is a quadruple $(X, C_*(X), HC_*, \varepsilon)$ *where:*

- X *is a locally effective object*
- $C_*(X)$ *is a (locally effective) chain complex canonically associated with* X, *which allows the study of the homological nature of* X
- HC_* *is an effective chain complex*
- ε *is a equivalence* $\varepsilon : C_*(X) \Lleftarrow\!\Rrightarrow HC_*$

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Theorem

Let an object with effective homology $(X, C_*(X), HC_*, \varepsilon)$ *then* $H_n(X) \cong H_n(HC_*)$ *for all* n.

Pushout

Definition

Let f, g morphisms, the pushout of f, g

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \\ Z & \end{tikzcd}$$

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$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xrightarrow{g'} & P_{(f,g)} \end{array}$$

is an object P for which the diagram:

- *commutes*

Pushout

Definition

Let f, g morphisms, the pushout of f, g

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "f'"] \arrow[ddr, bend left, "f''"] & \\
Z \arrow[r, "g'"] \arrow[drr, bend right, "g''"'] & P_{(f,g)} \arrow[dr, dashed, "p"] & \\
 & & Q
\end{tikzcd}
$$

is an object P for which the diagram:

- *commutes*
- *respects the universal property*

Pushout

Standard Construction

$P_{(f,g)} \cong (Y \amalg (X \times I) \amalg Z)/\sim$ *where:*

- I *is the unit interval*
- *for every* $x \in X$, $\sim$:
  - $(x, 0) \sim f(x) \in Y$
  - $(x, 1) \sim g(x) \in Z$

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Construction of the Effective Homology of the Pushout: Step 1

Given $f : X \to Y$ and $g : X \to Z$ simplicial morphisms where X, Y and Z are simplicial sets:

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Theorem (Algorithm: Standard Construction, Implementation: J. Heras)

Input: *two simplicial morphisms $f : X \to Y$ and $g : X \to Z$ where X, Y and Z are simplicial sets.*

Output: *the pushout $P_{(f,g)}$.*

Construction of the Effective Homology of the Pushout: Step 2

Given $f : X \to Y$ and $g : X \to Z$ simplicial morphisms where X, Y and Z are simplicial sets with effective homology:

$$\begin{tikzcd}
(X, C_*(X), HX_*, \varepsilon_X) \arrow[r, "f"] \arrow[d, "g"] & (Y, C_*(Y), HY_*, \varepsilon_Y) \arrow[d] \\
(Z, C_*(Z), HZ_*, \varepsilon_Z) \arrow[r] & (P_{(f,g)}, C_*(P_{(f,g)}), _, _)
\end{tikzcd}$$

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(Z, C_*(Z), HZ_*, \varepsilon_Z) \arrow[r] & (P_{(f,g)}, C_*(P_{(f,g)}), HP_*, \varepsilon_P)
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Construction of the Effective Homology of the Pushout: Step 2

Given $f : X \to Y$ and $g : X \to Z$ simplicial morphisms where X, Y and Z are simplicial sets with effective homology:

$$\begin{array}{ccc} (X, C_*(X), HX_*, \varepsilon_X) & \xrightarrow{f} & (Y, C_*(Y), HY_*, \varepsilon_Y) \\ \Big\downarrow g & & \Big\downarrow \\ (Z, C_*(Z), HZ_*, \varepsilon_Z) & \longrightarrow & (P_{(f,g)}, C_*(P_{(f,g)}), HP_*, \varepsilon_P) \end{array}$$

Theorem (Algorithm: F. Sergeraert, Implementation: J. Heras)

Input: *two simplicial morphisms $f : X \to Y$ and $g : X \to Z$ where X, Y and Z are simplicial sets with effective homology.*

Output: *the effective homology version of $P_{(f,g)}$, that is, an equivalence $C_*(P_{(f,g)}) \Leftrightarrow HP_*$, where HP_* is an effective chain complex.*

Effective Homology of the Pushout

Theorem

Input:

$$C_*(B) \text{ a chain complex;}$$

$$(C_*(A), HA_*, \varepsilon_A);$$

$$(C_*(C), HC_*, \varepsilon_C);$$

$$0 \xleftarrow{\ 0\ } C_*(A)_* \underset{j}{\overset{\sigma}{\rightleftarrows}} C_*(B) \underset{i}{\overset{\rho}{\rightleftarrows}} C_*(C) \longleftarrow 0$$

Output: $(C_*(B), HB_*, \varepsilon_B)$

$$0 \longleftarrow M \rightleftarrows CP_{(f,g)} \rightleftarrows CY \oplus CZ \longleftarrow 0$$

where $M = X \times I \setminus ((X \times \{0\}) \cup (X \times \{1\}))$

Effective Homology of the Pushout

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Theorem

Input: two simplicial sets X and Y with effective homology
Output: an equivalence $C_*(X \oplus Y) \Lleftarrow DD_* \Rrightarrow HD_*$, where HD_* is effective.

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Consider the short exact sequence:

$$0 \longleftarrow M \rightleftarrows C(X \times I) \rightleftarrows C(X \times \{0\}) \oplus C(X \times \{1\}) \longleftarrow 0$$

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Output: $(C_*(A), HA_*, \varepsilon_A)$

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Theorem (Eilenberg-Zilber Theorem)

Input: two simplicial sets X and Y with effective homology
Output: an equivalence $C_*(X \times Y) \Lleftarrow DC_* \Rrightarrow C_*(X) \otimes C_*(Y)$,
where $C_*(X) \otimes C_*(Y)$ are effective.

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$$0 \longleftarrow M \rightleftarrows C(X \times I) \rightleftarrows C(X \times \{0\}) \oplus C(X \times \{1\}) \longleftarrow 0$$

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$C_*(B)$ *a chain complex;*

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Output: $(C_*(B), HB_*, \varepsilon_B)$

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- Further Work:
  - Implementation of new constructions

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