Helping Students Learn to Write Proofs

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Self-referential statement

The slides for this talk are posted at

`mathcs.slu.edu/~stevens/projnext/proofs.pdf`

and will be posted on the Green16 drive

Courses that introduce students to writing proofs

Two types:

• "Transition" courses: Logic and proofs are main subject matter

• Courses focused on specific math content, but with attention to learning to write proofs.

Types of Courses: "Transition" Courses

Logic, Sets, and Proof (S. Dakota State U.)
Foundations (Kenyon C.)
Introduction to Proofs (Iowa State U.)
Intro. to Mathematical Reasoning (U. Wash.)
Strategies of Proof (Calif. State U. Fullerton)
Foundations of Adv. Math. (Willamette U.)
Principles of Mathematics (Saint Louis U.)

Sample Texts: “Transition” Courses

Mathematical Proofs (Chartrand *et al.*)

Chapter Zero (Schumacher)

A Transition to Advanced Mathematics (Smith *et al.*)

The Art of Proof (Beck/Geoghegan)

Reading, Writing, and Proving (Daepp/Gorkin)

An Intro. to Mathematical Thinking (Gilbert/Vastrone)

The Foundations of Mathematics (Sibley)

Bridge to Abstract Mathematics (Lawrence *et al.*)

Book of Proof (Hammack)

A Gentle Intro. to the Art of Mathematics (Fields)

Mathematical Reasoning (Sundstrom)

[blue = free] [red = Fellow]

Types of Courses: Specific Math Content

Discrete Mathematics (Shippensburg U.)

Discrete Mathematics: Gateway to Advanced Mathematics (Sarah Lawrence C.)

Intermediate Analysis (U. Houston)

Intro. to Number Theory (U. Texas Austin)

Linear Algebra (Clark U.)

Sample Texts: Specific Math Content

Discrete Mathematics with Applications (Epp)

Discrete Mathematics [Puzzles/Games] (Ensley/Crawley)

Discrete Mathematics with Ducks (belcastro)

Mathematical Thinking [Discrete, No. Theory, Combinatorics] (D'Angelo/West)

Mathematics: A Discrete Intro. (Scheinerman)

Analysis with an Introduction to Proof (Lay)

Number Theory Through Inquiry (Marshall *et al.*)

Linear Algebra as an intro. to abstract math (Lankham *et al.*)

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Sample Texts: Inquiry-based

Chapter Zero (Schumacher)

Number Theory Through Inquiry (Marshall *et al.*)

Distilling Ideas: An Intro. to Mathematical Thinking (Katz and Starbird)

Introduction to Proofs (Hefferon)

Notes for a Course on Proof (Jensen-Vallin)

Introduction to Proof (Taylor)

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In groups of 3 or 4:

1) Describe the course(s) in which you will be teaching students to write proofs.

e.g.: "I teach a sophomore-level course about logic and proofs at a medium-sized, private university. This course is required for all math majors and most math minors."

2) As a group, list three things that you want your students to learn in your course(s).

[Someone take notes.]

Course Goals

- Understand and apply important concepts involving logic, sets, relations, functions, and cardinality;
- Interpret mathematical notation correctly and express your own ideas in correct notation;
- Understand correct mathematical proofs and identify incorrect ones;
- Make and evaluate conjectures;
- Discover and write proofs of your own, using standard strategies of proof, including induction;
- Solve mathematical problems and provide convincing justifications for their solutions.


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Homework

The exercises are designed to

- **nip misconceptions in the bud;**
- **let students practice proofs in a low-risk environment.**

From a student who took this course in Fall 2013:

"To all future students: Do the homework! It will make your life easier!"

In groups, discuss these three examples of student work. What do you need to clarify for these students? How would you do that?

1. Let $A = \{1, \{2,3\}\}$. Decide whether each of the following statements is true or false:
$1 \in A$, $1 \subseteq A$, $\{1\} \in A$, $\{1\} \subseteq A$, $\{2,3\} \in A$, $\{2,3\} \subseteq A$.

Student response: **T, F, T, T, F, T**

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Student response: **T, F, T, T, F, T**

This student doesn't understand

- **the difference between elements and subsets of a set;**
- **sets that have sets as elements.**

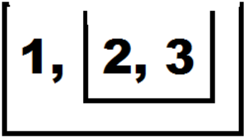
Pothole #1: Notation

Students are not used to reading notation carefully and interpreting it on their own.

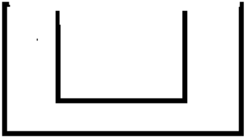
Sets

A SET =

$$= \{\quad\} = \emptyset$$



$$= \{1, \{2, 3\}\}$$



$$= \{\emptyset\}$$

Russian dolls as nested sets

Notational pothole: Typical student work #2

2. Let $X = \{n^2 + 3n - 50 : n \in \mathbb{N}\}$. Find three distinct elements of X.

Student response: **Using $n = 1, 2$, and 10, the student computes $n^2 + 3n - 50$, getting $-46, -40$, and 80. The student concludes that $80 \in X$, but discards -46 and -40.**

Notational pothole: Typical student work #2

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Student response: Using $n = 1, 2$, and 10, the student computes $n^2 + 3n - 50$, getting $-46, -40$, and 80. The student concludes that $80 \in X$, but discards -46 and -40.

This student thinks that $n^2 + 3n - 50$ (not just n) has to be a natural number.

Worksheet: Sets, subsets, and power sets

1. Let $A = \{1, \{2, 3\}\}$. Decide whether each of the following statements is true or false:

   a. $1 \in A$ b. $1 \subseteq A$ c. $\{1\} \in A$

   d. $\{1\} \subseteq A$ e. $\{2, 3\} \in A$ f. $\{2, 3\} \subseteq A$

2. List 5 distinct elements of $X = \{n^2 + 3n - 50 : n \in \mathbb{N}\}$.
3. Let $C = \{\dots, -7, -3, 1, 5, \dots\}$. One way to describe C is $C = \{4n + 1 : n \in \mathbb{Z}\}$. Describe C in two other ways.
4. Let A be as in #1, and let $B = \{x, 1\}$.
   - a. Find $A \cup B$.
   - b. List the elements of $\mathcal{P}(A \cup B)$.
   - c. Find $|\mathcal{P}(A \cup B)|$.

Finding another pothole: Typical student work #3

3. Prove: The sum of an odd integer and an even integer is odd.

Student response: **Let a be an odd integer and b be an even integer. Then a equals $1+$ an even integer, so $a+b$ equals an even integer plus an even integer plus 1. Thus $a+b$ is not even, so it must be odd.**

Finding another pothole: Typical student work #3

3. Prove: The sum of an odd integer and an even integer is odd.

Student response: **Let a be an odd integer and b be an even integer. Then a equals $1+$ an even integer, so $a+b$ equals an even integer plus an even integer plus 1. Thus $a+b$ is not even, so it must be odd.**

What definitions of "even" and "odd" is this student using?

Pothole: The role of formal definitions in proofs

An integer is odd if it

- **is not even;**
- **is not evenly divisible by** 2**;**
- **leaves a remainder of 1 when divided by** 2**;**
- **can be written in the form** $2k+1$ **for some integer** k**.**

Interpreting notation correctly

Understanding the role of definitions

What should you ask students to prove?

Prove: The sum of an odd integer and an even integer is odd.

How old were you when you realized that this is true?

Why prove things?

- That's what mathematicians do.
- Yes, it **will** be on the test.
- It's good to verify things that you already know are true.
- If something seems to be true, it's good to figure out whether it is **always** true.
- If something is only sometimes true, it's good to figure out **when** it's true.
- A proof tells you **why** something is true.
- When something surprises you, you want to know whether it **always** happens, and **why**.

Prove something surprising: A card trick

Take an ordinary deck of 52 playing cards. Shuffle it thoroughly, and then divide it into two stacks of 26 cards each. Count the number of black cards in the first stack and the number of red cards in the second stack. Magically, these numbers are the same!

Carefully explain why this trick always works. If you use any symbols (such as x, R, B, etc.) in your explanation, be sure to say what they represent. [To get full credit on this problem, your explanation must be clear and convincing. If I have to read it more than twice, in order to decide whether it is correct, then it's probably unclear and/or unconvincing.]

Prove something surprising: A coin trick

Some pennies are spread out on a table. Each coin is lying either heads up or tails up. You are blindfolded, so you cannot see the coins or the table. Someone tells you how many coins are lying heads up. You choose that many coins, turn each of them over, and move them to the side. You now have two groups of coins (the ones that you moved, and the ones that you did not move). Magically, the number of coins lying heads up in the first group is the same as the number of coins lying heads up in the second group.

More magic tricks at
`mathcs.slu.edu/~stevens/projnext/magic.pdf`

Obstacle: Mathematical language

If P, **then** Q.

P **implies** Q.

P **is sufficient for** Q.

Q **is necessary for** P.

Q **whenever** P.

For Q, **it is sufficient that** P.

For P, **it is necessary that** Q.

P **only if** Q.

Q, **if** P.

Q **when** P.

S: If f is differentiable, then f is continuous.

Fill in the blanks to get statements equivalent to S:

a) For f to be ____________, it is sufficient for f to be ____________.

b) For f to be____________, it is necessary for f to be ____________.

c) f is ____________ whenever f is ____________.

d) f is ____________ only if f is ____________.

“P if Q” means “$Q \Rightarrow P$.”

“P if and only Q” means

“$Q \Rightarrow P$ and $P \Rightarrow Q$.”

So “P only if Q” means “$P \Rightarrow Q$.”

A valid proof of what?

Theorem 1: Let $m \in \mathbb{Z}$. If ??????, then ???????

Proof 1: Assume m is even. Then $m = 2j$ for some integer j. Therefore $m^3 = (2j)^3 = 8j^3 = 2(4j^3)$. Let $k = 4j^3$. Then k is an integer and $m^3 = 2k$, so m^3 is even.

What is Theorem 1?

A: If m^3 is even, then m is even.

B: If m is even, then m^3 is even.

C: If m is even, then k is an integer.

D: None of these

Valid or not?

Theorem 2: Let $m \in \mathbb{Z}$. If m^2 is odd, then m is odd.

Proof 2: Assume m is odd. Then $m = 2j + 1$ for some integer j. Therefore m^2 equals $(2j+1)^2 = 4j^2 + 4j + 1 = 2(2j^2 + 4j) + 1$, which is odd, since $2j^2 + 4j$ is an integer.

What is wrong with Proof #2?

A: It starts by assuming the conclusion.

B: The algebra is wrong.

C: Both of the above.

D: Nothing

[Clickers, paper ballots, polleverywhere.com,
Near Pod, Mentimeter, SMSPoll, etc.]

Valid or not?

Theorem 3: Let $x, y \in \mathbb{Z}$. If x and y are integers of the same parity, then $3x + 5y$ is even.

Proof 3: The integers 3 and 5 are both odd. If x is odd and y is odd, then a previous theorem tells us that $3x$ and $5y$ are odd. From another previous theorem, it follows that $3x + 5y$ is even.

What is wrong with Proof #3?

A: You need to add "WOLOG."

B: It proves the converse of the theorem.

C: It needs another case.

D: Nothing

A valid proof of what?

Theorem 4: Let $x \in \mathbb{Z}$. If ??????, then ??????

Proof 4: Assume that x is even. Then $x = 2a$ for some integer a. Thus

$$3x^2 - 4x - 5 = 3(2a)^2 - 4(2a) - 5 = 12a^2 - 8a - 5 = 2(6a^2 - 4a - 3) + 1.$$

Since $6a^2 - 4a - 3 \in \mathbb{Z}$, it follows that $3x^2 - 4x - 5$ is odd.

What is Theorem 4?

A: If x is odd, then $3x^2 - 4x - 5$ is even.
B: If $3x^2 - 4x - 5$ is even, then x is odd.
C: x is odd only if $3x^2 - 4x - 5$ is even.
D: Something else.

Rephrasing the theorem

The proof shows that "If x is even, then $3x^2 - 4x - 5$ is odd." This is equivalent to

"If $3x^2 - 4x - 5$ is even, then x is odd" (contrapositive).

These are equivalent, respectively, to

"x is even only if $3x^2 - 4x - 5$ is odd,"

and

"$3x^2 - 4x - 5$ is even only if x is odd."

Rephrasing the theorem

The proof shows that "If x is even, then $3x^2 - 4x - 5$ is odd." This is equivalent to

"If $3x^2 - 4x - 5$ is even, then x is odd" (contrapositive). [Option B]

These are equivalent, respectively, to

"x is even only if $3x^2 - 4x - 5$ is odd,"

and

"$3x^2 - 4x - 5$ is even only if x is odd."

But make it easy for the reader!

Suppose you are asked to prove

Theorem 4: If $3x^2 - 4x - 5$ is even, then x is odd.

Start your proof like this:

Proof 4: We will prove the contrapositive, which says: "If x is even, then $3x^2 - 4x - 5$ is odd." Assume that x is even. Then ...

Valid or not?

Theorem 5: Let $a, b \in \mathbb{Z}$. If a and b have the same parity, then $a - b$ is even.

Proof 5: Without loss of generality, we assume that a and b are even. Then there exist integers k and j such that $a = 2k$ and $b = 2j$. Therefore $a - b = 2k - 2j = 2(k - j)$. Since $k - j \in \mathbb{Z}$, we conclude that $a - b$ is even.

What is wrong with Proof #5?

A: "Without loss of generality" should be abbreviated "WOLOG."

B: It needs another case.

C: Something else

D: Nothing

Theorem: If $f : A \to B$ and $g : B \to C$ are functions and $g \circ f$ is surjective, then g is surjective.

Question: Assume that the students have studied the definition of "surjective." What problems might they encounter in devising a proof of this theorem?

Setting the proof up

Given: $f : A \to B$ and $g : B \to C$ are functions.
$g \circ f$ is surjective.

Want: g is surjective.

Issues:

• Quantifiers (and they are hidden)
• How can you use the hidden “for all” statement in the hypothesis?

Prove: $g \circ f$ surjective $\Rightarrow g$ surjective

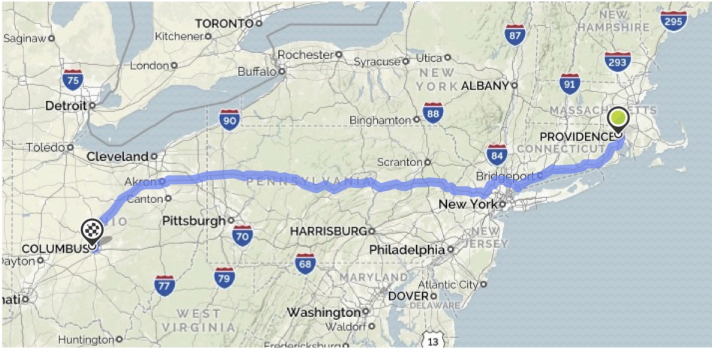
Rephrase what you want: $\forall c \in C, \exists b \in B, g(b) = c$

Start with: Let $c \in C$.

Objection! You're starting with the conclusion. Isn't that illegal?

No. Although the **proof** has to start with the hypothesis, your **thought process** often begins by thinking carefully about the conclusion.

Driving from Providence to Columbus, Ohio



Prove: $g \circ f$ surjective $\Rightarrow g$ surjective

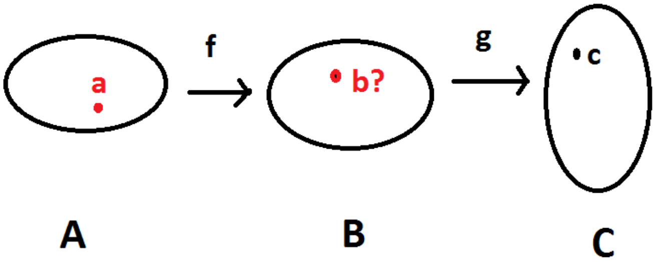
Rephrasing the conclusion in quantifiers often gives you a hint about how to set up the proof.

Want: $\forall c \in C, \exists b \in B, g(b) = c$

Outline of proof: Let $c \in C$. Choose b as follows ... (with b probably depending on c). Verify that $b \in B$, and then show that $g(b)$ really does equal c.

So the challenge is to figure out how, starting with $c \in C$, you should choose a $b \in B$ that makes $g(b) = c$.

Prove: $g \circ f$ surjective $\Rightarrow g$ surjective



c=g(f(some a)), so let b=f(that a)

Since $g \circ f$ is surjective, there exists $a \in A$ such that $g(f(a)) = c$. Let $b = f($that $a)$.

But wait!

Proof: Let $c \in C$....

Another objection! By letting c be an element of C, you assumed that $C \neq \emptyset$. Doesn't the proof need another case, to deal with the situation where $C = \emptyset$?

No, and there are many ways to explain why not. But it's nice that students ask this question.

Group Projects

Each group gets its own project that involves writing one or more proofs.

The format encourages students to read each other's projects critically.

Group projects: Format #1

Theorem: If A and B are sets and $\mathcal{C}$ is a partition of A and $\mathcal{D}$ is a partition of B, and if $A \cap B = \emptyset$, then $\mathcal{C} \cup \mathcal{D}$ is partition of $A \cup B$.

Write (a) a clear, complete, and valid proof of the theorem; and (b) an invalid "proof" of the theorem, making the invalid "proof" as convincing as possible. Provide an example that illustrates what your theorem says.

The class then has to identify the valid and invalid proofs.

Discussion format: oral presentation, or poster session

Rubric for student critiques of posters

1. Is the example a valid and helpful illustration of the theorem? Why or why not?

2. Is there a correct proof on the poster? If so, which one(s) are correct?

3. Is there an invalid proof on the poster? If so, which one(s) are incorrect? Explain why the incorrect proof(s) are incorrect.

4. Does the poster effectively present the example and the proofs? Is it easy to read and visually appealing? Rate the poster on a scale from 0 (ineffective) to 5 (very effective).

Group projects: Format #2

S: If A and B are sets, then $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

Decide whether S is true or false.

If S is **true**, give one or more examples to illustrate S, and then prove S.

If S is **false**, give an example to show that it is false. Then "salvage" S by changing the hypothesis or the conclusion to make a true statement. Finally, prove the modified version of S.

Discussion format: Two groups critique each other's projects.

Rubric for student critiques of projects

1. **Who are the members of your group?**

2. **Which project are you evaluating?**

3. **Has the group correctly determined whether the statement is true or false?**

4. **Are the examples and proofs clear and correct? If not, explain why they are unclear or incorrect.**

5. **What is the most important thing that this group could do to make its project better?**

6. **What is the best feature of this project?**

Challenge

In the context of a course that you expect to teach, make up three topics that are suitable for group projects that will help students learn to write proofs.

Some relevant research

Books

Making the Connection (ed. Carlson/Rasmussen), MAA, 2008

Mathematics Education Research: A Guide for the Research Mathematician (McKnight/Magid/Murphy), AMS, 2000 (*Full disclosure: AMS is my employer.*)

Website

Keith Weber: www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof

[red = Fellow]

Some research papers

The Role and Function of Proof in Mathematics (de Villiers), 1990

Validations of Proofs Considered as Texts (Selden/Selden), J. Research in Math Education, 2003

Student (Mis)use of Mathematical Definitions (Edwards/Ward), Amer. Math. Monthly, 2004

How Mathematicians Determine if an Argument is a Valid Proof (Weber), J. Res. in Math Education, 2008

More research papers

What Do *We* Mean By Mathematical Proof? (CadwalladerOlsker), J. Humanistic Math, 2011

Generating and Using Examples in the Proving Process (Sandefur *et al.*), Educ. Studies in Math, 2013

Investigating and Improving Undergraduate Proof Comprehension (Alcock *et al.*), AMS Notices, 2015

ANY QUESTIONS?

THANK YOU!

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Slides posted at

`mathcs.slu.edu/~stevens/projnext/proofs.pdf`