1 Triangles: Basics

This section will cover all the basic properties you need to know about triangles and the important points of a triangle. **You should know all of this by heart!** This is especially true when we cover more advanced topics in geometry later on because I will not be spending time in the future to cover basic material.

Let ABC be a (non-degenerate) triangle. Let G, I, O, H be the **centroid**, **incentre**, **circumcentre** and **orthocentre** of the triangle respectively. Let I_A, I_B, I_C be the excentres of the excircle opposite A, B, C respectively. Let r, R be the inradius and cirumradius of triangle ABC. Let K be the area of ABC.

Let us very briefly review these points, the proof of their existence and their properties.

Centroids:
Let X, Y, Z be the midpoints of BC, CA, AB respectively. Please solve the following basic problems.

1. AX, BY, CZ intersect at a point G. This point is called the **centroid** of ABC.
2. Prove that
$$\frac{AG}{GX} = \frac{BG}{GY} = \frac{CG}{GZ} = 2.$$
3. Let G' be the reflection of G across X. Prove that $BGCG'$ is a parallelogram.

Circumcentre:
Let X, Y, Z be the midpoints of BC, CA, AB respectively.

1. The three lines perpendicular to BC, CA, AB passing through X, Y, Z respectively, are concurrent at a point O and O is equidistant to A, B, C. This point is called the **circumcentre** of ΔABC, whose radius R is called the **circumradius** of ΔABC.
2. Prove that
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$
3. Prove that
$$K = \frac{abc}{4R}.$$

Orthocentre

Let D, E, F be the foot of the perpendicular from A, B, C on BC, CA, AB respectively. Please prove the following facts.

1. By noting that XYZ is similar to ABC, prove that AD, BE, CF are concurrent. This point is called the **orthocentre** of ΔABC.

2. From (1), conclude that
$$\frac{AH}{OX} = \frac{BH}{OY} = \frac{CH}{OZ} = 2.$$

3. From the section on centroids, conclude that H, G, O are collinear. This line is called the **Euler Line** of triangle ΔABC.

4. Let H_A, H_B, H_C be the midpoints of AH, BH, CH respectively. Let O' be the midpoint of HO. Prove that O' is equidistant to $D, E, F, X, Y, Z, H_A, H_B, H_C$. Conclude that these nine points are concyclic. The circle passing through these nine points is called (appropriately) the **nine-point circle** of ΔABC.

5. Amongst the nine points on the nine-point circle, find the pairs of points which form a diameter of this circle. (This should give you many angles which are equal to 90^o.)

6. Prove that amongst the points of A, B, C, H, the orthocentre of any three of these points is the fourth point.

7. Prove that the point which is the image of reflection of H across any side of the triangle, is on the circumcircle of the triangle.

Incentres:

1. The internal angle bisectors of angle A, B, C intersect at a point I. This point is called the **incentre** of ΔABC.

2. The incentre I is the centre of a circle which is tangent to the segments BC, CA, AB, say at P, Q, R respectively. [a] The **inradius** r is the radius of this circle. Express r in terms of the side lengths of the triangle a, b, c and its area K.

3. Let $s = (a + b + c)/2$ be the semi-perimeter of ΔABC. Prove that $|AQ| = |AR| = s - a$. $|BR| = |BP| = s - b$ and $|CP| = |CQ| = s - c$.

4. Let the internal angle bisector of A intersect BC at T. Prove that
$$\frac{|AB|}{|AC|} = \frac{|BT|}{|TC|}.$$

5. Let the internal angle bisector of angle A of ΔABC intersect the circumcircle of ABC at M. Prove that $|MB| = |MC| = |MI|$. i.e. M is on the midpoint of the arc BC not containing A, on the circumcircle of ΔABC and the circumcircle of ΔBIC has centre M.

[a] A common mistake is to think that AP, BQ, CR are the angle bisectors of A, B, C. This is not true, and in fact is never true for scalene triangles!

Excentres:

1. The internal angle bisector of A, and the external angle bisectors B, C intersect at a point I_A. This is called the **excentre opposite** A. Analogous definition follows for the excentre opposite B and C. The circle with centre I_A tangent to BC, AB, AC is called the **excircle opposite** A.

2. Prove that the excircle opposite A touches BC at a point P_A which is the reflection of P across the midpoint of BC. Conclude that $AB + BP_A = P_AC + CA$, meaning P_A splits the broken line AB, BC, CA in half.

3. From (2), prove that the excircle opposite A touches ray AB, AC at points whose distance from A is the semi-perimeter of ΔABC.

4. Let P' be the point on the incircle of ΔABC such that $P'P$ is a diameter of the incircle. Prove that A, P', P_A are collinear.

5. Prove that I_A, C, I_B are collinear. Similarly, I_B, A, I_C are collinear and I_C, B, I_A are collinear.

6. Prove that I_AP_A, I_BP_B, I_CP_C are concurrent. Prove that this point of concurrency is the circumcentre of $\Delta I_AI_BI_C$.

7. Prove that I is the orthocentre of $\Delta I_AI_BI_C$.

8. Let the external angle bisector of A intersect line BC at T'. Prove that

$$\frac{|AB|}{|AC|} = \frac{|BT'|}{|TC'|}.$$

Find an interpretation of this equation if this external angle bisector is parallel to BC. Compare this also for the analogous result for internal angle bisectors.

Exercises For Tuesday, September 23, 2008:

1. Given triangle ABC such that $\angle A = 60^o$, with orthocentre H, incentre I and circumcentre O. Prove that B, C, H, I, O are concyclic. In fact, if a triangle has the property such that B, C and two of these points are concyclic, then $\angle A = 60^o$.

   (This problem can be called, why problem proposers love setting an angle to be 60^o.)

2. Let $ABCD$ be a convex quadrilateral (with vertices appearing in that order) such that $\angle DAC = 80^o$, $\angle ACD = 50^o$, $\angle BDC = 30^o$ and $\angle DBC = 40^o$. Prove that ΔABC is equilateral.

3. Given a triangle ABC with $\angle A = 60^o$, let D be any point on side BC. Let O_1 be the circumcentre of ABD and O_2 be the circumcentre of ACD. Let M be the intersection of BO_1 and CO_2 and N be the circumcentre of DO_1O_2. Prove that MN passes through a point independent of D.

4. Given triangle ABC, let D be the foot of the perpendicular from A on BC and M be the midpoints of BC. Points P, Q are on rays AB and AC respectively such that $|AP| = |AQ|$ and M is on line PQ. Let S be the circumcentre of APQ. Prove that $|SD| = |SM|$.

5. Given an acute-angled triangle ABC, let H be the orthocentre of ABC, K be the midpoint of AH and M be the midpoint of BC. Prove that the intersection of the angle bisectors of $\angle ABH$ and the angle bisector of $\angle ACH$ lies on the line KM.

Exercises for Tuesday, September, 30, 2008:

1. Let ABC be a triangle with $|AC| > |AB|$. Let the X be the intersection of the perpendicular bisector of BC and the internal angle bisector of A. Let P, Q be the foot of the perpendicular from X on AB extended and AC. Let Z be the intersection of PQ and BC. Find the ratio BZ/ZC.

2. Given a triangle ABC with orthocentre H, centre of the nine-point circle O and altitude AD, let P be the midpoint of AH and Q be the midpoint of PD. Prove that OQ is parallel to BC.

3. Given an acute-angled triangle ABC, let H be the orthocentre of ABC, K be the midpoint of AH and M be the midpoint of BC. Prove that the intersection of the internal angle bisector of $\angle ABH$ and the internal angle bisector of $\angle ACH$ lies on the line KM. (From last week)

4. Let ABC be an acute-angled triangle with $|AB| < |AC|$, altitudes AD, BE, CF and orthocentre H. Let P be the intersection of BC and EF, M be the midpoint of BC and Q be the intersection of the circumcircle of MBF and MCE.

   (a) Prove that $\angle PQM = 90^o$.

   (b) Conclude that P, H, Q are collinear.

   (c) Let ω be the circle passing through B, C, E, F. What are the images each of the points $A, B, C, D, E, F, P, Q, M$, the midpoints of AB, BC, CA and the midpoints of AH, BH, CH under the inversion about ω?

5. A quadrilateral is said to be **bicentric** if it contains a circumcircle (i.e. $ABCD$ is cyclic) and an incircle. Let a, b, c, d be the side lengths of a bicentric quadrilateral $ABCD$, with the lengths appearing in that order around the quadrilateral. Let s be the semiperimeter of the quadrilateral.

   (a) Prove that $a + c = b + d$. (This is in fact a necessary and sufficient condition for a quadrilateral to have an incircle.)

   (b) Let r be the radius of the incircle of $ABCD$ and R be the radius of the circumcircle of $ABCD$. Prove that

   $$r = \frac{\sqrt{abcd}}{s}, \; R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{abcd}}.$$

   (c) Prove that the area of $ABCD$ is $\sqrt{abcd} = \frac{1}{2}\sqrt{p^2q^2 - (ac-bd)^2}$ where p, q are the lengths of the diagonals AC, BD respectively. [1]

[1] Do you recall what Ptolemy's Theorem states?

(d) Let O, I be the circumcentre and incentre of $ABCD$ respectively. Let P be the intersection of the diagonals AC and BD. Prove that P, O, I are collinear.

Concurrency: Ceva and Menelaus Theorem

Ceva's Theorem:
Given a triangle ABC and points D, E, F on segments BC, CA, AB respectively. Then AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Proof: Use areas. $\square$

We now generalize Ceva's Theorem to a hybrid of Ceva and Menelaus Theorem.

Ceva and Menelaus' Theorem
Given triangle ABC and points D, E, F on lines BC, CA, AB respectively. Suppose k of the points D, E, F are external of the segment BC, CA, AB respectively. Then

1. AD, BE, CF are concurrent if and only if $k = 0$ or 2, and
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$
2. D, E, F are collinear if and only if $k = 1$ or 3, and
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

An alternative statement of Ceva and Menelaus Theorem can be stated using signed lengths.

Using Ceva and Menalaus' Theorem, we can also prove Monge's Theorem. Given two non-intersecting circles, the **internal similitude** of the two circles is defined to be the intersection of their two common internal tangents. The **external similtude** of the two circles is defined to be the intersection of their two common external tangents.

Monge's Theorem: Given three pairwise non-intersecting circles $\omega_1, \omega_2, \omega_3$ with centres O_1, O_2, O_3. Let P_1 be a point of similitude of ω_2 and ω_3. Define P_2 and P_3 analogously.

(a) Suppose exactly k of these similitudes are external where $k \in \{1, 3\}$. Prove that P_1, P_2, P_3 are collinear.

(b) Suppose exactly k of these similitudes are external where $k \in \{0, 2\}$. Prove that O_1P_1, O_2P_2, O_3P_3 are concurrent.

We also introduce several important theorems related to collinearity and concurrency of points.

Desargue's Theorem: Given two triangles ABC, DEF, prove that AD, BE, CF are concurrent if and only if $AB \cap DE, BC \cap EF, CA \cap FD$ are collinear.

Proof: This can be done using repeated applications of Menelaus Theorem or by using projective coordinates by setting the line passing through $AB \cap DE, BC \cap EF, CA \cap FD$ as the line of infinity. We leave the details to you. $\square$

Pappus' Theorem: Given two lines l_1, l_2, let A, B, C be points on l_1 appearing in that order and D, E, F be points on l_2 appearing in that order. Then $AE \cap BD, BF \cap CE, CD \cap AF$ are collinear.

Proof: Use projective coordinates by setting the line passing through $AE \cap BD, BF \cap CE$ as the line of infinity. We leave the details to you. $\square$

Pascal's Theorem: Let A, B, C, D, E, F be vertices of a cyclic hexagon. Prove that $AB \cap DE, BC \cap EF, CD \cap FA$ are collinear.

Proof: Use projective coordinates. Blah. $\square$

It is important to also notice the degenerate cases of Pascal's Theorem. i.e. when two more of A, B, C, D, E, F are equal.

Brianchon's Theorem: Let $ABCDEF$ be a hexagon that circumscribes a circle. Then AD, BE, CF are concurrent.

Proof: Apply the dual of Pascal's Theorem. $\square$

Again, note the degenerate cases of Brianchon's Theorem, where a point of the hexagon can be a point of tangency with its incircle.

Corollary: Let $ABCD$ be a bicentric quadrilateral with incircle γ, touching AB, BC, CD, DA at P, Q, R, S respectively. Prove that AC, BD, PR, QS are concurrent.

Let ABC be a triangle and a point P on the same plane as the triangle. The **pedal triangle** of P (with respect to ABC) is defined to be the triangle with vertices which are the foot of the perpendicular from P on BC, CA, AB.

Simson's Theorem: Given triangle ABC and a point P in the same plane as ABC, the pedal triangle of P is degenerate (i.e. a line) if and only if P is on the ciccumcircle of ABC. This line is called the **Simson Line** of ABC for P.

Proof: Angle chase it. $\square$

Exercises for Tuesday, October 7, 2008 and Tuesday, October 14, 2008

1. Given ABC be a triangle, let P, Q, R be the points where the incircle of ABC touch on BC, CA, AB respectively and X, Y, Z be the points where the excircles opposite A, B, C respectively touch BC, CA, AB respectively.

   (a) Prove that AP, BQ, CR are concurrent. This point is called the **Gergonne point** of ΔABC.

   (b) Prove that AX, BY, CZ are concurrent. This point is called the **Nagel point** of ΔABC.

   (c) Let G, I, N be the centroid, incentre and Nagel point respectively of ΔABC. Prove that I, G, N are collinear and
$$\frac{IG}{GN} = \frac{1}{2}.$$

2. Given triangle ABC, let lines passing through B, C respectively tangent to the circumcircle of ABC meet at P.

   (a) Prove that AP is a symmedian of A. i.e. the reflection of the median from A about the internal angle bisector of A.

   (b) Prove that the three symmedians are concurrent. This point of concurrency is called the **Lemoine point** of ΔDEF.

   (c) Let L be the Lemoine point of ABC and D, E, F be the feet of the perpendicular on BC, CA, AB from L. Prove that L is the centroid of ABC.

3. Given three fixed pairwise distinct points A, B, C lying on one straight line in this order. Let G be a circle passing through A and C whose center does not lie on the line AC. The tangents to G at A and C intersect each other at a point P. The segment PB meets the circle G at Q. Show that the point of intersection of the angle bisector of the angle AQC with the line AC does not depend on the choice of the circle G.

4. Let ABC be a triangle and P, Q be distinct points on line BC such that $|AP| = |AQ| = s$ where s is the semi-perimeter of ABC. Prove that the excircle opposite A is tangent to the circumcircle of ΔAPQ.

5. Let ABC be a triangle with circumcircle ω. Let Γ_A be the circle tangent to AB and AC and internally tangent to ω, touching ω at A'. Define B', C' analogously. Prove that AA', BB', CC' are concurrent.

6. Given triangle ABC, let the incircle γ of ABC touch BC, CA, AB at P, Q, R respectively. Let X be any interior point of the γ and suppose PX, QX, RX intersect γ a second time at A', B', C' respectively. Prove that AA', BB', CC' are concurrent.

7. Given triangle ABC with incircle γ, let T_A be the point on γ such that the circumcircle γ_A of T_ABC is tangent to γ at T_A. Let the common tangent at T_A of γ and γ_A intersect BC at P_A. Define P_B, P_C, T_B, T_C analogously.

   (a) Prove that P_A, P_B, P_C are collinear.

   (b) Prove that AT_A, BT_B, CT_C are concurrent.

8. Let ABC be a triangle, and $\omega_1, \omega_2, \omega_3$ be circles inside ABC that are tangent (externally) one to each other, such that ω_1 is tangent to AB and AC, ω_2 is tangent to BC and BA and ω_3 is tangent to CA and CB . Let D be the common point of ω_2 and ω_3, E the common point of ω_3 and ω_1, and F the common point of ω_1 and ω_2. Show that the lines AD, BE, CF are concurrent.

9. Let ABC be a triangle with incentre I. Let D, E, F be incentres of IBC, ICA, IAB respectively. Prove that AD, BE, CF are concurrent.[2]

10. Let $ABCD$ be a cyclic quadrilateral, l_1, l_2 be lines passing through D perpendicular to AB and AC respectively. Let T be any point on line AD and suppose BT intersects l_1 at X and CT intersects l_2 at Y. Prove that XY passes through a point which is independent of the choice of T.

11. Let $ABCD$ be a convex quadrilateral with BA different from BC. Denote the incircles of triangle ABC and ADC by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to AD and CD.

    (a) Prove that $AB + AD = CB + CD$.

    (b) Prove that the common external tangent to k_1 and k_2 intersects on k.

12. A point P lies on the side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of the triangle CPD, and let I be its incentre. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let the lines AC and BD meet at E, and let the lines AK and BL meet at F. Prove that E, I, F are collinear.

13. Let ABC be a triangle and L be a point on side BC. Extend rays AB and AC to points M, N respectively such that $\angle ALC = 2\angle AMC$ and $\angle ALB = 2\angle ANB$. Let O be the circumcentre of ΔAMN. Prove that OL is perpendicular to BC.

14. Given a triangle ABC with incentre I that touches BC, CA, AB at D, E, F respectively. Let P be the intersection point of the circumcircle of AEF and ABC which is not A. Define Q, R analogously. Prove that PD, QE, RF are concurrent.

[2] I do not have a solution to this problem yet.