Fixed Income, continued

Reference: Bodie et al, Ch 16

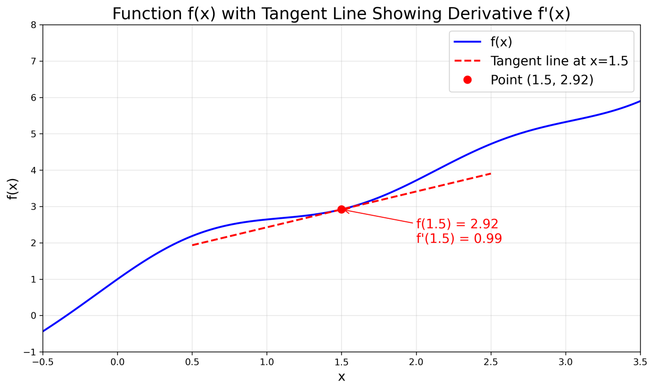
Econ 457

Week 10-b

Outline

1. Calculus
2. Duration
3. Bond Convexity
4. Applications

The first derivative of a function is equal to the slope of the tangent lines to that function.



1. Calculus

Foundation Material

Rules for Derivatives

Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Example: $\frac{d}{dx}[x^3] = 3x^2$

Product Rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Example: $\frac{d}{dx}[x^2 \cdot e^x] = 2x \cdot e^x + x^2 \cdot e^x$

Chain Rule:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Example: $\frac{d}{dx}[(1+r)^{-t}] = -(1+r)^{-t-1} \cdot 1 = -t(1+r)^{-t-1}$

1. Calculus

Foundation Material

Estimating the Derivative with Small Changes in X

Slope Approximation: For a function $f(x)$, the slope at point x can be approximated using small changes:

$$\text{Slope} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

As Δx becomes smaller, this approximation becomes more accurate.

Second Derivative

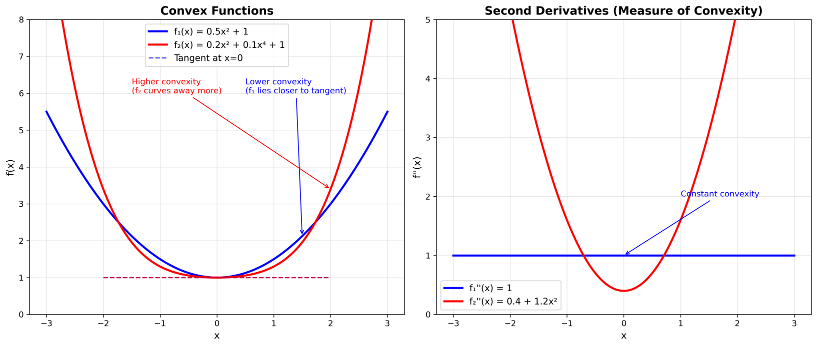
The second derivative measures the rate of change of the slope of the tangent lines.

- If $f''(x) > 0$ the function is convex
- Larger (smaller) values of $f''(x)$ indicate more (less) convexity
- If $f''(x) < 0$ the function is concave (function looks like a cave)

1. Calculus

Foundation Material

Second Derivative



2. Duration

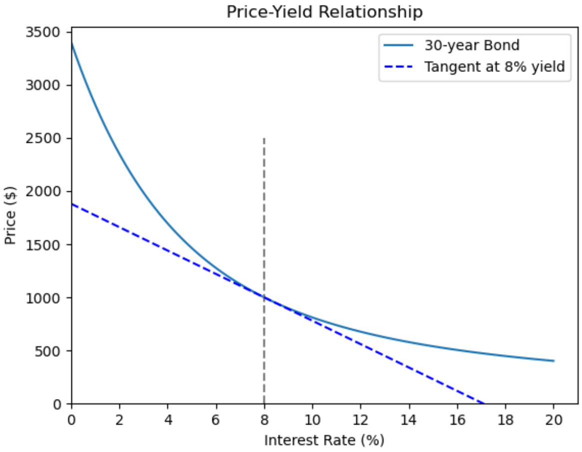
Definitions

The **duration** of a bond is the approximate percentage change in the bond price for a 100 basis point change in the yield to maturity.

- *Intuition:* Duration is the first derivative (slope) of the bond price-yield curve.
- *Convention:* Duration is positive for most bonds (even though the slope of change in the bond price is negative) and duration is referred to as 'years' (more on this later).
- *Example:* For a bond with 20 years of duration, the price will increase by 20% if yields fall by 100 basis points.

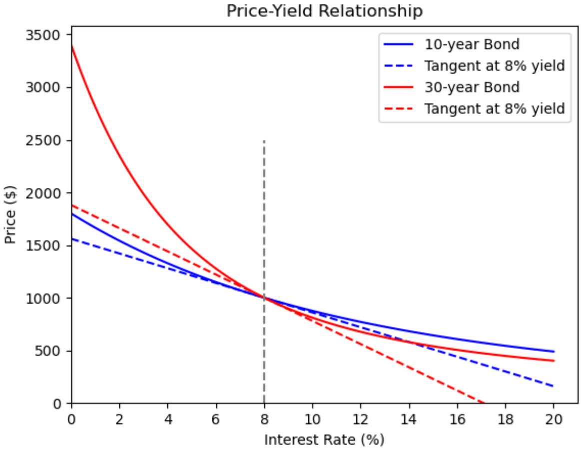
2. Duration

Chart of Price-Yield Relationship



2. Duration

Chart of Price-Yield Relationship



2. Duration

First Derivative, continued

In practice, bond duration is often estimated by 'shocking' the bond price equation with a small change in the yield, and then calculating the slope of the bond price-yield relationship in the normal way.

Example for Bond Pricing:

- Let $P(y)$ = bond price as function of yield y
- To estimate duration at yield $y = 5\%$:
- Calculate $P(5.01\%) - P(4.99\%)$ and divide by 0.02%
- This gives approximate percentage price change per basis point

2. Duration

Modified Duration

Define the **modified duration** of a bond as:

$$-\frac{P'(y)}{P(y)}$$

Where $P(y)$ is the bond price given a yield y and $P'(y)$ is the first derivative of the bond price with respect to the yield.

Note that, by convention, duration is expressed as a percent change (divide by $P(y)$), and that duration is positive (multiply by -1).

2. Duration

Maucalay Duration

Maucalay Duration is the weighted average of the time to maturity of the bond's cash flows, where the weight applied to each cash flow is the proportion of the total present value of the bond accounted for by that payment (i.e. the present value of the cash flow divided by the present value of the bond).

$$D = \sum_{t=1}^{T} t \times w_t$$

Where
t = Time to Maturity, $w_t = \frac{CF_t/(1+y)^t}{P}$, CF_t = Cash flow at time t,
P = Bond Price, y = Yield to maturity.

2. Duration

Maucalay Duration

Maucalay observed the maturity of a bond was not necessarily the most important time-characteristic. Maucaly said:

> *Let us use the word ”duration” to signify the essence of the time element in a loan.*

The weights applied to each cash flow are what he meant by the ”essence”.

Due to the initial insights of Maucaly regarding the ”essense” of the time element, now the convention is to refer to duration in ”years” even though that doesn't have any real relevance to the modified duration formulation.

See K. Winston, *Quantitative Risk and Portfollio Management* page 92

2. Duration

Maucalay Duration

Here is how Modified Duration is related to Maucalay Duration:

- Start with generic bond price equation

$$P(y) = \sum_{t=1}^{T} \frac{C}{(1+y)^t} + \frac{FV}{(1+y)^T}$$

- Write down the Modified Duration: take the first derivative of $P(y)$, divide by $P(y)$, and multiply by -1

$$\frac{-P'(y)}{P(y)} = \sum_{t=1}^{T} \frac{\frac{t \cdot C}{(1+y)^{(t+1)}}}{P} + T\frac{\frac{FV}{(1+y)^{(T+1)}}}{P}$$

- Factor out $\frac{1}{(1+y)}$ from the left hand side

$$\frac{-P'(y)}{P(y)} = \frac{1}{(1+y)}\left[\sum_{t=1}^{T} t\frac{\frac{C}{(1+y)^t}}{P} + T\frac{\frac{FV}{(1+y)^T}}{P}\right]$$

- The term in brackets is the Maucaly Duration from the previous slide, where the weights are $\frac{\frac{C}{(1+y)^t}}{P}$ and $\frac{\frac{FV}{(1+y)^T}}{P}$

2. Duration

Maucalay Duration and Modified Duration

Conclude that:

$$\text{Modified Duration} = -\frac{P'(y)}{P(y)} = \frac{\text{Mauculay Duration}}{(1+y)}$$

Maucaly Duration is usually is very similar to the Modified Duration, especially when y is small.

In practice its common to hear traders refer to ”duration” without specifying which measure. This usually doesn’t matter, because, as we’ll see below, these are used for estimates anyways.

2. Duration

Rules

The following rules for duration can be shown either by thinking about the weighted cash flows, or by taking a derivative.

1. The Macaulay Duration of a zero-coupon bond equals T (Only one cash flow!). The Modified Duration of a zero-coupon bond is $\frac{T}{1+y}$ (take the derivative...).
2. The Modified Duration of a perpetuity is equal to $\frac{1}{y}$, and the Macaulay Duration is $\frac{1+y}{y}$
3. Holding maturity constant, duration is lower when the coupon rate is higher.
4. Holding the coupon rate constant, duration generally increases with its time to maturity.
5. Holding other factors constant, duration higher when yield to maturity is lower

3. Bond Convexity

Definitions

The **convexity** of a bond measures the curvature of the bond price-yield relationship.

- *Mathematical Definition:* Convexity is the second derivative of the bond price with respect to yield
- *Intuition:* Duration (first derivative) gives the slope; convexity gives the curvature
- *Formula:*

$$\text{Convexity} = \frac{P''(y)}{P(y)}$$

- *Key Property:* Bond prices are generally convex with respect to yields (convexity > 0)

3. Bond Convexity

Definitions

- Duration alone underestimates price gains when yields fall
- Duration alone overestimates price losses when yields rise
- Convexity correction improves price change estimates

Convexity is generally valuable to investors. They are willing to pay more for convex bonds, and, conversely, may require a discount (higher yield) for negatively convex bonds.

3. Bond Convexity

Price Change Approximation

First-Order Approximation (Duration Only):

$$\Delta P \approx -D \times P \times \Delta y$$

Second-Order Approximation (Duration + Convexity):

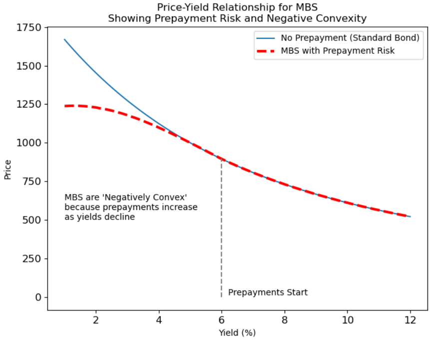
$$\Delta P \approx -D \times P \times \Delta y + \frac{1}{2} \times C \times P \times (\Delta y)^2$$

Where: D = Modified Duration, C = Convexity, P = Current Bond Price, Δy = Change in Yield

3. Bond Convexity

Mortgage - negative convexity

Mortgage-backed securities (MBS) are said to have ”negative convexity” due to prepayment risk.



3. Bond Convexity

Other sources of convexity

MBS are a specific example of a bond with an embedded option. The embedded option changes the convexity profile.

Callable corporate bonds are often negatively convex for the same reason.

Options can also be used to explicitly change the convexity profile of a bond portfolio (upcoming lecture)

The key is to draw the price-yield relationship of the bond (or the portfolio) and think about the curvature.

4. Two Applications

Practice Questions

Hedging a Bond Portfolio

You start with a portfolio of \$1,000,000 in on-the-run 10 year US Treasury bonds (10 years to maturity, 3.5% coupon, 3.5% yield) and \$1,000,000 in cash. [1]

You use cash to buy \$1,000,0000 of a high yield corporate bond: 4% coupon, 10 years to maturity, 15% yield. [2]

You want exposure to the default premium of the high yield bond but you do **NOT** want exposure to the interest rate risk of the high yield bond.

Question: How much 10 year US Treasury bonds do you sell to offset the purchase of the corporate bond?

[1] Cash has zero duration.

[2] Assume the high yield bond is purchased using cash.

4. Two Applications

Practice Questions

Hedging a Bond Portfolio

1. Durations for Different Bonds

	Cash	US Treasury Bond	High Yield Bond
Maturity	0	10	10
Coupon	0.00%	3.5%	4%
Yield	4.75%	3.5%	15%
Price		$ 100.00	$ 44.81
Macaulay Duration		8.60	7.48
Modified Duration v.1	0	8.31	6.50
Duration Ratio			0.78

Hedging Ratio

4. Two Applications

Practice Questions

Hedging a Bond Portfolio

2. Weights for Different Portfolios

	Cash	US Treasury Bond	High Yield Bond
Starting	50%	50%	0%
After buying HY	0%	50%	50%
After hedging HY Bond	39%	11%	50%

3. Duration and Yields for Different Portfolios

	Duration	Yield
Starting	4.16	4.1%
After buying HY	7.41	9.3%
After hedging HY Bond	4.16	9.7%

4. Two Applications

Practice Questions

Estimating Performace of a Bond Portfolio

Consider a 30 year zero coupon bond that yields 5% and a 10 year zero coupon bond that also yield 5%. Assume the yield curve is flat.

Investors start to worry about a recession, all yields drop to 4.5%.

What is your return on the 30 year bond? On the 10 year bond?

Then the recession actually occurs, all yields drop to 2.5%.

Question: What is your return on the 30 year bond? On the 10 year bond?

4. Two Applications

Practice Questions

Predicting Performance

1. Duration

	30y Zero	10y Zero
Maturity	30	10
Coupon	0	0
Yield	5%	5%
Price	23.16	61.41
Duration	30.0	10.0

2. Yield Changes

	Recession Worry	Recession Happens
Starting Yield	5.0%	5.0%
Ending Yield	4.5%	2.5%
Change (bps)	-50	-250

4. Two Applications

Practice Questions

Predicting Performance

3. Approximate Performance of Two bonds
Use: (Duration) X (-(Change in Yields, bps)/100)

	30y Zero	10y Zero
Recession Worry		
Approx. % Change	14.99	5.00
Recession Happens		
Approx. % Change	74.95	24.99

4. Actual Performance of Two Bonds

	30y Zero	10y Zero
Recession Worry		
Price	26.72	64.41
% Change	15.4%	4.9%
Recession Worry		
Price	47.70	78.13
% Change	105.9%	27.2%