# Diversification

## Reference: Bodie et al, Ch 7

Econ 457

Week 4-a

# Outline

1. Review: Data Construction
2. Sum of Two Random Variables
3. SPY and TLT
4. Optimization: SPY, TLT and Cash

# 1. Review: Data Construction

Total Return (%) = Price Return (%) + Dividend Yield (%)

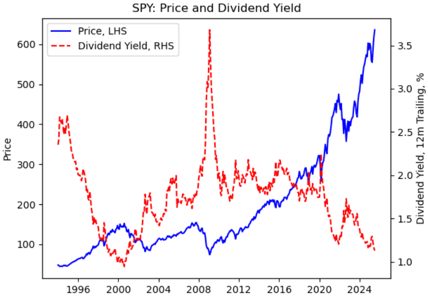
Where the Dividend Yield = Income / (Initial Price).

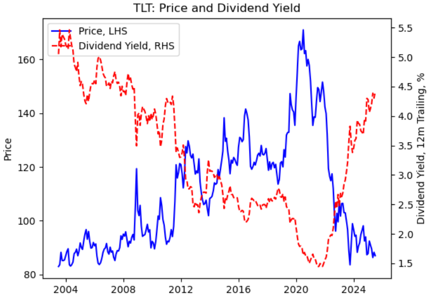
Compute Total Return Index:

1. Match income series to price series
2. Compute Total Return for Each Period: $r_{tr}$
3. Create an Index starting at 100, subsequent periods =
   $i_t = (1 + r_{tr}) \cdot i_{t-1}$

# 1. Review Data Construction

## Dividend Yield




Data Source: Yahoo finance
Dividend yield is calculated using the 12-month *trailing* dividends and the current price. While this is standard practice, it is also not great.

# 1. Review: Data Construction

## Sharpe Ratio

**Sharpe Ratio** = Excess Returns / Std Dev of Excess Returns

$$\frac{E[r] - r_f}{\sigma}$$

Where $E[r] - r_f$ is the *excess* return of the asset and $\sigma$ is the standard deviation.

Compute Historical Sharpe Ratios:

1. Match Total Return Series to Risk-free Return Series
2. For each period, compute Excess Returns = Total Return (%) - Risk-Free Return (%)
3. Compute Mean and Standard Deviation of Excess Returns
4. Sharpe Ration = Mean / Standard Deviation

# 1. Review: Data Construction

## Sharpe Ratio

Table: ETF Performance Statitics - Excess Returns

| Ticker | Start Date | Mean (%) | Std Dev (%) | Sharpe Ratio |
|---|---|---|---|---|
| SPY | 1993-01 | 8.440 | 14.820 | 0.569 |
| TLT | 2002-07 | 3.090 | 13.660 | 0.226 |
| GLD | 2004-11 | 9.180 | 16.680 | 0.550 |
| EWW | 1996-03 | 9.420 | 26.570 | 0.355 |
| EWD | 1996-03 | 8.640 | 24.470 | 0.353 |
| EWH | 1996-03 | 4.630 | 24.100 | 0.192 |
| EWI | 1996-03 | 6.440 | 23.820 | 0.271 |
| EWJ | 1996-03 | 1.170 | 17.700 | 0.066 |
| EWL | 1996-03 | 6.350 | 16.610 | 0.382 |
| EWP | 1996-03 | 7.980 | 23.650 | 0.338 |

# 2. Sum of Two Random Variable

**Foundation Material**

Adding two normally distributed variables $X$ and $Y$:

**Case 1:** $X$ and $Y$ are *independent*

$$X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

**Case 2:** $X$ and $Y$ are *correlated*

$$X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2 + 2Cov(X, Y))$$

Note, in a previou lecture we saw this using the correlation coefficient: $\rho_{x,y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_y}$.

# 2. Sum of Two Random Variable

Foundation Material

## Variance-Covariance Matrix

The **Variance-Covariance matrix** of $X$ and $Y$ is as follows:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_x^2 & \text{Cov}(X, Y) \\ \text{Cov}(X, Y) & \sigma_y^2 \end{pmatrix}$$

where:

- $\sigma_x^2$ is the variance of $X$
- $\sigma_y^2$ is the variance of $Y$
- $\text{Cov}(X, Y)$ is the covariance between $X$ and $Y$

# 2. Sum of Two Random Variable

Foundation Material

## Empirical Variance-Covariance Matrix

Given observations $(x_1, y_1), (x_2, y_2), \ldots, (x_T, y_T)$, the empirical variance-covariance matrix is:

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \hat{\sigma}_x^2 & \widehat{\text{Cov}}(X, Y) \\ \widehat{\text{Cov}}(X, Y) & \hat{\sigma}_y^2 \end{pmatrix}$$

where:

$$\hat{\sigma}_x^2 = \frac{1}{T-1} \sum_{t=1}^{T} (x_t - \bar{x})^2 \tag{1}$$

$$\hat{\sigma}_y^2 = \frac{1}{T-1} \sum_{t=1}^{T} (y_t - \bar{y})^2 \tag{2}$$

$$\widehat{\text{Cov}}(X, Y) = \frac{1}{T-1} \sum_{t=1}^{T} (x_t - \bar{x})(y_t - \bar{y}) \tag{3}$$

and $\bar{x} = \frac{1}{T} \sum_{t=1}^{T} x_t$, $\bar{y} = \frac{1}{T} \sum_{t=1}^{T} y_t$

# SPY and TLT

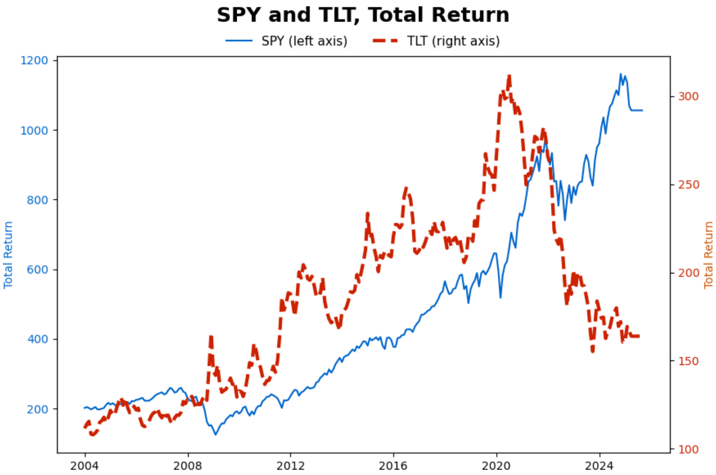
## Portfolio Construction

You are constructing a portfolio with SPY and TLT. For the moment, your only choice is how much to allocate to SPY. The remainder is allocated to TLT.

A note on terminology: The *risk premium* is the expected excess return of a security: $\mathbb{E}[r_{SPY}] - r_f$ The historic average excess returns is often used as an estimate of the risk premium.

# SPY and TLT

## Total Returns



# SPY and TLT

Empirical Means, Variance-Covariance Matrix

**Summary Statistics:**

$$\text{Excess Returns:}[\mu_{SPX}, \mu_{TLT}] = [9.4, 3.1]$$

$$\hat{\mathbf{\Sigma}} = \begin{pmatrix} \hat{\sigma}^2_{SPY} & \widehat{\text{Cov}}(SPY, TLT) \\ \widehat{\text{Cov}}(SPY, TLT) & \hat{\sigma}^2_{TLT} \end{pmatrix} = \begin{pmatrix} 219.61 & -21.09 \\ -21.09 & 186.68 \end{pmatrix}$$

Assume that the risk-free rate is 4%.

# SPY and TLT

Empirical Means, Variance-Covariance Matrix

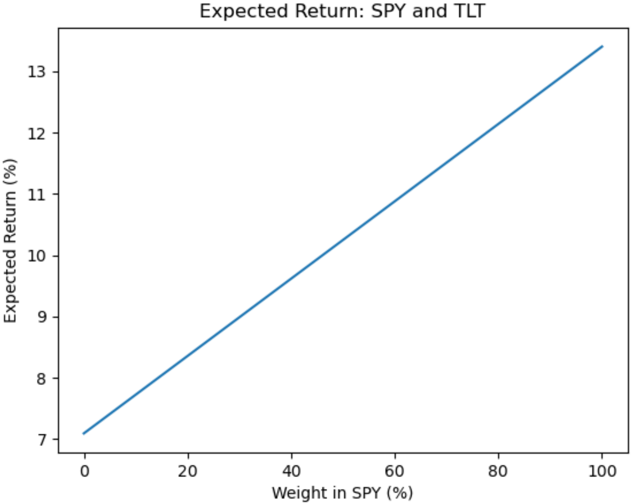
In words...

**Summary Statistics:**

- Risk Premium (SPY): $\mathbb{E}[r_{SPY}] - r_f = 9.4\%$
- Risk Premium (TLT): $\mathbb{E}[r_{TLT}] - r_f = 3.1\%$
- Expected Return (SPY): $\mathbb{E}[r_{SPY}] - r_f = 13.4\%$
- Expected Return (TLT): $\mathbb{E}[r_{TLT}] - r_f = 7.1\%$
- Standard Deviation (SPY): $\hat{\sigma}_{SPY} = 14.8\%$
- Standard Deviation (TLT): $\hat{\sigma}_{TLT} = 13.6\%$
- Covariance: $\widehat{\text{Cov}}(SPY, TLT) = -21.09$
- Correlation: $\hat{\rho}_{SPY,TLT} = -0.104$

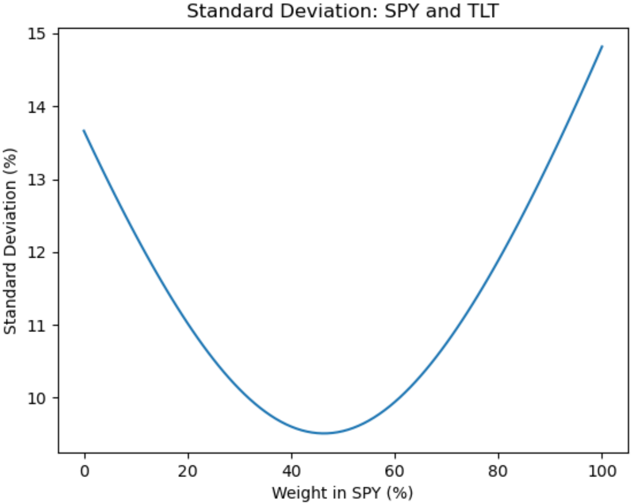
# SPY and TLT

## Portfolio Construction, cont'd



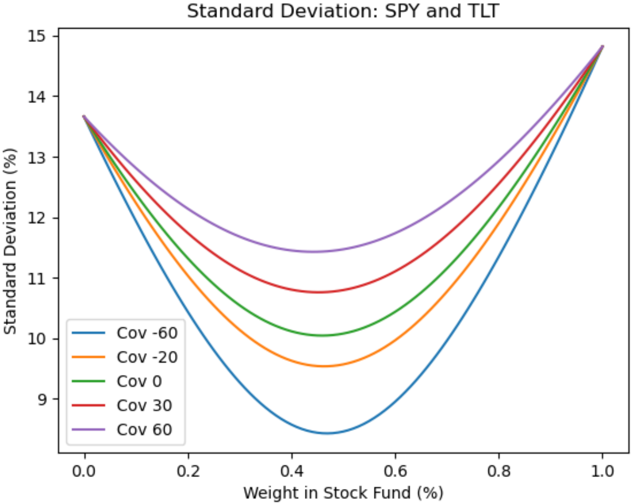
# SPY and TLT

## Portfolio Construction, cont'd



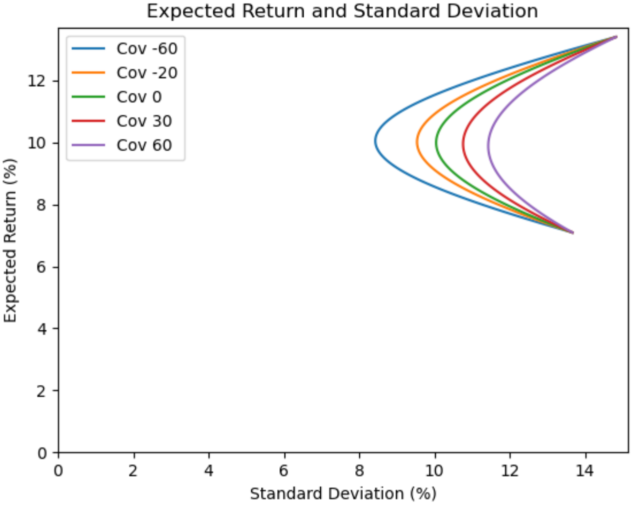
# SPY and TLT

## Portfolio Construction, cont'd



# SPY and TLT

## Portfolio Construction, cont'd



# Optimization: SPY, TLT, and Cash

## Step 1: Maximize the Sharpe Ratio

Maximize the Sharpe ratio of the portfolio, which is given by

$$\max_{w_i} S_p = \frac{\mathbb{E}(r_p) - r_f}{\sigma_p}$$

s.t.

$$w_{SPY} + w_{TLT} = 1$$

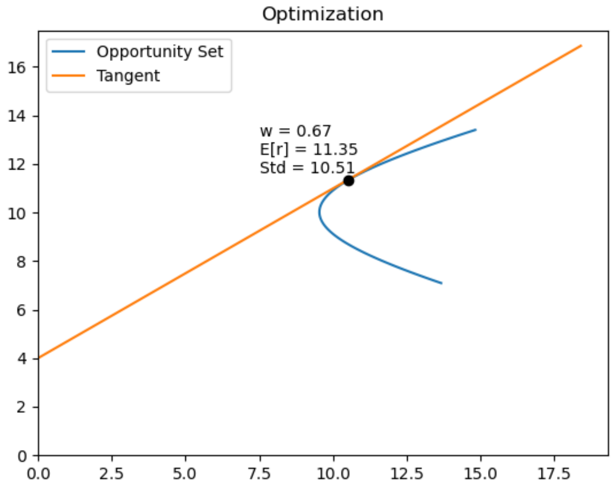
Substitute expressions for $\mathbb{E}(r_p)$, $\sigma_p$ and $w_{TLT}$

$$\max_{w_{SPY}} = \frac{w_{SPY} * \mathbb{E}(R_{SPY}) + (1 - w_{SPY})\mathbb{E}(R_{TLT}) - r_f}{w_{SPY}^2 * \sigma_{SPY}^2 + (1 - w_{SPY})^2 * \sigma_{TLT}^2 + 2 * (w_{SPY}) * (1 - w_{SPY}) * Cov(r_{SPY}, r_{TLT})}$$

and then take the derivative with respect to $w$, set the derivative equal to zero, and finally solve for $w$.

# Optimization: SPY, TLT, and Cash

Step 1: Maximize the Sharpe Ratio



# Optimization: SPY, TLT, and Cash

## Step 1: Maximize the Sharpe Ratio

Note that the Sharpe Ratio of the portfolio with $w = 0.67$ is *higher* than the Sharpe Ratio of the S&P:

Table: Sharpe Ratios

| | Risk Premium | Std Dev | Sharpe Ratio |
|---|---|---|---|
| SPY | 9.4 | 14.8 | 0.63 |
| TLT | 3.1 | 13.6 | 0.22 |
| 67% SPY + 33% TLT | 7.35 | 10.51 | 0.69 |

# Optimization: SPY, TLT, and Cash

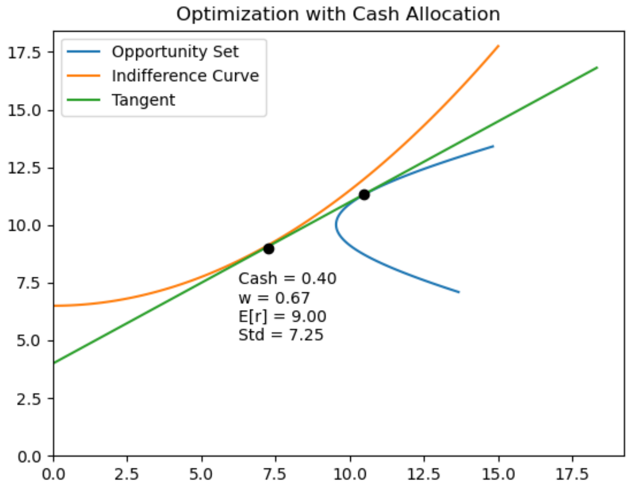
## Step 2: Decide on Cash Allocation

Now that you have determined the optimal mix of SPY and TLT, the next decision is how much risk you want to take in your portfolio. This is accomplished using the capital allocation process we discussed last week, with the only difference being that you are now allocating between cash and the optimal portfolio, rather than cash and the S&P.

Note that the allocation line is once again the *capital allocation line (CAL)*, as the Capital Markets Line (CML) only referred to the case where the risky portfolio was the S&P.

# Optimization: SPY, TLT, and Cash

## Step 2: Decide on Cash Allocation



# Optimization: SPY, TLT, and Cash

## Review

In order to find the optimal allocation between three assets (SPY, TLT, and cash), follow these steps:

1. Estimate return and risk characteristics
2. Find optimal allocation between SPY and TLT, maximize the Sharpe Ratio
3. Find optimal allocation to cash, using indifference curves