Capital Allocation

Reference: Bodie et al, Ch 6

Econ 457

Week 3-a

Economics 457

Course Outline Covered

Subject	Weeks	Sub-topics
Intro	1 & 2	Measuring Returns, Distribution of Returns, Evaluating Returns
Portfolio Construction	3, 4, 5	Capital Allocation, Diversification, Index Model
Market Equilibrium	6 & 7	CAPM, Fama-French Factors
Fixed Income	8 & 9	Prices, Yields, Yield Curve, Duration and Convexity
Equity	10 & 11	Dividend Discount Models, Price-Earnings Ratios, Efficient Markets, Equity Risk Premium
Derivatives	12, 13, 14	Futures, Swaps, Options
Financial Crisis of 2008	15	Causes, What Happened, Aftermath

Outline

1. Utility and Indifference Curves
2. Preferences over Return and Risk
3. Capital Allocation Line (budget constraint)
4. Optimization
5. Practice

Concave Utility

Utility functions are generally *concave* with respect to wealth.

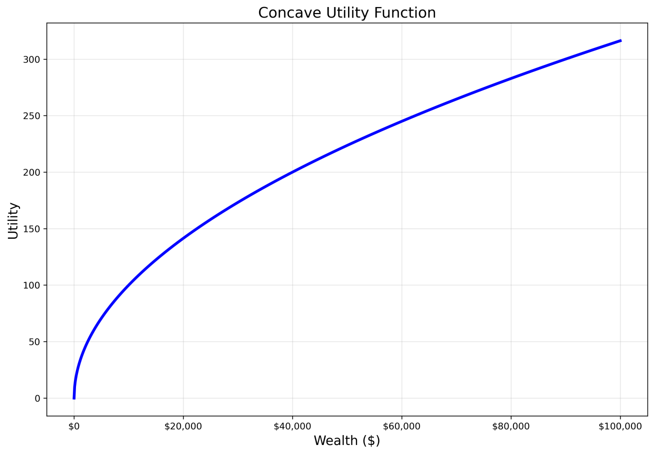
This reflects *diminishing marginal utility* with respect wealth.

Intuitively, if a household goes from, say, making $50,000 to $150,000 annually that can result in a material change in standard of living. In contrast, moving from $100,000,000 to $100,100,000 probably doesn't matter as much.

1. Utility and Indifference Curves

Foundation Material

Concave Utility



Expected Values

Definition: Expected value is the probability-weighted average of all possible outcomes

$$E[X] = \sum_i p_i \cdot x_i$$

where p_i is the probability of outcome x_i

- **Linearity:** $E[aX + bY] = aE[X] + bE[Y]$
- **Constant:** $E[c] = c$ for any constant c

1. Utility and Indifference Curves

Foundation Material

Expected Values and Concave Utility

It follows from the concavity of the utility function that for a given set of outcomes represented by x:

$$\mathbb{E}[u(x)] < u(\mathbb{E}[x])$$

In words, the expected utility over a set of uncertain outcomes is lower than the utility of the expected value over those same outcomes.

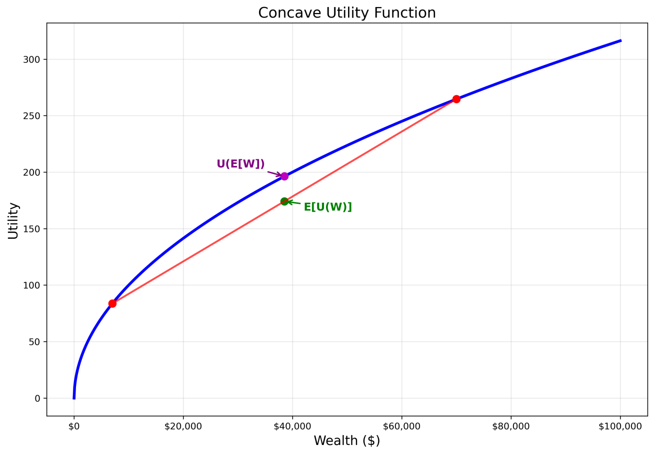
In even simpler words: people prefer certain outcomes over uncertain outcomes.

This is known as 'Jensen's inequality' and holds for any concave function.

1. Utility and Indifference Curves

Foundation Material

Concave Utility



Outline

Quick note on the numbers in this lecture

The charts in this lecture assume the following:

- Expected excess returns of the S&P: 8.5%
- Expected standard deviation of the S&P: 12.5%
- Expected Sharpe Ratio of S&P: 0.7
- Risk-free rate: 4%

These expectations are roughly in-line with the historic S&P returns and volatility since 2004.

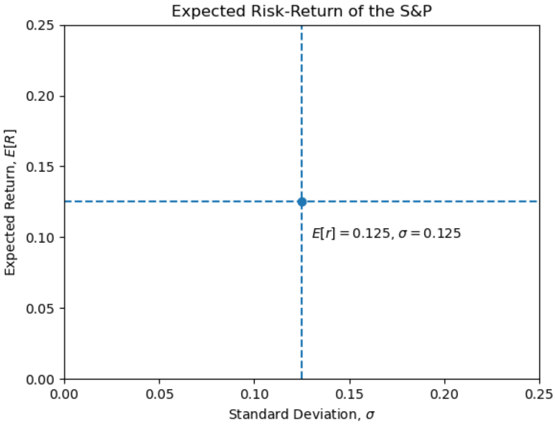
At the same time, they are better than the historic Sharpe Ratio since 1926, which is closer to 0.5.

Anyways, this lecture is mostly to illustrate the approach, rather than precisely estimate these numbers.

2. Preferences over Return and Risk

Preferences and Utility

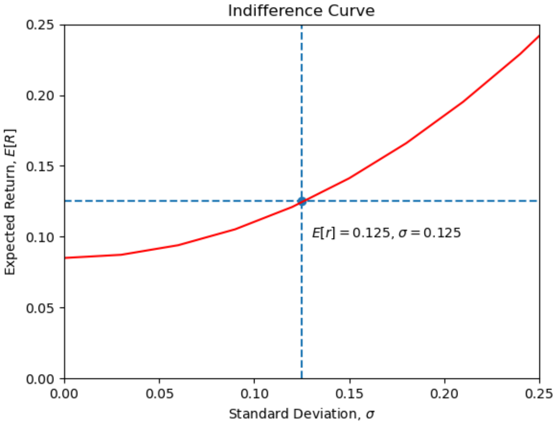
Standard preferences prefer more return and less volatility.



2. Preferences over Return and Risk

Indifference Curves

An indifference curves trace the risk-return combinations that yield the same utility.



2. Preferences over Return and Risk

Indifference Curves

Assume the following equation for utility:

$$U = E[r] - 0.5 \cdot A \cdot \sigma^2$$

Utility is increasing in $E[r]$ and decreasing in σ^2. The convexity comes from the squared-term.

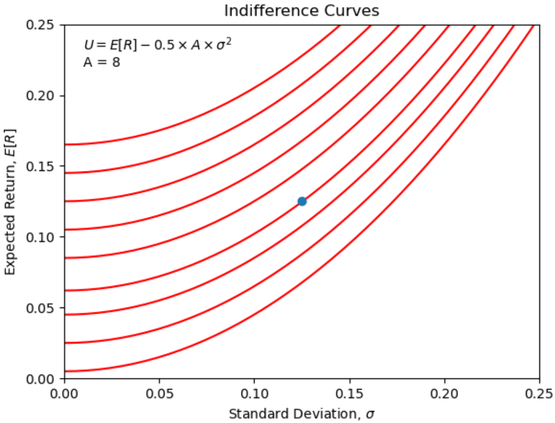
To get an indifference curve, set U equal to a constant, then plot the function $E[R] = U - 0.5 \cdot A \cdot \sigma^2$

What happens for larger (smaller) values of A? How do the indifference curves change?

2. Preferences over Return and Risk

Indifference Curves

Higher indifference curves (up and to the left) correspond to more utility.



3. Capital Allocation Line

Assume, for the moment, that you have two options: cash or a broad portfolio of risky assets. Accordingly, your only decision is how much of your portfolio to allocate to the risky asset.

The **Capital Allocation Line (CAL)** depicts all possible combiations in risk-return space available from this decision.

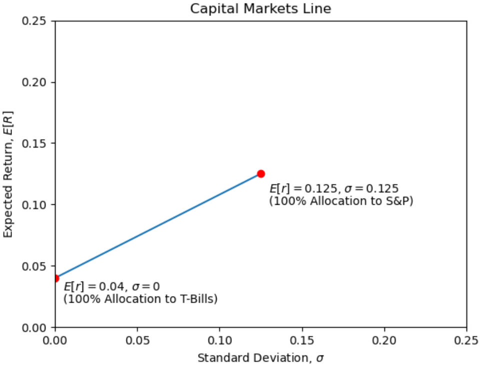
The CAL intersects the y-axis at $y = r_{risk-free}$ and goes through the point $(x = \sigma_{risky}, y = (r_{risky} + r_{risk-free}))$

3. Capital Allocation Line

Capital Markets Line and the Sharpe Ratio

When the portfolio of risky assets is a basket of broad-based stocks, the CAL has a specific name: **Capital Markets Line (CML)**.

Note that the slope of the CML is the Sharpe Ratio.



4. Optimization

We can now solve for the optimal portfolio using standard maximization techniques:

$$\max_{w} \quad U = E[r] - 0.5 \cdot A \cdot \sigma^2$$

Subject to:

$$E[r] = w \cdot E[r_{\text{risky}}] + (1 - w) \cdot r_f$$

$$\sigma^2 = w^2 \cdot \sigma^2_{\text{risky}}$$

where w is the weight in the risky asset.

4. Optimization

To solve this, take the first derivative with respect to w:

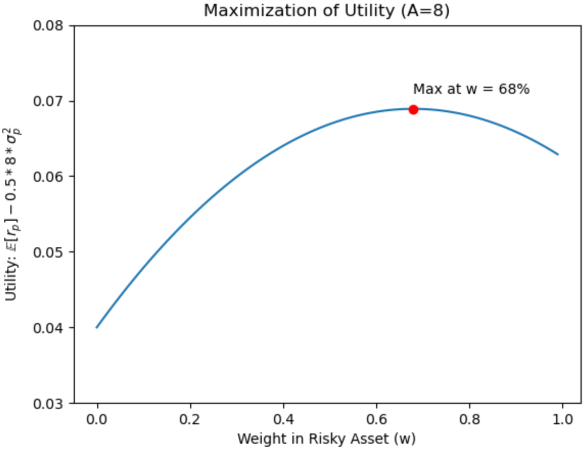
$$
\begin{aligned}
\frac{dU}{dw} &= w \cdot E[r_{\text{risky}}] + (1 - w) \cdot r_f - 0.5A \cdot w^2 \cdot \sigma^2_{\text{risky}} \\
&= 0
\end{aligned}
$$

Solving for w:

$$w^* = \frac{E[r_{\text{risky}}] - r_f}{A \cdot \sigma^2_{\text{risky}}}$$

Be careful: this is not the Sharpe Ratio. The denominator has a σ^2 and also the cofficient A.

4. Optimization



4. Optimization