Diversification

Reference: Bodie et al, Ch 7

Econ 457

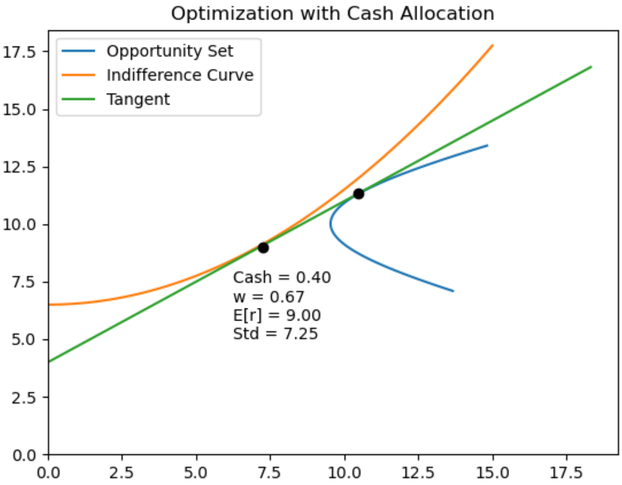
Week 4-b

Outline

1. Diversification - Review
   - Set-up
   - Standard Deviation - Cov term
   - Maximization - Sharpe Ratio
2. Optimization
3. Three Asset Optimization: SPY, TLT, and GLD
4. Many Asset Optimization
5. Excel
   - Correlations (last week's homework)
   - Solver (this week's homework)
6. Practice

1. Diversification - Review

Set up



1. Diversification - Review

Standard Deviation - COV term

Generic Case:

X and *Y* are *correlated*

$$X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2 + 2Cov(X, Y))$$

Specific Case for Portfolio Diversification:

Combining two assets in a portfolio, X and Y, where the weights of each asset (w_x and w_y) sums to one.

$$w_x X + w_y Y \sim N(w_x\mu_x + w_y\mu_y, w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y Cov(X, Y))$$

Notice how the weights get treated in the variance term. Continued...

1. Diversification - Review

Standard Deviation - COV term

...Continued

The general formula for the variance of a portfolio composed of n risky assets:

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j)$$

In the case that the portfolio is equally weighted (all $w = 1/n$) this becomes:

$$\sigma_p^2 = \sum_{i=1}^{n} \frac{1}{n^2} \sigma_i^2 + \sum_{j=1, j \neq i}^{n} \sum_{i=1}^{n} \frac{1}{n^2} Cov(r_i, r_j)$$

1. Diversification - Review

Standard Deviation - COV term

There is a benefit to diversification for any correlation coefficient $\rho_{x,y} < 1$. Remember that $\rho_{x,y} = \frac{COV(X,Y)}{\sigma_x \sigma_y}$

Consider the case when $w_x = w_y = 0.5$ and $\sigma_x = \sigma_y = \sigma$. The variance is then given by

$$0.5^2\sigma^2 + 0.5^2\sigma^2 + 2 * 0.5 * 0.5 * COV(X, Y)$$

If $\rho_{x,y} = 1$ then this evaluates to $0.25\sigma^2 + 0.25\sigma^2 + \frac{\sigma^2}{0.5} = \sigma^2$ and there is no benefit from diversification.

It follows that if $\rho_{x,y} < 1$ then the variance term will be smaller, and therefore will be smaller and there will be a benefit from diversification.

1. Diversification - Review

Standard Deviation - COV term

Also note that if $w_x = w_y = 0.5$, $\sigma_x = \sigma_y = \sigma$, and $\rho_{x,y} = 0$ then the variance term is given by

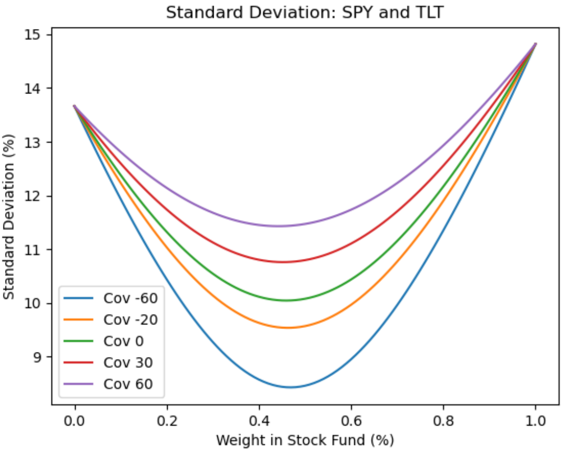
$$0.25\sigma^2 + 0.25\sigma^2 = 0.5\sigma^2$$

And the standard deviation is given by $\sqrt{0.5}\sigma$.

Any correlation less than 0 will further reduce the variance and standard deviation terms.

1. Diversification - Review

Standard Deviation - COV term



1. Diversification - Review

Maximization - Sharpe Ratio

We define the objective function that we are maximizing to be the Sharpe Ratio of the portfolio:

$$\frac{\mathbb{E}[r_p] - r_f}{\sigma_p}$$

How to think about this? The Sharpe ratio is the expected excess return per unit of volatility. Alternatively, the Sharpe ratio is a measure of the efficiency of the portfolio.

You could imagine defining the objective function differently, but the Sharpe ratio is standard.

2. Optimization Review

Foundation Material

Typical Optimization Steps

1. Define the objective function (e.g., maximize the Utility or Sharpe Ratio).
2. Specify constraints (e.g., weights sum to 1, no short selling, minimum/maximum allocations).
3. If possible, rewrite the objective function in terms of the variables you are choosing (e.g. rewrite $U(E[r], \sigma)$ as $U(w)$ where w is the weight of the risky asset in the portfolio.)
4. Use optimization tools:
   - Take first derivative with respect to choice variable, set derivative to zero, solve for choice variable.
   - Use Lagrange multipliers
   - Excel Solver, Python's `scipy.optimize`, R's `optim`, etc.
5. Analyze and interpret the resulting portfolio.

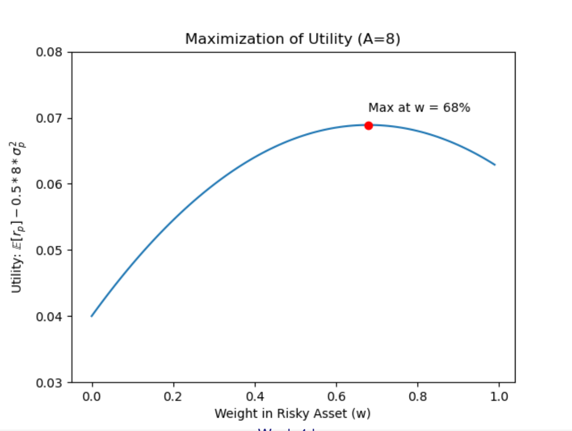
2. Optimization Review

Foundation Material

Maximize Utility in Capital Allocation Problem

Rewrite utility as a function of the weight in the risky asset (w):

$$U(w) = w \cdot E[r_{\text{risky}}] + (1 - w) \cdot r_f - 0.5A \cdot w^2 \cdot \sigma^2_{\text{risky}}$$



2. Optimization Review

Foundation Material

Maximize Sharpe Ratio in Diversification Problem

For a problem with only two assets, we can rewrite Sharpe Ratio as a function of the weight in the SPY (w_{SPY}):

$$\text{Sharpe Ratio}(w_{SPY}) =$$

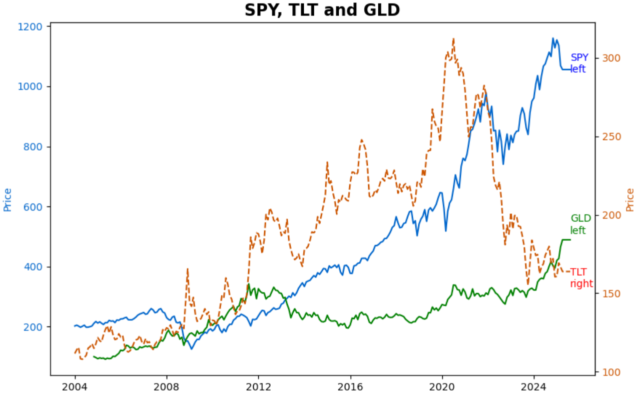
$$= \frac{w_{SPY} * \mathbb{E}(R_{SPY}) + (1 - w_{SPY})\mathbb{E}(R_{TLT}) - r_f}{w_{SPY}^2 * \sigma_{SPY}^2 + (1 - w_{SPY})^2 * \sigma_{TLT}^2 + 2 * (w_{SPY}) * (1 - w_{SPY}) * Cov(r_{SPY}, r_{TLT})}$$

Then take the first derivative with respect to w_{SPY} and set equal to zero.

For a portfolio with many assets, you can't simply the problem in the same way. Need to use a solver (or Lagrange multipliers)

3. Three Asset Optimization: SPY, TLT, and GLD

Total Return Series



Data Source: Yahoo Finance

3. Three Asset Optimization: SPY, TLT, and GLD

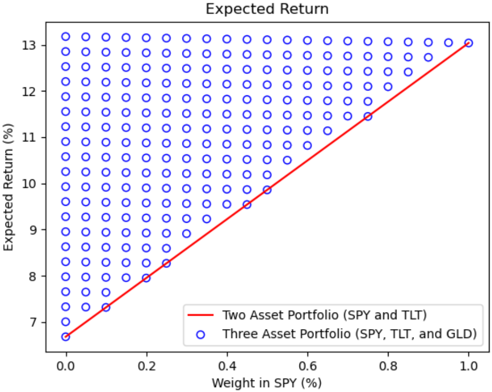
Sharpe Ratios

Table: ETF Performance Statitics - Excess Returns

Ticker	Start Date	Mean (%)	Std Dev (%)	Sharpe Ratio
SPY	1993-01	8.440	14.820	0.569
TLT	2002-07	3.090	13.660	0.226
GLD	2004-11	9.180	16.680	0.550
EWW	1996-03	9.420	26.570	0.355
EWD	1996-03	8.640	24.470	0.353
EWH	1996-03	4.630	24.100	0.192
EWI	1996-03	6.440	23.820	0.271
EWJ	1996-03	1.170	17.700	0.066
EWL	1996-03	6.350	16.610	0.382
EWP	1996-03	7.980	23.650	0.338

3. Three Asset Optimization: SPY, TLT, and GLD

Expected Return



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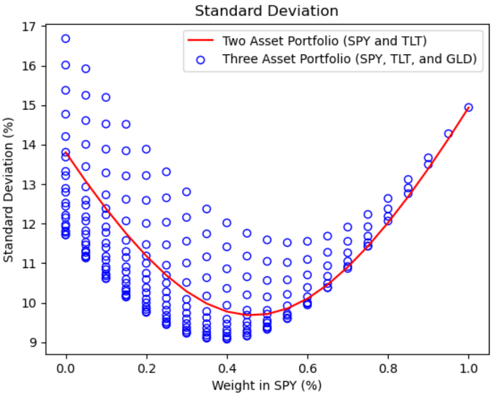
3. Three Asset Optimization: SPY, TLT, and GLD

Standard Deviation



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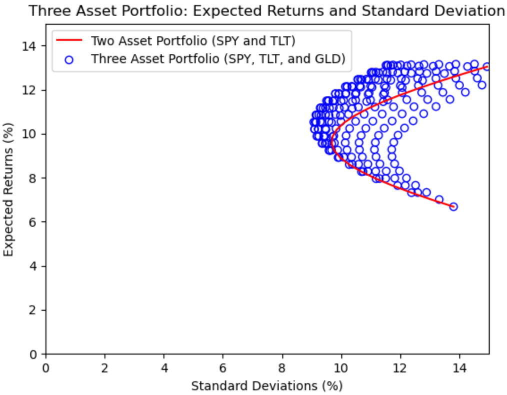
3. Three Asset Optimization: SPY, TLT, and GLD

Expected Return and Standard Deviation



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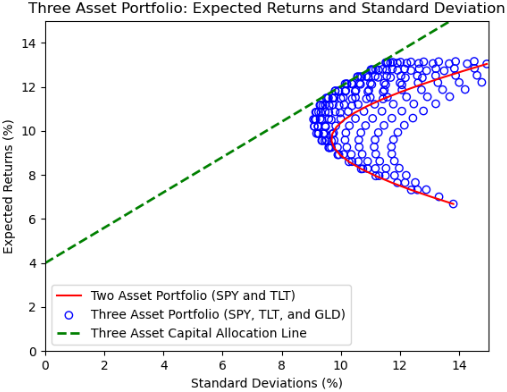
3. Three Asset Optimization: SPY, TLT, and GLD

Capital Allocation Line



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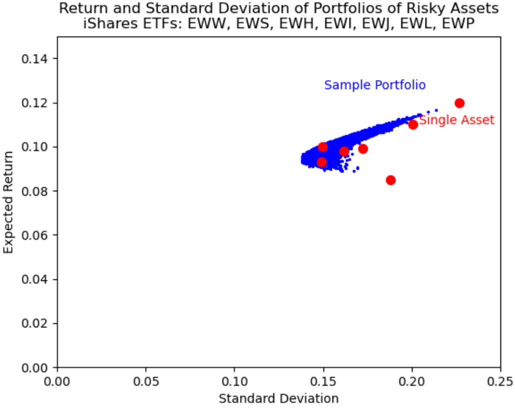
4. Many Asset Optimization

Expected Return and Standard Deviations



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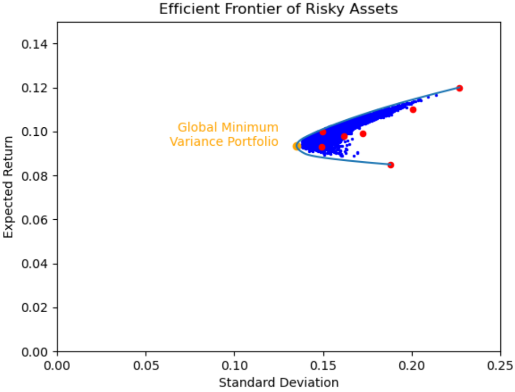
4. Many Asset Optimization

Global Minimum Variance Portfolio



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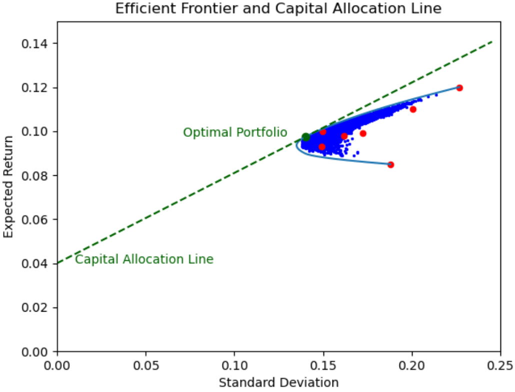
4. Many Asset Optimization

Capital Allocation Line and Optimal Portfolio



Data Source: Yahoo Finance

4. Many Asset Optimization

Review

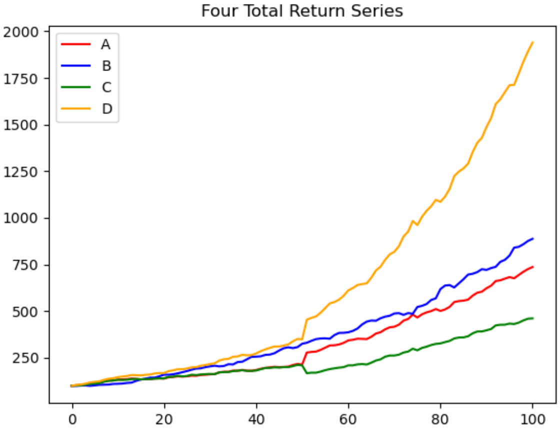
Steps for finding the optimal portfolio with many assets:

1. Estimate expected returns and Var-Cov matrix
2. (Optional: plot various portfolios, find the Global Minimum Variance Portfolio)
3. Maximize the Sharpe Ratio
4. Draw the Capital Allocation Line

5. Excel

MS Excel

Correlations (last week's homework)



5. Excel

MS Excel

Correlations (last week's homework)

Takeaways

- Correlations are based on [monthly] returns (not a total return index). The levels may look similar – they all go up – but it's the changes that matter.
- Correlations take out the mean. Series A-C have a mean increase of 2%, but that gets removed in the correlation calculation. Series D has a mean increase of 4%, but that also gets removed.
- Correlations are disproportionately impacted by the observations with the largest moves. Series A and C are identical except for observation 50. That single observation, which is the one with the largest change, drives the negative correlation.

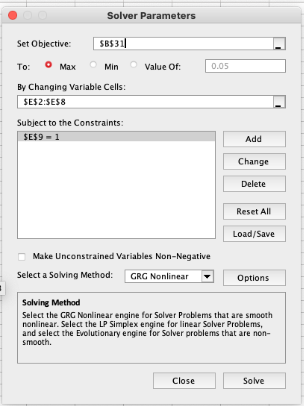
5. Excel

MS Excel

Maximization (this week's homework)

=B29/B30

	A	B	C	D	E	F	G	H
1	Risk Premiums			Weights			Book Weights	
2	US	0.06		US	0.69746945		0.6112	
3	UK	0.053		UK	0.1253189		0.8778	
4	France	0.07		France	-0.0845063		-0.214	
5	Germany	0.08		Germany	0.01356116		-0.5097	
6	Australia	0.058		Australia	0.12945108		0.0695	
7	Japan	0.045		Japan	0.12981478		0.2055	
8	Canada	0.059		Canada	-0.0106561		-0.0402	
9				Total	1			
10	Var/Cov							
11		US	UK	France	Germany	Australia	Japan	Canada
12	US	0.0224	0.0184	0.025	0.0288	0.0195	0.0121	0.0205
13	UK	0.0184	0.0223	0.0275	0.0299	0.0204	0.0124	0.0206
14	France	0.025	0.0275	0.0403	0.0438	0.0259	0.0177	0.0273
15	Germany	0.0288	0.0299	0.0438	0.0515	0.0301	0.0183	0.0305
16	Australia	0.0195	0.0204	0.0259	0.0301	0.0261	0.0147	0.0234
17	Japan	0.0121	0.0124	0.0177	0.0183	0.0147	0.0353	0.0158
18	Canada	0.0205	0.0206	0.0273	0.0305	0.0234	0.0158	0.0298
19								
20								
21	w'*VARCOV							
22		US	UK	France	Germany	Australia	Japan	Canada
23		0.02008369	0.0177406	0.02343098	0.02677826	0.01941424	0.01506277	0.01974896
24								
25	w'*VARCOV*w							
26		0.01887219						
27								
28	Sharpe Ratio							
29	Risk-Premiur	0.0564	=SUMPRODUCT(B2:B8,B21:B27)					
30	Standard Dev	0.1374	=SQRT(SUM(C32:I32))					
31	Sharpe Ratio	0.4104						



6. Practice

Practice Questions

1. Stocks offer an expected rate of return of 18% with a standard deviation of 22%. Gold offers an expected return of 10% with a standard deviation of 30%.
   - Would anyone hold gold? Demonstrate why or why not with a graph.
   - Reanswer the question above if the correlation coefficient between gold and stocks equal 1.
   - Could the set of assumptions in the second part (where correlation equals 1) be an equilibrium? Why or why not?
2. True or false: The standard deviation of the portfolio is always equal to the weighted average of the standard deviations of the assets in the portfolio.

6. Practice

Practice Questions

3 The correlation coefficients between pairs of stocks are as follows: Corr(A,B) = 0.85, Corr(A,C) = 0.6, Corr(A,D) = 0.45. Each stock has an expected return of 8% and a standard deviation of 20%.

- If your entire portfolio is now composed of stock A and you can add one stock to your portfolio, which do you add?
- Would the answer change for more risk averse investors?
- Suppose that you could also add T-Bills, which have a rate of 8%. How would your portfolio look now?

6. Practice

Practice Questions

4 Abigail Grace starts with \$900k and then inherits \$100k in portfolio ABC stock. She has the following estimates:

Table: Risk and Return Characteristics

	$E[r]$, monthly	σ
Original Portfolio	0.67%	2.37%
ABC	1.25	2.95

The correlation coefficient of ABC with her original portfoio is 0.4. Answer the following questions:

- Calculate the expected return of the new portfolio, the Covariance of ABC stock returns with her original portfolio, and the standard deviation of her new portfolio.
- Grace sells ABC stock and buys T-Bills at a rate of 0.42% monthly. Recalculate the expected return and standard deviation.
- Grace is considering selling ABC and buying XYZ, which has the same expected return and standard deviation as ABC. Does it matter whether Grace owns ABC or XZY? Why or why not?