# Financial Market Returns

## Reference: Bodie et al, Ch 5

Econ 457

Week 1-b

# Outline

1. Exponentials and Logarithms
2. Measuring Returns
3. Income
4. Measurment Issues:
    - Compounding, Geo v. Arth Mean, Total Return Series
5. Excel

# 1. Exponentials and Logarthims

**Foundation Material**

## Some basic facts

Exponential Function:

$$10^4 = 10 \times 10 \times 10 \times 10$$

First derivative:

$$\frac{d(e^x)}{dx} = e^x$$

Negative Exponents:

$$a^{-x} = \frac{1}{a^x}$$

Fractional Exponents:

$$a^{1/x} = \sqrt[x]{a}$$

## Some basic facts, cont'd

Logarithim is the inverse of the exponential function. $log_{10}()$ is the inverse of $10^x$ and $ln()$ is the inverse of $e^x$

First derivative:

$$\frac{d(ln(x))}{dx} = \frac{1}{x}$$

Logarithms of expressions with exponents:

$$log(x^a) = a \cdot log(x)$$

Logarithm of expressions with products:

$$log(x \cdot y) = log(x) + log(y)$$

# 1. Exponentials and Logarthims

Foundation Material

## Some basic facts, cont'd

Two more facts about logarthims we are going to use in this lecture:

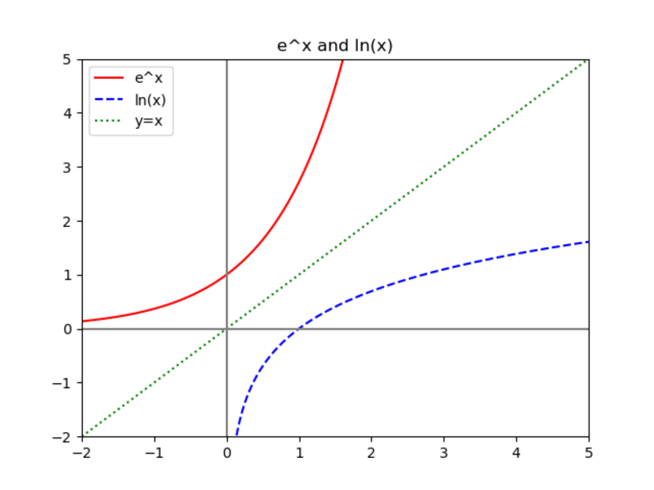
1. In a log-linear graph the scale of the y-axis is $log(y)$ or $ln(y)$. The slope of the log-linear graph is the average (continously compounded) growth rate.
2. Logarithim is a concave function:

$$log((a+b)/2) > (log(a) + log(b))/2$$

# 1. Exponentials and Logarthims

**Foundation Material**

## Some basic facts, cont'd



# 2. Measuring Returns

## Total Return - Definition

Total Return: A performance metric that accounts for capital gains (i.e. changes in price) and income (e.g. dividends, coupons, etc) paid to the owner of a financial security.

Total return is typically expressed as a percentage of the initial price.

$$\text{Total Return} = \frac{\text{Income}}{\text{Initial Price}} + \frac{\text{Change in Price}}{\text{Initial Price}}$$

The first term is sometimes referred to as either the ”current yield” or ”dividend yield”, and the second term is referred to as the ”capital gain rate.”

# 2. Measuring returns

## Holding Period Return and Annual Percentage Rate

Holding Period Return (HPR):

$$\text{HPR} = r(T) = \frac{\text{Price Increase + Income}}{\text{P(T)}}$$

Effective Annual Rate (EAR)

$$\text{EAR} = (1 + \text{HPR})^{1/T} - 1$$

- Where T is the length of time the investment was held in years
- Express total return as a *rate* of return over a year
- Notice this assumes annual compounding
- T could be positive number, greater than 1 or a fraction

# 3. Income

## Types of Income

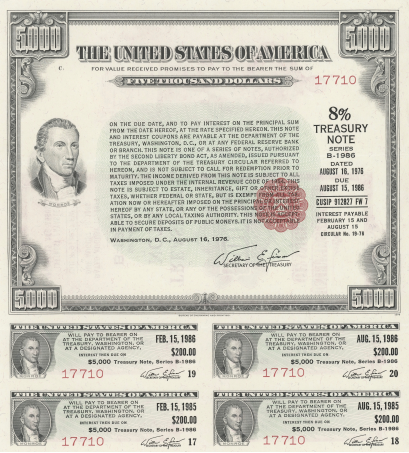
Table: Income Sources by Investment Type

| Investment Type | Income Source | Frequency |
|---|---|---|
| Bonds | Coupon payments | Semi-annual (usually) |
| Stocks | Dividends | Quarterly |
| ETFs | Distributions | Quarterly/Annual |

ETF distributions are often composed of dividends or coupons of the underlying assets, but occassionally will include capital gains as well.

# 3. Income

Treasury Bond: Note the “coupons” at the bottom




Source: The Joe I. Herbstman Memorial Collection of American Finance

# 3. Income

## Timing of Income

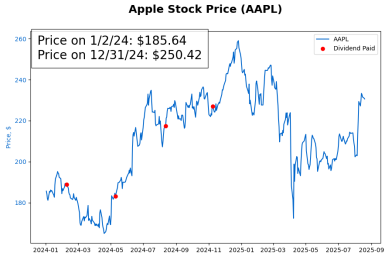
Be careful with the timing of income. Often a problem will say ”you purchase a stock that just paid a dividend.” Do NOT count that dividend in the total return.

# 3. Income

## Example: AAPL

Table: AAPL Dividends in 2024

| date | dividend |
|---|---|
| 2024-02-09 | 0.24 |
| 2024-05-10 | 0.25 |
| 2024-08-12 | 0.25 |
| 2024-11-08 | 0.25 |

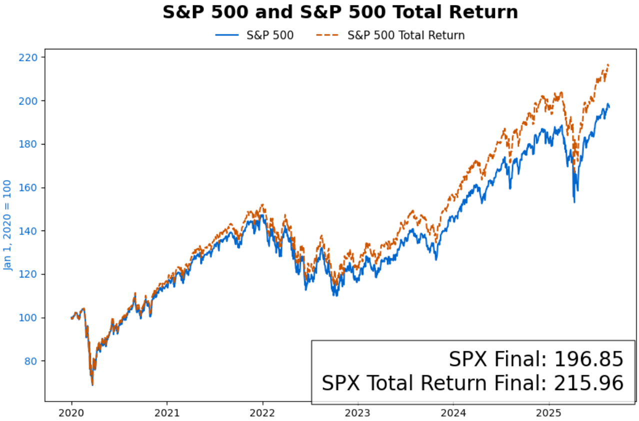



The price of AAPL was 185.64 on January 2, 2024. The price of AAPL rose to 250.42 on December 31, 2024. What was the Holding Period Return from owning AAPL stock throughout 2024?

Data source: Yahoo finance

# 3. Income

## Example: S&P Total Return




Data source: Interactive Brokers (SPX and SPXT)

# 4. Measurement Issues

Compounding: APR

Annual Percentage Rates of Return

$$(1 + \text{EAR}) = \left(1 + \frac{APR}{n}\right)^n$$

$$\text{APR} = n \times \left[(1 + \text{EAR})^{1/n} - 1\right]$$

- Where $n$ is the number of compounding periods per year
- After calculating the rate ($\frac{APR}{n}$) then compound to get annual rate
- This is the convention used in short term investments
- Bonds typically have semi annual compounding (2 times per year), and the coupon is given by $\frac{\text{Coupon Rate}}{2}$

# 4. Measurement Issues

Compounding: Definition of e

The exponential function $e^x$ can be defined as a limit:

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

Notice the similarity to the formula for the APR.

As $n \to \infty$ the formula for the $APR \to e^{r_{cc}}$, where $r_{cc}$ is the ”continuously compounded” rate of return.

# 4. Measurement Issues

Compounding

Continuously compounded rate of return ($r_{cc}$):

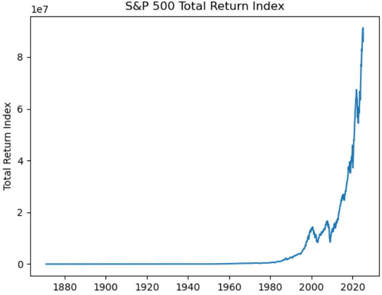
$$1 + \text{EAR} = exp(r_{cc}) = e^{r_{cc}}$$

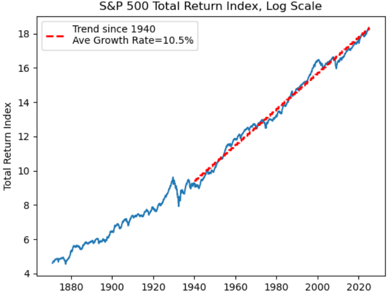
- If you are given $e^{r_{cc}}$, then with a little rearranging you can find *EAR*: $\text{EAR} = e^{r_{cc}} - 1$
- Conversely, if you are given *EAR* then you can take *ln*() of both sides to find $r_{cc}$

Most of the time in Econ 457 the compounding is either annual, or at the same frequency as the income (e.g. coupons, dividends). Occassionally we'll use continous compounding makes the math easier (e.g. option pricing).

# 4. Measurement Issues

## Compounding: Log scale




Data Source: Robert Shiller, www.shillerdata.com

# 4. Measurment Issues

Compounding: Reinvestment

What did you do with the income? The initial expression for HPR assumed you did nothing.

The expressions for EAR and APR assumed income was reinvested regularly.

The correct choice depends on the context. It’s easy to reinvest income from an ETF, it may be harder in a privately held stock.

# 4. Measuring returns

## Geometric v. Arithmetic Mean

For a series of returns, the *arithmetic* average is:

$$r_{ave} = \frac{1}{T} \times \sum_{i=1}^{T} (1 + r_i)$$

The *geometric* average is:

$$r_{geo-ave} = \sqrt[T]{\prod_{i=1}^{T} (1 + r_i)}$$

# 4. Measuring returns

## Geometric v. Arithmetic Mean

The arithmetic average is always greater than the geometric average.

You can show this using the fact that the *ln*() function is concave (next slide).

The difference between the arithmetic and geometric averages is increasing in the variance of the return series.

# 4. Measuring returns

## Geometric v. Arithmetic Mean

Showing that the Geometric Mean $<$ the Arithmetic Mean

$$\begin{aligned}
\text{Geometric Mean} &= \sqrt[T]{\prod_{i=1}^{T} x_i} && (1)\\
\ln(\text{Geometric Mean}) &= \frac{1}{T}\sum_{i=1}^{T} \ln(x_i) && (2)\\
&< \ln\left(\frac{1}{T}\sum_{i=1}^{T}(x_i)\right) && (3)\\
&< \ln(\text{Arithmetic Mean}) && (4)\\
\implies \text{Geometric Mean} &< \text{Arithmetic Mean} && (5)
\end{aligned}$$

Where (2) applies properties of *ln*(), (3) is because *ln*() is a concave function (Jensen's inequality), and (4) is because *ln*() is monotonically increasing.

# 4. Measuring returns

## Geometric v. Arithmetic Mean, Example

Stock ABC doesn't pay dividends. In year 1 the price of ABC declines by 20%. In year 2 the price rebounds by 20%.

What is the arithmetic average of the annual returns? What is the geometric average? Which measure is more appropriate?

# 4. Measurement Issues

## Total Return Series

A total return series reflects the growth of an investment due to both price changes and income.

The convention is to incorporate the income in the period that it is received. This is most appropriate for monthly series. May consider alternatives for a daily series.

Bonds have a different convention, which we will discuss later.

# 4. Measurement Issues

## Total Return Series

Constructing a total return index:

1. Start:

$$index_{t0} = 100$$

2. Calculate holding period return:

$$\text{HPR}_{t1} = \frac{(\text{income} + \text{price change})}{\text{Initial Price}} - 1$$

3. The value for the next period is given by:

$$index_{t+1} = (1 + \text{HPR}_{t1}) \cdot index_t$$

4. Repeat...

# 4. Measurement Issues

## Total Return Series, Arthmetic v. Geometric Mean

Table: S&P 500 Total Returns by Decade, (% Annualized)

| Decade | Arithmetic Return | Geometric Mean | Standard Deviation |
|---|---|---|---|
| 1870s | 8.11 | 7.46 | 10.86 |
| 1880s | 6.29 | 5.83 | 9.62 |
| 1890s | 6.24 | 5.55 | 11.84 |
| 1900s | 10.62 | 9.82 | 12.70 |
| 1910s | 4.99 | 4.44 | 10.51 |
| 1920s | 15.46 | 14.25 | 15.08 |
| 1930s | 4.39 | 0.03 | 30.63 |
| 1940s | 9.47 | 8.58 | 13.23 |
| 1950s | 18.24 | 17.74 | 10.08 |
| 1960s | 8.05 | 7.51 | 10.31 |
| 1970s | 6.56 | 5.69 | 13.23 |
| 1980s | 16.88 | 16.04 | 12.98 |
| 1990s | 17.19 | 16.66 | 10.45 |
| 2000s | 0.38 | -0.73 | 14.68 |
| 2010s | 13.02 | 12.56 | 9.55 |
| 2020s | 13.72 | 12.11 | 14.15 |

Data Source: Robert Shiller, Ken French. Monthly data annualized.

# 5. Excel

Practice Questions

## Some basic commands

vlookup(): Used to match data from different tabs

isna(): Tell Excel how to treat missing values

$: Used to ’freeze’ the cell row or column

Excel dates: can change the format

Charts: mostly use line chart in this class