Method of Characteristics

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1 Introduction

Method of Characteristics is a geometric approach to solving first order quasi-linear (coefficients can be functions of dependent variable) PDEs. It reduces a PDE to a system of ODEs.

2 Method of Characteristics

Consider the following PDE, which is often obtained when applying Lie's symmetry methods to solve ODEs.

$$\xi(x,y)r_x + \eta(x,y)r_y = 0$$

The solution is a function r(x,y). The above equation can be written as,

$$(\xi,\eta).(r_x,r_y) = 0$$

This means the gradient of $r(x,y)$ is orthogonal to the vector field (ξ,η). Therefore, the level sets of $r(x,y) = f$ are tangential to the vector field (ξ,η). Or equivalently the the field lines (ξ,η) are tangential to the plane curves $r(x,y) = c$. The ODE describing these set of curves are,

$$dx/ds = \xi$$
$$dy/ds = \eta$$

for some parameter $s \in \mathcal{R}$. Rearranging the above, we get,

$$dx/\xi = dy/\eta$$

$$dy/dx = \eta/\xi$$

Integrating the above equation gives the curve $y = f(x)+c$ along which $r(x,y) = c$. Therefore, $r(x,y) = y - f(x)$.

Similarly, let us consider the above equation with unity on the r.h.s,

$$\xi(x,y)s_x + \eta(x,y)s_y = 1$$

we can re-write this as,

$$(\xi, \eta, 1).(s_x, s_y, -1) = 0$$

This means that, $(s_x, s_y, 1)$ is normal to the surface $s(x,y) - z = c$ and the vector field $(\xi, \eta, 1)$ lies on the surface.

The curves on the surface are given by,

$$dx/dt = \xi$$
$$dy/dt = \eta$$
$$dz/dt = 1$$

for $t \in R$. Rearranging the above set of equations,

$$dx/\xi = dy/\eta = ds = dz$$

$r(x,y)$ is obtained by integrating the above equations as,

$$\int ds = \int dy/\eta$$

or

$$\int ds = \int dx/\xi$$

3 Examples

3.1 Example-1

Consider the differential equation,

$$dy/dx = y/x + x$$

The infinitesimals, $\xi = 0$ and $\eta = x$ can be verified to be the vector field corresponding to the symmetry orbit.

Hence the partial differential equations to be solved in this case are,

$$xr_y = 0$$

for some function r(x,y) =c, and

$$xs_y = 1$$

for some function s(x,y).

The first equation implies, that r is a function of x only, that is $r(x) = c$. The simplest function satisfying the requirement is $r = x$. Equivalently from

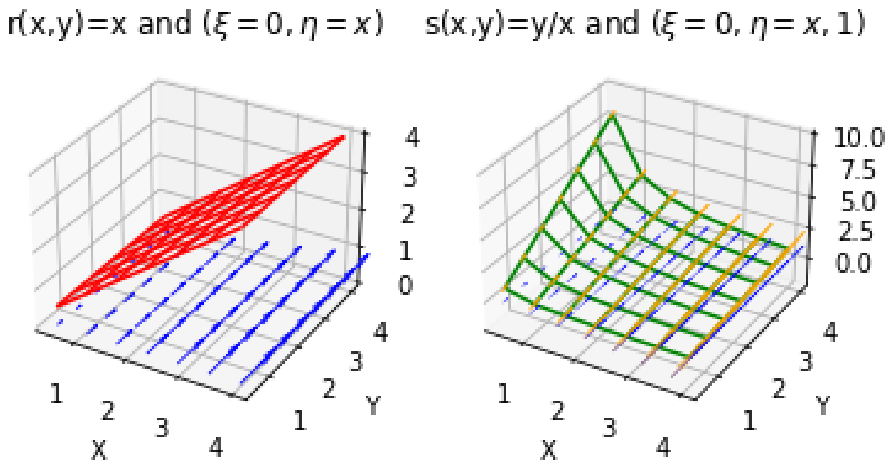

Figure 1: Characteristic lines and surfaces for Example -1

the figure 1, the orbit fields are vectors along the y direction. So the level sets of $r = x$ are along these vectors.

The characteristics equation for the second equations is,

$$ds = dy/x$$

$$s = y/x + c$$

To make the connection to Lie symmetry method, it is observed that the the PDEs discussed arise as conditions on the canonical coordinates. The functions ξ and η are the tangent vectors in the infinitesimal symmetry generators $\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial x}$ of a given differential equation. The canonical coordinates are usually obtained by the MOC, which in this case is,

$$(r, s) = (x, y/x)$$

3.2 Example-2

$$dy/dx = e^{-x}y^2 + y + e^x$$

The ansatz, $\xi = 1$ and $\eta = y$ can be verified to be the vector filed of the symmetry orbit.

Hence the partial differential equations to be solved in this case are,

$$r_x + yr_y = 0$$

for some function r(x,y) =c, and

$$r_x + yr_y = 1$$

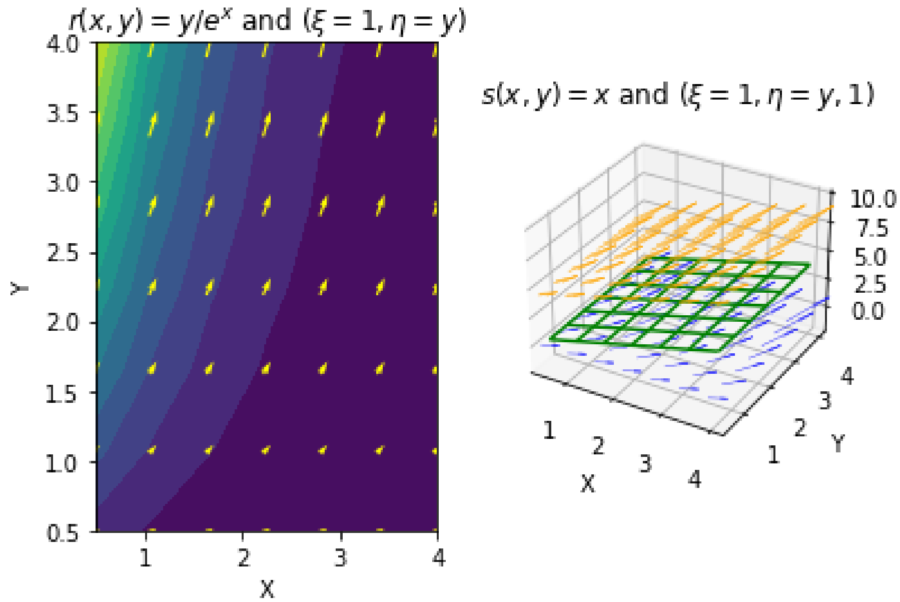

Figure 2: Characteristic lines and surfaces for Example -2

for some function s(x,y).

The characteristics equation for the first equations is,

$$dx = dy/y$$

Integrating we get $x = ln(y) + c$, or $y/e^x = c$. Therefore, $r = y/e^x$.

The l.h.s of figure 2 depicts the level sets of r(x,y) and the field (ξ, η).

Integrating the characteristics equation $dx = ds$ for the second PDE, we get,

$$s = x + c$$

The r.h.s of figure 2 depicts the wire-frame of surface s(x,y) and the field (ξ, η) by blue lines and the field $(\xi, \eta, 1)$ by orange lies. The field lines have been shifted above the surface s for visibility.

Therefore the canonical coordinates as mentioned in the previous example are,

$$(r, s) = (y/e^x, x)$$

To summarize, the method of characteristics can be applied to linear or quasi linear PDEs. Given a PDE of the form,

$$\xi(x, y, z)z_x + \eta(x, y, z)z_y = \delta(x, y, z)$$

which is,

$$(\xi(x, y, z), \eta(x, y, z), \delta(x, y, z)).(z_x, z_y, -1) = 0$$

This is geometrically equivalent to the statement that the vector field (ξ, η, δ) is tangential to the surface $z = z(x, y)$. The solution is a union of the integral curves to the vector field defined by,

$$dx/dt = \xi$$
$$dy/dt = \eta$$
$$dz/dt = \delta$$

known as Lagrange-Charpit equations. The characteristics equation can be written in a parameter invariant form as,

$$dx/\xi(x, y, z) = dy/\eta(x, y, z) = dz/\delta(x, y, z)$$

and integrated.

References

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