The Adjoints of Matrices

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Summer 2024

Abstract.

This paper explores the adjoints of square matrices, defining the adjoint (or adjugate) as the transpose of the cofactor matrix. We prove that for any $n \times n$ matrix A, the product $A \cdot \text{adj}(A)$ equals $\det(A)I_n$, verifying this with both invertible and non-invertible matrices. Additionally, we demonstrate that $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A$ and derive a formula for the inverse of an invertible matrix using its adjoint. The paper also shows that a non-zero determinant implies matrix invertibility.

1 The Adjoint Matrix

Before defining an adjoint matrix, we need to define some concepts first.

Definition 1.1. The cofactor matrix Cof(A) of a square matrix A is the matrix whose (i,j)-entry is the (i,j)-cofactor C_{ij} of the matrix A.

Example 1.1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}.$$

To find the cofactor matrix Cof(A), we need to compute each (i, j)-cofactor C_{ij} of the matrix A.

Let's calculate the cofactor C_{11} as an example:

- Remove the first row and first column from A. The resulting 2×2 submatrix is:
$$\begin{pmatrix} 4 & 5 \\ 0 & 6 \end{pmatrix}$$
- Calculate the determinant of this submatrix:
$$\det \begin{pmatrix} 4 & 5 \\ 0 & 6 \end{pmatrix} = (4 \cdot 6) - (5 \cdot 0) = 24$$
- The sign factor for C_{11} is given by $(-1)^{1+1} = 1$.
- Thus, the cofactor C_{11} is:
$$C_{11} = 1 \cdot 24 = 24$$

Similarly, we can compute the other cofactors. The cofactor matrix Cof(A) of A is then:

$$\mathrm{Cof}(A) = \begin{pmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{pmatrix}.$$

Definition 1.2. An adjoint of a square matrix A, denoted $\mathrm{adj}(A)$, is defined to be $\mathrm{adj}(A) = [\mathrm{Cof}(A)]^T$. The adjoint is also known as adjugate and adjunct.

Example 1.2. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}.$$

To find the adjoint $\mathrm{adj}(A)$, we first need to determine the cofactor matrix $\mathrm{Cof}(A)$:

$$\mathrm{Cof}(A) = \begin{pmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{pmatrix}.$$

The adjoint $\mathrm{adj}(A)$ is the transpose of the cofactor matrix:

$$\mathrm{adj}(A) = [\mathrm{Cof}(A)]^T = \begin{pmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{pmatrix}.$$

Theorem 1.1. For any $n \times n$ matrix A, the result of multiplying the matrix A by the adjoint matrix from the right equals multiplying the identity matrix by the determinant of A. In other words:

$$A \cdot \mathrm{adj}(A) = \det(A) I_n.$$

Proof. Consider the matrix $A = (a_{ij})$ and let $\mathrm{Cof}(A) = (C_{ij})$ be its cofactor matrix. The adjoint (or adjugate) of A is $\mathrm{adj}(A) = \mathrm{Cof}(A)^T$. Thus, $\mathrm{adj}(A) = (C_{ji})$.

The matrix $\det(A)I_n$ is defined such that its (i, j)-entry $= \det(A)$ when $i = j$ and (i, j)-entry $= 0$ otherwise.

To prove that $A \cdot \mathrm{adj}(A) = \det(A)I_n$, we need to show that:

$$\sum_{k=1}^{n} a_{ik} C_{jk} = \begin{cases} \det(A) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

First, consider the case where $i = j$:

$$\sum_{k=1}^{n} a_{ik} C_{ik}.$$

This sum represents the expansion of the determinant of A along the i-th row. Therefore,

$$\sum_{k=1}^{n} a_{ik} C_{ik} = \det(A).$$

Next, consider the case where $i \neq j$:

$$\sum_{k=1}^{n} a_{ik} C_{jk}.$$

In this case, let's construct a new matrix B by replacing the j-th row of A with its i-th row. Notice that this new matrix B will have two identical rows: the i-th row and the j-th row. The determinant of any matrix with two identical rows is zero.

Now, the (i, j)-cofactor C_{jk} of A corresponds to the determinant of the $(n-1)\times(n-1)$ submatrix obtained by deleting the j-th row and k-th column from A, multiplied by $(-1)^{j+k}$. The sum $\sum_{k=1}^{n} a_{ik} C_{jk}$ thus represents the cofactor expansion of the determinant of the matrix B along the j-th row (which is the same as the i-th row of A).

Because B has two identical rows, its determinant is zero:

$$\det(B) = \sum_{k=1}^{n} a_{ik} C_{jk} = 0.$$

Therefore,

$$\sum_{k=1}^{n} a_{ik} C_{jk} = 0.$$

Combining both cases, we have:

$$A \cdot \operatorname{adj}(A) = \begin{pmatrix} \sum_{k=1}^{n} a_{1k} C_{1k} & 0 & \cdots & 0 \\ 0 & \sum_{k=1}^{n} a_{2k} C_{2k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=1}^{n} a_{nk} C_{nk} \end{pmatrix} = \det(A) I_n.$$

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Example 1.3. Let's consider the matrix A:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}.$$

First, we find the cofactor matrix $\mathrm{Cof}(A)$:

$$\mathrm{Cof}(A) = \begin{pmatrix} (-24) & 20 & (-5) \\ 18 & (-15) & 4 \\ 5 & (-4) & 1 \end{pmatrix}.$$

The adjoint (adjugate) of A is:

$$\mathrm{adj}(A) = \mathrm{Cof}(A)^T = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}.$$

The determinant of A is:

$$\det(A) = 1 \cdot (1 \cdot 0 - 4 \cdot 6) - 2 \cdot (0 \cdot 0 - 4 \cdot 5) + 3 \cdot (0 \cdot 6 - 1 \cdot 5) = 1.$$

We verify the theorem $A \cdot \mathrm{adj}(A) = \det(A) I_n$:

$$A \cdot \mathrm{adj}(A) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix} \cdot \begin{pmatrix} -24 & 4 & -4 \\ 20 & -15 & 5 \\ -5 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det(A) I_n.$$

Example 1.4. Now consider the matrix B:

$$B = \begin{pmatrix} 2 & 4 & 1 \\ 6 & 8 & 2 \\ 4 & 8 & 2 \end{pmatrix}.$$

First, we find the cofactor matrix $\mathrm{Cof}(B)$:

$$\mathrm{Cof}(B) = \begin{pmatrix} 0 & -4 & 16 \\ 0 & 0 & 0 \\ 0 & 2 & -8 \end{pmatrix}.$$

The adjoint (adjugate) of B is:

$$\mathrm{adj}(B) = \mathrm{Cof}(B)^T = \begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 2 \\ 16 & 0 & -8 \end{pmatrix}.$$

The determinant of B is:

$$\det(B) = 2 \cdot (8 \cdot 2 - 2 \cdot 8) - 4 \cdot (6 \cdot 2 - 2 \cdot 4) + 1 \cdot (6 \cdot 8 - 8 \cdot 4) = 0.$$

We verify the theorem $B \cdot \operatorname{adj}(B) = \det(B) I_n$:

$$B \cdot \operatorname{adj}(B) = \begin{pmatrix} 2 & 4 & 1 \\ 6 & 8 & 2 \\ 4 & 8 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 2 \\ 16 & 0 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det(B) I_n.$$

Theorem 1.2. *For any $n \times n$ matrix A, it holds that $A \cdot adj(A) = adj(A) \cdot A$.*

Proof. We know from theorem 1.1 that $A \cdot \operatorname{adj}(A) = \det(A) I_n$, so we need to prove that $\operatorname{adj}(A) \cdot A = \det(A) I_n$. We will use a similar approach to the one used in the proof of theorem 1.1.

Consider the matrix $A = (a_{ij})$ and let $\operatorname{Cof}(A) = (C_{ij})$ be its cofactor matrix. The adjoint (or adjugate) of A is $\operatorname{adj}(A) = \operatorname{Cof}(A)^T$. Thus, $\operatorname{adj}(A) = (C_{ji})$.

The matrix $\det(A) \cdot I_n$ is defined such that its (i, j)-entry is $\det(A)$ when $i = j$ and 0 otherwise.

To prove that $\operatorname{adj}(A) \cdot A = \det(A) \cdot I_n$, we need to show that:

$$\sum_{k=1}^{n} C_{ki} a_{kj} = \begin{cases} \det(A) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Case 1: $i = j$

To show:

$$\sum_{k=1}^{n} C_{ki} a_{ki} = \det(A)$$

Explanation:

The (i, i)-entry of $\operatorname{adj}(A) \cdot A$ is:

$$(\operatorname{adj}(A) \cdot A)_{ii} = \sum_{k=1}^{n} C_{ki} a_{ki}$$

This sum represents the cofactor expansion of the determinant of A along the i-th column. By definition of the determinant:

$$\sum_{k=1}^{n} C_{ki} a_{ki} = \sum_{k=1}^{n} a_{ki} C_{ki} = \det(A)$$

where C_{ki} is the cofactor of a_{ki} in A.

Case 2: $i \neq j$

To show:

$$\sum_{k=1}^{n} C_{ki} a_{kj} = 0$$

Explanation:

In this case, construct a new matrix B by replacing the j-th column of A with the i-th column of A. Notice that this new matrix B will have two identical columns: the i-th column and the j-th column. The determinant of any matrix with two identical columns is zero.

The (i, j)-cofactor C_{ki} of A corresponds to the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting the k-th row and i-th column from A, multiplied by $(-1)^{i+k}$.

The sum $\sum_{k=1}^{n} C_{ki} a_{kj}$ thus represents the cofactor expansion of the determinant of matrix B along the j-th column (which is the same as the i-th column of A).

Because B has two identical columns, its determinant is zero:

$$\det(B) = \sum_{k=1}^{n} C_{ki} a_{kj} = 0$$

Therefore:

$$\sum_{k=1}^{n} C_{ki} a_{kj} = 0$$

Combining Both Cases

Combining the results from both cases, we have:

$$\operatorname{adj}(A) \cdot A = \begin{pmatrix} \sum_{k=1}^{n} C_{k1} a_{k1} & 0 & \cdots & 0 \\ 0 & \sum_{k=1}^{n} C_{k2} a_{k2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=1}^{n} C_{kn} a_{kn} \end{pmatrix} = \det(A) \cdot I_n$$

Thus, $A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A) \cdot I_n$.

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Theorem 1.3. *Given that A is an invertible square matrix of size $n \times n$. The inverse of A, denoted A^{-1}, can be derived from the relationship:*

$$A \cdot adj(A) = \det(A) I_n$$

Where:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Proof. 1. Start with the given formula:

$$A \cdot \operatorname{adj}(A) = \det(A) I_n$$

2. Multiply both sides of the equation by $\frac{1}{\det(A)}$:

$$\frac{1}{\det(A)} A \cdot \operatorname{adj}(A) = \frac{1}{\det(A)} \det(A) I_n$$

3. Simplify the right-hand side:

$$\frac{1}{\det(A)} A \cdot \operatorname{adj}(A) = I_n$$

4. Since I_n is the identity matrix, and $A \cdot A^{-1} = I_n$, it follows that:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Thus, the formula for the inverse of an invertible matrix A using its adjoint is:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

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Theorem 1.4. *For any square matrix A of size $n \times n$, If $\det(A) \neq 0$, then A is invertible.*

Proof. We use the theorem:

$$A \cdot \operatorname{adj}(A) = \det(A) I_n$$

1. Given: $\det(A) \neq 0$.
2. Divide both sides of the equation by $\det(A)$:
$$\frac{1}{\det(A)} A \cdot \operatorname{adj}(A) = \frac{1}{\det(A)} \det(A) I_n$$
3. Simplify the right-hand side:
$$\frac{1}{\det(A)} A \cdot \operatorname{adj}(A) = I_n$$
4. By the definition of the inverse of matrices, a square matrix A of size $n \times n$ is invertible if and only if there exists a matrix B such that $AB = I_n$ and $BA = I_n$.
5. Let $B = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$. Then we have:
$$A \cdot B = A \cdot \left(\frac{1}{\det(A)} \cdot \operatorname{adj}(A) \right) = I_n$$
6. Since $A \cdot B = I_n$, this shows that A is invertible, and the inverse of A is:
$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$
7. **Otherwise:** If $\det(A) = 0$:
$$A \cdot B = A \cdot \left(\frac{1}{0} \cdot \operatorname{adj}(A) \right) \neq I_n$$
8. Therefore, if $\det(A) = 0$, A does not have an inverse because there does not exist a matrix B such that $A \cdot B = I_n$.

In conclusion, since we have shown that $A \cdot B = I_n$ with $B = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$, it follows that A is invertible if $\det(A) \neq 0$. Conversely, if $\det(A) = 0$, then A is not invertible because $A \cdot B \neq I_n$

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Coding Part:

```python
import numpy as np

def is_complete(label_matrix): #checks if all entries in label matrix are covered.
    for element in np.nditer(label_matrix):
        if element == 0:
            return False
    return True
```

```python
def operations(M): #core calculations for the rref,determinant, and inverse functions below.
    M = M.astype(float)
    A = M.copy() #A is the matrix to be converted to RREF
    # label_matrix is responsible for checking if each entry is covered or not.
    label_matrix = np.zeros_like(A)
    identity_matrix = np.identity(4)
    possible_inverse_matrix = identity_matrix #inverse starts as identity then is reduced to the inverse if det != 0


    #track changes from Type-1 and Type-2 EROS to calculate the determinant later.
    row_swap_num = 0
    determinant = 1

    # head is defined to be the index uppermost element in a "non-covered" column.
    while not is_complete(label_matrix):
        row, col = label_matrix.shape
        found_head = False

        # Find the head (the first non-covered element in a non-covered column)
        for j in range(col):
            for i in range(row):
                if label_matrix[i, j] == 0:
                    head_x, head_y = i, j
                    found_head = True
                    break
            if found_head:
                break
```

```python
    # Find the first non-zero element in the column
    for tail_x in range(head_x, row):
        if A[tail_x, head_y] != 0:
            break

    if A[tail_x, head_y] == 0:
        label_matrix[:, head_y] = 1  # Mark the entire column as covered
    else:
        # Type-1 REF: switch rows head_x and tail_x
        if head_x != tail_x:
            A[[head_x, tail_x]] = A[[tail_x, head_x]]
            possible_inverse_matrix[[head_x, tail_x]] = possible_inverse_matrix[[tail_x, head_x]]
            row_swap_num += 1

        # Type-2 REF: normalize the row so that the leading coefficient is 1
        determinant = determinant * A[head_x, head_y]
        possible_inverse_matrix[head_x] = possible_inverse_matrix[head_x] / A[head_x, head_y]
        A[head_x] = A[head_x] / A[head_x, head_y]

        # Type-3 REF: eliminate all other entries in the column
        for i in range(row):
            if i != head_x:
                possible_inverse_matrix[i] = possible_inverse_matrix[i] - A[i, head_y] * possible_inverse_matrix[head_x]
                A[i] = A[i] - A[i, head_y] * A[head_x]

        # Update determinant based on number of row swaps.
        if(row_swap_num % 2 != 0): #odd
            determinant = -determinant

        # Mark the row and column as covered
        label_matrix[:, head_y] = 1
        label_matrix[head_x, :] = 1

# Multiply the gathered constants in determinant calculation by the main diagonal since it is now an upper triangle
for i in range(row):
    for j in range(col):
        if i == j:
            determinant = determinant*A[i,j]

return A, determinant, possible_inverse_matrix
```

```python
def RREF(M):
    rref,_,_ = operations(M)
    return rref
def determinant(M):
    _,determinant,_ = operations(M)
    return determinant
def inverse(M):
    if determinant(M) == 0:
        print("Matrix is Singular, therefore it is not invertible.")
    else:
        _,_,inverse_matrix = operations(M)
        return inverse_matrix
```

```
# Main function
matrixA = np.zeros((4,4))
print("Enter a 4x4 matrix: ")
for i in range(4):
    for j in range(4):
        matrixA[i,j] = input(f"Entry at position {i+1,j+1}:\n")


print(f"RREF is \n{RREF(matrixA)}\n")
print(f"Determinant is \n{determinant(matrixA)}\n")
print(f"Inverse is \n{inverse(matrixA)}\n")



#---------the code below is for testing purposes -----------#

#you can test rref using an online rref calculator or use sympy library.
# print(f"Determinant is \n{np.linalg.det(matrixA)}\n")
# print(f"Inverse is \n{np.linalg.inv(matrixA)}\n")
```