Positional Notation

1 Numbers and Notation

We begin by trying to justify a distiction between numbers and their notation. If you are asked to write a random number, you might write "10." That peice of writing is not, itself, the number you are thinking of. Rather, the symbols you write *refer* to a certain number. By "symobls," I mean certain shapes which are given meaning. Letters, and other glyphs are all symbols. To see this reference-base relationship between symbols and numbers, consider that the same number could be written differently. Using Roman numerals, the symobl "X" refers to the same number as "10." Alternatively, notice that different languages use unique speech to speak of the same number. We don't say that an enligh-speaker's "ten" and a french-speaker's "dix" are different numbers. Instead, we know that English and French have different methods of referring to the same number.

So, there is a difference between a number and how it is written and said. When a number is written or spoken, there is a reference to the number. A system for writting representations of something is sometimes called a "notation."

We can think of the reference-based relationship between notations and numbers with functions. For example, consider the set of symbols $\mathbb{S}$ of some notation which represents real numbers. For this notation to be useful, there must be some function $f : \mathbb{S} \to \mathbb{R}$ which gives the number refered to for each symbol.

Here, we invesitgate positional notation.

1.1 Notation Requirement

In our case, it is impossible to consider numbers without notation systems. As I write equations here, I must use some writing to insert numbers into your mind. When I write "0," you read that as zero and then the number appears in your mind. There is no other way to do this in writing other than to use a system which you and I both understand. I commonly use what is called the decimal system.

Here, I intend to dicsuss symbols such as "0" separately from the numbers they bring up in your mind; the numbers they reference. But then, how should I bring up the actual number referenced by "0?" To do so, let us define a new notation for numbers. This will be the same system as we have always used, but with bars on top of the symbols. For example, the symbols $\overline{34}$ refer to the same number as 34. I will carefully use this system for when a symbol should refer to a number. Thus, when writing "45," I am not trying to refer to any number, but am just writing certain symbols.

2 The Decimal System

One approach to discuss positional notation is to outright define it. In this section, I add some motivaion to the definition. We will analyze how we write numbers to get some insight in the sytem we use.

Consider writing down numbers successively. Only worrying about the natural numbers $\mathbb{N}$, we start with zero ($\overline{0}$) and write each number on a sheet of paper. The first few numbers are written with single symbols:

$$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9) \tag{1}$$

Once we have reached nine, how do we write the next number? We combine a "1" with a "0." Next, we keep the first "1," and iterate the second symbol through the sybols in (1):

$$\mathbf{1}0 \to \mathbf{1}1 \to \mathbf{1}2 \to \cdots \to \mathbf{1}9 \tag{2}$$

After we exaust all posiibilities for the second symbol, we increment the first:

$$\mathbf{2}0 \to \mathbf{2}1 \to \mathbf{2}2 \to \cdots \to \mathbf{2}9 \tag{3}$$

And so on until we reach "99," at which point we add a third symbol and return the others to "0":

$$\mathbf{9}9 \to \mathbf{10}0 \to \mathbf{10}1 \to \mathbf{10}2 \to \dots \tag{4}$$

Notice now that as we continue, we keep the first two symbols in place until we exaust all possibilities for the last symbol.

In general, we start with a single symbol that incrments through the possibilities in (1). Then, add a symbol in front and keep it in place as the second symbol increments. Once a symbol has gone through its possibilities, work on the one before it. If there isn't one before, it, add one.

Notice that there are $\overline{10}$ symbols in (1). So, once we've exausted a single symbol, we have gone through $\overline{10}$ elements of the natural numbers. That is, we have written symbols for the numbers from $\overline{0}$ to $\overline{9}$. Adding a second symbol in our writing gets us to the number $\overline{10}$. For each possibility of the new symbol, we cycle through each of the $\overline{10}$ possibilities from (1), so that "99" represents the $\left(\overline{10} + \overline{9} \cdot \overline{10}\right)^{\text{th}}$ element $\mathbb{N}$ (the number $\overline{99}$ since $\mathbb{N}$ starts with $\overline{0}$).

When we get to three symbols, each increment of the first has $\overline{100}$ possibilities for the second and third symbols ("00" to "99"). This follows from the previous paragraph, with the addition of a leading "0" to the first $\overline{10}$ number representations. Thus, "x00," where is one of our symbols, represnts the number $\overline{x} \cdot \overline{100}$.

We can generalize from here. The addition of a new symbol gives the chance to cycle through the possibilites of the other symbols with a leading "0." For example, to get to "4321", we've gone through each possibility of the first symbol from 1 to 4, each possibility of the second symbol from 1 to 3, and so on.

Since we are working with ten symbols, each additional symbol in a representation multiplies the possible representations by ten.

This exploration has shown us that in our common notation system, we use the order of symbols to represent greater and greater numbers. Us having used ten symbols is an arbitrary descision, though. This is addressed by the definitions in the next section.

3 Positional Notation

In this section, we give a more rigorous definition of positional notation. We start by defining numeral systems in general. To do that, we must define glyphs. This is a rather philosohpical task. In this context, we take a glyph to mean something like a written "2" or that same shape on a screen. We are focused mainly on writting, but even a spoken word can be a glyph. This allows combinations of words to represent numbers in the same was as written glyphs.

Definition 5. *A glyph is an object.*

Remark. This is an extremely broad defintion, but I find it necessary when considering, in the broadest of terms, what can be a glyph. In most cases, glyphs are written (or displayed, for computer screens and in general) or spoken objects. When displayed, a glyph is a goemtric shape that has certain distinguishing characteristics (e.g. the circular nature of "0"). When spoke, a glyph is a series of pressure waves in a medium that, when they affect an ear drum, are understood, given some language knowledge of the hearer.

These cover most cases, but any object could be a glyph. There is nothing stopping us from calling, say, a dog a glyph. The dog, in a certain numeral system, might repsresent the number $\overline{22}$. Even a dream could be chosen to represent a certain number.

Definition 6. *A numeral system for a set of numbers X is a tuple*

$$(G, V)$$

where G is a set of glyphs and V is a function from (tuples of any length (except empty tuples) containing only elements of G) to X.

With the above, we can now define positional notation.

Definition 7. *Let $b \in \mathbb{N}^*$, and G_b be a b-ary tuple of glyphs. Then base b positional notation is a numeral system with $X = \mathbb{R}$, $G = G_b$ and V defined by*