

Positional Notation

1 Numbers and Notation

We begin by trying to justify a distinction between numbers and their notation. If you are asked to write a random number, you might write “10.” That piece of writing is not, itself, the number you are thinking of. Rather, the symbols you write *refer* to a certain number. By “symbols,” I mean certain shapes which are given meaning. Letters, and other glyphs are all symbols. To see this reference-based relationship between symbols and numbers, consider that the same number could be written differently. Using Roman numerals, the symbol “X” refers to the same number as “10.” Alternatively, notice that different languages use unique speech to speak of the same number. We don’t say that an English-speaker’s “ten” and a French-speaker’s “dix” are different numbers. Instead, we know that English and French have different methods of referring to the same number.

So, there is a difference between a number and how it is written and said. When a number is written or spoken, there is a reference to the number. A system for writing representations of something is sometimes called a “notation.”

We can think of the reference-based relationship between notations and numbers with functions. For example, consider the set of symbols \mathbb{S} of some notation which represents real numbers. For this notation to be useful, there must be some function $f : \mathbb{S} \rightarrow \mathbb{R}$ which gives the number referred to for each symbol.

Here, we investigate positional notation.

1.1 Notation Requirement

In our case, it is impossible to consider numbers without notation systems. As I write equations here, I must use some writing to insert numbers into your mind. When I write “0,” you read that as zero and then the number appears in your mind. There is no other way to do this in writing other than to use a system which you and I both understand. I commonly use what is called the decimal system.

Here, I intend to discuss symbols such as “0” separately from the numbers they bring up in your mind; the numbers they reference. But then, how should I bring up the actual number referenced by “0?” To do so, let us define a new notation for numbers. This will be the same system as we have always used, but with bars on top of the symbols. For example, the symbols $\overline{34}$ refer to the same number as 34. I will carefully use this system for when a symbol should refer to a number. Thus, when writing “45,” I am not trying to refer to any number, but am just writing certain symbols.

2 The Decimal System

One approach to discuss positional notation is to outright define it. In this section, I add some motivation to the definition. We will analyze how we write numbers to get some insight in the system we use.

Consider writing down numbers successively. Only worrying about the natural numbers \mathbb{N} , we start with zero, $\overline{0}$ and write each number on a sheet of paper. The first few numbers are written with single symbols:

$$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9) \tag{1}$$

Once we have reached nine, how do we write the next number? We combine a “1” with a “0.” Next, we keep the first “1,” and iterate the second symbol through the symbols in (1):

$$10 \rightarrow 11 \rightarrow 12 \rightarrow \dots \rightarrow 19 \quad (2)$$

After we exhaust all possibilities for the second symbol, we increment the first:

$$20 \rightarrow 21 \rightarrow 22 \rightarrow \dots \rightarrow 29 \quad (3)$$

And so on until we reach “99,” at which point we add a third symbol and return the others to “0”:

$$99 \rightarrow 100 \rightarrow 101 \rightarrow 102 \rightarrow \dots \quad (4)$$

Notice now that as we continue, we keep the first two symbols in place until we exhaust all possibilities for the last symbol.

In general, we start with a single symbol that increments through the possibilities in (1). Then, add a symbol in front and keep it in place as the second symbol increments. Once a symbol has gone through its possibilities, work on the one before it. If there isn’t one before, it, add one.

Notice that there are $\overline{10}$ symbols in (1). So, once we’ve exhausted a single symbol, we have gone through $\overline{10}$ elements of the natural numbers. That is, we have written symbols for the numbers from $\overline{0}$ to $\overline{9}$. Adding a second symbol in our writing gets us to the number $\overline{10}$. For each possibility of the new symbol, we cycle through each of the $\overline{10}$ possibilities from (1), so that “99” represents the $(\overline{10} + \overline{9} \cdot \overline{10})^{\text{th}}$ element \mathbb{N} (the number $\overline{99}$ since \mathbb{N} starts with $\overline{0}$).

When we get to three symbols, each increment of the first has $\overline{100}$ possibilities for the second and third symbols (“00” to “99”). This follows from the previous paragraph, with the addition of a leading “0” to the first $\overline{10}$ number representations. Thus, “x00,” where x is one of our symbols, represents the number $\overline{x} \cdot \overline{100}$.

We can generalize from here. The addition of a new symbol gives the chance to cycle through the possibilities of the other symbols with a leading “0.”