

# Report

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## 1 Introduction

Decoder is one of the key components of Multiple-Input Multiple-Output (MIMO) systems.

Designing of high performance and low complexity detector has become a bottleneck of Large

MIMO systems. Recently, an interesting property of large MIMO channel called channel

hardening phenomenon has been found [1] [2] [3] [4], which states the phenomenon that

with the number of receiving or transmitting antennas increasing, the variances of channel

mutual information decrease. Simply saying, the channel "hardens". An useful aspect of

channel hardening phenomenon is that off diagonal elements of  $\mathbf{H}^H\mathbf{H}$  matrix becomes more and more ignorable comparing to diagonal elements when the number of receiving antennas or transmitting antennas becomes large.

Channel hardening phenomenon provides an opportunity for designing computationally economical detection algorithms. For example, linear detectors (LDs) such as zero forcing (ZF) and minimum mean square error (MMSE) detectors need to do costly matrix operations such as multiplication and inverse. With channel hardening phenomenon, this process can be accelerated by approximate matrix inversion using series expansion techniques and deterministic approximations in large dimensions [5]. Channel hardening phenomenon can also provide computational advances to machine learning based detectors, e.g. probability data association (PDA) [6] [7] and message passing detectors [8] [9] [10] and Monte-Carlo Markov Chain (MCMC) MIMO detectors [11] [12]. This type of algorithms work well in large sparse systems.

In this report, we propose some useful insights of the orthogonal properties of channel. We define orthogonality measurement (om), which is a well round metric that considers both diagonal and off diagonal elements of  $\mathbf{H}^H\mathbf{H}$ .

The rest of the report is organized as follows, section 2 introduces system model. Section

3 provides some preliminaries. Section 4 discusses the derivation of logarithmic expectation of orthogonality measurement. In section 5 we obtain probability density function of orthogonality measurement. Some computer simulation results are presented in section 6.

## 2 System Model

We consider a complex uncoded spatial multiplexing MIMO system with  $N_r$  receive and  $N_t$  transmit antennas,  $N_r \geq N_t$ , over a flat fading channel. Using a discrete time model,  $\mathbf{y} \in \mathbb{C}^{N_r \times 1}$  is the received symbol vector written as:

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}, \quad (1)$$

where  $\mathbf{s} \in \mathbb{C}^{N_t \times 1}$  is the transmitted symbol vector, with components that are mutually independent and taken from a finite signal constellation alphabet  $\mathbb{O}$  (e.g. 4-QAM, 16-QAM, 64-QAM) of size  $M$ . The possible transmitted symbol vectors  $\mathbf{s} \in \mathbb{O}^{N_t}$ , satisfy  $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathbf{I}_{N_t}E_s$ , where  $E_s$  denotes the symbol average energy, and  $\mathbb{E}[\cdot]$  denotes the expectation operation. Furthermore  $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$  denotes the Rayleigh fading channel propagation matrix with independent identically distributed (i.i.d) circularly symmetric complex Gaussian components

with unit variance. Finally,  $\mathbf{n} \in \mathbb{C}^{N_r \times 1}$  is the additive white Gaussian noise (AWGN) vector with zero mean components and  $\mathbb{E}[\mathbf{n}\mathbf{n}^H] = \mathbf{I}_{N_r}N_0$ , where  $N_0$  denotes the noise power spectrum density, and hence  $\frac{E_s}{N_0}$  is the signal to noise ratio (SNR).

Assume the receiver has perfect channel state information (CSI), meaning that  $\mathbf{H}$  is known, as well as the SNR. The task of the MIMO decoder is to recover  $\mathbf{s}$  based on  $\mathbf{y}$  and  $\mathbf{H}$ .

### 3 Preliminaries

Orthogonality deficiency measures the how orthogonal a matrix is [13], which is defined by

$$\phi_{od} = 1 - \frac{\det(\mathbf{W})}{\prod_{i=1}^{N_t} \|\mathbf{h}_i\|^2}, \quad (2)$$

where  $\mathbf{W} = \mathbf{H}^H \mathbf{H}$  denotes Wishart matrix,  $\mathbf{h}_i$  denotes the  $i$  th column of  $\mathbf{H}$ ,  $\det(\cdot)$  denotes determinant operation,  $\|\cdot\|^2$  denotes 2-norm operation. In (2),  $\|\mathbf{h}_i\|^2 = \sum_{j=1}^{N_r} |\mathbf{H}_{ji}|^2$ ,  $\mathbf{H}_{ji}$  denotes the component of  $\mathbf{H}$  at  $j$  th row and  $i$  th column.  $|\mathbf{H}_{ji}| \sim \text{Rayleigh}(1/\sqrt{2})$ , therefore  $\|\mathbf{h}_i\|^2 \sim \text{Gamma}(N_r, 1)$  [14].  $\text{Gamma}(k, \theta)$  denotes Gamma distribution, with  $k$  degrees of

freedom and scale  $\theta$ . Furthermore, we have:

$$2\|\mathbf{h}_i\|^2 \sim \text{Gamma}(N_r, 2) \sim \chi_{2N_r}^2, \quad (3)$$

$\chi_k^2$  denotes chi-square distribution with  $k$  degrees of freedom. For the sake of simplicity, (2)

can be changed to:

$$\phi_{om} = \frac{\det(\mathbf{W})}{\prod_{i=1}^{N_t} \|\mathbf{h}_i\|^2} = \frac{2^{N_t} \det(\mathbf{W})}{\prod_{i=1}^{N_t} 2\|\mathbf{h}_i\|^2}. \quad (4)$$

Taking logarithmic operation to  $\phi_{om}$  we have

$$\ln(\phi_{om}) = N_t \ln(2) + \ln(\det(\mathbf{W})) - \sum_{i=1}^{N_t} \ln(2\|\mathbf{h}_i\|^2), \quad (5)$$

$\phi_{om}$  in (4) is defined as Orthogonality Measure. Based on Hadamard's inequality ( $\prod_{i=1}^{N_t} \|\mathbf{h}_i\| \geq$

$\det(\mathbf{H})$ ).  $\phi_{om} \in [0, 1]$ . If  $\phi_{om}$  is more closer to 1,  $\mathbf{H}$  is closer to orthogonal matrix.

Because  $\mathbf{W} = \mathbf{H}^H \mathbf{H}$ , do QR factorization to  $\mathbf{H}$

$$\mathbf{H} = \mathbf{Q}\mathbf{R}, \quad (6)$$

where  $\mathbf{Q} \in \mathbb{C}^{N_r \times N_t}$  is a unitary matrix and  $\mathbf{R} \in \mathbb{C}^{N_t \times N_t}$  is the upper triangular matrix.

Using (6), we have  $\mathbf{W} = \mathbf{R}^H \mathbf{R}$ .  $r_{ii}$  denotes the  $i$  th diagonal component of  $\mathbf{R}$ , thus  $\det(\mathbf{W})$

can be rewritten as:

$$\det(\mathbf{W}) = \det(\mathbf{R}^H \mathbf{R}) = \det(\mathbf{R}^H) \det(\mathbf{R}) = \prod_{i=1}^{N_t} r_{ii}^H \prod_{i=1}^{N_t} r_{ii} = \prod_{i=1}^{N_t} |r_{ii}|^2. \quad (7)$$

Notice that  $\mathbf{R}$  can be viewed as the Cholesky factorization of  $\mathbf{W}$ . Therefore, we have

$$\|\mathbf{h}_i\|^2 = \mathbf{W}_{ii} = \sum_{j=1}^{i-1} |r_{ji}|^2 + |r_{ii}|^2, \quad (8)$$

where  $\mathbf{W}_{ii}$  denotes the  $i$  th diagonal element of  $\mathbf{W}$ . Thus based on (7) and (8), (4) can be

rewritten as:

$$\phi_{om} = \prod_{i=1}^{N_t} \frac{|r_{ii}|^2}{|r_{ii}|^2 + \sum_{j=1}^{i-1} |r_{ji}|^2}. \quad (9)$$

## 4 Logarithmic Expectation of Orthogonality Measurement

Taking expectation of (5), we have

$$\mathbb{E}[\ln(\phi_{om})] = N_t \ln(2) + \mathbb{E}[\ln(\det(\mathbf{W}))] - \sum_{i=1}^{N_t} \mathbb{E}[\ln(2\|\mathbf{h}_i\|^2)]. \quad (10)$$

Consider  $\mathbf{H} = [\mathbf{h}'_1, \mathbf{h}'_2, \dots, \mathbf{h}'_{N_t}]'$ , where  $\mathbf{h}_i$  denotes the  $i$  th row of  $\mathbf{H}$ , because each component of  $\mathbf{H}$  is mutually independent and subject to circularly symmetric complex Gaussian distribution, i.e.  $h_i \sim \mathbb{CN}(\mathbf{0}, \mathbf{I}_{N_t})$ . Therefore,  $\mathbf{W} = \mathbf{H}^H \mathbf{H} \sim \mathbb{CW}(N_r, \mathbf{I}_{N_t})$ ,  $\mathbb{CW}(n, \mathbf{\Sigma})$  denotes complex Wishart distribution with  $n$  degrees of freedom and covariance matrix  $\mathbf{\Sigma}$ . The logarithmic expectation of  $\mathbf{W}$  can be rewritten as

$$\mathbb{E}[\ln(\det(\mathbf{W}))] = \frac{\tilde{\Gamma}'_{N_t}(N_r)}{\tilde{\Gamma}_{N_t}(N_r)} = \sum_{i=1}^{N_t} \psi(N_r - i + 1), \quad (11)$$

where  $\tilde{\Gamma}_m(n)$  denotes the multivariate Gamma function. Proof: see Appendix A.

Because the logarithmic expectation of a Gamma distribution variable  $\tilde{\vartheta} \sim \text{Gamma}(n, \theta)$

can be written as:

$$\mathbb{E}[\ln(\tilde{\theta})] = \psi(n) + \ln(\theta), \quad (12)$$

where  $\psi(n)$  denotes Digamma function. Thus according to (3), we have:

$$\mathbb{E}[\ln(2||\mathbf{h}_i||^2)] = \psi(N_r) + \ln(2). \quad (13)$$

Proof: see Appendix B.

Based on (10)(11)(13), The logarithmic expectation of  $\phi_{om}$  can be written as:

$$\begin{aligned} \mathbb{E}[\ln(\phi_{om})] &= N_t \ln(2) + \sum_{i=1}^{N_t} \psi(N_r - i + 1) - N_t \psi(N_r) - N_t \ln(2) \\ &= \sum_{i=1}^{N_t} \psi(N_r - i + 1) - N_t \psi(N_r) \end{aligned} \quad (14)$$

## 5 Probability Density Function of Orthogonality Measurement

Recall (9)

$$\phi_{om} = \prod_{i=1}^{N_t} \frac{|r_{ii}|^2}{|r_{ii}|^2 + \sum_{j < i} |r_{ji}|^2}. \quad (15)$$



All the components in  $\mathbf{R}$  are independently distributed and  $r_{ji} \sim \mathbb{CN}(0, 1)$ ,  $|r_{ii}|^2 \sim \text{Gamma}(N_r - i + 1, 1)$  [15]. Because  $|r_{ji}| \sim \text{Rayleigh}(1/\sqrt{2})$ ,  $\sum_{j < i} |r_{ji}|^2 \sim \text{Gamma}(i - 1, 1)$ . Defining  $\alpha_i = \sum_{j < i} |r_{ji}|^2$  and  $\beta_i = |r_{ii}|^2$ ,  $\alpha_i$  and  $\beta_i$  are mutually independent, therefore (9) can be rewritten as

$$\phi_{om} = \prod_{i=1}^{N_t} \frac{\beta_i}{\beta_i + \alpha_i}, \quad (16)$$

From [16], if  $X \sim \text{Gamma}(k_1, \theta)$  and  $Y \sim \text{Gamma}(k_2, \theta)$ , then  $\frac{X}{X+Y} \sim B(k_1, k_2)$ , where  $B$  denotes Beta distribution. Therefore  $\frac{\beta_i}{\beta_i + \alpha_i} \sim B(k_1^i, k_2^i)$ , where  $k_1^i = N_r - i + 1$ ,  $k_2^i = i - 1$ . we define  $\eta_i = \frac{\beta_i}{\beta_i + \alpha_i}$ , it is obvious that  $\eta_i$  are independently distributed. Based on (16), we have

$$\phi_{om} = \prod_{i=1}^{N_t} \eta_i. \quad (17)$$

Therefore the density function of  $\phi_{om}$  can be defined as

$$f_{\phi_{om}}(x) = \frac{1}{x} \sum_{\mathbf{j}} \left( \prod_{i=1}^{N_t} c(k_1^i, k_2^i, j^i) \right) f(\ln(x) | \mathbf{k}_1 + \mathbf{j}), \quad (18)$$

where  $\sum_{\mathbf{j}} = \sum_{j^1} \sum_{j^2} \cdots \sum_{j^{N_t}}$ , the range of  $j^i \in [0, k_2^i - 1]$ ,  $c(k_1^i, k_2^i, j^i) = (-1)^{j^i} \binom{k_2^i - 1}{j^i} [(k_1^i + k_2^i) \mathbb{B}(k_1^i, k_2^i)]^{-1}$ ,  $\mathbb{B}(\alpha, \beta)$  denotes beta function.  $f(\ln(x) | \mathbf{k}_1 + \mathbf{j}) = (\prod_{i=1}^{N_t} (k_1^i + j^i)) \sum_{i=1}^{N_t} [\exp((k_1^i + j^i) \ln(x)) / \prod_{j=1, j \neq i}^{N_t} (k_1^j + j^j - k_1^i - j^i)]$ .  $\mathbf{k}_1 + \mathbf{j} = [k_1^1 + j^1, \dots, k_1^{N_t-1} + j^{N_t-1}, k_1^{N_t} + j^{N_t}]$ . Proof:

see Appendix C.

Consider logarithmic expectation of  $\phi_{om}$ , we have

$$E[\ln(\phi_{om})] = \sum_{i=1}^{N_t} E[\ln(\eta_i)], \quad (19)$$

where  $E[\ln(\eta_i)] = \psi(k_1^i) - \psi(k_1^i + k_2^i)$ , thus we have

$$E[\ln(\phi_{om})] = \sum_{i=1}^{N_t} \psi(N_r - i + 1) - N_t \psi(N_r). \quad (20)$$

we can find (20) is consistent with (14).

## 6 Computer Simulations

Computer simulations are made for different sizes of V-BLAST MIMO systems, with  $5 \leq N_r \leq 100, 5 \leq N_t \leq N_r$ , the empirical estimation of logarithmic expectation of  $\phi_{om}$ ,  $E[\ln(\phi_{om})]_{em}$ , is calculated by taking average over  $1e4$  channel realizations for each size of MIMO systems, as shown in Fig.1, the Theoretical logarithmic expectation of  $\phi_{om}$   $E[\ln(\phi_{om})]_t$  in (20) is plotted in Fig.2. Average deviation between  $E[\ln(\phi_{om})]_{em}$  and  $E[\ln(\phi_{om})]_t$  is also calculated,  $V_{em-t} = 7.3043e - 04$ .

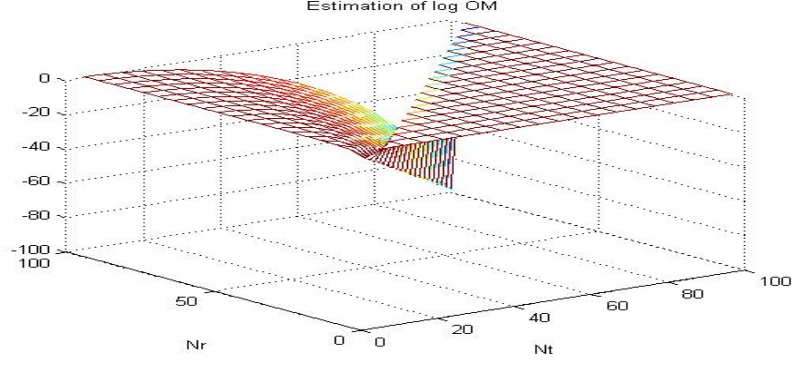


Figure 1: Empirical Estimation  $E[\ln(\phi_{om})]_{em}$

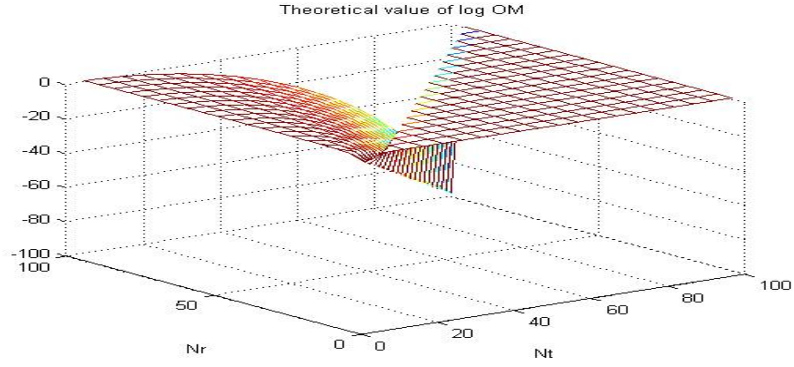


Figure 2: Theoretical  $E[\ln(\phi_{om})]_t$

Fig.3 demonstrates the relation between the number of users ( $N_t$ ) and  $E[\ln(\phi_{om})]_t$  under cases of different numbers of antennas at base station ( $N_r$ ). From Fig.3, we can see, on the one hand, with  $N_r$  fixed,  $E[\ln(\phi_{om})]$  decreases while  $N_t$  increases, however the gradient of each curve becomes more and more gentle. On the other hand, when  $N_r$  becomes larger  $E[\ln(\phi_{om})]$  becomes more insensitive to variation of  $N_t$ .

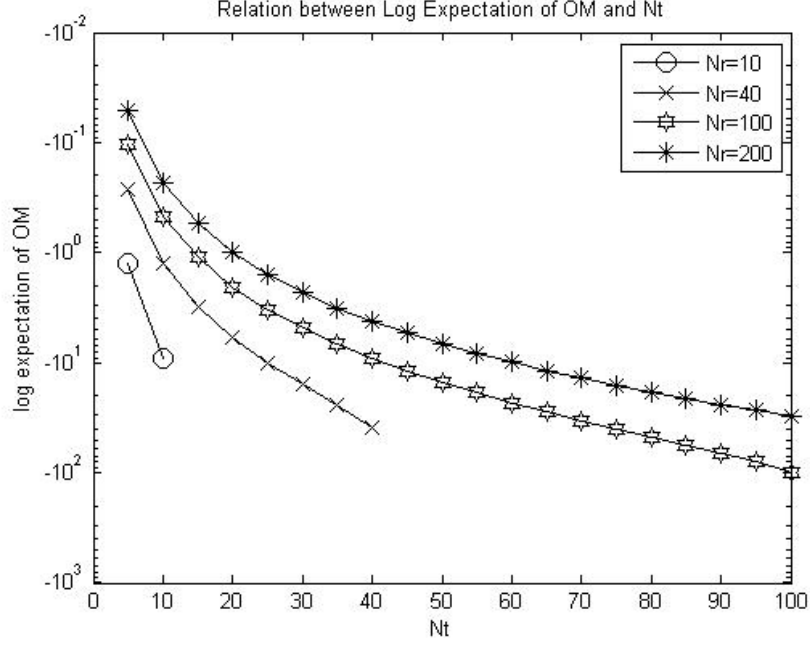


Figure 3: Relation between  $N_t$  and  $E[\ln(\phi_{om})]_t$

## A Appendix A

Let  $\mathbf{A} \in \mathbb{C}^{m \times m}$ ,  $A \sim \mathbb{CW}(n, \Sigma)$ ,  $\mathbb{CW}(n, \Sigma)$  denotes complex Wishart distribution with  $n$  degrees of freedom and covariance matrix  $\Sigma$ . It is obvious  $\mathbf{A}$  is Hermitian positive definite matrix,  $\mathbf{A} = \mathbf{A}^H > 0$ .

The pdf of  $\mathbf{A}$  can be written as [15]:

$$f(\mathbf{A}) = \{\tilde{\Gamma}_m(n) \det(\Sigma)^n\}^{-1} \det(\mathbf{A})^{n-m} \text{etr}(-\Sigma^{-1} \mathbf{A}), \quad (21)$$

where  $\tilde{\Gamma}_m(\beta)$  denotes multivariate complex Gamma function defined by:

$$\tilde{\Gamma}_m(\beta) = \pi^{\frac{m(m-1)}{2}} \prod_{i=1}^m \Gamma(\beta - i + 1) \quad \text{Re}(\beta) > m - 1. \quad (22)$$

Furthermore, from [15], we have

$$\tilde{\Gamma}_m(\beta) = \int_{\mathbf{X}=\mathbf{X}^H>0} \text{etr}(-\mathbf{X}) \det(\mathbf{X})^{\beta-m} d\mathbf{X} \quad \text{Re}(\beta) > m - 1. \quad (23)$$

We derive logarithmic expectation of  $\det(\mathbf{A})$

$$\begin{aligned} E[\ln(\det(\mathbf{A}))] &= \int_{\mathbf{A}=\mathbf{A}^H>0} \ln(\det(\mathbf{A})) f(\mathbf{A}) d\mathbf{A} \\ &= \int_{\mathbf{A}=\mathbf{A}^H>0} \ln(\det(\mathbf{A})) \{\tilde{\Gamma}_m(n) \det(\boldsymbol{\Sigma})^n\}^{-1} \det(\mathbf{A})^{n-m} \text{etr}(-\boldsymbol{\Sigma}^{-1} \mathbf{A}) d\mathbf{A} \\ &= \frac{\det(\boldsymbol{\Sigma})^{-n}}{\tilde{\Gamma}_m(n)} \int_{\mathbf{A}=\mathbf{A}^H>0} \ln(\det(\mathbf{A})) \det(\mathbf{A})^{n-m} \text{etr}(-\boldsymbol{\Sigma}^{-1} \mathbf{A}) d\mathbf{A}, \end{aligned} \quad (24)$$

if  $\boldsymbol{\Sigma} = \mathbf{I}$ , (24) can be written as

$$E[\ln(\det(\mathbf{A}))] = \frac{1}{\tilde{\Gamma}_m(n)} \int_{\mathbf{A}=\mathbf{A}^H>0} \ln(\det(\mathbf{A})) \det(\mathbf{A})^{n-m} \text{etr}(-\mathbf{A}) d\mathbf{A}. \quad (25)$$

Because  $\frac{d}{dn}[\det(\mathbf{A})]^{n-m} = \ln(\det(\mathbf{A}))\det(\mathbf{A})^{n-m}$ , (25) can be rewritten as

$$E[\ln(\det(\mathbf{A}))] = \frac{1}{\tilde{\Gamma}_m(n)} \frac{d}{dn} \int_{\mathbf{A}=\mathbf{A}^H>0} \text{etr}(-\mathbf{A}) \det(\mathbf{A})^{n-m} d\mathbf{A}, \quad (26)$$

using (23), (26) can be rewritten as

$$E[\ln(\mathbf{A})] = \frac{\tilde{\Gamma}'_m(n)}{\tilde{\Gamma}_m(n)}. \quad (27)$$

Based on (22), we have

$$\tilde{\Gamma}'_m(n) = \pi^{\frac{m(m-1)}{2}} \sum_{i=1}^m [\Gamma'(n-i+1) \prod_{j=1, j \neq i}^m \Gamma(n-j+1)], \quad (28)$$

Thus we have

$$E[\ln(\det(\mathbf{A}))] = \frac{\tilde{\Gamma}'_m(n)}{\tilde{\Gamma}_m(n)} = \sum_{i=1}^m \frac{\Gamma'(n-i+1)}{\Gamma(n-i+1)} = \sum_{i=1}^m \psi(n-i+1), \quad (29)$$

where  $\psi$  denotes Digamma function.

## B Appendix B

If  $x \sim \text{Gamma}(n, \theta)$ , with shape parameter  $k$  and scale parameter  $\theta$ ,  $x > 0$ ,  $\Gamma(k)$  denotes

Gamma function, the density function of Gamma distribution is

$$f(x, k, \theta) = \frac{x^{k-1} e^{-x/\theta}}{\Gamma(k) \theta^k}. \quad (30)$$

Thus we have

$$E[\ln(x)] = \frac{1}{\Gamma(k)} \int_0^\infty \ln(x) x^{k-1} e^{-x/\theta} \theta^{-k} dx, \quad (31)$$

define  $z = x/\theta$  and since  $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$ , (31) can be rewritten as

$$E[\ln(x)] = \ln(\theta) + \frac{1}{\Gamma(k)} \int_0^\infty \ln(z) z^{k-1} e^{-z} dz. \quad (32)$$

Because  $\frac{d(z^{k-1})}{dk} = \ln(z) z^{k-1}$ , (32) can be rewritten as

$$\begin{aligned} E[\ln(z)] &= \ln(\theta) + \frac{1}{\Gamma(k)} \frac{d}{dk} \int_0^\infty z^{k-1} e^{-z} dz \\ &= \ln(\theta) + \frac{\Gamma'(k)}{\Gamma(k)} \\ &= \ln(\theta) + \psi(k), \end{aligned}$$

where  $\psi(k)$  denotes Digamma function.

## C Appendix C

$x_1, x_2, \dots, x_{N_t}$  are independent beta variables, the probability density function (pdf) can be written as:

$$f(x_i) = \frac{1}{\mathbb{B}(k_1^i, k_2^i)} x_i^{k_1^i-1} (1-x_i)^{k_2^i-1}, \quad (33)$$

define  $y_i = -\ln(x_i) = g(x_i)$ , Based on Jacobian transformation, we have

$$f_{y_i}(\rho) = \left| \frac{dy_i}{dx_i} \right|^{-1} f_{x_i}(g^{-1}(\rho)) = \frac{1}{\mathbb{B}(k_1^i, k_2^i)} e^{-k_1^i \rho} (1 - e^{-\rho})^{k_2^i-1}. \quad (34)$$

where (34) can be alternatively expressed as [17]

$$f_{y_i}(\rho) = \sum_{j^i=0}^{k_2^i-1} c(k_1^i, k_2^i, j^i) (k_1^i + j^i) \exp(-(k_1^i + j^i)\rho), \quad (35)$$

where  $c(k_1^i, k_2^i, j^i) = (-1)^{j^i} \binom{k_2^i-1}{j^i} [(k_1^i + k_2^i) \mathbb{B}(k_1^i, k_2^i)]^{-1}$ ,  $\mathbb{B}(\alpha, \beta)$  denotes beta function. Based on the lemma 1 of [17], if  $a_1, a_2, \dots, a_n$  are independent exponentially distributed random



variables, with pdf given by

$$t_i \exp(-t_i a_i) \quad (36)$$

then pdf of  $a = \sum_{i=1}^n a_i$  can be written as

$$f(a|\mathbf{t}) = \prod_{i=1}^n t_i \sum_{i=1}^n [\exp(-t_i a) / \prod_{j=1, j \neq i}^n (t_j - t_i)], \quad (37)$$

where  $t = [t_1, t_2, \dots, t_n]$ . The pdf of  $y_i$  can be viewed as the weighting summation of exponential distribution functions, define  $y = \sum_{i=1}^n y_i$ , based on (37), the pdf of  $y$  is given by

$$f_y(m) = \sum_{\mathbf{j}} \{ [\prod_{i=1}^n c(k_1^i, k_2^i, j^i)] f(m|\mathbf{k}_1 + \mathbf{j}) \}, \quad (38)$$

where  $\sum_{\mathbf{j}} = \sum_{j^1} \sum_{j^2} \dots \sum_{j^n}$ , the range of  $j^i$  is defined by  $j^i \in [0, k_2^i]$ ,  $f(m|\mathbf{k}_1 + \mathbf{j}) = (\prod_{i=1}^{N_t} (k_1^i + j^i)) \sum_{i=1}^{N_t} [\exp(-(k_1^i + j^i)m) / \prod_{j=1, j \neq i}^{N_t} (k_1^j + j^j - k_1^i - j^i)]$ ,  $\mathbf{k}_1 + \mathbf{j} = [k_1^1 + j^1, k_1^2 + j^2, \dots, k_1^n + j^n]$ . we define  $U = \exp(-y) = \prod_{i=1}^n x_i$ , using Jacobian transformation, the pdf of  $U$  is given by

$$f_U(u) = \left| \frac{du}{dy} \right|^{-1} f_y(-\ln(u)) = \frac{1}{u} \sum_{\mathbf{j}} \{ [\prod_{i=1}^n c(k_1^i, k_2^i, j^i)] f(-\ln(u)|\mathbf{k}_1 + \mathbf{j}) \}. \quad (39)$$

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