Element-Based Lattice Reduction Algorithms for Large MIMO Detection

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***Abstract*—Large multi-input multi-output (MIMO) systems with tens or hundreds of antennas have shown great potential for next generation of wireless communications to support high spectral efficiencies. However, due to the non-deterministic polynomial hard nature of MIMO detection, large MIMO systems impose stringent requirements on the design of reliable and computationally efficient detectors. Recently, lattice reduction (LR) techniques have been applied to improve the performance of low-complexity detectors for MIMO systems without increasing the complexity dramatically. Most existing LR algorithms are designed to improve the orthogonality of channel matrices, which is not directly related to the error performance. In this paper, we propose element-based lattice reduction (ELR) algorithms that reduce the diagonal elements of the noise covariance matrix of linear detectors and thus enhance the asymptotic performance of linear detectors. The general goal is formulated as solving a "shortest longest vector reduction" or a stronger version, "shortest longest basis reduction," both of which require high complexity to find the optimal solution. Our proposed ELR algorithms find sub-optimal solutions to the reductions with low complexity and high performance. The fundamental properties of the ELR algorithms are investigated. Simulations show that the proposed ELR-aided detectors yield better error performance than the existing low-complexity detectors for large MIMO systems while maintaining lower complexity.**

***Index Terms*—Lattice reduction, linear detector, MIMO, orthogonality deficiency, Gaussian reduction algorithm**

I. Introduction

MULTI-INPUT MULTI-OUTPUT (MIMO) systems with multiple antennas have been adopted in modern wireless systems (e.g., IEEE 802.11ac/n). As the dramatic increase of the demands on data rate and throughput, the number of antennas has been scaled up to tens or hundreds to fulfill the performance requirements [1], [2]. However, a critical challenge of large MIMO systems is to design reliable and computationally efficient detectors. The well-known maximum likelihood detector (MLD) provides optimal error performance, but it suffers from exponential complexity in terms of the number of transmit antennas [3], [4]. To address the stringent needs of large MIMO detection, several machine learning-/artificial intelligence-based detectors that achieve near-optimal performance for large MIMO systems have been developed [5]–[8]. Two local neighborhood search methods–likelihood ascent search (LAS) [5] and reactive tabu search (RTS) [7] obtain near-optimal performance for BPSK or QPSK modulations but exhibit considerable performance degradation for higher-order QAM. To further improve performance for higher-order QAM, layered tabu search (LTS) was proposed in [8]; however, this algorithm involves considerably high complexity when the problem size and/or the constellation size is large.

For the sake of low-complexity detection, linear detectors (LDs) such as zero forcing (ZF) and minimum-mean-square-error (MMSE) LDs have been widely adopted because of their polynomial computational complexity. However, in MIMO systems, these detectors suffer from inferior performance compared to the MLD and lose diversity due to their sensitivity to ill-conditioned channel matrices. Recently, to bridge the performance gap between LDs and the MLD, [9]–[14] have proposed lattice reduction (LR) techniques that collect the same diversity as the MLD for MIMO systems with low complexity. In addition, compared to the search-based detectors in [5], [7], [8], LR-aided detector is beneficial in that its instantaneous complexity does not depend on constellation size and noise realizations, which is preferable for hardware implementations.

Technically, as finding the optimal basis of a lattice is computationally demanding, lattice reduction is generally a difficult task. For a two-dimensional system, the Gaussian reduction algorithm (GRA) is optimal in the sense that it generates the shortest basis vectors in a lattice. For the system with higher dimensions, several lattice reduction algorithms were proposed with various complexity and performance tradeoff [15]–[17]. Because of its polynomial complexity in the average case [18] and near-optimal performance, the Lenstra, Lenstra, and Lovász (LLL) algorithm [15] has been widely adopted. In addition, studies in [12], [13], [19] showed that LLL-aided LDs achieve the same diversity as the MLD for MIMO systems. The dual LLL (D-LLL) algorithm, which reduces the lattice in the dual space, was studied in [16], [20], which showed that the D-LLL-aided LDs achieve better performance than the LLL-aided LDs. As an alternative to the LLL algorithms, Seysen's algorithms (SAs) [11], [17], [21], [22] were proposed to minimize Seysen's metric, which quantifies the orthogonality of the basis in both primal and dual spaces. The equivalence of the SA and the GRA in two-dimensional systems has been shown in [23]. As illustrated in [22], the SA-aided LD provides better bit-error-rate (BER) performance than the LLL-aided LD, especially for large MIMO systems. However, SAs require higher computational complexity over the LLL algorithms mainly because of the high computational cost for each basis update [22], and LLL-

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aided LDs do not perform well when the problem size is large [14], [24]. For large MIMO systems, one is looking for LR techniques which further reduce the complexity while keeping or enhancing the performance. The element-based lattice reduction (ELR) algorithm was first introduced in [24]. However, there is no rigid problem formulation and deep analysis in [24]. This paper targets the general ELR problem, reveals the properties of ELR-reduced basis, thoroughly compares with the existing LR algorithms and other alternatives for large MIMO systems.

In this paper, we propose element-based lattice reduction (ELR) algorithms for large MIMO detection. Compared to the existing LLL methods and SAs, the proposed ELR algorithm has a different designing goal: to minimize the diagonal elements in noise covariance matrix. First, we show the relationship between the diagonal elements in noise covariance matrix with symbol-wise asymptotic pairwise error probability (PEP) and develop two LRs to minimize the asymptotic PEP, which are called "shortest longest vector (SLV) reduction" and a stronger version, "shortest longest basis (SLB) reduction," respectively, in this paper. However, because of the high computational complexity of the SLV and the SLB reductions, we propose two ELR algorithms, namely, the ELR-SLV and ELR-SLB algorithms, which find sub-optimal solutions to the SLV and SLB reductions, respectively. Properties of the ELR-reduced basis are analyzed. Finally, simulations are conducted to show that the proposed ELR methods attain better error performance than the state-of-the-art LR algorithms for large MIMO systems while requiring lower complexity in terms of the number of arithmetic operations.

The rest of the paper is organized as follows. Section II introduces the system model and the corresponding linear detectors. Section III describes lattice reduction and LR-aided linear detectors. Section IV studies the SLV and SLB reductions and proposes ELR algorithms. Section V discusses the proposed reductions. Section VI shows the numerical results for the proposed ELR-aided detectors. Section VII concludes the paper.

Notation: Superscript $^{\mathcal{H}}$ denotes the Hermitian, the * conjugate, and the ^{T} transpose. The real and imaginary parts of a complex number are denoted as $\Re[\cdot]$ and $\Im[\cdot]$. Upper- and lower-case boldface letters indicate matrices and column vectors, respectively. $A_{i,k}$ indicates the (i,k)th entry of matrix $\mathbf{A}$. $\mathbf{I}_N$ denotes the $N\times N$ identity matrix, $\mathbf{0}_{N\times L}$ is the $N\times L$ matrix with all entries zero, and $\mathbf{1}_{N\times L}$ is the $N\times L$ matrix with all entries one. $\mathbb{Z}$ is the integer set, and $j=\sqrt{-1}$. $\mathbb{Z}[j]$ denotes the Gaussian integer ring whose elements have a form $\mathbb{Z}+j\mathbb{Z}$. $a\leftarrow b$ means "replace a with b." The inner product between two complex vectors is defined as $\langle \mathbf{a},\mathbf{b}\rangle=\mathbf{a}^{\mathcal{H}}\mathbf{b}$. $E\{\cdot\}$ denotes the statistical expectation. $\|\cdot\|$ denotes the 2-norm and $|\cdot|$ represents the absolute value of a scalar and the cardinality for a set.

II. System Model

Consider a V-BLAST MIMO transmission model with N transmit antennas and M receive antennas as

$$\mathbf{x}=\mathbf{H}\mathbf{s}+\mathbf{w}, \tag{1}$$

where $\mathbf{H}$ is an $M\times N, (M\geq N)$ channel matrix, $\mathbf{x}=[x_1,x_2,\cdots,x_M]^T$ is the received signal vector, $\mathbf{s}=[s_1,s_2,\cdots,s_N]^T, (s_i\in\mathcal{S})$ are the information symbols drawn from the QAM constellation set $\mathcal{S}$, and $\mathbf{w}=[w_1,w_2,\cdots,w_M]^T$ is the white complex Gaussian variable vector with zero mean and covariance $\sigma_w^2\mathbf{I}_M$. The entries of $\mathbf{H}$ are modeled as independent and identically distributed (i.i.d.) complex Gaussian variables with zero mean and unit variance, and the real and imaginary parts of s_i are drawn from the set $\{2m+1-\sqrt{\mathcal{M}}, m=0,1,\cdots,\sqrt{\mathcal{M}}-1\}$ with $\mathcal{M}$ being the constellation size of $\mathcal{S}$. We assume a quasi-static channel environment, i.e., channel matrix $\mathbf{H}$ is invariant during a block and changes independently from block to block. In addition, we assume that channel matrix $\mathbf{H}$ is known at the receiver, but unknown at the transmitter.

Given the model in (1), the maximum likelihood detector (MLD) is

$$\hat{\mathbf{s}}^{\mathrm{ML}}=\arg\min_{\mathbf{s}\in\mathcal{S}^N}\|\mathbf{x}-\mathbf{H}\mathbf{s}\|^2. \tag{2}$$

In general, finding the solution of (2) is a non-deterministic polynomial hard (NP-hard) problem, which suggests that no existing algorithm is able to find the optimal solution efficiently, especially when N and/or the constellation size $|\mathcal{S}|$ is large. Since $\mathcal{S}$ is a finite set, one could resort to the exhaustive search to solve (2), which suffers from exponential complexity with respect to problem size N. Sphere decoding algorithms (SDAs) have been proposed to reduce the searching complexity, but the variance of the complexity is still high and the performance is degraded by fixing the complexity for large MIMO systems [2], [3].

To alleviate the high complexity of the MLD and SDAs, linear detectors (LDs) are proposed for their polynomial complexity with degraded error performance. When $M\geq N$, the zero-forcing linear detector (ZF-LD) for the model in (1) is

$$\hat{\mathbf{s}}^{\mathrm{ZF}}=\mathcal{Q}\left(\mathbf{H}^{\dagger}\mathbf{x}\right)=\mathcal{Q}\left(\left(\mathbf{H}^{\mathcal{H}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathcal{H}}\mathbf{x}\right), \tag{3}$$

where $\mathbf{H}^{\dagger}$ is the Moore-Penrose pseudo inverse of channel matrix $\mathbf{H}$ and $\mathcal{Q}(\cdot)$ is the symbol-wise quantizer to the constellation set $\mathcal{S}$.

The minimum-mean-square-error linear detector (MMSE-LD) aims at minimizing $E\{\|\mathbf{x}-\mathbf{H}\mathbf{s}\|^2\}$ as

$$\hat{\mathbf{s}}^{\mathrm{MMSE}}=\mathcal{Q}\left(\left(\mathbf{H}^{\mathcal{H}}\mathbf{H}+\frac{\sigma_w^2}{\sigma_s^2}\mathbf{I}_N\right)^{-1}\mathbf{H}^{\mathcal{H}}\mathbf{x}\right), \tag{4}$$

where $E\{\mathbf{s}\mathbf{s}^{\mathcal{H}}\}=\sigma_s^2\mathbf{I}_N$.

Note that, although linear detectors (ZF-LD and MMSE-LD) have lower complexity than the MLD and SDAs, their error performance degrades significantly for MIMO systems by only collecting diversity order $M-N+1$ [25], [26].

III. Lattice-Reduction-aided Linear Detectors

The gap between the MLD and LDs is mainly due to the non-orthogonality of the channel matrix $\mathbf{H}$. The motivation of LR-aided LDs is based on the fact that if channel matrix $\mathbf{H}$ is "close" to orthogonal, the decision region of LDs is also "close" to that of the MLD [14]. Hence, to improve the error performance of LDs, we employ lattice reduction to find another more orthogonal basis $\tilde{\mathbf{H}}$ that defines the same lattice as $\mathbf{H}$. As a result, LR-aided LDs yield error performance close to the MLD and has the same error performance as the MLD if the lattice-reduced basis is orthogonal.

A. Lattice Reduction Algorithms

A lattice is defined as

$$\mathbb{L} = \left\{ \sum_{i=1}^{N} a_i \mathbf{h}_i | a_i \in \mathbb{Z}[j] \right\}, \tag{5}$$

where $\mathbf{h}_i, i = 1, \cdots, N$ are the basis vectors of lattice $\mathbb{L}$. If information symbols $\mathbf{s}$ are drawn from the $\mathcal{M}$-QAM constellation set $\mathcal{S}$ with the real and imaginary parts being $\{2m + 1 - \sqrt{\mathcal{M}}, m = 0, 1, \cdots, \sqrt{\mathcal{M}} - 1\}$, then $\mathbf{Hs} \in \mathbb{L}$, where the basis is the columns of $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_N]$. Furthermore, the dual basis of $\mathbb{L}$ is defined as

$$\mathbf{H}' = [\mathbf{h}'_1, \mathbf{h}'_2, \cdots, \mathbf{h}'_N] = (\mathbf{H}^{\mathcal{H}})^{\dagger}, \tag{6}$$

and the dual lattice is denoted as $\mathbb{L}'$.

Given basis $\mathbf{H}$, LR algorithms reduce the lattice basis to find a more "orthogonal" channel matrix $\tilde{\mathbf{H}}$ while lattice $\mathbb{L}$ remains the same for the new basis. Accordingly, reducing basis $\mathbf{H}$ is equivalent to finding $\tilde{\mathbf{H}} = \mathbf{HT}$ [27], where $\mathbf{T}$ is a unimodular matrix, such that all the entries of $\mathbf{T}$ and $\mathbf{T}^{-1}$ are Gaussian integers, and the determinant of $\mathbf{T}$ is ± 1 or $\pm j$.

In order to find the unimodular matrix $\mathbf{T}$, the Gaussian reduction algorithm (GRA) [9] is the optimal LR for two-dimensional lattices ($N = 2$). When $N \geq 2$, Hermite proposed a criterion of reduction in [28], called the "Hermite criterion[1]." Minkowski relaxed the Hermite criterion and proposed Minkowski reduction in [30]. Recently, to mitigate the high complexity of Minkowski reduction, several LR algorithms were proposed and widely used because of their computational merits over Minkowski reduction [27]. These algorithms include Korkin-Zolotarev (KZ) reduction [27], the LLL algorithm and its complex valued counterparts in [12], [13], [15], and Seysen's algorithm (SA) [17], [22].

B. Lattice-Reduction-Aided Linear Detectors

Given unimodular matrix $\mathbf{T}$, LR-aided ZF-LDs perform ZF equalization with lattice-reduced channel matrix $\tilde{\mathbf{H}}$ as

$$\mathbf{y} = \tilde{\mathbf{H}}^{\dagger}\mathbf{x} = \mathbf{T}^{-1}\mathbf{s} + \tilde{\mathbf{H}}^{\dagger}\mathbf{w} = \mathbf{z} + \mathbf{n}. \tag{7}$$

By applying scaling and shifting on $\mathbf{s}$, i.e., $(\mathbf{s} - (1 + j)\mathbf{1}_{N\times 1})/2$, we transfer the real and imaginary parts of the constellation set $\mathcal{S}$ to consecutive integer sets, which are further transferred to the lattice-reduction domain as $\mathbf{b} = \mathbf{T}^{-1}(\mathbf{s} - (1 + j)\mathbf{1}_{N\times 1})/2$, whose real and imaginary parts are in the consecutive integer sets as well. By removing the boundary constraint on $\mathbf{b}$, we obtain the estimate of $\mathbf{z}$ as [10], [12]

$$\hat{\mathbf{z}} = 2\hat{\mathbf{b}} + (1 + j)\mathbf{T}^{-1}\mathbf{1}_{N\times 1}, \tag{8}$$

[1]In this paper, we use the Hermite criterion to distinguish from "Hermite reduction," which was also referred to as KZ reduction in the recent literature [27], [29].

where $\hat{\mathbf{b}} = \lfloor (\mathbf{y} - (1 + j)\mathbf{T}^{-1}\mathbf{1}_{N\times 1})/2 \rceil$ and $\lfloor \cdot \rceil$ being a rounding function. Finally, LR-aided ZF-LDs obtain the estimate of $\hat{\mathbf{s}}$ by rounding $\mathbf{T}\hat{\mathbf{z}}$ to the constellation set $\mathcal{S}$ as

$$\hat{\mathbf{s}} = \mathcal{Q}(\mathbf{T}\hat{\mathbf{z}}) = \mathcal{Q}\left(\mathbf{s} + 2\mathbf{T}\left\lfloor \frac{1}{2}\tilde{\mathbf{H}}^{\dagger}\mathbf{w} \right\rceil\right). \tag{9}$$

For the LR-aided MMSE-LD, we formulate an MMSE-extended system with $\bar{\mathbf{H}} = [\mathbf{H}^T, \sigma_w/\sigma_s \mathbf{I}_N]^T$ and $\bar{\mathbf{x}} = [\mathbf{x}^T, \mathbf{0}_{1\times N}]^T$ (see [31], [32]). After applying LR algorithms to $\bar{\mathbf{H}}$, the LR-aided MMSE-LD replaces $\tilde{\mathbf{H}}^{\dagger}$ and $\mathbf{x}$ in Eq. (7) with $\tilde{\bar{\mathbf{H}}}^{\dagger}$ and $\bar{\mathbf{x}}$, and then follows Eqs. (8) and (9).

IV. Element-Based Lattice Reduction Algorithm

In this section, we develop element-based lattice reduction (ELR) algorithms that reduce the asymptotic PEP of linear detectors.

A. Problem Statement

Based on the ZF-LD in (3), the PEP that the ith transmitted symbol s_i is erroneously detected as $\hat{s}_i \neq s_i$ given channel matrix $\mathbf{H}$ is [12, Eq. (12)]

$$P(s_i \to \hat{s}_i | \mathbf{H}) = Q\left(\sqrt{\frac{|e_{s_i}|^2}{2\sigma_w^2 C_{i,i}}}\right), \tag{10}$$

where $e_{s_i} = s_i - \hat{s}_i$, $Q(x) = (2\pi)^{-\frac{1}{2}} \int_x^{\infty} \exp(-t^2/2)dt$, $\mathbf{C} = (\mathbf{H}^{\mathcal{H}}\mathbf{H})^{-1}$ is the scaled covariance matrix of the noise after equalization, and $C_{i,i}$ denotes the ith diagonal element of $\mathbf{C}$.

Similarly, based on the LR-aided ZF-LD in Eq. (7), the PEP that z_i is erroneously detected as $\hat{z}_i \neq z_i$ given lattice-reduced channel matrix $\tilde{\mathbf{H}}$ is

$$P(z_i \to \hat{z}_i | \tilde{\mathbf{H}}) = Q\left(\sqrt{\frac{|e_{z_i}|^2}{2\sigma_w^2 \tilde{C}_{i,i}}}\right), \tag{11}$$

with $e_{z_i} = z_i - \hat{z}_i$ and $\tilde{\mathbf{C}} = (\tilde{\mathbf{H}}^{\mathcal{H}}\tilde{\mathbf{H}})^{-1} = \mathbf{T}^{-1}\mathbf{C}(\mathbf{T}^{-1})^{\mathcal{H}}$. Since the PEP for the ith symbol z_i is determined by $\tilde{C}_{i,i}$, the lattice-based method improves the PEP performance of z_i if the diagonal elements of $\tilde{\mathbf{C}}$ are reduced. In addition, as the SNR increases, the detecting term with the largest $\tilde{C}_{i,i}$ becomes the dominant term on PEP, and thus, the largest $\tilde{C}_{i,i}$ becomes the one we are going to minimize [14], [20]. Note that, although the PEP performance of z_i does not directly relate to the PEP performance of s_i due to the re-quantization step in Eq. (9), the PEP performance of z_i relates to the symbol error rate of $\mathbf{z}$, which is the upper bound of the symbol error rate of $\mathbf{s}$, i.e., $P(\mathbf{s} \neq \hat{\mathbf{s}}|\mathbf{H}) \leq P(\mathbf{z} \neq \hat{\mathbf{z}}|\tilde{\mathbf{H}}) = 1 - P\left(\left\lfloor \frac{1}{2}\tilde{\mathbf{H}}^{\dagger}\mathbf{w} \right\rceil = 0 | \tilde{\mathbf{H}}\right)$.

Mathematically, we formulate the optimization problem as finding a unimodular matrix $\mathbf{T}$ by minimizing the largest diagonal element of $\tilde{\mathbf{C}}$ as

$$\begin{aligned} \min \quad & \max_i (\tilde{C}_{i,i}) \\ \text{s.t.} \quad & \tilde{\mathbf{C}} = \mathbf{T}^{-1}\mathbf{C}(\mathbf{T}^{-1})^{\mathcal{H}} \\ & \mathbf{T} \in GL_N(\mathbb{Z}[j]), \end{aligned} \tag{12}$$

where $GL_N(\mathbb{Z}[j])$ is the group of all $N \times N$ unimodular matrices. Since $\tilde{C}_{i,i} = \|\tilde{\mathbf{h}}'_i\|^2$ is the squared 2-norm of the ith vector in the dual basis (defined in (6)) after the reduction, the optimization in (12) is equivalent to minimizing the longest basis vector in the dual basis. In this paper, we refer to the optimization (12) as the "dual shortest longest vector (D-SLV) reduction."

Note that the optimal solution to (12) is generally not unique, i.e., there may exist two or more bases of lattice $\mathbb{L}$ that satisfy the D-SLV reduction. Therefore, once we have the optimal solution to Eq. (12), we could improve PEP performance by minimizing the second largest diagonal element of $\tilde{\mathbf{C}}$ with respect to $\mathbf{T}$ as

$$\begin{aligned} \min \quad & \max_i^2(\tilde{C}_{i,i}) \\ \text{s.t.} \quad & \tilde{\mathbf{C}} = \mathbf{T}^{-1}\mathbf{C}(\mathbf{T}^{-1})^{\mathcal{H}} \\ & \max_i(\tilde{C}_{i,i}) = \tilde{C}^{(\mathrm{o})} \\ & \mathbf{T} \in GL_N(\mathbb{Z}[j]), \end{aligned} \tag{13}$$

which generally relies on the optimization result of (12) with $\tilde{C}^{(\mathrm{o})}$ being the optimal objective value of (12) and $\max_i^2(\tilde{C}_{i,i})$ being the second largest diagonal element of $\tilde{\mathbf{C}}$. After solving optimization problem (13), we can further optimize the third largest diagonal element, the fourth one, and so on. This procedure continues until all the diagonal elements of $\tilde{\mathbf{C}}$ are minimized. In this paper, we call this process the "dual shortest longest basis (D-SLB) reduction." Similarly, we can derive the primal SLV (P-SLV) and primal SLB (P-SLB) reductions by replacing matrix $\mathbf{C}$ with the Gram matrix $\mathbf{G} = \mathbf{H}^{\mathcal{H}}\mathbf{H}$.

B. ELR Algorithms

Since finding the global optimal solution to the D-SLV reduction in (12) and the D-SLB reduction are computationally demanding (if possible), directly solving the D-SLV and D-SLB reductions is prohibitive for large MIMO systems. To alleviate the high complexity of these reductions, in the following, we develop ELR algorithms to find sub-optimal solutions for the D-SLV and D-SLB reductions in an iterative fashion.

Because $\mathbf{T}$ is a unimodular matrix, $\mathbf{T}' = (\mathbf{T}^{-1})^{\mathcal{H}}$ is also a unimodular matrix that can be represented as the product of a series of column-addition operation matrices. For each column-addition operation, ELR algorithms choose an index pair (i, k), calculate $\lambda_{i,k} \in \mathbb{Z}[j]$, and update the kth column of matrix $\mathbf{T}'$ as,

$$\mathbf{t}'_k \leftarrow \mathbf{t}'_k + \lambda_{i,k}\mathbf{t}'_i, \tag{14}$$

where $\mathbf{t}'_i$ is the ith column of $\mathbf{T}'$. Following the column-addition operation, the kth column and the kth row of $\tilde{\mathbf{C}}$, namely, $\tilde{\mathbf{c}}_k$ and $\tilde{\mathbf{c}}^{(k)}$, are sequentially updated as

$$\begin{aligned} \tilde{\mathbf{c}}_k &\leftarrow \tilde{\mathbf{c}}_k + \lambda_{i,k}\tilde{\mathbf{c}}_i, \\ \tilde{\mathbf{c}}^{(k)} &\leftarrow \tilde{\mathbf{c}}^{(k)} + \lambda_{i,k}^*\tilde{\mathbf{c}}^{(i)}. \end{aligned} \tag{15}$$

Thus, the current basis $\tilde{\mathbf{H}}$ is updated as

$$\tilde{\mathbf{h}}_i \leftarrow \tilde{\mathbf{h}}_i - \lambda_{i,k}^*\tilde{\mathbf{h}}_k, \tag{16}$$

where $\tilde{\mathbf{h}}_i$ is the ith column of $\tilde{\mathbf{H}}$.

It is readily checked that the value of $\tilde{C}_{k,k}$ is updated as

$$\tilde{C}_{k,k} \leftarrow \tilde{C}_{k,k} + |\lambda_{i,k}|^2\tilde{C}_{i,i} + \lambda_{i,k}^*\tilde{C}_{i,k} + \lambda_{i,k}\tilde{C}_{k,i}, \tag{17}$$

and the remaining diagonal elements of the updated matrix $\tilde{C}_{m,m}$, $m \in \{1, \cdots, N\}$, $m \neq k$ are unchanged.

Since $\tilde{C}_{k,k}$ and $\tilde{C}_{i,i}$ are real and positive, and $\tilde{C}_{k,i} = \tilde{C}_{i,k}^*$, Eq. (17) can be rewritten as

$$\begin{aligned} \tilde{C}_{k,k} \leftarrow \tilde{C}_{k,k} &+ (\Re[\lambda_{i,k}]^2 + \Im[\lambda_{i,k}]^2)\tilde{C}_{i,i} \\ &+ 2\Re[\lambda_{i,k}]\Re[\tilde{C}_{i,k}] + 2\Im[\lambda_{i,k}]\Im[\tilde{C}_{i,k}]. \end{aligned} \tag{18}$$

By taking the partial derivative of (18) with respect to the real and imaginary parts of $\lambda_{i,k}$, $\tilde{C}_{k,k}$ is minimized when

$$\lambda_{i,k} = -\left\lfloor \frac{\tilde{C}_{i,k}}{\tilde{C}_{i,i}} \right\rceil. \tag{19}$$

The amount that $\tilde{C}_{k,k}$ decreases is

$$\Delta_{i,k} = -|\lambda_{i,k}|^2\tilde{C}_{i,i} - \lambda_{i,k}^*\tilde{C}_{i,k} - \lambda_{i,k}\tilde{C}_{k,i} \geq 0. \tag{20}$$

Definition 1 (Reducible diagonal element) *A diagonal element $\tilde{C}_{k,k}$ is reducible if and only if there exists $i \neq k$, such that $\lambda_{i,k} \neq 0$ (i.e., $\Delta_{i,k} > 0$).*

The procedure of an ELR algorithm is briefly summarized below. Given the initial matrix $\tilde{\mathbf{C}} = (\mathbf{H}^{\mathcal{H}}\mathbf{H})^{-1}$, for each iteration, the algorithm selects a reducible $\tilde{C}_{k,k}$ and i such that $\Delta_{i,k} > 0$. Then Eq. (15) is applied to update $\tilde{\mathbf{C}}$. For next iteration, the algorithm just picks up another reducible $\tilde{C}_{n,n}$ from updated $\tilde{\mathbf{C}}$. This procedure repeats until the termination condition is satisfied. Two natural questions are how to choose index pair (i, k) for each iteration, and when to terminate the algorithm. In the following, we propose two ELR algorithms for the D-SLV and D-SLB reductions, respectively.

Algorithm 1 (D-ELR-SLV) *The dual ELR-SLV (D-ELR-SLV) algorithm selects the largest $\tilde{C}_{k,k}$ and chooses $i = \arg\max_{\tilde{i}=1,\tilde{i}\neq k}^{N} \Delta_{\tilde{i},k}$. The algorithm terminates when the largest diagonal element of $\tilde{\mathbf{C}}$ is irreducible.*

Algorithm 2 (D-ELR-SLB) *The dual ELR-SLB (D-ELR-SLB) algorithm finds the largest reducible $\tilde{C}_{k,k}$ and chooses $i = \arg\max_{\tilde{i}=1,\tilde{i}\neq k}^{N} \Delta_{\tilde{i},k}$. This procedure repeats until no reduction can be made on the diagonal elements of $\tilde{\mathbf{C}}$.*

The detailed pseudo-codes of the ELR algorithms can be found in Table I.

Remark 1 (D-ELR and P-ELR): Since the proposed ELR algorithms in this section aim at minimizing the length of the basis in the dual space, we refer to them as the "dual ELR (D-ELR) algorithms." Correspondingly, one could derive the "primal ELR algorithms" (called the "P-ELR algorithms") to minimize the length of the basis vectors in the primal space by replacing matrix $\mathbf{C}$ with Gram matrix $\mathbf{G} = \mathbf{H}^{\mathcal{H}}\mathbf{H}$ in Table I.

Remark 2 (Comparisons between D-ELR-SLV and D-ELR-SLB): Obviously, the D-ELR-SLV is equivalent to the D-ELR-SLB when $N = 2$, and the D-ELR-SLV is an early-terminated version of the D-ELR-SLB when $N \geq 3$. As a result, for large MIMO systems, the D-ELR-SLV algorithm requires fewer iterations than the D-ELR-SLB algorithm. In addition, the complexity of the D-ELR-SLV algorithm for each

TABLE I
THE DUAL ELEMENT-BASED LATTICE REDUCTION ALGORITHMS.

Input: $\mathbf{H}$, **Output:** $\tilde{\mathbf{H}}$, $\mathbf{T}$	
(1)	$\tilde{\mathbf{C}} = (\mathbf{H}^{\mathcal{H}}\mathbf{H})^{-1}$, $\mathbf{T}' = \mathbf{I}_N$
(2)	**Do**
(3)	$\lambda_{i,k} \leftarrow -\left\lfloor \frac{\tilde{C}_{i,k}}{\tilde{C}_{i,i}} \right\rceil, \forall i \neq k$
(4a)	**For the D-ELR-SLV**: **If** the largest element of $\tilde{\mathbf{C}}$ is irreducible, **goto** 11
(4b)	**For the D-ELR-SLB**: **If** all $\lambda_{i,k} = 0, \forall i \neq k$, **goto** 11
(5)	Find the largest reducible $\tilde{C}_{k,k}$
(6)	Choose $i = \arg\max_{\tilde{i}=1,\tilde{i}\neq k}^{N} \Delta_{\tilde{i},k}$
(7)	$\mathbf{t}'_k \leftarrow \mathbf{t}'_k + \lambda_{i,k}\mathbf{t}'_i$
(8)	$\tilde{\mathbf{c}}_k \leftarrow \tilde{\mathbf{c}}_k + \lambda_{i,k}\tilde{\mathbf{c}}_i$
(9)	$\tilde{\mathbf{c}}^{(k)} \leftarrow \tilde{\mathbf{c}}^{(k)} + \lambda^*_{i,k}\tilde{\mathbf{c}}^{(i)}$
(10)	**While (true)**
(11)	$\mathbf{T} = (\mathbf{T}'^{-1})^{\mathcal{H}}, \tilde{\mathbf{H}} = \mathbf{H}\mathbf{T}$

iteration is lower than that of the D-ELR-SLB algorithm. This is because, although the D-ELR-SLV and the D-ELR-SLB have the same cost on basis updates (lines 7-9 in Table I), the D-ELR-SLV algorithm requires, at most, $\mathcal{O}(N)$ to find the largest $C_{k,k}$ and $i = \arg\max_{\tilde{i}} \Delta_{\tilde{i},k}$, while, the D-ELR-SLB algorithm requires $\mathcal{O}(N^2)$ to find the index pair (i,k) at the worst case. In summary, we would expect that the D-ELR-SLB yields better error performance than the D-ELR-SLV at the cost of higher complexity.

Remark 3 (D-ELR-aided MMSE-LDs) Although the D-ELR algorithms developed in this section are motivated for LR-aided ZF-LDs, the algorithms are also applicable to LR-aided MMSE-LDs by reducing the mean square error (MSE) of the information symbol in the lattice-reduced domain $\mathbf{z}$.

Given $\mathbf{s} = \mathbf{T}\mathbf{z}$ and the model in Eq. (1), the MMSE solution of $\mathbf{z}$ is [32]

$$\begin{aligned}\hat{\mathbf{z}}^{\text{MMSE}} &= (\mathbf{T}^{\mathcal{H}}\mathbf{H}^{\mathcal{H}}\mathbf{H}\mathbf{T} + \sigma_w^2\mathbf{R}_{zz}^{-1})^{-1}\mathbf{T}^{\mathcal{H}}\mathbf{H}^{\mathcal{H}}\mathbf{y} \\ &= \mathbf{T}^{-1}\left(\mathbf{H}^{\mathcal{H}}\mathbf{H} + \frac{\sigma_w^2}{\sigma_s^2}\mathbf{I}_N\right)^{-1}\mathbf{H}^{\mathcal{H}}\mathbf{x},\end{aligned} \tag{21}$$

where $\mathbf{R}_{zz} = E\{\mathbf{z}\mathbf{z}^{\mathcal{H}}\} = \sigma_s^2(\mathbf{T}^{\mathcal{H}}\mathbf{T})^{-1}$.

The autocorrelation matrix of the estimation error in the lattice-reduced domain $\mathbf{z} - \hat{\mathbf{z}}^{\text{MMSE}}$ is

$$\tilde{\mathbf{R}}_e = (\sigma_w^{-2}\mathbf{T}^{\mathcal{H}}\mathbf{H}^{\mathcal{H}}\mathbf{H}\mathbf{T} + \sigma_s^{-2}\mathbf{T}^{\mathcal{H}}\mathbf{T})^{-1} = \mathbf{T}^{-1}\mathbf{R}_e(\mathbf{T}^{-1})^{\mathcal{H}}, \tag{22}$$

with $\mathbf{R}_e = (\sigma_w^{-2}\mathbf{H}^{\mathcal{H}}\mathbf{H} + \sigma_s^{-2}\mathbf{I}_N)^{-1}$ being the autocorrelation matrix of the estimation error of $\mathbf{s}$. Since the diagonal elements of $\tilde{\mathbf{R}}_e$ are the MSEs for $\mathbf{z}$, D-ELR algorithms can reduce the MSEs for $\mathbf{z}$ by reducing the diagonal elements of $\tilde{\mathbf{R}}_e$. Note that, because $\mathbf{R}_e = \sigma_w^2(\bar{\mathbf{H}}^{\mathcal{H}}\bar{\mathbf{H}})^{-1}$, D-ELR algorithms for MMSE essentially perform reduction on the basis of the MMSE-extended matrix $\bar{\mathbf{H}}$. We refer to the LR-aided MMSE-LDs using the D-ELR algorithms on basis $\bar{\mathbf{H}}$ as "the D-ELR-aided MMSE-LDs."

Remark 4 (D-ELR-aided SIC Detectors with Detection Ordering): Previous studies on LR-aided detectors suggest that LR algorithms can also significantly improve the error performance of successive interference cancelation (SIC) detectors [14], [20], [29], [32], which generally show better error performance than LDs. In this paper, we will study the error performance of D-ELR-aided SIC detectors via simulations. In addition, since the diagonal elements $\tilde{C}_{k,k}$ are known after D-ELR, we can used these diagonal elements to determine the symbol detection order for LR-aided SIC detectors. This can be done by sorting diagonal elements $\tilde{C}_{k,k}$; then the ordered SIC detection starts from the symbol z_i with the smallest variance, the second smallest variance, and so on. The cost for determining this order is to rank all diagonal elements $\tilde{C}_{k,k}$, whose cost is almost negligible relative to the cost of other steps in SIC detection (e.g., QR decomposition of the channel matrix). We call this ordered SIC detector using sorted variances the "SV-SIC detector."

V. DISCUSSION OF THE SLV AND SLB REDUCTIONS AND THE ELR ALGORITHMS

A. Relationship Between the SLB Reduction and the Hermite Criterion

Let us first introduce the Hermite criterion.

Definition 2 (The Hermite criterion [28]) *Basis $[\tilde{\mathbf{h}}_1, \cdots, \tilde{\mathbf{h}}_N]$ of a lattice $\mathbb{L}$ satisfies the Hermite criterion if the following condition holds: For all bases $[\mathbf{h}_1, \cdots, \mathbf{h}_N]$ of lattice $\mathbb{L}$, there must exist $i \in [1, \cdots, N]$, such that for any $k = 1, \cdots, i-1$, $\|\tilde{\mathbf{h}}_k\| = \|\mathbf{h}_k\|$ and $\|\tilde{\mathbf{h}}_i\| \leq \|\mathbf{h}_i\|$.*

In other words, if we order all possible basis vector sequences $(\|\mathbf{h}_1\|, \cdots, \|\mathbf{h}_N\|)$ of lattice $\mathbb{L}$ lexicographically, $(\|\tilde{\mathbf{h}}_1\|, \cdots, \|\tilde{\mathbf{h}}_N\|)$ is the first one in the list. The criterion guarantees that $\tilde{\mathbf{h}}_1$ is the shortest non-zero vector in lattice $\mathbb{L}$.

Meanwhile, the procedure of the P-SLB reduction described in Section IV-A is summarized as follows:

Definition 3 (The P-SLB reduction) *Basis $[\tilde{\mathbf{h}}_1, \cdots, \tilde{\mathbf{h}}_N]$ of a lattice $\mathbb{L}$ satisfies the P-SLB reduction if the following condition holds: For all bases $[\mathbf{h}_1, \cdots, \mathbf{h}_N]$ of lattice $\mathbb{L}$, there must exist $i \in [1, \cdots, N]$, such that for any $k = i+1, \cdots, N$, $\|\tilde{\mathbf{h}}_k\| = \|\mathbf{h}_k\|$ and $\|\tilde{\mathbf{h}}_i\| \leq \|\mathbf{h}_i\|$.*

Similar to the Hermite criterion, Definition 3 indicates that if we order all possible sequences $(\|\mathbf{h}_N\|, \cdots, \|\mathbf{h}_1\|)$ of lattice $\mathbb{L}$ lexicographically, $(\|\tilde{\mathbf{h}}_N\|, \cdots, \|\tilde{\mathbf{h}}_1\|)$ is the first one in the list. Note that the main difference between the P-SLB reduction and the Hermite criterion is the choice of k, and the basis vectors of the P-SLB reduction in the sequence are in reverse order of the Hermite criterion.

B. Properties of the ELR-SLB-Reduced Basis

In this subsection, we study the properties of the ELR-SLB-reduced basis, which are useful to the discussion on the relationship with the GRA and the orthogonality deficiency in the following subsections. After obtaining lattice-reduced basis $\tilde{\mathbf{H}}$, the D-ELR-SLB algorithm yields the following property of the entries of matrix $\tilde{\mathbf{C}}$:

$$\begin{aligned} -\frac{1}{2}\tilde{C}_{k,k} < \Re[\tilde{C}_{i,k}] < \frac{1}{2}\tilde{C}_{k,k}, \forall i \neq k, \\ -\frac{1}{2}\tilde{C}_{k,k} < \Im[\tilde{C}_{i,k}] < \frac{1}{2}\tilde{C}_{k,k}, \forall i \neq k. \end{aligned} \tag{23}$$

Similarly, the P-ELR-SLB-reduced basis has the following property

$$\begin{aligned} -\frac{1}{2}\tilde{G}_{k,k} < \Re[\tilde{G}_{i,k}] < \frac{1}{2}\tilde{G}_{k,k}, \forall i \neq k, \\ -\frac{1}{2}\tilde{G}_{k,k} < \Im[\tilde{G}_{i,k}] < \frac{1}{2}\tilde{G}_{k,k}, \forall i \neq k, \end{aligned} \tag{24}$$

where $\tilde{\mathbf{G}} = \tilde{\mathbf{H}}^{\mathcal{H}}\tilde{\mathbf{H}}$ is the Gram matrix after reduction.

C. Relationship Between the ELR Algorithms and the Gaussian Reduction Algorithm

The pseudo-code of the GRA can be found in [23, Table I]. When the iteration stops, the GRA yields the following properties of basis $\tilde{\mathbf{h}}_1, \tilde{\mathbf{h}}_2$:

$$\|\tilde{\mathbf{h}}_1\| \leq \|\tilde{\mathbf{h}}_2\|; \tag{25}$$

$$|\Re[\langle \tilde{\mathbf{h}}_1, \tilde{\mathbf{h}}_2 \rangle]| \leq \frac{1}{2}\|\tilde{\mathbf{h}}_1\|^2 \quad \text{and} \quad |\Im[\langle \tilde{\mathbf{h}}_1, \tilde{\mathbf{h}}_2 \rangle]| \leq \frac{1}{2}\|\tilde{\mathbf{h}}_1\|^2. \tag{26}$$

We call basis $[\tilde{\mathbf{h}}_1, \tilde{\mathbf{h}}_2]$ a "Gaussian-reduced basis" if the basis satisfies the properties in Eqs. (25) and (26). It is shown in [9, Proposition 1] that, $\tilde{\mathbf{h}}_1$ is the shortest (non-zero) vector in a lattice, and $\tilde{\mathbf{h}}_2$ is the shortest vector that is not a multiple of $\tilde{\mathbf{h}}_1$. In addition, the Gaussian-reduced basis of a given lattice is unique (up to signs) [33, page 244].

Given the aforementioned properties of the Gaussian-reduced basis in Eqs. (25) and (26) and the ELR-reduced basis in Eqs. (23) and (24), we establish the relationship between the ELR algorithms and the GRA that, for any two-dimensional lattice, the P-ELR-reduced basis, the D-ELR-reduced basis, and the GRA reduced basis are the same (up to signs and permutation). Note that, when $N = 2$, the ELR-SLV is equivalent to the ELR-SLB.

D. Relationship Between the ELR Algorithms and the Existing LR Algorithms

The iterative procedure of ELR algorithms is similar to the one for the SA [11], [21], [23] and a generalized SA, namely, $C^{(p,k)}$ algorithm in [34]. Instead of minimizing the diagonal elements of matrix $\mathbf{C}$, the SA aims at minimizing Seysen's metric that quantifies the orthogonality of a matrix $\tilde{\mathbf{H}}$ as

$$S(\tilde{\mathbf{H}}) = \sum_{n=1}^{N} \|\tilde{\mathbf{h}}_n\|^2 \|\tilde{\mathbf{h}}'_n\|^2 = \sum_{n=1}^{N} G_{n,n} C_{n,n}, \tag{27}$$

where $G_{n,n}$ is the (n, n)th entry of the Gram matrix $\mathbf{G} = \mathbf{H}^{\mathcal{H}}\mathbf{H}$.

Similar to the ELR algorithm, starting with $\tilde{\mathbf{H}} = \mathbf{H}$ and $\mathbf{T} = \mathbf{I}_N$, the SA performs a column-addition operation for each iteration by selecting an index pair (i, k) and

$$\lambda_{i,k} = -\left\lfloor \frac{1}{2}\left(\frac{\tilde{C}_{i,k}}{\tilde{C}_{i,i}} - \frac{\tilde{G}_{i,k}}{\tilde{G}_{k,k}}\right)\right\rceil,$$

such that $S(\tilde{\mathbf{H}})$ is minimized (see [11], [21], [23] for details). This procedure repeats until no reduction can be made on $S(\tilde{\mathbf{H}})$.

The Seysen's metric is further extended in [34], which proposed a two-stage LR algorithm. The first stage is a standard LR algorithm (e.g., LLL or SA), and the second stage is the $C^{(p,k)}$ algorithm, which aims at minimizing the following generalized metric

$$C^{(p,k)}(\tilde{\mathbf{H}}) = \sum_{n=1}^{N} \|\tilde{\mathbf{h}}_n\|^k \|\tilde{\mathbf{h}}'_n\|^{(p+k)}. \tag{28}$$

Similar to the SA, for each iteration, the $C^{(p,k)}$ algorithm chooses an index pair (i, k) and $\lambda_{i,k}$ that minimize $C^{(p,k)}(\tilde{\mathbf{H}})$. The $C^{(p,k)}$ algorithm stops when $C^{(p,k)}(\tilde{\mathbf{H}})$ cannot be reduced by any column-addition operation. When $p = 0$ and $k = 2$, the $C^{(p,k)}$ algorithm becomes the SA, and when $p = 2$ and $k = 0$, the $C^{(p,k)}$ metric in Eq. (28) becomes

$$C^{(2,0)}(\tilde{\mathbf{H}}) = \sum_{n=1}^{N} \|\tilde{\mathbf{h}}'_n\|^2 = \sum_{n=1}^{N} \tilde{C}_{n,n}, \tag{29}$$

which is the sum of the diagonal elements of $\tilde{\mathbf{C}}$. For each iteration, the $C^{(2,0)}$ algorithm chooses $\lambda_{i,k}$ similar in Eq. (19) and the index pair $(i, k) = \arg\max_{\tilde{i},\tilde{k}} \Delta_{\tilde{i},\tilde{k}}$ such that $C^{(2,0)}(\tilde{\mathbf{H}})$ is minimized by the column-addition operation.

However, our problem and approaches are different from the existing ones in the following aspects: *i*) Compared to the SA, our proposed D-ELR algorithms have a different objective function, resulting in different rules on choosing $\lambda_{i,k}$ and the index pair (i, k). *ii*) Our proposed D-ELR algorithms perform reduction directly on the channel matrix $\mathbf{H}$, while the LR algorithm in [34] is a two-stage algorithm, whose first stage is a standard LR algorithm. Since we demonstrate in Sec. VI that standard LR algorithm (e.g., LLL or SA) exhibits higher complexity than our proposed D-ELR algorithms, the LR algorithm in [34] requires much higher complexity than our proposed D-ELR algorithms for MIMO systems. *iii*) Compared to the $C^{(p,k)}$ algorithm, whose goal is to minimize a generalized metric, the D-ELR algorithms have a different goal, which is to minimize the asymptotic error performance of LR-aided LDs by approximately solving the SLV and SLB reductions. *iv*) Compared to the $C^{(2,0)}$ algorithm, the proposed D-ELR-SLV and D-ELR-ELB algorithms use different strategies on choosing the index pair (i, k). Note that the $C^{(2,0)}$ algorithm generally requires $\mathcal{O}(N^2)$ operations for finding $(i, k) = \arg\max_{\tilde{i},\tilde{k}} \Delta_{\tilde{i},\tilde{k}}$, while the D-ELR-SLV requires only $\mathcal{O}(N)$ operations. In addition, the D-ELR-SLV uses a different termination condition compared to the $C^{(2,0)}$ algorithm. *v*) The proposed ELR algorithms directly work for the complex-valued basis, while the $C^{(2,0)}$ algorithm only works for real-valued basis.

E. Orthogonality Deficiency of the ELR Algorithms

The orthogonality deficiency (od) measures the orthogonality of a matrix, defined in [12][2]:

$$od(\mathbf{H}) = 1 - \frac{\det(\mathbf{H}^{\mathcal{H}}\mathbf{H})}{\prod_{n=1}^{N} \|\mathbf{h}_n\|^2}. \tag{30}$$

Note that, based on Hadamard's inequality $\left(\prod_{n=1}^{N} \|\mathbf{h}_n\| \geq \det(\mathbf{H})\right)$, we have $0 \leq od(\mathbf{H}) \leq 1$. If $\mathbf{H}$ is singular, $od(\mathbf{H}) = 1$, and if $\mathbf{H}$ is orthogonal, $od(\mathbf{H}) = 0$. An important result of od related to wireless communications is that if there exists a constant $\varepsilon > 0$, such that $od(\tilde{\mathbf{H}}) \leq 1 - \varepsilon$ for all possible realizations of channel matrix $\mathbf{H}$, then LR-aided LDs enable the same diversity as the MLD for MIMO systems [12], [13], [19].

To illustrate the od of $\tilde{\mathbf{H}}$ after ELR, we have the following proposition:

Proposition 1 *For each iteration of a D-ELR (P-ELR) algorithm, the od of dual basis $\tilde{\mathbf{H}}'$ (primal basis $\tilde{\mathbf{H}}$) decreases monotonically.*

Proof: See Appendix A. ■

Thus, from Proposition 1, we conclude that the od of a D-ELR (P-ELR)-reduced basis is smaller than or equal to that of original dual basis $\mathbf{H}'$ (primal basis $\mathbf{H}$).

In contrast to the LLL and the SA, the following result shows that the od of ELR-SLB-reduced basis can be arbitrarily close to 1.

Proposition 2 *If $N \geq 3$, then there exists a D-ELR-SLB-reduced basis $\tilde{\mathbf{H}}'$, such that for any given $0 < \varepsilon < 1$, the od of the reduced basis $od(\tilde{\mathbf{H}}')$ and the od of the corresponding primal basis $od(\tilde{\mathbf{H}})$ are greater than ε.*

Proof: See Appendix B. ■

From Proposition 2, it follows that the P-ELR-SLB cannot yield a bounded od for the basis in the dual or primal space either. Here, we further illustrate the effects of the ELR algorithms on od using a numerical example.

Fig. 1 depicts the ods of the lattice-reduced channel matrices for different LR algorithms in a 4×4 MIMO system with i.i.d. Gaussian channels. For each LR algorithm, 10^7 channel realizations are simulated, and the empirical complementary cumulative distribution functions (CCDFs) of $od(\tilde{\mathbf{H}})$ are plotted in Fig. 1. First of all, ods of the reduced bases by the SA and the LLL are bounded by a number strictly less than 1, which is consistent with the results in [12], [15], [19], [22]. Second, although the ods of the reduced bases by the D-ELR-SLB and the P-ELR-SLB are "better" than the original channel matrix, all ods of the channel matrices before reduction, the D-ELR-SLB-reduced basis, and the P-ELR-SLB-reduced basis can be very close to one.

[2]Another measurement of the orthogonality of a matrix is the orthogonality defect $\frac{\prod_{n=1}^{N} \|\mathbf{h}_n\|^2}{\det(\mathbf{H}^{\mathcal{H}}\mathbf{H})}$ in [35], which is equivalent to (30), but is bounded from below by 1.

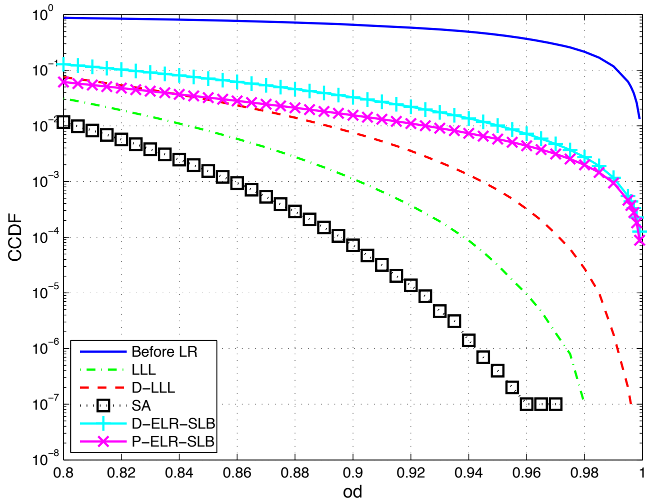


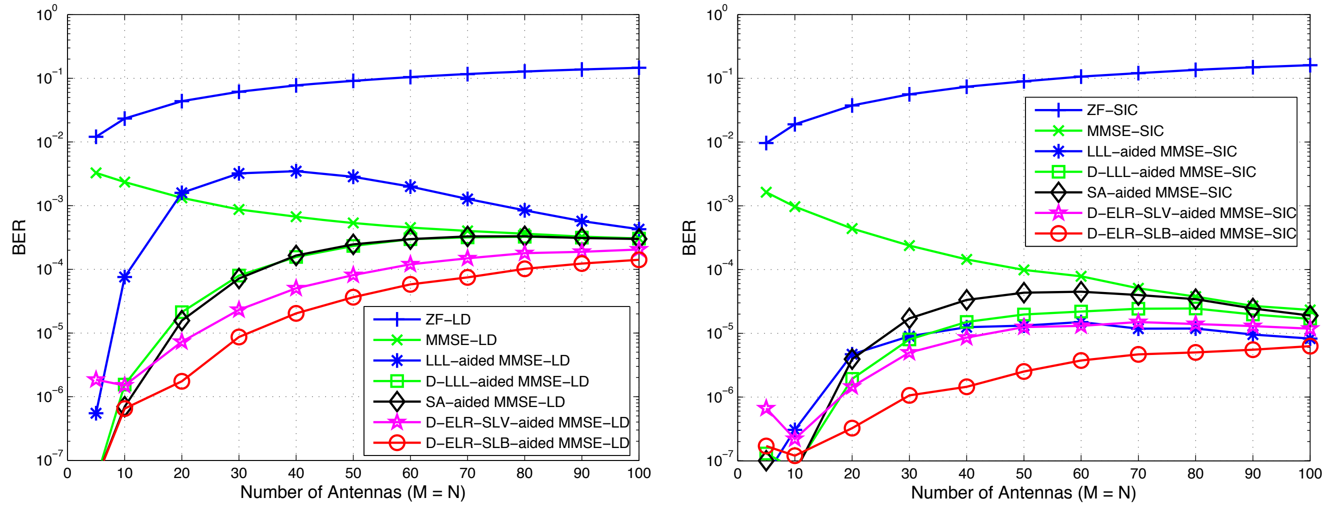
Fig. 1. CCDFs of $od(\mathbf{H})$ and $od(\tilde{\mathbf{H}})$ of different LR algorithms for MIMO systems with $N = M = 4$.

VI. Numerical Results

In this section, we thoroughly compare the error performance and the complexity of the proposed ELR-aided detectors to those of *i)* the state-of-the-art LR-aided detectors, *ii)* the MMSE iterative soft interference cancelation (MMSE-ISIC) method [36], and *iii)* the LTS algorithm [8] via simulations. The state-of-the-art LR algorithms include the LLL algorithm in [12], the D-LLL algorithm in [16], and the SA with greedy implementation in [21]. The LLL and D-LLL algorithms use the reduction quality parameter $\delta = 3/4$, and all the LR-aided MMSE detectors perform LR on the extended channel matrix $\bar{\mathbf{H}}$. In addition, we do not consider the fixed complexity sphere decoding algorithm in [37], which exhibits degraded error performance and high complexity for large MIMO systems [2], [8]. Unless stated otherwise, the entries of $\mathbf{H}$ are modeled as i.i.d. complex Gaussian variables with zero mean and unit variance. The SNR is defined as the received information bit energy versus noise variance.

A. Performance Comparisons of the LR-aided Detectors

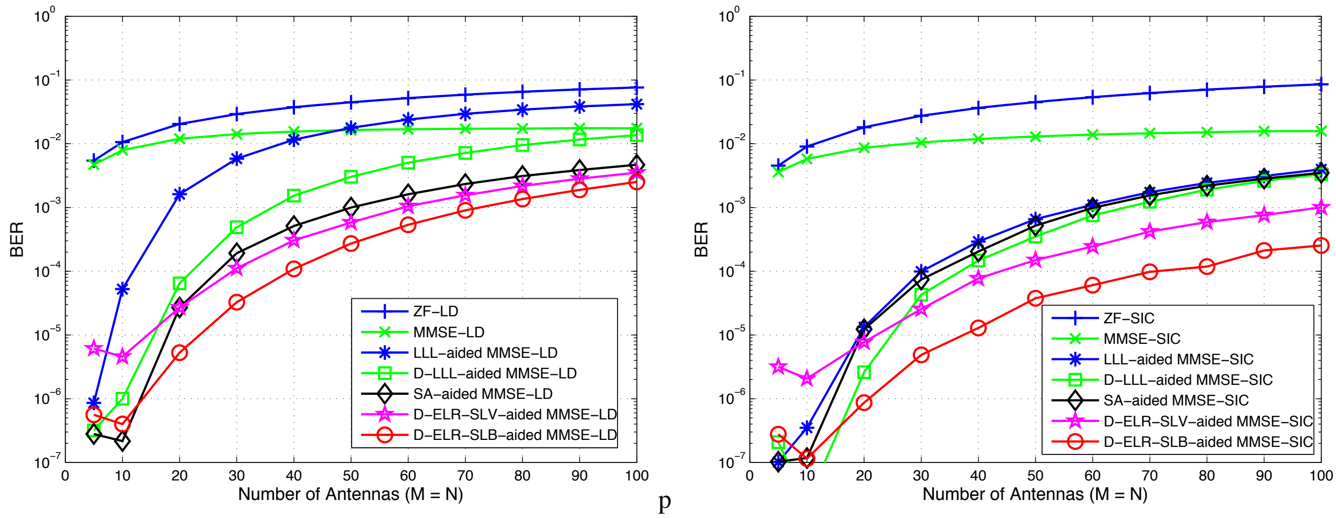
1) Error Performance Comparisons with i.i.d. Channels: Fig. 2 displays the error performance of the LR-aided detectors with 4QAM, SNR at 20 dB, and different numbers of antennas $N = M$. Unless stated otherwise, all the LR-aided MMSE-SIC detectors do not employ any pre-/post- ordering techniques. Several observations can be made here: *i)* All the LR-aided MMSE detectors except the LLL-aided MMSE-LD obtain significant performance gain over the MMSE detectors when $N \leq 50$, while the LLL-aided MMSE-LD shows degraded performance relative to the MMSE-LD when $N > 20$ (see Fig. 2(a)). The D-LLL-aided MMSE-LD and the SA-aided MMSE-LD exhibit almost the same error performance, which is better than that of the MMSE-LD. However, the gain of the D-LLL-aided MMSE-LD and the SA-aided MMSE-LD relative to the MMSE-LD decreases as the number of antennas increases and vanishes when the number of antennas is large ($N \geq 80$). *ii)* The proposed D-ELR-SLB-aided MMSE-LD show significant error performance improvement



(a) Performance of LR-aided LDs.

(b) Performance of LR-aided SIC detectors.

Fig. 2. Performance comparisons of the different detectors for MIMO systems with 4QAM, SNR = 20 dB, and different numbers of antennas.



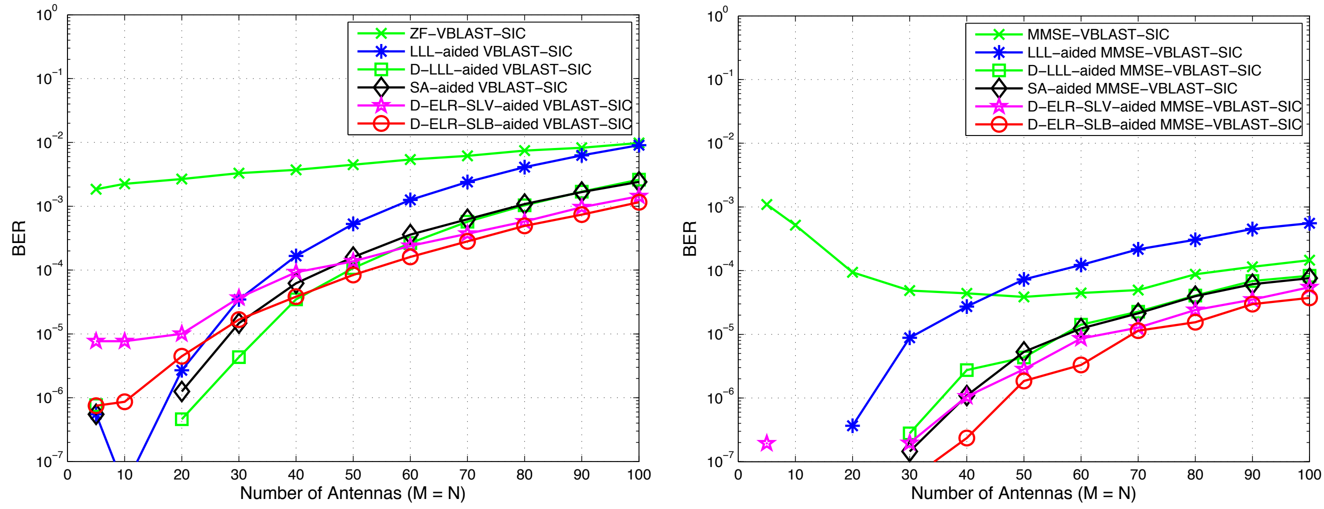
(a) Performance of LR-aided LDs.

(b) Performance of LR-aided SIC detectors.

Fig. 3. Performance comparisons of the different detectors for MIMO systems with 64QAM, SNR = 30 dB, and different numbers of antennas.

over the MMSE-LD and existing LR-aided LDs for large MIMO systems ($N \geq 20$). For example, the proposed D-ELR-SLB-aided LD has an about one order gain relative to the SA-aided MMSE-LD in terms of BER when $N = 50$. *iii)* The performance of the LR-aided MMSE-LDs is improved by the LR-aided MMSE-SIC detectors, which outperform the MMSE-SIC detector (see Fig. 2(b)). However, similar to the case for LR-aided MMSE-LD, as the number of antennas N increases, the performance of the MMSE-SIC approaches the performance of the existing LR-aided MMSE-SICs, and they exhibit close performance when $N = 100$. *iv)* For LR-aided MMSE-SICs, the proposed D-ELR-SLB-aided MMSE-SIC still exhibits superior error performance to that of the existing LR-aided MMSE-SICs, where the proposed D-ELR-SLB-aided SIC detector outperforms the MMSE-SIC by three orders of BER performance when $N = M = 20$, and it outperforms the D-LLL-aided MMSE-SIC detectors and the SA-aided MMSE-SIC detectors by about one order of BER for large MIMO systems (e.g., 4.3×10^{-5} BER for the SA-aided MMSE-SIC versus 2.5×10^{-6} BER for the D-ELR-SLB-aided MMSE-SIC in the 50×50 MIMO system). *v)* From Figs. 2(a)-(b), the proposed D-ELR-SLV-aided detectors also shows better error performance than the existing LR-aided detectors when $N > 20$. However, for small MIMO systems, i.e., $N \leq 10$, the performance of the D-ELR-SLV-aided detectors is worse than that of the existing LR-aided detectors. This indicates that the D-ELR-SLV-aided detector is more suitable for large MIMO systems relative to small systems. *vi)* The ZF detectors show significantly inferior error performance compared to the MMSE and LR-aided MMSE detectors.

Fig. 3 demonstrates the error performance of the LR-aided detectors with 64QAM, SNR at 30 dB, and $N = M$. In this case, The performance gain of all the LR-aided MMSE detectors is clearer. First of all, as shown in Fig. 3(a), pthe LLL-aided MMSE-LD still shows inferior error performance to the MMSE-LD when $N \geq 50$, and the SA-aided MMSE-LD outperforms the D-LLL MMSE-LD. In all, for large



(a) Performance of LR-aided VBLAST-SIC detectors.

(b) Performance of LR-aided MMSE-VBLAST-SIC detectors.

Fig. 4. Performance comparisons of the different VBLAST-SIC detectors for MIMO systems with 64QAM, SNR = 30 dB, and different numbers of antennas.

MIMO detection ($N \geq 20$), the proposed D-ELR-SLB-aided MMSE-LD still achieves considerable gain over the existing LR-aided LDs, and the performance of the D-ELR-SLV-aided MMSE-LD is slight better than that of the SA-aided MMSE-LD. Next, as shown in Fig. 3(b), the LR-aided MMSE-SIC detectors generally exhibit better error performance than their corresponding LR-aided MMSE-LDs. Furthermore, all the LR-aided MMSE-SIC detectors exhibit significantly better performance than the MMSE-SIC detector, especially when the problem size is small to medium. As an example, the LLL-aided MMSE-SIC has an about two-order BER gain over MMSE-SIC when $N = M = 30$. In addition, compared to the case for 4QAM in Fig. 2(b), the LLL-aided MMSE-SIC, the D-LLL-aided MMSE-SIC, and the SA-aided MMSE-SIC are approaching one and another in the level of performance as N increases. In contrast, the proposed D-ELR-SLB-aided SIC detector obtains greater than one-order gain, and the proposed D-ELR-SLV-aided SIC detector obtains about half-order gain in BER over the existing LR-aided MMSE SIC detectors for large MIMO ($N \geq 40$).

Fig. 4 show the error performance of the LR-aided SIC detectors with post-ordering techniques, 64QAM, SNR at 30 dB, and $N = M$. We adopt the V-BLAST ordering method in [31], [38] and refer to the SIC using the V-BLAST order as "VBLAST-SIC." From Fig. 4(a), we observe that with V-BLAST order, the existing LR-aided VBLAST-SICs show better performance than the proposed D-ELR-aided VBLAST-SICs when $N \leq 20$, while the gain of the proposed D-ELR-SLB-aided VBLAST-SIC shows up when the number of antennas is large ($N \geq 50$). Compared Fig. 4(a) to 4(b), the MMSE-VBLAST-SICs show significant performance gain over the VBLAST-SIC. In addition, compared to the results without ordering in Fig. 3(b), the V-BLAST order significantly improves the error performance, especially for the MMSE-VBLAST-SIC. However, the price paid for the V-BLAST order is the extra complexity with order $\mathcal{O}(N^3)$ [38]. It is interesting to see that the LLL-aided MMSE-VBLAST-SIC shows worse error performance than the MMSE-VBLAST-SIC for large MIMO ($N \geq 50$). The D-LLL-aided MMSE-VBLAST-SIC, the SA-aided MMSE-VBLAST-SIC, and the proposed D-ELR-SLV-aided MMSE-VBLAST-SIC exhibit very close error performance, which is better than that of the MMSE-VBLAST-SIC, especially the number of antennas is relative small (e.g., $N \leq 40$). The proposed D-ELR-SLB-aided MMSE-VBLAST-SIC still achieves the best error performance among all LR-aided MMSE-VBLAST-SICs for large MIMO detection.

2) Complexity Comparisons with i.i.d. Channels: In this subsection, we illustrate the complexity of the LR methods. Besides the LLL, the D-LLL, the SA, and the proposed D-ELR-SLB and D-ELR-SLV, we also consider the complexity-reduced LLL (called "the LLL with SQRD") in [32], which applies sorted QR decomposition (SQRD) as the preprocessing step of the LLL. Table II summarizes the average number of basis updates, i.e., the average number of column addition and swapping operations on $\tilde{\bar{\mathbf{H}}}$, of the LR algorithms for the MMSE detectors in the MIMO systems with 64QAM, SNR at 30 dB, and $N = M$. It shows that the LLL algorithm requires the highest number of basis updates among the five LR algorithms when $N \geq 20$, and the D-LLL algorithm is the highest when $N < 20$. The SQRD preprocessing technique significantly reduces the number of basis updates of the LLL, but the number of basis updates of the LLL with SQRD is still higher is higher than that of the SA. Our proposed D-ELR-SLB algorithm requires lower number of basis updates than the SA, except for the case when $N = 20, 40$, and the proposed D-ELR-SLV algorithm requires the lowest number of basis updates for all N for MIMO systems.

The numbers of arithmetic operations (real additions and real multiplications) for basis updates of the LLL algorithm, the LLL algorithm with SQRD, the D-LLL algorithm, the SA, the D-ELR-SLB, and the D-ELR-SLV for MIMO systems are also plotted in Fig. 5. The SA has the highest number of operations among the five LR algorithms. This is because the SA requires $(128n - 18)$ arithmetic operations per basis update [11], [22], which is much higher than those of the LLL,

TABLE II
AVERAGE NUMBER OF BASIS UPDATES OF LLL, LLL WITH SQRD, D-LLL, SA, D-ELR-SLV, AND D-ELR-SLB FOR MIMO SYSTEMS.

N = M	5	10	20	40	60	80	100
LLL	11.00	44.33	133.15	326.58	534.21	741.95	961.08
LLL with SQRD	9.19	35.31	111.91	285.07	471.56	656.19	851.12
D-LLL	12.47	52.19	133.01	253.76	371.88	485.83	604.60
SA	8.23	22.07	41.14	82.95	132.70	185.68	243.45
D-ELR-SLV	5.39	11.95	23.89	46.65	69.46	91.835	114.43
D-ELR-SLB	7.72	19.42	41.58	84.62	127.74	170.69	213.89

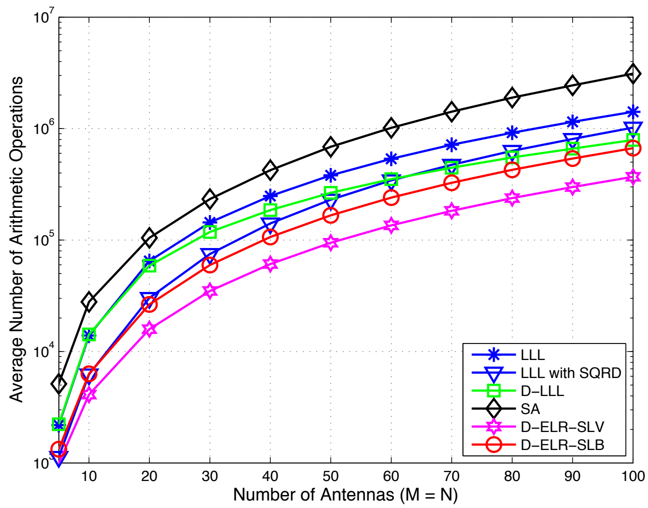


Fig. 5. Average number of arithmetic operations for basis updates of LLL, D-LLL, SA, D-ELR-SLV, and D-ELR-SLB with different numbers of antennas N for MIMO systems.

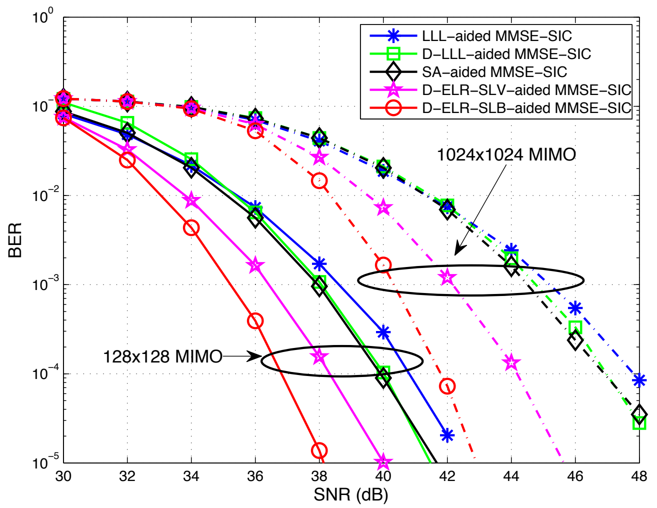


Fig. 6. Performance comparisons of the different detectors in very large MIMO systems with 256QAM.

the LLL with SQRD, the D-LLL, and the D-ELR algorithms. The proposed D-ELR algorithms require $(31n-7)$ arithmetic operations per basis update, including $16n+8$ for Eqs. (14) and (15) and $15(n-1)$ to update $\lambda_{i,k}$ and $\Delta_{i,k}$. The small number of basis updates and the low number of arithmetic operations for each basis update yield the lowest complexity of the D-ELR algorithms for large MIMO systems.

3) Error Performance Comparisons for Very Large MIMO Systems: Fig. 6 depicts the error performance of different LR-aided MMSE-SIC detectors for very large channel matrices ($N = 128, 1024$) with 256QAM. In this case, the proposed D-ELR-aided MMSE-SIC detectors shows significantly superior error performance to the state-of-the-art LR-aided MMSE-SIC detectors. Furthermore, the advantage of the proposed D-ELR-aided MMSE-SIC detectors increases as the number of antennas of the MIMO system increases, where the D-ELR-SLB-aided MMSE-SIC achieves more than a 3.5 dB gain and a 5 dB gain over the SA-aided MMSE-SIC for $N = 128, 1024$ at BER $= 10^{-5}$, respectively, and the D-ELR-SLV-aided MMSE-SIC achieves more than a 1.5 dB gain and a 2 dB gain over the SA-aided MMSE-SIC for $N = 128, 1024$ at BER $= 10^{-5}$, respectively. The SA-aided MMSE-SIC performs almost the same as the D-LLL-aided MMSE-SIC and slightly better than the LLL-aided MMSE-SIC.

4) Error Performance Comparisons for Correlated Channels: This test case considers the error performance of the LR-aided detectors with correlated MIMO channels. The MIMO channel is modeled in [39] as

$$\mathbf{H} = \mathbf{\Sigma}_{\mathrm{rx}}^{\frac{1}{2}}\mathbf{H}'\mathbf{\Sigma}_{\mathrm{tx}}^{\frac{1}{2}}, \tag{31}$$

where $\mathbf{H}'$ is an $M \times N$ matrix having i.i.d. zero-mean and unit-variance Gaussian variables, and $\mathbf{\Sigma}_{\mathrm{rx}}$ and $\mathbf{\Sigma}_{\mathrm{tx}}$ are $M \times M$ and $N \times N$ correlation matrices of the receive and transmit antennas, respectively. We adopt channel scenario A in [39] with the uniform linear array (ULA) antenna configuration. The spacing of the antennas is adjusted such that the absolute value of the correlation between the adjacent antennas is 0.3. Fig. 7 shows that our proposed D-ELR-aided MMSE-SIC detectors still achieve significant error performance gain over the existing LR-aided detectors in the large correlated MIMO channels ($N > 30$). The LLL-aided MMSE-SIC, the D-LLL-aided MMSE-SIC, and the SA-aided MMSE-SIC show similar BER performance, and they exhibit error performance closer to the MMSE-SIC compared to those in the absence of the correlations in Fig. 3.

5) Performance Comparisons with Different Ordering Methods: In this test case, the error performance of the SIC detectors using different post-ordering methods is compared in a 64×64 MIMO system with 256QAM. For the comparisons, we adopt the V-BLAST ordering method in [31], [38]. From Fig. 8, the proposed LR-aided MMSE-SV-SICs can achieve close performance to their corresponding LR-aided MMSE-VBLAST-SICs, which require the extra complexity $\mathcal{O}(N^3)$ for the V-BLAST ordering when $N = M$ given $\tilde{\mathbf{R}}_e$. For example, the LLL-aided MMSE-VBLAST-SIC has an about 0.4 dB gain over the LLL-aided MMSE-SV-SIC at BER $= 10^{-5}$.

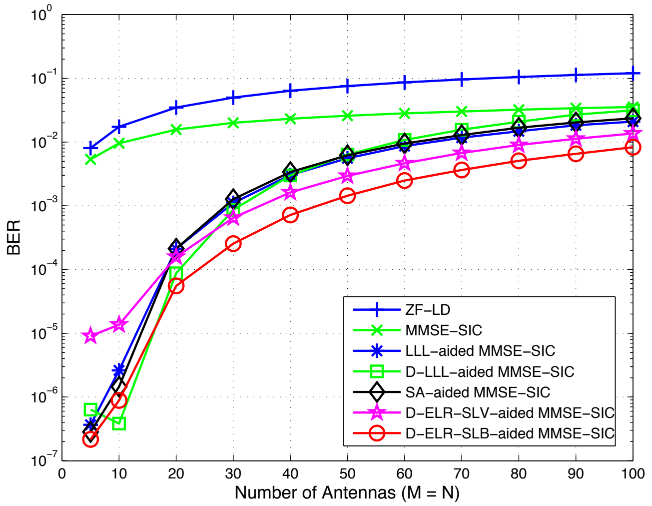


Fig. 7. Performance comparisons of the different detectors with 64QAM, SNR = 30 dB, and correlated MIMO channels.

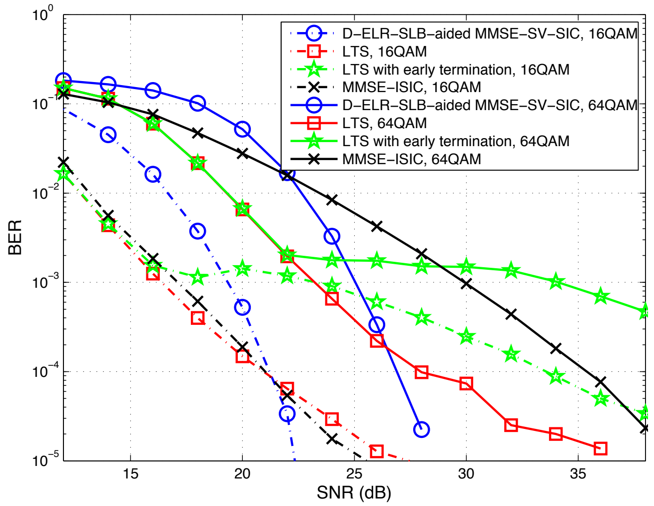


Fig. 9. Performance comparisons of different detectors with $M = N = 32$ and different constellation sizes.

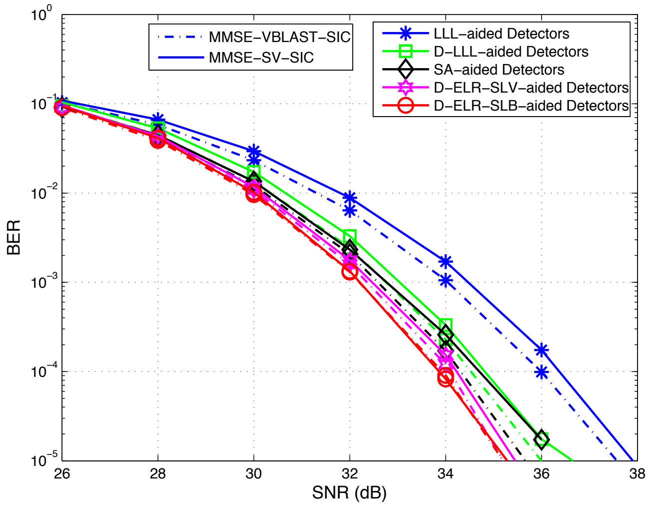


Fig. 8. Performance comparisons of the detectors with different ordering methods, $M = N = 64$, and 256QAM.

In addition, both the D-ELR-SLV-aided MMSE-SV-SIC and the D-ELR-SLB-aided MMSE-SV-SIC show almost the same performance as the D-ELR-SLV-aided and D-ELR-SLB-aided MMSE-VBLAST-SICs, which outperform the existing LR-aided MMSE-VBLAST-SICs at BER $= 10^{-5}$.

B. Comparisons of the ELR-aided Detector with Other Detectors

The error performance and the complexity of the D-ELR-SLB-aided MMSE-SV-SIC, the LTS, and the MMSE-ISIC with 5 iterations are compared in a 32×32 MIMO system. From Figs. 9 and 10, one can observe that the LTS exhibits better error performance in the low-to-medium SNR region (SNR $<$ 26 dB) at much higher complexity than the D-ELR-SLB-aided MMSE-SV-SIC for 64QAM (see Fig. 10). In addition, as shown in Fig. 10, the complexity of the LTS varies considerably for different realizations of noise and channel, where the worst-case complexity (indicated as 99.9% upper percentile) is much higher than the average complexity. Thus, in low-to-medium SNRs, the complexity of the LTS in worst case is about two orders higher than that of the D-ELR-SLB-aided MMSE-SV-SIC. In addition, the performance gain of the LTS over our proposed ones decreases as SNR increases, where the LTS shows an about 6 dB loss to the D-ELR-SLB aided MMSE-SV-SIC at BER = 2×10^{-5}. Furthermore, to demonstrate how the worst-case instantaneous complexity affects LTS performance, we also plot the performance of the LTS with early termination in Fig. 9, which stops the tabu search and outputs the estimates when the instantaneous complexity exceeds the 99% upper percentile of the LTS complexity. As a result of the incomplete (early terminated) tabu search, significant performance degradation is found when the 99.9% upper percentile complexity of the LTS is much higher than the 99% upper percentile one, e.g., SNR $>$ 22 dB for 64QAM. Additionally, since the 99% upper percentile complexity of the LTS is very close to the worst-case complexity (e.g., the 99.9% upper percentile complexity) for SNR $<$ 22 dB and 64QAM, no performance loss occurs for the LTS with early termination. Hence, all these results suggest that the worst-case high-complexity tabu search makes a critical contribution to the error performance of the LTS.

The MMSE-ISIC shows better performance for 64QAM in the low SNR region (SNR $<$ 22 dB), while the D-ELR-SLB-aided MMSE-SV-SIC, which requires much lower complexity than the MMSE-ISIC, significantly outperforms the MMSE-ISIC in medium-to-high SNRs (SNR $>$ 22 dB). For the case of 16QAM, the LTS and the MMSE-ISIC exhibit similar performance but still show worse performance than the D-ELR-SLB-aided MMSE-SV-SIC in high SNRs while requiring much higher complexity (not shown in this paper).

Besides the aforementioned comments about the complexity of all the methods, from Fig. 10, two additional remarks about the complexity of the D-ELR-SLB-aided MMSE-SV-SIC can be drawn: *i*) Although the instantaneous complexity of D-ELR-SLB depends on the channel realizations, the worst-case complexity (99.9% upper percentile) of the D-ELR-SLB-aided MMSE-SV-SIC is almost the same as the average complexity; and *ii*) the complexity of the D-ELR-SLB-aided MMSE-SV-

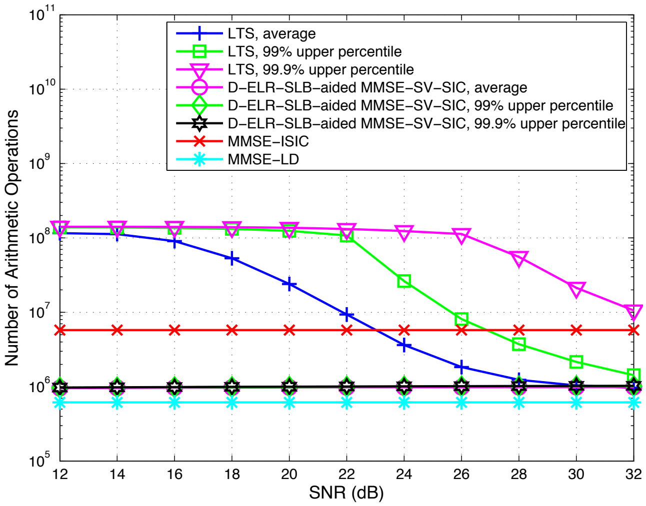


Fig. 10. Number of arithmetic operations of different detectors with $M = N = 32$, 64QAM, and different SNRs for MIMO systems.

SIC is only slightly higher than that of the low-complexity MMSE-LD. The reason for the similar levels of complexity is that, while the complexity of D-ELR-SLB constitutes only a small portion of the overall detection complexity (c.f. Figs. 5 and 10), the main computational complexity of the D-ELR-SLB-aided MMSE-SV-SIC consists of the preprocessing operations (i.e., matrix inversion of the Gram matrix and QR decomposition), which requires the same polynomial order complexity of the MMSE-LD.

VII. Conclusion

In this paper, we proposed two element-based lattice reduction algorithms that achieve high performance with low complexity. For large MIMO systems, we have shown that: *i*) The D-ELR-SLV-aided and D-ELR-SLB-aided detectors significantly outperform the linear detectors and SIC detectors with a small amount of extra complexity; *ii*) the performance of the D-ELR-SLV-aided and D-ELR-SLB-aided detectors is superior to those of the state-of-the-art LR-aided detectors, and their complexity is lower for large MIMO systems; *iii*) compared to the LTS and the MMSE-ISIC detectors, the ELR-SLB-aided detector attains better performance in medium-to-high SNRs and requires much lower complexity; *iv*) the complexity of the proposed D-ELR-SLB-aided detector is almost independent from SNRs, constellations, and channel realizations; and *v*) the proposed sorted variance ordering method significantly enhances the error performance of the LR-aided SIC detectors without requiring extra complexity. In addition, the ELR algorithms improve the orthogonality deficiency of the original channel matrix in the primal or dual space. For two-dimensional system, the proposed D-ELR-SLB and D-ELR-SLV algorithms yield the same basis (up to signs and permutation) as the Gaussian reduction algorithm. Furthermore, the performance of the D-ELR-aided detectors is robust to correlated MIMO channels. Because of these benefits, the D-ELR-aided detectors are promising for future large MIMO systems with affordable complexity. Some future topics to improve ELR include applying ELR to grouped large MIMO systems and soft-input soft-output ELR-aided detectors for the coded case.

Appendix A

Since the determinant of the dual lattice $\det(\mathbb{L}') = \sqrt{\det(\mathbf{H}'^{\mathcal{H}}\mathbf{H}')}$ does not change, from the definition of od in Eq. (30), we obtain

$$od(\tilde{\mathbf{H}}') = 1 - \frac{(\det(\mathbb{L}'))^2}{\prod_{n=1}^{N} \tilde{C}_{n,n}}. \quad (32)$$

Based on the fact that, for each iteration, one of the diagonal elements $\tilde{C}_{n,n}$ is decreased by $\Delta_{i,k}$, and $\Delta_{i,k}$ is strictly larger than zero, we conclude that the od is monotonically decreasing for each iteration.

Appendix B

Suppose $N = 3$ and $\tilde{\mathbf{C}}$ is

$$\tilde{\mathbf{C}} = \begin{bmatrix} 1 & 0.5-\epsilon & 0.5-\epsilon \\ 0.5-\epsilon & 1 & -0.5+\epsilon \\ 0.5-\epsilon & -0.5+\epsilon & 1 \end{bmatrix}, \quad (33)$$

with $0 < \epsilon < 0.5$. It is easily checked that the dual basis corresponding to the matrix $\tilde{\mathbf{C}}$ is a D-ELR-reduced basis. From Eq. (32), the od of the dual basis is $od(\tilde{\mathbf{H}}') = 1 - \det(\tilde{\mathbf{C}}) = 2(0.5-\epsilon)^3 + 3(0.5-\epsilon)^2$, yielding

$$\lim_{\epsilon \to 0^+} od(\tilde{\mathbf{H}}') = 1. \quad (34)$$

When $N > 3$, one can construct the matrix as

$$\tilde{\mathbf{C}}_{N \times N} = \begin{bmatrix} \tilde{\mathbf{C}} & \mathbf{0}_{3\times(N-3)} \\ \mathbf{0}_{(N-3)\times 3} & \mathbf{I}_{N-3} \end{bmatrix}, \quad (35)$$

whose dual basis has the same od as that when $N = 3$.

For the corresponding primal basis $\tilde{\mathbf{H}}$, if $N = 3$, from Eq. (33), the corresponding Gram matrix is

$$\tilde{\mathbf{G}} = \tilde{\mathbf{C}}^{-1} = G_c \begin{bmatrix} 1 & -\frac{0.5-\epsilon}{0.5+\epsilon} & -\frac{0.5-\epsilon}{0.5+\epsilon} \\ -\frac{0.5-\epsilon}{0.5+\epsilon} & 1 & \frac{0.5-\epsilon}{0.5+\epsilon} \\ -\frac{0.5-\epsilon}{0.5+\epsilon} & \frac{0.5-\epsilon}{0.5+\epsilon} & 1 \end{bmatrix}, \quad (36)$$

where G_c is a constant that is not of interest. Then, $od(\tilde{\mathbf{H}}) = 3\left(\frac{0.5-\epsilon}{0.5+\epsilon}\right)^2 - 2\left(\frac{0.5-\epsilon}{0.5+\epsilon}\right)^3$ and

$$\lim_{\epsilon \to 0^+} od(\tilde{\mathbf{H}}) = 1. \quad (37)$$

When $N > 3$, the Gram matrix corresponding to the matrix in Eq. (35) has the same od as that in (36).

As a summary, if $N \geq 3$, then there exists a D-ELR-SLB-reduced basis $\tilde{\mathbf{H}}'$ with $od(\tilde{\mathbf{H}}') = 2(0.5-\epsilon)^3 + 3(0.5-\epsilon)^2$, $od(\tilde{\mathbf{H}}) = 3\left(\frac{0.5-\epsilon}{0.5+\epsilon}\right)^2 - 2\left(\frac{0.5-\epsilon}{0.5+\epsilon}\right)^3$, and $0 < \epsilon < 0.5$ such that for any given $0 < \varepsilon < 1$, we have $\varepsilon < \min(od(\tilde{\mathbf{H}}'), od(\tilde{\mathbf{H}})) = \min\left(2(0.5-\epsilon)^3 + 3(0.5-\epsilon)^2, 3\left(\frac{0.5-\epsilon}{0.5+\epsilon}\right)^2 - 2\left(\frac{0.5-\epsilon}{0.5+\epsilon}\right)^3\right)$.

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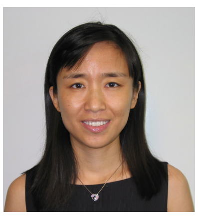

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