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Expectations of Functions of Complex Wishart Matrix

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Abstract In this article, we study lower and upper triangular factorizations of the complex Wishart matrix. Further, using these factorizations, we obtain several expected values of scalar and matrix valued functions of the complex Wishart matrix. We also generalize Muirhead's identity for the complex case which gives a number of interesting special cases.

Keywords Bartlett's decomposition · Complex random matrix · Jacobian · Moments · Complex multivariate gamma function · Transformation · Wishart distribution

Mathematics Subject Classification (2000) Primary 62H10 · Secondary 62E20

1 Introduction

An $m \times m$ random Hermitian positive definite matrix A is said to have a Wishart distribution with parameters m, n $(n \geq m)$ and Σ, written as $A \sim \mathbb{C}W_m(n, \Sigma)$, if its p.d.f. is given by

$$\{\tilde{\Gamma}_m(n) \det(\Sigma)^n\}^{-1} \det(A)^{n-m} \operatorname{etr}(-\Sigma^{-1} A), \tag{1.1}$$

where Σ is an $m \times m$ Hermitian positive definite matrix and $\tilde{\Gamma}_m(\alpha)$ is the complex multivariate gamma function defined by

$$\tilde{\Gamma}_m(\alpha) = \pi^{m(m-1)/2} \prod_{i=1}^{m} \Gamma(\alpha - i + 1), \quad \operatorname{Re}(\alpha) > m - 1. \tag{1.2}$$

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The complex Wishart distribution is the joint distribution of sample variances and covariances from complex multivariate normal population (Goodman [7], Wooding [29]) and plays an important role in multivariate statistical analysis. Several test statistics in complex multivariate statistical analysis are functions of complex Wishart matrices. Complex Wishart and complex inverse Wishart distributed random matrices are used in applications like radar, sonar, or seismic in order to model the statistical properties of complex sample covariance matrices and complex inverse sample covariance matrices, respectively (Maiwald and Kraus [19]).

The complex matrix variate distributions play an important role in various fields of research. Applications of complex random matrices can be found in multiple time series analysis, nuclear physics and radio communications (Carmeli [2], Krishnaiah [17], Mehta [20] and Smith and Gao [25]). A number of results on the distributions of complex random matrices have also been derived. The complex matrix variate Gaussian distribution was introduced by Wooding [29], Turin [28], and Goodman [7]. The complex Wishart distribution was studied by Goodman [7, 8], Srivastava [26], Hayakawa [11], Chikuse [3] and Gupta and Kabe [9]. James [12] and Khatri [13] derived the complex central as well as the noncentral matrix variate beta distributions. Distributional results on quadratic forms involving complex normal variables were given by Khatri [14] and Conradie and Gupta [4]. Nagar and Arias [22] defined complex matrix Cauchy distribution and studied its properties. Systematic treatment of the distributions of complex random matrices was given by Tan [27] which included the Gaussian, Wishart, beta, and Dirichlet distributions.

In this article, we obtain several expected values of scalar and matrix valued functions of the complex Wishart matrix. To obtain these expectations we use lower and upper triangular factorizations of the complex Wishart matrix and Muirhead's identity (Muirhead [21]) for the complex case. Section 2 gives definitions and results on matrix algebra, complex multivariate Gaussian distribution, complex Wishart distribution and Jacobians of certain matrix transformations. Section 3 deals with several expected values of scalar and matrix valued functions of the complex Wishart matrix. Finally, in Sect. 4, using triangular factorizations of the complex Wishart matrix, we obtain the distribution of signal to noise ratio.

2 Some Useful Results

In this section we will define complex Wishart distribution and derive some of its properties. We first state the following notations and results (Khatri [13], Srivastava [26], Andersen, Højbjerre, Sørensen and Eriksen [1]) that will be used in this and subsequent sections. Let $A = (a_{ij})$ be an $m \times m$ matrix of complex numbers. Then, A' denotes the transpose of A; $\bar{A}$ denotes conjugate of A; A^H denotes conjugate transpose of A; $\operatorname{tr}(A) = a_{11} + \cdots + a_{mm}$; $\operatorname{etr}(A) = \exp(\operatorname{tr}(A))$; $\det(A) =$ determinant of A; $\det(A)_+ =$ absolute value of $\det(A)$; $A = A^H > 0$ means that A is Hermitian positive definite and $A^{1/2}$ denotes the unique Hermitian positive definite square root of $A = A^H > 0$. For the partition $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{smallmatrix}\right)$, $\det(A_{ii}) \neq 0$, $i = 1, 2$, the Schur compliments of A_{22} and A_{11} are defined as $A_{11\cdot 2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ and $A_{22\cdot 1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$, respectively. Further, assuming that A_{11}^{-1}, A_{22}^{-1}, $A_{11\cdot 2}^{-1}$ and $A_{22\cdot 1}^{-1}$ exist, we have

$$A^{-1} = \begin{pmatrix} A_{11\cdot 2}^{-1} & -A_{11\cdot 2}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}A_{11\cdot 2}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}A_{11\cdot 2}^{-1}A_{12}A_{22}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} A_{22\cdot 1}^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22\cdot 1}^{-1} \\ -A_{22\cdot 1}^{-1} A_{21} A_{11}^{-1} & A_{22\cdot 1}^{-1} \end{pmatrix}.$$

Lemma 2.1 (Khatri [13]) *Let Z and W be $m \times m$ Hermitian positive definite matrices. If $Z = GWG^H$, where $G\,(m \times m)$ is a complex nonsingular matrix, then $J(Z \to W) =$* $\det(GG^H)^m$.

Lemma 2.2 (Goodman [7]) *Let W be a Hermitian positive definite matrix and $W = TT^H$ where T is an $m \times m$ complex triangular matrix with positive diagonal elements. Then, (i) $J(W \to T) = 2^m \prod_{i=1}^m t_{ii}^{2(m-i)+1}$ if T is lower triangular, and (ii) $J(W \to T) = 2^m \prod_{i=1}^m t_{ii}^{2i-1}$ if T is upper triangular.*

Definition 2.1 The complex multivariate gamma function, denoted by $\tilde{\Gamma}_m(\alpha)$, is defined by

$$\tilde{\Gamma}_m(\alpha) = \int_{X=X^H>0} \operatorname{etr}(-X) \det(X)^{\alpha-m}\, dX, \quad \operatorname{Re}(\alpha) > m-1. \tag{2.1}$$

By evaluating the integral in (2.1), the complex multivariate gamma function can be expressed as product of ordinary gamma functions given by (1.2).

Theorem 2.1 *Let $X\,(m \times n)$ be a complex matrix of rank $m\,(\le n)$ and $f(X)$ be a function of X which depends on X through XX^H only. That is $f(X) = g(XX^H)$ for some function g. Then,*

$$\int_{XX^H=W} f(X)\, dX = \frac{\pi^{nm}}{\tilde{\Gamma}_m(n)} \det(W)^{n-m} g(W), \quad W = W^H > 0. \tag{2.2}$$

Proof See Srivastava [26]. □

Now, we give definition of the complex multivariate normal distribution and certain properties of the complex Wishart distribution that are used in deriving results in this and subsequent sections.

Definition 2.2 The m-dimensional complex random vector $\mathbf{z}$ is said to have a complex multivariate normal distribution with complex mean vector $\boldsymbol{\mu}$ and Hermitian positive definite covariance matrix Σ, denoted as $\mathbf{z} \sim \mathbb{C}N_m(\boldsymbol{\mu}, \Sigma)$, if its p.d.f. is given by

$$\pi^{-m} \det(\Sigma)^{-1} \exp[-(z-\boldsymbol{\mu})^H \Sigma^{-1} (z-\boldsymbol{\mu})], \quad z \in \mathbb{C}^m.$$

Theorem 2.2 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$ and C be any $m \times m$ complex nonsingular matrix. Then, $CAC^H \sim \mathbb{C}W_m(n, C\Sigma C^H)$.*

Proof The result follows by making the transformation $V = CAC^H$ with the Jacobian $J(A \to V) = \det(CC^H)^{-m}$ in the density of A given by (1.1). □

Corollary 2.1 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$ and $\Sigma^{-1} = C^H C$ where C is an $m \times m$ non-singular matrix. Then, $CAC^H \sim \mathbb{C}W_m(n, I_m)$.*

Theorem 2.3 *Let $A \sim \mathbb{C}W_m(n, I_m)$ and U $(m \times m)$ be an unitary matrix, whose elements are either constants or random variables distributed independently of A. Then, the distribution of A is invariant under the transformation $A \to UAU^H$ and is independent of U in the latter case.*

Proof First, let U be a non-random unitary matrix. Then, from Theorem 2.2, $UAU^H \sim \mathbb{C}W_m(n, I_m)$. If, however, U is a random unitary matrix, then $UAU^H|U \sim \mathbb{C}W_m(n, I_m)$. Since this distribution does not depend on U, $UAU^H \sim \mathbb{C}W_m(n, I_m)$. □

The following result is of importance in complex multivariate analysis and is known as Bartlett's decomposition.

Theorem 2.4 *Let $A \sim \mathbb{C}W_m(n, I_m)$, and $A = TT^H$, where $T = (t_{ij})$ is a complex upper triangular matrix with $t_{ii} > 0$ and $t_{ij} = t_{1ij} + \sqrt{-1}\, t_{2ij}$, $i < j$. Then t_{ij}, $1 \le i \le j \le m$ are independently distributed, $t_{ii}^2 \sim G(n - m + i)$, $1 \le i \le m$ and $t_{ij} \sim \mathbb{C}N(0, 1)$, $1 \le i < j \le m$.*

Proof The density of A is

$$\{\tilde{\Gamma}_m(n)\}^{-1} \det(A)^{n-m} \operatorname{etr}(-A), \quad A = A^H > 0. \tag{2.3}$$

Transforming $A = TT^H$, with the Jacobian $J(A \to T) = 2^m \prod_{i=1}^m t_{ii}^{2i-1}$, in (2.3), we get the joint density of $t_{11}, t_{12}, \ldots, t_{1m}, t_{22}, \ldots, t_{2m}, \ldots, t_{mm}$ as

$$\begin{aligned} &\{\tilde{\Gamma}_m(n)\}^{-1} 2^m \prod_{i=1}^m (t_{ii}^2)^{n-m+i-1/2} \exp\left(-\sum_{1\le i\le j\le m} |t_{ij}|^2\right) \\ &= \prod_{1\le i<j\le m} \left\{\frac{\exp(-|t_{ij}|^2)}{\pi}\right\} \prod_{i=1}^m \left\{\frac{2(t_{ii}^2)^{n-m+i-1/2}\exp(-t_{ii}^2)}{\Gamma(n-m+i)}\right\}, \\ &\quad t_{ii} > 0,\ 1 \le i \le m,\ -\infty < t_{1ij}, t_{2ij} < \infty,\ 1 \le i < j \le m. \end{aligned} \tag{2.4}$$

From (2.4), it is easily seen that t_{ij}, $1 \le i \le j \le m$, are independently distributed and $t_{ij} \sim \mathbb{C}N(0, 1)$, $1 \le i < j \le m$. By substituting $y_{ii} = t_{ii}^2$, one can show that $t_{ii}^2 \sim G(n - m + i)$, $1 \le i \le m$. □

The univariate gamma distribution denoted by $G(a)$ in the above theorem is defined by the p.d.f.

$$\{\Gamma(a)\}^{-1} x^{a-1} \exp(-x), \quad x > 0.$$

A similar result can also be proved for a lower triangular factorization of A, as given in the next theorem.

Theorem 2.5 *Let $A \sim \mathbb{C}W_m(n, I_m)$ and $A = TT^H$, where $T = (t_{ij})$ is a complex lower triangular matrix with $t_{ii} > 0$ and $t_{ij} = t_{1ij} + \sqrt{-1}\, t_{2ij}$, $j < i$. Then, t_{ij}, $1 \le j \le i \le m$ are independently distributed, $t_{ii}^2 \sim G(n - i + 1)$, $1 \le i \le m$ and $t_{ij} \sim \mathbb{C}N(0, 1)$, $1 \le j < i \le m$.*

Proof Similar to the proof of Theorem 2.4. □

Theorem 2.6 *Let $A \sim \mathbb{C}W_m(n, I_m)$, and $\mathbf{x} \sim \mathbb{C}N_m(\mathbf{0}, I_m)$ be independent. Then,*

$$\mathbf{x}^H (C^H C)^{-1} \mathbf{x} \sim IB(m, n-m+1),$$

where $A = CC^H$, the complex matrix C being either triangular or nonsingular, and the inverted beta distribution designated by $IB\,(\alpha, \beta)$ is defined by the p.d.f.

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}, \quad \alpha > 0,\ \beta > 0,\ x > 0.$$

Proof Let $\mathbf{y} = (C^{-1})^H \mathbf{x}$. Then $\mathbf{y}|C \sim \mathbb{C}N_m(\mathbf{0}, (C^{-1})^H C^{-1})$. Denote the conditional and the unconditional densities of $\mathbf{y}$ by $f(\mathbf{y}|C)$ and $f(\mathbf{y})$ respectively. Then,

$$\begin{aligned} f(\mathbf{y}) &= E_C[f(\mathbf{y}|C)] = E_C[f(\mathbf{y}|CC^H)] \\ &= E_{CC^H}[f(\mathbf{y}|CC^H)] = \int_{A=A^H>0} f(\mathbf{y}|A) g(A)\, dA, \end{aligned}$$

where $g(A)$ is the p.d.f. of A. Now,

$$\begin{aligned} f(\mathbf{y}) &= \frac{1}{\pi^m \tilde{\Gamma}_m(n)} \int_{A=A^H>0} \det(A)^{n-m+1} \operatorname{etr}(-A - A\mathbf{y}\mathbf{y}^H)\, dA \\ &= \frac{\tilde{\Gamma}_m(n+1)}{\pi^m \tilde{\Gamma}_m(n)} \det(I_m + \mathbf{y}\mathbf{y}^H)^{-(n+1)} \\ &= \frac{\Gamma(n+1)}{\pi^m \Gamma(n-m+1)} \det(I_m + \mathbf{y}^H\mathbf{y})^{-(n+1)}. \end{aligned}$$

Finally, using Theorem 2.1, we get the density of $\mathbf{y}^H\mathbf{y} = \mathbf{x}^H (C^H C)^{-1} \mathbf{x} = v$ (say) as

$$\frac{\Gamma(n+1)}{\Gamma(m)\Gamma(n-m+1)} \frac{v^{m-1}}{(1+v)^{n+1}},\ v > 0,$$

which is the desired result. □

3 Expected Values

Theorem 3.1 *If $A \sim \mathbb{C}W_m(n, \Sigma)$, then $\det(\Sigma^{-1}A) \sim \prod_{i=1}^m u_i$, where u_i's are independent, $u_i \sim G(n-i+1)$, $i = 1, \ldots, m$, and*

$$E[\det(A)^h] = \det(\Sigma)^h \prod_{i=1}^m \frac{\Gamma(n-i+1+h)}{\Gamma(n-i+1)}, \quad \operatorname{Re}(h) \geq -n+m. \tag{3.1}$$

Proof Let $V = \Sigma^{-1/2} A \Sigma^{-1/2}$, then from Corollary 2.1, $V \sim \mathbb{C}W_m(n, I_m)$. Now, from Theorem 2.5, V can be written as TT^H and $\det(V) = \det(A)\det(\Sigma)^{-1} = \prod_{i=1}^m u_i$, where $u_i = t_{ii}^2$ are independently distributed as $G(n-i+1)$, $i = 1, \ldots, m$. Further,

$$E[\det(V)^h] = E\left(\prod_{i=1}^m u_i\right)^h = \prod_{i=1}^m \left\{\frac{\Gamma(n-i+1+h)}{\Gamma(n-i+1)}\right\}, \quad \operatorname{Re}(h) \geq -n+m.$$

The final result is obtained by noting that $E[\det(A)^h] = \det(\Sigma)^h E[\det(V)^h]$. □

Alternately, $E[\det(A)^h]$ can be evaluated by integrating over the density of A as

$$E[\det(A)^h] = \int_{A=A^H>0} \det(A)^h \frac{\det(A)^{n-m}\,\mathrm{etr}(-\Sigma^{-1}A)}{\det(\Sigma)^n \tilde{\Gamma}_m(n)}\, dA$$
$$= \det(\Sigma)^h \frac{\tilde{\Gamma}_m(n+h)}{\tilde{\Gamma}_m(n)}, \quad \mathrm{Re}(h) \geq -n+m. \tag{3.2}$$

Writing complex multivariate gamma functions in terms of ordinary gamma functions we get (3.1).

Theorem 3.1 was first derived by Goodman [8] using characteristic function approach.

Lemma 3.1 *Let $A \sim \mathbb{C}W_m(n, I_m)$. Then, for a positive integer r, (i) $E(A^r) = \tilde{c}(r,n,m)I_m$, and (ii) if $E(A^{-r})$ exists then it is given by $E(A^{-r}) = \tilde{d}(r,n,m)I_m$, where $\tilde{c}(r,n,m)$ and $\tilde{d}(r,n,m)$ are constants depending on r, n and m.*

Proof Since, for any $m \times m$ unitary matrix U, the complex matrices A and UAU^H have same distribution, we have

$$E(A^r) = E[(UAU^H)^r] = UE(A^r)U^H,$$

which implies that

$$E(A^r)U = UE(A^r).$$

Thus, $E(A^r)$ must be a scalar multiple of the identity matrix. Similar argument holds for $E(A^{-r})$. □

Lemma 3.2 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$. Then, (i) $E(A) = \tilde{c}(1,n,m)\,\Sigma$ (ii) $E(A^{-1}) = \tilde{d}(1,n,m)\,\Sigma^{-1}$ (iii) $E(A\Sigma^{-1}A) = \tilde{c}(2,n,m)\,\Sigma$ (iv) $E(A^{-1}\Sigma A^{-1}) = \tilde{d}(2,n,m)\,\Sigma^{-1}$ (v) $E(A\Sigma^{-1}A\Sigma^{-1}A) = \tilde{c}(3,n,m)\,\Sigma$ and (vi) $E(A^{-1}\Sigma A^{-1}\Sigma A^{-1}) = \tilde{d}(3,n,m)\,\Sigma^{-1}$, where $\tilde{c}(r,n,m)$ and $\tilde{d}(r,n,m)$ are defined in Lemma 3.1.*

Proof Let $\Sigma = QQ^H$ where Q is an $m \times m$ non-singular matrix. Define $V = Q^{-1}A(Q^H)^{-1}$ and observe that $V \sim \mathbb{C}W_m(n, I_m)$. Now

$$E(A) = E(QVQ^H) = \tilde{c}(1,n,m)QQ^H = \tilde{c}(1,n,m)\Sigma,$$

$$E(A^{-1}) = E[(Q^H)^{-1}V^{-1}Q^{-1}]$$
$$= \tilde{d}(1,n,m)(QQ^H)^{-1} = \tilde{d}(1,n,m)\Sigma^{-1},$$

and

$$E(A^{-1}\Sigma A^{-1}) = E[(Q^H)^{-1}V^{-2}Q^{-1}]$$
$$= \tilde{d}(2,n,m)(QQ^H)^{-1} = \tilde{d}(2,n,m)\Sigma^{-1}.$$

Similarly, one can check that $E(A\Sigma^{-1}A) = \tilde{c}(2,n,m)\Sigma$, $E(A\Sigma^{-1}A\Sigma^{-1}A) = E(QV^3Q^H)$ $= \tilde{c}(3,n,m)\Sigma$ and $E(A^{-1}\Sigma A^{-1}\Sigma A^{-1}) = E[(Q^H)^{-1}V^{-3}Q^{-1}] = \tilde{d}(3,n,m)\Sigma^{-1}$. □

Theorem 3.2 *Let the constants $\tilde{c}(r,n,m)$ and $\tilde{d}(r,n,m)$ be defined as in Lemma 3.1. Then,*

$$\begin{aligned}
&\tilde{c}(1,n,m)=n,\\
&\tilde{c}(2,n,m)=n(n+m),\\
&\tilde{d}(1,n,m)=(n-m)^{-1}, \quad n>m,\\
&\tilde{d}(2,n,m)=n[(n-m-1)(n-m)(n-m+1)]^{-1}, \quad n>m+1.
\end{aligned}$$

Proof Let $A=(a_{ij})$, then $A^2=(\sum_{k=1}^m a_{ik}\bar{a}_{jk})$. Further, let $A=TT^H$, where $T=(t_{ij})$ is a complex lower triangular matrix with $t_{ii}>0$. Partition T as

$$T=\begin{pmatrix} t_{11} & 0\\ \mathbf{t}_{21} & T_{22}\end{pmatrix}, \quad T_{22}\,((m-1)\times(m-1)).$$

Then $a_{11}=t_{11}^2$ and $\sum_{k=1}^m a_{1k}\bar{a}_{1k}=t_{11}^4+t_{11}^2\mathbf{t}_{21}^H\mathbf{t}_{21}$. From Theorem 2.5, t_{11} and $\mathbf{t}_{21}$ are independent, $t_{11}^2\sim G(n)$ and $\mathbf{t}_{21}\sim \mathbb{C}N_{m-1}(\mathbf{0},I_m)$. Hence

$$\tilde{c}(1,n,m)=E(a_{11})=E(t_{11}^2)=n,$$

and

$$\begin{aligned}
\tilde{c}(2,n,m)&=E\left[\sum_{k=1}^m a_{1k}\bar{a}_{1k}\right]=E(t_{11}^4+t_{11}^2\mathbf{t}_{21}^H\mathbf{t}_{21})\\
&=n(n+1)+n(m-1)=n(n+m),
\end{aligned}$$

where we have used the result $\mathbf{t}_{21}^H\mathbf{t}_{21}\sim G(m-1)$. Similarly if $A^{-1}=(a^{ij})$, then $A^{-2}=(\sum_{k=1}^m a^{ik}\bar{a}^{jk})$. Further, let $A=TT^H$, where $T=(t_{ij})$ is a complex upper triangular matrix with $t_{ii}>0$. Partition T as

$$T=\begin{pmatrix} t_{11} & \mathbf{t}^H\\ 0 & T_{22}\end{pmatrix}, \quad T_{22}\,((m-1)\times(m-1)).$$

Then,

$$T^{-1}=\begin{pmatrix} t_{11}^{-1} & -t_{11}^{-1}\mathbf{t}^H T_{22}^{-1}\\ 0 & T_{22}^{-1}\end{pmatrix},$$

and

$$\begin{aligned}
A^{-1}&=(TT^H)^{-1}=(T^H)^{-1}T^{-1}\\
&=\begin{pmatrix} t_{11}^{-2} & -t_{11}^{-2}\mathbf{t}^H T_{22}^{-1}\\ -t_{11}^{-2}(T_{22}^H)^{-1}\mathbf{t} & (T_{22}T_{22}^H)^{-1}+t_{11}^{-2}(T_{22}^H)^{-1}\mathbf{t}\mathbf{t}^H T_{22}^{-1}\end{pmatrix}.
\end{aligned}\tag{3.3}$$

From (3.3) it follows that $a^{11}=t_{11}^{-2}$ and $\sum_{k=1}^m a^{1k}\bar{a}^{1k}=t_{11}^{-4}+t_{11}^{-4}\mathbf{t}^H(T_{22}^H T_{22})^{-1}\mathbf{t}$. From Theorem 2.4, t_{11}, $\mathbf{t}$ and T_{22} are independent, $t_{11}^2\sim G(n-m+1)$, $\mathbf{t}\sim\mathbb{C}N_{m-1}(\mathbf{0},I_m)$ and $T_{22}T_{22}^H\sim \mathbb{C}W_{m-1}(n,I_{m-1})$. Further, from Theorem 2.6, $\mathbf{t}^H(T_{22}^H T_{22})^{-1}\mathbf{t}\sim IB(m-1,n-m+2)$. Hence

$$\tilde{d}(1,n,m)=E(a^{11})=E(t_{11}^{-2})=(n-m)^{-1}, \quad n>m,$$

and

$$\tilde{d}(2,n,m) = E\left[\sum_{k=1}^{m} a^{1k}\bar{a}^{1k}\right] = E(t_{11}^{-4})E[1+\mathbf{t}^H(T_{22}^H T_{22})^{-1}\mathbf{t}]$$
$$= \frac{n}{(n-m+1)(n-m)(n-m-1)}, \quad n > m+1$$

where the last line has been obtained by using the result $E(t_{11}^{-4}) = [(n-m)(n-m-1)]^{-1}$, $n > m+1$ and $E[1+\mathbf{t}^H(T_{22}^H T_{22})^{-1}\mathbf{t}] = n(n-m+1)^{-1}$. $\square$

Theorem 3.3 *Let $A \sim \mathbb{C}W_m(n, I_m)$, and $\mathbf{c} \in \mathbb{C}^m$, $\mathbf{c} \neq \mathbf{0}$. Then*

$$E(A\mathbf{c}\mathbf{c}^H A) = n^2\mathbf{c}\mathbf{c}^H + n\mathbf{c}^H\mathbf{c}I_m,$$

and

$$E(A^{-1}\mathbf{c}\mathbf{c}^H A^{-1}) = \frac{\mathbf{c}^H\mathbf{c}I_m + (n-m)\mathbf{c}\mathbf{c}^H}{(n-m+1)(n-m)(n-m-1)}, \quad n > m+1.$$

Proof Since $\mathbf{c}\mathbf{c}^H$ is an $m \times m$ complex matrix of rank one, we can find an unitary matrix Q such that $\mathbf{c}\mathbf{c}^H = \mathbf{c}^H\mathbf{c}Q\mathbf{e}_1\mathbf{e}_1'Q^H$, where $\mathbf{e}_1$ is the first vector of the standard basis of $\mathbb{R}^m$. Substituting for $\mathbf{c}\mathbf{c}^H$, we have

$$E(A\mathbf{c}\mathbf{c}^H A) = \mathbf{c}^H\mathbf{c}E[AQ\mathbf{e}_1\mathbf{e}_1'Q^H A] = \mathbf{c}^H\mathbf{c}QE[V\mathbf{e}_1\mathbf{e}_1'V]Q^H = \mathbf{c}^H\mathbf{c}QE(\mathbf{v}_1\mathbf{v}_1^H)Q^H \quad (3.4)$$

where $Q^H AQ = V = (\mathbf{v}_1, \ldots, \mathbf{v}_m) \sim \mathbb{C}W_m(n, I_m)$. Now, let $V = TT^H$ where $T = (t_{ij})$ is a complex lower triangular matrix with $t_{ii} > 0$. Partition T as

$$T = \begin{pmatrix} t_{11} & 0 \\ \mathbf{t}_{21} & T_{22} \end{pmatrix}, \quad T_{22}\,((m-1)\times(m-1)).$$

Then $\mathbf{v}_1^H = (t_{11}^2, t_{11}\mathbf{t}_{21}^H)$. From Theorem 2.5, t_{11} and $\mathbf{t}_{21}$ are independent, $t_{11}^2 \sim G(n)$ and $\mathbf{t}_{21} \sim \mathbb{C}N_{m-1}(\mathbf{0}, I_m)$. Hence

$$E(\mathbf{v}_1\mathbf{v}_1^H) = E\left[\begin{pmatrix} t_{11}^2 \\ t_{11}\mathbf{t}_{21} \end{pmatrix}(t_{11}^2, t_{11}\mathbf{t}_{21}^H)\right] = \begin{pmatrix} E(t_{11}^4) & E(t_{11}^3)E(\mathbf{t}_{21}^H) \\ E(t_{11}^3)E(\mathbf{t}_{21}) & E(t_{11}^2)E(\mathbf{t}_{21}\mathbf{t}_{21}^H) \end{pmatrix}$$
$$= \begin{pmatrix} n(n+1) & \mathbf{0}' \\ \mathbf{0} & nI_{m-1} \end{pmatrix} = nI_m + n^2\mathbf{e}_1\mathbf{e}_1'. \quad (3.5)$$

Now, substituting (3.5) in (3.4) and simplifying, we get the desired result. Similarly,

$$E(A^{-1}\mathbf{c}\mathbf{c}^H A^{-1}) = \mathbf{c}^H\mathbf{c}QE[\mathbf{v}^{(1)}(\mathbf{v}^{(1)})^H]Q^H \quad (3.6)$$

where $V = (\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(m)})$. Now, let $V = TT^H$, where T is a complex upper triangular matrix with positive diagonal elements and partition T as

$$T = \begin{pmatrix} t_{11} & \mathbf{t}^H \\ 0 & T_{22} \end{pmatrix}, \quad T_{22}\,((m-1)\times(m-1)).$$

Then, from (3.3) it is clear that $(\mathbf{v}^{(1)})^H = (t_{11}^{-2}, -t_{11}^{-2}\mathbf{t}^H T_{22}^{-1})$ and

$$\begin{aligned} E[\mathbf{v}^{(1)}(\mathbf{v}^{(1)})^H] &= E\left[\begin{pmatrix} t_{11}^{-2} \\ -t_{11}^{-2}(T_{22}^H)^{-1}\mathbf{t}\end{pmatrix}(t_{11}^{-2}, -t_{11}^{-2}\mathbf{t}^H T_{22}^{-1})\right] \\ &= E(t_{11}^{-4})\begin{pmatrix} 1 & E(\mathbf{t}^H)E(T_{22}^{-1}) \\ E(T_{22}^H)^{-1}E(\mathbf{t}) & E((T_{22}^H)^{-1}\mathbf{t}\mathbf{t}^H T_{22}^{-1})\end{pmatrix} \\ &= \frac{I_m + (n-m)\mathbf{e}_1\mathbf{e}_1'}{(n-m+1)(n-m)(n-m-1)}, \quad n > m+1. \end{aligned} \tag{3.7}$$

Now, substituting (3.7) in (3.6) and simplifying, we get the result. □

Theorem 3.4 *Let* A *and* $\mathbf{y}$ *be distributed independently,* $A \sim \mathbb{C}W_m(n, I_m)$ *and* $\mathbf{y} \sim \mathbb{C}N_m(\boldsymbol{\eta}, I_m)$. *Define* $\mathbf{d} = A^{-1}\mathbf{y}$. *Then*

$$E(\mathbf{d}) = (n-m)^{-1}\boldsymbol{\eta}, \quad n > m,$$

and

$$\mathrm{Cov}(\mathbf{d}) = \frac{(n-m)(\boldsymbol{\eta}^H\boldsymbol{\eta} + n)I_m + \boldsymbol{\eta}\boldsymbol{\eta}^H}{(n-m+1)(n-m)^2(n-m-1)}, \quad n > m+1.$$

Proof Using independence of A and $\mathbf{y}$, we get

$$E(\mathbf{d}) = E(A^{-1})E(\mathbf{y}) = (n-m)^{-1}\boldsymbol{\eta}, \quad n > m.$$

From Theorem 3.3,

$$E(\mathbf{d}\mathbf{d}^H) = E(A^{-1}\mathbf{y}\mathbf{y}^H A^{-1}) = \frac{E(\mathbf{y}^H\mathbf{y})I_m + (n-m)E(\mathbf{y}\mathbf{y}^H)}{(n-m+1)(n-m)(n-m-1)}, \quad n > m+1.$$

Note that $E(\mathbf{y}\mathbf{y}^H) = \boldsymbol{\eta}\boldsymbol{\eta}^H + I_m$ and $E(\mathbf{y}^H\mathbf{y}) = E[\mathrm{tr}(\mathbf{y}\mathbf{y}^H)] = \mathrm{tr}(\boldsymbol{\eta}\boldsymbol{\eta}^H + I_m) = \boldsymbol{\eta}^H\boldsymbol{\eta} + m$. Hence

$$\begin{aligned} \mathrm{Cov}(\mathbf{d}) &= E(\mathbf{d}\mathbf{d}^H) - E(\mathbf{d})E(\mathbf{d}^H) \\ &= \frac{(\boldsymbol{\eta}^H\boldsymbol{\eta} + m)I_m + (n-m)(\boldsymbol{\eta}\boldsymbol{\eta}^H + I_m)}{(n-m+1)(n-m)(n-m-1)} - \frac{\boldsymbol{\eta}\boldsymbol{\eta}^H}{(n-m)^2}. \end{aligned}$$

Now, the simplification of the above expression yields the result. □

Theorem 3.5 *Let* A^* *and* $\mathbf{x}$ *be independent,* $A^* \sim \mathbb{C}W_m(n, \Sigma)$ *and* $\mathbf{x} \sim \mathbb{C}N_m(\boldsymbol{\mu}, \Sigma)$. *Define* $\mathbf{d}^* = (A^*)^{-1}\mathbf{x}$. *Then*

$$E(\mathbf{d}^*) = (n-m)^{-1}\Sigma^{-1}\boldsymbol{\mu}, \quad n > m,$$

and

$$\mathrm{Cov}(\mathbf{d}^*) = \frac{(n-m)(\boldsymbol{\mu}^H\Sigma^{-1}\boldsymbol{\mu} + n)\Sigma^{-1} + \Sigma^{-1}\boldsymbol{\mu}\boldsymbol{\mu}^H\Sigma^{-1}}{(n-m+1)(n-m)^2(n-m-1)}, \quad n > m+1.$$

Proof Define $A = \Sigma^{-1/2}A^*\Sigma^{-1/2}$ and $\mathbf{y} = \Sigma^{-1/2}\mathbf{x}$. Then, $\mathbf{d}^* = \Sigma^{-1/2}A^{-1}\mathbf{y} = \Sigma^{-1/2}\mathbf{d}$ with $A \sim \mathbb{C}W_m(n, I_m)$ and $\mathbf{y} \sim \mathbb{C}N_m(\boldsymbol{\eta}, I_m)$ where $\boldsymbol{\eta} = \Sigma^{-1/2}\boldsymbol{\mu}$. Now, $E(\mathbf{d}^*) = \Sigma^{-1/2}E(\mathbf{d})$ and

$\mathrm{Cov}(\mathbf{d}^*) = \Sigma^{-1/2}\,\mathrm{Cov}(\mathbf{d})\Sigma^{-1/2}$. Substituting for $E(\mathbf{d})$ and $\mathrm{Cov}(\mathbf{d})$ from Theorem 3.4 and replacing $\boldsymbol{\eta}$ by $\Sigma^{-1/2}\boldsymbol{\mu}$ we get the desired result. □

Theorems 3.2–3.5, in the real case, were obtained by Das Gupta [5]. Several results, namely, $E(V^{-1})$, $\mathrm{Cov}[\mathrm{Vec}(V^{-1})]$, $\mathrm{Var}(v_{ii})$, $\mathrm{Var}(v_{i-1,i})$, $E(A^{-1})$, and $\mathrm{Cov}[\mathrm{Vec}(A^{-1})]$ where $V \sim \mathbb{C}W_m(n, I_m)$ and $A \sim \mathbb{C}W_m(n, \Sigma)$ are given in Shaman [24]. In the next theorem, we will evaluate certain expected values using results on zonal polynomials of Hermitian matrix (James [12], Khatri [14, 15]). It is well known that if $A \sim \mathbb{C}W_m(n, \Sigma)$, then

$$E[\tilde{C}_\kappa(AB)] = [n]_\kappa \tilde{C}_\kappa(\Sigma B), \tag{3.8}$$

and

$$E[\tilde{C}_\kappa(A^{-1}B)] = \frac{(-1)^k}{[-n+m]_\kappa}\tilde{C}_\kappa(\Sigma^{-1}B), \quad n > m-1+k_1, \tag{3.9}$$

where $\tilde{C}_\kappa(X)$ is the zonal polynomial of the Hermitian $m \times m$ matrix X corresponding to the partition $\kappa = (k_1, \ldots, k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$, $k_1 + \cdots + k_m = k$ and the *complex generalized hypergeometric coefficient* $[a]_\kappa$ is defined as

$$[a]_\kappa = \prod_{i=1}^{m}(a-i+1)_{k_i}, \tag{3.10}$$

with $(a)_r = a(a+1)\cdots(a+r-1) = (a)_{r-1}(a+r-1)$ for $r = 1, 2, \ldots,$ and $(a)_0 = 1$. For $k = 2, 3$, explicit formulas for $\tilde{C}_\kappa(X)$ are available as (Khatri [15]),

$$\tilde{C}_{(2)}(X) = \frac{1}{2}\left[(\mathrm{tr}\, X)^2 + \mathrm{tr}(X^2)\right], \tag{3.11}$$

$$\tilde{C}_{(1^2)}(X) = \frac{1}{2}[(\mathrm{tr}\, X)^2 - \mathrm{tr}(X^2)], \tag{3.12}$$

$$\tilde{C}_{(3)}(X) = \frac{1}{6}[(\mathrm{tr}\, X)^3 + 3(\mathrm{tr}\, X)(\mathrm{tr}\, X^2) + 2\,\mathrm{tr}(X^3)], \tag{3.13}$$

$$\tilde{C}_{(2,1)}(X) = \frac{2}{3}[(\mathrm{tr}\, X)^3 - \mathrm{tr}(X^3)], \tag{3.14}$$

$$\tilde{C}_{(1^3)}(X) = \frac{1}{6}[(\mathrm{tr}\, X)^3 - 3(\mathrm{tr}\, X)(\mathrm{tr}\, X^2) + 2\,\mathrm{tr}(X^3)]. \tag{3.15}$$

From the above results, it is straightforward to show that

$$\mathrm{tr}(X^2) = \tilde{C}_{(2)}(X) - \tilde{C}_{(1^2)}(X), \tag{3.16}$$

and

$$\mathrm{tr}(X^3) = \tilde{C}_{(3)}(X) - \frac{1}{2}\tilde{C}_{(2,1)}(X) + \tilde{C}_{(1^3)}(X). \tag{3.17}$$

Also, substituting $X = I_m$ in (3.11)–(3.15), it is easy to see that $\tilde{C}_{(2)}(I_m) = m(m+1)/2$, $\tilde{C}_{(1^2)}(I_m) = m(m-1)/2$, $\tilde{C}_{(3)}(I_m) = m(m+1)(m+2)/6$, $\tilde{C}_{(2,1)}(I_m) = 2m(m^2-1)/3$ and $\tilde{C}_{(1^3)}(I_m) = m(m-1)(m-2)/6$.

Theorem 3.6 *Let the constants* $\tilde{c}(r,n,m)$ *and* $\tilde{d}(r,n,m)$ *be defined as in Lemma 3.1. Then,*

$$\tilde{c}(3,n,m) = n(1+3mn+m^2+n^2),$$

and for $n > m+2$,

$$\tilde{d}(3,n,m) = \frac{n(n+m)}{(n-m+2)(n-m+1)(n-m)(n-m-1)(n-m-2)}.$$

Proof Since, $E(A^r) = \tilde{c}(r,n,m)I_m$, we have $E[\mathrm{tr}(A^r)] = \tilde{c}(r,n,m)m$. Thus, the coefficient of m in $E[\mathrm{tr}(A^r)]$ is $\tilde{c}(r,n,m)$. Now, using (3.17), the expected value of $\mathrm{tr}(A^3)$ is obtained as

$$\begin{aligned} E[\mathrm{tr}(A^3)] &= E[\tilde{C}_{(3)}(A)] - \frac{1}{2}E[\tilde{C}_{(2,1)}(A)] + E[\tilde{C}_{(1^3)}(A)] \\ &= [n]_{(3)}\tilde{C}_{(3)}(I_m) - \frac{1}{2}[n]_{(2,1)}\tilde{C}_{(2,1)}(I_m) + [n]_{(1^3)}\tilde{C}_{(1^3)}(I_m), \end{aligned}$$

where the last line has been obtained by using (3.8). Similarly, applying (3.9), we get

$$\begin{aligned} E[\mathrm{tr}(A^{-3})] &= E[\tilde{C}_{(3)}(A^{-1})] - \frac{1}{2}E[\tilde{C}_{(2,1)}(A^{-1})] + E[\tilde{C}_{(1^3)}(A^{-1})] \\ &= \frac{(-1)^3\tilde{C}_{(3)}(I_m)}{[-n+m]_{(3)}} - \frac{1}{2}\frac{(-1)^3\tilde{C}_{(2,1)}(I_m)}{[-n+m]_{(2,1)}} + \frac{(-1)^3\tilde{C}_{(1^3)}(I_m)}{[-n+m]_{(1^3)}}. \end{aligned}$$

Now, using the results $[n]_{(3)} = n(n+1)(n+2)$, $[n]_{(2,1)} = n(n+1)(n-1)$, $[n]_{(1^3)} = n(n-1)(n-2)$, $[-n+m]_{(3)} = -(n-m)(n-m-1)(n-m-2)$, $[-n+m]_{(2,1)} = -(n-m)(n-m-1)(n-m+1)$, $[-n+m]_{(1^3)} = -(n-m)(n-m+1)(n-m+2)$, $\tilde{C}_{(3)}(I_m) = m(m+1)(m+2)/6$, $\tilde{C}_{(2,1)}(I_m) = 2m(m^2-1)/3$ and $\tilde{C}_{(1^3)}(I_m) = m(m-1)(m-2)/6$ in the above expressions and simplifying, we obtain

$$E[\mathrm{tr}(A^3)] = mn(1+3mn+m^2+n^2),$$

and for $n > m+2$,

$$E[\mathrm{tr}(A^{-3})] = \frac{mn(n+m)}{(n-m+2)(n-m+1)(n-m)(n-m-1)(n-m-2)}.$$

Finally, computation of coefficients of m in the above expressions yields $\tilde{c}(3,n,m)$ and $\tilde{d}(3,n,m)$. □

Similarly, applying the technique described above, expected values of $(\mathrm{tr}\,A^{-1})^2$, $(\mathrm{tr}\,A^{-1})^3$ and $\mathrm{tr}(A^{-1})\,\mathrm{tr}(A^{-2})$ are evaluated as

$$\begin{aligned} E[(\mathrm{tr}\,A^{-1})^2] &= E[\tilde{C}_{(2)}(A^{-1})] + E[\tilde{C}_{(1^2)}(A^{-1})] \\ &= \frac{m(mn-m^2+1)}{(n-m+1)(n-m)(n-m-1)}, \quad n > m+1, \end{aligned} \tag{3.18}$$

$$
\begin{aligned}
E[(\operatorname{tr} A^{-1})^3] &= E[\tilde{C}_{(3)}(A^{-1})] + E[\tilde{C}_{(2,1)}(A^{-1})] + E[\tilde{C}_{(1^3)}(A^{-1})] \\
&= \frac{m[m^2(n-m)^2 - 5m^2 + 3mn + 4]}{(n-m+2)(n-m+1)(n-m)(n-m-1)(n-m-2)}, \\
&\quad n > m+2, \qquad (3.19)
\end{aligned}
$$

and

$$
\begin{aligned}
E[\operatorname{tr}(A^{-1})\operatorname{tr}(A^{-2})] &= E[\tilde{C}_{(3)}(A^{-1})] - E[\tilde{C}_{(1^3)}(A^{-1})] \\
&= \frac{mn[m(n-m)+2]}{(n-m+2)(n-m+1)(n-m)(n-m-1)(n-m-2)}, \\
&\quad n > m+2. \qquad (3.20)
\end{aligned}
$$

Further, using (3.18), (3.19) and (3.20), it is straightforward to show that

$$
E[(\operatorname{tr} A^{-1})A^{-1}] = \frac{(mn - m^2 + 1)I_m}{(n-m+1)(n-m)(n-m-1)}, \quad n > m+1,
$$

$$
E[(\operatorname{tr} A^{-1})^2 A^{-1}] = \frac{[m^2(n-m)^2 - 5m^2 + 3mn + 4]I_m}{(n-m+2)(n-m+1)(n-m)(n-m-1)(n-m-2)},
$$

where $n > m+2$ and

$$
\begin{aligned}
E[\operatorname{tr}(A^{-1})A^{-2}] &= E[\operatorname{tr}(A^{-2})A^{-1}] \\
&= \frac{n[m(n-m)+2]I_m}{(n-m+2)(n-m+1)(n-m)(n-m-1)(n-m-2)},
\end{aligned}
$$

where $n > m+2$.

Now, following Muirhead [21], we give an identity involving complex Wishart distribution which will be used to compute a number of expected values of functions of complex Wishart matrix.

Theorem 3.7 *Suppose that $A \sim \mathbb{C}W_m(n, \Sigma)$. Let $h(A)$ be a real valued measurable function of A such that the function $f(t; A) = h(t^{-1}A)$, $t > 0$, is differentiable at $t = 1$. Let $f'(t) = \frac{\partial}{\partial t} f(t; A)$. Then,*

$$
E[\operatorname{tr}(\Sigma^{-1}A)h(A)] = mnE[h(A)] - E[f'(1; A)], \qquad (3.21)
$$

provided the expectations involved exist.

Proof For $t > 0$ define the function $g(t)$ as

$$
g(t) = \frac{\det(\Sigma)^{-n} t^{mn}}{\tilde{\Gamma}_m(n)} \int_{A=A^H>0} \exp(-t \operatorname{tr}(\Sigma^{-1}A)) \det(A)^{n-m} h(A)\, dA, \qquad (3.22)
$$

and note that $g(1) = E[h(A)]$. Differentiating (3.22) with respect to t and putting $t = 1$ gives

$$
g'(1) = mnE[h(A)] - E[\operatorname{tr}(\Sigma^{-1}A)h(A)]. \qquad (3.23)
$$

Now, put $X = tA$ in (3.22), then $g(t)$ can be written as

$$g(t) = \frac{\det(\Sigma)^{-n}}{\tilde{\Gamma}_m(n)} \int_{X=X^H>0} \exp(-\operatorname{tr}(\Sigma^{-1}X)) \det(X)^{n-m} f(t; X)\, dX,$$

from which it follows that

$$g'(1) = E[f'(1; A)]. \tag{3.24}$$

Equating (3.23) and (3.24) we get the desired result. □

In many interesting applications the function $h(\cdot)$ has the property that, for $x > 0$, $h(xA) = x^{\ell} h(A)$ for some real ℓ. Then, $f(t; A) = h(t^{-1}A) = t^{-\ell} h(A)$, so that

$$f'(1, A) = -\ell\, h(A).$$

This yields the following result.

Corollary 3.1 *If $h(xA) = x^{\ell} h(A)$ for some ℓ, then*

$$E[\operatorname{tr}(\Sigma^{-1}A)h(A)] = (mn + \ell) E[h(A)]. \tag{3.25}$$

Let $h(A) = (\operatorname{tr} A)^{-1}$, so that $\ell = -1$, and

$$E\left[\frac{\operatorname{tr}(\Sigma^{-1}A)}{\operatorname{tr}(A)}\right] = (mn - 1) E\left[\frac{1}{\operatorname{tr} A}\right]. \tag{3.26}$$

Theorem 3.8 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$. Then, for $k = 0, 1, 2, \ldots,$*

$$E[\operatorname{tr}(\Sigma^{-1}A)]^k = (mn)_k.$$

Proof Put $h(A) = \operatorname{tr}(\Sigma^{-1/2}A\Sigma^{-1/2}) = \operatorname{tr}(\Sigma^{-1}A)$ in (3.25), so that $\ell = 1$. This gives

$$E[\operatorname{tr}(\Sigma^{-1}A)]^2 = (mn + 1) E[\operatorname{tr}(\Sigma^{-1}A)] = (mn + 1)mn$$

where we have used the fact that $E(A) = n\Sigma$. Next, by taking $h(A) = [\operatorname{tr}(\Sigma^{-1}A)]^2$, with $\ell = 2$, we get

$$E[\operatorname{tr}(\Sigma^{-1}A)]^3 = (mn + 2) E[\operatorname{tr}(\Sigma^{-1}A)]^2 = mn(mn + 1)(mn + 2).$$

The result for arbitrary k is trivial by induction. □

Theorem 3.9 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$. Then, for $k = 0, 1, 2, \ldots,$*

$$E[\operatorname{tr}(\Sigma^{-1}A)]^{-k} = \frac{(-1)^k}{(-mn + 1)_k}, \quad mn > k.$$

Proof Take $h(A) = [\operatorname{tr}(\Sigma^{-1/2}A\Sigma^{-1/2})]^{-1} = [\operatorname{tr}(\Sigma^{-1}A)]^{-1}$ in (3.25), so that $\ell = -1$ and

$$E[\operatorname{tr}(\Sigma^{-1}A)]^{-1} = \frac{1}{mn - 1}, \quad mn > 1.$$

Next, by substituting $h(A) = [\operatorname{tr}(\Sigma^{-1}A)]^{-2}$ and $\ell = -2$, we obtain

$$E[\operatorname{tr}(\Sigma^{-1}A)]^{-2} = \frac{1}{(mn - 1)(mn - 2)}, \quad mn > 2.$$

By induction

$$E[\mathrm{tr}(\Sigma^{-1}A)]^{-k} = \frac{1}{(mn-1)(mn-2)\cdots(mn-k)}, \quad mn > k.$$

Now, the result follows by noting that $(mn-1)(mn-2)(mn-k) = (-1)^k\,(-mn+1)_k$. □

Theorem 3.10 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$ and B be an $m \times m$ non-random Hermitian matrix. Then, for $k = 0, 1, 2, \ldots,$*

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^k\,\mathrm{tr}(\Sigma^{-1}AB)] = n(mn+1)_k\,\mathrm{tr}(B).$$

Proof Take $h(A) = \mathrm{tr}(\Sigma^{-1}AB)$ and $\ell = 1$ in (3.25) to obtain

$$E[\mathrm{tr}(\Sigma^{-1}A)\,\mathrm{tr}(\Sigma^{-1}AB)] = (mn+1)E[\mathrm{tr}(\Sigma^{-1}AB)] = n(mn+1)\,\mathrm{tr}(B).$$

Next, taking $h(A) = \mathrm{tr}(\Sigma^{-1}A)\,\mathrm{tr}(\Sigma^{-1}AB)$, with $\ell = 2$, gives

$$\begin{aligned} E[\{\mathrm{tr}(\Sigma^{-1}A)\}^2\,\mathrm{tr}(\Sigma^{-1}AB)] &= (mn+2)E[\mathrm{tr}(\Sigma^{-1}A)\,\mathrm{tr}(\Sigma^{-1}AB)] \\ &= n(mn+1)(mn+2)\,\mathrm{tr}(B). \end{aligned}$$

By using an inductive argument, we have

$$\begin{aligned} E[\{\mathrm{tr}(\Sigma^{-1}A)\}^k\,\mathrm{tr}(\Sigma^{-1}AB)] &= n(mn+1)(mn+2)\cdots(mn+k)\,\mathrm{tr}(B) \\ &= n(mn+1)_k\,\mathrm{tr}(B), \end{aligned}$$

and the proof is complete. □

Corollary 3.2 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$. Then, for $k = 0, 1, 2, \ldots,$*

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^k\,\mathrm{tr}(A)] = n(mn+1)_k\,\mathrm{tr}(\Sigma),$$

and

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^k A] = n(mn+1)_k\,\Sigma.$$

Theorem 3.11 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$ and B be an $m \times m$ non-random Hermitian matrix. Then, for $k = 0, 1, 2, \ldots,$*

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^k\,\mathrm{tr}(\Sigma A^{-1}B)] = \frac{(mn-1)_k\,\mathrm{tr}(B)}{n-m}, \quad n > m.$$

Proof Let $h(A) = \mathrm{tr}(\Sigma A^{-1}B)$ in (3.25), so that $\ell = -1$. This yields

$$\begin{aligned} E[\mathrm{tr}(\Sigma^{-1}A)\,\mathrm{tr}(\Sigma A^{-1}B)] &= (mn-1)E[\mathrm{tr}(\Sigma A^{-1}B)] \\ &= \frac{(mn-1)\,\mathrm{tr}(B)}{(n-m)}, \quad n > m. \end{aligned}$$

where we have used $E(A^{-1}) = (n-m)^{-1}\Sigma^{-1}, n > m$. Next, for $h(A) = \operatorname{tr}(\Sigma^{-1}A) \times \operatorname{tr}(\Sigma A^{-1}B)$, with $\ell = 0$, (3.25) yields

$$\begin{aligned} E[\{\operatorname{tr}(\Sigma^{-1}A)\}^2 \operatorname{tr}(\Sigma A^{-1}B)] &= mnE[\operatorname{tr}(\Sigma^{-1}A)\operatorname{tr}(\Sigma A^{-1}B)] \\ &= \frac{(mn-1)mn\operatorname{tr}(B)}{n-m}, \quad n > m. \end{aligned}$$

Hence, by induction

$$\begin{aligned} E[\{\operatorname{tr}(\Sigma^{-1}A)\}^k \operatorname{tr}(\Sigma A^{-1}B)] &= \frac{(mn-1)mn\cdots(mn+k-2)\operatorname{tr}(B)}{n-m} \\ &= \frac{(mn-1)_k\operatorname{tr}(B)}{n-m}, \quad n > m, \end{aligned}$$

which is the desired result. □

Corollary 3.3 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$. Then, for $k = 0, 1, 2, \ldots,$*

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^k A^{-1}] = \frac{(mn-1)_k \Sigma^{-1}}{n-m}, \quad n > m,$$

and

$$E[\operatorname{tr}(\Sigma A^{-1})A] = \frac{(mn-1)\Sigma}{n-m}, \quad n > m.$$

Theorem 3.12 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$. Then, for $k = 0, 1, 2, \ldots,$*

$$\begin{aligned} &E[\{\operatorname{tr}(\Sigma^{-1}A)\}^k \det(A)^h] \\ &= (mn+mh)_k \det(\Sigma)^h \frac{\tilde{\Gamma}_m(n+h)}{\tilde{\Gamma}_m(n)}, \quad \operatorname{Re}(n+h) > m-1. \end{aligned}$$

Proof Let $h(A) = \det(A)^h$ in (3.25), so that $\ell = mh$. Then, from (3.25), we have

$$E[\operatorname{tr}(\Sigma^{-1}A)\det(A)^h] = (mn+mh)E[\det(A)^h].$$

Now, using (3.2) the above expression is rewritten as

$$E[\operatorname{tr}(\Sigma^{-1}A)\det(A)^h] = (mn+mh)\det(\Sigma)^h \frac{\tilde{\Gamma}_m(n+h)}{\tilde{\Gamma}_m(n)}, \quad n+h > m-1.$$

Next, taking $h(A) = \operatorname{tr}(\Sigma^{-1}A)\det(A)^h$, with $\ell = mh+1$, gives

$$\begin{aligned} E[\{\operatorname{tr}(\Sigma^{-1}A)\}^2 \det(A)^h] &= (mn+mh+1)E[\operatorname{tr}(\Sigma^{-1}A)\det(A)^h] \\ &= (mn+mh)(mn+mh+1) \\ &\quad \times \det(\Sigma)^h \frac{\tilde{\Gamma}_m(n+h)}{\tilde{\Gamma}_m(n)}, \quad n+h > m-1. \end{aligned}$$

Now, the result follows by induction. □

Theorem 3.13 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$ and B be an $m \times m$ non-random Hermitian matrix. Then, for $r = 0, 1, 2, \ldots,$*

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^r \tilde{C}_\kappa(AB)] = (mn+k)_r [n]_\kappa \tilde{C}_\kappa(\Sigma B)$$

and

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^{-r} \tilde{C}_\kappa(AB)] = \frac{(-1)^r [n]_\kappa \tilde{C}_\kappa(\Sigma B)}{(-mn-k+1)_r}, \quad r < mn+k.$$

Proof Let $h(A) = \tilde{C}_\kappa(BA)$ in (3.25), so that $\ell = k$ and

$$E[\operatorname{tr}(\Sigma^{-1}A)\tilde{C}_\kappa(AB)] = (mn+k)E[\tilde{C}_\kappa(AB)] = (mn+k)[n]_\kappa \tilde{C}_\kappa(\Sigma B),$$

where we have used (3.8). Next, by taking $h(A) = \operatorname{tr}(\Sigma^{-1}A)\tilde{C}_\kappa(AB)$, with $\ell = k+1$, we arrive at

$$\begin{aligned} E[\{\operatorname{tr}(\Sigma^{-1}A)\}^2 \tilde{C}_\kappa(AB)] &= (mn+k+1)E[\operatorname{tr}(\Sigma^{-1}A)\tilde{C}_\kappa(AB)] \\ &= (mn+k)(mn+k+1)[n]_\kappa \tilde{C}_\kappa(\Sigma B). \end{aligned}$$

Finally, the desired result is obtained by induction. To prove second part, we substitute $h(A) = \{\operatorname{tr}(\Sigma^{-1}A)\}^{-r}\tilde{C}_\kappa(BA)$ and $\ell = k - r$ in (3.25), to obtain

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^{-r} \tilde{C}_\kappa(AB)] = \frac{E[\{\operatorname{tr}(\Sigma^{-1}A)\}^{-r+1} \tilde{C}_\kappa(AB)]}{mn+k-r},$$

where $mn + k > r$. Now, using the above recurrence relation it is easy to see that

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^{-r} \tilde{C}_\kappa(AB)] = \frac{E[\tilde{C}_\kappa(AB)]}{(mn+k-r)\cdots(mn+k-1)}.$$

Now, the result follows by substituting for $E[\tilde{C}_\kappa(AB)]$ and observing that $(mn+k-r)\cdots(mn+k-2)(mn+k-1) = (-1)^r(-mn-k+1)_r$. □

Theorem 3.14 *Let $A \sim \mathbb{C}W_m(n, \Sigma)$ and B be an $m \times m$ non-random Hermitian matrix. Then, for $r = 0, 1, 2, \ldots,$ and $n \geq m + k_1$,*

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^r \, \tilde{C}_\kappa(A^{-1}B)] = \frac{(-1)^k (mn-k)_r \tilde{C}_\kappa(\Sigma^{-1}B)}{[-n+m]_\kappa}, \quad mn + r > k + 1,$$

and

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^{-r} \tilde{C}_\kappa(A^{-1}B)] = \frac{(-1)^{r+k} \, \tilde{C}_\kappa(\Sigma^{-1}B)}{(-mn+k+1)_r [-n+m]_\kappa}, \quad r + k < mn + k.$$

Proof Let $h(A) = \tilde{C}_\kappa(A^{-1}B)$ in (3.25), so that $\ell = -k$. This gives

$$E[\operatorname{tr}(\Sigma^{-1}A)\tilde{C}_\kappa(A^{-1}B)] = (mn-k)E[\tilde{C}_\kappa(A^{-1}B)].$$

Now, using (3.9) we obtain

$$E[\operatorname{tr}(\Sigma^{-1}A)\tilde{C}_\kappa(A^{-1}B)] = (mn-k)\frac{(-1)^k}{[-n+m]_\kappa}\tilde{C}_\kappa(\Sigma^{-1}B).$$

Next, taking $h(A) = \operatorname{tr}(\Sigma^{-1}A)\tilde{C}_\kappa(A^{-1}B)$, with $\ell = -k+1$, gives

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^2\tilde{C}_\kappa(A^{-1}B)] = (mn-k)(mn-k+1)\frac{(-1)^k}{[-n+m]_\kappa}\tilde{C}_\kappa(\Sigma^{-1}B).$$

Continuing this process, we get

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^r\tilde{C}_\kappa(A^{-1}B)] = (mn-k)\cdots(mn-k+r-1)\frac{(-1)^k}{[-n+m]_\kappa}\tilde{C}_\kappa(\Sigma^{-1}B),$$

where $k+1 < mn+r$. Now, the result follows by observing that $(mn-k)_r = (mn-k)\cdots(mn-k+r-1)$. To prove second part, we take $h(A) = \{\operatorname{tr}(\Sigma^{-1}A)\}^{-r}\tilde{C}_\kappa(A^{-1}B)$ in (3.25), so that $\ell = -k-r$ and we get

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^{-r}\tilde{C}_\kappa(A^{-1}B)] = \frac{E[\{\operatorname{tr}(\Sigma^{-1}A)\}^{-r+1}\tilde{C}_\kappa(A^{-1}B)]}{mn-k-r},$$

where $r+k < mn$. Now, the above recurrence relation yields

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^{-r}\tilde{C}_\kappa(A^{-1}B)] = \frac{(-1)^k\tilde{C}_\kappa(\Sigma^{-1}B)}{(mn-k-r)\cdots(mn-k-1)[-n+m]_\kappa},$$

where $n \ge m+k_1$. Finally, the result follows by noting that $(mn-k-r)\cdots\ (mn-k-1) = (-1)^r(-mn-k+1)_r$. □

Theorem 3.15 *Let $A \sim \mathbb{C}W_m(n,\Sigma)$ and B be an $m\times m$ non-random Hermitian matrix. Then, for $r = 0, 1, 2, \ldots,$*

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^r\operatorname{tr}(A\Sigma^{-1}AB)] = (mn+2)_r n(n+m)\operatorname{tr}(\Sigma B)$$

and for $n > m+1$,

$$E[\{\operatorname{tr}(\Sigma^{-1}A)\}^r\operatorname{tr}(A^{-1}\Sigma A^{-1}B)] = \frac{n(mn-2)_r\operatorname{tr}(\Sigma^{-1}B)}{(n-m-1)(n-m)(n-m+1)}.$$

Proof Let $h(A) = \operatorname{tr}(A\Sigma^{-1}AB)$ in (3.25), so that $\ell = 2$. This gives

$$\begin{aligned}E[\operatorname{tr}(\Sigma^{-1}A)\operatorname{tr}(A\Sigma^{-1}AB)] &= (mn+2)E[\operatorname{tr}(A\Sigma^{-1}AB)]\\ &= (mn+2)n(n+m)\operatorname{tr}(\Sigma B),\end{aligned}$$

where the last line has been derived by applying the result $E[\operatorname{tr}(A\Sigma^{-1}AB)] = n(n+m)\operatorname{tr}(\Sigma B)$ obtained from Lemma 3.2 and Theorem 3.2. Next, by taking $h(A) = \operatorname{tr}(\Sigma^{-1}A)\times \operatorname{tr}(A\Sigma^{-1}AB)$, with $\ell = 3$, we get

$$\begin{aligned}E[\{\operatorname{tr}(\Sigma^{-1}A)\}^2\operatorname{tr}(A\Sigma^{-1}AB)] &= (mn+3)E[\operatorname{tr}(\Sigma^{-1}A)\operatorname{tr}(A\Sigma^{-1}AB)]\\ &= (mn+3)(mn+2)n(n+m)\operatorname{tr}(\Sigma B).\end{aligned}$$

Finally, the desired result is obtained by induction. The proof of the second part is similar. □

Corollary 3.4 *Let* $A \sim \mathbb{C}W_m(n, \Sigma)$. *Then, for* $r = 0, 1, 2, \ldots,$

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^r A\Sigma^{-1}A] = n(mn+2)_r(n+m)\Sigma,$$

$$E[\mathrm{tr}(A\Sigma^{-1}A\Sigma^{-1})A] = n(mn+2)(n+m)\Sigma,$$

and for $n > m+1$,

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^r A^{-1}\Sigma A^{-1}] = \frac{n(mn-2)_r\Sigma^{-1}}{(n-m-1)(n-m)(n-m+1)},$$

$$E[\mathrm{tr}(A^{-1}\Sigma A^{-1}\Sigma)A] = \frac{n(mn-2)\Sigma}{(n-m-1)(n-m)(n-m+1)}.$$

Theorem 3.16 *Let* $A \sim \mathbb{C}W_m(n, \Sigma)$ *and* B *be an* $m \times m$ *non-random Hermitian matrix. Then, for* $r = 0, 1, 2, \ldots,$

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^r \,\mathrm{tr}(A\Sigma^{-1}A\Sigma^{-1}AB)] = n(mn+3)_r(1+3nm+n^2+m^2)\,\mathrm{tr}(\Sigma B)$$

and for $n > m+2$,

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^r \,\mathrm{tr}(A^{-1}\Sigma A^{-1}\Sigma A^{-1}B)]$$
$$= \frac{n(mn-3)_r(n+m)\,\mathrm{tr}(\Sigma^{-1}B)}{(n-m-2)(n-m-1)(n-m)(n-m+1)(n-m+2)}.$$

Proof Similar to the proof of the Theorem 3.15. □

Corollary 3.5 *Let* $A \sim \mathbb{C}W_m(n, \Sigma)$. *Then, for* $r = 0, 1, 2, \ldots,$

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^r A\Sigma^{-1}A\Sigma^{-1}A] = n(mn+3)_r(1+3nm+n^2+m^2)\Sigma,$$

and

$$E[\mathrm{tr}(A\Sigma^{-1}A\Sigma^{-1}A\Sigma^{-1})A] = n(mn+3)(1+3nm+n^2+m^2)\Sigma.$$

Corollary 3.6 *Let* $A \sim \mathbb{C}W_m(n, \Sigma)$. *Then, for* $n > m+2$ *and* $r = 0, 1, 2, \ldots,$

$$E[\{\mathrm{tr}(\Sigma^{-1}A)\}^r A^{-1}\Sigma A^{-1}\Sigma A^{-1}]$$
$$= \frac{n(mn-3)_r(n+m)\Sigma^{-1}}{(n-m-2)(n-m-1)(n-m)(n-m+1)(n-m+2)},$$

and

$$E[\mathrm{tr}(A^{-1}\Sigma A^{-1}\Sigma A^{-1}\Sigma)A]$$
$$= \frac{n(mn-3)(n+m)\Sigma}{(n-m-2)(n-m-1)(n-m)(n-m+1)(n-m+2)}.$$

Theorem 3.17 *Let* $A = TT^H \sim \mathbb{C}W_m(n, I_m)$, *where* $T = (t_{ij})$ *is a complex lower triangular matrix with positive diagonal elements, then*

$$E(T^HT)^{-1} = B$$

where $B = \mathrm{diag}(b_1, \ldots, b_m)$ *with*

$$b_1 = \frac{1}{n-1}$$

and

$$b_j = \frac{n}{(n-j)(n-j+1)}, \quad j = 2, \ldots, m.$$

Proof From Theorem 2.5, it is known that t_{ij}'s $(1 \le j \le i \le m)$ are independent, with $t_{ij} \sim \mathbb{C}N(0, 1)$, $1 \le j < i \le m$ and $t_{ii}^2 \sim G(n-i+1)$, $i = 1, \ldots, m$. Let

$$d_i = E\left(\frac{1}{t_{ii}^2}\right) = \frac{1}{n-i}, \quad i = 1, \ldots, m. \tag{3.27}$$

For any diagonal matrix D, with diagonal elements ± 1, DTD and T have the same distribution and therefore,

$$B = E(T^H T)^{-1} = E[(DTD)^H (DTD)]^{-1} = DBD,$$

which implies that B is a diagonal matrix. Writing

$$T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix}$$

we get

$$T^{-1} = \begin{pmatrix} T_{11}^{-1} & 0 \\ -T_{22}^{-1} T_{21} T_{11}^{-1} & T_{22}^{-1} \end{pmatrix}$$

and

$$(T^H T)^{-1} = \begin{pmatrix} T_{11}^{-1} (T_{11}^H)^{-1} & T_{11}^{-1} R_{21}^H \\ R_{21} (T_{11}^H)^{-1} & R_{21} R_{21}^H + T_{22}^{-1} (T_{22}^H)^{-1} \end{pmatrix},$$

where $R_{21} = -T_{22}^{-1} T_{21} T_{11}^{-1}$. Now, taking expectation we get

$$E(T^H T)^{-1} = B = \begin{pmatrix} E(T_{11}^H T_{11})^{-1} & 0 \\ 0 & E(R_{21} R_{21}^H + (T_{22}^H T_{22})^{-1}) \end{pmatrix}. \tag{3.28}$$

Letting T_{11} be $(m-1) \times (m-1)$, $T_{22} = t_{mm}$, $T_{21}(1 \times (m-1)) = \mathbf{t}^H$ and using the independence of T_{11}, t_{mm} and $\mathbf{t}^H$, we get

$$b_m = E\left[\frac{1}{t_{mm}^2} \{1 + \mathbf{t}^H (T_{11}^H T_{11})^{-1} \mathbf{t}\}\right] = E\left[\frac{1}{t_{mm}^2}\right] E\{1 + \mathbf{t}^H (T_{11}^H T_{11})^{-1} \mathbf{t}\}. \tag{3.29}$$

Since $\mathbf{t} \sim \mathbb{C}N_{m-1}(\mathbf{0}, I_{m-1})$, and from (3.28),

$$E(T_{11}^H T_{11})^{-1} = \mathrm{diag}(b_1, \ldots, b_{m-1}),$$

we have

$$E[\mathbf{t}^H (T_{11}^H T_{11})^{-1} \mathbf{t}] = E[\mathrm{tr}\{\mathbf{t}\mathbf{t}^H (T_{11}^H T_{11})^{-1}\}] = \mathrm{tr}[E(\mathbf{t}\mathbf{t}^H) E(T_{11}^H T_{11})^{-1}] = \sum_{j=1}^{m-1} b_j. \tag{3.30}$$

Now, using (3.30) in (3.29), one obtains

$$b_m = d_m \left(1 + \sum_{i=1}^{m-1} b_i\right).$$

By an inductive process, it is straightforward to show that

$$b_j = d_j \left(1 + \sum_{i=1}^{j-1} b_i\right), \quad j = 2, \ldots, m \tag{3.31}$$

and

$$b_1 = d_1.$$

Solving (3.31), in terms of b_j's, we get

$$b_1 = d_1,$$

$$b_j = d_j \prod_{i=1}^{j-1} (1 + d_i), \quad j = 2, \ldots, m,$$

and using (3.27), we finally get

$$b_1 = \frac{1}{n-1},$$

and

$$b_j = \frac{n}{(n-j)(n-j+1)}, \quad j = 2, \ldots, m.$$

□

The above result, in the real case, was derived by Eaton and Olkin [6]. Using this procedure one can derive a similar result for an upper triangular factorization of A as given in the next theorem.

Theorem 3.18 *Let $A = TT^H \sim \mathbb{C}W_m(n, I_m)$, where $T = (t_{ij})$ is a complex upper triangular matrix with positive diagonal elements. Then*

$$E(T^H T)^{-1} = B$$

where $B = \operatorname{diag}(b_1, \ldots, b_m)$ with

$$b_j = \frac{n}{(n-m+j)(n-m+j-1)}, \quad j = 1, 2, \ldots, m-1,$$

and

$$b_m = \frac{1}{n-1}.$$

Proof Similar to the proof of Theorem 3.17. □

4 Distribution of Signal to Noise Ratio

If Σ, the noise covariance matrix, is known, then the optimum filter for the detection of a signal $\boldsymbol{\delta}$ is $\boldsymbol{\delta}^H\Sigma^{-1}\mathbf{x}$ when $\mathbf{x}$ is a complex vector observation. The signal to noise ratio, which is an index to the efficiency of discrimination, in such a case is $\boldsymbol{\delta}^H\Sigma^{-1}\boldsymbol{\delta}$. If Σ is not known but an estimate $n^{-1}S$ based on n d.f. is available, we may use the estimated filter $n\boldsymbol{\delta}^H S^{-1}\mathbf{x}$. The signal to noise ratio in such a case is

$$\tilde{\rho} = \frac{(\boldsymbol{\delta}^H S^{-1}\boldsymbol{\delta})^2}{\boldsymbol{\delta}^H S^{-1}\Sigma S^{-1}\boldsymbol{\delta}} \tag{4.1}$$

which by the Cauchy-Schwartz inequality is less than $\boldsymbol{\delta}^H\Sigma^{-1}\boldsymbol{\delta}$. The efficiency of the estimated filter can be examined by considering the ratio $B = (\boldsymbol{\delta}^H\Sigma^{-1}\boldsymbol{\delta})^{-1}\tilde{\rho}$. Further, let

$$G = \frac{\boldsymbol{\delta}^H\Sigma^{-1}\boldsymbol{\delta}}{\boldsymbol{\delta}^H S^{-1}\boldsymbol{\delta}}. \tag{4.2}$$

The distribution of B was first obtained by Reed, Mallett and Brennan [23]. Khatri and Rao [16] derived the distributions of B and G using results on marginal and conditional distributions of a complex Wishart matrix, distributions of regression coefficient matrix and its quadratic forms and showed that B and G are independent. They also derived the density of the product BG in terms of confluent hypergeometric function. Here we will derive these results using triangular factorization of the complex Wishart matrix (Gupta and Nagar [10]). Note that B and G can be rewritten as

$$B = \frac{(\mathbf{c}^H A^{-1}\mathbf{c})^2}{(\mathbf{c}^H\mathbf{c})(\mathbf{c}^H A^{-2}\mathbf{c})}, \tag{4.3}$$

and

$$G = \frac{\mathbf{c}^H\mathbf{c}}{\mathbf{c}^H A^{-1}\mathbf{c}}, \tag{4.4}$$

respectively, where $\mathbf{c} = \Sigma^{-1/2}\boldsymbol{\delta}$ and $A = \Sigma^{-1/2}S\Sigma^{-1/2} \sim \mathbb{C}W_m(n, I_m)$.

Theorem 4.1 *Let $A \sim \mathbb{C}W_m(n, I_m)$, and $\mathbf{c} \in \mathbb{C}^m$, $\mathbf{c} \neq \mathbf{0}$. Then, B and G are independent, $B \sim B^I(n-m+2, m-1)$ and $G \sim G(n-m+1)$.*

Proof From Theorem 2.3, it is known that for any $m \times m$ unitary matrix G, $GAG^H \sim \mathbb{C}W_m(n, I_m)$. Now, let $V = (v_{ij}) = GAG^H$ and choose the unitary matrix G as $G^H = ((\mathbf{c}^H\mathbf{c})^{-\frac{1}{2}}\mathbf{c}, G_1^H)$. Then,

$$\begin{aligned} &A^{-1} = G^H V^{-1} G, \qquad A^{-2} = G^H V^{-2} G, \\ &\mathbf{c}^H A^{-1}\mathbf{c} = (\mathbf{c}^H\mathbf{c})v^{11}, \end{aligned} \tag{4.5}$$

and

$$\mathbf{c}^H A^{-2}\mathbf{c} = (\mathbf{c}^H\mathbf{c})\left[(v^{11})^2 + \sum_{k=2}^{m}|v^{1k}|^2\right], \tag{4.6}$$

where $V^{-1} = (v^{jk})$. By substituting from (4.5) and (4.6) in (4.3) and (4.4), we get

$$B = \frac{(v^{11})^2}{(v^{11})^2 + \sum_{k=2}^{m} |v^{1k}|^2}, \qquad G = (v^{11})^{-1}.$$

Now, let $V = TT^H$, where T is a complex upper triangular matrix with positive diagonal elements and partition T as

$$T = \begin{pmatrix} t_{11} & \mathbf{t}^H \\ 0 & T_{22} \end{pmatrix}, \qquad T_{22}\,((m-1)\times(m-1)).$$

Then, from (3.3) it follows that

$$B = \frac{1}{1 + \mathbf{t}^H (T_{22}^H T_{22})^{-1}\mathbf{t}}, \qquad G = t_{11}^2.$$

From Theorem 2.4 and Theorem 2.6, it follows that t_{11}^2 and $\mathbf{t}^H (T_{22}^H T_{22})^{-1}\mathbf{t}$ are independent, with $t_{11}^2 \sim G(n-m+1)$ and $\mathbf{t}^H (T_{22}^H T_{22})^{-1}\mathbf{t} \sim IB(m-1, n-m+2)$. Since,

$$\frac{1}{1 + \mathbf{t}^H (T_{22}^H T_{22})^{-1}\mathbf{t}} \sim B^I(n-m+2, m-1),$$

the theorem follows. □

Let $Z = GB = (\mathbf{c}^H A^{-2}\mathbf{c})^{-1}\mathbf{c}^H A^{-1}\mathbf{c}$. From Theorem 4.1, the h-th moment of Z is obtained as

$$E(Z^h) = \frac{\Gamma(n-m+2+h)\Gamma(n-m+1+h)\Gamma(n+1)}{\Gamma(n-m+2)\Gamma(n-m+1)\Gamma(n+1+h)}.$$

Now, using the inverse Mellin transform and the above moment expression, the density of Z is derived as

$$f(z) = \frac{\Gamma(n+1)}{\Gamma(n-m+2)\Gamma(n-m+1)}(2\pi\iota)^{-1} \times \int_C \frac{\Gamma(n-m+2+h)\Gamma(n-m+1+h)}{\Gamma(n+1+h)} z^{-1-h}\,dh, \quad z > 0,$$

where C is a suitable contour, $\iota = \sqrt{-1}$, and $0 < z < \infty$. Now, using the definition of G-function (Luke [18, p. 144]), the above density can be written as

$$f(z) = \frac{\Gamma(n+1)z^{-1}}{\Gamma(n-m+2)\Gamma(n-m+1)} G_{1,2}^{2,0}\left[z \,\middle|\, \begin{matrix} n+1 \\ n-m+1, n-m+2 \end{matrix} \right], \quad z > 0,$$

Finally, using (11) of Luke [18, p. 231]), we get

$$f(z) = \frac{\Gamma(n+1)z^{n-m}\exp(-z/2)}{\Gamma(n-m+2)\Gamma(n-m+1)} W_{1-m,1/2}(z), \quad z > 0,$$

where $W_{\alpha,\beta}$ is the Whitaker's function.

From the above analysis it can also be observed that $\mathbf{c}^H A^{-1}\mathbf{c} = (\mathbf{c}^H\mathbf{c})v^{11}$, $\mathbf{c}^H A\mathbf{c} = (\mathbf{c}^H\mathbf{c})v_{11}$ where $v_{11} = t_{11}^2 + \sum_{j=2}^m |t_{1j}|^2$, $v^{11} = t_{11}^{-2}$. Further, $t_{11}^2 \sim G(n-m+1)$ and $\sum_{j=2}^m |t_{1j}|^2 \sim G(m-1)$ are independent. Now, using standard results on statistical distribution theory, it can be seen that $(\mathbf{c}^H\mathbf{c})^{-1}\mathbf{c}^H A\mathbf{c} \sim G(n)$ and $(\mathbf{c}^H\mathbf{c})^2[(\mathbf{c}^H A\mathbf{c})(\mathbf{c}^H A^{-1}\mathbf{c})]^{-1} \sim B^I(n-m+1, m-1)$.

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