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THE DISTRIBUTION OF PRODUCT OF INDEPENDENT BETA RANDOM VARIABLES WITH APPLICATION TO MULTIVARIATE ANALYSIS

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Summary

Schatzoff [9] obtained the forms of the probability density function (pdf) and the cumulative distribution function (cdf) of the product of independent beta random variables when their parameters had some special values. The forms, however, did not indicate the constants explicitly. In this paper his approach is modified so as to allow presentation of explicit expressions for the pdf and cdf of the product of independent beta random variables (without restriction to the values of the parameters) in neat forms. Applications in multivariate analysis are given for the central and the non-central cases.

1. Introduction

In this paper we shall not, generally, distinguish between the random variables and the values taken by them. The context will make the meaning clear. By $U\sim\beta(x, a, b)$, we shall mean that the random variable U has a beta distribution with parameters $a>0$, $b>0$, viz, the pdf of U at x is $\beta[(a, b)]^{-1}x^{a-1}(1-x)^{b-1}$. Define $x\equiv\prod_{i=1}^{p} x_i$, $x^*\equiv\prod_{i=1}^{p} x_i^{l_i}$ ($l_i>0$, $i=1,\cdots,p$), where $x_i\sim\beta(x_i, a_i, b_i)$, $i=1,\cdots,p$ and $x_1,\cdots,x_p$ are independent. Schatzoff [9] obtained the structural form (i.e., the constants were not stated explicitly) of the pdf and cdf of x in closed form when $a_i=(n-i+1)/2$, $b_i=q/2$ and p or q was even. The constants could be calculated recursively when p and q were not large; otherwise their determination in Schatzoff's own words was not an easy task. For the case when both p and q were odd, Schatzoff expressed the cdf of x in an integral form.

Gupta [3] applied Schatzoff's method to obtain the structural form

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of the non-central distribution of Wilks' statistic in the linear case. Both Schatzoff and Gupta were dealing with the likelihood ratio criterion for MANOVA model and as such were concerned with some restricted values for a_i, b_i, $i=1,\cdots,p$.

Mathai [8] obtained the pdf of x in series form using contour integration. His series expressions, however, appear to be very complicated.

In this paper we modify Schatzoff's approach to obtain explicit neat expressions for the pdf and cdf of x and x^* where a_i and b_i, $i=1,\cdots,p$ are arbitrary. These are given in Theorem 1 and Corollary 1 under Section 3. Note that the distribution of many likelihood ratio test statistics under the null hypothesis is that of x^* for monotone samples. (See Bhargava [1], [2].) In Corollary 2, Section 3, we give the pdf and cdf of a mixture of populations each of which has the distribution of product of independent beta random variables. Corollary 2 enables to write the non-central distributions of some Wilks' statistics in Section 4. Pillai et al. [5] obtained the non-central distributions of Wilks' criteria for the three cases, namely, a) covariances, b) MANOVA and c) canonical correlations in terms of G-functions. Pillai and Jouris [7] obtained similar distribution results for the complex Wishart case. In Section 4 we obtain the non-central distributions of Pillai et al. [5] as mixtures of products of independent beta random variables as given in Corollary 2. The results relating to MANOVA case generalize the results of Gupta [3] mentioned above. We also remark that Pillai and Jouris [7] results can be expressed similarly. Section 2 defines notations and some functions and limiting operations in terms of which the pdf and cdf of x and x^* can be described. It also gives some preliminary results as Lemmas 1 and 2.

2. Notations and preliminaries

Matrices and vectors will be denoted by bold face letters and the transpose of $\boldsymbol{A}$ by $\boldsymbol{A}'$. If $\boldsymbol{A}$ is a square matrix, then $|\boldsymbol{A}|$=determinant of $\boldsymbol{A}$.

We now define some functions and limiting operations in terms of which our pdf and cdf will be expressed. Let $\boldsymbol{t}=(t_1,\cdots,t_p)'$, be a $p\times 1$ vector. In our context $t_i>0$, $i=1,\cdots,p$. Let us define the determinants:

$$(2.1)\qquad D(\boldsymbol{t})=\begin{vmatrix} 1 & t_1 & \cdots & t_1^{p-2} & t_1^{p-1} \\ 1 & t_2 & \cdots & t_2^{p-2} & t_2^{p-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & t_p & \cdots & t_p^{p-2} & t_p^{p-1} \end{vmatrix}=\prod_{i=2}^{p}\prod_{j=1}^{i-1}(t_i-t_j)\ ,$$

$N_{(1)}(y|\boldsymbol{t})$ = Determinant obtained from $D(\boldsymbol{t})$ by replacing its last column by $(\exp(-t_1y), \exp(-t_2y), \cdots, \exp(-t_py))'$

$$=(-1)^{p-1}D(\boldsymbol{t})\sum_{i=1}^{p}\left[\exp(-t_iy)\Big/\prod_{\substack{j=1\\(j\neq i)}}^{p}(t_j-t_i)\right], \tag{2.2}$$

$$N_1(y|\boldsymbol{t})=(-1)^{p-1}(t_1t_2\cdots t_p)N_{(1)}(y|\boldsymbol{t}), \tag{2.3}$$

$N_{(2)}(y|\boldsymbol{t})$ = Determinant obtained from $D(\boldsymbol{t})$ by replacing its last column by $(\exp(-t_1y)/t_1, \exp(-t_2y)/t_2, \cdots, \exp(-t_py)/t_p)'$

$$=(-1)^{p-1}D(\boldsymbol{t})\sum_{i=1}^{p}\left[\exp(-t_iy)\Big/t_i\prod_{\substack{j=1\\(j\neq i)}}^{p}(t_j-t_i)\right] \tag{2.4}$$

and

$$N_2(y|\boldsymbol{t})=(-1)^{p-1}(t_1t_2\cdots t_p)N_{(2)}(y|\boldsymbol{t})=D(\boldsymbol{t})-\int_0^y N_1(y|\boldsymbol{t})dy\,. \tag{2.5}$$

Notice that the determinants, $D(\boldsymbol{t})$, $N_i(y|\boldsymbol{t})$ and $N_{(i)}(y|\boldsymbol{t})$ for $i=1, 2$ are zeroes when any two of $t_1, \cdots, t_p$ are equal. We now define two functions

$$f_i(y|\boldsymbol{t})=N_i(y|\boldsymbol{t})/D(\boldsymbol{t}) \qquad \text{for } i=1, 2\,, \tag{2.6}$$

with the understanding that when any two of the t_i's are equal, their values will be given with the help of La Hospital rule (i.e., through the limiting process). To explain this, suppose $t_1=t_2=\cdots=t_r=t_1^*$, and $t_1^*, t_{r+1}, \cdots, t_p$ are all different. By La Hospital rule, we get

$$f_i(y|(t_1^*, \cdots, t_1^*, t_{r+1}, \cdots, t_p))=\frac{\prod_{j=1}^{r}\left[\left(\frac{\partial}{\partial t_j}\right)^{j-1}N_i(y|\boldsymbol{t})\right]_{t_1=\cdots=t_r=t_1^*}}{\prod_{j=1}^{r}\left[\left(\frac{\partial}{\partial t_j}\right)^{j-1}D(\boldsymbol{t})\right]_{t_1=\cdots=t_r=t_1^*}} \qquad \text{for } i=1, 2\,. \tag{2.7}$$

This procedure tells us to change the first r rows of $N_i(y|\boldsymbol{t})$ and $D(\boldsymbol{t})$ by new r rows, of which the jth row of $D(\boldsymbol{t})$, $N_{(1)}(y|\boldsymbol{t})$ and $N_{(2)}(y|\boldsymbol{t})$ are respectively given by

$$\Big(\underbrace{0\cdots 0}_{(j-1)\text{ times}}, 1, \binom{j}{j-1}t_1^*, \binom{j+1}{j-1}(t_1^*)^2, \cdots, \binom{p-2}{j-1}(t_1^*)^{p-j-1}, \binom{p-1}{p-1}(t_1^*)^{p-j}\Big) \qquad \text{for } D(\boldsymbol{t})\,, \tag{2.8}$$

$$\Big(\underbrace{0\cdots 0}_{(j-1)\text{ times}}, 1, \binom{j}{j-1}t_1^*, \binom{j+1}{j-1}(t_1^*)^2, \cdots, \binom{p-2}{j-1}(t_1^*)^{p-j-1}, (-y)^{j-1}\exp(-t_1^*y)/(j-1)!\Big) \qquad \text{for } N_{(1)}(y|\boldsymbol{t})\,, \tag{2.9}$$

and

$$(2.10)\quad \Big(\underbrace{0\cdots 0}_{(j-1)\text{ times}}, 1, \binom{j}{j-1}t_1^*, \cdots, \binom{p-2}{j-1}(t_1^*)^{p-j-1},$$
$$(-1)^{j-1}\sum_{\alpha=0}^{j-1}(yt_1^*)^\alpha \exp(-t_1^* y)/\alpha!(t_1^*)^j\Big) \qquad \text{for } N_{(2)}(y\,|\,\boldsymbol{t}),$$
$$\text{for } j=1, 2, \cdots, r, \binom{n}{x}=n!/x!(n-x)! \qquad \text{and}$$

$$(2.11)\qquad N_i(y\,|\,\boldsymbol{t})=(-1)^{p-1}(t_1^*)^r t_{r+1}\cdots t_p N_{(i)}(y\,|\,\boldsymbol{t}), \qquad (i=1, 2).$$

From the above definition of the replacement of rows, we get the required values of the functions $f_i(y\,|\,t_1^*, \cdots, t_1^*, t_{r+1}, \cdots, t_p)$. For brevity, we will call this procedure (of replacement of rows) as limiting operation for t_1^*.

If the t_i's are equal in groups of the type: $t_1=\cdots=t_{r_1}=t_1^*$, $t_{r_1+1}=\cdots=t_{r_2}=t_2^*, \cdots, t_{r_1+\cdots+r_{k-1}+1}=\cdots=t_{r_1+\cdots+r_k}=t_k^*$, where $\sum_{j=1}^k r_j=p$ and $t_1^*, \cdots, t_p^*$ are all different, we can write down the expressions for

$$f_i(y\,|\,\underbrace{t_1^*, \cdots, t_1^*}_{r_1}, \underbrace{t_2^*, \cdots, t_2^*}_{r_2}, \cdots, \underbrace{t_k^*, \cdots, t_k^*}_{r_k}), \qquad i=1, 2$$

in determinant forms by following the rules for replacement of rows as mentioned above. One can obtain the results by applying the limiting operations for $t_1^*, t_2^*, \cdots, t_k^*$ in succession.

Further define

$$(2.12)\qquad N_3(U\,|\,\boldsymbol{t})=U^{-1}N_1(-\log_e U\,|\,\boldsymbol{t}),$$

$$(2.13)\qquad N_4(U\,|\,\boldsymbol{t})=N_2(-\log_e U\,|\,\boldsymbol{t}),$$

and

$$(2.14)\qquad g_i(U\,|\,\boldsymbol{t})=N_{2+i}(U\,|\,\boldsymbol{t})/D(\boldsymbol{t}) \qquad \text{for } i=1, 2.$$

Note that $g_i(U\,|\,\boldsymbol{t})$, $i=1, 2$ are well defined for all values of $\boldsymbol{t}$ (whether some of the t_i's are equal or not) since right-hand side of (2.14) is well defined.

We now define some notations which will be used in Section 4. If $\boldsymbol{S}$ is distributed as Wishart such that $\mathrm{E}(\boldsymbol{S})=n\Sigma$ where $\boldsymbol{S}$ is a $p\times p$ positive definite matrix (i.e., $S>0$), then this will be denoted by $\boldsymbol{S}\sim W_p(n, \boldsymbol{\Sigma})$. $\boldsymbol{S}\sim W_p(n, \boldsymbol{\Sigma}, \boldsymbol{\Omega})$ means $\boldsymbol{S}$ is distributed as non-central Wishart with non-central parameters $\boldsymbol{\Omega}$. $C_K(S)$ denotes zonal polynomial for $S>0$. (See James [4].) Let us denote

$$\Gamma_p(n, K)=\Pi^{p(p-1)/4}\prod_{i=1}^p \Gamma\Big(n+k_i-\frac{i-1}{2}\Big), \qquad \Gamma_p(n)=\Gamma_p(n, 0),$$

and

$$(n_K)=\Gamma_p(n, K)/\Gamma_p(n) ,$$

where $K=\{k_1, k_2, \cdots, k_p\}$, $k_1\geqq k_2\geqq\cdots\geqq k_p\geqq 0$, $\sum_{i=1}^{p} k_i=k$ is a partition of the positive integer k.

We now obtain some preliminary results which will be utilized in Section 3 to obtain the distribution of product of independent beta variables.

LEMMA 1. *Let $y_1, y_2, \cdots, y_p$ be independent exponential random variables with the probability density function (pdf) of y_i being given as*

$$t_i \exp(-t_i y_i) \qquad \text{for } y_i>0,\ t_i>0 \text{ and } i=1, 2, \cdots, p .$$

Then, the pdf and cdf (cumulative distribution function) of $y=\sum_{i=1}^{p} y_i$ are respectively given by $f_1(y\,|\,\boldsymbol{t})$ and $1-f_2(y\,|\,\boldsymbol{t})$ where $f_1(y\,|\,\boldsymbol{t})$ and $f_2(y\,|\,\boldsymbol{t})$ are defined by (2.6).

PROOF. First, let us consider the case when all the t_i's are different. Then the pdf of $y_1+y_2=y_1'$ is

$$t_1 t_2 \exp(-t_1 y_1') \int_0^{y_1'} \exp((t_1-t_2)x)dx$$

or

$$t_1 t_2[(t_2-t_1)^{-1} \exp(-t_1 y_1')+(t_1-t_2)^{-1} \exp(-t_2 y_1')] .$$

Now, extending the same arguments, we can get the pdf of $y_1+y_2+y_3=y'$ as

$$t_1 t_2 t_3[\{(t_2-t_1)(t_3-t_1)\}^{-1} \exp(-t_1 y')+\{(t_1-t_2)(t_3-t_2)\}^{-1} \cdot \exp(-t_2 y')+\{(t_1-t_3)(t_2-t_3)\}^{-1} \exp(-t_3 y')] .$$

Proceeding in this way, we can prove the required results for the pdf and cdf of y by using (2.1) to (2.3) and (2.1), (2.4) and (2.5) respectively when all the t_i's are different. When some of the t_i's are equal, we can obtain the pdf and cdf of y by the limiting process as explained in Section 2. This completes the proof of Lemma 1. From Lemma 1 we can establish the following lemma by a simple transformation.

LEMMA 2. *Let $y_1, y_2, \cdots, y_p$ be independent variables, each having an exponential distribution as defined in Lemma* 1. *Then the pdf and cdf of $U=\prod_{i=1}^{p}\{\exp(-y_i)\}$ are given by $g_1(U\,|\,\boldsymbol{t})$ and $g_2(U\,|\,\boldsymbol{t})$ respectively, which are defined by* (2.14).

3. Distribution of product of independent beta random variables

Let $x_i \sim \beta(x_i; a_i, b_i)$ and $y_i = -\log_e x_i$. Then the pdf of y_i is

$$(3.1) \qquad [B(a_i, b_i)]^{-1} e^{-a_i y_i} (1-e^{-y_i})^{b_i-1} \qquad \text{for } 0<y_i<\infty .$$

Alternately this can be expressed as

$$(3.2) \qquad \sum_{j_i=0}^{\infty} C(a_i, b_i, j_i)(a_i+j_i) e^{-(a_i+j_i)y_i} , \qquad \text{for } 0<y_i<\infty ,$$

where

$$(3.3) \qquad (a_i, b_i, j_i) = (-1)^{j_i} \binom{b_i-1}{j_i} [(a_i+j_i)B(a_i, b_i)]^{-1} .$$

Note that for $0<b_i<1$, (3.2) is a mixture of exponential distributions. When b_i is a positive integer, (3.2) is the sum of a finite series with j_i varying from 0 to b_i-1.

THEOREM 1. *Let $x_i \sim \beta(x_i; a_i, b_i)$, $i=1, \cdots, p$ and let $x_1, \cdots, x_p$ be independent. Let $y_i = -\log_e x_i$ and $y = \sum_1^p y_i$. Then the pdf and the cdf of y are respectively given by*

$$(3.4) \qquad f(y, \boldsymbol{a}, \boldsymbol{b}) = \sum_{j} \left(\prod_{i=1}^{p} c(a_i, b_i, j_i) \right) f_1(y \mid \boldsymbol{a}+\boldsymbol{j})$$

and

$$(3.5) \qquad F(y, \boldsymbol{a}, \boldsymbol{b}) = 1 - \sum_{j} \left(\prod_{i=1}^{p} c(a_i, b_i, j_i) \right) f_2(y \mid \boldsymbol{a}+\boldsymbol{j})$$

and the pdf and the cdf of $x \equiv \sum_1^p x_i$ are respectively given by

$$(3.6) \qquad g(x, \boldsymbol{a}, \boldsymbol{b}) = \sum_{j} \left(\prod_{i=1}^{p} c(a_i, b_i, j_i) \right) g_1(x \mid \boldsymbol{a}+\boldsymbol{j})$$

and

$$(3.7) \qquad G(x, \boldsymbol{a}, \boldsymbol{b}) = \sum_{j} \left(\prod_{i=1}^{p} c(a_i, b_i, j_i) \right) g_2(x \mid \boldsymbol{a}+\boldsymbol{j})$$

where

$$\boldsymbol{j} \equiv (j_1, j_2, \cdots, j_p)' ,$$

$$\boldsymbol{a} = (a_1, a_2, \cdots, a_p)' , \qquad \boldsymbol{b} = (b_1, b_2, \cdots, b_p)'$$

$$(\boldsymbol{a}+\boldsymbol{j}) \equiv (a_1+j_1, a_2+j_2, \cdots, a_p+j_p)' ,$$

and

$$\sum_{j} \text{ is used for } \sum_{j_1}\sum_{j_2}\cdots\sum_{j_p} ,$$

where the range of summation of j_i is from 0 to ∞ unless b_i is a positive integer in which case it is from 0 to b_i-1.

Also, $(f_1(y|\boldsymbol{a}+\boldsymbol{j}), f_2(y|\boldsymbol{a}+\boldsymbol{j}))$ and $(g_1(y|\boldsymbol{a}+\boldsymbol{j}), g_2(y|\boldsymbol{a}+\boldsymbol{j}))$ are respectively defined by (2.6) and (2.14).

PROOF. The proofs of (3.4) and (3.5) follow from (3.2) and Lemma 1 while those of (3.6) and (3.7) follow from Lemma 2.

COROLLARY 1. *Let y_i be as defined in Theorem 1. Let $y^*=\sum_{i=1}^{p} l_i y_i$, $l_i>0$. Then the pdf and cdf of y^* are respectively given by*

$$\sum_{j}\left(\prod_{i=1}^{p} C(a_i, b_i, j_i)\right) f_1(y^*|(\boldsymbol{a}+\boldsymbol{j})/\boldsymbol{l}) , \tag{3.8}$$

and

$$1-\sum_{j}\left(\prod_{i=1}^{p} C(a_i, b_i, j_i)\right) f_2(y^*|(\boldsymbol{a}+\boldsymbol{j})/\boldsymbol{l}) , \tag{3.9}$$

and the pdf and the cdf of $x^*=\prod_{i=1}^{p} x_i^{l_i}$, $l_i>0$ are respectively given by

$$\sum_{j}\left(\prod_{i=1}^{p} C(a_i, b_i, j_i)\right) g_1(x^*|(\boldsymbol{a}+\boldsymbol{j})/\boldsymbol{l}) , \tag{3.10}$$

and

$$\sum_{j}\left(\prod_{i=1}^{p} C(a_i, b_i, j_i)\right) g_2(x^*|(\boldsymbol{a}+\boldsymbol{j})/\boldsymbol{l}) , \tag{3.11}$$

where

$$(\boldsymbol{a}+\boldsymbol{j})/\boldsymbol{l}=\left(\frac{a_1+j_1}{l_1}, \frac{a_2+j_2}{l_2}, \cdots, \frac{a_p+j_p}{l_p}\right)' . \tag{3.12}$$

The distribution of many likelihood ratio test statistics under the null hypotheses is that of the product of positive powers of independent beta random variables, when the sample obtained are monotone (See Bhargava [1], [2]). This corollary enables us to obtain them in series form. This result is new. In the past the pdf of the likelihood ratio statistic for the Hotelling's T^2 problem for a monotone sample has been expressed in terms of H-functions. (See Bhargava [2].)

COROLLARY 2. *The pdf and cdf of a mixture of populations with* *the (k, K)th population having the density $g(x, \boldsymbol{a}^{(k,K)}, \boldsymbol{b}^{(k,K)})$ and its weight as $\alpha(k, K)$, with k varying from 0 to ∞ and $K=(k_1, \cdots, k_p)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_p \geqq 0$, $\sum_{i=1}^{p} k_i = k$, a partition of the positive integer k, are respectively given by*

$$\sum_{k=0}^{\infty} \sum_{K} \alpha(k, K) g(x, \boldsymbol{a}^{(k,K)}, \boldsymbol{b}^{(k,K)}) \tag{3.13}$$

and

$$\sum_{k=0}^{\infty} \sum_{K} \alpha(k, K) G(x, \boldsymbol{a}^{(k,K)}, \boldsymbol{b}^{(k,K)}) , \tag{3.14}$$

where

$$\boldsymbol{a}^{(k,K)} = (a_1^{(k,K)}, \cdots, a_p^{(k,K)})' ; \qquad \boldsymbol{b}_1^{(k,K)} = (b_1^{(k,K)}, \cdots, b_p^{(k,K)})' ;$$
$$a_i^{(k,K)} > 0 ; \; b_i^{(k,K)} > 0 .$$

In Section 4, we use (3.13) and (3.14) to express the non-central distributions of some Wilks' criteria. One can easily establish the following result directly, and hence, its proof is not given.

THEOREM 2. *Let $x_i \sim \beta(x_i ; a_i, b_i)$, $i=1, 2$ and let x_1 and x_2 be independently distributed and further $a_1 = a_2 + b_2 + n$ where $n \geqq 0$ is an integer. Then the pdf of $x = \prod_1^2 x_i$ is given by*

$$\sum_{r=0}^{n} \binom{n}{r} \frac{B(a_2+r, b_2+n-r)}{B(a_2, b_2)} B(x ; a_2+r, b_1+b_2+n-r) \tag{3.15}$$

that is, the density of x is a mixture of beta densities.

Theorem 2 when applicable enables some simplifications in (3.6) and (3.7).

4. Applications

The non-central distributions of Wilks' criteria for the three situations, namely, (a) covariances, (b) MANOVA and (c) canonical correlations were obtained by Pillai et al. [5] in terms of G-functions. Here we first show that the distribution in each case is a mixture of the distributions of products of independent beta random variables and then obtain them in series form from Corollary 2.

For (a), S_1 and S_2 are independent Wisharts, $W_p(n_i, \Sigma_i)$, $i=1, 2$; $W = |S_2|/|\delta S_1 + S_2|$ (if $\delta > 0$ is given) and Λ is a diagonal matrix whose diagonal elements are the characteristic (Ch.) roots of $\Sigma_1 \Sigma_2^{-1}$. Then in the notation of James [4], we have from Pillai et al. [5]

$$\text{(4.1)} \qquad \begin{aligned} \mathrm{E}\, W^h &= \sum_{k=0}^{\infty} \sum_{K} \alpha(k, K) \frac{\Gamma_p((n_1+n_2)/2, K)\Gamma_p(n_2/2+h)}{\Gamma_p(n_2/2)\Gamma_p((n_1+n_2)/2+h, K)} \\ &= \sum_{k=0}^{\infty} \sum_{K} \alpha(k, K)\, \mathrm{E}\left[\prod_{i=1}^{p} x_i(k_i)\right]^h, \end{aligned}$$

where

$$\text{(4.2)} \qquad \alpha(k, K) = |\delta \Lambda|^{-n_1/2}(n_1/2)_K C_K(I_p - (\delta\Lambda)^{-1})(k!)^{-1}$$

and $x_i(k_i)$, $i=1,\cdots,p$, are independent beta random variables with $x_i(k_i) \sim B(x_i(k_i), (n_2-i+1)/2, n_1/2+k_i)$, $i=1,\cdots,p$. Since $0<W<1$, the pdf and cdf of W are given by (3.13) and (3.14) respectively where $\alpha(k, K)$ is given by (4.2) and

$$\text{(4.3)} \qquad \begin{aligned} \boldsymbol{a}^{(k,K)} &= \left(\frac{n_2}{2}, \frac{n_2-1}{2}, \cdots, \frac{n_2-(p-1)}{2}\right)', \quad \text{and} \\ \boldsymbol{b}^{(k,K)} &= \left(\frac{n_1}{2}+k_1, \frac{n_1}{2}+k_2, \cdots, \frac{n_1}{2}+k_p\right). \end{aligned}$$

Notice that $\boldsymbol{a}^{(k,K)}$ is a constant vector. It does not depend on k or K.

We may note that the above given expressions for the pdf and cdf of W are true even when $n_1<p$ but $n_2 \geqq p$. (When $n_1<p$, one can consider $S_1 = XX'$, where the column vectors of X: $p \times n_1$ are distributed as independent normals $N(0, \Sigma_1)$ and one obtains the same expressions.)

(b) For the MANOVA case, S_1 and S_2 are independent, $S_2 \sim W_p(n_2, \Sigma)$ and $S_1 \sim W_p(n_1, \Sigma, \Omega)$, where Ω is the non-central parameter. (Here also for $n_1<p$, we use the same type of convention as mentioned above and get the same final result.) Let $W=|S_2|/|S_1+S_2|$. Then, using the moments of Pillai et al. [5] and the procedure mentioned above, the pdf and cdf of W are given by (3.13) and (3.14) respectively where

$$\text{(4.4)} \qquad \alpha(k, K) = \exp(-\operatorname{tr}\Omega)C_K(\Omega)/k!$$

and $\boldsymbol{a}^{(k,K)}$ and $\boldsymbol{b}^{(k,K)}$ are given by (4.3).

(c) For the canonical correlation case, $S = \begin{array}{cc} \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{bmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array}$ is distributed as $W_{p+q}\left[n_1, \Sigma = \begin{array}{cc} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array}\right]$ with $n_1 \geqq p+q$ (only) and $n_2 = n_1 - q \geqq p$. Let P^2 be a diagonal matrix with diagonal elements as the ch. roots of $[\Sigma_{11}^{-1}\, \Sigma_{12}\, \Sigma_{22}^{-1}\, \Sigma_{12}']$, and $W=|S|/|S_{11}||S_{22}|$. Then, using the moments of Pillai et al. [5] and the procedure mentioned above, the pdf and cdf of W are given by (3.13), and (3.14) where

$$\text{(4.5)} \qquad \alpha(k, K) = |I_p - P^2|^{n_1/2}(n_1/2)_K C_K(P^2)(k!)^{-1},$$

and $\boldsymbol{a}^{(k,K)}$ and $\boldsymbol{b}^{(k,K)}$ are given by (4.3).

Similar results hold for the complex Wishart case for the above three situations. The moments are given by Pillai and Jouris [7]. They enable us to obtain $\alpha(k, K)$, $\boldsymbol{a}^{(k,K)}$ and $\boldsymbol{b}^{(k,K)}$.

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