Report

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1 System Model

We consider a complex uncoded spatial multiplexing MIMO system with N_r receive and N_t transmit antennas, $N_r \geq N_t$, over a flat fading channel. Using a discrete time model, $\mathbf{y} \in \mathbb{C}^{N_r \times 1}$ is the received symbol vector written as:

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}, \tag{1}$$

where $\mathbf{s} \in \mathbb{C}^{N_t \times 1}$ is the transmitted symbol vector, with components that are mutually independent and taken from a signal constellation $\mathbb{O}$ (4-QAM, 16-QAM, 64-QAM) of size

M. The possible transmitted symbol vectors $\mathbf{s} \in \mathbb{O}^{N_t}$, satisfy $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathbf{I}_{N_t} E_s$, where E_s denotes the symbol average energy, and $\mathbb{E}[\cdot]$ denotes the expectation operation. Furthermore $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ denotes the Rayleigh fading channel propagation matrix with independent identically distributed (i.i.d) circularly symmetric complex Gaussian components of zero mean and unit variance. Finally, $\mathbf{n} \in \mathbb{C}^{N_r \times 1}$ is the additive white Gaussian noise (AWGN) vector with zero mean components and $\mathbb{E}[\mathbf{n}\mathbf{n}^H] = \mathbf{I}_{N_r} N_0$, where N_0 denotes the noise power spectrum density, and hence $\frac{E_s}{N_0}$ is the signal to noise ratio (SNR).

Assume the receiver has perfect channel state information (CSI), meaning that $\mathbf{H}$ is known, as well as the SNR. The task of the MIMO decoder is to recover $\mathbf{s}$ based on $\mathbf{y}$ and $\mathbf{H}$.

2 Modification of Orthogonality Deficiency

Original definition of orthogonality deficiency:

$$\phi_{od} = 1 - \frac{det(\mathbf{W})}{\prod_{i=1}^{N_t} ||\mathbf{h}_i||^2}, \tag{2}$$

where $\mathbf{W} = \mathbf{H}^H\mathbf{H}$ denotes Wishart matrix, $\mathbf{h}_i$ denotes the i th column of $\mathbf{H}$, $det(\cdot)$ denotes determinant operation, $||\cdot||^2$ denotes 2-norm operation. In (2), $||\mathbf{h}_i||^2 = \sum_{i=1}^{N_t} |\mathbf{H}_{ij}|^2$, $\mathbf{H}_{ij}$ denotes the component of $\mathbf{H}$ at i th row and j th column. $\mathbf{H}_{ij} \sim Rayleigh(1/\sqrt{2})$, therefore $||\mathbf{h}_i||^2 \sim \Gamma(N_r, 1)$ [1]. $\Gamma(k, \theta)$ denotes Gamma distribution, with k degrees of freedom. Furthermore, we have:

$$2||\mathbf{h}_i||^2 \sim Gamma(N_r, 2) \sim \chi^2_{2N_r}, \tag{3}$$

χ^2_k denotes chi-square distribution with k degrees of freedom. Because $\ln(\chi^2)$ converges much faster than χ^2 [2] [3] as well as for simplicity, (2) can be changed to:

$$\phi_{om} = \frac{2^{N_t} det(\mathbf{W})}{\prod_{i=1}^{N_t} 2||\mathbf{h}_i||^2} \longrightarrow \frac{N_t \ln 2 + \ln det(\mathbf{W})}{\sum_{i=1}^{N_t} \ln 2||\mathbf{h}_i||^2}, \tag{4}$$

ϕ_{om} in (4) is defined as Orthogonality Measure. Based on Hadamard's inequality ($\prod_{i=1}^{N_t} ||\mathbf{h}_i|| \geq det(\mathbf{H})$) $\phi_{om} \in [0, 1]$. If ϕ_{om} is more closer to 1, $\mathbf{H}$ is closer to orthogonal matrix.

3 Derivation of Marginal Probability Density Functions (PDFs)

First we consider the marginal PDFs of the components in (4), define

$$V = \sum_{i=1}^{N_t} \ln 2||\mathbf{h}_i||^2, \tag{5}$$

$$U = N_t \ln 2 + \ln det\mathbf{W}, \tag{6}$$

where V is the sum of components $\ln 2||\mathbf{h}_i||^2 \quad i \in [1, N_t]$. Since each component converges to normality rapidly, it can be easily proved that V converges to normality.

Considering U, $\mathbf{W} = \mathbf{H}^H\mathbf{H}$, do QR factorization:

$$\mathbf{H} = \mathbf{Q}\mathbf{R}, \tag{7}$$

where $\mathbf{Q} \in \mathbb{C}^{N_r \times N_t}$ is a unitary matrix and $\mathbf{R} \in \mathbb{C}^{N_t \times N_t}$ is the upper triangular matrix. Using (7), we have $\mathbf{W} = \mathbf{R}^H\mathbf{R}$. r_{ii} denotes the i th diagonal component of $\mathbf{R}$, thus U can be rewritten as:

$$U = N_t \ln 2 + \ln(det \mathbf{R}^H \mathbf{R}) = N_t \ln 2 + \ln det(\mathbf{R}^H) det(\mathbf{R}) = N_t \ln 2 + \ln \prod_{i=1}^{N_t} r_{ii}^H \prod_{i=1}^{N_t} r_{ii} = N_t \ln 2 + \sum_{i=1}^{N_t} \ln |r_{ii}|^2. \tag{8}$$

The next step that can be take into account is to find the distribution of $|r_{ii}|^2$.

An alternative is to consider Wishart distribution of $det(\mathbf{W})$.

4 Derivation of Probability of Orthogonality Measure

An alternative modification of (4) can be written as:

$$\phi_{om} = \frac{\prod_{i=1}^{N_t} |r_{ii}|^2}{\prod_{i=1}^{N_t} ||\mathbf{h}_i||^2}, \tag{9}$$

Take the logarithm of (9), we have

$$\log \phi_{om} = \sum_{i=1}^{N_t} \log \frac{|r_{ii}|^2}{||\mathbf{h}_i||^2}. \tag{10}$$

Notice that $\mathbf{R}$ can be viewed as the Cholesky factorization of $\mathbf{W}$. Based on Cholesky factorization, we have $||\mathbf{h}_i||^2 = \sum_{j=1}^{i-1} |r_{ji}|^2 + |r_{ii}|^2$. Thus (10) can be rewritten as:

$$\log \phi_{om} = \sum_{i=1}^{N_t} \log \frac{1}{\sum_{j=1}^{i-1} |r_{ji}|^2/|r_{ii}|^2 + 1}. \tag{11}$$

From (11), the lattice reduction we proposed should aim to reduce the component $\sum_{j=1}^{i-1} |r_{ji}|^2/|r_{ii}|^2$.

5 Derivation of Logarithmic Expectation of Orthogonality Measurement

Taking the logarithmic of ϕ_{om} in (4), we have

$$\ln(\phi_{om}) = N_t \ln(2) + \ln(det(\mathbf{W})) - \sum_{i=1}^{N_t} \ln(2||\mathbf{h}_i||^2), \tag{12}$$

taking expectation, we have

$$\mathbb{E}[\ln(\phi_{om})] = N_t \ln(2) + \mathbb{E}[\ln(det(\mathbf{W}))] - \sum_{i=1}^{N_t} \mathbb{E}[\ln(2||\mathbf{h}_i||^2)]. \tag{13}$$

Consider Rayleigh fading channel, $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \cdots \mathbf{h}_{N_t}]$, where $\mathbf{h}_i$ denotes the i th column of $\mathbf{H}$, because each component of $\mathbf{H}$ is zero mean, and $\mathbf{h}_i$ are mutually independent. Therefore, $\mathbf{W} = \mathbf{H}^H\mathbf{H} \sim W(n, \mathbf{\Sigma})$ denotes Wishart distribution with n degrees of freedom and covariance matrix $\mathbf{\Sigma}$. The logarithmic expectation of $\mathbf{W}$ can be rewritten as [4]

$$\mathbb{E}[\ln(det(\mathbf{W}))] = \sum_{i=1}^{N_t} \psi(\frac{N_r + 1 - i}{2}) + N_t \ln(2) + \ln(det(\mathbf{I})) = \sum_{i=1}^{N_t} \psi\frac{N_r + 1 - i}{2} + N_t \ln(2), \tag{14}$$

where $\psi(n)$ denotes the Digamma function. $\psi(n+1)$ can be expressed as $\psi(n+1) = -\gamma + \int_0^1 \frac{1-x^n}{1-x}$, in which γ denotes the Euler-Mascherioni constant. Thus (14) can be rewritten as:

$$\mathbb{E}[\ln(det(\mathbf{W}))] = -N_t\gamma + \sum_{i=1}^{N_t} \int_0^1 \frac{1 - x^{\frac{(N_r - i)}{2}}}{1 - x} dx + N_t \ln(2). \tag{15}$$

Because the expectation of logarithmic of Gamma function can be written as:

$$\mathbb{E}[\ln(Gamma(n, \theta))] = \psi(n) + \ln(\theta), \tag{16}$$

thus according to (3), we have:

$$\mathbb{E}[\ln(2||\mathbf{h}_i||^2)] = \psi(N_r) + \ln(2). \tag{17}$$

Based on (13)(14)(17), The logarithmic expectation of ϕ_{om} can be written as:

$$\mathbb{E}[\ln(\phi_{om})] = N_t \ln(2) + \sum_{i=1}^{N_t} \psi \frac{N_r + 1 - i}{2} + N_t \ln(2) - \sum_{i=1}^{N_t} \psi(N_r) - N_t \ln(2) = N_t \ln(2) + \sum_{i=1}^{N_t} [\psi(\frac{N_r - i + 1}{2}) - \tag{18}$$

6 Derivation of Joint PDF of Logarithmic Orthogonality Measurement

Based on [5], the distribution of lower triangular matrix $\mathbf{R}$ is

$$\begin{aligned} p_{\mathbf{R}}(\mathbf{R}) = \frac{2^{N_t}}{\prod_{i=1}^{N_t} \Gamma(N_t - i + 1)} \pi^{\frac{-N_t}{2}(N_t - 1)} \times \qquad & (a) \\ \prod_{i=1}^{N_t} exp(-|r_{ii}|^2) \times \qquad & (b) \\ \prod_{i=1}^{N_t} \prod_{j<i} exp(-|r_{ij}|^2) \times \qquad & (c) \\ \prod_{i=1}^{N_t} (|r_{ii}|^{2(N_r - i) + 1}). \qquad & (d) \end{aligned} \tag{19}$$

Combining sub term (b) and (d), we have

$$\prod_{i=1}^{N_t} exp(-|r_{ii}|^2) \prod_{i=1}^{N_t} (|r_{ii}|^{2(N_r - i) + 1}) = \prod_{i=1}^{N_t} \Gamma(2(N_r - i + 1)) \frac{(|r_{ii}|^{(2(N_r - i + 1) - 1)} e^{-e^{|r_{ii}|^2}})}{\Gamma(2(N_r - i + 1))}, \tag{20}$$

combining the rest terms, we have

$$\frac{2^{N_t}}{\prod_{i=1}^{N_t}\Gamma(N_t-i+1)}\pi^{\frac{-N_t}{2}(N_t-1)}\prod_{i=1}^{N_t}\prod_{j<i}exp(-|r_{ij}|^2)=\frac{2^{N_t}}{\prod_{i=1}^{N_t}\Gamma(N_t-i+1)}\pi^{\frac{-N_t}{4}(N_t-1)}\prod_{i=1}^{N_t}\prod_{j<i}\frac{1}{\sqrt{\pi}}exp(-|r_{ij}|^2) \tag{21}$$

Finally based on (20), (21), (19) can be rewritten as:

$$p_{\mathbf{R}}(\mathbf{R})=C\prod_{i=1}^{N_t}\prod_{j<i}\frac{1}{\sqrt{\pi}}exp(-|r_{ij}|^2)\prod_{i=1}^{N_t}\frac{(|r_{ii}|^{(2(N_r-i+1)-1)}e^{-e^{|r_{ii}|^2}})}{\Gamma(2(N_r-i+1))}, \tag{22}$$

where C denotes constant

$$C=\frac{2^{N_t}}{\prod_{i=1}^{N_t}\Gamma(N_t-i+1)}\pi^{\frac{-N_t}{4}(N_t-1)}\prod_{i=1}^{N_t}\Gamma(2(N_r-i+1)). \tag{23}$$

From (22), we can find all the components in $\mathbf{R}$ are mutually independent, furthermore, $|r_{ij}|\sim N(0,1/2)$, $|r_{ii}|^2\sim Gamma(2(N_r-i+1),1)$. Because $|r_{ij}|$ are independent Gaussian, $\sum_{j<i}|r_{ij}|^2\sim Gamma(\frac{i-1}{2},1)$. Defining $\alpha_i=\sum_{j<i}|r_{ij}|^2$ and $\beta_i=|r_{ii}|^2$, α_i and β_i are mutually independent, therefore (11) can be rewritten to

$$\ln(\phi_{om})=\sum_{i=1}^{N_t}\ln(\frac{\beta_i}{\alpha_i+\beta_i}), \tag{24}$$

From [6], if $X \sim Gamma(k_1, \theta)$ and $Y \sim Gamma(k_2, \theta)$, then $\frac{X}{X+Y} \sim B(k_1, k_2)$, where B denotes Beta distribution. Therefore $\frac{\beta_i}{\beta_i+\alpha_i} \sim B(k_1, k_2)$, where $k_1 = 2(N_r - i + 1)$, $k_2 = \frac{i-1}{2}$. we define $\eta_i = \frac{\beta_i}{\beta_i+\alpha_i}$, based on (24), we have

$$\ln(\phi_{om}) = \sum_{i=1}^{N_t} \ln(\eta_i), \tag{25}$$

where $\ln(\eta_i)$ are independent but not identically distributed random variables.

$$\mu_i = \mathbb{E}[\ln(\eta_i)] = \psi(k_1) + \psi(k_1 + k_2) \tag{26}$$

$$\delta_i^2 = Var[\ln(\eta_i)] = \psi_1(k_1) + \psi_1(k_1 + k_2) \tag{27}$$

where ψ_1 denotes trigamma function $\psi_1(x) = \frac{d\psi(x)}{dx}$. Based on Lyapunov Central Limit Theorem [7], we have

$$\ln(\phi_{om}) \sim N(\mu, \delta^2), \tag{28}$$

in (28), $\delta^2 = \sum_{i=1}^{N_t} \delta_i^2$, $\mu = \sum_{i=1}^{N_t} \mu_i$. However we need to check Lindeberg's condition:

define $s_n = \sum_{i=1}^{n} \sigma^2$ if $\vartheta > 0$ exists and (29) is satisfied

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\vartheta}} \sum_{i=1}^{n} \mathbb{E}[|x_i - \mu_i|^{2+\vartheta}] = 0, \tag{29}$$

References

[1] A. Papoulis, "Stochastic processes," *McGra. w*, 1996.

[2] L. H. Shoemaker, "Fixing the f test for equal variances," *The American Statistician*, vol. 57, no. 2, pp. 105–114, 2003.

[3] M. S. Bartlett and D. Kendall, "The statistical analysis of variance-heterogeneity and the logarithmic transformation," *Supplement to the Journal of the Royal Statistical Society*, pp. 128–138, 1946.

[4] C. M. Bishop *et al.*, *Pattern recognition and machine learning.* springer New York, 2006, vol. 4, no. 4.

[5] M. S. Srivastava, "On the complex wishart distribution," *The Annals of Mathematical Statistics*, pp. 313–315, 1965.

[6] A. K. Gupta and S. Nadarajah, *Handbook of beta distribution and its applications.* CRC Press, 2004.

[7] J. Jacod and A. N. Shiryaev, *Limit theorems for stochastic processes.* Springer Berlin, 1987, vol. 1943877.