Report

Tianpei Chen

Department of Electrical and Computer Engineering

McGill University

May 13, 2015

1 Introduction

Decoder is one of the key components of Multiple-Input Multiple-Output (MIMO) systems. Designing of high performance and low complexity detector has become a bottleneck of Large MIMO systems. Recently, an interesting property of large MIMO channel called channel hardening phenomenon has been found [1] [2] [3] [4], which states the phenomenon that with the number of receiving or transmitting antennas increasing, the variances of channel mutual information decrease. Simply saying, the channel "hardens". An useful aspect of channel hardening phenomenon is that off diagonal elements of $\mathbf{H}^H\mathbf{H}$ matrix become more and more negligible comparing to diagonal elements when the number of receiving antennas or transmitting antennas becomes large.

Channel hardening phenomenon provides an opportunity for designing computationally economical detection algorithms. For example, linear detectors (LDs) such as zero forcing (ZF) and minimum mean square error (MMSE) detectors need to do costly matrix operations such as matrix multiplication and inverse. With channel hardening phenomenon, this process can be accelerated by approximate matrix inversion using series expansion techniques and deterministic approximations in large dimensions [5]. Channel hardening phenomenon can also provide computational advances to machine learning based detectors, e.g. probability data association (PDA) [6] [7] and message passing detectors [8] [9] [10] and Monte-Carlo Markov Chain (MCMC) MIMO detectors [11] [12]. This type of algorithms work well in large sparse systems.

In this report, we propose some useful insights of the orthogonal properties of channel. We define orthogonality measurement (om), which is a well round metric that considers both diagonal and off diagonal elements of $\mathbf{H}^H\mathbf{H}$.

The rest of the report is organized as follows, section 2 introduces system model. Section

3 provides some preliminaries. Section 4 discusses the derivation of logarithmic expectation of orthogonality measurement. In section 5 we obtain probability density function of orthogonality measurement. Some computer simulation results are presented in section 6.

2 System Model

We consider a complex uncoded spatial multiplexing MIMO system with N_r receive and N_t transmit antennas, $N_r \geq N_t$, over a flat fading channel. Using a discrete time model, $\mathbf{y} \in \mathbb{C}^{N_r \times 1}$ is the received symbol vector written as:

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}, \tag{1}$$

where $\mathbf{s} \in \mathbb{C}^{N_t \times 1}$ is the transmitted symbol vector, with components that are mutually independent and taken from a finite signal constellation alphabet $\mathbb{O}$ (e.g. 4-QAM, 16-QAM, 64-QAM) of size M. The possible transmitted symbol vectors $\mathbf{s} \in \mathbb{O}^{N_t}$, satisfy $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathbf{I}_{N_t} E_s$, where E_s denotes the symbol average energy, and $\mathbb{E}[\cdot]$ denotes the expectation operation. Furthermore $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ denotes the Rayleigh fading channel propagation matrix with independent identically distributed (i.i.d) circularly symmetric complex Gaussian components with unit variance. Finally, $\mathbf{n} \in \mathbb{C}^{N_r \times 1}$ is the additive white Gaussian noise (AWGN) vector with zero mean components and $\mathbb{E}[\mathbf{n}\mathbf{n}^H] = \mathbf{I}_{N_r} N_0$, where N_0 denotes the noise power spectrum density, and hence $\frac{E_s}{N_0}$ is the signal to noise ratio (SNR).

Assume the receiver has perfect channel state information (CSI), meaning that $\mathbf{H}$ is known, as well as the SNR. The task of the MIMO decoder is to recover $\mathbf{s}$ based on $\mathbf{y}$ and $\mathbf{H}$.

3 Preliminaries

Orthogonality deficiency measures the how orthogonal a matrix is [13], which is defined by

$$\phi_{od} = 1 - \frac{det(\mathbf{W})}{\prod_{i=1}^{N_t} ||\mathbf{h}_i||^2}, \tag{2}$$

where $\mathbf{W} = \mathbf{H}^H \mathbf{H}$ denotes Wishart matrix, $\mathbf{h}_i$ denotes the i th column of $\mathbf{H}$, $det(\cdot)$ denotes determinant operation, $|| \cdot ||^2$ denotes 2-norm operation. In (2), $||\mathbf{h}_i||^2 = \sum_{j=1}^{N_r} |\mathbf{H}_{ji}|^2$, $\mathbf{H}_{ji}$ denotes the component of $\mathbf{H}$ at j th row and i th column. $|\mathbf{H}_{ji}| \sim Rayleigh(1/\sqrt{2})$, therefore $||\mathbf{h}_i||^2 \sim Gamma(N_r, 1)$ [14]. $Gamma(k, \theta)$ denotes Gamma distribution, with k degrees of freedom and scale θ. Furthermore, we have:

$$2||\mathbf{h}_i||^2 \sim Gamma(N_r, 2) \sim \chi^2_{2N_r}, \tag{3}$$

χ^2_k denotes chi-square distribution with k degrees of freedom. For the sake of simplicity, (2) can be changed to:

$$\phi_{om} = \frac{det(\mathbf{W})}{\prod_{i=1}^{N_t} ||\mathbf{h}_i||^2} = \frac{2^{N_t} det(\mathbf{W})}{\prod_{i=1}^{N_t} 2||\mathbf{h}_i||^2}. \tag{4}$$

Taking logarithmic operation to ϕ_{om} we have

$$\ln(\phi_{om}) = N_t \ln(2) + \ln(det(\mathbf{W})) - \sum_{i=1}^{N_t} \ln(2||\mathbf{h}_i||^2), \tag{5}$$

ϕ_{om} in (4) is defined as Orthogonality Measure. Based on Hadamard's inequality ($\prod_{i=1}^{N_t} ||\mathbf{h}_i|| \geq det(\mathbf{H})$). $\phi_{om} \in [0, 1]$. If ϕ_{om} is more closer to 1, $\mathbf{H}$ is closer to orthogonal matrix.

Because $\mathbf{W} = \mathbf{H}^H\mathbf{H}$, do QR factorization to $\mathbf{H}$

$$\mathbf{H} = \mathbf{QR}, \tag{6}$$

where $\mathbf{Q} \in \mathbb{C}^{N_r \times N_t}$ is a unitary matrix and $\mathbf{R} \in \mathbb{C}^{N_t \times N_t}$ is the upper triangular matrix.

Using (6), we have $\mathbf{W} = \mathbf{R}^H\mathbf{R}$. r_{ii} denotes the i th diagonal component of $\mathbf{R}$, thus $det(\mathbf{W})$ can be rewritten as:

$$det(\mathbf{W}) = det(\mathbf{R}^H\mathbf{R}) = det(\mathbf{R}^H)det(\mathbf{R}) = \prod_{i=1}^{N_t} r_{ii}^H \prod_{i=1}^{N_t} r_{ii} = \prod_{i=1}^{N_t} |r_{ii}|^2. \tag{7}$$

Notice that $\mathbf{R}$ can be viewed as the Cholesky factorization of $\mathbf{W}$. Therefore, we have

$$||\mathbf{h}_i||^2 = \mathbf{W}_{ii} = \sum_{j=1}^{i-1} |r_{ji}|^2 + |r_{ii}|^2, \tag{8}$$

where $\mathbf{W}_{ii}$ denotes the i th diagonal element of $\mathbf{W}$. Thus based on (7) and (8), (4) can be rewritten as:

$$\phi_{om} = \prod_{i=1}^{N_t} \frac{|r_{ii}|^2}{|r_{ii}|^2 + \sum_{j=1}^{i-1} |r_{ji}|^2}. \tag{9}$$

4 Logarithmic Expectation of Orthogonality Measurement

Taking expectation of (5), we have

$$\mathbb{E}[\ln(\phi_{om})] = N_t \ln(2) + \mathbb{E}[\ln(det(\mathbf{W}))] - \sum_{i=1}^{N_t} \mathbb{E}[\ln(2||\mathbf{h}_i||^2)]. \tag{10}$$

Consider $\mathbf{H} = [\mathbf{h}_1^{'}, \mathbf{h}_2^{'}, \cdots \mathbf{h}_{N_r}^{'}]^{'}$, where $\mathbf{h}_i$ denotes the i th row of $\mathbf{H}$, because each component of $\mathbf{H}$ is mutually independent and subject to circularly symmetric complex Gaussian distribution, i.e. $\mathbf{h}_i \sim \mathbb{C}N(\mathbf{0}, \mathbf{I}_{N_t})$. Therefore, $\mathbf{W} = \mathbf{H}^H\mathbf{H} \sim \mathbb{C}W(N_r, \mathbf{I}_{N_t})$, $\mathbb{C}W(n, \mathbf{\Sigma})$ denotes complex Wishart distribution with n degrees of freedom and covariance matrix $\mathbf{\Sigma}$. The logarithmic expectation of $\mathbf{W}$ can be rewritten as

$$\mathbb{E}[\ln(det(\mathbf{W}))] = \frac{\tilde{\Gamma}'_{N_t}(N_r)}{\tilde{\Gamma}_{N_t}(N_r)} = \sum_{i=1}^{N_t} \psi(N_r - i + 1), \tag{11}$$

where $\tilde{\Gamma}_m(n)$ denotes the multivariate Gamma function and $\psi(n)$ denotes Digamma function. Proof: see Appendix A.

Because the logarithmic expectation of a Gamma distribution variable $\eth \sim Gamma(n, \theta)$

can be written as:

$$\mathbb{E}[\ln(\eth)] = \psi(n) + \ln(\theta), \tag{12}$$

where $\psi(n)$ denotes Digamma function. Thus according to (3), we have:

$$\mathbb{E}[\ln(2||\mathbf{h}_i||^2)] = \psi(N_r) + \ln(2). \tag{13}$$

Proof: see Appendix B.

Based on (10)(11)(13), The logarithmic expectation of ϕ_{om} can be written as:

$$\begin{aligned} \mathbb{E}[\ln(\phi_{om})] &= N_t \ln(2) + \sum_{i=1}^{N_t} \psi(N_r - i + 1) - N_t \psi(N_r) - N_t \ln(2) \\ &= \sum_{i=1}^{N_t} \psi(N_r - i + 1) - N_t \psi(N_r) \end{aligned} \tag{14}$$

5 Probability Density Function of Orthogonality Measurement

Recall (9)

$$\phi_{om} = \prod_{i=1}^{N_t} \frac{|r_{ii}|^2}{|r_{ii}|^2 + \sum_{j<i} |r_{ji}|^2}. \tag{15}$$

All the components in $\mathbf{R}$ are independently distributed and $r_{ji} \sim \mathbb{C}N(0,1)$, $|r_{ii}|^2 \sim Gamma(N_r - i + 1, 1)$ [15]. Because $|r_{ji}| \sim Rayleigh(1/\sqrt{2})$, $\sum_{j<i} |r_{ji}|^2 \sim Gamma(i-1, 1)$. Defining $\alpha_i = \sum_{j<i} |r_{ji}|^2$ and $\beta_i = |r_{ii}|^2$, α_i and β_i are mutually independent, therefore (9) can be rewritten as

$$\phi_{om} = \prod_{i=1}^{N_t} \frac{\beta_i}{\beta_i + \alpha_i}, \tag{16}$$

From [16], if $X \sim Gamma(k_1, \theta)$ and $Y \sim Gamma(k_2, \theta)$, then $\frac{X}{X+Y} \sim B(k_1, k_2)$, where B denotes Beta distribution. Therefore $\frac{\beta_i}{\beta_i + \alpha_i} \sim B(k_1^i, k_2^i)$, where $k_1^i = N_r - i + 1$, $k_2^i = i - 1$. we define $\eta_i = \frac{\beta_i}{\beta_i + \alpha_i}$, it is obvious that η_i are independently distributed. Based on (16), we have

$$\phi_{om} = \prod_{i=1}^{N_t} \eta_i. \tag{17}$$

Therefore the density function of ϕ_{om} can be defined as

$$f_{\phi_{om}}(x) = \frac{1}{x} \sum_{\mathbf{j}} (\prod_{i=1}^{N_t} c(k_1^i k, k_2^i, j^i)) f(-\ln(x)|\mathbf{k_1} + \mathbf{j}), \tag{18}$$

where $\sum_{\mathbf{j}} = \sum_{j^1} \sum_{j^2} \cdots \sum_{j^{N_t}}$, the range of $j^i \in [0, k_2^i - 1]$, $c(k_1^i, k_2^i, j_i) = (-1)^{j^i} \binom{k_2^i - 1}{j^i}$ $[(k_1^i + k_2^i)\mathbb{B}(k_1^i, k_2^i)]^{-1}$, $\mathbb{B}(\alpha, \beta)$ denotes beta function. $f(-\ln(x)|\mathbf{k_1} + \mathbf{j}) = (\prod_{i=1}^{N_t}(k_1^i + j^i)) \sum_{i=1}^{N_t}[exp((k_1^i + j^i)\ln(x)) / \prod_{j=1 j \neq i}^{N_t}(k_1^j + j^j - k_1^i - j^i)]$. $\mathbf{k_1} + \mathbf{j} = [k_1^1 + j^1, \cdots k_1^{N_t - 1} +$

$j^{N_t-1}, k_1^{N_t} + j^{N_t}]$. Proof: see Appendix C.

Consider logarithmic expectation of ϕ_{om}, we have

$$E[\ln(\phi_{om})] = \sum_{i=1}^{N_t} E[\ln(\eta_i)], \tag{19}$$

where $E[\ln(\eta_i)] = \psi(k_1^i) - \psi(k_1^i + k_2^i)$, thus we have

$$E[\ln(\phi_{om})] = \sum_{i=1}^{N_t} \psi(N_r - i + 1) - N_t\psi(N_r). \tag{20}$$

we can find (20) is consistent with (14).

6 Computer Simulations

Computer simulations are made for different sizes of V-BLAST MIMO systems, with $5 \leq N_r \leq 100, 5 \leq N_t \leq N_r$, the empirical estimation of logarithmic expectation of ϕ_{om}, $E[\ln(\phi_{om})]_{em}$, is calculated by taking average over $1e4$ channel realizations for each size of MIMO systems, as shown in Fig.1, the Theoretical logarithmic expectation of ϕ_{om} $E[\ln(\phi_{om})]_t$ in (20) is plotted in Fig.2. Average deviation between $E[\ln(\phi_{om})]_{em}$ and $E[\ln(\phi_{om})]_t$ is also calculated, $V_{em-t} = 7.3043e-04$.

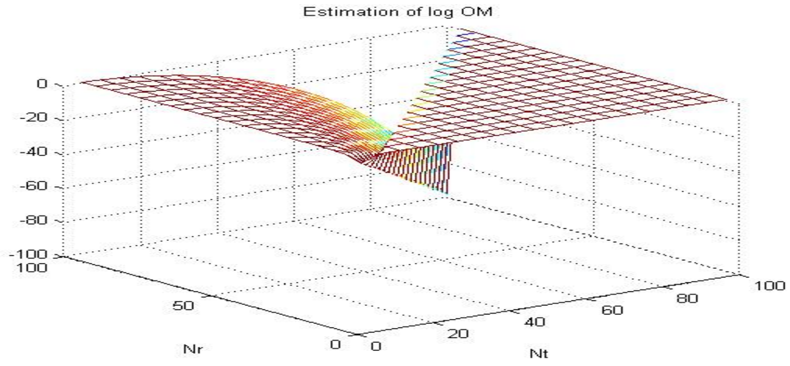

Figure 1: Empirical Estimation $E[\ln(\phi_{om})]_{em}$

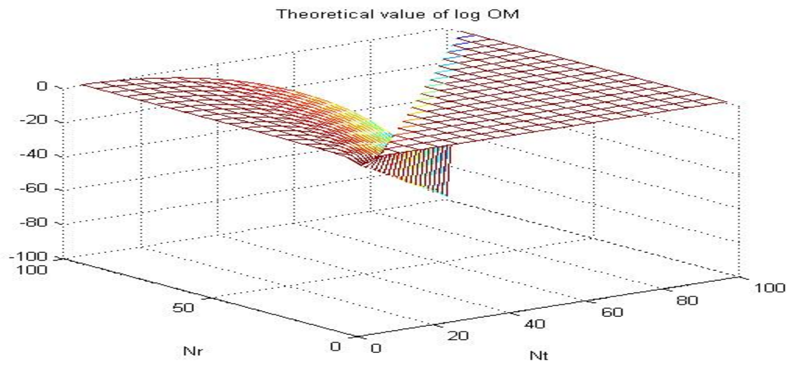

Figure 2: Theoretical $E[\ln(\phi_{om})]_t$

Fig.3 demonstrates the relation between the number of users (N_t) and $E[\ln(\phi_{om})]_t$ under cases of different numbers of antennas at base station (N_r). From Fig.3, we can see, on the one hand, with N_r fixed, $E[\ln(\phi_{om})]$ decreases while N_t increases, however the gradient of each curve becomes more and more gentle. On the other hand, when N_r becomes larger $E[\ln(\phi_{om})]$ becomes more insensitive to variation of N_t.

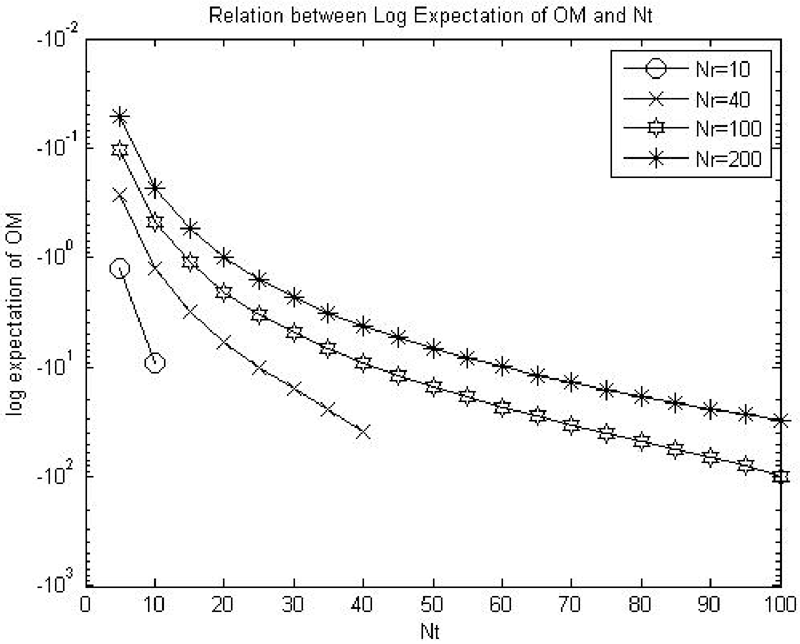

Figure 3: Relation between N_t and $E[\ln(\phi_{om})]_t$

A Appendix A

Let $\mathbf{A} \in \mathbb{C}^{m \times m}$, $A \sim \mathbb{C}W(n, \mathbf{\Sigma})$, $\mathbb{C}W(n, \mathbf{\Sigma})$ denotes complex Wishart distribution with n degrees of freedom and covariance matrix $\mathbf{\Sigma}$. It is obvious $\mathbf{A}$ is Hermition positive definite matrix, $\mathbf{A} = \mathbf{A}^H > 0$.

The pdf of $\mathbf{A}$ can be written as [15]:

$$f(\mathbf{A}) = \{\tilde{\Gamma}_m(n) det(\mathbf{\Sigma})^n\}^{-1} det(\mathbf{A})^{n-m} etr(-\mathbf{\Sigma}^{-1}\mathbf{A}), \tag{21}$$

where $\tilde{\Gamma}_m(\beta)$ denotes multivariate complex Gamma function defined by:

$$\tilde{\Gamma}_m(\beta) = \pi^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} \Gamma(\beta - i + 1) \quad Re(\beta) > m - 1. \tag{22}$$

Furthermore, from [15], we have

$$\tilde{\Gamma}_m(\beta) = \int_{\mathbf{X}=\mathbf{X}^H>0} etr(-\mathbf{X}) det(\mathbf{X})^{\beta-m} d\mathbf{X} \quad Re(\beta) > m - 1. \tag{23}$$

We derive logarithmic expectation of $det(\mathbf{A})$

$$\begin{aligned} E[\ln(det(\mathbf{A}))] &= \int_{\mathbf{A}=\mathbf{A}^H>0} \ln(det(\mathbf{A})) f(\mathbf{A}) d\mathbf{A} \\ &= \int_{\mathbf{A}=\mathbf{A}^H>0} \ln(det(\mathbf{A})) \{\tilde{\Gamma}_m(n) det(\mathbf{\Sigma})^n\}^{-1} det(\mathbf{A})^{n-m} etr(-\mathbf{\Sigma}^{-1}\mathbf{A}) d\mathbf{A} \\ &= \frac{det(\mathbf{\Sigma})^{-n}}{\tilde{\Gamma}_m(n)} \int_{\mathbf{A}=\mathbf{A}^H>0} \ln(det(\mathbf{A})) det(\mathbf{A})^{n-m} etr(-\mathbf{\Sigma}^{-1}\mathbf{A}) d\mathbf{A}, \end{aligned} \tag{24}$$

if $\mathbf{\Sigma} = \mathbf{I}$, (24) can be written as

$$E[\ln(det(\mathbf{A}))] = \frac{1}{\tilde{\Gamma}_m(n)} \int_{\mathbf{A}=\mathbf{A}^H>0} \ln(det(\mathbf{A})) det(\mathbf{A})^{n-m} etr(-\mathbf{A}) d\mathbf{A}. \tag{25}$$

Because $\frac{d}{dn}[det(\mathbf{A})]^{n-m} = \ln(det(\mathbf{A}))det(\mathbf{A})^{n-m}$, (25) can be rewritten as

$$E[\ln(det(\mathbf{A}))] = \frac{1}{\tilde{\Gamma}_m(n)} \frac{d}{dn} \int_{\mathbf{A}=\mathbf{A}^H>0} etr(-\mathbf{A})det(\mathbf{A})^{n-m} d\mathbf{A}, \tag{26}$$

using (23), (26) can be rewritten as

$$E[\ln(\mathbf{A})] = \frac{\tilde{\Gamma}'_m(n)}{\tilde{\Gamma}_m(n)}. \tag{27}$$

Based on (22), we have

$$\tilde{\Gamma}'_m(n) = \pi^{\frac{m(m-1)}{2}} \sum_{i=1}^{m} [\Gamma'(n-i+1) \prod_{j=1, j\neq i}^{m} \Gamma(n-j+1)], \tag{28}$$

Thus we have

$$E[\ln(det(\mathbf{A}))] = \frac{\tilde{\Gamma}'_m(n)}{\tilde{\Gamma}_m(n)} = \sum_{i=1}^{m} \frac{\Gamma'(n-i+1)}{\Gamma(n-i+1)} = \sum_{i=1}^{m} \psi(n-i+1), \tag{29}$$

where ψ denotes Digamma function.

B Appendix B

If $x \sim Gamma(n, \theta)$, with shape parameter k and scale parameter θ, $x > 0$, $\Gamma(k)$ denotes Gamma function, the density function of Gamma distribution is

$$f(x, k, \theta) = \frac{x^{k-1} e^{-x/\theta}}{\Gamma(k)\theta^k}. \tag{30}$$

Thus we have

$$E[\ln(x)] = \frac{1}{\Gamma(k)} \int_0^\infty \ln(x) x^{k-1} e^{-x/\theta} \theta^{-k} dx, \tag{31}$$

define $z = x/\theta$ and since $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$, (31) can be rewritten as

$$E[\ln(x)] = \ln(\theta) + \frac{1}{\Gamma(k)} \int_0^\infty \ln(z) z^{k-1} e^{-z} dz. \tag{32}$$

Because $\frac{d(z^{k-1})}{dk} = \ln(z) z^{k-1}$, (32) can be rewritten as

$$\begin{aligned} E[\ln(z)] &= \ln(\theta) + \frac{1}{\Gamma(k)} \frac{d}{dk} \int_0^\infty z^{k-1} e^{-z} dz \\ &= \ln(\theta) + \frac{\Gamma'(k)}{\Gamma(k)} \\ &= \ln(\theta) + \psi(k), \end{aligned}$$

where $\psi(k)$ denotes Digamma function.

C Appendix C

$x_1, x_2, \cdots x_{N_t}$ are independent beta variables, the probability density function (pdf) can be written as:

$$f(x_i) = \frac{1}{\mathbb{B}(k_1^i, k_2^i)} x_i^{k_1^i - 1} (1 - x_i)^{k_2^i - 1}, \tag{33}$$

define $y_i = -\ln(x_i) = g(x_i)$, Based on Jacobian transformation, we have

$$f_{y_i}(\rho) = |\frac{dy_i}{dx_i}|^{-1} f_{x_i}(g^{-1}(\rho)) = \frac{1}{\mathbb{B}(k_1^i, k_2^i)} e^{-k_1^i \rho} (1 - e^{-\rho})^{k_2^i - 1}. \tag{34}$$

where (34) can be alternatively expressed as [17]

$$f_{y_i}(\rho) = \sum_{j^i=0}^{k_2^i - 1} c(k_1^i, k_2^i, j^i)(k_1^i + j^i) exp(-(k_1^i + j^i)\rho), \tag{35}$$

where $c(k_1^i, k_2^i, j_i) = (-1)^{j^i} \binom{k_2^i - 1}{j^i} [(k_1^i + k_2^i)\mathbb{B}(k_1^i, k_2^i)]^{-1}$, $\mathbb{B}(\alpha, \beta)$ denotes beta function. Based on the lemma 1 of [17], if $a_1, a_2, \cdots a_n$ are independent exponentially distributed random variables, with pdf given by

$$t_i exp(-t_i a_i) \tag{36}$$

then pdf of $a = \sum_{i=1}^{n} a_i$ can be written as

$$f(a|\mathbf{t}) = \prod_{i=1}^{n} t_i \sum_{i=1}^{n} [exp(-t_i a) / \prod_{j=1 j\neq i}^{j=n} (t_j - t_i)], \tag{37}$$

where $t = [t_1, t_2, \cdots t_n]$. The pdf of y_i can be viewed as the weighting summation of exponential distribution functions, define $y = \sum_{i=1}^{n} y_i$, based on (37), the pdf of y is given by

$$f_y(m) = \sum_{\mathbf{j}} \{ [\prod_{i=1}^{n} c(k_1^i, k_2^i, j^i)] f(m|\mathbf{k_1} + \mathbf{j}) \}, \tag{38}$$

where $\sum_{\mathbf{j}} = \sum_{j^1} \sum_{j^2} \cdots \sum_{j^n}$, the range of j^i is defined by $j^i \in [0, k_2^i]$, $f(m|\mathbf{k_1} + \mathbf{j}) = (\prod_{i=1}^{N_t} (k_1^i + j^i)) \sum_{i=1}^{N_t} [exp(-(k_1^i + j^i)m) / \prod_{j=1 j\neq i}^{N_t} (k_1^j + j^j - k_1^i - j^i)]$, $\mathbf{k_1} + \mathbf{j} = [k_1^1 + j^1, k_1^2 + j^2 \cdots k_1^n + j^n]$. we define $U = exp(-y) = \prod_{i=1}^{n} x_i$, using Jacobian transformation, the pdf of U is given by

$$f_U(u) = |\frac{du}{dy}|^{-1} f_y(-\ln(u)) = \frac{1}{u} \sum_{\mathbf{j}} \{ [\prod_{i=1}^{n} c(k_1^i, k_2^i, j^i)] f(-\ln(u)|\mathbf{k_1} + \mathbf{j}) \}. \tag{39}$$

References

[1] B. M. Hochwald, T. L. Marzetta, and V. Tarokh, "Multiple-antenna channel hardening and its implications for rate feedback and scheduling," *Information Theory, IEEE Transactions on*, vol. 50, no. 9, pp. 1893–1909, 2004.

[2] D. Tse and P. Viswanath, *Fundamentals of wireless communication.* Cambridge university press, 2005.

[3] V. A. Marčenko and L. A. Pastur, "Distribution of eigenvalues for some sets of random matrices," *Sbornik: Mathematics*, vol. 1, no. 4, pp. 457–483, 1967.

[4] A. M. Tulino and S. Verdú, "Random matrix theory and wireless communications," *Communications and Information theory*, vol. 1, no. 1, pp. 1–182, 2004.

[5] M. Wu, B. Yin, A. Vosoughi, C. Studer, J. R. Cavallaro, and C. Dick, "Approximate matrix inversion for high-throughput data detection in the large-scale MIMO uplink," in *Circuits and Systems (ISCAS), 2013 IEEE International Symposium on.* IEEE, 2013, pp. 2155–2158.

[6] J. C. Fricke, M. Sandell, J. Mietzner, and P. A. Hoeher, "Impact of the Gaussian approximation on the performance of the probabilistic data association MIMO decoder," *EURASIP Journal on Wireless Communications and Networking*, vol. 2005, no. 5, pp. 796–800, 1900.

[7] D. Pham, K. R. Pattipati, P. K. Willett, and J. Luo, "A generalized probabilistic data association detector for multiple antenna systems," *IEEE Communications Letters*, vol. 8, no. 4, pp. 205–207, 2004.

[8] P. Som, T. Datta, N. Srinidhi, A. Chockalingam, and B. S. Rajan, "Low-complexity detection in large-dimension MIMO-ISI channels using graphical models," *Selected Topics in Signal Processing, IEEE Journal of*, vol. 5, no. 8, pp. 1497–1511, 2011.

[9] J. Goldberger and A. Leshem, "MIMO detection for high-order QAM based on a Gaussian tree approximation," *Information Theory, IEEE Transactions on*, vol. 57, no. 8, pp. 4973–4982, 2011.

[10] T. Lakshmi Narasimhan and A. Chockalingam, "Channel hardening-exploiting message passing (CHEMP) receiver in large-scale MIMO systems," 2013.

[11] B. Farhang-Boroujeny, H. Zhu, and Z. Shi, "Markov chain Monte Carlo algorithms for CDMA and MIMO communication systems," *Signal Processing, IEEE Transactions on*, vol. 54, no. 5, pp. 1896–1909, 2006.

[12] T. Datta, N. Ashok Kumar, A. Chockalingam, and B. S. Rajan, "A novel MCMC algorithm for near-optimal detection in large-scale uplink mulituser MIMO systems," in *Information Theory and Applications Workshop (ITA), 2012.* IEEE, 2012, pp. 69–77.

[13] X. Ma and W. Zhang, "Performance analysis for MIMO systems with lattice-reduction aided linear equalization," *Communications, IEEE Transactions on*, vol. 56, no. 2, pp. 309–318, 2008.

[14] A. Papoulis, "Stochastic processes," *McGra. w*, 1996.

[15] D. K. Nagar and A. K. Gupta, "Expectations of functions of complex Wishart matrix," *Acta applicandae mathematicae*, vol. 113, no. 3, pp. 265–288, 2011.

[16] A. K. Gupta and S. Nadarajah, *Handbook of beta distribution and its applications.* CRC Press, 2004.

[17] R. Bhargava and C. Khatri, "The distribution of product of independent beta random variables with application to multivariate analysis," *Annals of the Institute of Statistical Mathematics*, vol. 33, no. 1, pp. 287–296, 1981.