2025-tower-paper

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1 Introduction

This paper studies the rate at which measures converge in a tower-like Markov chain [2]. Specifically, we investigate polynomial decay of correlations, which arise often in dynamical systems. Our main contribution is a sharp first-order bound on the error terms of the renewal measure when delay is present, expanding on the work of Rogozin in [1]., as well as a result giving bounds on the total variation distance. We also provide possible refinements to the bound when the delay terms and the renewal rates are of the same order.

Just the standard abstract here. What has been done before, and what is this related to?

- Rogozin Paper
- That other guy who came up with the complex analysis stuff?
- Young Tower
- Decay of Correlations
- Renewal Theory

2 Preliminaries

Just put some initial results here. What did Rogozin do? What did Young do? The helper lemmas for the convergence results can go here, although I feel like theres room for improvement here.

3 Markov Results on Tower

Here I'll show the results on the tower like stationarity, what the stationary measure is, etc. I also need to give a proof of the renewal equation with the delay.

4 Convergence Results for Bases and Measures

Here, I'll give the proofs of the order estimates for the terms un, Un, Vn, and Rn. Split the cases for alpha and beta. Give the special case when I have monotonic sequences. Suppose $P_n = O(n^{-\alpha}\ell_1(n))$ and $u_n = O(n^{-\beta}\ell_2(n))$ where $1 \le \alpha < \beta$, and ℓ_1 and ℓ_2 are two slowly varying functions. If $\alpha = 1$, then

$$\left| a_n - \frac{1}{\mu} \right| = \frac{1}{\mu^2} V_n + o\left(\ell_1^*(n)\right),$$

and if $\alpha > 1$, then

$$\left| a_n - \frac{1}{\mu} \right| = \frac{1}{\mu^2} V_n + o(n^{-\alpha+1}\ell_1(n))$$

where

$$V_n = \{\, O\left(\ell_1^*(n)\right) if\ \alpha = 1, O\left(n^{-\alpha+1}\ell_1(n)\right) if\ \alpha > 1.$$

We recall from Lemma **??** that

$$a_n = \frac{1}{\mu} - \frac{1}{\mu} \sum_{k=n+1}^{\infty} u_k + \frac{1}{\mu^2} \sum_{k=0}^{n} u_k Q_{n-k} + R_n.$$

Since $u_n = O(n^{-\beta}\ell_2(n))$, and $\beta > 1$, by assumption, we have by Lemma **??** that

$$U_n = \sum_{k=n+1}^{\infty} u_k = O(n^{-\beta+1}\ell_2(n)).$$

To bound the V_n terms, we first consider the case when $\alpha = 1$. In this case, since $P_n = O(n^{-\alpha}\ell_1(n))$, we use Lemma **??** to see

$$Q_n = \sum_{k=n+1}^{\infty} P_n = O\left(\sum_{k=n+1}^{\infty} k^{-1}\ell_1(k) \right) = O(\ell_1^*(n)).$$

Using this bound, we from Lemma **??**, that

$$V_n = \sum_{k=0}^{n} u_k Q_{n-k} = O(\ell_1^*(n)).$$

Lastly, we consider the terms R_n. From Rogozin's result in Theorem **??**, we have

$$R_n(1) = \{\, O\left(\ell_1^*(n)\right) if\ \alpha = 1, O(n^{-2\alpha+2}\ell_1(n)) if\ 1 < \alpha < 2, O(n^{-2}\ell_1^-(n)) if\ \alpha = 2,\ and O(n^{-\alpha}\ell_1(n)) if\ \alpha > 2.$$

Let $\gamma = \min(2\alpha - 2, \beta)$. Then again from Lemma **??**, we see

$$R_n = \sum_{k=0}^{n} u_k R_{n-k}(1) = \{\, O\left((\ell_1^+(n))^2\right) if\ \alpha = 1, O(n^{-\gamma}\ell_1(n)) if\ 1 < \alpha < 2, O(n^{-2}\ell_1^-(n)) if\ \alpha = 2, O(n^{-\alpha}\ell_1(n)) i$$

In all cases, $R_n = o(\ell_1^+(n))$. Thus we see that V_n in the largest order term, and so

$$a_n = \frac{1}{\lambda} + \frac{1}{\lambda^2} V_n + o\left(\sum_{k=1}^{n} k^{-1}\ell_1(k) \right).$$

5 Correlation (if I have time)

What if the returns aren't independent but depend on the section they left from?

6 Conclusion

Summary of what I did, and maybe some future areas to work on.

References

[1] B. A. Rogozin. "An Estimate of the Remainder Term in Limit Theorems of Renewal Theory". In: *Theory of Probability and its Applications* XVIII.4 (1973), pp. 662–677.

[2] Lai-Sang Young. "Statistical Properties of Dynamical Systems with Some Hyperbolicity". In: *The Annals of Mathematics* 147.3 (May 1998), p. 585. ISSN: 0003-486X. DOI: 10.2307/120960. URL: http://dx.doi.org/10.2307/120960.