Chapter 5 : Uncertain Data Fusion

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Chapter organization

- In this chapter, each datum delivered by a source is (possibly) tainted with **uncertainty**.

- The chapter is organized into sections w.r.t. the chosen **uncertainty** representation framework.

Uncertainty theories in the literature :

- The probability theory is by far the most frequently used and renowned one.
- Second most renowned is the possibility theory [8] which relies on Zadeh's fuzzy sets theory [7].
- Possibilistic approaches gained popularity in the second half of the 20th century because they provide a simple and flexible framework to describe uncertain situations that probabilities fail to fully grasp.
- Ignorance, and more generally imprecise proposals, are encoded awkwardly by probabilities.

Uncertainty theories :

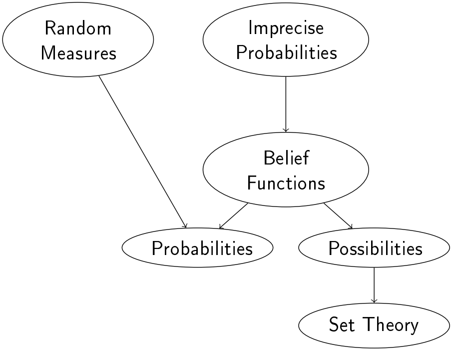
- Imprecision and uncertainty can be jointly handled by a more general framework introduced in the 90's by Walley [6] : imprecise probabilities.
- Imprecise probabilities are actually interval of probabilities. For instance, the statement $\mathbb{P}(A) \leq 0.9$ is an imprecise probability.
- Imprecise probabilities are also in correspondence with sets of probability measures. In our example, this is the set of all probability measures such that the measure of A is less than 0.9.

Uncertainty theories :

- We will also investigate a special case of imprecise probabilities known as belief functions.
- The belief function theory encompasses the probability theory, the possibility theory and Cantor's set theory.
- This latter framework is also known as Dempster-Shafer theory of evidence [1, 4].
- There are also circumstances allowing us to derive probabilities of probabilities which are captured by random measures.

Uncertainty theories :

- The hierarchy of uncertainty theories is given in the following figure :



Uncertainty theories : notations

- Let $\mathcal{P}_{\mathcal{X}}$ denote the set of probability measures expressing the odds that a candidate value in $\mathcal{X}$ is x.

- Let $\mathcal{B}_{\mathcal{X}}$ denote the set of belief functions expressing the odds that a candidate value in $\mathcal{X}$ is x.

- Finally, let $2^{\mathcal{P}_{\mathcal{X}}}$ denote the power set of $\mathcal{P}_{\mathcal{X}}$.

- The hierarchy displayed in the previous figure implies that :

$$\mathcal{P}_{\mathcal{X}} \subsetneq \mathcal{B}_{\mathcal{X}} \subsetneq 2^{\mathcal{P}_{\mathcal{X}}}. \tag{1}$$

Generalized definitions of **conjunctive** and **disjunctive** operators :

- Time for a broader definition of the conjunctive and disjunctive nature of fusion operators.

Definition

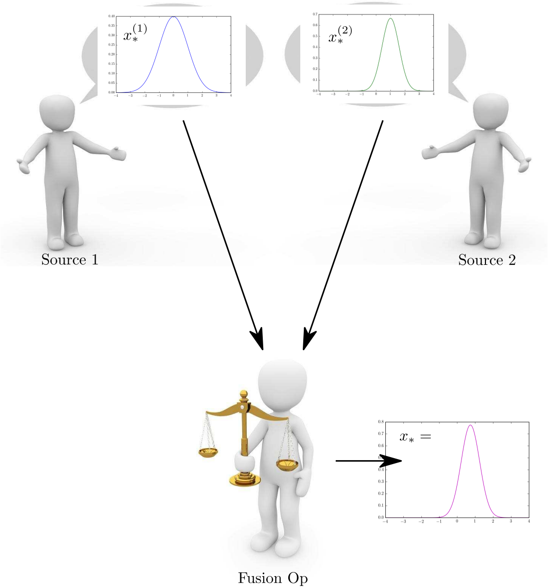
Let us consider a (general) data fusion problem with $\hat{\hat{x}}$ is a given fusion operator. $\hat{\hat{x}}$ is said to be **conjunctive** if for any source with advocacy stating that $x = x'$ is (almost surely) impossible, then the aggregate also states so.

Definition

Let us consider a (general) data fusion problem with $\hat{\hat{x}}$ is a given fusion operator. $\hat{\hat{x}}$ is said to be **disjunctive** if for any source with advocacy stating that $x = x'$ is not (almost surely) impossible, then the aggregate also states so.

Chapter organization

Probability fusion setting :



Probability fusion :

- Let us give a formal definition of probabilistic data fusion problems :

Definition

Probabilistic data fusion is a subclass of data fusion where advocacies live in $\mathbb{X} = \mathcal{P}_{\mathcal{X}}$.

- In such problems, informations sources deliver a discrete distribution or a density [1].

1. existence of densities, or Radon Nikodym derivatives w.r.t.Lebesgue is always assumed in this course.

Probability fusion : paradox ?

- Suppose, one has only two sources to aggregate.

- The 1st one delivers a distribution $x_*^{(1)} = P_x^{(1)}$ and the 2nd one delivers $x_*^{(2)} = P_x^{(2)}$.

- It does not mean that $P_x^{(1)} = P_x^{(2)} = P_x$!

$\rightarrow$ $P_x^{(i)}$ is just an estimate of P_x given the data seen by source $y^{(i)}$ and the source model $h^{(i)}$.

These are subjective conditional probabilities of x : $P_x^{(i)} = P_{x|y^{(i)},h^{(i)}}$.

Probability fusion : **how to** ?

- In a few cases, one can derive a principled [2] fusion rule for distributions (assuming some hypotheses).

- In general, rules are hard to justify based on probabilistic calculus alone. We are then left with two options :
    - by listing desirable axiomatic properties of the rule,
    - by selecting a model and fitting it to data.

      (usually both)

2. In the sense that the rule is justified by theorems of probability theory or probabilistic calculus.

Principled probability fusion : histogram weighted average

- $\mathcal{X}$ is a finite set.
- $y^{(i)}$ is a dataset of iid samples drawn from P_x.
- Each dataset $y^{(i)}$ has cardinality n_i.
- $x_*^{(i)} = P_x^{(i)}$ is an empirical histogram obtained from counting occurrences in $y^{(i)}$.
- Then, we have the following convergence in probability :

$$\frac{1}{n_1 + \ldots n_{N_s}} \sum_{i=1}^{N_s} n_i \times P_x^{(i)} \underset{N_s \to \infty}{\longrightarrow} P_x. \tag{2}$$

Principled probability fusion : histogram average

Explanation :

The weighted average of histograms is in this case the maximum likelihood estimate of a multinomial distribution derived from the union of the datasets

$$\bigcup_{i=1}^{N_s} y^{(i)}.$$

Principled probability fusion : a Bayesian scenario (TP 2)

- For simplicity, we assume $N_s = 2$.
- $y^{(1)}$ is a dataset of samples drawn from P_1 and x is a parameter of this distribution.
- $y^{(2)}$ is a dataset of samples drawn from P_2 and x is a also parameter of this other distribution.
- The parametric models of P_1 and P_2 are known.
- From a Bayesian standpoint, we seek the posterior distribution

$$P_{x|y^{(1)},y^{(2)}}.$$

Principled probability fusion : a Bayesian scenario (TP 2)

- Assuming we have a prior P_x, for any $a \in \mathcal{X}$ we can write

$$P_{x|y^{(1)},y^{(2)}}(a) \propto P_{Y^{(1)},Y^{(2)}|x=a}\left(y^{(1)},y^{(2)}\right) \times P_x(a) \quad (3)$$

- Assuming conditional independence, we have

$$P_{x|y^{(1)},y^{(2)}}(a) \propto P_{Y^{(1)}|x=a}\left(y^{(1)}\right) \times P_{Y^{(2)}|x=a}\left(y^{(2)}\right) \times P_x(a) \quad (4)$$

- Each $P_{Y^{(i)}|x=a}\left(y^{(i)}\right)$ is a likelihood function of parameter x.

Principled probability fusion : a Bayesian scenario (TP 2)

- Each source output is a conditional distribution

$$x_*^{(i)} = P_{x|y^{(i)}}.$$

- Applying Bayes theorem to each likelihood term, we finally obtain the rule

$$\begin{aligned} P_{x|y^{(1)},y^{(2)}}(a) &\propto \frac{P_{x|y^{(1)}}(a)}{P_x(a)} \times \frac{P_{x|y^{(2)}}(a)}{P_x(a)} \times P_x(a), && (5) \\ &\propto \frac{P_{x|y^{(1)}}(a) \times P_{x|y^{(2)}}(a)}{P_x(a)}. && (6) \end{aligned}$$

Principled probability fusion : a Bayesian scenario (TP 2)

Definition

Each advocacy is a conditional probability law : $x_*^{(i)} = P_{x|y_i}$. The **Bayes operator** $\hat{\hat{x}}_{bay}$ is defined for each $a \in \mathcal{X}$ as follows :

$$\hat{\hat{x}}_{bay} \quad \left(P_{x|y_1}, .., P_{x|y_{N_s}}\right)(a) \to \frac{1}{Z} \times \frac{1}{P_x\left(a\right)^{N_s-1}} \prod_{i=1}^{N_s} P_{x|y_i}\left(a\right), \qquad (7)$$

with Z a normalization constant so that x_* is a probability distribution.

- When the prior is uniform, we retrieve the geometric mean.

Important remark in connection with TP 2 :

- When each distribution $P^{(i)}$ belongs to the same parametric family, then we can substitute them with their corresponding vectors of sufficient statistics.

- The problem can then be reshaped as a vector fusion problem.

Bayes op :

Property

The Bayes operator $\hat{\hat{x}}_{bay}$ is a conjunctive fusion operator.

Proof

The proof is trivial. If one source is sure that x' is not a possible value for x, then this means that $x_*^{(i)}(x') = P_{x|S_i}(x') = 0$. According to equation (7), this implies that $x_*(x') = P_{x|S_1,..,S_{N_s}}(x') = 0$, which of course means that x' is not a possible value for x.

Bayes op : limitations

- The Bayes operator consists in multiplying rather small values. When the number of sources N_s grows, machine precision may be reached.
- This is usually circumvented by using programming tricks like log-probabilities.
- Due to its conjunctive nature, the way advocacies are obtained by the sources must be handled with great care. Indeed, if a source states that $x = x'$ is impossible while this is untrue, then x' is permanently eliminated from the set of solutions.

Parametric/Axiomatic probability fusion :

- Let P_* denote the aggregate distribution.

- Axiom (i) : weak set wise function property (WSFP)

Definition (WSFP)

For all subset $A \subseteq \mathcal{X}$,

$$P_*(A) = g^{(A)}\left(P_x^{(1)}(A), \ldots, P_x^{(N_s)}(A)\right), \tag{8}$$

for some function $g^{(A)} : [0;1]^{N_s} \to [0;1]$.

- Interpretation : the aggregated opinion on the chances of event A are depending solely on the source opinions on the same event A.

Parametric/Axiomatic probability fusion :

- Axiom (ii) : strong set wise function property (SSFP)

Definition (SSFP)

For all subset $A \subseteq \mathcal{X}$,

$$P_* (A) = g\left(P_x^{(1)} (A), \ldots, P_x^{(N_s)} (A) \right), \tag{9}$$

for some function $g : [0;1]^{N_s} \to [0;1]$.

- Interpretation : same as before but the combination rule is the same for each event otherwise relabeling the elements of $\mathcal{X}$ would impact the fusion.

Parametric/Axiomatic probability fusion :

- Axiom (iii) : unanimity (or idempotence)

Definition (Unanimity)

If $P^{(i)} = P_0$ for all i, then $P_* = P_0$.

- Interpretation : if the sources are unanimous, then the aggregate distribution is a copy of the input ones.

Parametric/Axiomatic probability fusion :

- Axiom (ii) and (iii) combined :

Proposition

If $|\mathcal{X}| \geq 3$, a probability distribution fusion operator satisfies SSFP and unanimity iff, the aggregate distribution writes

$$P_* = \sum_{i=1}^{N_s} w_i P_x^{(i)},$$

where coefficients w_i are non-negative and sum to one : $\sum_{i=1}^{N_s} w_i = 1$.

- These operators are known as linear opinion pool operators and will be denoted by $\hat{\hat{x}}_{lop}$.

Parametric/Axiomatic probability fusion :

- Axiom (iv) : independence preservation (IP)

Definition (IP)

For any two subsets A and B of $\mathcal{X}$ s.t. $P_x^{(i)}(A \cap B) = P_x^{(i)}(A) \times P_x^{(i)}(B)$ $\forall i$, then $P_*(A \cap B) = P_*(A) \times P_*(B)$

- Interpretation : when sources are unanimous about the independence of a pair of events, the aggregate distribution should encode the same piece of information.

- No linear opinion pool operator achieves IP except of $w_i = 1$ for some i (dictatorship or selection).

Parametric/Axiomatic probability fusion :

- In the Bayesian setting, the posterior P' is an update of the prior P through the likelihood function L : $P'(x) \propto L(x) P(x)$.

- Axiom (v) : Bayesian externality (EB)

Definition (EB)

Let $(P')_*$ denote the combination of the updated distribution $P'^{(i)}$ using likelihood function L and $(P_*)'$ denote the updated combination of the distributions $P^{(i)}$ using the same likelihood function. Then

$$(P')_* = (P_*)'.$$

- Interpretation : Bayesian update and fusion commute.

- The time at which some information arrives does not matter.

Parametric/Axiomatic probability fusion :

- Axiom (iii) and (v) combined : unanimity + EB

Proposition

If a probability distribution fusion operator writes

$$P_* = \frac{1}{Z} \prod_{i=1}^{N_s} \left(P_x^{(i)} \right)^{w_i},$$

where coefficients w_i are non-negative and sum to one : $\sum_{i=1}^{N_s} w_i = 1$ and Z is a normalization constant, then this it achieves unanimity and EB.

- These operators are known as logarithmic opinion pool operators and will be denoted by $\hat{\hat{x}}_{log}$.

Parametric/Axiomatic probability fusion :

- Another nice property of $\hat{\hat{x}}_{log}$:

Proposition

$\hat{\hat{x}}_{log}$ is the solution of the minimization problem :

$$\arg\min_{P} \sum_{i=1}^{N_s} w^{(i)} d_{KL}\left(P, P^{(i)}\right),$$

where d_{KL} is the Kullback-Leibler divergence.

Parametric/Axiomatic probability fusion :

- How to set the weights for $\hat{\hat{x}}_{lop}$ and $\hat{\hat{x}}_{log}$ ops ?
- Grid search on a validation set does not scale well when N_s grows.
- Meta-data based weights as in TP 1 makes sense.
- Fitting the operator weights on a validation set is possible (with approximations) in a supervised learning context :
  - directly from data for $\hat{\hat{x}}_{log}$ [2],
  - For $\hat{\hat{x}}_{lop}$ [3], by solving

$$\begin{array}{c} \arg\min_{\mathbf{w}} \\ \text{subject to} \\ w_i \geq 0 \\ \sum_{i=1}^{N_s} w_i = 1 \end{array} \sum_{i=1}^{N_s} w_i \left({E_{\text{out}}}^{(i)} - \Delta E^{(i)} \right),$$

with ${E_{\text{out}}}^{(i)} = \mathbb{E}_y \left[L \left(f^* (y), f^{(i)} (y) \right) \right]$ the expected loss of the i^{th} classifier (source) as compared to the oracle f^* and $\Delta E^{(i)} = \mathbb{E}_y \left[L \left(\sum_{i=1}^{N_s} w_i f^{(i)} (y), f^{(i)} (y) \right) \right]$ the expected loss discrepancies with the ensemble.

Chapter organization

The need for sets of probabilities : $\mathcal{X} = \{a, b, c\}$

- Probabilities are not fully expressive when it comes to represent a certain but imprecise piece of information like

$$x \in A = \{a, b\}.$$

- This event belongs to the σ-algebra $\sigma_{\mathcal{X}}$ on which probability measures are defined.

- We must have $P_x(A) = 1$ and $P_x(A^c) = 0.$.

- But what can I say about $P_x(a)$? One can chose any $p \in [0; 1]$ and assign probabilities as $P_x(a) = p$ and $P_x(b) = 1 - p$.

$\rightarrow$ There are infinitely many candidate distributions, *i.e.* a set of distributions.

The need for sets of probabilities : $\mathcal{X} = \{a, b, c\}$

- 1st idea : circumvent this issue by conditioning, *i.e.* replacing the set of possible values $\mathcal{X}$ with A (and update probabilities).

- Conditioning succeeds to deal with this situation but there is still no actual mean to represent $\{x \in A\}$ with a probability measure on $\sigma_{\mathcal{X}}$.

- Conditioning does not apply to advocacies like

  « $x \in A$ with prob. .8 and $x = c$ with prob. .2 ».

- 2nd idea : use another σ-algebra : $\sigma' = \{\emptyset, A, A^c, \mathcal{X}\} \subset \sigma_{\mathcal{X}}$. The information can then be adequatly represented but how can we combine measures defined on different σ-algebras ?

- Let's give it a try !

Example : Dice throw

Suppose $\mathcal{X} = \{1; 2; 3; 4; 5; 6\}$ is the set of outcomes of a dice throw. Three sources provide a (conditional) probability distribution over a probability space $(\Omega, \sigma_\Omega, \mathbb{P})$:

- S_1 provides $P_{x|S_1}$ whose codomain the following measurable space $(\mathcal{X}, \sigma_1)$,
- S_2 provides $P_{x|S_2}$ whose codomain the following measurable space $(\mathcal{X}, \sigma_2)$,
- S_3 provides $P_{x|S_3}$ whose codomain the following measurable space $(\mathcal{X}, \sigma_3)$,

Suppose also that $\sigma_1 = \sigma_2 = \sigma_\mathcal{X} = 2^\mathcal{X}$ while $\sigma_3 = \{\emptyset, \{2; 4; 6\}, \{1; 3; 5\}, \mathcal{X}\}$. In other words, S_1 and S_2 can express the odds of any outcome and S_3 can only discriminate odd numbers from even ones.

Example : Dice throw

In addition, we have

element $i \in \mathcal{X}$	1	2	3	4	5	6
$P_{x\mid S_1}(i)$	0	0.3	0.4	0.2	0.1	0
$P_{x\mid S_2}(i)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

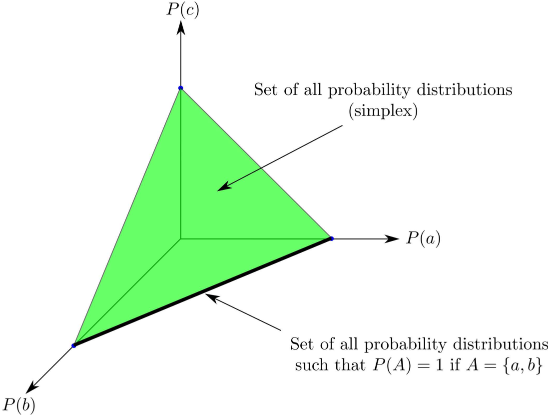
set A	$\{2;4;6\}$	$\{1;3;5\}$
$P_{x\mid S_3}(A)$	0.8	0.2

Example : 1^{st} solution - Break the problem into pieces !

Example : 2$^{\text{nd}}$ solution - Choose a distribution on $\mathcal{X}$ that best represents the 3$^{\text{rd}}$ source.

The need for sets of probabilities :

- We came to the conclusion that such a source supporting $x \in A$ is adequatly represented by the set of probability measures such that A has probability 1.
- In our example, this set is $\{P \in \mathcal{P}_{\mathcal{X}} \mid P(a) = p, P(b) = 1 - p, p \in [0;1]\}$.



The need for sets of probabilities :

- We will therefore now consider that advocacies are sets of probability measures, *i.e.* $\mathbb{X} = 2^{\mathcal{P}_{\mathcal{X}}}$, hence the following definition :

Definition

Probabilistic set-valued data fusion is a subclass of data fusion where advocacies live in $\mathbb{X} = 2^{\mathcal{P}_{\mathcal{X}}}$ [a].

a. $2^{\mathcal{P}_{\mathcal{X}}}$ is the power set of probability measures defined on $(\mathcal{X}, \sigma_{\mathcal{X}})$.

Probabilistic set-valued data fusion :

- We focus on the countable finite case $|\mathcal{X}| < \infty$ and $\sigma_{\mathcal{X}} = 2^{\mathcal{X}}$.
- In general, we will denote by $\mathcal{P}_i$ a set of probability measures : $\mathcal{P}_i \in 2^{\mathcal{P}_{\mathcal{X}}}$.
- We will call such sets **p.m.-sets** for short.

Probabilistic set-valued data fusion : 2 altenative representations

- In the dice throwing example, a source delivered the following advocacy :

$$« \ x \in A \text{ with prob. } .8 \text{ and } x = c \text{ with prob. } .2 \ ».$$

- The corresponding p.m.-set is $\mathcal{P}_i = \{P \in \mathcal{P}_{\mathcal{X}} \mid P(a) = p, P(b) = .8 - p, p \in [0; .8]\}$.

- Observe that the same information can be encoded by stating that

$$0 \leq P(a) \leq .8, \quad 0 \leq P(b) \leq .8 \quad \text{and} \quad P(c) = .2.$$

- In this case, the p.m.-set can be equivalently represented by probability bounds.

Convenient objects to encode probability bounds : Capacities

Definition

Let ν denote a set-function from $2^{\mathcal{X}}$ to $\mathbb{R}$. ν is said to be a **capacity** if it has the following properties :

- $\nu(\emptyset) = 0$,
- $\nu(\mathcal{X}) = 1$,
- $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$, for any A, B in $2^{\mathcal{X}}$ (monotony).

Capacities vs Probability measures :

Probability measure :

- $\mu(\emptyset) = 0$,
- $\mu(\mathcal{X}) = 1$,
- $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$,
- $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$.

Capacity :

- $\nu(\emptyset) = 0$,
- $\nu(\mathcal{X}) = 1$,
- $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$.

- Probability measures = additive capacities.
- Capacities = non-additive measures, (bit awkward because the additivity property belongs to the definition of measures).

Capacities : properties

Definition

A probability measure $P : (\mathcal{X}, \sigma_{\mathcal{X}} = 2^{\mathcal{X}}) \to [0;1]$ is said to **dominate** a capacity $\nu : 2^{\mathcal{X}} \to [0;1]$ if for all $A \in 2^{\mathcal{X}}$, $\nu(A) \leq P(A)$.

Definition

The **core** $\mathcal{P}_\nu$ of a capacity ν is the set of probability measures dominating ν.

Property

All cores are convex closed subsets of $\mathcal{P}_{\mathcal{X}}$.

Capacities : properties

Definition

A capacity $\nu : 2^{\mathcal{X}} \to [0;1]$ is said to be **n-monotonic**, with $n \in \mathbb{N}^*$, if and only if for any family of n events $\mathcal{A} = (A_i)_{i=1}^{n}$, one has :

$$\sum_{I \subseteq \mathcal{A}} (-1)^{|I|+1} \nu \left(\bigcap_{A \in I} A \right) \leq \nu \left(\bigcup_{1 \leq i \leq n} A_i \right) \quad \text{(def. for } n \geq 2\text{).} \tag{10}$$

- This property will help us to identify classes of capacities with interesting properties.
- Note that 1-monotonicity is just monotonicity as described in the definition of capacities.

Capacities : Mass functions

Definition

Given a capacity $\nu : 2^{\mathcal{X}} \to [0;1]$, its **Möbius transform** is the set-function m from the power set $2^{\mathcal{X}}$ to the real line $\mathbb{R}$ defined as follows :

$$m : 2^{\mathcal{X}} \to \mathbb{R}, \tag{11}$$

$$A \to \sum_{B \subseteq A} (-1)^{|A \setminus B|} \nu(B). \tag{12}$$

The function m is called mass function.

- Due to the boundary conditions in the definition of capacities, any mass function m is such that :
  - $\sum_{A \subseteq \mathcal{X}} m(A) = 1$,
  - $m(\emptyset) = 0$.

Capacities : Mass functions

- The Möbius transform can be reversed. For any capacity ν and its mass function m, we have :

$$\nu(A) = \sum_{B \subseteq A} m(B), \forall A \subseteq \mathcal{X} \text{(inverse Möbius)}. \tag{13}$$

- This equation looks very much alike a Cramer system of $2^{|\mathcal{X}|}$ equations which intuitively guarantee that the bijective correspondence between capacities and mass functions.

Lemma

Let $\nu : 2^{\mathcal{X}} \to [0;1]$ denote an ∞-monotonic capacity. Then the codomain of its mass function m is $[0;1]$: $\forall A \subseteq \mathcal{X}$,

$$0 \leq m(A) \leq 1.$$

Upper and lower probabilities :

- Next : How do we relate capacities with probability bounds ?

Definition

Let $\mathcal{P}_i$ denote a p.m.-set. The **lower enveloppe** $\underline{\nu}_{\mathcal{P}_i}$ of $\mathcal{P}_i$ is a mapping defined as follows :

$$\begin{aligned} \underline{\nu}_{\mathcal{P}_i} : 2^{\mathcal{X}} &\rightarrow [0;1]\,, \\ A &\rightarrow \min_{\mu \in \mathcal{P}_i} \{\mu(A)\}\,. \end{aligned} \tag{14}$$

Upper and lower probabilities :

Definition

Let $\mathcal{P}_i$ denote a p.m.-set. The **upper enveloppe** $\overline{\nu}_{\mathcal{P}_i}$ of $\mathcal{P}_i$ is a mapping defined as follows :

$$\begin{aligned} \overline{\nu}_{\mathcal{P}_i} : 2^{\mathcal{X}} &\rightarrow [0;1], \\ A &\rightarrow \max_{\mu \in \mathcal{P}_i} \{\mu(A)\}. \end{aligned} \tag{15}$$

- This means that for any $A \in 2^{\mathcal{X}}$ and any $\mu \in \mathcal{P}_i$, we have :

$$\underline{\nu}_{\mathcal{P}_i}(A) \leq \mu(A) \leq \overline{\nu}_{\mathcal{P}_i}(A). \tag{16}$$

Upper and lower probabilities :

- In this context, lower enveloppes are often called lower probabilities and upper enveloppes are called upper probabilties.

Theorem

Let $\mathcal{P}_i$ denote a p.m.-set with lower enveloppe $\underline{\nu}_{\mathcal{P}_i}$. Then the upper enveloppe $\overline{\nu}_{\mathcal{P}_i}$ of $\mathcal{P}_i$ is the conjugate of its lower enveloppe :

$$\overline{\nu}_{\mathcal{P}_i} = \left(\underline{\nu}_{\mathcal{P}_i}\right)^c . \tag{17}$$

Upper and lower probabilities : Example with $\mathcal{X} = \{a, b, c\}$
Source i delivers the p.m.-set $\mathcal{P}_i = \{\mu_1, \mu_2\}$.

set $A \in 2^{\mathcal{X}}$	$\mu_1(A)$	$\mu_2(A)$	$\underline{\nu}_{\mathcal{P}_i}(A)$	$\overline{\nu}_{\mathcal{P}_i}(A)$	$\overline{\nu}_{\mathcal{P}_i}(A^c)$	$(\overline{\nu}_{\mathcal{P}_i})^c(A) = 1 - \overline{\nu}_{\mathcal{P}_i}(A^c)$
$\emptyset$	0	0	0	0	1	0
$\{a\}$	$1/3$	$1/2$	$1/3$	$1/2$	$2/3$	$1/3$
$\{b\}$	$1/3$	$1/2$	$1/3$	$1/2$	$2/3$	$1/3$
$\{a, b\}$	$2/3$	1	$2/3$	1	$1/3$	$2/3$
$\{c\}$	$1/3$	0	0	$1/3$	1	0
$\{a, c\}$	$2/3$	$1/2$	$1/2$	$2/3$	$1/2$	$1/2$
$\{b, c\}$	$2/3$	$1/2$	$1/2$	$2/3$	$1/2$	$1/2$
$\{a, b, c\} = \mathcal{X}$	1	1	1	1	0	1

The 4^{th} column = lower enveloppe of $\mathcal{P}_i$.
The 5^{th} column = upper enveloppe of $\mathcal{P}_i$.
Last column = conjugate of the upper enveloppe = lower enveloppe.

Upper and lower probabilities :

- The preceding theorem is very interesting in the sense that it is unnecessary to study both the lower and upper enveloppes since they are in bijective correspondence.

- In general, a p.m.-set $\mathcal{P}_i$ is included in the core $\mathcal{P}_{\underline{\nu}_{\mathcal{P}_i}}$ induced by its lower enveloppe :

$$\mathcal{P}_i \subseteq \mathcal{P}_{\underline{\nu}_{\mathcal{P}_i}}. \tag{18}$$

- However, they are not always equal.

Upper and lower probabilities : Example with $\mathcal{X} = \{a, b, c\}$
Source i delivers the p.m.-set $\mathcal{P}_i = \{\mu_1, \mu_2\}$.

set $A \in 2^{\mathcal{X}}$	$\mu_1(A)$	$\mu_2(A)$	$\underline{\nu}_{\mathcal{P}_i}(A)$	$\mu_3(A)$
$\emptyset$	0	0	0	0
$\{a\}$	$1/3$	$1/2$	$1/3$	$5/12$
$\{b\}$	$1/3$	$1/2$	$1/3$	$5/12$
$\{a, b\}$	$2/3$	1	$2/3$	$5/6$
$\{c\}$	$1/3$	0	0	$1/6$
$\{a, c\}$	$2/3$	$1/2$	$1/2$	$7/12$
$\{b, c\}$	$2/3$	$1/2$	$1/2$	$7/12$
$\{a, b, c\} = \mathcal{X}$	1	1	1	1

4th column = lower enveloppe of $\mathcal{P}_i$.
Last column = values of the measure $\mu_3 = \frac{\mu_1+\mu_2}{2}$.

We have $\mu_3 \notin \mathcal{P}_i$ while $\mu_3 \in \mathcal{P}_{\underline{\nu}_{\mathcal{P}_i}}$ because μ_3 dominates $\underline{\nu}_{\mathcal{P}_i}$.

Fusion op for p.m.-sets :

- Since the source advocacies are p.m.-sets, they are also just sets and consequently, all fusion operators introduced in chapter 3 apply.

Definition

Each advocacy is a p.m.-set : $x_*^{(i)} = \mathcal{P}_i$. The **conjunctive operator** $\hat{\hat{x}}_\cap$ is defined as follows :

$$\hat{\hat{x}}_\cap \qquad (\mathcal{P}_1, .., \mathcal{P}_{N_s}) \to \bigcap_{i=1}^{N_s} \mathcal{P}_i. \tag{19}$$

Conjunctive op for p.m.-sets :

Property

The conjunctive operator $\hat{\hat{x}}_{\cap}$ is conjunctive.

Property

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two cores with respective lower enveloppes ν_1 and ν_2, then the p.m.-set $\mathcal{P} = \hat{\hat{x}}_{\cap}(\mathcal{P}_1, \mathcal{P}_2)$ is also a core whose lower probabilities are

$$\nu = \max\{\nu_1; \nu_2\} \text{ (entrywise max).}$$

The upper probabilities are obtained using an entrywise min.

Chapter organization

Theory of belief functions :

- Belief functions are ∞-monotonic lower probabilities of a p.m.-set.
- However, they were introduced by plugging a probability measure into a multi-valued mapping.

Definition

Evidential data fusion is a subclass of data fusion where advocacies live in the set of belief functions $\mathbb{X} = \mathcal{B}_{\mathcal{X}}$.

- All p.m.-sets encoded by belief functions (seen as a lower enveloppe) is a core.
- If a source is reliable, this core should contain the (true) probability distribution of x.

Belief functions : basics

- Recalling lemma 1, mass functions obtained via Möbius transform from a belief function are positive.

- Mass functions are then reminiscent of probability distribution except that masses are distributed on $2^{\mathcal{X}}$ instead of $\mathcal{X}$.

- In the theory of belief functions, a focal element of a mass function m_i is a set $A \subseteq \mathcal{X}$ such that $m_i(A) > 0$ meaning that the i^{th} collected evidence supports the event $\{\theta \in A\}$.

- This evidence viewpoint of belief functions accounts for the fact that the theory of belief functions is also frequently called the evidence theory. Approaches developped within this framework are thus called evidential.

Evidential data fusion :

Definition

Each advocacy is a mass function : $x_*^{(i)} = m_i$.
The **Dempster's rule operator** $\hat{\hat{x}}_\otimes$ is defined as follows :

$$\hat{\hat{x}}_\otimes \quad : \quad (m_1, .., m_{N_s}) \to m_*, \tag{20}$$

with m_* a mass function such that for any $A \in 2^{\mathcal{X}}$, one has :

$$m_*(A) \quad = \quad \begin{cases} \frac{1}{1-\kappa} \sum\limits_{\substack{A_1,..,A_{N_s} \in 2^{\mathcal{X}} \\ \text{s.t. } \bigcap\limits_{i=1}^{N_s} A_i = A}} \prod\limits_{i=1}^{N_s} m_i(A_i) & \text{if } \kappa < 1 \\ \emptyset & \text{otherwise} \end{cases} . \tag{21}$$

Evidential data fusion :

Definition

The parameter κ is called **Dempster's degree of conflict**. It is defined as follows :

$$\kappa = \sum_{\substack{A_1,..,A_{N_s} \in 2^{\mathcal{X}} \\ \text{s.t. } \bigcap_{i=1}^{N_s} A_i = \emptyset}} \prod_{i=1}^{N_s} m_i\left(A_i\right). \tag{22}$$

- This parameter is the total mass assigned to incompatible pieces of evidence.

Dempster's rule : properties

Property

- Dempster's rule is **commutative** : $\hat{\hat{x}}_{\otimes}(m_1, m_2) = \hat{\hat{x}}_{\otimes}(m_1, m_2)$, for any mass functions m_1 and m_2.
- Dempster's rule is **associative** : $\hat{\hat{x}}_{\otimes}(m_1, \hat{\hat{x}}_{\otimes}(m_2, m_3)) = \hat{\hat{x}}_{\otimes}(\hat{\hat{x}}_{\otimes}(m_1, m_2), m_3)$, for any mass functions m_1, m_2 and m_3.
- Unique **neutral element** : If $m(\mathcal{X}) = 1$ then $\hat{\hat{x}}_{\otimes}(m_1, m) = m_1$, for any mass function m_1.

- Note that, however, Dempster rule is in general not idempotent : $\hat{\hat{x}}_{\otimes}(m, m) \neq m$. This latter point is sometimes criticized because when two sources are saying the exact same thing, it may be desirable that the aggregate is equal to their proposal.

Dempster's rule : properties

- Since Dempster's rule is associative and commutative, it is unnecessary to compute it in batch mode.
- It is far less time consuming to do pairwise combinations. In addition, when $N_s = 2$, Dempster's rule is a bit more easy to grasp : let $m_{1\otimes 2} = \hat{\hat{x}}_{\otimes}\left(m_1, m_2\right)$. For any $A \in 2^{\mathcal{X}}$, we have :

$$m_{1\otimes 2}\left(A\right) = \begin{cases} \frac{1}{1-\kappa} \sum\limits_{\substack{A_1, A_2 \in 2^{\mathcal{X}} \\ \text{s.t. } A_1 \cap A_2 = A}} m_1\left(A_1\right) m_2\left(A_2\right) & \text{if } \kappa < 1 \\ \emptyset & \text{otherwise} \end{cases} \tag{23}$$

with $\kappa = \sum\limits_{\substack{A_1, A_2 \in 2^{\mathcal{X}} \\ \text{s.t. } A_1 \cap A_2 = \emptyset}} m_1\left(A_1\right) m_2\left(A_2\right)$.

Dempster's rule : properties

Property

Curse of conflict :
Let $(m_i)_{i=1}^N$ be a sequence of mass functions. Let κ_n denote Dempster's degree of conflict computed from the n 1^{st} members of the sequence.
The sequence κ_n is non-decreasing.

Dempster's rule : justifications

- Suppose $m_2(B) = 1$, the result of the conjunctive combination of m_1 and m_2 is denoted by $m_{1|B}$. For any $A \in 2^{\mathcal{X}}$, one has :

$$m_{1|B}(A) = \frac{\kappa}{1-\kappa} \sum_{\substack{C \in 2^{\mathcal{X}} \\ \text{s.t. } B \cap C = A}} m_1(C) \text{ (ev. conditioning)}. \quad (24)$$

- 1st justification : Evidential conditioning is compliant with Bayesian conditioning in the sense that if m_1 is a Bayesian mass function with $\mathcal{P}_{bel_1} = \{P_1\}$, then the only element in the p.m.-set induced by $bel_{1|B}$ is the conditional probability measure $P_{1|x \in B}$.

- 2nd justification : Dempster's rule just relies on the (usual) statistical independence assumption among probability measures of the sources.

Decision making with belief functions :

- If the aggregate x_* is a mass function, usual decision making do not apply.

- Solutions :
  - In practice, the restriction of upper probabilities on singletons[3] work well.
  - I can look for the probability distribution that best represents the p.m.-set (center of gravity - pignistic transform).
  - I can compute lower and upper expectations and end up with and interval-valued solution.

3. A singleton is a set with unit cardinality.

Conjunctive op and Dempster rule in action :

Example : the trial

3 suspects for a murder case : $\mathcal{X} = \{peter; paul; mary\}$ but only 1 is guilty.
10 witnesses :

- 1st source : 8 are supporting the fact that the culprit is a man. 1 supports the opposite. 1 is undecided.
- 2nd source : 5 are supporting the fact that the culprit is dark haired. 1 is supporting the fact that the culprit is red haired. 4 are undecided. Peter and Mary are dark haired while Paul is red haired.

What can be inferred about the murderer identity ?

Chapter organization

Capacities : properties

- Capacities can have many properties, some of which are given in the sequel.

Definition

A capacity ν is called **super-additive** if for all sets A and B in $2^{\mathcal{X}}$, one has :

$$\nu(A) + \nu(B) \leq \nu(A \cup B). \tag{25}$$

A capacity ν is called **sub-additive** if for all sets A and B in $2^{\mathcal{X}}$, one has :

$$\nu(A) + \nu(B) \geq \nu(A \cup B). \tag{26}$$

A capacity that is both super and sub-additive is a probability measure.

Capacities : properties

- It is noteworthy that in particular :
  - if ν is super-additive, $\nu(A) + \nu(A^c) \leq 1$,
  - if ν is sub-additive, $\nu(A) + \nu(A^c) \geq 1$,

Definition

The **conjugate capacity** ν^c of a capacity ν is such that for all $A \in 2^{\mathcal{X}}$, $\nu^c(A) = 1 - \nu(A^c)$.

Capacities : properties

Property

Any n-monotonic capacity is also $(n-1)$-monotonic.

- The above property obviously implies that the set of ∞-monotonic capacities is included in the set of n-monotonic capacities which in turn is included in the set of 1-monotonic capacities.

Lemma

For any 2-monotonic capacity ν, we have $\mathcal{P}_\nu \neq \emptyset$.

- Since the above lemma holds for 2-monotonic capacities, it is also true for n-monotonic capacities ($n \geq 2$).
- This property is quite handy in some circumstances.

Upper and lower probabilities :

- Enveloppes are characterizing p.m.-sets but we would like preferably that they entirely encode the same information as p.m.-sets.
- This is tantamount to have $\mathcal{P}_i = \mathcal{P}_{\underline{\nu}_{\mathcal{P}_i}}$ because $\underline{\nu}_{\mathcal{P}_i}$ uniquely defines $\mathcal{P}_{\underline{\nu}_{\mathcal{P}_i}}$.

Definition

A p.m.-set $\mathcal{P}_i$ with lower enveloppe $\underline{\nu}_{\mathcal{P}_i}$ is said to be **coherent** if $\mathcal{P}_i = \mathcal{P}_{\underline{\nu}_{\mathcal{P}_i}}$.

Theorem

If the lower enveloppe $\underline{\nu}_{\mathcal{P}_i}$ of a p.m.-set $\mathcal{P}_i$ is 2-monotonic, then this p.m.-set is coherent.

- From the above theorem, we know that 2-monotonic super-additive capacities entirely characterize a closed convex p.m.-set.

Upper and lower probabilities :

Theorem

Any lower (resp. upper) enveloppe of p.m.-set is a super-additive (resp. sub-addtive) capacity.

- In this context, super-additive capacities are often called lower probabilities and sub-additive capacities are called upper probabilties.

Theorem

Let $\mathcal{P}_i$ denote a p.m.-set with lower enveloppe $\underline{\nu}_{\mathcal{P}_i}$. Then the upper enveloppe $\overline{\nu}_{\mathcal{P}_i}$ of $\mathcal{P}_i$ is the conjugate of its lower enveloppe :

$$\overline{\nu}_{\mathcal{P}_i} = \left(\underline{\nu}_{\mathcal{P}_i}\right)^c . \tag{27}$$

Belief functions : basics

- There are several noteworhty sub-classes of mass functions.
  - A mass function having only one focal element A is called a categorical mass function and it is denoted by m_A. The categorical mass function $m_{\mathcal{X}}$ is called the vacuous mass function because it carries no information.
  - A simple mass function m_A^w is the convex combination of $m_{\mathcal{X}}$ with a categorical mass function m_A with $A \neq \Omega$: $m_A^w = (1 - w)\, m_A + w\, m_{\mathcal{X}}$ with $w \in [0; 1]$.
  - A mass function whose focal elements have unit cardinality is a bayesian mass function. Such a mass function is formally equivalent to a probability distribution. The underlying p.m.-set has thus also unit cardinality. This also shows that the theory of belief functions encompasses the probability theory.
  - A consonant mass function is such that for any pair of focal elements (A, B), one has either $A \subseteq B$ or $B \subseteq A$. The inclusion is thus a total order relation for focal elements of consonant mass functions.

Belief functions : basics

- Several alternatives for evidence representation are commonly used :
  - the belief bel_i which is the inverse Möbius transform of the mass function m_i.
  - the commonality function q_i which is the inverse co-Möbius transforms of the mass function m_i. We have that :

$$q_i(A) = \sum_{B \supseteq A} m_i(B), \forall A \in 2^{\mathcal{X}}. \tag{28}$$

  - the plausibility function pl_i is the conjugate of bel_i : $pl_i = (bel_i)^c$. The plausibility function is consequently viewed as an upper probability. For any $A \in 2^{\mathcal{X}}$ and any $\mu \in \mathcal{P}_{bel_i}$, we have :

$$bel_i(A) \leq \mu(A) \leq pl_i(A). \tag{29}$$

    In addition, we also have :

$$pl_i(A) = \sum_{\substack{B \subseteq \Omega \\ B \cap A \neq \emptyset}} m_i(B), \forall A \in 2^{\mathcal{X}}. \tag{30}$$

Belief functions : basics

- Several alternatives for evidence representation are commonly used :
  - the implicability function b_i is such that $\forall A \subseteq \mathcal{X}$, $b_i(A) = bel_i(A) + m_i(\emptyset)$.
- There is a one-to-one correspondence between a mass function m_i and any of these four functions.
- If the thruthfulness of the evidence encoded in a mass function can be evaluated through a coefficient $\alpha \in [0, 1]$, then a so-called discounting operation on m can be performed. A discounted mass function is denoted by m^α and we have :

$$m^\alpha = (1 - \alpha)m + \alpha m_{\mathcal{X}}. \tag{31}$$

- α is called the discounting rate. Since $m_{\mathcal{X}}$ represents a state of ignorance, setting $\alpha = 1$ discards it from further processing. Note that the notation is coherent with that of simple mass functions. Indeed, a simple mass function m_A^w is the categorical mass function m_A discounting with rate w.

Belief functions : basics

- Another useful concept is the complement $\overline{m}_i$ of a mass function m_i. The function $\overline{m}_i$ is such that $\forall A \subseteq \mathcal{X}$, $\overline{m}_i(A) = m_i(A^c)$.
- Important remark : many authors relax the constraint $m_i(\emptyset) = 0$. A mass function such that $m_i(\emptyset) > 0$ is said to be unnormalized.
- In this case, the set of belief functions is not a subset of capacities anymore because the boundary condition $bel_i(\mathcal{X}) = 1$ is not no longer valid.
- Nonetheless, this is not much a problem because one can retrieve a regular mass function by deleting the mass allocated to the empty set and re-normalize the function so that it sums to one.
- The mass assigned to the empty set is often regarded as an indicator that the solution of the problem does not belong to $\mathcal{X}$, which implies that the problem is ill-posed.

Evidential data fusion :

- The unnormalized version of Dempster's rule is known as the conjunctive rule (same rule without normalizing constant $\frac{1}{1-\kappa}$).

Definition

Let us consider a data fusion problem as defined in 19. Each advocacy is a mass function : $x_*^{(i)} = m_i$. The **conjunctive rule operator** $\hat{\hat{x}}_{\cap}$ is defined as follows :

$$\hat{\hat{x}}_{\cap} \quad : \quad (m_1, .., m_{N_s}) \to m_*, \tag{32}$$

with m_* a mass function such that for any $A \in 2^{\mathcal{X}}$, one has :

$$m_* (A) \quad = \sum_{\substack{A_1,..,A_{N_s} \in 2^{\mathcal{X}} \\ \text{s.t. } \bigcap_{i=1}^{N_s} A_i = A}} \prod_{i=1}^{N_s} m_i (A_i) \,. \tag{33}$$

Evidential data fusion :

- The conjunctive rule has the same properties as Dempster's rule, plus one more :

Property

$m_{\emptyset}$ is the unique **absorbing element** of $\hat{\hat{x}}_{\bigcirc\!\!\!\!\cap}$: $\hat{\hat{x}}_{\bigcirc\!\!\!\!\cap}(m, m_{\emptyset}) = m_{\emptyset}$, for any mass function m.
In addition, the conjunctive rule has the **conflict curse** : for any mass functions m_1 and m_2, their conjunctive combination $m_{1\cap 2} = \hat{\hat{x}}_{\bigcirc\!\!\!\!\cap}(m_1, m_2)$ is such that :

$$\max\left\{m_1(\emptyset); m_2(\emptyset)\right\} \leq m_{1\cap 2}(\emptyset). \tag{34}$$

- This means that conflict can only grow as N_s increases.

Evidential data fusion :

- A mass function $m_{1\cap 2}$ obtained by pairwise combination of two mass functions m_1 and m_2 is such that for any $A \in 2^{\mathcal{X}}$, one has :

$$m_{1\cap 2}(A) = \sum_{\substack{A_1, A_2 \in 2^{\mathcal{X}} \\ \text{s.t. } A_1 \cap A_2 = A}} m_1(A_1)\, m_2(A_2)\,. \tag{35}$$

Property

Let q_1 and q_2 denote two commonality functions computed from two mass functions m_1 and m_2 using equation (28). Let $q_{1\cap 2}$ denote the commonality function obtained using the same equation on the mass function $m_{1\cap 2} = \hat{\hat{x}}_{\textcircled{\cap}}(m_1, m_2)$. For any $A \in 2^{\mathcal{X}}$, we have :

$$q_{1\cap 2}(A) = q_1(A)\, q_2(A)\,. \tag{36}$$

- This last property shows that conjunctive combination is very easy to perform in the commonality space.

Fusion with belief functions :

- Unsurprisingly, a disjunctive rule also exists in the theory of belief functions. It is often seen as the *dual* rule of the conjunctive one.

Definition

Let us consider a data fusion problem as defined in 19. Each advocacy is a mass function : $x_*^{(i)} = m_i$. The **disjunctive rule operator** $\hat{\hat{x}}_{\textcircled{\cup}}$ is defined as follows :

$$\hat{\hat{x}}_{\textcircled{\cup}} \quad : \quad \left(x_*^{(1)}, .., x_*^{(N_s)}\right) \to x_*, \tag{37}$$

$$(m_1, .., m_{N_s}) \to m_*, \tag{38}$$

with m_* a mass function such that for any $A \in 2^{\mathcal{X}}$, one has :

$$m_*(A) = \sum_{\substack{A_1,..,A_{N_s} \in 2^{\mathcal{X}} \\ \text{s.t. } \bigcup_{i=1}^{N_s} A_i = A}} \prod_{i=1}^{N_s} m_i(A_i). \tag{39}$$

Fusion with belief functions :

- The disjunctive rule can process standard or unnormalized mass functions.

Property

We have the following properties for the disjunctive rule :

- The disjunctive rule is commutative : $\hat{\hat{x}}_{\textcircled{\cup}}(m_1, m_2) = \hat{\hat{x}}_{\textcircled{\cup}}(m_1, m_2)$, for any mass functions m_1 and m_2.
- The disjunctive rule is associative : $\hat{\hat{x}}_{\textcircled{\cup}}(m_1, \hat{\hat{x}}_{\textcircled{\cup}}(m_2, m_3)) = \hat{\hat{x}}_{\textcircled{\cup}}(\hat{\hat{x}}_{\textcircled{\cup}}(m_1, m_2), m_3)$, for any mass functions m_1, m_2 and m_3.
- $m_\emptyset$ is the unique neutral element of the disjunctive : $\hat{\hat{x}}_{\textcircled{\cup}}(m_1, m_\emptyset) = m_1$, for any mass function m_1.

Fusion with belief functions :

Property

We have the following properties for the disjunctive rule :

- $m_{\mathcal{X}}$ is the unique absorbing element of the disjunctive : $\hat{\hat{x}}_{\bigcirc\!\!\!\!\cup}\left(m_1, m_{\mathcal{X}}\right) = m_{\mathcal{X}}$, for any mass function m_1.
  In addition, the disjunctive rule has the **ignorance curse** : for any mass functions m_1 and m_2, their disjunctive combination $m_{1\cup 2} = \hat{\hat{x}}_{\bigcirc\!\!\!\!\cup}\left(m_1, m_2\right)$ is such that :

$$\max\left\{m_1\left(\mathcal{X}\right); m_2\left(\mathcal{X}\right)\right\} \leq m_{1\cup 2}\left(\mathcal{X}\right). \tag{40}$$

- This means that ignorance can only grow as N_s increases.

Fusion with belief functions :

Property

Let b_1 and b_2 denote two implicability functions computed from two mass functions m_1 and m_2. Let $b_{1\cap 2}$ denote the commonality function obtained from the mass function $m_{1\cup 2} = \hat{\hat{x}}_{\textcircled{\cup}}(m_1, m_2)$. For any $A \in 2^{\mathcal{X}}$, we have :

$$b_{1\cup 2}(A) = b_1(A)\, b_2(A)\,. \tag{41}$$

- This last property shows that disjunctive combination is very easy to perform in the implicability space.

Fusion with belief functions :

- The underlying duality between the conjunctive and disjunctive rules has its roots in the following property :

Property

De Morgan laws property

Let $\bigcirc\!\!\!\!\cap$ and $\bigcirc\!\!\!\!\cup$ denote the binary operations corresponding to the conjunctive and disjunctive rules respectively. For any mass function m_1 and m_2, we have :

$$\overline{m_1 \bigcirc\!\!\!\!\cap\, m_2} = \overline{m}_1 \bigcirc\!\!\!\!\cup\, \overline{m}_2, \quad (42)$$

$$\overline{m_1 \bigcirc\!\!\!\!\cup\, m_2} = \overline{m}_1 \bigcirc\!\!\!\!\cap\, \overline{m}_2. \quad (43)$$

Fusion with belief functions :

Property

The conjunctive rule is **distributive** over the disjunctive rule. Let $\textcircled{\cap}$ and $\textcircled{\cup}$ denote the binary operations corresponding to the conjunctive and disjunctive rules respectively. For any mass function m_1, m_2 and m_3, we have :

$$m_1 \textcircled{\cap} (m_2 \textcircled{\cup} m_3) = (m_1 \textcircled{\cap} m_2) \textcircled{\cup} (m_1 \textcircled{\cap} m_3) . \quad (44)$$

Property

The conjunctive operator $\hat{\hat{x}}_{\textcircled{\cap}}$ and Dempster's operator $\hat{\hat{x}}_{\otimes}$ are conjunctive fusion operators. The disjunctive operator $\hat{\hat{x}}_{\textcircled{\cup}}$ is a disjunctive fusion operator.

Decision making with belief functions :

- Most decision making approaches are fed with a probability measure, so the lead idea in evidence theory is to compute a probability measure μ_* that is the best representative of the p.m.-set $\mathcal{P}_{bel_*}$.

- The notion of *best representative* is of course subject to debate and is usually justified with respect to a given criterion.

- Smets [5] introduced the pignistic transform which maps the set of belief functions $\mathcal{B}_{\mathcal{X}}$ to the set of probability measures $\mathcal{P}_{\mathcal{X}}$.

Decision making with belief functions :

Definition

Given a belief function $bel : 2^{\mathcal{X}} \rightarrow [0;1]$ with associated mass function m_i, its **Pignistic transform** is the probability measure *betp* from the power set $2^{\mathcal{X}}$ to $[0;1]$ defined as follows :

$$\begin{aligned} betp : 2^{\mathcal{X}} &\rightarrow [0;1]\,, \\ A &\rightarrow \sum_{x_0 \in A} \sum_{B \subseteq \mathcal{X} \,|\, x_0 \in B} \frac{1}{|B|} \frac{m_i(B)}{1 - m_i(\emptyset)}. \end{aligned} \tag{45}$$

The probability measure *betp* is called the **pignistic measure** of *bel* and it is the barycenter of $\mathcal{P}_{bel}$.

- Since *betp* is a probability measure on a finite space, it is sufficient to compute its probability distribution solely.

Decision making with belief functions :

- Since $\mathcal{P}_{\mathcal{X}} \subsetneq \mathcal{B}_{\mathcal{X}}$, the pignistic transform is lossy.
- There are many belief functions sharing the same pignistic measure.

Property

For any belief function *bel* with associated pignistic measure *betp*, it holds that $betp \in \mathcal{P}_{bel}$, *i.e.* for all $A \in 2^{\mathcal{X}}$,

$$bel\,(A) \leq betp\,(A) \leq pl\,(A)\,. \tag{46}$$

Decision making with belief functions : example

Suppose the solution set $\mathcal{X} = \{a, b, c\}$. The advocacy of the i^{th} source is a belief function bel_i.

set $A \in 2^{\mathcal{X}}$	$bel_i(A)$	$pl_i(A)$	$m_i(A)$	$betp_i(A)$
$\emptyset$	0	0	0	0
$\{a\}$	$1/3$	$1/2$	$1/3$	$5/12$
$\{b\}$	$1/3$	$1/2$	$1/3$	$5/12$
$\{a, b\}$	$2/3$	1	0	$5/6$
$\{c\}$	0	$1/3$	0	$1/6$
$\{a, c\}$	$1/2$	$2/3$	$1/6$	$7/12$
$\{b, c\}$	$1/2$	$2/3$	$1/6$	$7/12$
$\{a, b, c\} = \mathcal{X}$	1	1	0	1

The 4^{th} column contains the pignistic measure $betp_i$.

Under equal decision costs, the maximum *a posteriori* solution is either $x = a$ or $x = b$.

Decision making with belief functions :

- There are also decision processes that can be designed from plausibility functions.
- More generally, it is also possible to compute lower and upper expectation for p.m.-sets which can lead to robust decisions too.

Definition

Let $\mathcal{P}_i$ denote a p.m.-set. The **lower expectation** $\underline{E}_{\mathcal{P}_i}$ of $\mathcal{P}_i$ is defined as follows :

$$\underline{E}_{\mathcal{P}_i}\left[f\right] = \min_{\mu \in \mathcal{P}_i} \left\{ E_\mu \left[f\right] \right\}, \tag{47}$$

with f a measurable mapping.

Decision making with belief functions :

Definition

Let $\mathcal{P}_i$ denote a p.m.-set. The **upper expectation** $\overline{E}_{\mathcal{P}_i}$ of $\mathcal{P}_i$ is defined as follows :

$$\overline{E}_{\mathcal{P}_i}[f] = \max_{\mu \in \mathcal{P}_i} \left\{ E_\mu [f] \right\}, \quad (48)$$

with f a measurable mapping.

Decision making with belief functions :

- In particular, when the p.m.-set $\mathcal{P}_i$ is governed by a belief function *bel*, or equivalently by its conjugate plausibility function *pl*, it holds that :

$$\overline{E}_{\mathcal{P}_i}\left[1_{\{x_k\}}\right] = pl\left(\{x_k\}\right), \tag{49}$$

  for any $x_k \in \mathcal{X}$.

- This is all the more interesting as making decision with the upper expectation requires only to compute plausibility values on $\mathcal{X}$ instead of $2^{\mathcal{X}}$.

Fusion with belief functions :

- There are of course many other combination rules in the literature (convex combination rules, idempotent rules, *etc.*).
- Note that the rules introduced in this section have nice justifications but these justifications are not expressed in terms of operations on the underlying p.m.-sets induced by the belief functions involved in the fusion. For example, the operator $\hat{\hat{x}}_{\bigcirc\!\!\!\!\cap}$ is not equivalent to the operator $\hat{\hat{x}}_{\cap}$ for p.m.-sets.

Belief functions and random sets

- It is possible to relate the belief function theory with random set theory.
- Let us first give a definition of a finite random set.

Definition

Let $(\Omega, \sigma_\Omega, \mu)$ denote a probability space. Let $2^{\mathcal{X}}$ denote the power set of a space $\mathcal{X}$. A **random set** Γ is a multi-valued mapping from Ω to $2^{\mathcal{X}}$ such that for any $B \in 2^{\mathcal{X}}$, one has

$$\{\omega : \Gamma(\omega) \cap B \neq \emptyset\} \in \sigma_\Omega. \tag{50}$$

The above property is called **strong measurability**.

Belief functions and random sets

- The preceding definition implies that random sets (r.s.) are a generalization of random variables.
- A random set is a random variable if $|\Gamma(\omega)| = 1$ for $\omega \in \Omega$.
- For any mass function m defined within the framework of the theory of belief functions, it is always possible to suppose the existence of a given probability space such that there exists a random set Γ mapping this probability space with $2^{\mathcal{X}}$.
- Conversely, for any random set Γ, the funtion $m = \mu \circ \Gamma^{-1}$ is a mass function as described in the theory of belief functions.
- Suppose a r.v. $\phi : \Omega \to \mathcal{X}$ is imprecisely known, *i.e.* we know that $\phi(\omega) \in B$ but we do not know which element of B is the image of ω through ϕ. Such information can be encoded using a random set Γ. (Dempster)

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