Towards a Categorical Model of the Lilac Separation Logic

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x and y point to disjoint heap locations

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X and Y are independent random variables

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- We used Lilac to verify a weighted sampling algorithm
- For more, see:

Lilac: A Modal Separation Logic for Conditional Probability

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PLDI'23

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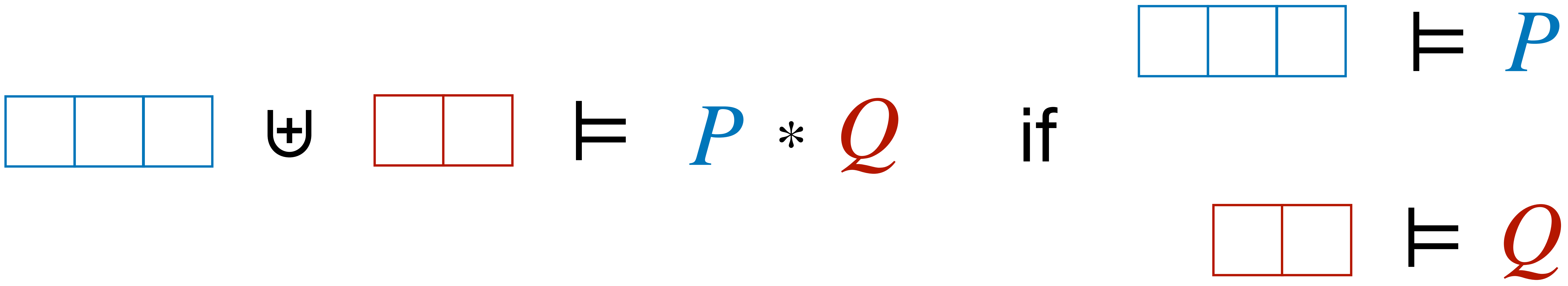

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- Separate probability spaces into independent subspaces:

$$(\mathscr{F}, \mu) \bullet (\mathscr{G}, \nu) \vDash P * Q \quad \text{if} \quad \begin{array}{l} (\mathscr{F}, \mu) \vDash P \\ (\mathscr{G}, \nu) \vDash Q \end{array}$$

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independent combination
("disjoint union for spaces")

The fine print

Proof. $\mathbf{1}$ is indeed a unit: if $(\mathcal{F}, \mu)$ is some other probability space on Ω then $\langle \mathcal{F}, \mathcal{F}_{\mathbf{1}} \rangle = \mathcal{F}$ and μ witnesses the independent combination of itself with $\mu_{\mathbf{1}}$. And the relation "$\mathcal{P}$ is an independent combination of Q and $\mathcal{R}$" is clearly symmetric in Q and $\mathcal{R}$, so $(\bullet)$ is commutative. We just need to show $(\bullet)$ is associative and respects $(\sqsubseteq)$.

For associativity, suppose $(\mathcal{F}_1, \mu_1) \bullet (\mathcal{F}_2, \mu_2) = (\mathcal{F}_{12}, \mu_{12})$ and $(\mathcal{F}_{12}, \mu_{12}) \bullet (\mathcal{F}_3, \mu_3) = (\mathcal{F}_{(12)3}, \mu_{(12)3})$. There are three things to check:

- Some μ_{23} witnesses the combination of $(\mathcal{F}_2, \mu_2)$ and $(\mathcal{F}_3, \mu_3)$.
- Some $\mu_{1(23)}$ witnesses the combination of $(\mathcal{F}_1, \mu_1)$ and $(\mathcal{F}_{23}, \mu_{23})$.
- $(\langle \mathcal{F}_1, \langle \mathcal{F}_2, \mathcal{F}_3 \rangle \rangle, \mu_{1(23)}) = (\langle \langle \mathcal{F}_1, \mathcal{F}_2 \rangle, \mathcal{F}_3 \rangle, \mu_{(12)3})$.

We'll show this as follows:

(1) $\langle \mathcal{F}_1, \langle \mathcal{F}_2, \mathcal{F}_3 \rangle \rangle = \langle \langle \mathcal{F}_1, \mathcal{F}_2 \rangle, \mathcal{F}_3 \rangle$.
(2) Define $\mu_{23} := \mu_{(12)3}|_{\mathcal{F}_{23}}$. This is a witness for $(\mathcal{F}_2, \mu_2)$ and $(\mathcal{F}_3, \mu_3)$.
(3) Define $\mu_{1(23)} := \mu_{(12)3}$. This is a witness for $(\mathcal{F}_1, \mu_1)$ and $(\mathcal{F}_{23}, \mu_{23})$.

To show the left-to-right inclusion for (1): by the universal property of freely-generated σ-algebras, we just need to show $\langle \langle \mathcal{F}_1, \mathcal{F}_2 \rangle, \mathcal{F}_3 \rangle$ is a σ-algebra containing $\mathcal{F}_1$ and $\langle \mathcal{F}_2, \mathcal{F}_3 \rangle$. It clearly contains $\mathcal{F}_1$. To show it contains $\langle \mathcal{F}_2, \mathcal{F}_3 \rangle$, we just need to show it contains $\mathcal{F}_2$ and $\mathcal{F}_3$ (by the universal property again), which it clearly does. The right-to-left inclusion is similar.

For (2), if $E_2 \in \mathcal{F}_2$ and $E_3 \in \mathcal{F}_3$ then $\mu_{23}(E_2 \cap E_3) = \mu_{(12)3}(E_2 \cap E_3) = \mu_{(12)3}((\Omega \cap E_2) \cap E_3) = \mu_{12}(\Omega \cap E_2)\mu_3(E_3) = \mu_1(\Omega)\mu_2(E_2)\mu_3(E_3) = \mu_2(E_2)\mu_3(E_3)$ as desired.

For (3), we need $\mu_{(12)3}(E_1 \cap E_{23}) = \mu_1(E_1)\mu_{23}(E_{23})$ for all $E_1 \in \mathcal{F}_1$ and $E_{23} \in \langle \mathcal{F}_2, \mathcal{F}_3 \rangle$. For this we use the π-λ theorem. Let $\mathcal{E}$ be the set $\{E_2 \cap E_3 \mid E_2 \in \mathcal{F}_2, E_3 \in \mathcal{F}_3\}$ of intersections of events in $\mathcal{F}_2$ and $\mathcal{F}_3$. $\mathcal{E}$ is a π-system that generates $\langle \mathcal{F}_2, \mathcal{F}_3 \rangle$ (lemma B.2). Let $\mathcal{G}$ be the set of events E_{23} such that $\mu_{(12)3}(E_1 \cap E_{23}) = \mu_1(E_1)\mu_{23}(E_{23})$ for all $E_1 \in \mathcal{F}_1$. We are done if $\langle \mathcal{E} \rangle \subseteq \mathcal{G}$. By the π-λ theorem, we just need to check that $\mathcal{E} \subseteq \mathcal{G}$ and that $\mathcal{G}$ is a λ-system. We have $\mathcal{E} \subseteq \mathcal{G}$ because if $E_2 \in \mathcal{F}_2$ and $E_3 \in \mathcal{F}_3$ then $\mu_{(12)3}(E_1 \cap (E_2 \cap E_3)) = \mu_1(E_1)\mu_2(E_2)\mu_3(E_3) = \mu_1(E_1)\mu_{23}(E_2 \cap E_3)$. To see that $\mathcal{G}$ is a λ-system, note that $\mu_1(E_1)\mu_{23}(E_{23}) = \mu_{(12)3}(E_1)\mu_{(12)3}(E_{23})$ and so $\mathcal{G}$ is actually equal to $\mathcal{F}_1^{\perp}$ (the set of events independent of $\mathcal{F}_1$), a λ-system by Lemma B.3.

To show $(\bullet)$ respects $(\sqsubseteq)$, suppose $(\mathcal{F}, \mu) \sqsubseteq (\mathcal{F}', \mu')$ and $(\mathcal{G}, \nu) \sqsubseteq (\mathcal{G}', \nu')$ and $(\mathcal{F}', \mu') \bullet (\mathcal{G}', \nu') = (\langle \mathcal{F}', \mathcal{G}' \rangle, \rho')$. We need to show (1) $(\mathcal{F}, \mu) \bullet (\mathcal{G}, \nu) = (\langle \mathcal{F}, \mathcal{G} \rangle, \rho)$ and (2) $(\langle \mathcal{F}, \mathcal{G} \rangle, \rho) \sqsubseteq (\langle \mathcal{F}', \mathcal{G}' \rangle, \rho')$ for some ρ. Define ρ to be the restriction of ρ' to $\langle \mathcal{F}, \mathcal{G} \rangle$. Now (1) holds because $\rho(F \cap G) = \rho'(F \cap G) = \rho'(F)\rho'(G) = \rho(F)\rho(G)$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$ (the second step follows from $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{G} \subseteq \mathcal{G}'$). For (2), $\langle \mathcal{F}, \mathcal{G} \rangle \subseteq \langle \mathcal{F}', \mathcal{G}' \rangle$ because $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{G} \subseteq \mathcal{G}'$, and $\rho = \rho'|_{\langle \mathcal{F}, \mathcal{G} \rangle}$ by construction. □

The fine print

THEOREM B.25. *Let $\mathcal{M}_{\text{disintegrable}}$ be the set of countably-generated probability spaces $\mathcal{P}$ that have finite footprint and can be extended to a Borel measure on the entire Hilbert cube. The restriction of the KRM given by Theorem 2.4 to $\mathcal{M}_{\text{disintegrable}}$ is still a KRM.*

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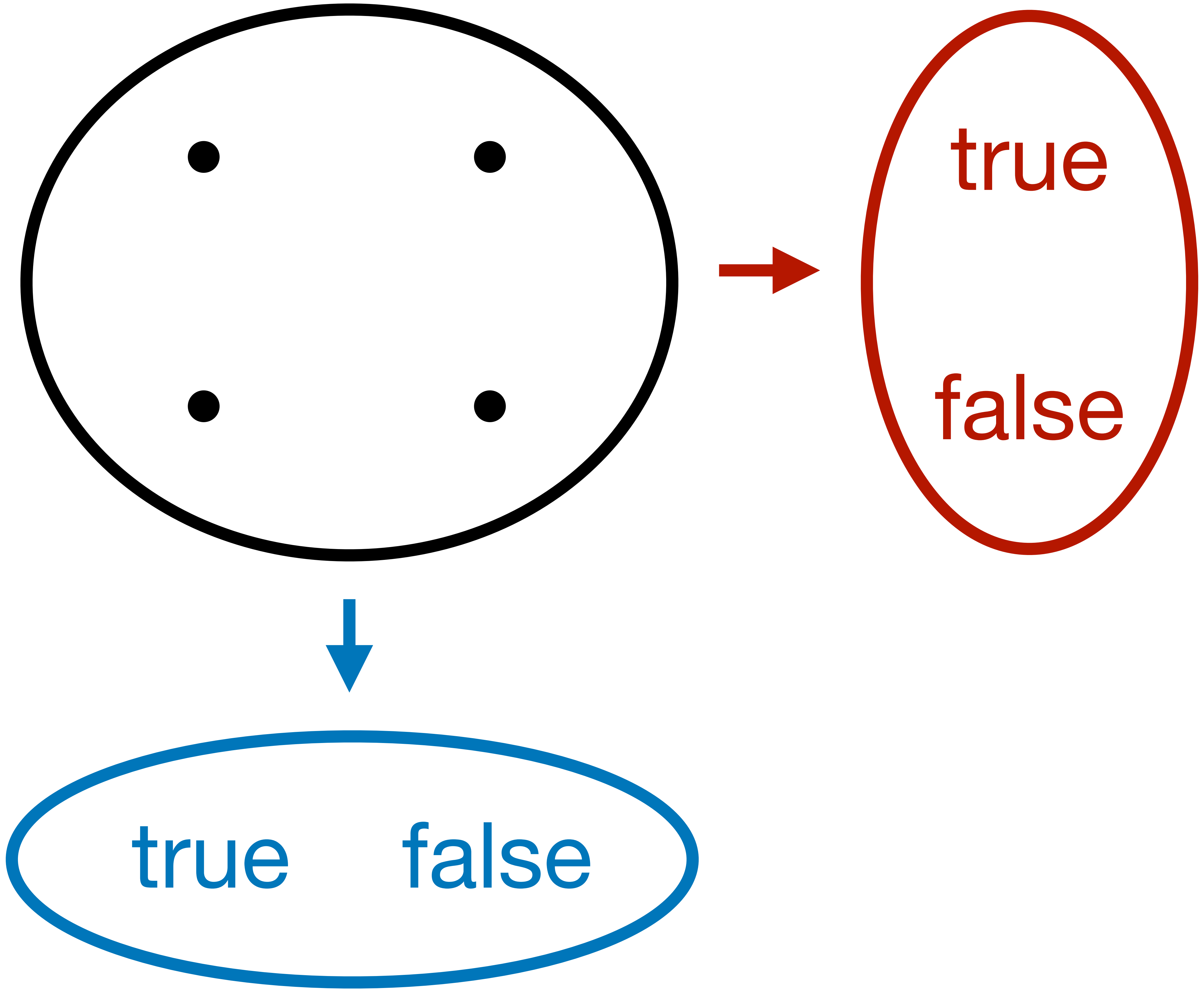

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Towards a categorical answer

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A Model for Syntactic Control of Interference

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MSCS'93

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6.1. The Tensor Product

The bifunctor $\otimes$ on $\mathbf{K}$ is a subfunctor of the categorical product $\times$, restricted so that different components are independent of one another.

If A, B are $\mathbf{K}$-objects then

$$
\begin{aligned}
(A \otimes B)X &= \{(a, b) \in A(X) \times B(X) \mid a \triangle b\}, \text{ ordered componentwise} \\
(A \otimes B)\, f\, (a, b) &= (A(f)\, a, B(f)\, b)
\end{aligned}
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Day convolution w.r.t. coproduct of heap shapes

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the Schanuel topos* **Sch**

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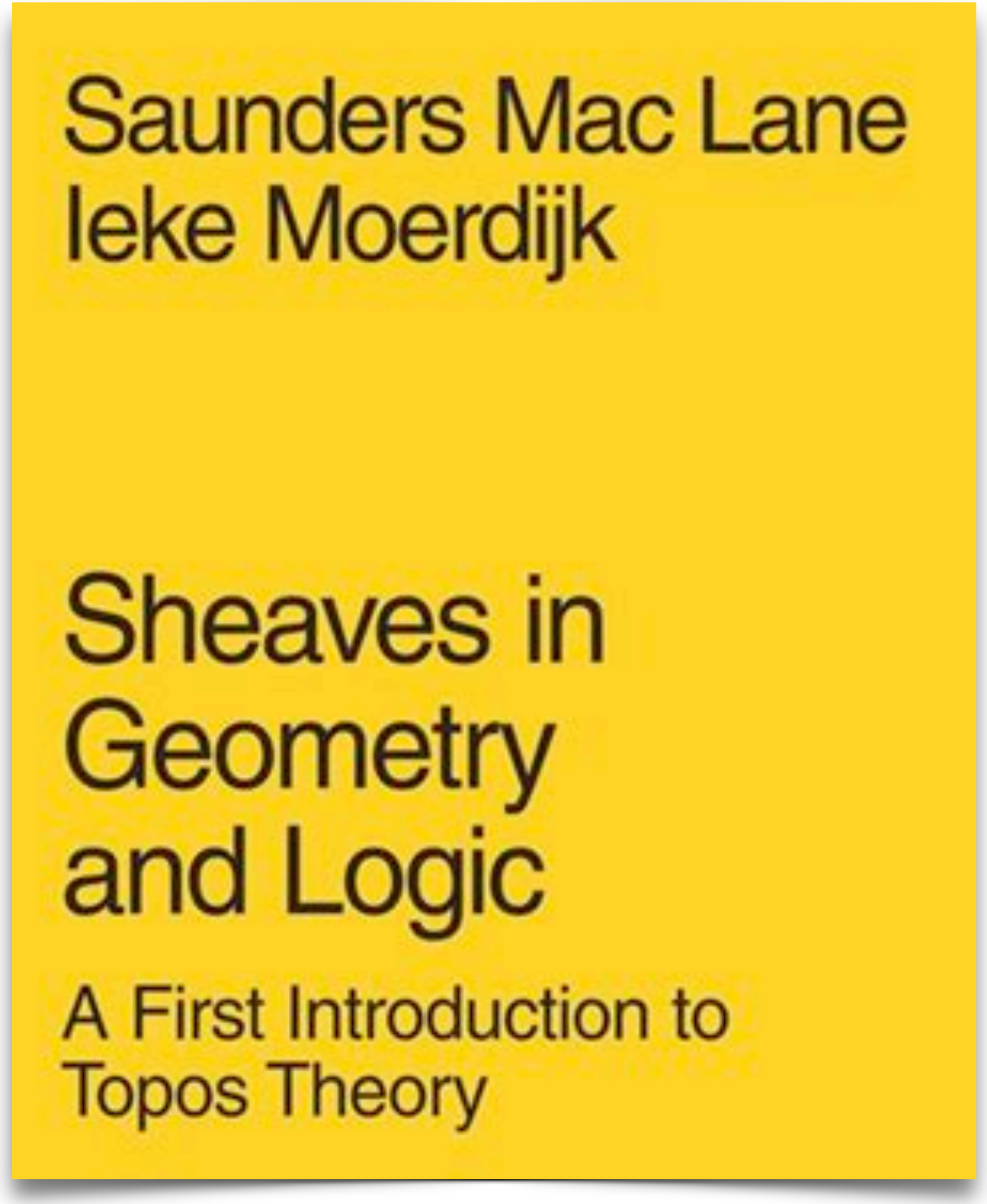

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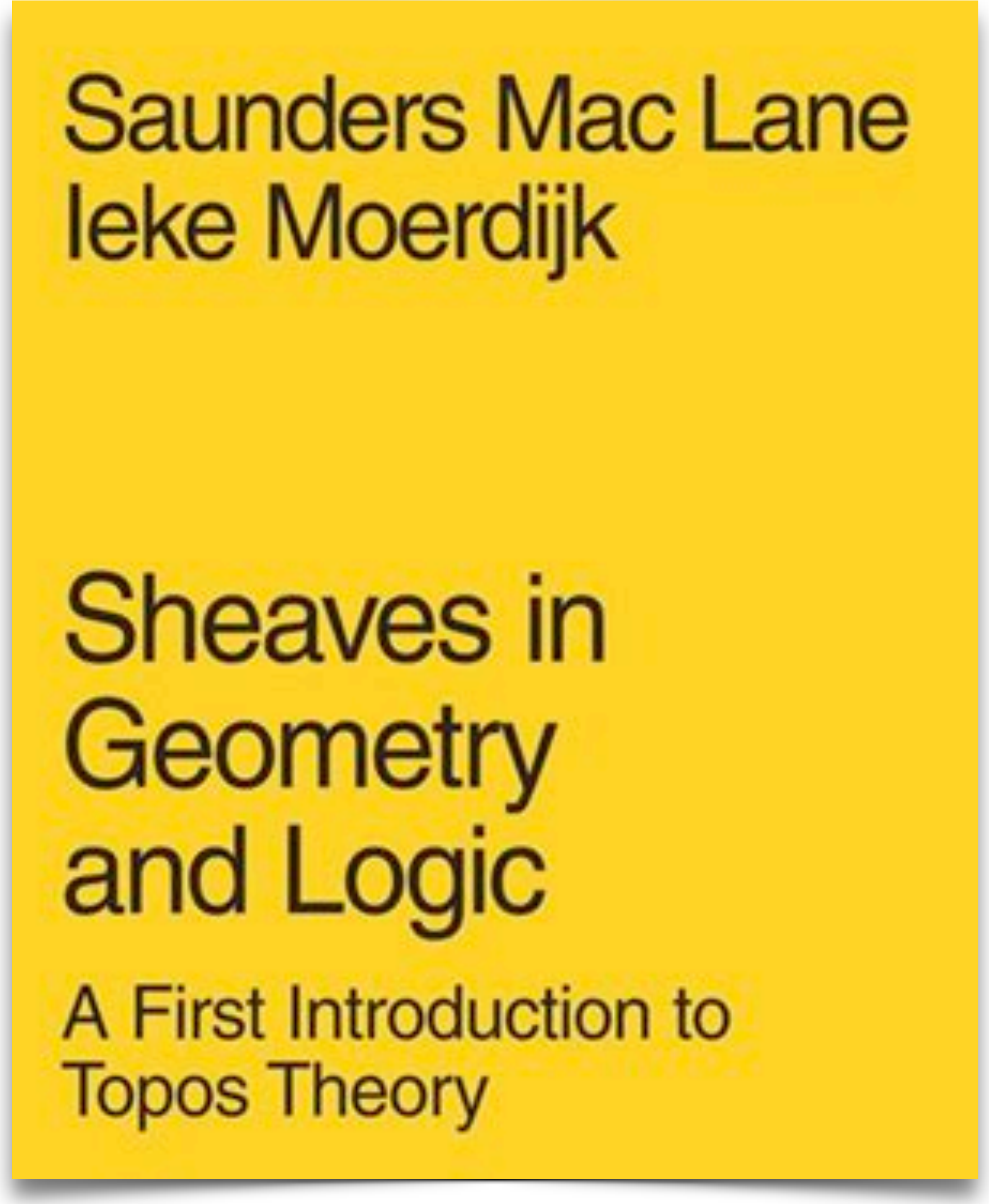
the category of nominal sets



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- Across this equivalence,

Day conv. w.r.t. coproduct in **Sch**	$\simeq$	pairs of disjoint heaps in **Nom**

Back to the present day

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ProbNom

"Probabilistic nominal sets"

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$$\mathbf{ProbSch} \simeq \mathbf{ProbNom}$$

ProbSch: A "probabilistic Schanuel topos"

ProbNom: "Probabilistic nominal sets"

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- Across this equivalence,

$$\begin{array}{ccc} \text{Day conv. w.r.t. product} & & \text{independent combination} \\ \text{in} & \simeq & \text{in} \\ \textbf{ProbSch} & & \textbf{ProbNom} \end{array}$$

Upshot

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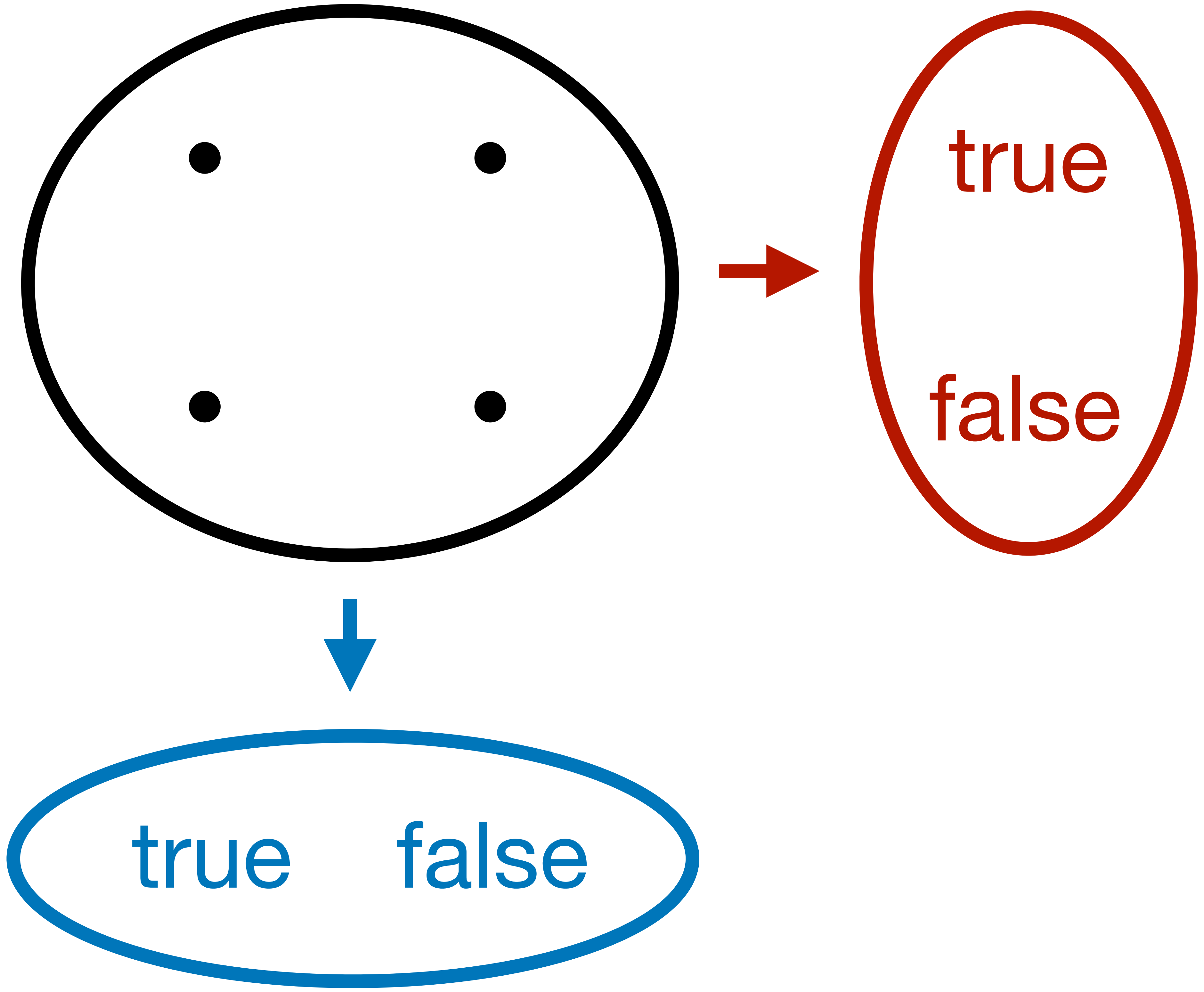

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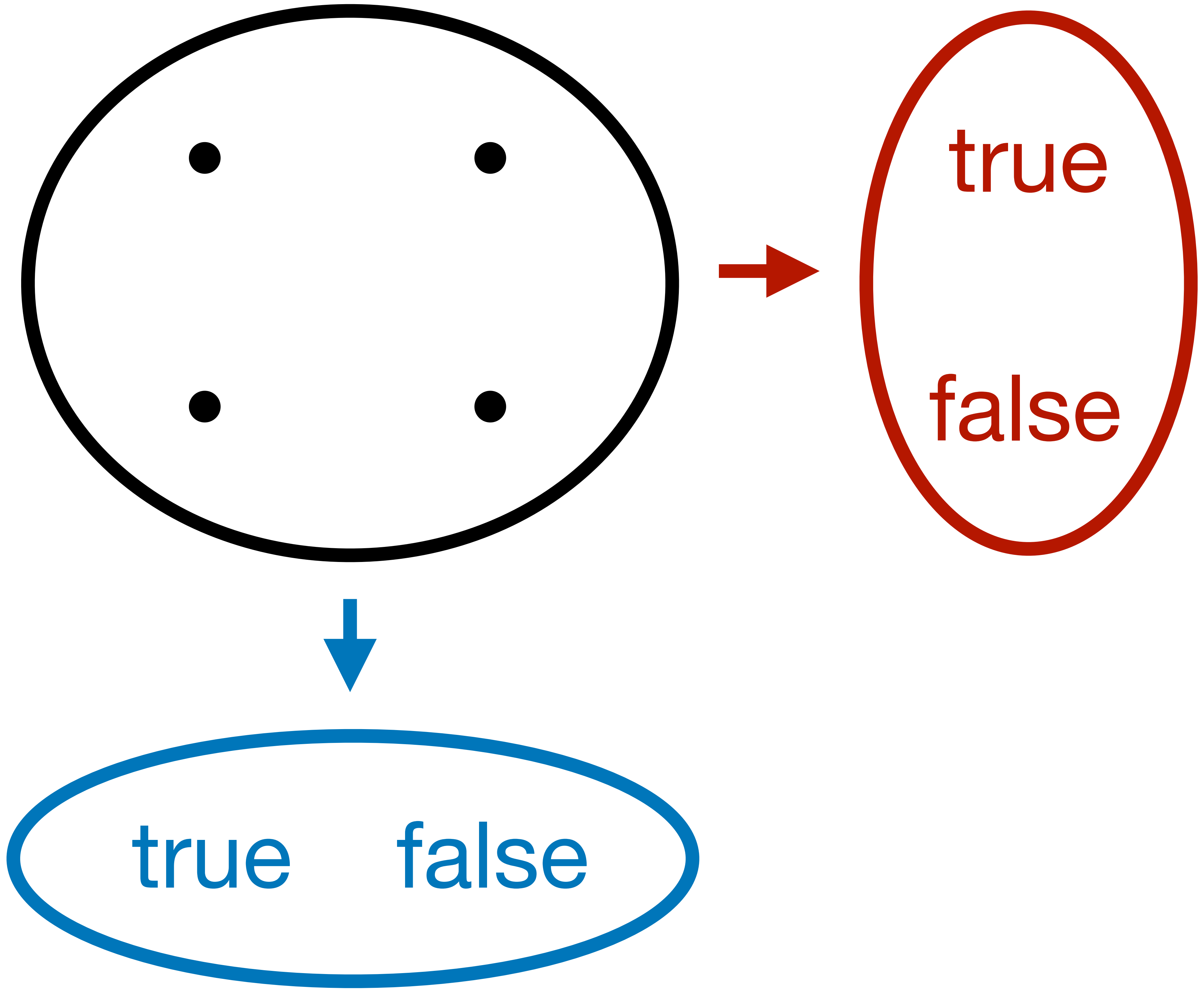

- And independent combination is right too!

Upshot

- Corroborates recent work linking probability to names

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Probabilistic Programming Semantics for Name Generation

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DARIO STEIN, University of Oxford, United Kingdom
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Probability Sheaves and the Giry Monad*

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Probability theory		**Nominal sets**
Measurable space	$\sim$	Support
Measurability	$\sim$	Supportedness
Probability space	$\sim$	Store
Probabilistic independence	$\sim$	Disjointness of stores

Upshot

https://johnm.li/lafi24.pdf

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- ==> maybe nominal techniques apply to probability?