

Lilac: A Modal Separation Logic for Conditional Probability

John Li

li.john@northeastern.edu

Amal Ahmed

amal@ccs.neu.edu

Steven Holtzen

s.holtzen@northeastern.edu



<https://johnm.li/lilac.pdf>

How to reason about complex probabilistic systems?

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How to reason about complex probabilistic systems?



Is my car safe?



How to reason about complex probabilistic systems?



Is my car safe?



Is this decision fair?



How to reason about complex probabilistic systems?



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Is this decision fair?



Is my result significant?

How to reason about complex probabilistic systems?

- Reasoning should be *modular*:

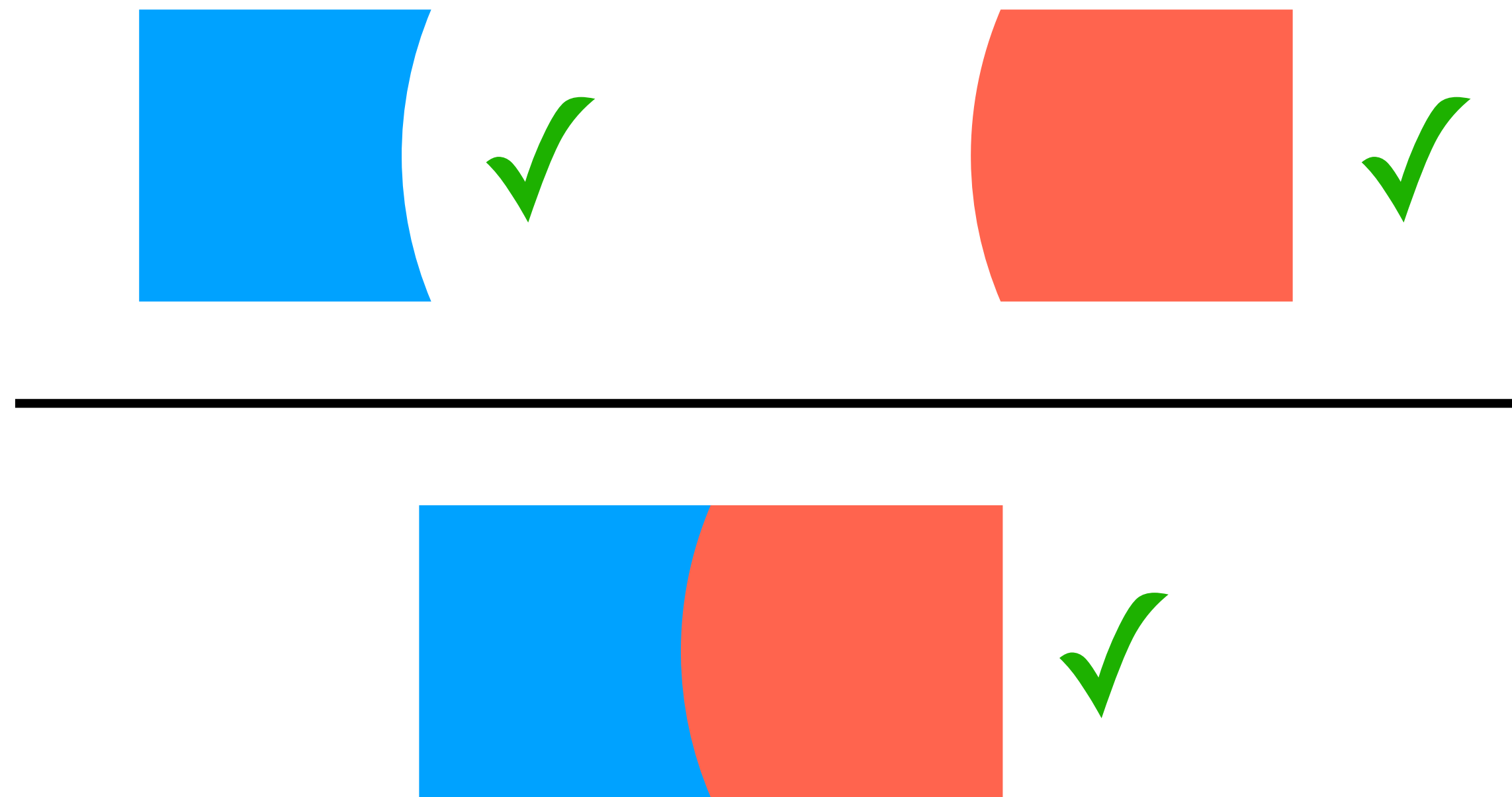
How to reason about complex probabilistic systems?

- Reasoning should be *modular*:



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- Reasoning should be *modular*:



Modularity comes from probabilistic independence

- Independence arises frequently and naturally:

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```
weights = np.random.rand(1000)
```

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weights = np.random.rand(1000)
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$weights[0], \dots, weights[999] \sim \text{Unif}[0,1]$ mutually independent

Modularity comes from probabilistic independence

- Independence arises frequently and naturally:

```
result = np.mean(data)
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Modularity comes from probabilistic independence

- Independence arises frequently and naturally:

if each `data[i]` is an independent estimate of ν ...

```
result = np.mean(data)
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...then `result` is a more accurate estimate of ν

Modularity comes from probabilistic independence

- Independence arises frequently and naturally.
- Idea: capture independence using *separation logic*

Ordinary separation logic is about disjointness

$x = \text{new } 0;$

$y = \text{new } 1;$

Ordinary separation logic is about disjointness

$x = \text{new } 0;$

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$(x \mapsto 0) * (y \mapsto 1)$

Ordinary separation logic is about disjointness

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x and y point to disjoint heap locations

Ordinary separation logic is about disjointness

$$\frac{\{P\} \ e \ \{x. Q(x)\}}{\{P * F\} \ e \ \{x. Q(x) * F\}} \text{ (Frame)}$$

Ordinary separation logic is about disjointness

When verifying $e...$


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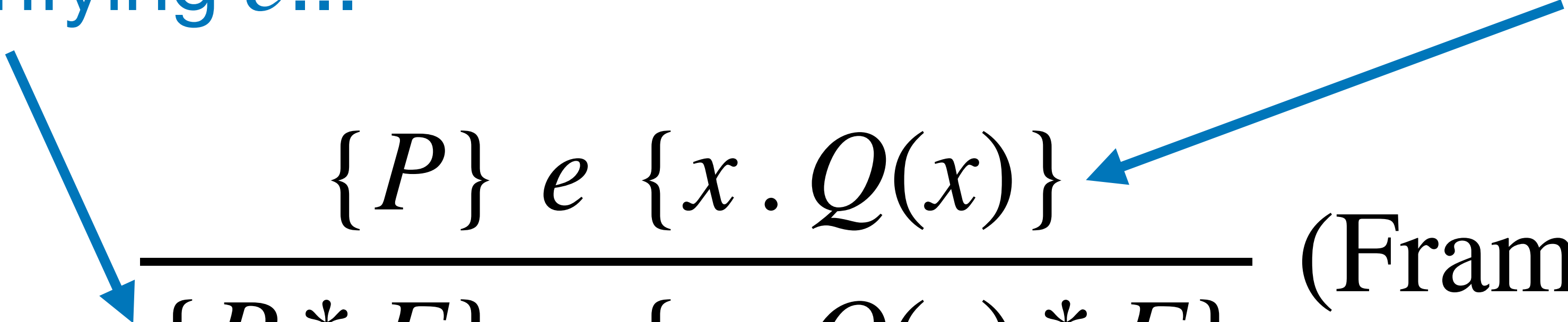
When verifying $e...$

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Ordinary separation logic is about disjointness

When verifying $e...$...I can ignore disjoint subheaps F

$$\frac{\{P\} \ e \ \{x. Q(x)\}}{\{P * F\} \ e \ \{x. Q(x) * F\}} \text{ (Frame)}$$


- This has enabled modular heap-based reasoning at scale.¹

¹C. Calcagno and D. Distefano. Infer: An automatic program verifier for memory safety of C programs. NFM 2011.

Lilac's separation is about independence

$X \leftarrow \text{flip } 1/2;$

$Y \leftarrow \text{flip } 1/2;$

Lilac's separation is about independence

$X \leftarrow \text{flip } 1/2;$

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$X \sim \text{Ber}(1/2) \quad * \quad Y \sim \text{Ber}(1/2)$

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X and Y are independent random variables

Wait, hasn't this been done before?

Wait, hasn't this been done before?

A Probabilistic Separation Logic

GILLES BARTHE, MPI for Security and Privacy, Germany and IMDEA Software Institute, Spain

JUSTIN HSU, University of Wisconsin–Madison, USA

KEVIN LIAO, MPI for Security and Privacy, Germany and University of Illinois Urbana-Champaign, USA

POPL'20

New in Lilac

New in Lilac: a simple frame rule

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$$\frac{\{P\} \ e \ \{x. Q(x)\}}{\{P * F\} \ e \ \{x. Q(x) * F\}} \text{ (Frame)}$$

New in Lilac: a simple frame rule

$$\frac{\{P\} \ e \ \{x . Q(x)\}}{\{P * F\} \ e \ \{x . Q(x) * F\}} \text{ (Frame)}$$

- Just like in ordinary separation logic!

New in Lilac: separation is independence

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```
weights = np.random.rand(1000)
```

$\text{weights}[0], \dots, \text{weights}[999] \sim \text{Unif}[0,1]$ mutually independent

New in Lilac: separation is independence

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weights = np.random.rand(1000)
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```
(weights[0] ~ Unif[0,1]) * ... * (weights[999] ~ Unif[0,1])
```

New in Lilac: separation is independence

```
weights = np.random.rand(1000)
```

$(\text{weights}[0] \sim \text{Unif}[0,1]) * \dots * (\text{weights}[999] \sim \text{Unif}[0,1])$

Inexpressible in PSL



New in Lilac: separation is independence

```
weights = np.random.rand(1000)
```

$(\text{weights}[0] \sim \text{Unif}[0,1]) * \dots * (\text{weights}[999] \sim \text{Unif}[0,1])$



Completely captures independence (Lemma 2.5)

New in Lilac: quantitative reasoning

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if each `data[i]` is an independent estimate of v ...

```
result = np.mean(data)
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...then `result` is a more accurate estimate of v

New in Lilac: quantitative reasoning

if each `data[i]` independent
and for all i we have $\mathbb{E}[\text{data}[i]] = \nu$ and $\text{Var}(\text{data}[i]) \leq \varepsilon \dots$

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New in Lilac: quantitative reasoning

if $\bigstar_{0 \leq i < |\text{data}|} \left(\mathbb{E}[\text{data}[i]] = v \text{ and } \text{Var}(\text{data}[i]) \leq \varepsilon \right) \dots$

`result = np.mean(data)`

$\dots \text{then } \mathbb{E}[\text{result}] = v \text{ and } \text{Var}(\text{result}) \leq \frac{\varepsilon}{|\text{data}|}$

New in Lilac: good interop with normal math

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An ordinary random variable

New in Lilac: good interop with normal math

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`result = np.mean(data)`

$\dots \text{then } \mathbb{E}[\text{result}] = v \text{ and } \text{Var}(\text{result}) \leq \frac{\varepsilon}{|\text{data}|}$

Ordinary expectation and variance

New in Lilac: good interop with normal math

if $\bigstar_{0 \leq i < |\text{data}|} \left(\mathbb{E}[\text{data}[i]] = v \text{ and } \text{Var}(\text{data}[i]) \leq \varepsilon \right) \dots$

`result = np.mean(data)`

$\dots \text{then } \mathbb{E}[\text{result}] = v \text{ and } \text{Var}(\text{result}) \leq \frac{\varepsilon}{|\text{data}|}$

\Rightarrow textbook proofs remain textbook

Key idea

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- Probability spaces are the heaps of probability theory.

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$x = \text{new } 0;$

$y = \text{new } 1;$

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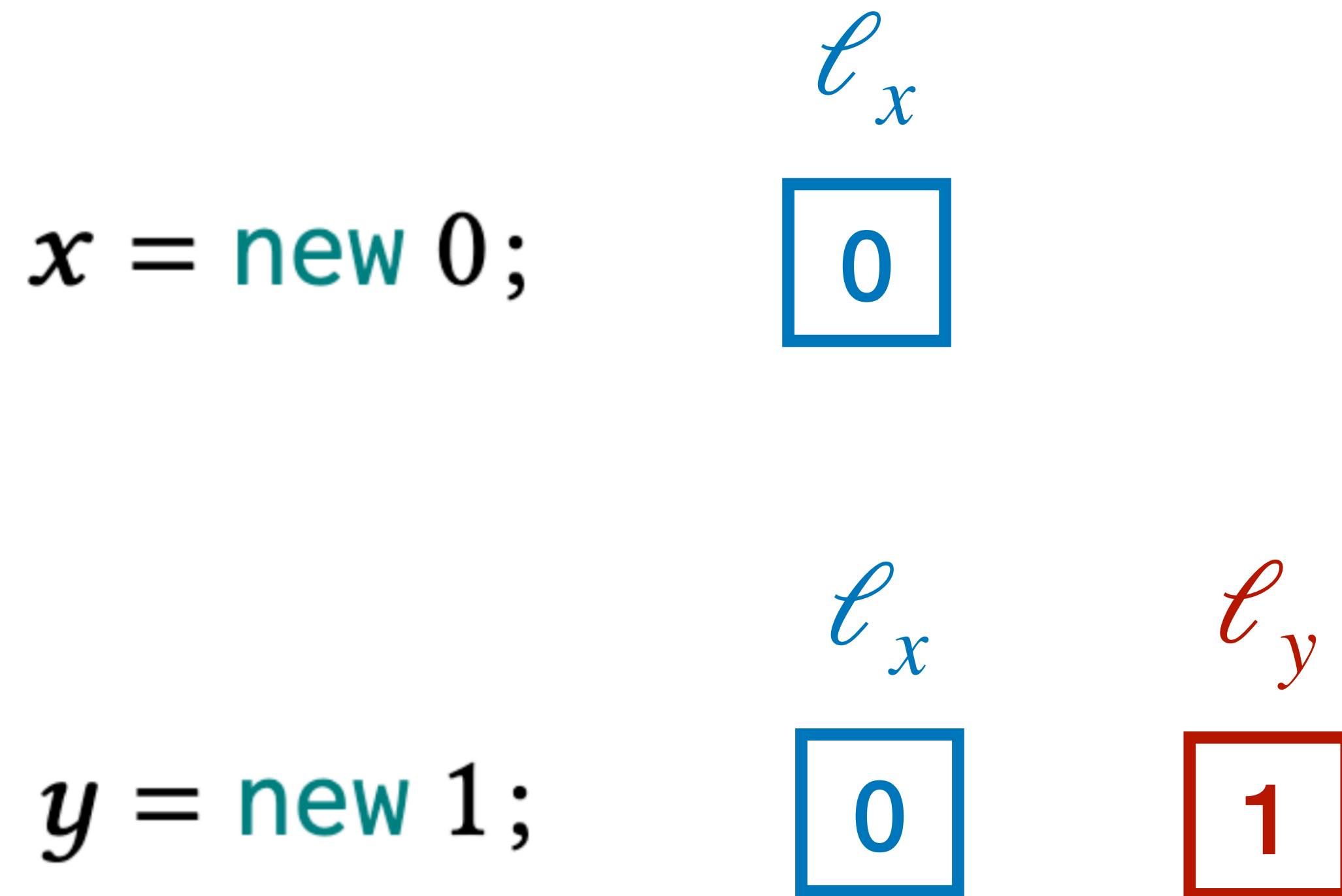
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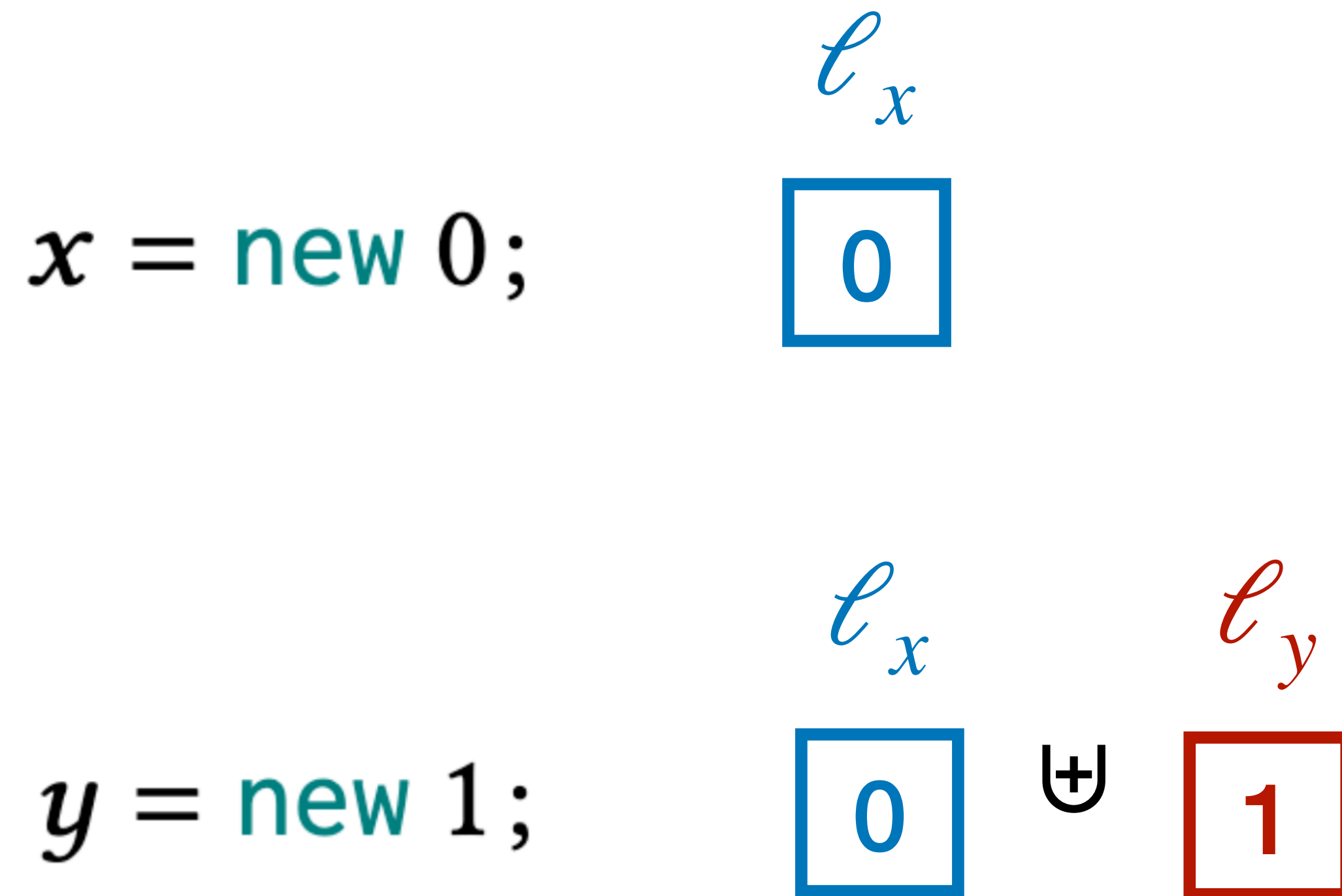
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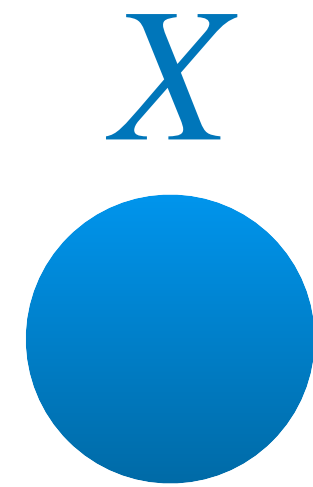
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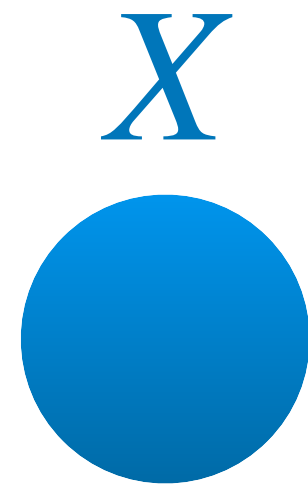


$Y \leftarrow \text{flip } 1/2;$

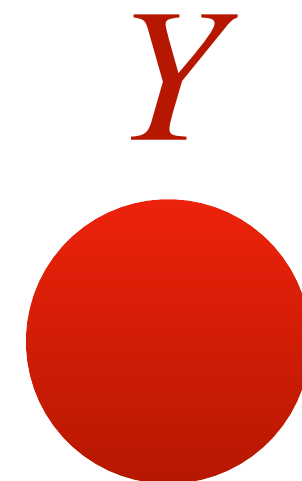
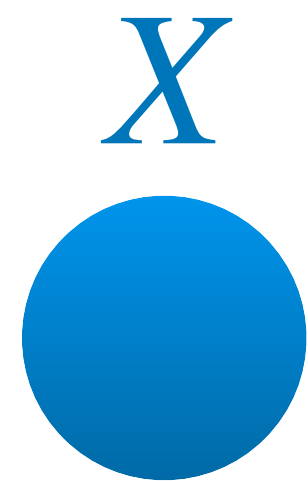
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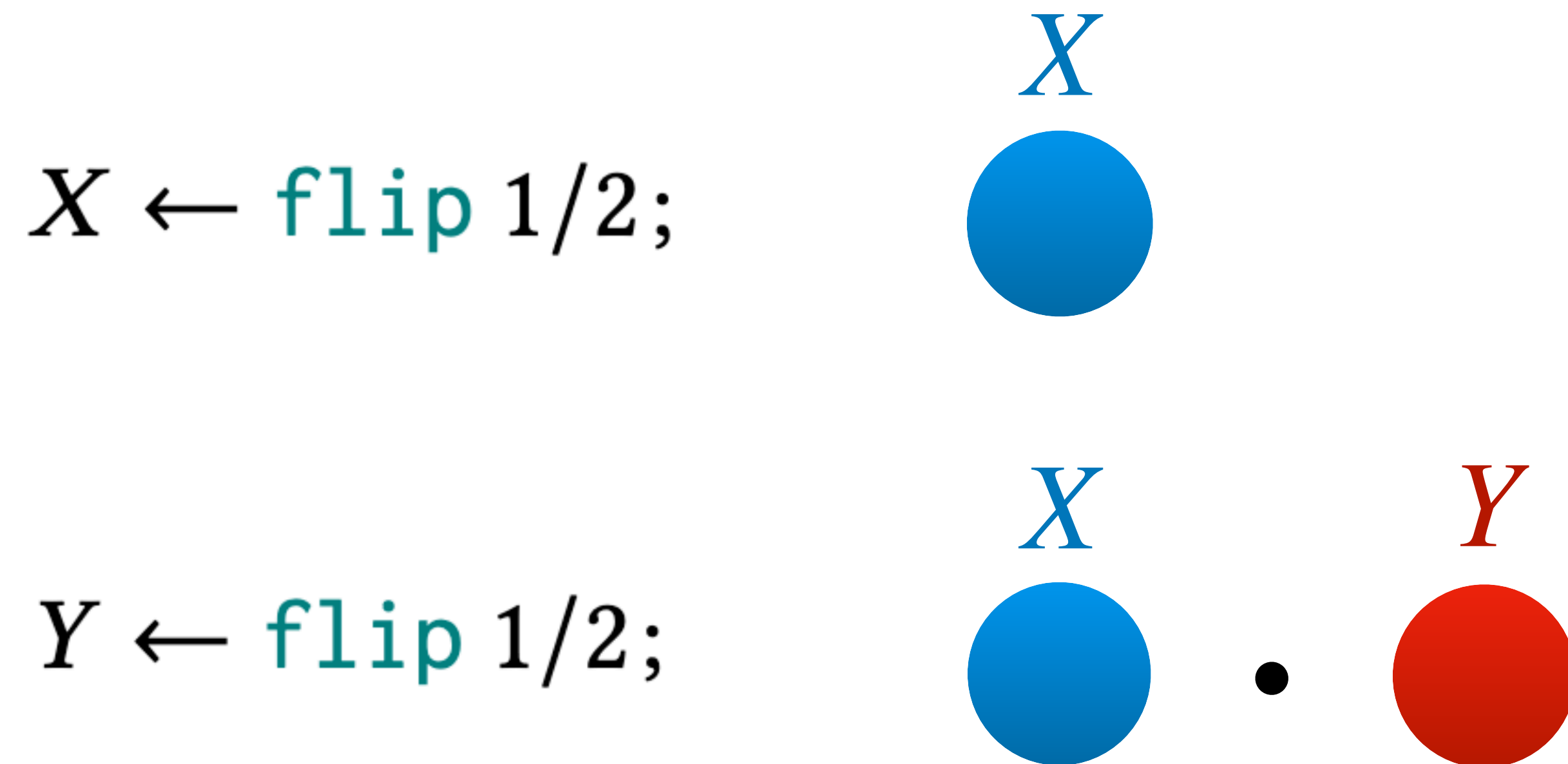


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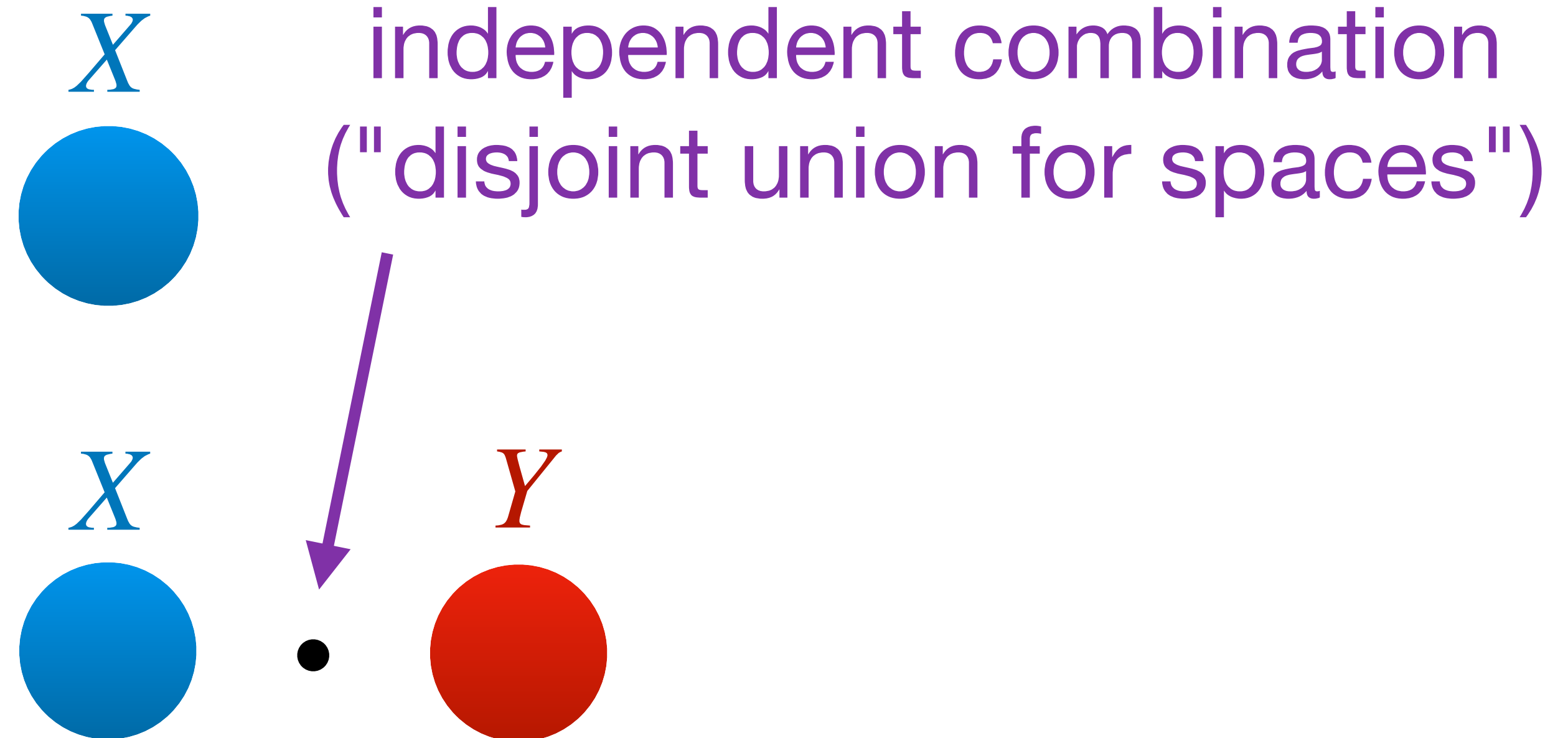


Key idea

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- Separating conjunction decomposes probability spaces:

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- Separating conjunction decomposes probability spaces:

$$\begin{array}{c} \boxed{} \boxed{} \boxed{} \uplus \boxed{} \boxed{} \models P * Q \quad \text{if} \quad \begin{array}{l} \boxed{} \boxed{} \boxed{} \models P \\ \boxed{} \boxed{} \models Q \end{array} \end{array}$$

Key idea

- Probability spaces are the heaps of probability theory.
- Separating conjunction decomposes probability spaces:

$$\text{blue circle} \cdot \text{red circle} \models P * Q \quad \text{if} \quad \begin{array}{l} \text{blue circle} \models P \\ \text{red circle} \models Q \end{array}$$

Probability spaces as heaps

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$$X \sim \text{Ber}(1/2)$$

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$X \sim \text{Ber}(1/2)$ means $\Pr[X = \text{true}] = \Pr[X = \text{false}] = 1/2$

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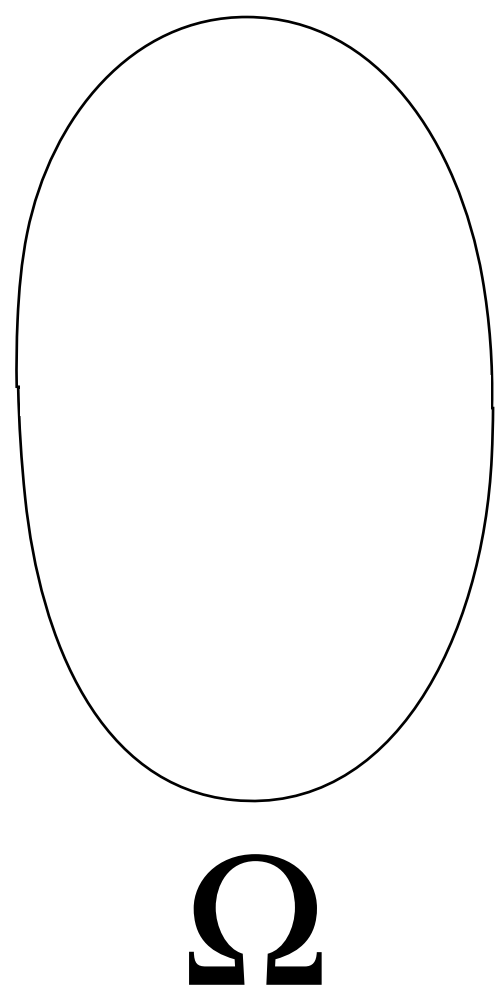
This hides a lot of machinery...

Probability spaces as heaps

$X \sim \text{Ber}(1/2)$ really means...

Probability spaces as heaps

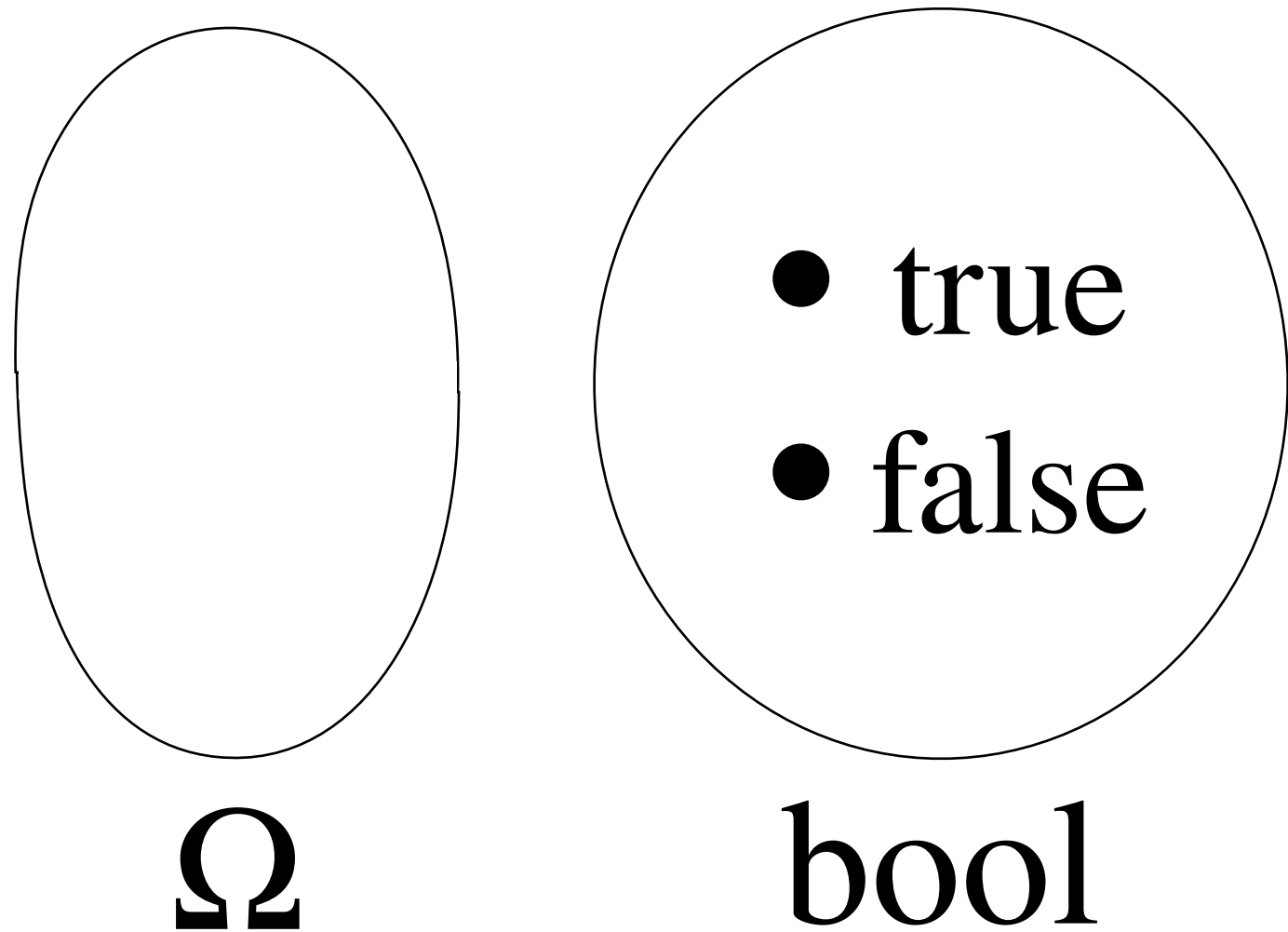
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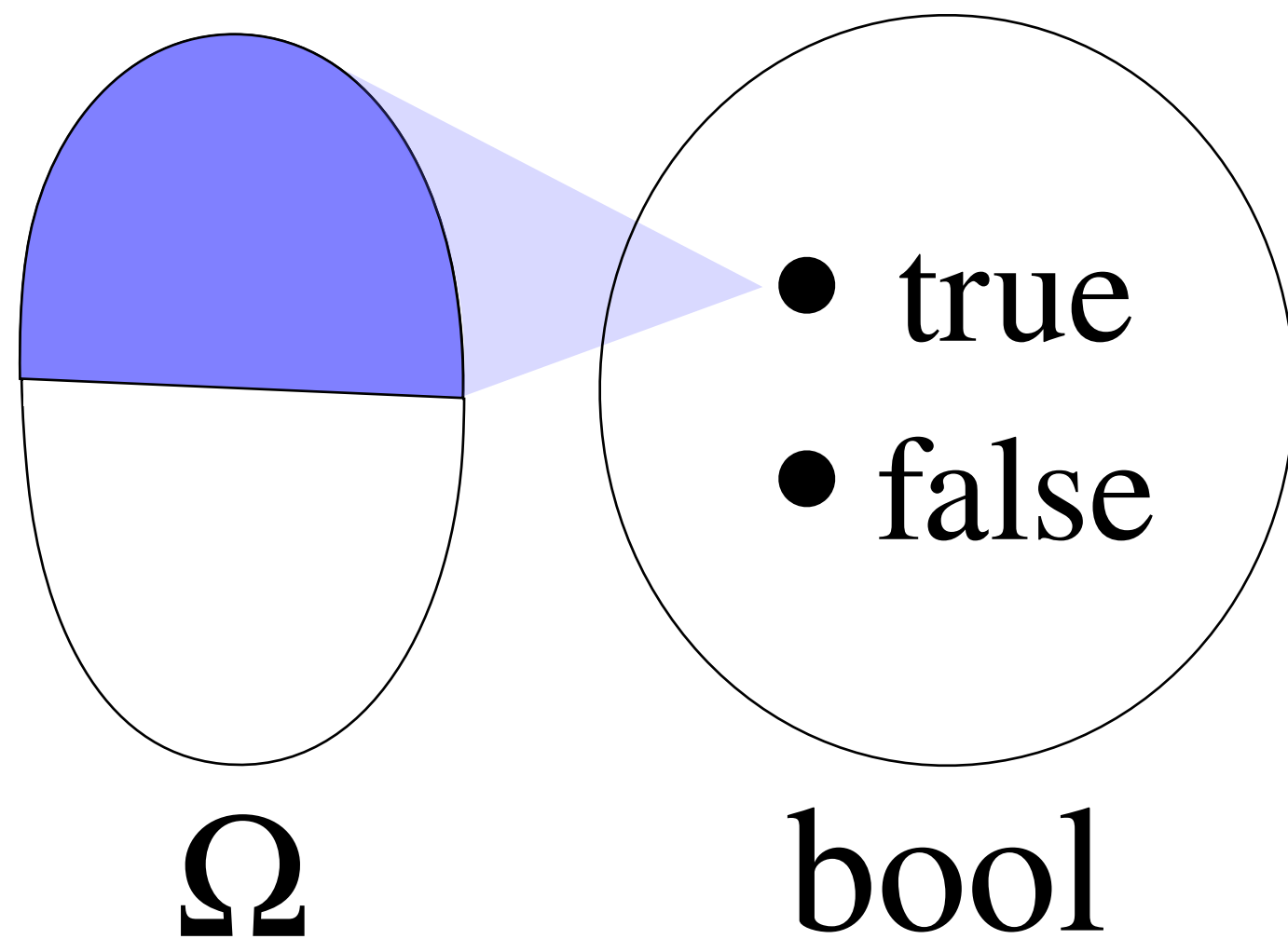
$X : \Omega \rightarrow \text{bool}$



Probability spaces as heaps

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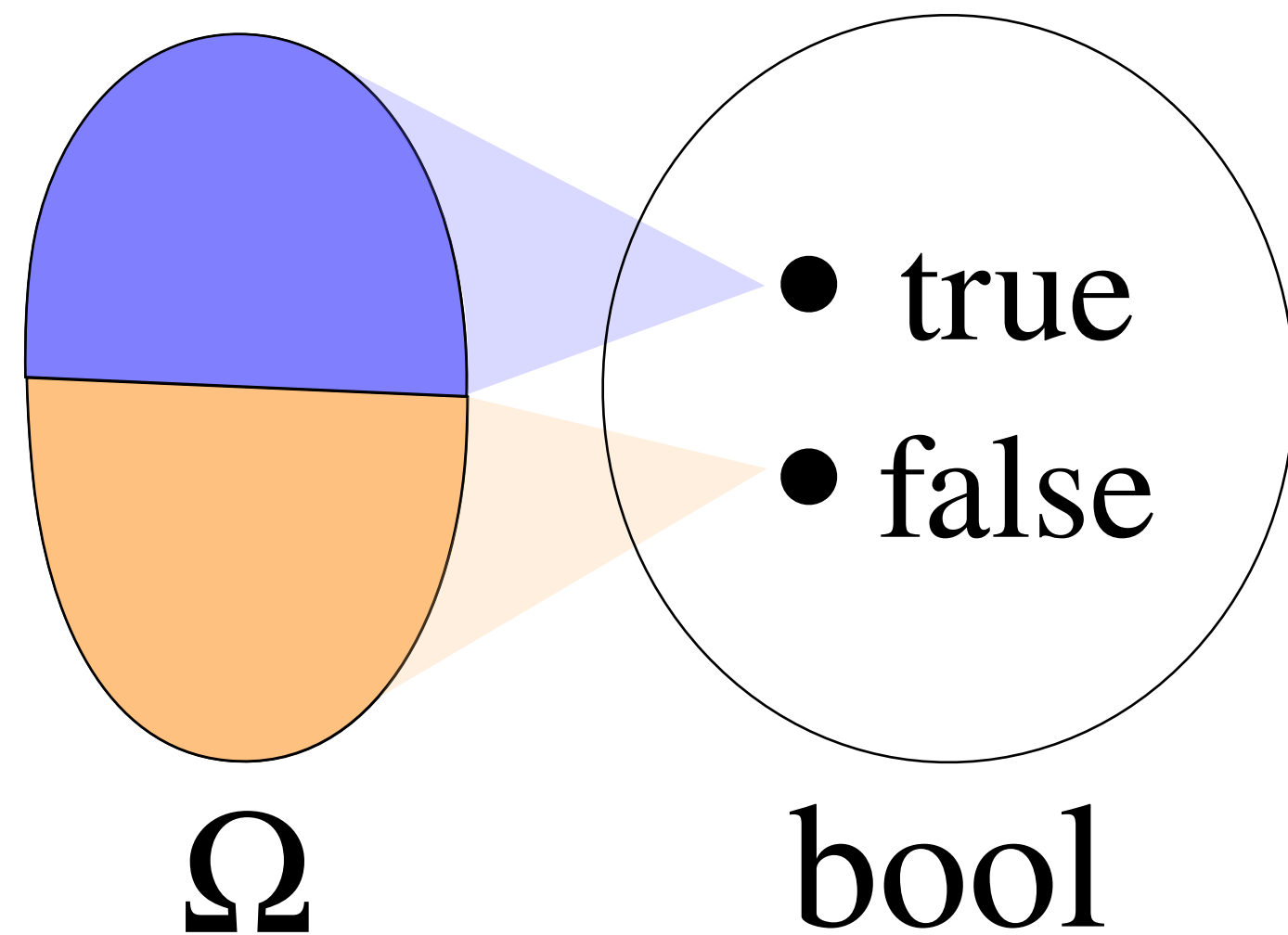
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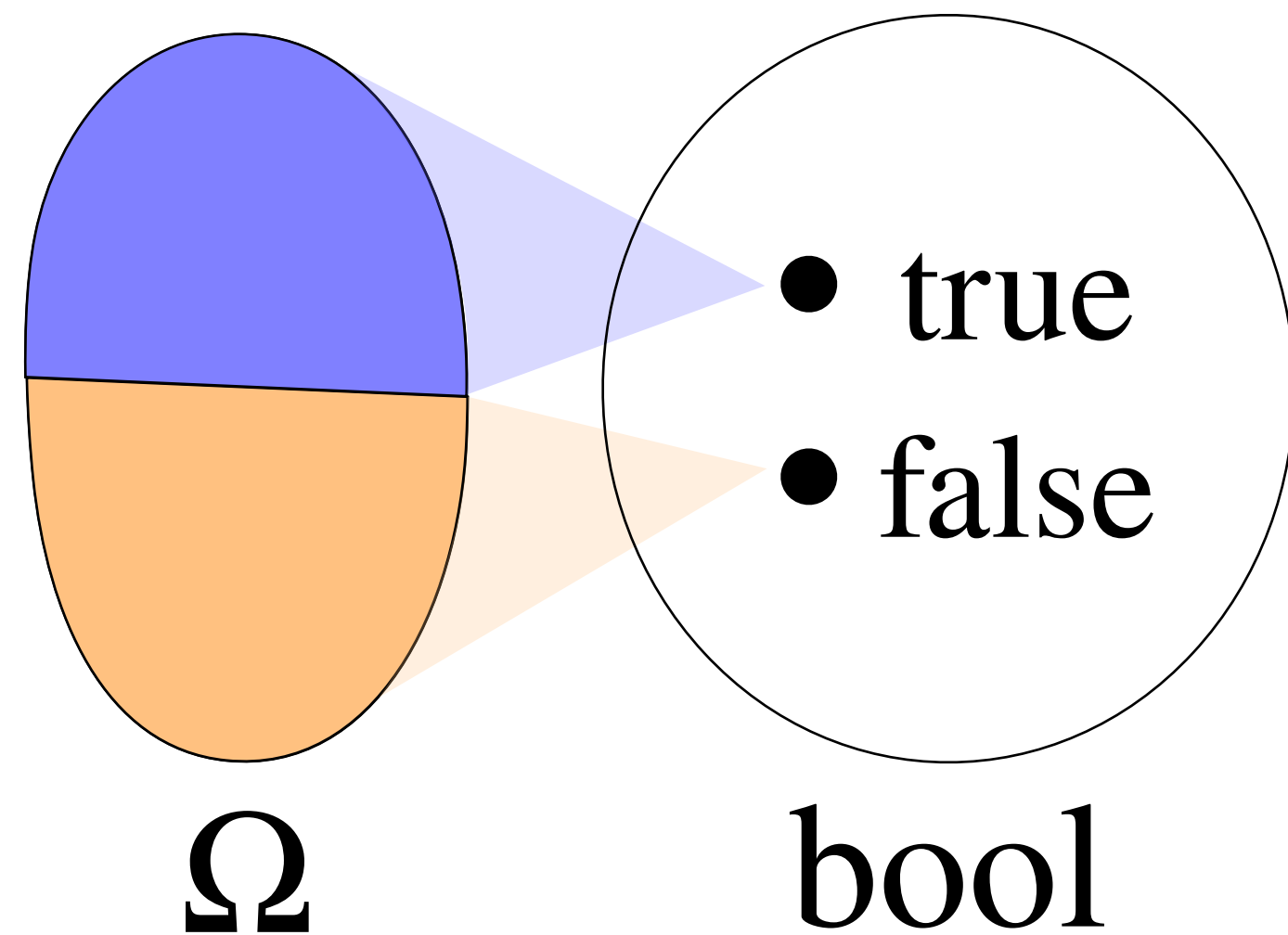
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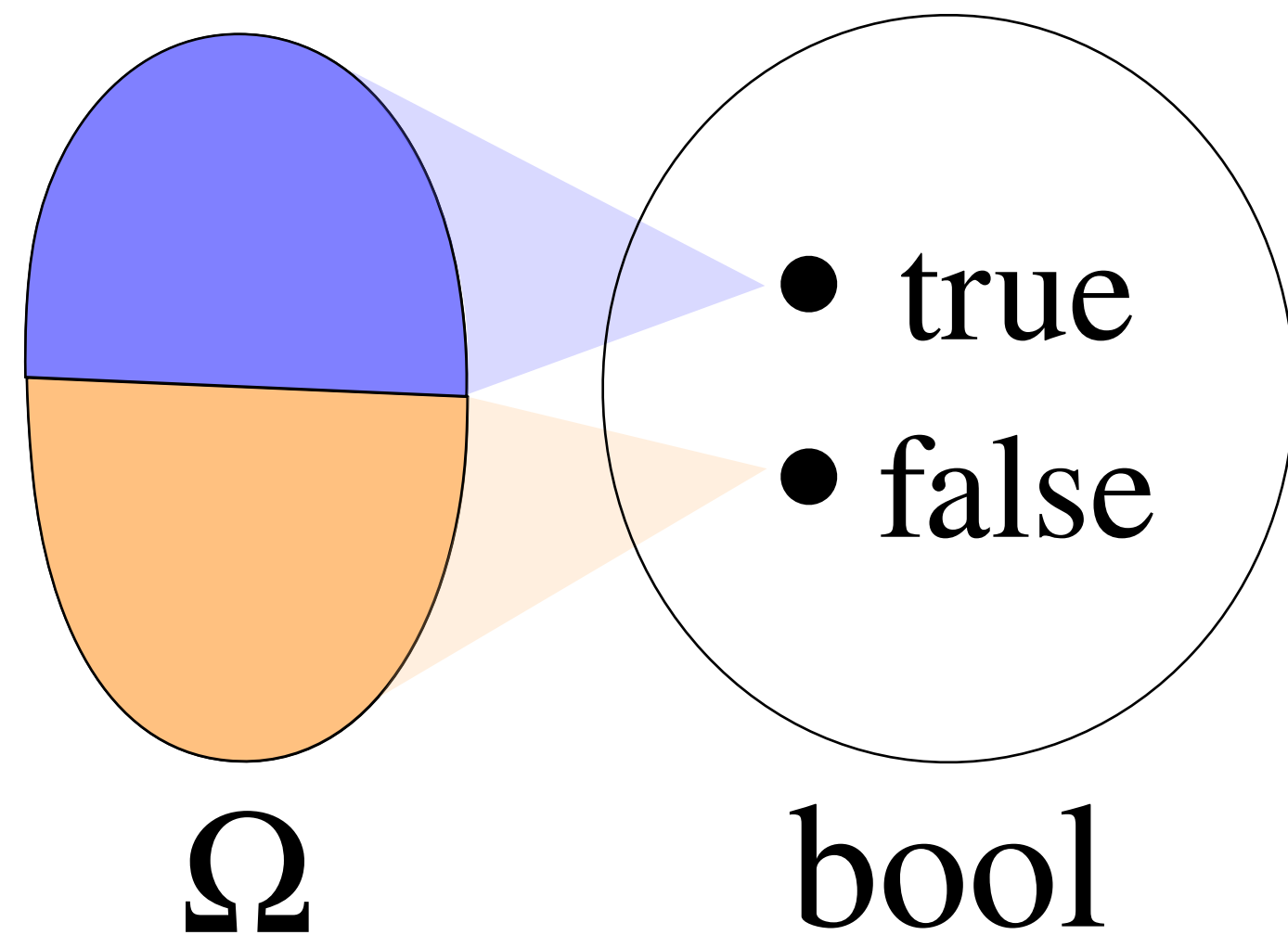


$$\mu : \text{events} \rightarrow [0,1]$$

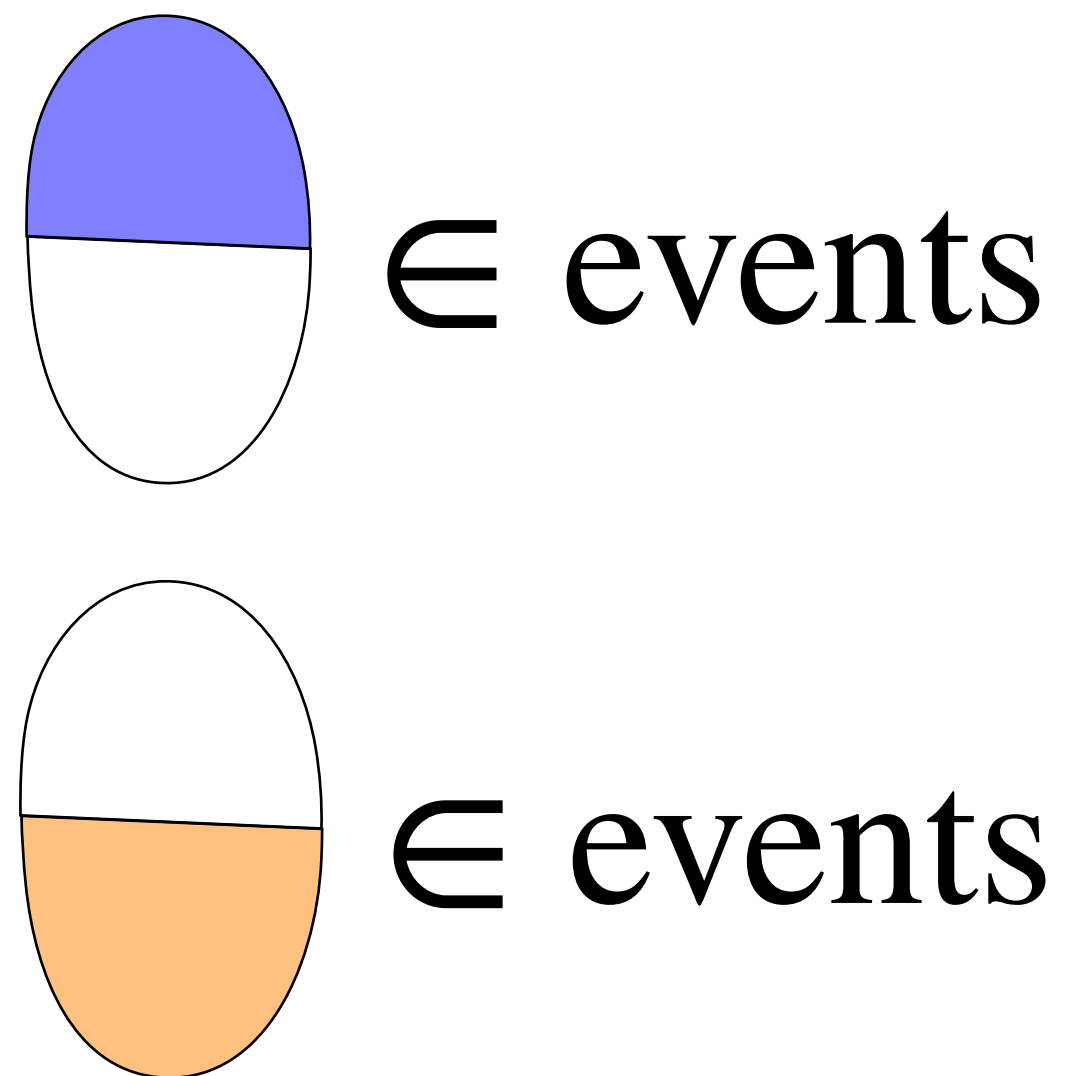
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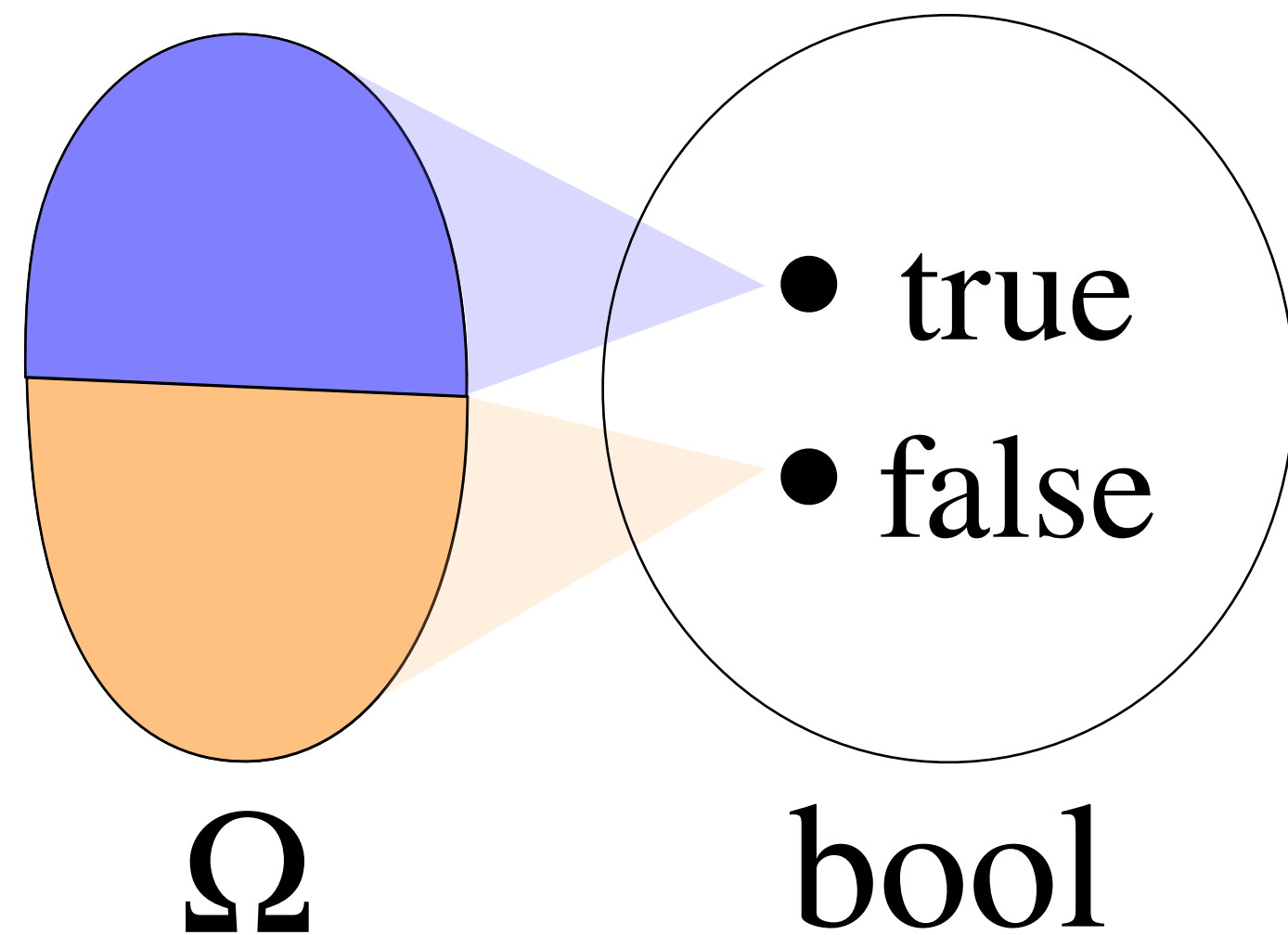
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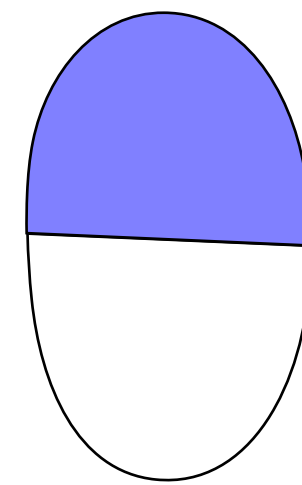
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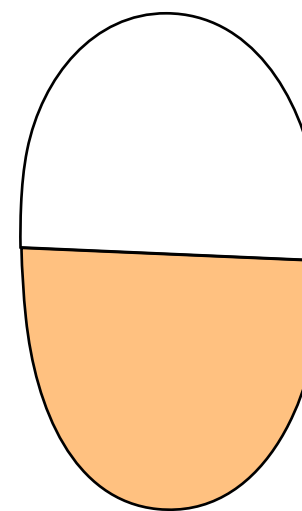


$$\mu : \text{events} \rightarrow [0,1]$$



$\in \text{events}$

$$\mu \left(\begin{array}{c} \text{blue top half} \\ \text{white bottom half} \end{array} \right) = 1/2$$



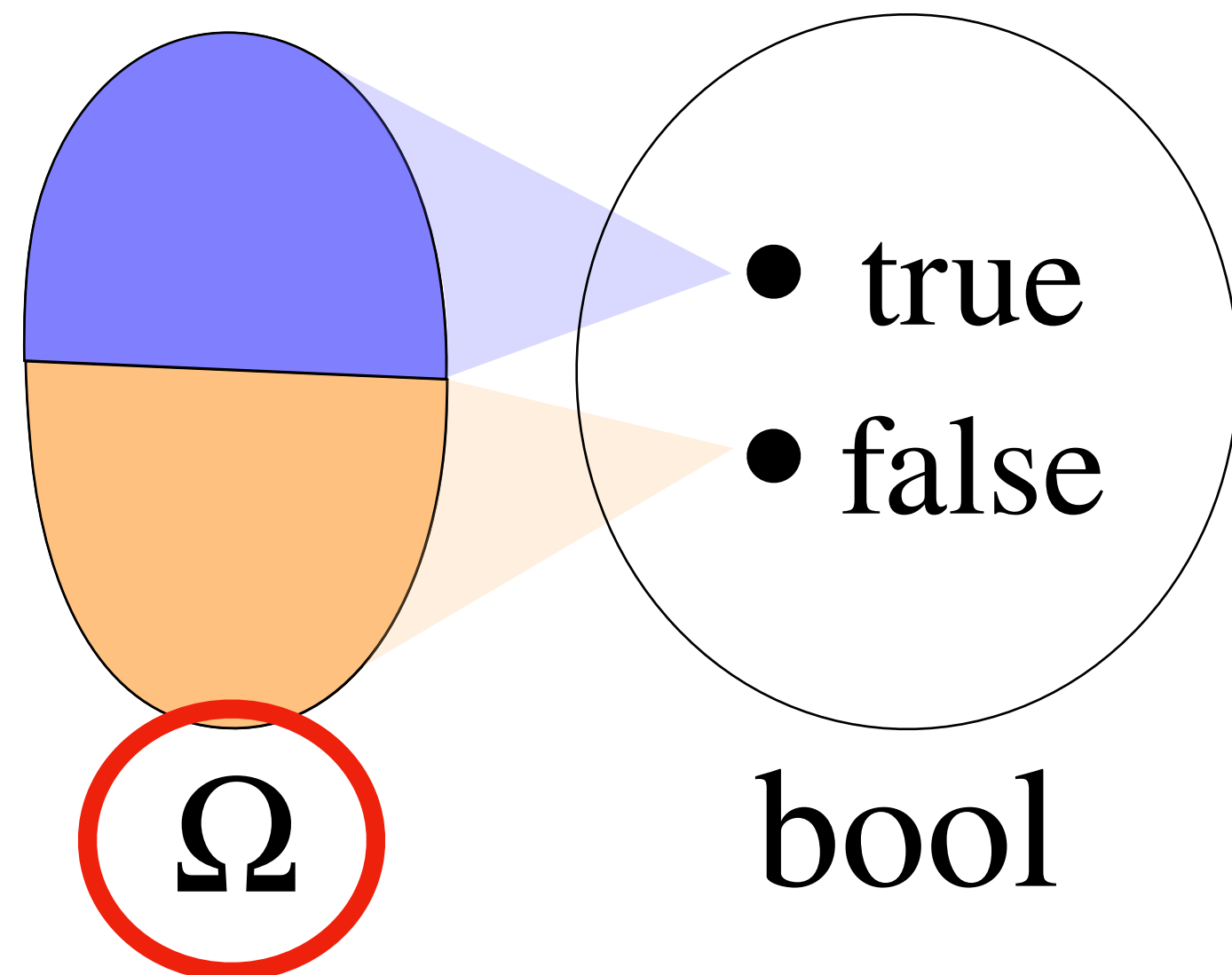
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$$\mu \left(\begin{array}{c} \text{white top half} \\ \text{orange bottom half} \end{array} \right) = 1/2$$

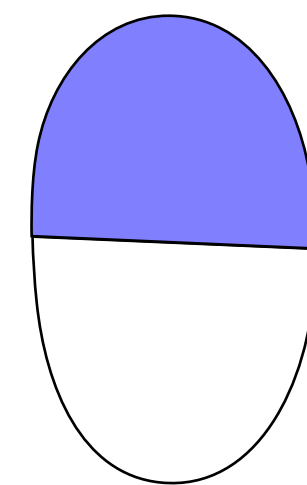
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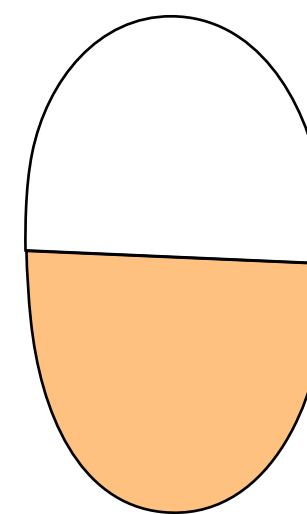


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$\in \text{events}$

$$\mu \left(\text{true event} \right) = 1/2$$



$\in \text{events}$

$$\mu \left(\text{false event} \right) = 1/2$$

Probability spaces as heaps

$X \sim \text{Ber}(1/2)$ really means...

events

μ

Ω

Probability spaces as heaps

$X \sim \text{Ber}(1/2)$ really means...

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Only accessed indirectly through X

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Probability spaces as heaps

$X \sim \text{Ber}(1/2)$ really means...

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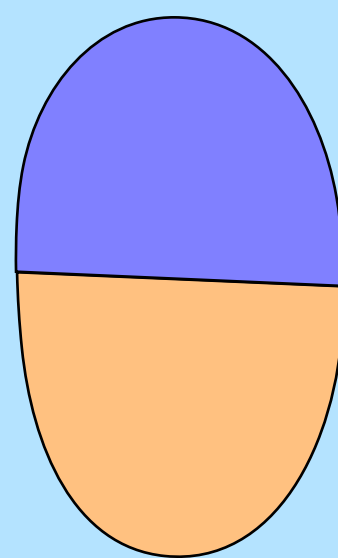
Together, form a probability space

Ω

Probability spaces as heaps

Probability theory

X



$(\Omega, \text{events}, \mu)$

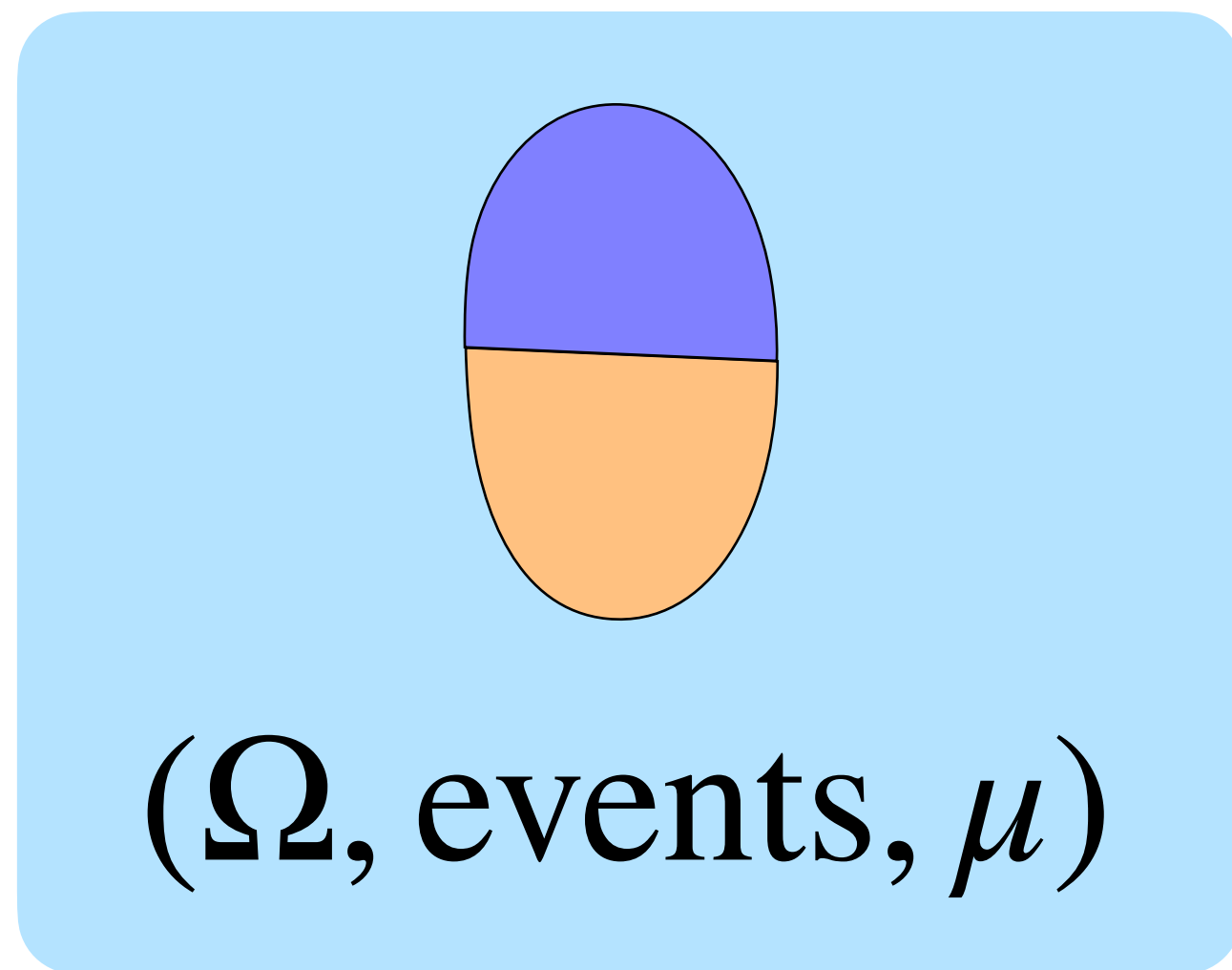
Probability spaces as heaps

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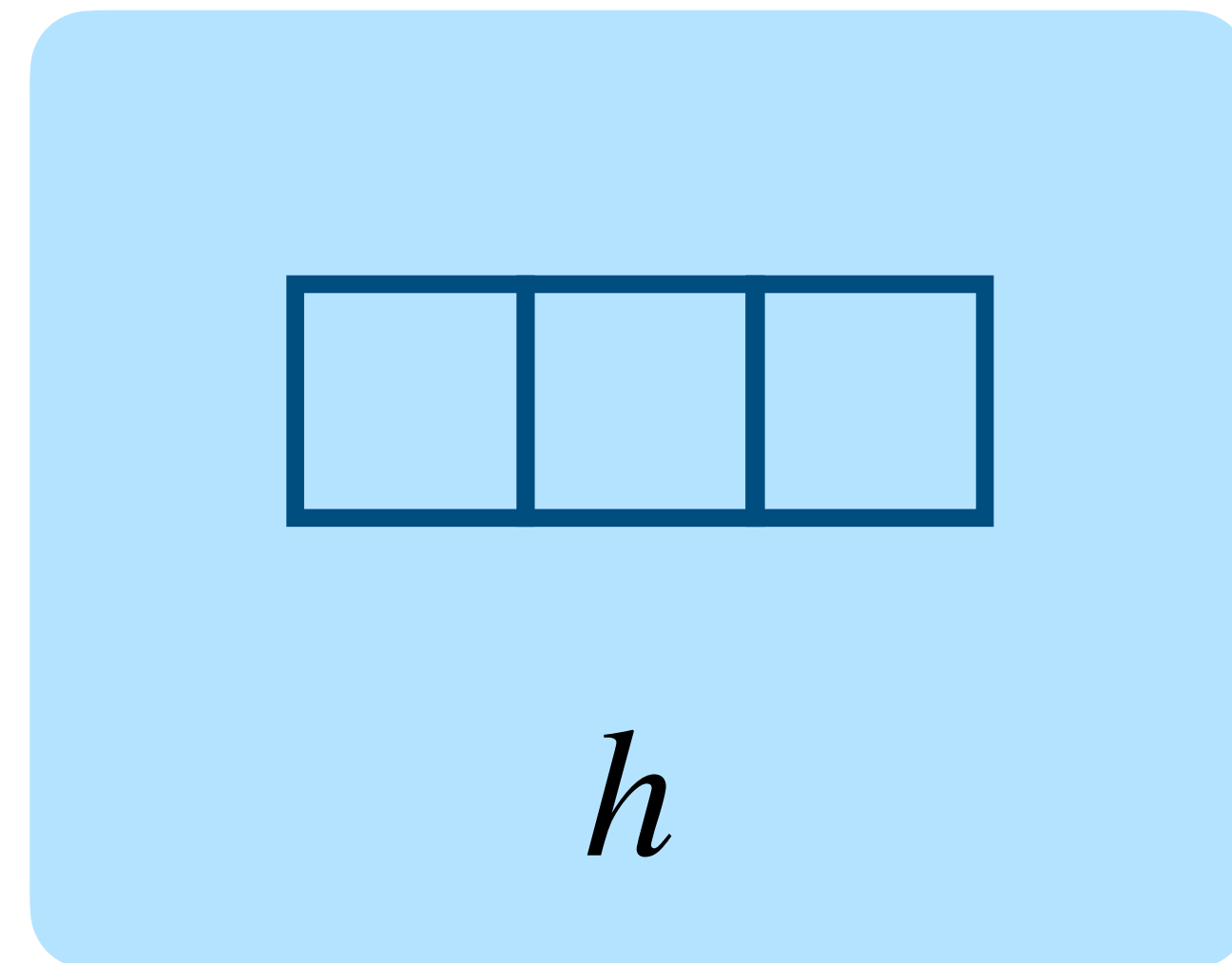
\simeq

Mutable references

X



ℓ



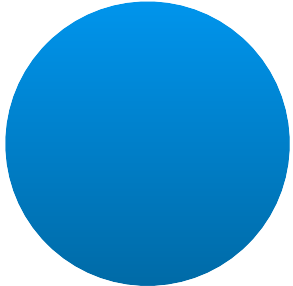
Lilac: a separation logic for probability

- Probability spaces are the heaps of probability theory.

Lilac: a separation logic for probability

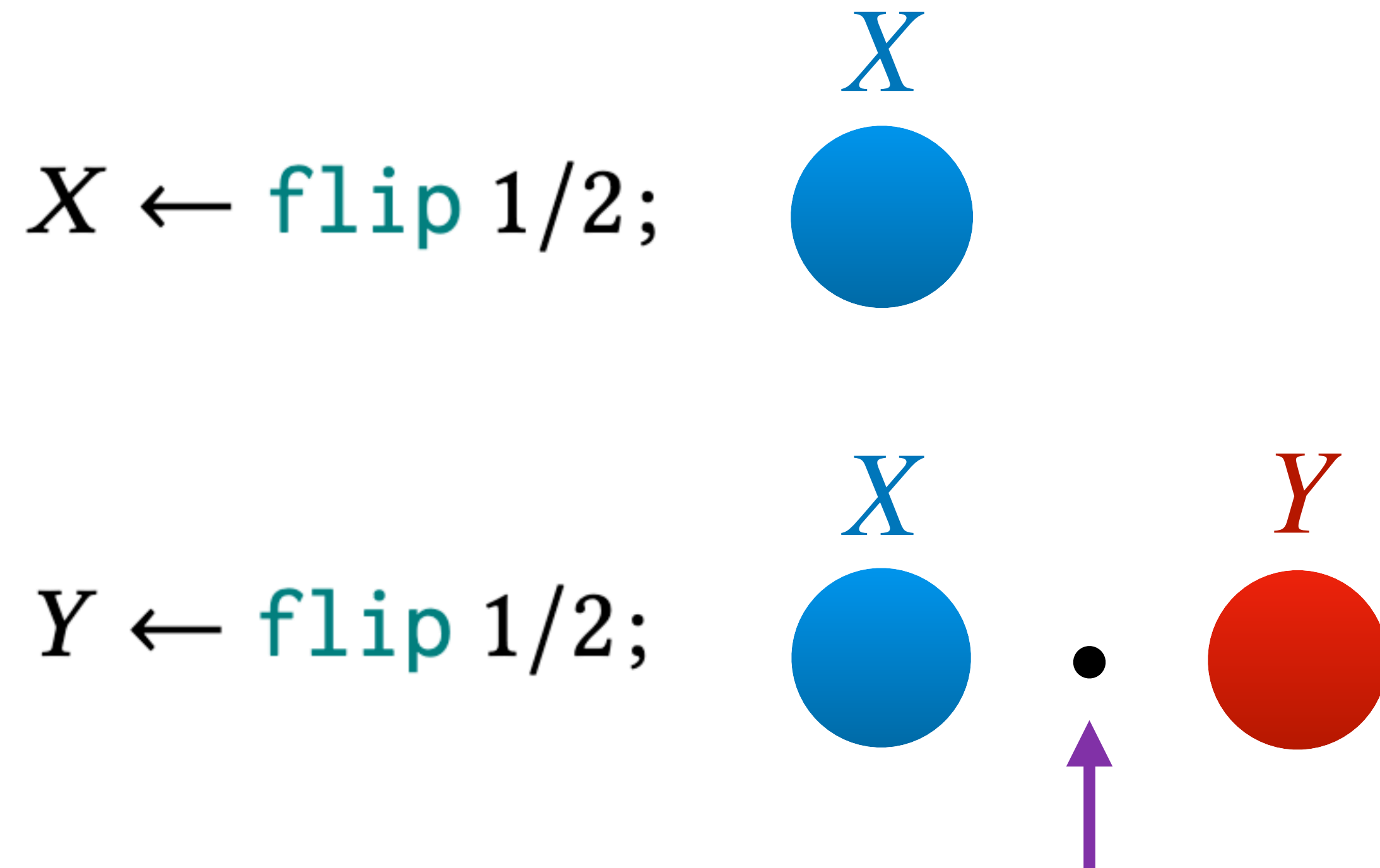
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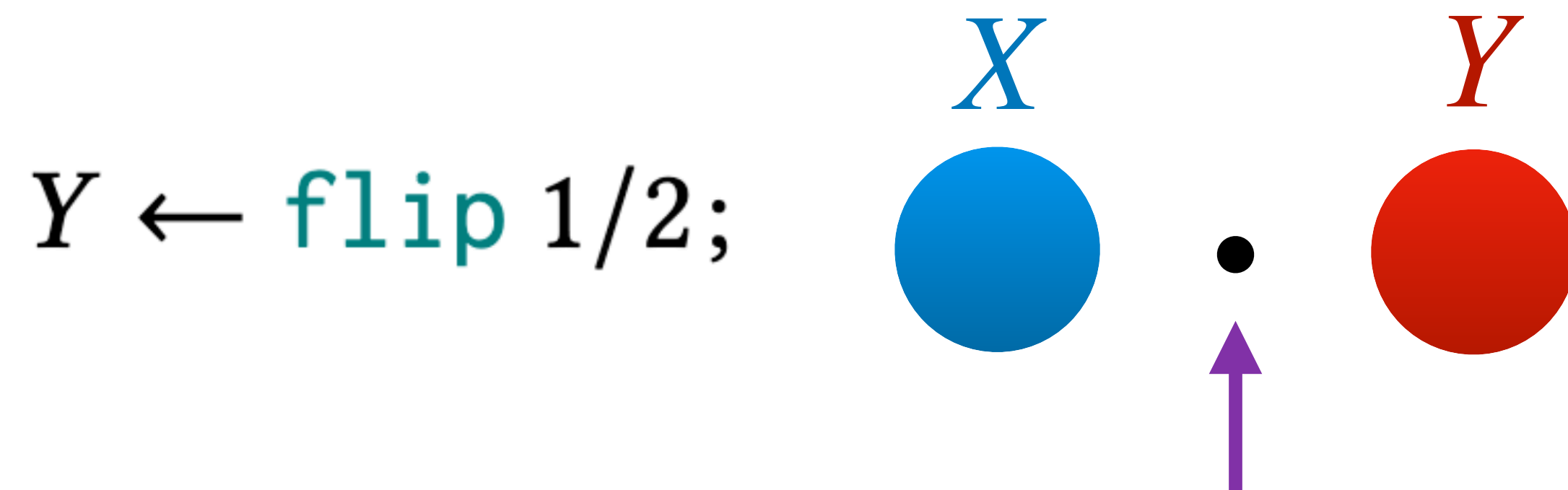


Captures independence (Lemma 2.5)

Lilac: a separation logic for probability

- Probability spaces are the heaps of probability theory.

$$X \leftarrow \text{flip } 1/2; \quad \begin{array}{c} X \\ \bullet \end{array} \quad \frac{\{P\} \ e \ \{x. Q(x)\}}{\{P * F\} \ e \ \{x. Q(x) * F\}} \quad (\text{Frame})$$

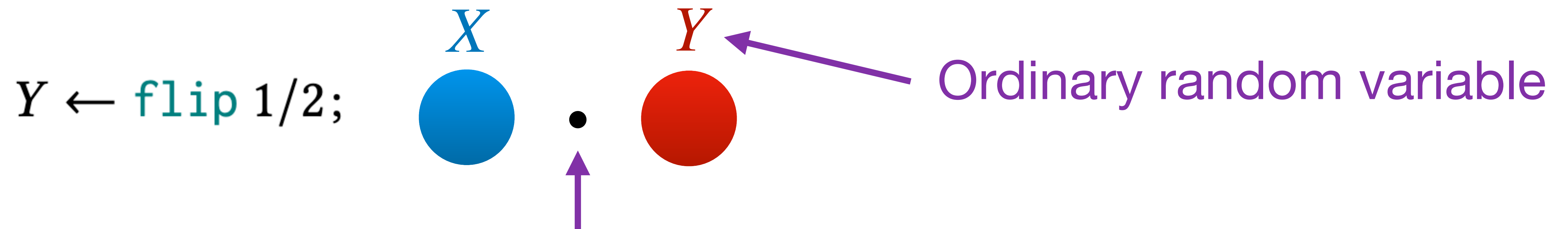


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Captures independence (Lemma 2.5)

Lilac: a modal separation logic for conditional probability

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- Conditioning as a *modality*:

Lilac: a **modal** separation logic for **conditional** probability

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$$\mathbf{C}_{x \leftarrow X} P$$

Lilac: a **modal** separation logic for **conditional** probability

- Conditioning as a *modality*:

$$\mathbf{C}_{x \leftarrow X} P$$

P holds conditional on $X = x$ for all x

Lilac: a **modal** separation logic for **conditional** probability

- Conditioning as a *modality*:

$$X \sim \text{Ber}(1/2) \quad * \quad Y \sim \text{Ber}(1/2)$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

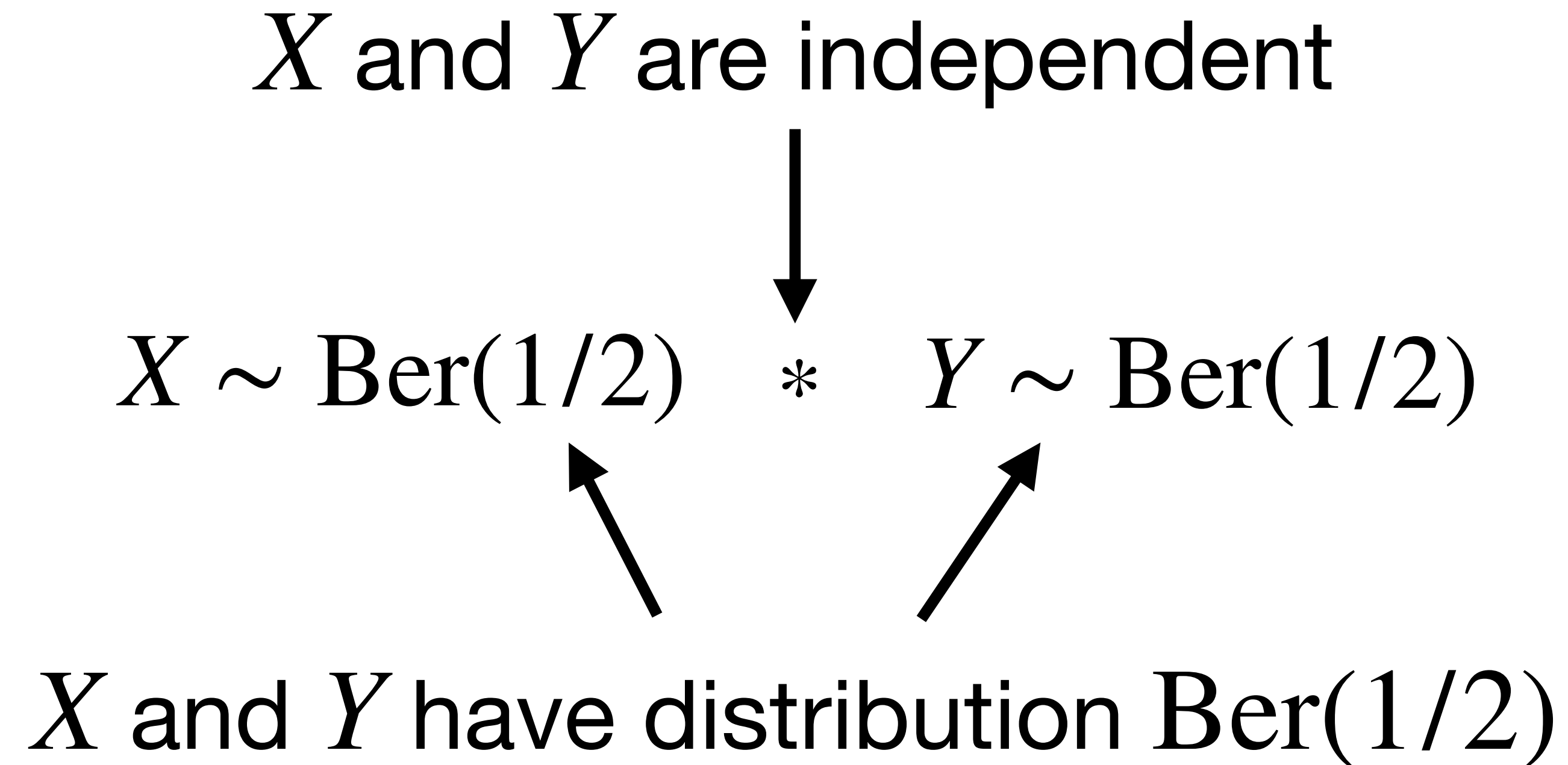
X and Y are independent



$X \sim \text{Ber}(1/2) \quad * \quad Y \sim \text{Ber}(1/2)$

Lilac: a **modal** separation logic for **conditional** probability

- Conditioning as a *modality*:



Lilac: a **modal** separation logic for **conditional** probability

- Conditioning as a *modality*:

$$\mathbf{C}_{z \leftarrow Z} \left(X \sim \text{Ber}(1/2) \quad * \quad Y \sim \text{Ber}(1/2) \right)$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

X and Y are conditionally independent given Z

$$\underset{z \leftarrow Z}{\mathbf{C}} \left(X \sim \text{Ber}(1/2) \quad \downarrow \quad * \quad Y \sim \text{Ber}(1/2) \right)$$

Lilac: a **modal** separation logic for **conditional** probability

- Conditioning as a *modality*:

X and Y are **conditionally** independent **given** Z

$$\mathbf{C}_{z \leftarrow Z} \left(X \sim \text{Ber}(1/2) * Y \sim \text{Ber}(1/2) \right)$$

X and Y have **conditional** distribution $\text{Ber}(1/2)$ **given** Z

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$\Pr[E] = 1/2$ E has probability $1/2$

$\mathbf{E}[X] = 0$ X has expectation 0

Lilac: a **modal** separation logic for **conditional** probability

- Conditioning as a *modality*:

$\mathbf{C}_{x \leftarrow X} \left(\text{Pr}[E] = 1/2 \right)$ E has probability $1/2$ given $X = x$

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Lilac: a **modal** separation logic for **conditional** probability

- Conditioning as a *modality*:

$\mathbf{C}_{x \leftarrow X} \left(\text{Pr}[E] = 1/2 \right)$ E has probability $1/2$ given $X = x$

$\mathbf{C}_{y \leftarrow Y} \left(\mathbf{E}[X] = 0 \right)$ X has conditional expectation 0

Lilac: a **modal** separation logic for **conditional** probability

- Conditioning as a *modality*
- Laws express intuitive facts and standard theorems:

Lilac: a **modal** separation logic for **conditional** probability

- Conditioning as a *modality*
- Laws express intuitive facts and standard theorems:

C-TOTAL-EXPECTATION

$$\mathbf{C}_{x \leftarrow X} \left(\mathbb{E}[E] = e \right) \wedge \mathbb{E}[e[X/x]] = v \vdash \mathbb{E}[E] = v$$

We used Lilac to verify

- Examples from prior work (cryptographic protocols)
- A tricky weighted sampling algorithm exercising
 - Continuous random variables
 - Quantitative reasoning
 - Separation as independence
 - Conditioning modality

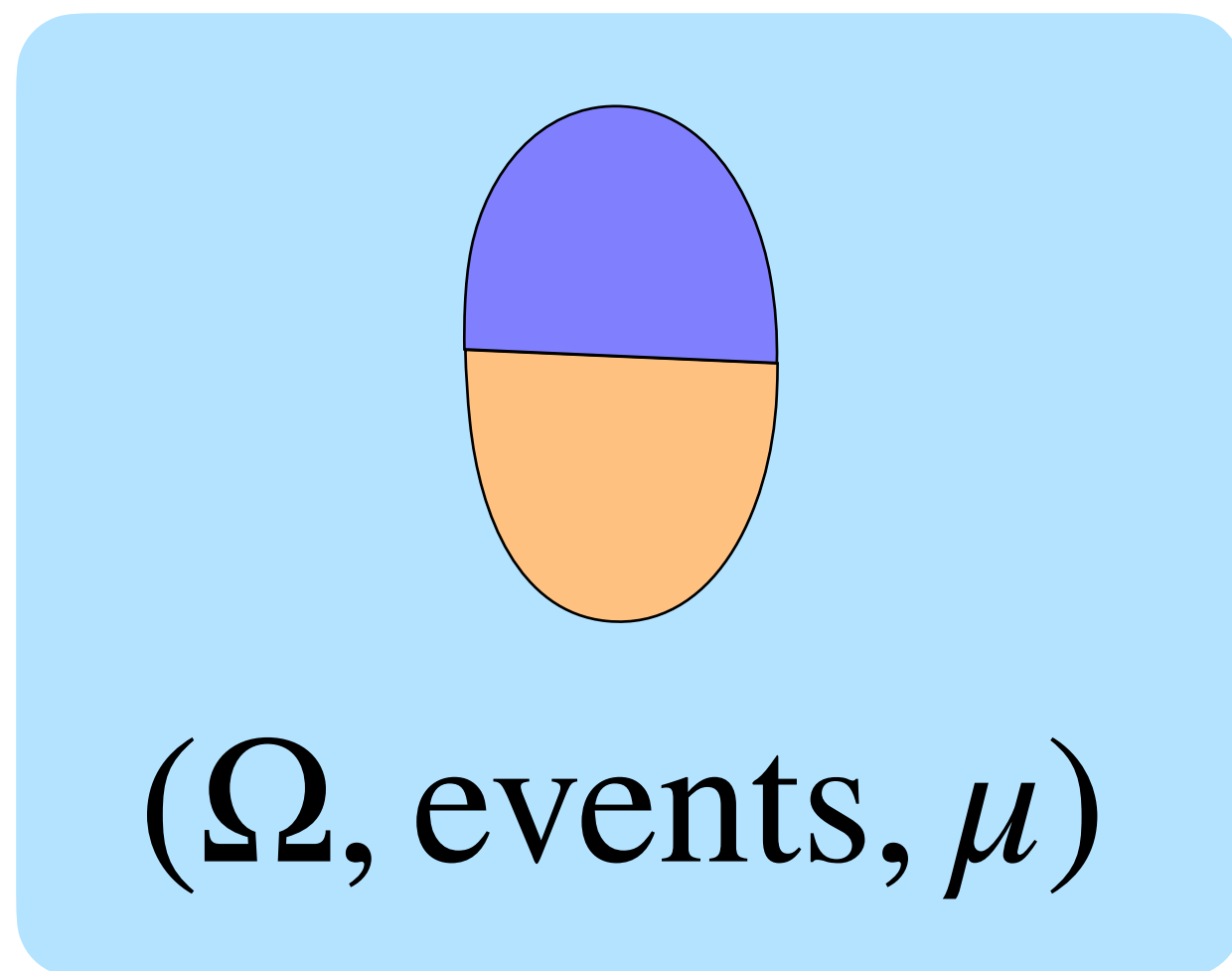
Also in the paper

- Conditioning modality
- Ownership is measurability
- Worked examples
- Almost-sure equality $X =_{\text{a.s.}} Y$

Thanks!

Probability theory

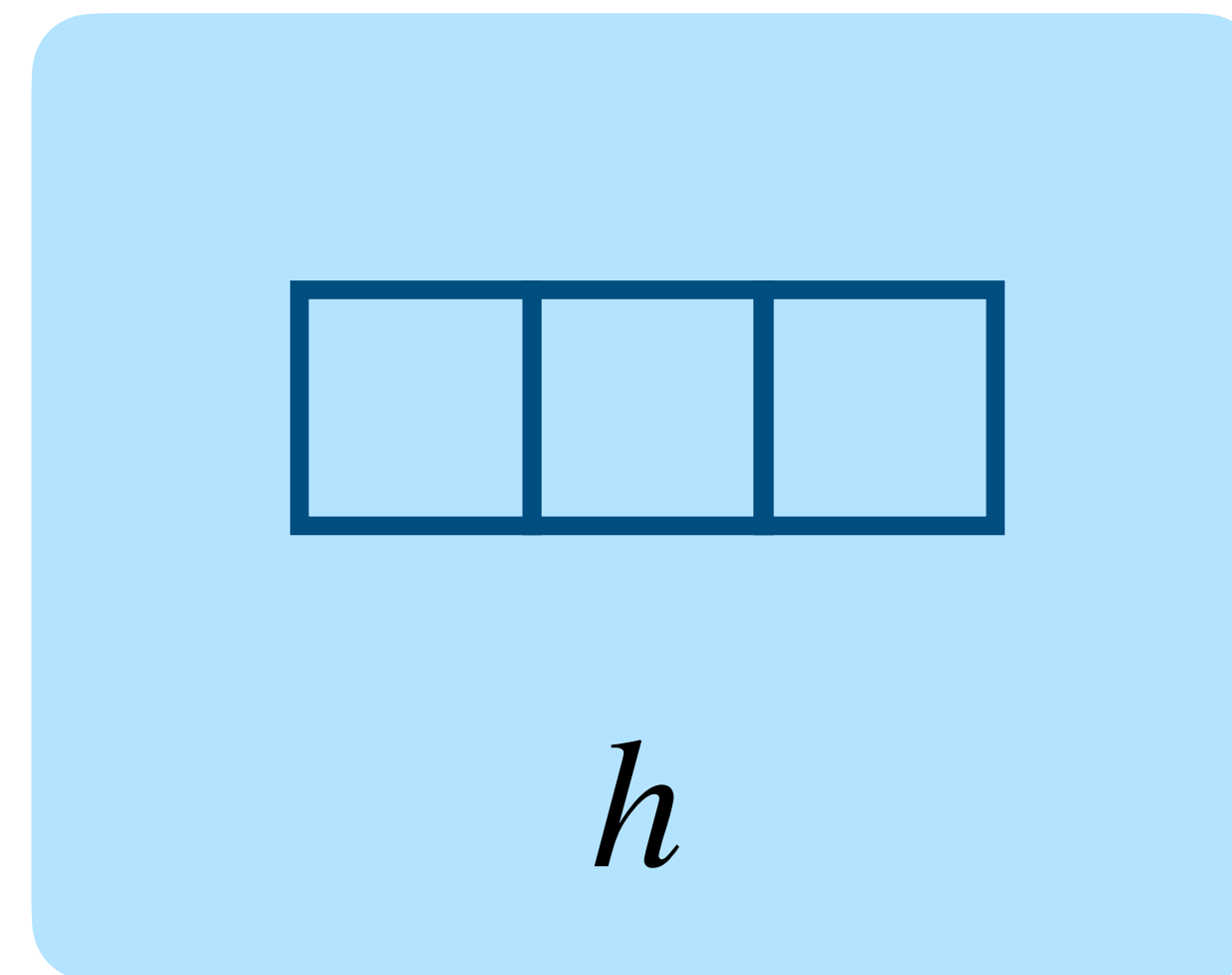
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Mutable references

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<https://johnm.li/lilac.pdf>