Lilac: A Modal Separation Logic for Conditional Probability

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https://johnm.li/lilac.pdf

How to reason about complex probabilistic systems?

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Is my car safe?

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Is this decision fair?

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Is my result significant?

How to reason about complex probabilistic systems?

- Reasoning should be *modular*:

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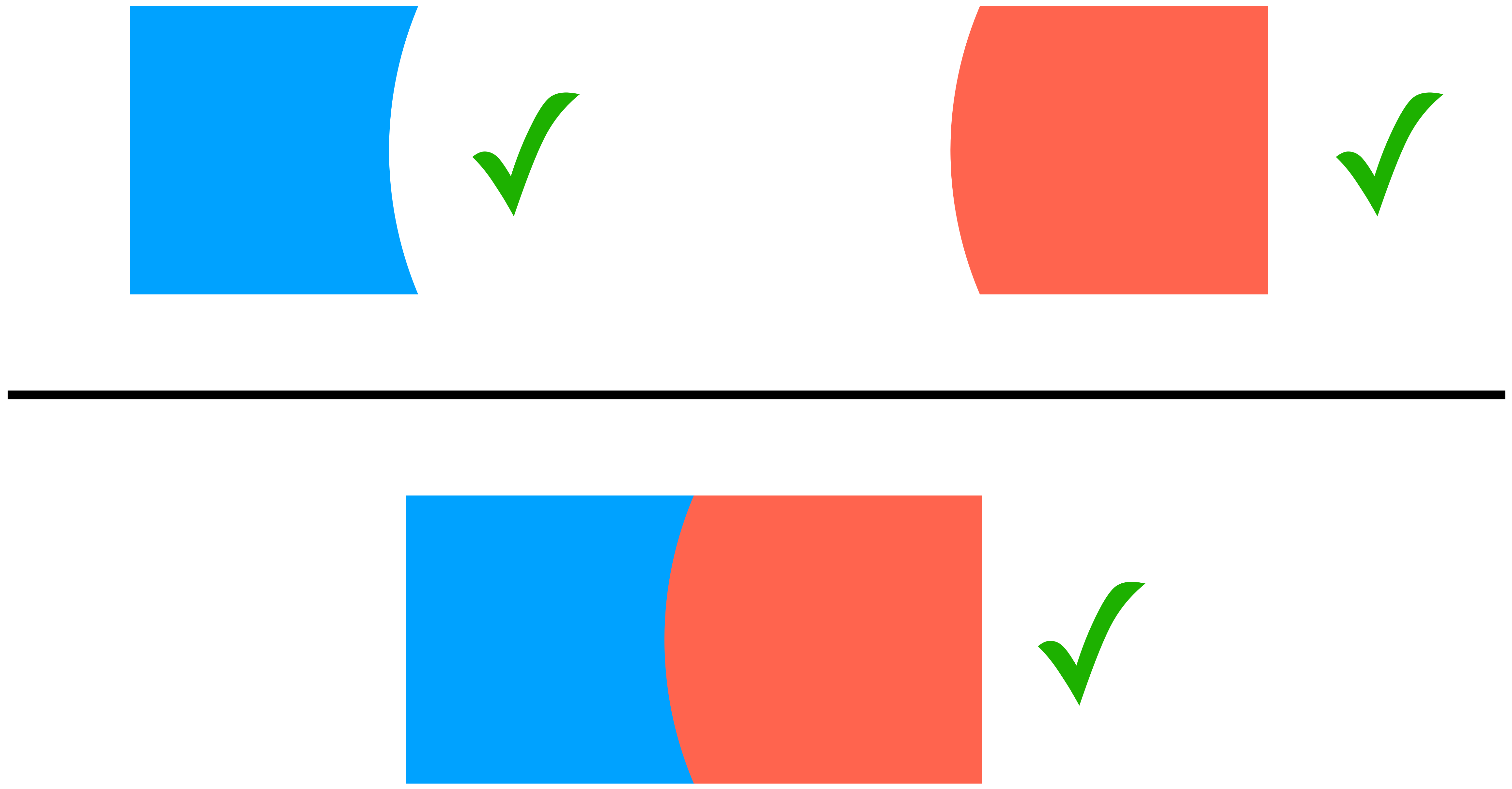

Modularity comes from probabilistic independence

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$$\texttt{weights}[0], \ldots, \texttt{weights}[999] \sim \text{Unif}[0,1] \text{ mutually independent}$$

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Modularity comes from probabilistic independence

- Independence arises frequently and naturally:

if each `data[i]` is an independent estimate of v...

$$\texttt{result} = \texttt{np.mean(data)}$$

...then `result` is a more accurate estimate of v

Modularity comes from probabilistic independence

- Independence arises frequently and naturally.
- Idea: capture independence using *separation logic*

Ordinary separation logic is about disjointness

$$x = \mathsf{new}\ 0;$$

$$y = \mathsf{new}\ 1;$$

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x and y point to disjoint heap locations

Ordinary separation logic is about disjointness

$$\frac{\{P\}\ e\ \{x\,.\,Q(x)\}}{\{P * F\}\ e\ \{x\,.\,Q(x) * F\}}\ \text{(Frame)}$$

Ordinary separation logic is about disjointness

When verifying e...

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- This has enabled modular heap-based reasoning at scale.[1]

[1] C. Calcagno and D. Distefano. Infer: An automatic program verifier for memory safety of C programs. NFM 2011.

Lilac's separation is about independence

$$X \leftarrow \texttt{flip}\ 1/2;$$

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X and Y are independent random variables

Wait, hasn't this been done before?

Wait, hasn't this been done before?

A Probabilistic Separation Logic

GILLES BARTHE, MPI for Security and Privacy, Germany and IMDEA Software Institute, Spain
JUSTIN HSU, University of Wisconsin–Madison, USA
KEVIN LIAO, MPI for Security and Privacy, Germany and University of Illinois Urbana-Champaign, USA

POPL'20

Wait, hasn't this been done before?

A Bunched Logic for Conditional Independence

Jialu Bao
University of Wisconsin–Madison

Simon Docherty
University College London

Justin Hsu
University of Wisconsin–Madison

Alexandra Silva
University College London

LICS'21

A Separation Logic for Negative Dependence

JIALU BAO, Cornell University, USA
MARCO GABOARDI, Boston University, USA
JUSTIN HSU, Cornell University, USA
JOSEPH TASSAROTTI, Boston College, USA

POPL'22

New in Lilac

New in Lilac: a simple frame rule

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- PSL's frame rule:

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- PSL's frame rule:

$$\frac{\vdash \{\phi\}\ c\ \{\psi\} \qquad FV(\eta) \cap MV(c) = \emptyset \\ FV(\psi) \subseteq T \cup RV(c) \cup WV(c) \qquad \models \phi \to \mathbf{D}[T \cup RV(c)]}{\vdash \{\phi * \eta\}\ c\ \{\psi * \eta\}}$$

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extra side conditions

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Completely captures independence (Lemma 2.5)

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$$(\texttt{weights}[0] \sim \mathrm{Unif}[0,1]) \;\ast\; \cdots \;\ast\; (\texttt{weights}[999] \sim \mathrm{Unif}[0,1])$$

Inexpressible in PSL

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$$\texttt{result} = \texttt{np.mean}(\texttt{data})$$

...then `result` is a more accurate estimate of v

New in Lilac: richer assertion language

if each $\mathtt{data[i]}$ independent
and for all i we have $\mathbb{E}[\mathtt{data[i]}] = v$ and $\mathrm{Var}(\mathtt{data[i]}) \leq \varepsilon$...

$$\mathtt{result} = \mathtt{np.mean}(\mathtt{data})$$

...then $\mathbb{E}[\mathtt{result}] = v$ and $\mathrm{Var}(\mathtt{result}) \leq \dfrac{\varepsilon}{|\mathtt{data}|}$

New in Lilac: richer assertion language

$$\text{if} \quad \mathop{\circledast}_{0 \le i < |\texttt{data}|} \left(\mathbb{E}[\texttt{data[i]}] = v \text{ and } \mathrm{Var}(\texttt{data[i]}) \le \varepsilon \right) \ldots$$

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New in Lilac: good interop with normal math

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An ordinary random variable

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Ordinary expectation and variance

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Ordinary comparison of real numbers

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$\Longrightarrow$ textbook proofs remain textbook

Key idea

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- Probability spaces are the heaps of probability theory.

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$\boxed{0}$

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$$X \leftarrow \texttt{flip}\ 1/2;$$

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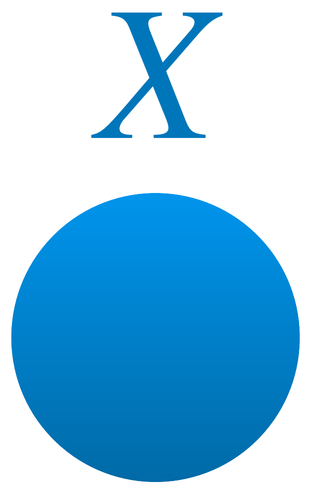

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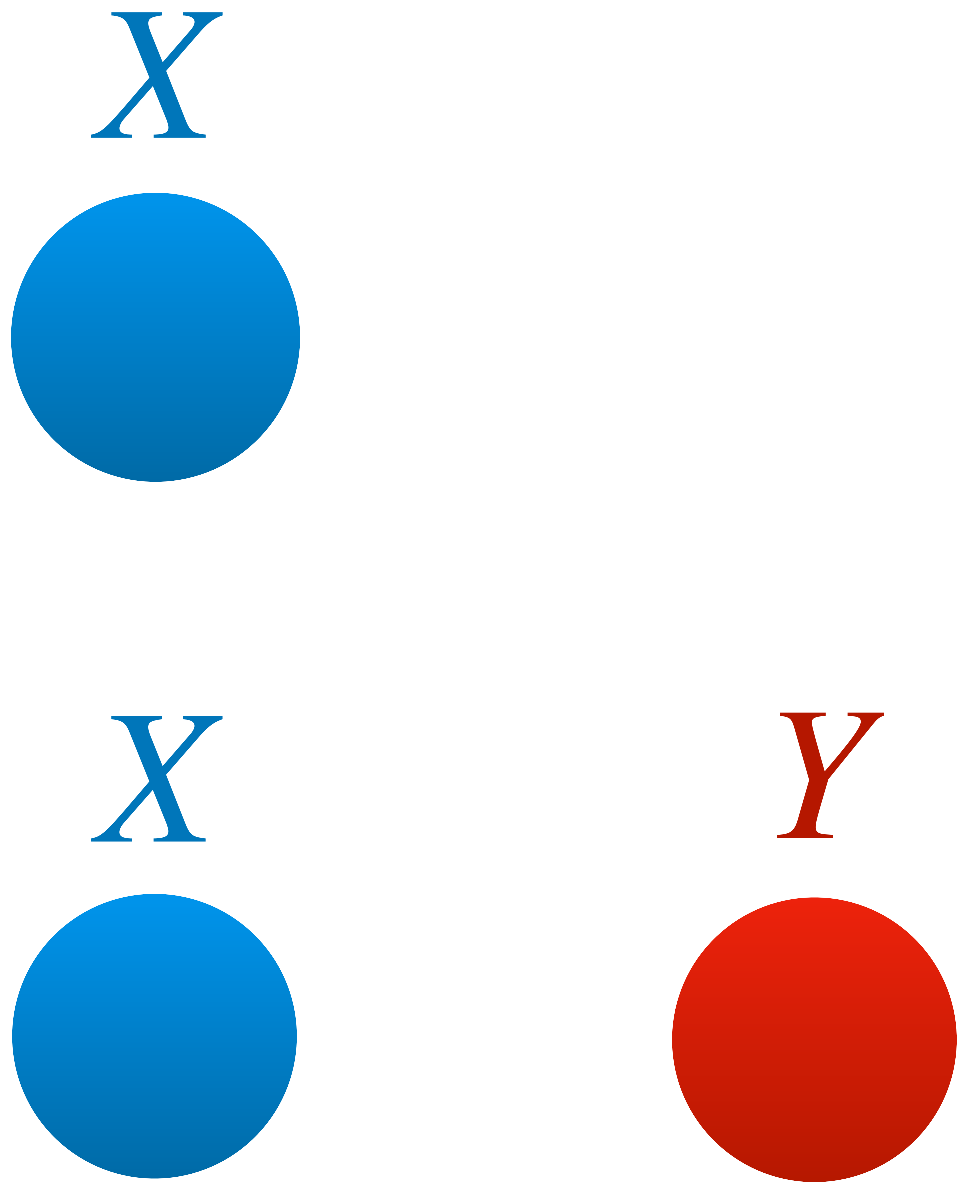

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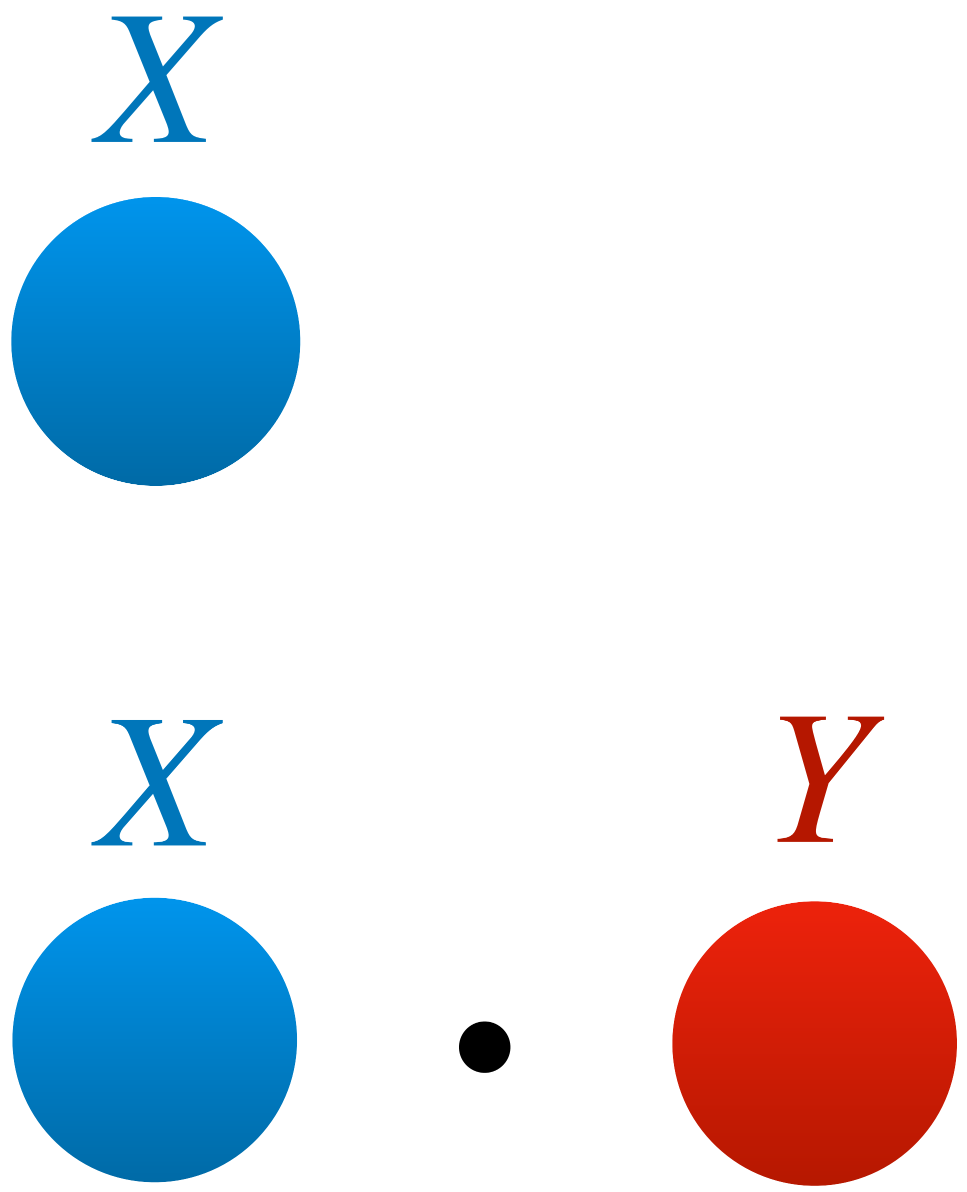

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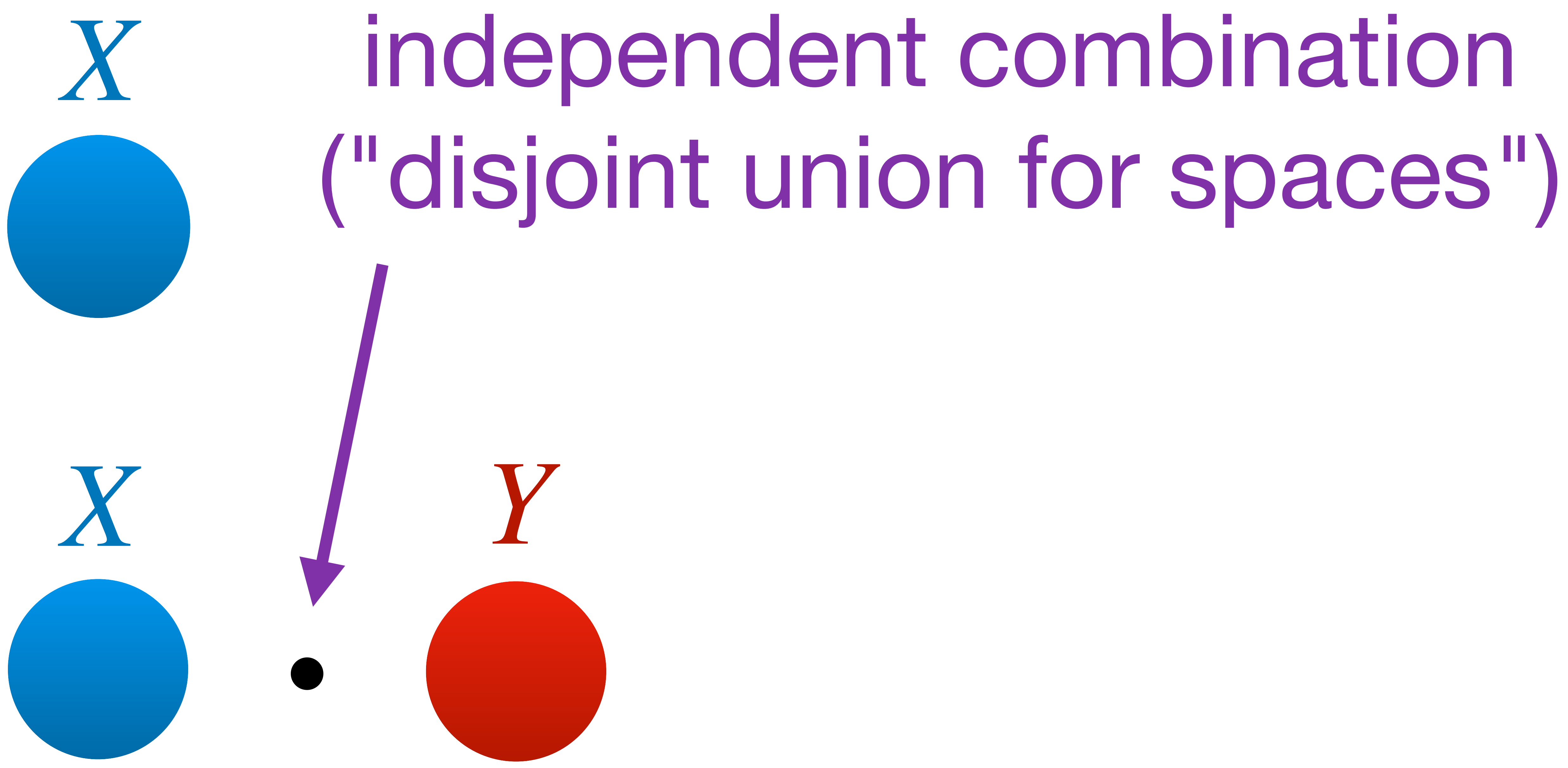

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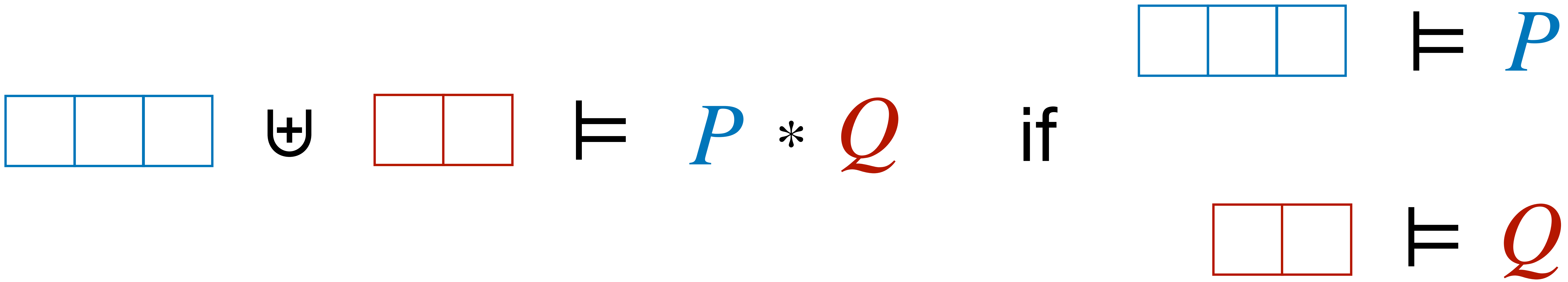

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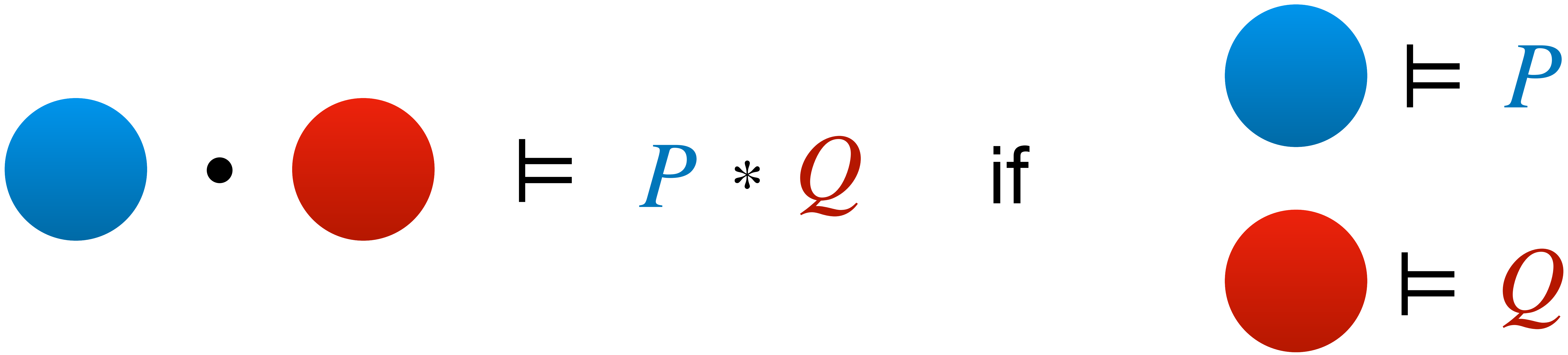

Probability spaces as heaps

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This hides a lot of machinery...

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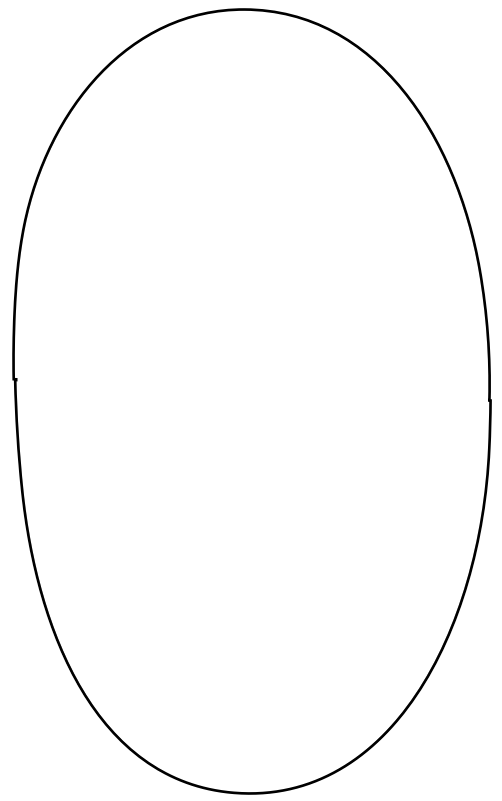

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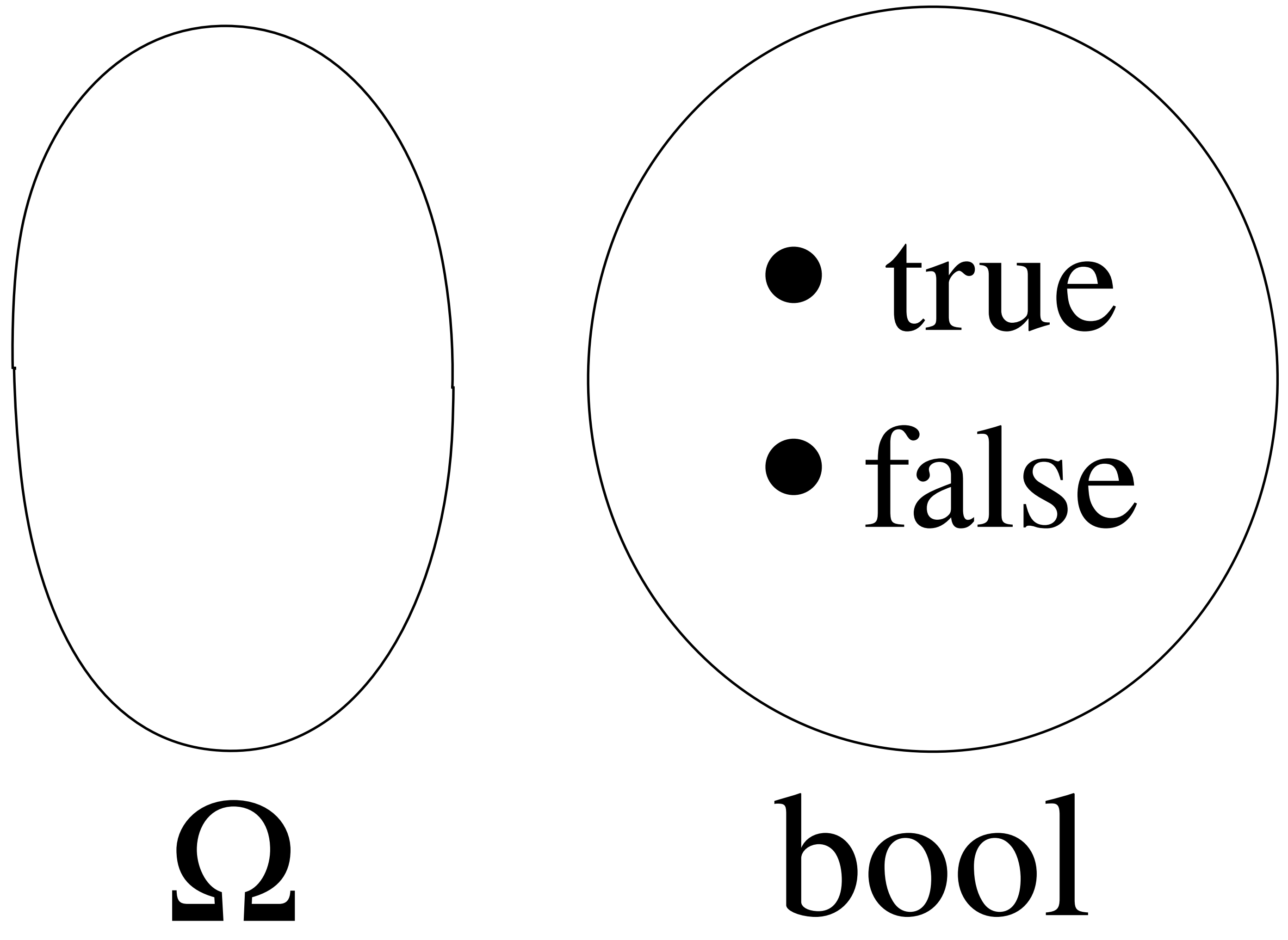
$X \sim \mathrm{Ber}(1/2)$ really means...

$$X : \Omega \to \mathrm{bool}$$



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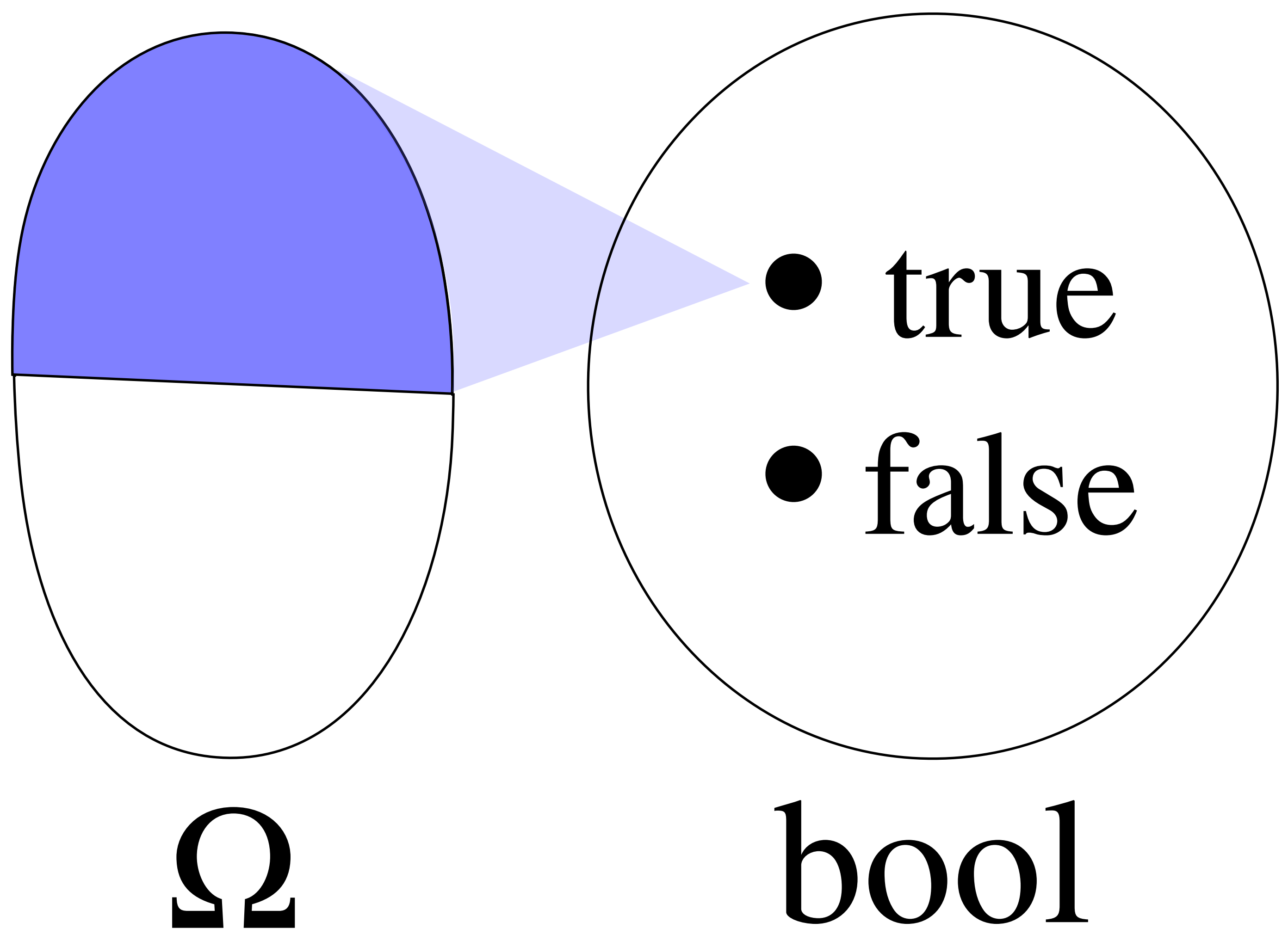

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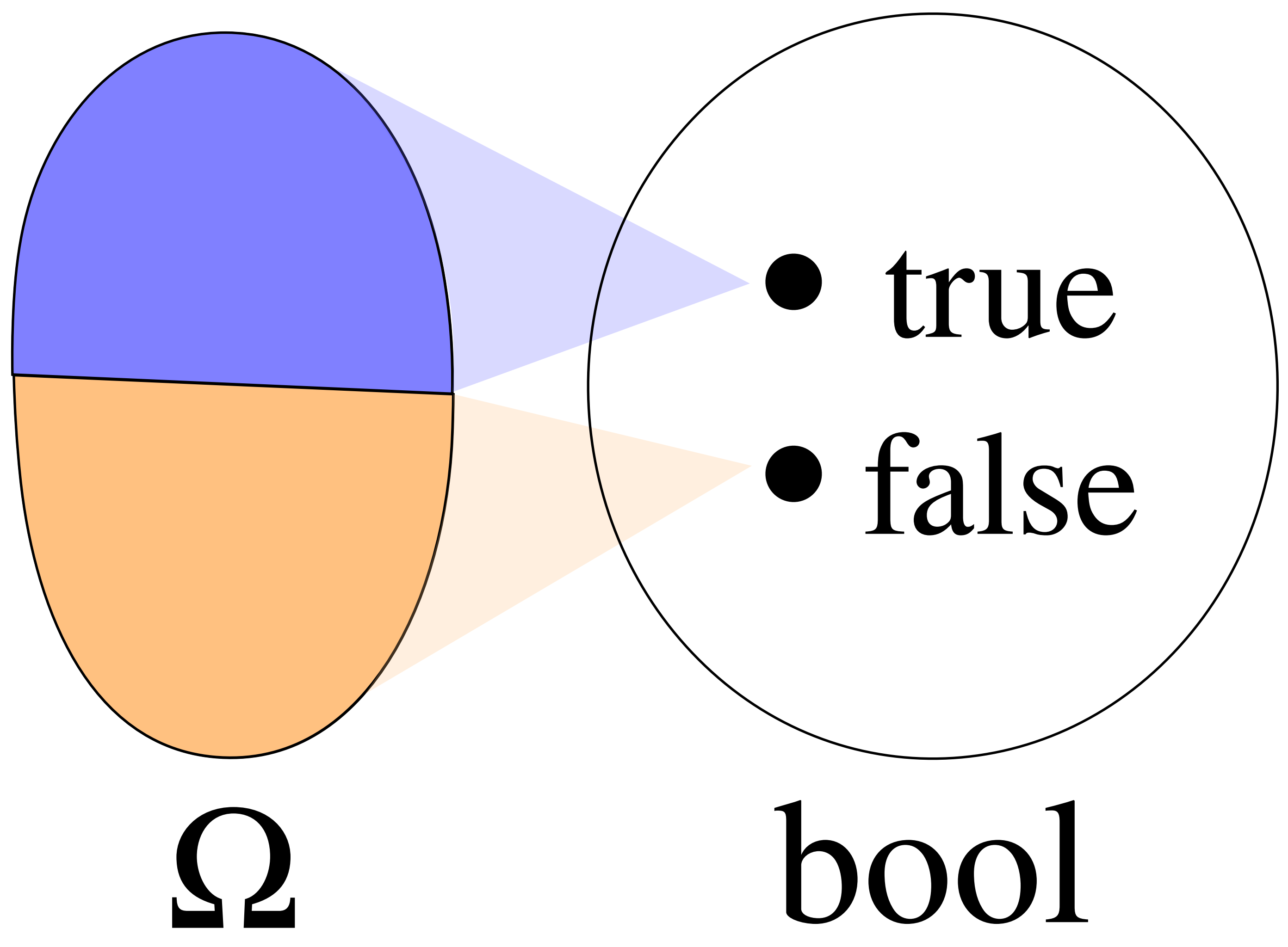

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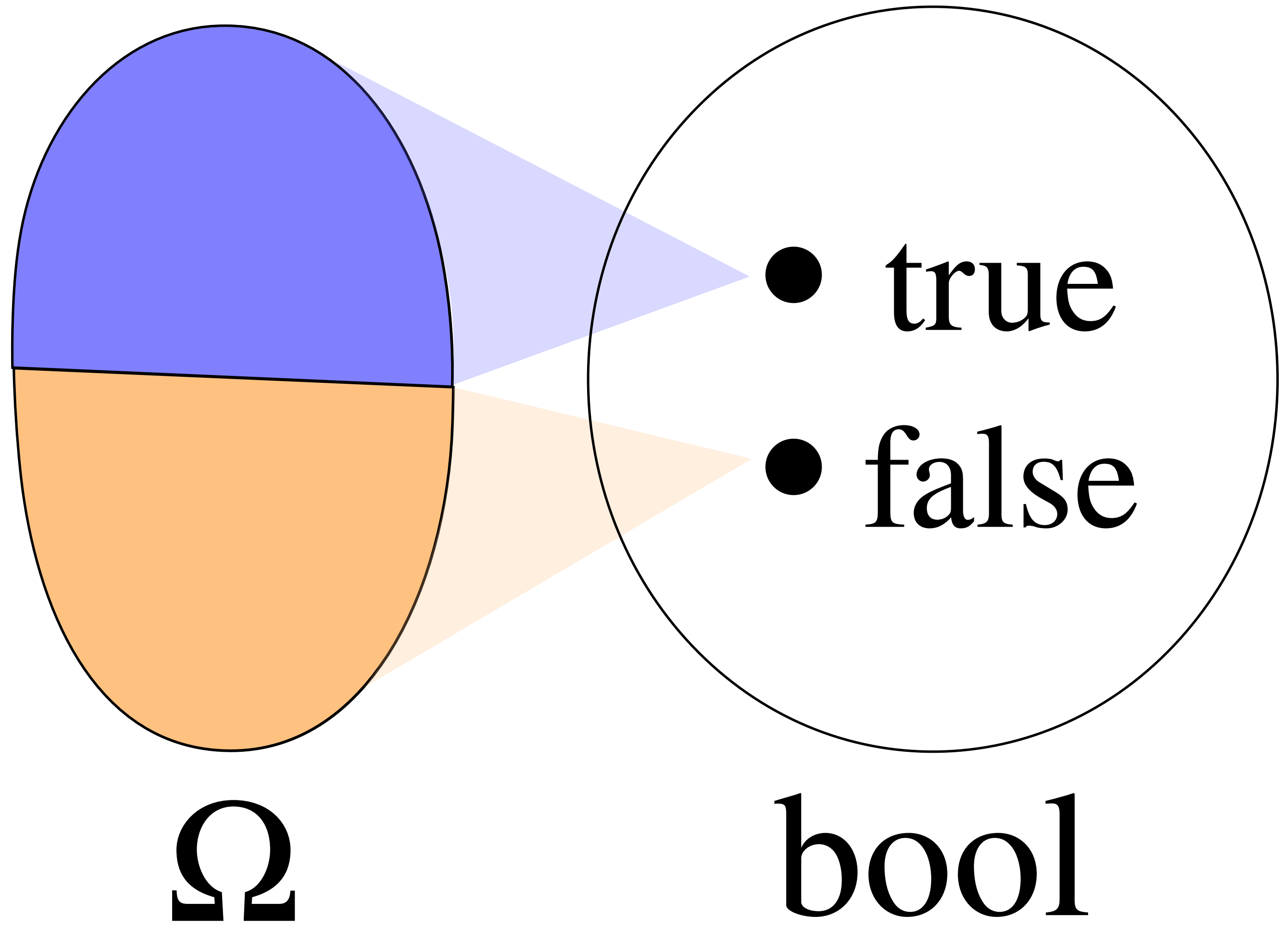

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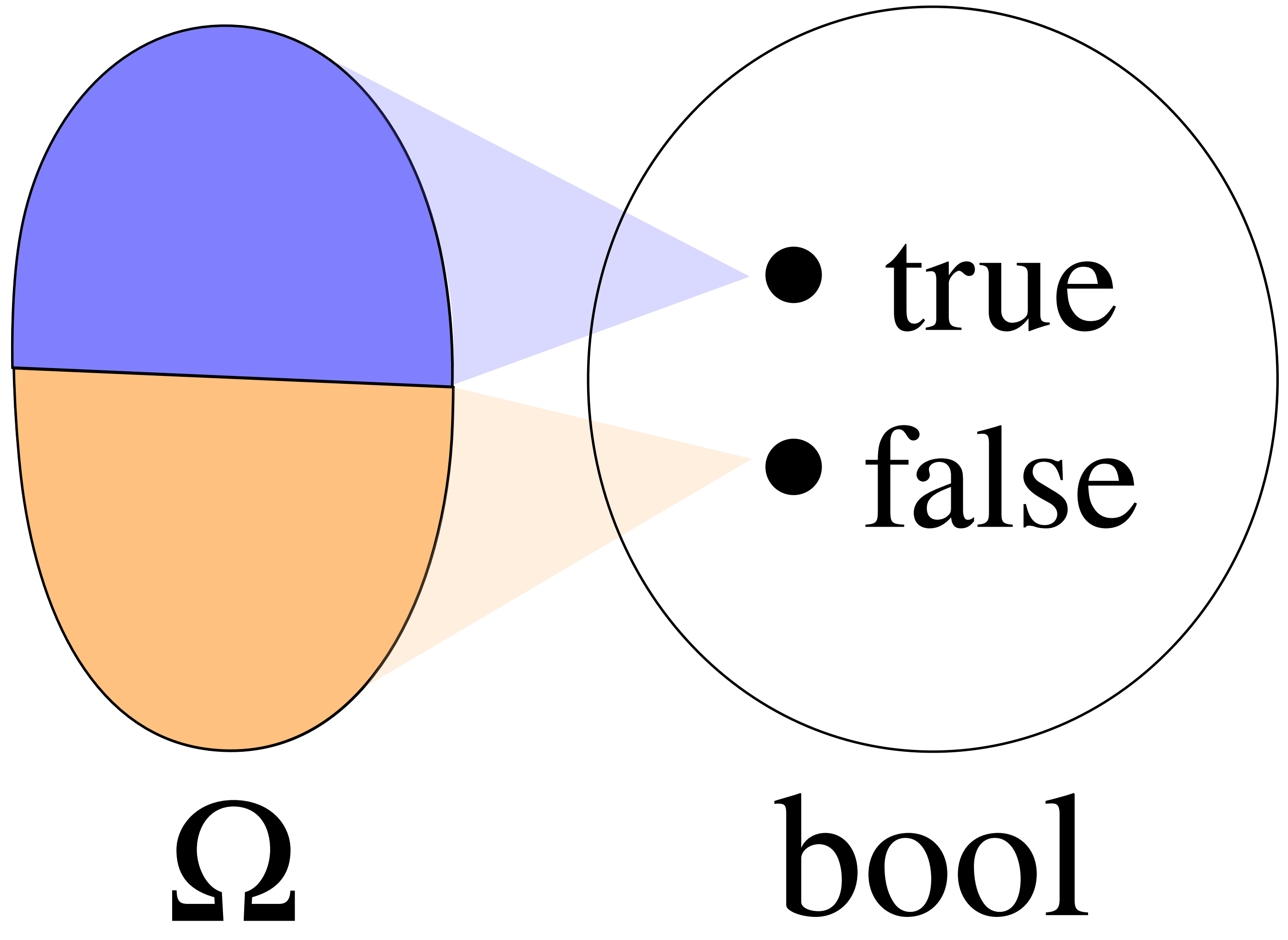

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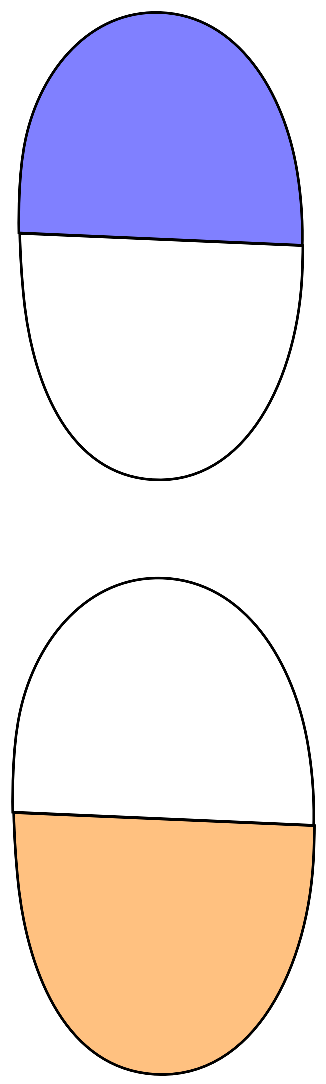

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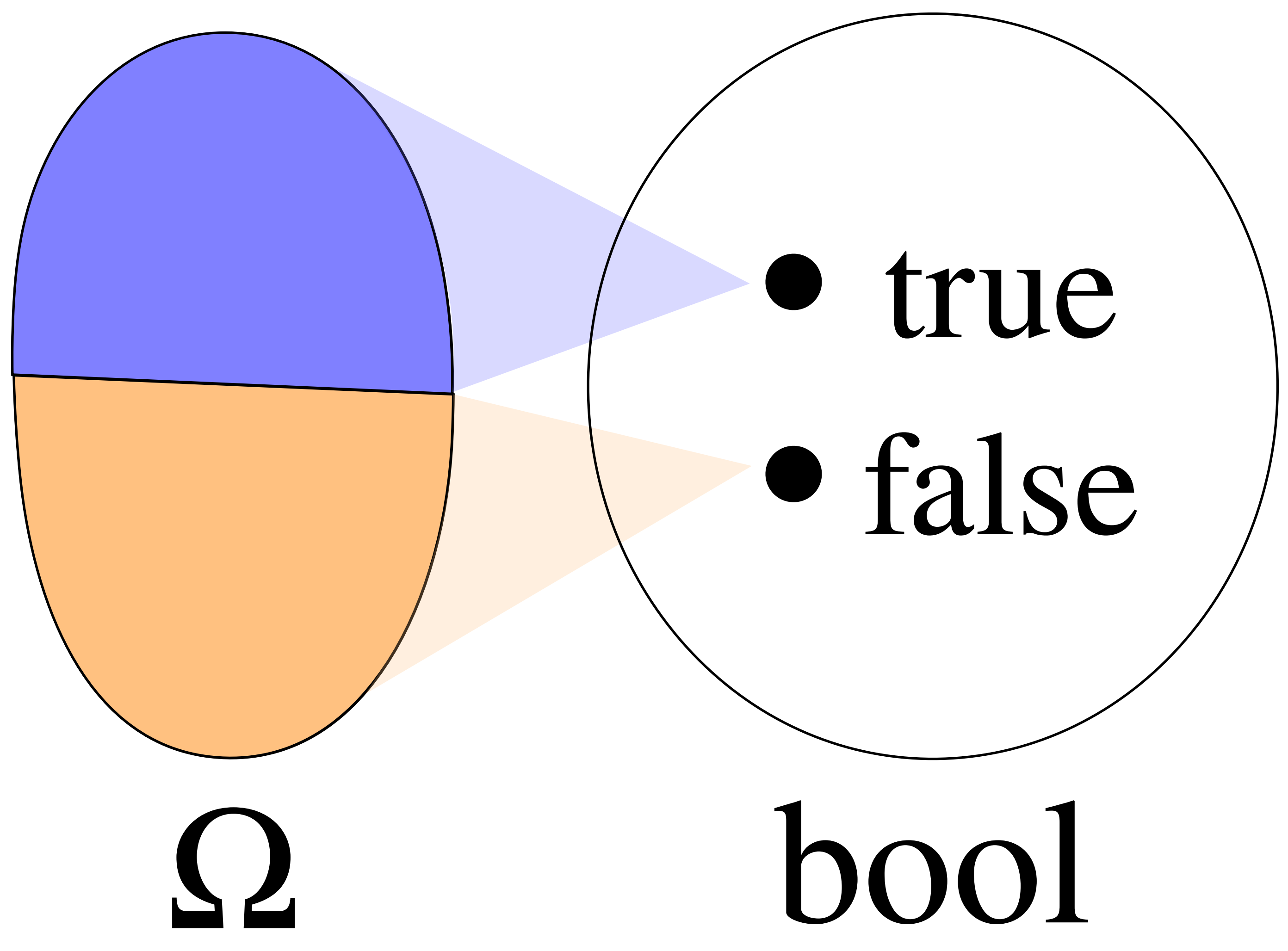

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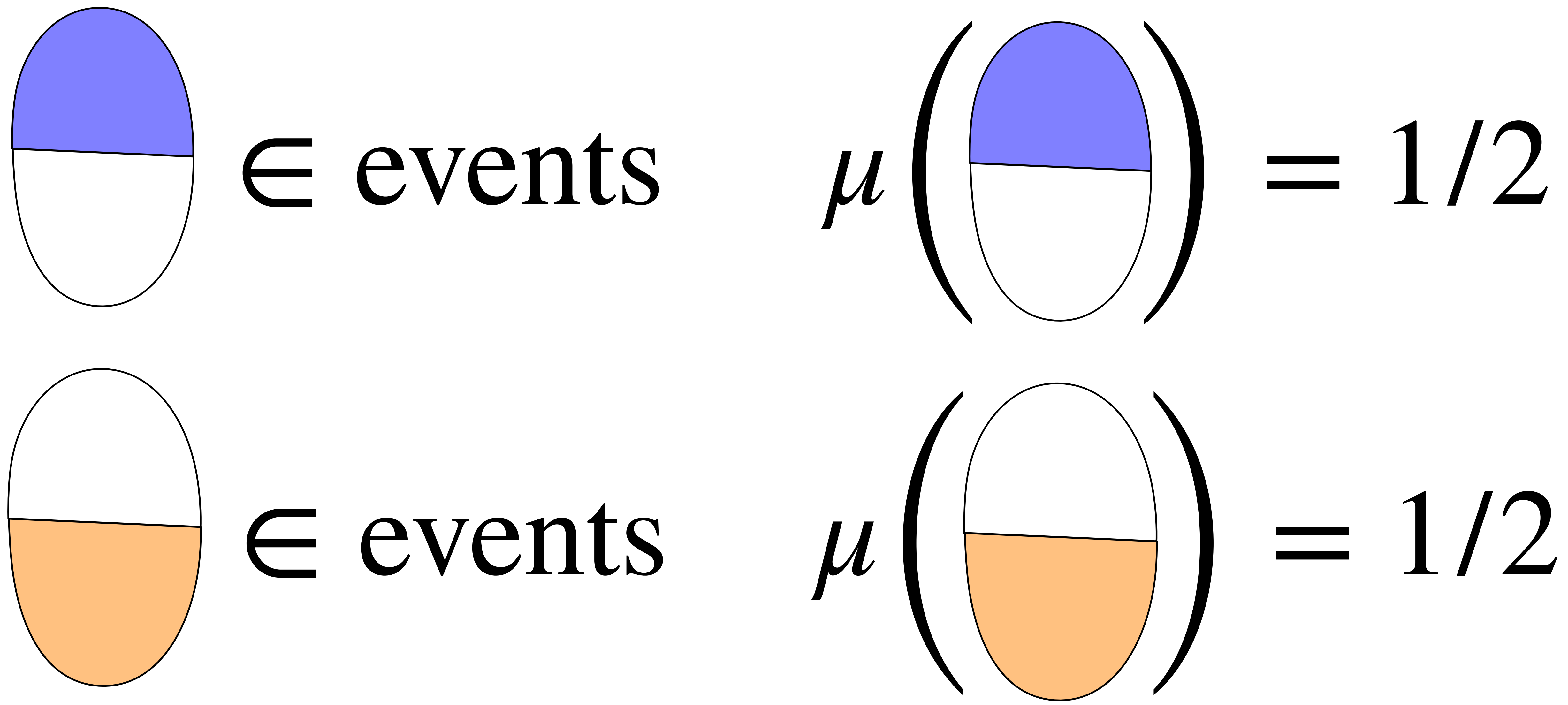

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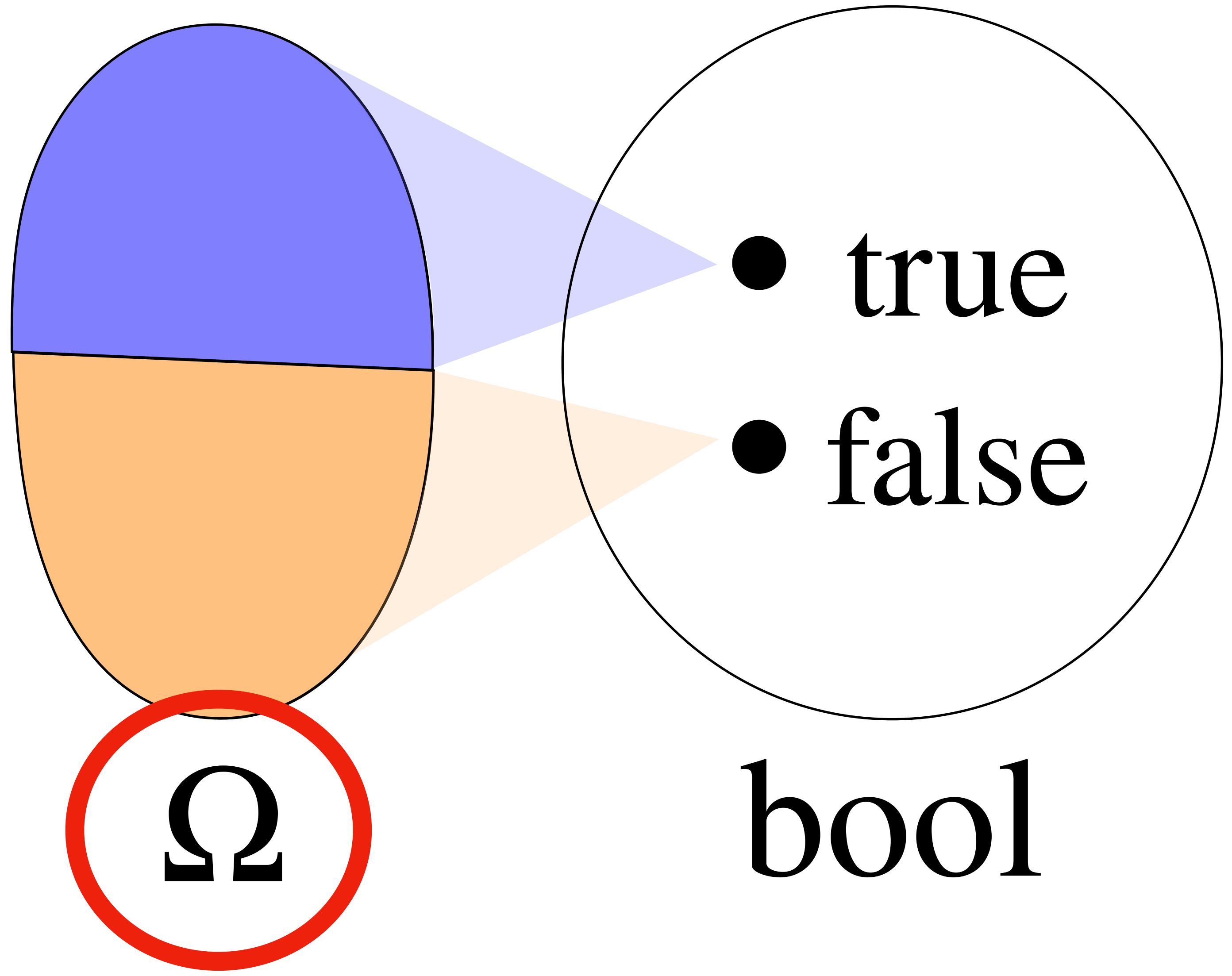

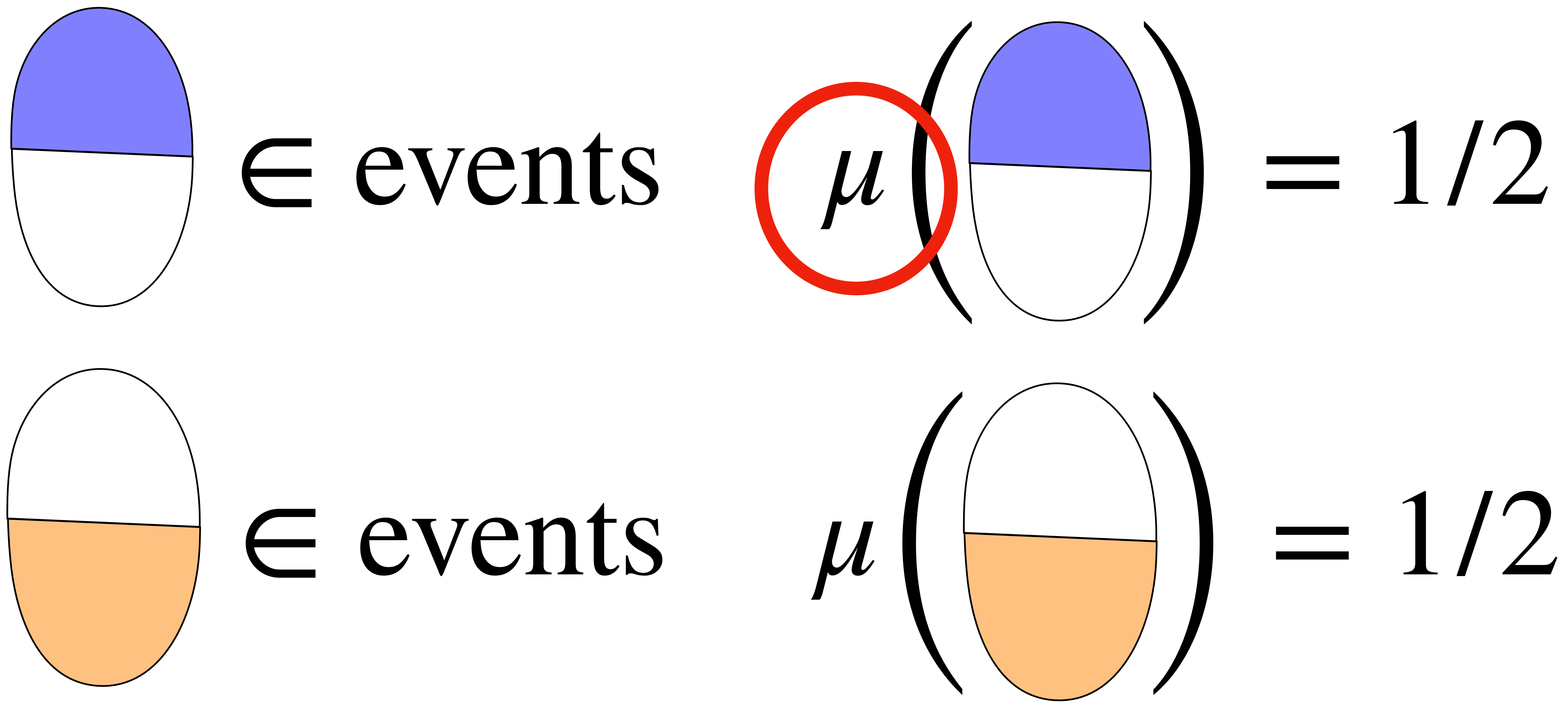

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Only accessed indirectly, through X

μ

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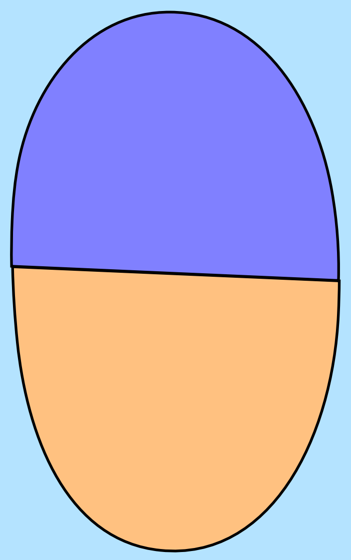
Kind of like a resource...

Ω

Probability spaces as heaps

Probability theory

X

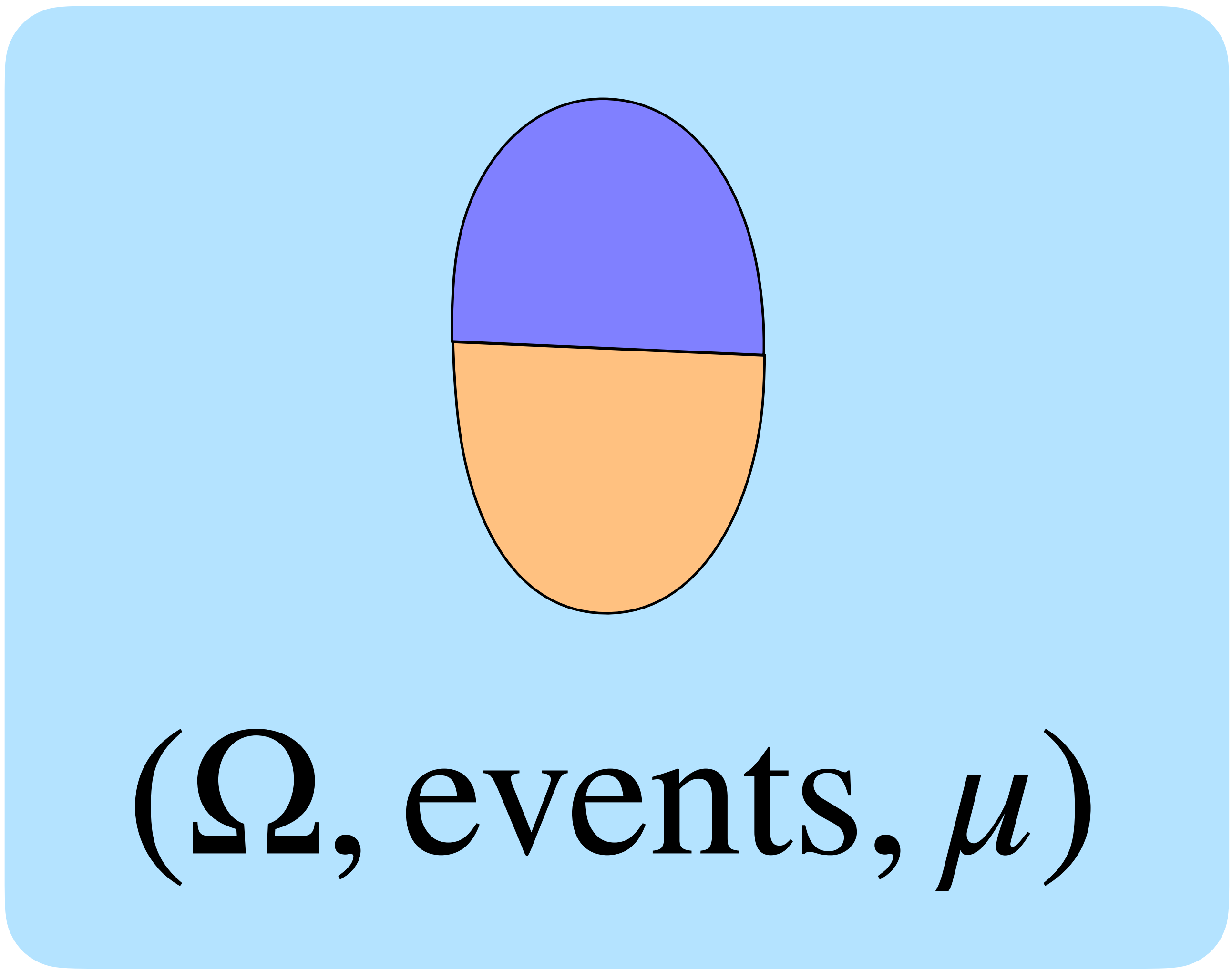


$(\Omega, \text{events}, \mu)$

Probability spaces as heaps

Probability theory $\simeq$ Mutable references

X



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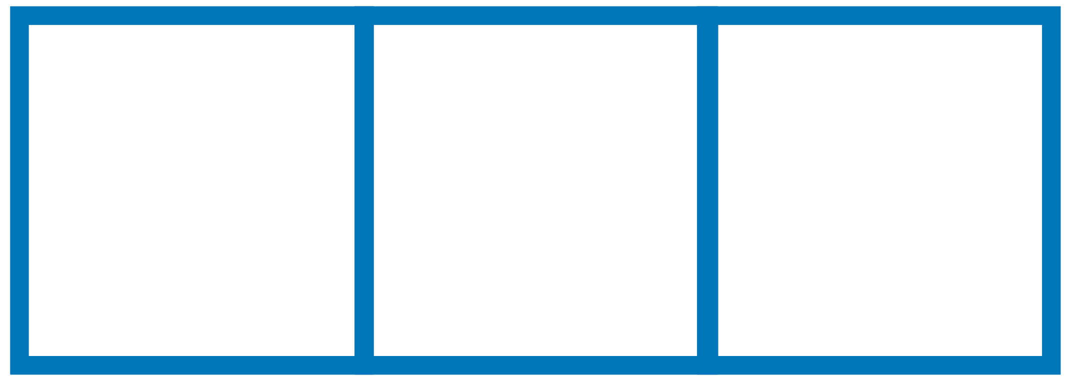

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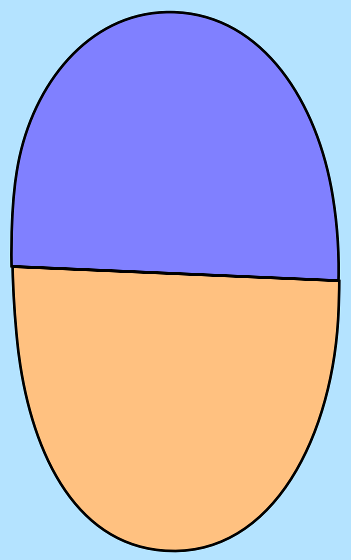
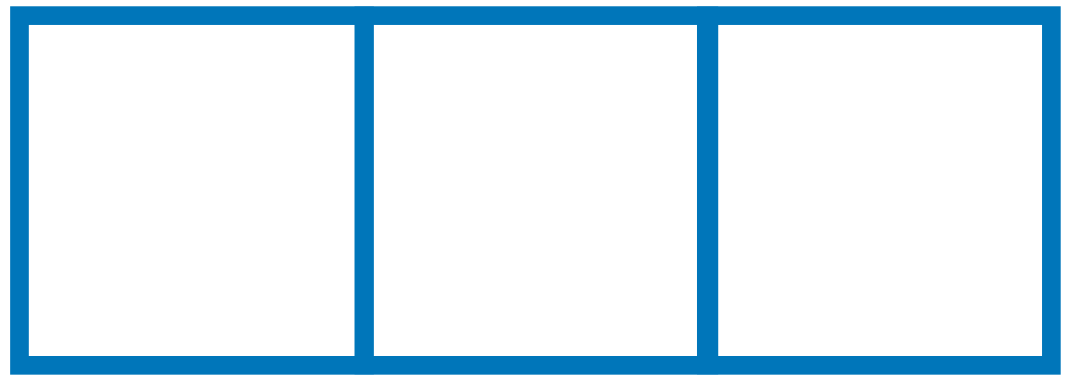


$(\Omega, \text{events}, \mu)$

This is a probability space!

ℓ



Lilac: a separation logic for probability

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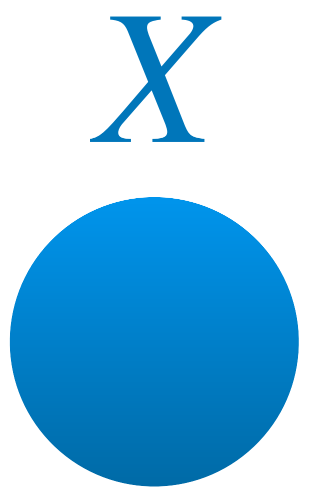

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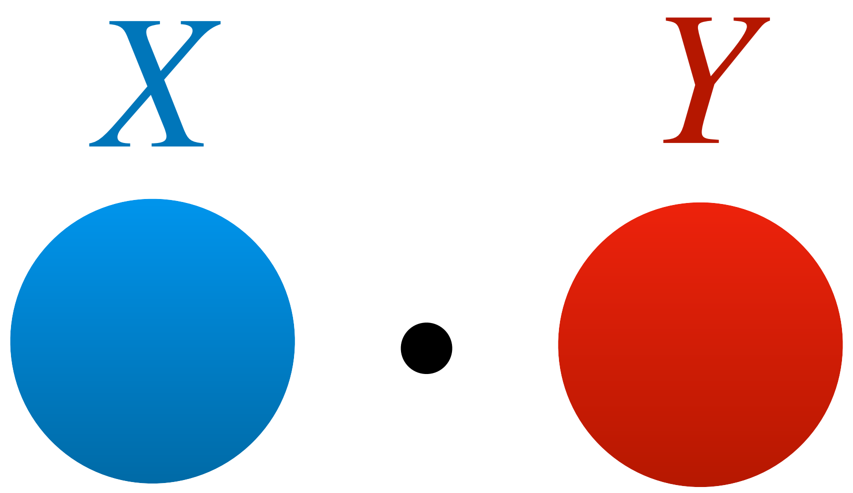

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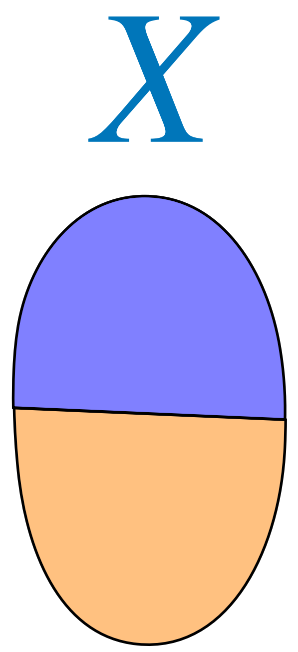

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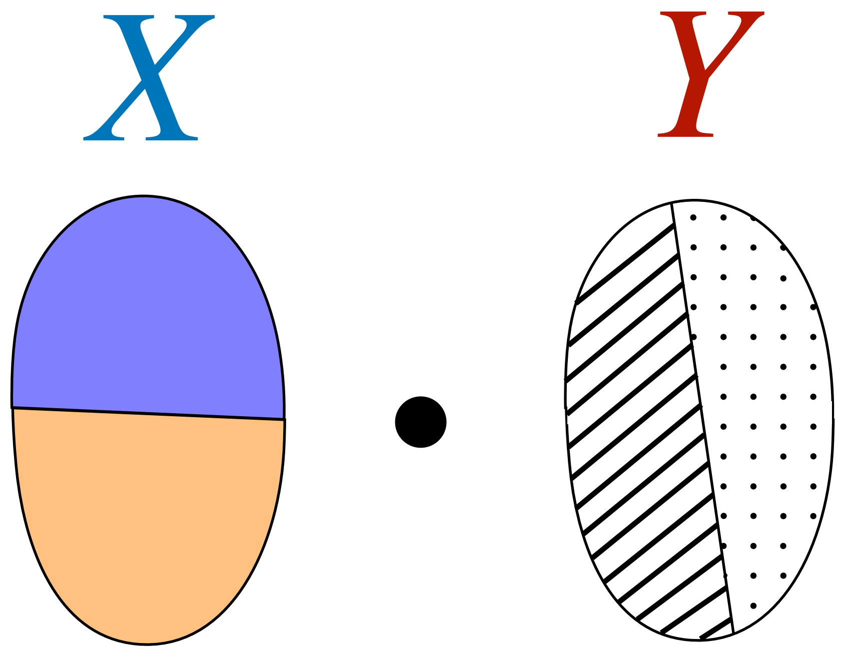

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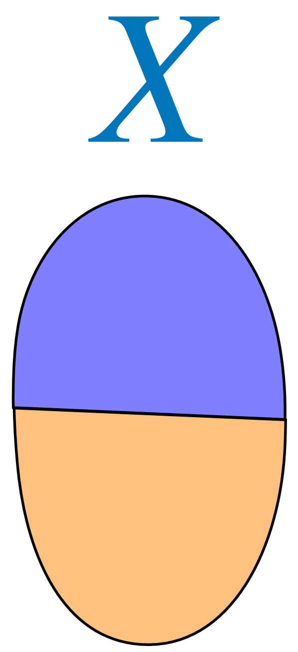

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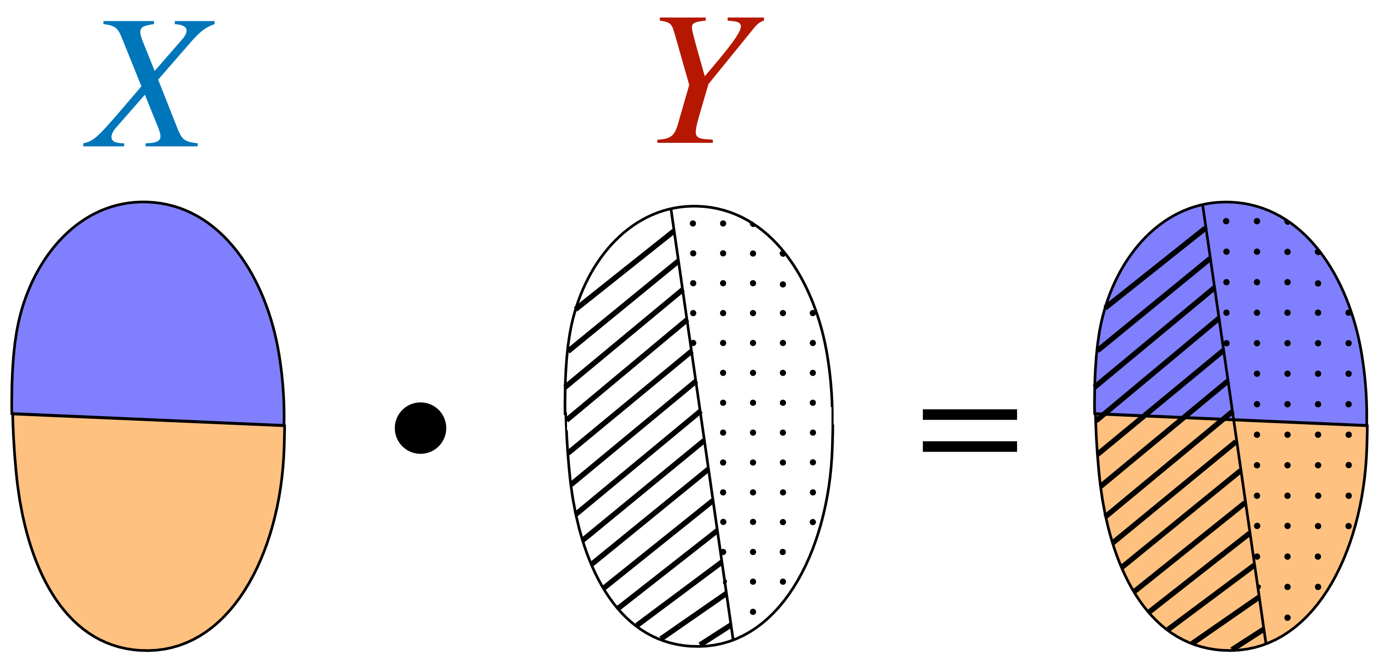

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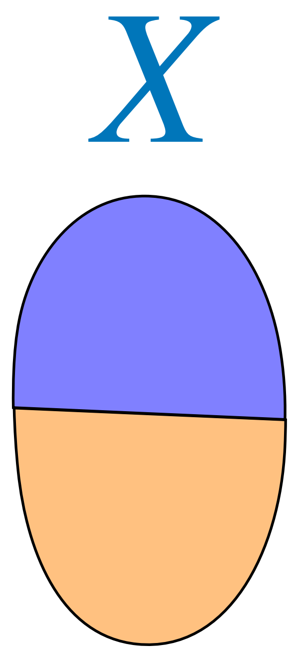
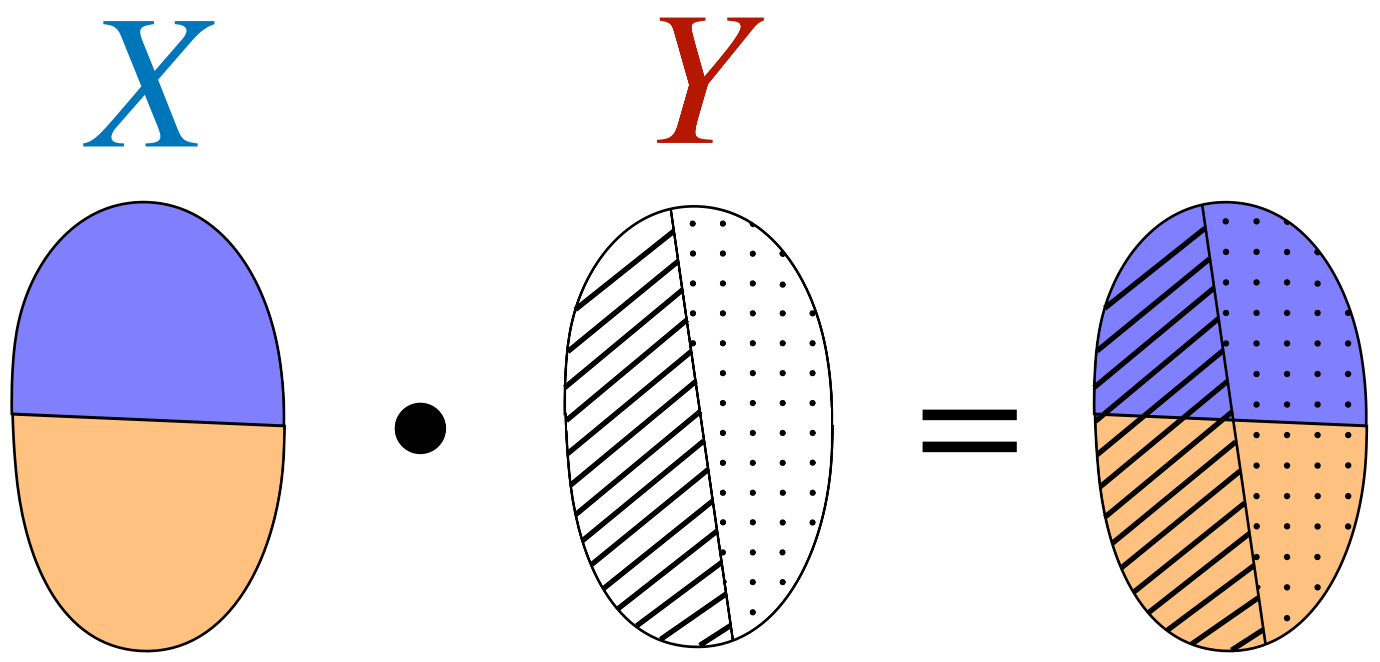


$Y \leftarrow \texttt{flip}\ 1/2;$



$\models \ X \sim \mathrm{Ber}(1/2) \ * \ Y \sim \mathrm{Ber}(1/2)$

Lilac: a modal separation logic for conditional probability

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- Conditioning as a *modality*:

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$$\underset{x \leftarrow X}{\mathbf{C}}\; P$$

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$$\underset{x \leftarrow X}{\mathbf{C}} \; P$$

P holds conditional on $X = x$ for all x

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$$X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2)$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

X and Y are independent

$$\Big\downarrow$$

$$X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2)$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

X and Y are independent

$$X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2)$$

X and Y have distribution $\mathrm{Ber}(1/2)$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$$\underset{z \leftarrow Z}{\mathbf{C}} \left(X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2) \right)$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

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Lilac: a modal separation logic for conditional probability

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$$\underset{z \leftarrow Z}{\mathbf{C}} \left(X \sim \mathrm{Ber}(1/2) \quad * \quad Y \sim \mathrm{Ber}(1/2) \right)$$

X and Y have conditional distribution $\mathrm{Ber}(1/2)$ given Z

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$\Pr[E] = 1/2$ $\qquad$ E has probability $1/2$

$\mathbf{E}[X] = 0$ $\qquad$ X has expectation 0

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$$\underset{x \leftarrow X}{\mathbf{C}}\left(\Pr[E] = 1/2\right) \qquad E \text{ has probability } 1/2 \text{ given } X = x$$

$$\mathbf{E}[X] = 0 \qquad X \text{ has expectation } 0$$

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*:

$$\underset{x \leftarrow X}{\mathbf{C}}\left(\Pr[E] = 1/2\right)$$

E has probability $1/2$ given $X = x$

$$\underset{y \leftarrow Y}{\mathbf{C}}\left(\mathbf{E}[X] = 0\right)$$

X has conditional expectation 0

Lilac: a modal separation logic for conditional probability

- Conditioning as a *modality*
- Laws express intuitive facts and standard theorems:

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- Conditioning as a *modality*
- Laws express intuitive facts and standard theorems:

C-Total-Expectation

$$\underset{x \leftarrow X}{\mathbf{C}} \left(\mathbb{E}[E] = e \right) \;\wedge\; \mathbb{E}[e[X/x]] = v \vdash \mathbb{E}[E] = v$$

We used Lilac to verify

- Examples from prior work (cryptographic protocols)
- A tricky weighted sampling algorithm exercising
  - Continuous random variables
  - Quantitative reasoning
  - Separation as independence
  - Conditioning modality

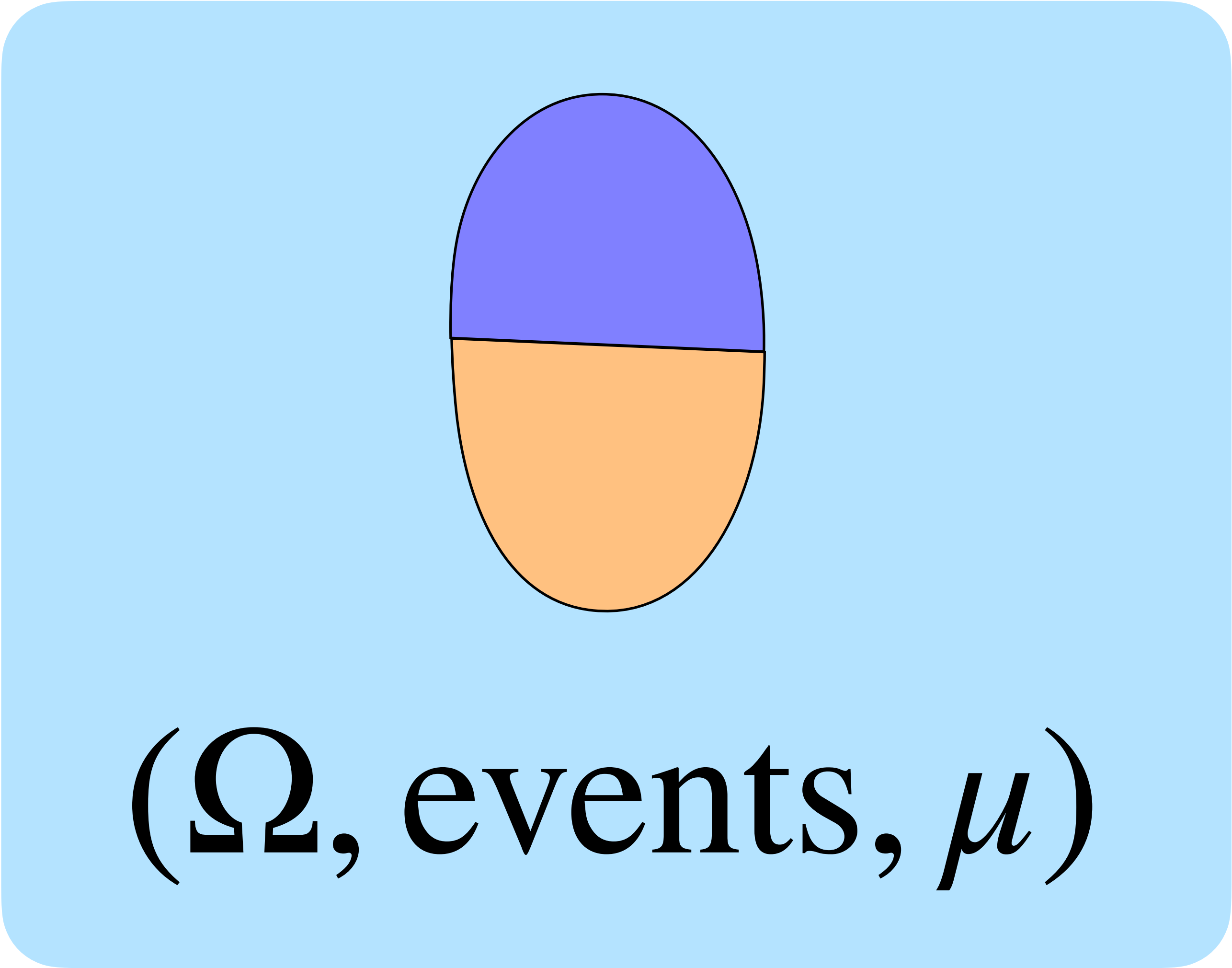
Also in the paper

- Conditioning modality
- Ownership is measurability
- Worked examples
- Almost-sure equality $X =_{\text{a.s.}} Y$

Thanks!

Probability theory $\simeq$ Mutable references

X



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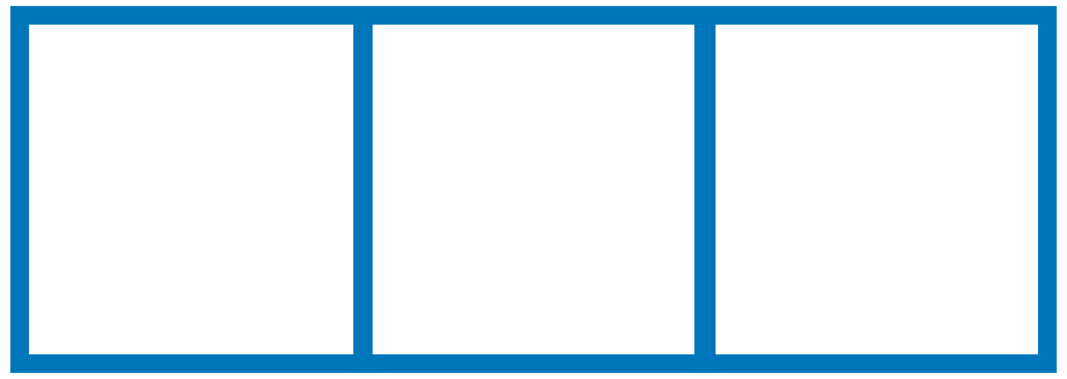

https://johnm.li/lilac.pdf