11-10-23

Content Review:

- homogeneity of cost function

def a function $f(x)$ is homogenous of degree k
if $f(a \cdot x) = a^k f(x) \quad \forall a \in \mathbb{R}$

$$c(q, r, w) = \min_{K, L} r \cdot K + w \cdot L \quad \text{s.t. } q = F(K, L)$$

$$= rK^* + wL^* \qquad MRTS_{K,L} = -\frac{r}{w}$$

$$c(q, ar, aw) = \min a \cdot rK + a \cdot wL \quad \text{s.t. } q = F(K, L)$$

$$MRTS_{K,L} = -\frac{ar}{aw} = -\frac{r}{w} \Rightarrow K^*, L^* \text{ unchanged}$$

$$= a \, c(q, r, w)$$

$c(q, r, w)$ is homogenous of degree 1 in input prices r, w

- Cost minimization across factories

a firm has n factories with cost functions $c_1(q_1)$, $c_2(q_2)$, ..., $c_n(q_n)$.
Total production $\bar{q}$

$$c(\bar{q}) = \min_{q_1, q_2, \dots, q_n} c_1(q_1) + c_2(q_2) + \dots + c_n(q_n)$$

$$\text{s.t. } q_1 + q_2 + \dots + q_n = \bar{q}$$

$$\Rightarrow \text{①} \; MC_1 = MC_2 = \dots = MC_n$$

$$\text{②} \; q_1 + q_2 + \dots + q_n = \bar{q}$$

System of equations to solve for $q_1, \dots, q_n$ as a function of $\bar{q}$

- Expected profit maximization

A random variable Y takes possible values $y_1, y_2, \dots, y_n$ with respective probabilities $x_1, \dots, x_n$, such that $\sum_{i=1}^{n} x_i = 1$.

$$E[Y] = \sum_{i=1}^{n} x_i y_i$$

ex: coin flip

$$Y = \begin{cases} 1, & \text{Heads} \\ 0, & \text{Tails} \end{cases} \Rightarrow E[Y] = \tfrac{1}{2}(1) + \tfrac{1}{2}(0)$$

$$= \tfrac{1}{2}$$

Consider a firm who does not observe prices before production. Prices take values $P_1, P_2, \dots, P_n$ with probabilities $x_1, \dots, x_n$.

$$E[\pi(q)] = \sum_{i=1}^{n} x_i \left(P_i q - c(q)\right)$$

$$= \left(\sum_{i=1}^{n} x_i P_i\right) q - c(q) = E[P]q - c(q)$$

FOC: $$\frac{\partial E[\pi(q)]}{\partial q} = 0$$

$$\Rightarrow E[P] = mc$$

Practice Questions:

1. $c(2, r, w) = 10$

$c(q, r, w)$ is HI in $r, w \Rightarrow c(q, a\cdot r, a\cdot w) = a\, c(q, r, w)$

$$c(2, 3r, 3w) = 3\, c(2, r, w) = 3 \cdot 10 = 30$$

2. 2 factories $\quad c_1(q_1) = 2q_1^2, \; c_2(q_2) = \frac{1}{2} q_2^2$

$\bar{q} = 5$ total output

① $MC_1 = MC_2$

$$\Rightarrow \frac{\partial c_1(q_1)}{q_1} = \frac{\partial c_2(q_2)}{q_2}$$

$$\Rightarrow 4q_1 = q_2$$

② $q_1 + q_2 = 5 = \bar{q}$

2 equations in 2 unknowns
use substitution or elimination

$$q_1 + 4q_1 = 5 \Rightarrow 5q_1 = 5 \Rightarrow q_1 = 1$$

$q_2 = 4 \cdot 1 = 4$

3. Prob $\frac{1}{3}$ $\quad P_1 = 1$ $\qquad C(q) = \frac{1}{2}q^2$

Prob $\frac{2}{3}$ $\quad P_2 = 4$

$$
\begin{aligned}
E[\pi(q)] &= E[\, p \cdot q - c(q)] \\
&= \tfrac{1}{3}\left(1 \cdot q - \tfrac{1}{2}q^2\right) + \tfrac{2}{3}\left(4q - \tfrac{1}{2}q^2\right) \\
&= \left(\tfrac{1}{3}\cdot 1 + \tfrac{2}{3}\cdot 4\right) q - \tfrac{1}{2}q^2 \\
&= \left(\tfrac{1}{3} + \tfrac{8}{3}\right) q - \tfrac{1}{2}q^2 \\
&= 3q - \tfrac{1}{2}q^2
\end{aligned}
$$

FOC: $\dfrac{\partial E[\pi(q)]}{\partial q} = 3 - q = 0$

$\Rightarrow q = 3$

$MC = E[P] \Rightarrow q = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 4 = 3$

4. $F(K,L) = \min\{4K, \frac{1}{2}L\}$

a. firm is minimizing costs, use $K=3$

$4K = \frac{1}{2}L \Rightarrow L = 8K = 8 \cdot 3 = 24$

b. $c(q) = \min\limits_{K,L} \; 20 \cdot K + 5 \cdot L$

s.t. $q = \min\{4K, \frac{1}{2}L\}$

$q = 4K = \frac{1}{2}L$

$\Rightarrow K^* = \frac{q}{4}, \quad L^* = 2q$

$$c(q) = 20K^* + 5L^* = 20 \cdot \frac{q}{4} + 5 \cdot 2q$$

$$= 5q + 10q$$

$$= 15q$$