

# Conditional Probability and Bayes Theorem

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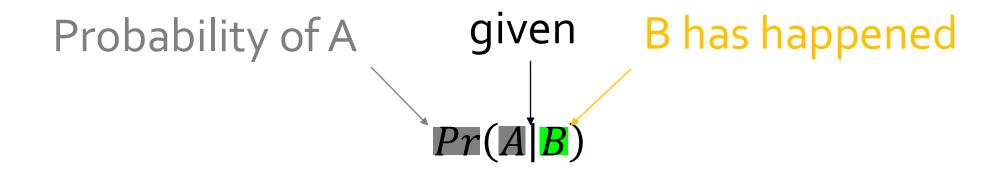
■ Let p(A|B) denote the probability of the event A occurring given that the event B has occurred. This is termed a conditional probability (i.e., event B has already happened or is sure to happen)

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

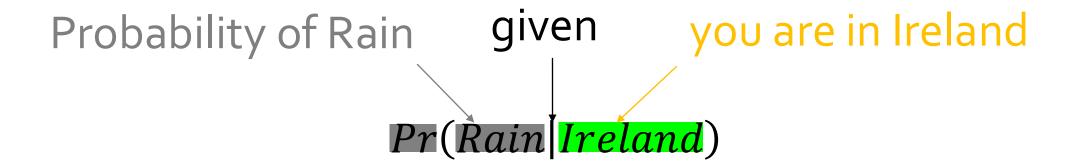
And similarly

$$p(B|A) = \frac{p(A \cap B)}{p(A)}$$











This can also be written as a multiplication law

$$p(A \cap B) = p(B)p(A|B) = p(A)p(B|A)$$

- In p(A|B), event B has the role of a reduced sample space
- The reduced sample space are just those comprising B



Extending this to three events

$$p(A \cap B \cap C) = p(A)p(B|A)p(C|A \cap B)$$

There results are very useful in problems involving sequential operations



# Independent Conditional probability

$$p(A \cap B) = p(A)p(B)$$

$$p(A \cap C) = p(A)p(C)$$

$$p(B \cap C) = p(B)p(C)$$

#### AND

$$p(A \cap B \cap C) = p(A)p(B)p(C)$$



# **Conditional Probability - Example**

In a population of 1000 people, 10% are left handed, 5% are colour-blind, and of these 10 are left-handed. A person is selected randomly from the population what is the probability of them being color-blind or left handed or both?



# Conditional Probability - Example

- What is the probability if rolling a two dice that sum to 5.
- What is the probability of rolling two dice and at least one of the die has a 2.
- Calculate the probability of rolling two dice which sum to 5 given that one or both of the dice rolled is a 2.





$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)}$$

Posterior probability  $\propto Likelihood \times Prior probability$ 



$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)}$$

Future  $\propto Present \times Past$ 



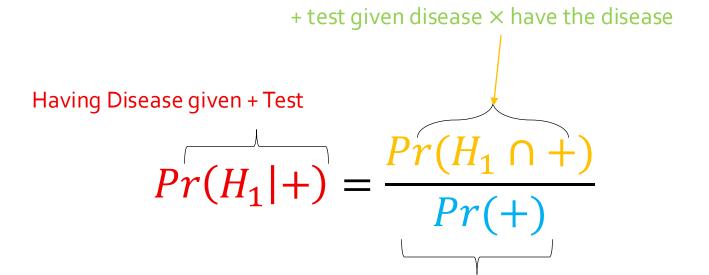
■ Let  $H_1$  and  $H_2$ , be two mutually exclusive exhaustive and possible events in a sample space S,

$$H_1 \cap H_2 = \emptyset$$
, and  $H_1 \cup H_2 = S$ 

- E.g.  $H_1$  person has a disease,  $H_2$  person does not have disease
- Then

$$p(H_1|+) = \frac{p(H_1 \cap +)}{p(+)}$$





Everyone with a + Test



# Example

- A clinical test, designed to diagnose a specific illness, comes out positive for a certain patient
- We are told that
- 1. The test is 79 percent reliable (that is, it misses 21 percent of actual cases)
- 2. On average, this illness affects 1 percent of the population in the same age group as the patient
- 3. The test has a false positive rate of 10 percent.
- Taking this into account and assuming you know nothing about the patient's symptoms or signs, what is the probability that this patient actually has the illness?

	Has illness	No illness	Totals
Totals	1,000	99,000	100,000



	Has illness	No illness	Totals
Test Positive	790		
Test Negative	210		
Totals	1,000		



	Has illness	No illness	Totals
Test Positive		9,900	
Test Negative		89,100	
Totals		99,000	



	Has illness	No illness	Totals
Test Positive	790	9,900	10,690
Test Negative	210	89,100	89,310
Totals	1,000	99,000	100,000

$$p(Illness|+) = \frac{Test\ Positive\ \&\ Has\ Illness}{\#\ Test\ Positive} = \frac{790}{10690}$$



+ test given disease × have the disease

$$Pr(H_1|+) = \frac{Pr(H_1 \cap +)}{Pr(+)}$$

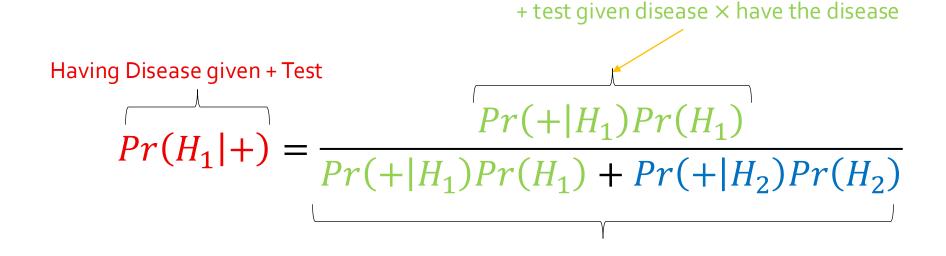
Everyone with a + Test

$$Pr(+) = Pr(+|H_1)Pr(H_1) + Pr(+|H_2)Pr(H_2)$$

+ test given no disease × do not have the disease



$$Pr(H_1|+) = \frac{Pr(H_1 \cap +)}{Pr(+)} = \frac{Pr(+|H_1)Pr(H_1)}{Pr(+|H_1)Pr(H_1) + Pr(+|H_2)Pr(H_2)}$$



Everyone with a + Test



# Example

prob of illness given pos. test =

```
prob of pos test when Prevalence illness is present of illness

prob of pos test when prob of pos test when illness is present of illness is not present of illness.
```



	Has illness	No illness	Totals
Test Positive	True Positive	False Positive	
Test Negative	False Negative	True Negative	
Totals			

$$p(Illness|Positive\ Test) = \frac{\text{True\ Positive}}{\text{True\ Positive}\ + False\ Positive} = \frac{790}{10690}$$



# Sensitivity and Specificity

	Has illness	No illness	Totals
Test Positive	True Positive	False Positive	
Test Negative	False Negative	True Negative	
Totals			

Sensitivity (TP) = among people with disease, the probability of a positive test

Specificity(TN) = among people without disease, the probability of a negative test

**1-specificity (1-TN)**= among people without disease, the probability of a positive test

	Has illness	No illness	Totals
Test Positive	790	9,900	10,690
Test Negative	210	89,100	89,310
Totals	1,000	99,000	100,000

**sensitivity** = 
$$\frac{790}{1000}$$
 = 0.79

**1-specificity** = 
$$1 - \frac{89100}{99000} = \frac{9900}{99000} = 0.1$$



# Example

prob of illness given pos. test =

(Sensitivity \* Prevelance) + (1 - Prevelance) \* (1 - Specificity)



# Example

prob of illness given pos. test =

```
prob of pos test when Prevalence illness is present of illness

prob of pos test when prob of pos test when illness is present of illness is not present of illness.
```



#### General

	Has illness	No illness	Totals
Test Positive	True Positive (TP)	False Positive (FP)	TP+FP
Test Negative	False Negative (FN)	True Negative (TN)	FN+TN
Totals	TP+FN	FP+TN	TP+FP+FN+TN

sensitivity = 
$$\frac{TP}{TP + FN}$$

specificity = 
$$\frac{TN}{FP+TN}$$



#### General

	Has illness	No illness	Totals
Test Positive	True Positive (TP)	False Positive (FP)	TP+FP
Test Negative	False Negative (FN)	True Negative (TN)	FN+TN
Totals	TP+FN	FP+TN	TP+FP+FN+TN

sensitivity = 
$$\frac{TP}{TP+FN}$$
 =  $P(Positive\ Test|Disease)$ 

$$specificity = \frac{TN}{FP + TN} = P(Negative \ Test | No \ Disease)$$



# Takeaway Point

- Conditional probability and Bayes' Theorem allow us to update beliefs based on new evidence.
- The past impacts the present to change the probability of the future

