# Conditional Probability and Bayes Theorem

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# Conditional Probability

# Conditional Probability

- Let p(A|B) denote the probability of the event A occurring given that the event B has occurred. This is termed a conditional probability (i.e., event B has already happened or is sure to happen)

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

- And similarly

$$p(B|A) = \frac{p(A \cap B)}{p(A)}$$

# Conditional Probability

Probability of A

given

B has happened

$$Pr(A|B)$$

# Conditional Probability

Probability of Rain given you are in Ireland

$$Pr(Rain|Ireland)$$

# Conditional probability

- This can also be written as a **multiplication law**

$$p(A \cap B) = p(B)p(A|B) = p(A)p(B|A)$$

- In p(A|B), event B has the role of a reduced sample space
- The reduced sample space are just those comprising B

# Conditional Probability

- Extending this to three events

$$p(A \cap B \cap C) = p(A)p(B|A)p(C|A \cap B)$$

- There results are very useful in problems involving sequential operations

# Independent Conditional probability

- $p(A \cap B) = p(A)p(B)$
- $p(A \cap C) = p(A)p(C)$
- $p(B \cap C) = p(B)p(C)$

$AND$

- $p(A \cap B \cap C) = p(A)p(B)p(C)$

# Conditional Probability - Example

In a population of 1000 people, 10% are left handed, 5% are colour-blind, and of these 10 are left-handed. A person is selected randomly from the population what is the probability of them being color-blind or left handed or both?

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# Conditional Probability - Example

- What is the probability if rolling a two dice that sum to 5.
- What is the probability of rolling two dice and at least one of the die has a 2.
- Calculate the probability of rolling two dice which sum to 5 given that one or both of the dice rolled is a 2.

# Bayes Theorem

# Bayes Theorem

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)}$$

Posterior probability $\propto$ $Likelihood\ \times Prior\ probability$

# Bayes Theorem

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)}$$

$$Future \propto Present \times Past$$

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# Bayes Theorem

- Let $H_1$ and $H_2$, be two mutually exclusive exhaustive and possible events in a sample space $S$,

$$H_1 \cap H_2 = \emptyset, \text{ and } H_1 \cup H_2 = S$$

- E.g. $H_1$ person has a disease, $H_2$ person does not have disease
- Then

$$p(H_1|+) = \frac{p(H_1 \cap +)}{p(+)}$$

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# Bayes Theorem

$$\overbrace{Pr(H_1|+)}^{\text{Having Disease given + Test}} = \frac{\overbrace{Pr(H_1 \cap +)}^{+\text{ test given disease} \times \text{have the disease}}}{\underbrace{Pr(+)}_{\text{Everyone with a + Test}}}$$

# Example

- A clinical test, designed to diagnose a specific illness, comes out positive for a certain patient
- We are told that

1. The test is 79 percent reliable (that is, it misses 21 percent of actual cases)
2. On average, this illness affects 1 percent of the population in the same age group as the patient
3. The test has a false positive rate of 10 percent.

- Taking this into account and assuming you know nothing about the patient’s symptoms or signs, **what is the probability that this patient actually has the illness?**

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# Alternative Calculation

| | Has illness | No illness | Totals |
|---|---|---|---|
| | | | |
| | | | |
| Totals | 1,000 | 99,000 | 100,000 |

# Alternative Calculation

| | Has illness | No illness | Totals |
|---|---|---|---|
| Test Positive | 790 | | |
| Test Negative | 210 | | |
| Totals | 1,000 | | |

# Alternative Calculation

| | Has illness | No illness | Totals |
|---|---|---|---|
| Test Positive | | 9,900 | |
| Test Negative | | 89,100 | |
| Totals | | 99,000 | |

# Alternative Calculation

| | Has illness | No illness | Totals |
|---|---|---|---|
| Test Positive | 790 | 9,900 | 10,690 |
| Test Negative | 210 | 89,100 | 89,310 |
| Totals | 1,000 | 99,000 | 100,000 |

$$p(Illness|+) = \frac{Test\ Positive\ \&\ Has\ Illness}{\#\ Test\ Positive} = \frac{790}{10690}$$

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# Bayes Theorem

$$Pr(H_1|+) = \frac{Pr(H_1 \cap +)}{Pr(+)}$$

Everyone with a + Test

$$Pr(+) = \underbrace{Pr(+|H_1)Pr(H_1)} + \underbrace{Pr(+|H_2)Pr(H_2)}$$

+ test given disease × have the disease

+ test given no disease × do not have the disease

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# Bayes Theorem

$$Pr(H_1|+) = \frac{Pr(H_1 \cap +)}{Pr(+)} = \frac{Pr(+|H_1)Pr(H_1)}{Pr(+|H_1)Pr(H_1) + Pr(+|H_2)Pr(H_2)}$$

$$\overbrace{Pr(H_1|+)}^{\text{Having Disease given + Test}} = \frac{\overbrace{Pr(+|H_1)Pr(H_1)}^{\text{+ test given disease} \times \text{have the disease}}}{\underbrace{Pr(+|H_1)Pr(H_1) + Pr(+|H_2)Pr(H_2)}_{\text{Everyone with a + Test}}}$$

# Example

- prob of illness given pos. test $=$

$$\frac{\text{prob of pos test when illness is present} * \text{Prevalence of illness}}{\text{prob of pos test when illness is present} * \text{Prevalence of illness} + \text{prob of pos test when illness is not present} * (1 - \text{Prevalence of illness})}$$

# Alternative Calculation

| | Has illness | No illness | Totals |
|---|---|---|---|
| Test Positive | True Positive | False Positive | |
| Test Negative | False Negative | True Negative | |
| Totals | | | |

$$p(Illness|Positive\ Test) = \frac{\text{True Positive}}{\text{True Positive } + False\ Positive} = \frac{790}{10690}$$

# Sensitivity and Specificity

| | Has illness | No illness | Totals |
|---|---|---|---|
| Test Positive | True Positive | False Positive | |
| Test Negative | False Negative | True Negative | |
| Totals | | | |

**Sensitivity (TP)** = among people with disease, the probability of a positive test

**Specificity(TN)** = among people without disease, the probability of a negative test

**1-specificity (1-TN)**= among people without disease, the probability of a positive test

# Alternative Calculation

| | Has illness | No illness | Totals |
|---|---|---|---|
| Test Positive | 790 | 9,900 | 10,690 |
| Test Negative | 210 | 89,100 | 89,310 |
| Totals | 1,000 | 99,000 | 100,000 |

**sensitivity** = $\frac{790}{1000} = 0.79$

**1-specificity** = $1 - \frac{89100}{99000} = \frac{9900}{99000} = 0.1$

# Example

- prob of illness given pos. test =

$$\frac{\text{Sensitivity} \ * \text{Prevelance}}{(\text{Sensitivity} * \text{Prevelance}) + (1 - \text{Prevelance}) * (1 - \text{Specificity})}$$

# Example

- prob of illness given pos. test =

$$\frac{\begin{matrix}\text{prob of pos test when} \\ \text{illness is present}\end{matrix} * \begin{matrix}\text{Prevalence} \\ \text{of illness}\end{matrix}}{\begin{matrix}\text{prob of pos test when} \\ \text{illness is present}\end{matrix} * \begin{matrix}\text{Prevalence} \\ \text{of illness}\end{matrix} + \begin{matrix}\text{prob of pos test when} \\ \text{illness is not present}\end{matrix} * (1 - \begin{matrix}\text{Prevalenc} \\ \text{of illness}\end{matrix}}$$

# General

| | Has illness | No illness | Totals |
|---|---|---|---|
| Test Positive | True Positive (TP) | False Positive (FP) | TP+FP |
| Test Negative | False Negative (FN) | True Negative (TN) | FN+TN |
| Totals | TP+FN | FP+TN | TP+FP+FN+TN |

**sensitivity** = $\frac{TP}{TP+FN}$

**specificity** = $\frac{TN}{FP+TN}$

# General

| | Has illness | No illness | Totals |
|---|---|---|---|
| Test Positive | True Positive (TP) | False Positive (FP) | TP+FP |
| Test Negative | False Negative (FN) | True Negative (TN) | FN+TN |
| Totals | TP+FN | FP+TN | TP+FP+FN+TN |

**sensitivity** = $\frac{TP}{TP+FN} = P(Positive\ Test|Disease)$

**specificity** = $\frac{TN}{FP+TN} = P(Negative\ Test|No\ Disease)$

# Takeaway Point

- **Conditional probability and Bayes' Theorem** allow us to update beliefs based on new evidence.

- The past impacts the present to change the probability of the future