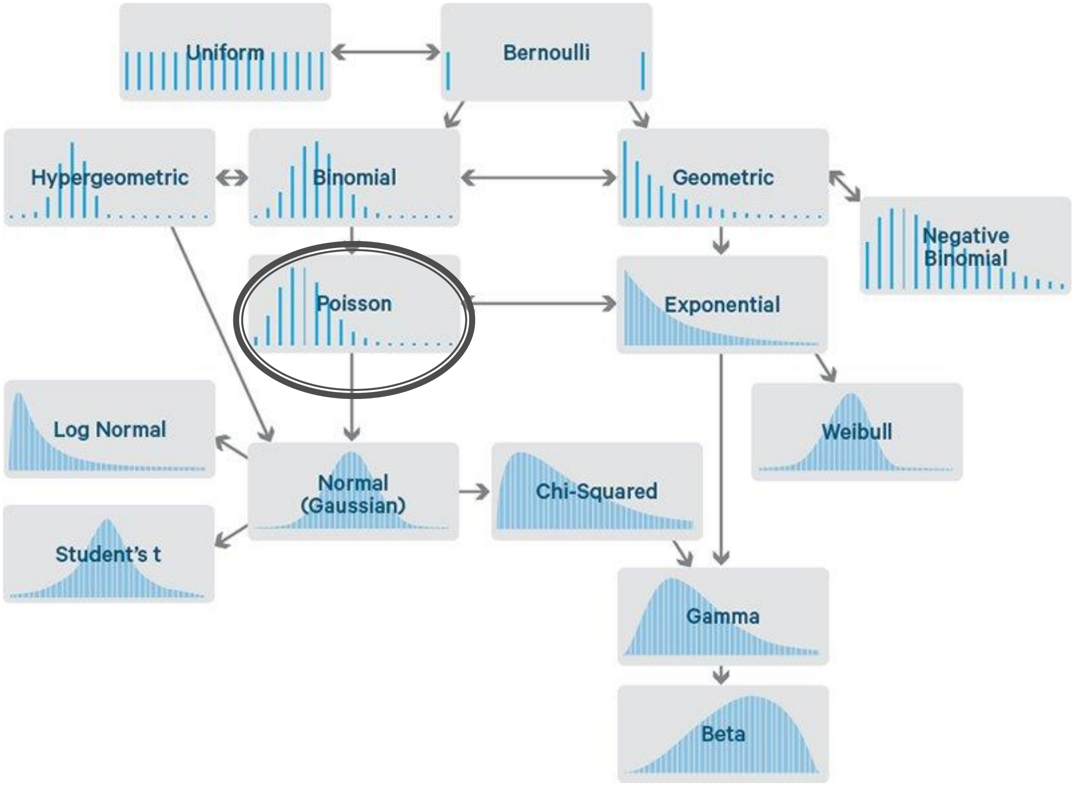
Poisson Distributions

Dr. John S. Butler

Poisson Distribution



Example Poisson Distributions

- The number of phone calls received at an exchange or call centre
- The number of customers arriving at a toll booth per day
- The number of flaws on a length of cable
- The number of cars passing using a stretch of road during a day
- The number of times a neuron fires
- The number of deaths by horse kicking in the Prussian army (original distribution)

All theses can be modeled using the Poisson Distribution

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Features of the Poisson Experiment

1. The experiment consists of a number of events happening randomly over time or space
2. The events occur independently of each other
3. The rate of occurrence of the events is a well defined average per unit/space
4. The Random Variable is the number of events occurring in a given interval

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Poisson Distribution

We assume the following

- The random variable X denotes the number of successes in the whole interval
- λ is the mean number of successes in the interval

X has a Poisson Distribution with parameter λ and

$$Pr(X = k) = f(k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \dots$$

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Let’s go to Prussia

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Lets go to Prussia

- Ten army corps were observed over 20 years, giving a total of 200 observations of one corps for a one year period.
- The total deaths from horse kicks were 122, and the average number of deaths by horse per year per Prussian cavalry corps was thus 122/200 = 0.61.

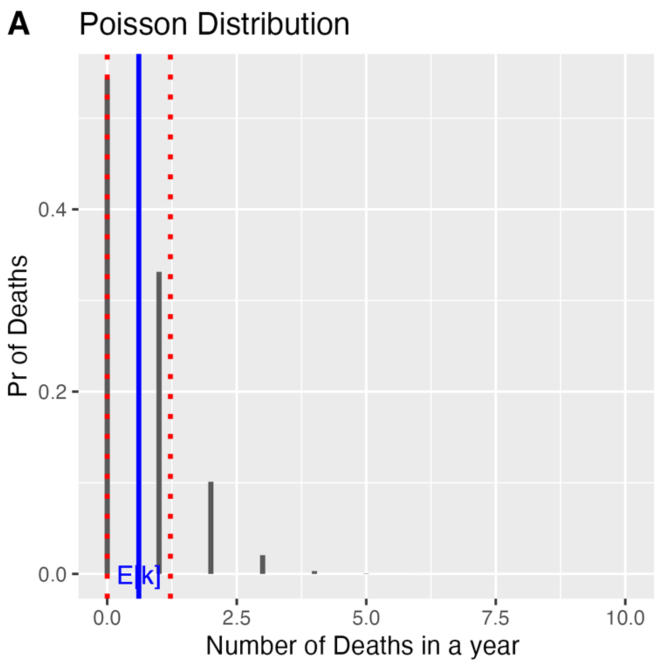
$$Pr(X = k) = f(k) = \frac{0.61^k e^{-0.61}}{k!}, k = 0, 1, 2, \ldots$$

- k is the number of deaths by horse per year per Prussian cavalry corps

https://www.umass.edu/wsp/resources/poisson/

Probability Mass Function

K	0	1	2	3	4
Pr(X=k)	$\frac{0.61^0 e^{-0.61}}{0!}$	$\frac{0.61^1 e^{-0.61}}{1!}$	$\frac{0.61^2 e^{-0.61}}{2!}$	$\frac{0.61^3 e^{-0.61}}{3!}$	$\frac{0.61^4 e^{-0.61}}{4!}$
Pr(X=k)	0.54335	0.3314	0.1010	0.02055	0.0031



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Lets go to Prussia

- What is the probability of exactly one death in a year?

$$Pr(X = 1) = f(1) = \frac{0.61^1 e^{-0.61}}{1!} = 0.3314$$

Lets go to Prussia

- What is the probability of at least one death in a year?

- $Pr(X \geq 1) = 1 - Pr(x = 0) = 1 - \frac{0.61^{0} e^{-0.61}}{0!}$= 1-0.5433=0.46

Lets go to Prussia

- What is the probability of exactly one death in two years?
- That is the probability of one death in the first year and no deaths in the second year $Pr(X = 1) * Pr(X = 0)$ or the probability of no death in the first year and one death in the second year $Pr(X = 0) * Pr(X = 1)$

$$P(X = 1) \times P(X = 0) + P(X = 0) \times P(X = 1) =$$

$$\frac{0.61^1 e^{-0.61}}{1!} \times \frac{0.61^0 e^{-0.61}}{0!} + \frac{0.61^0 e^{-0.61}}{0!} \times \frac{0.61^1 e^{-0.61}}{1!}$$

$$= 0.5433 \times 0.3312 + 0.3312 \times 0.5433 = 0.361324$$

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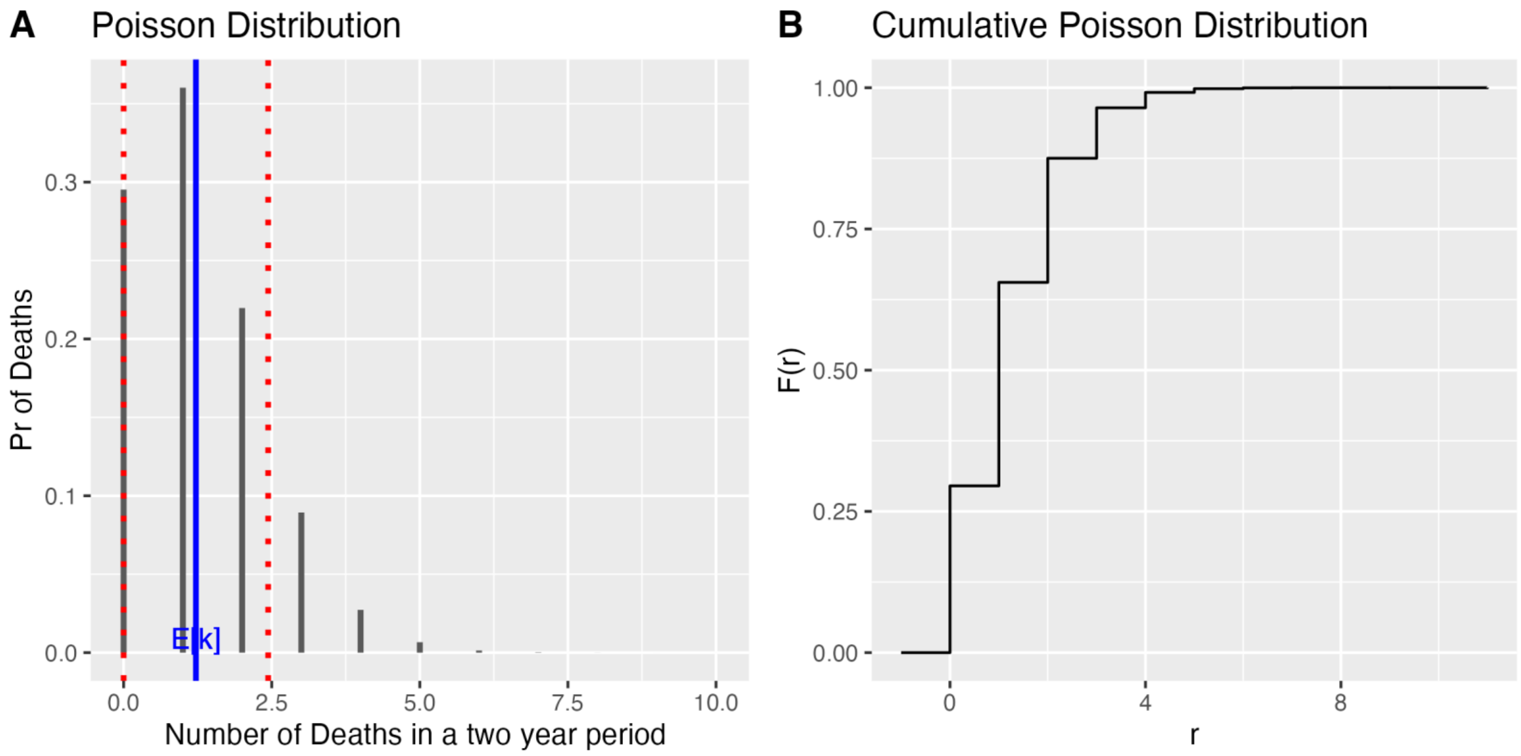
Lets go to Prussia (Two Years)

- What is the probability of exactly one death in two years?
- How about we do it another way, what is the average number of deaths over two years 0.61*2=1.22

$$Pr(X = 1|Over\ two\ years) = \frac{1.22^1 e^{-1.22}}{1!} = 0.36132$$

Lets go to Prussia (Two Years)



Lets go to Prussia (Three Years)

- What is the probability of exactly one death in three years?
- How about we do it another way, what is the average number of deaths over three years 0.61*3=1.83

$$Pr(X = 1|Over\ three\ years) = \frac{1.83^1 e^{-1.83}}{1!} = 0.29355$$

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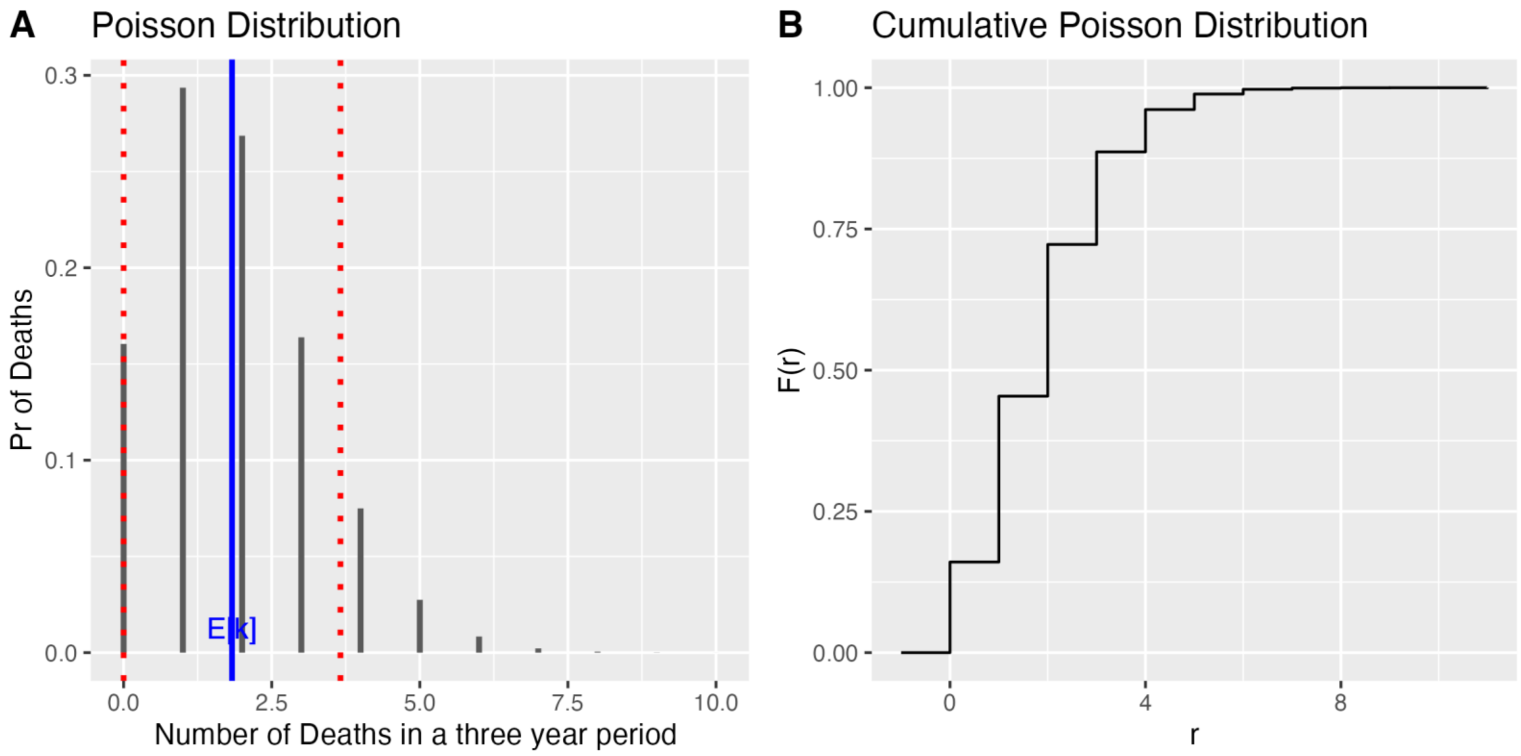
Lets go to Prussia (Three Years)



What is the expected number of deaths per year

K	0	1	2	3	4
Pr(X=k)	0.54335	0.3314	0.1010	0.02055	0.0031

$$E[X] = \sum_{k=0}^{\infty} kPr(x_k)$$

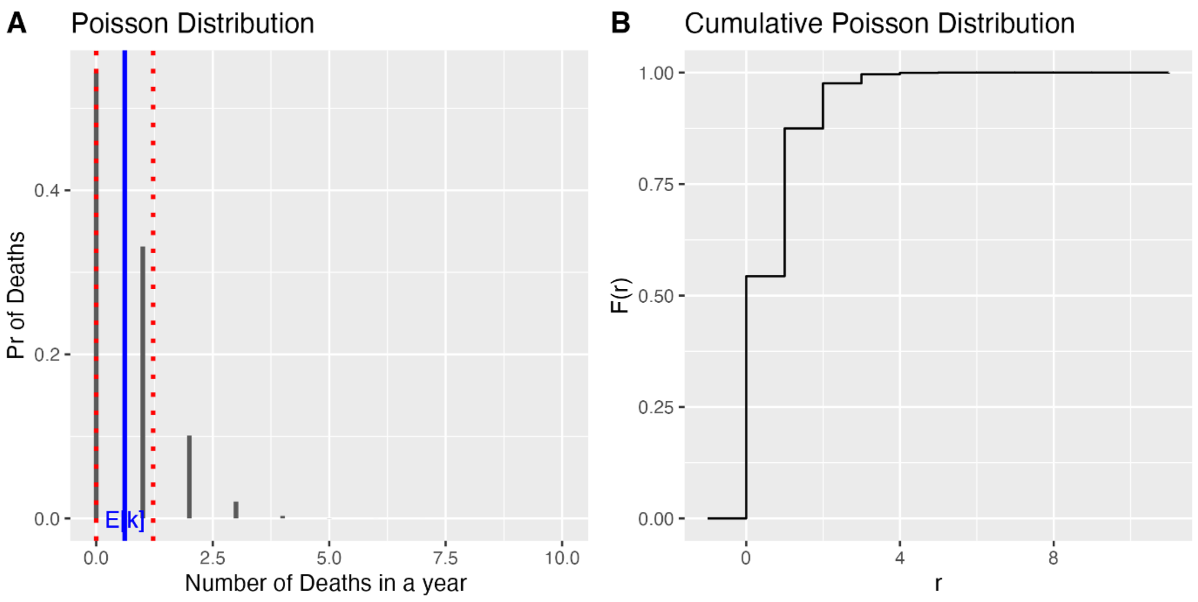
What is the expected number of deaths per year

K	0	1	2	3	4
Pr(X=k)	0.54335	0.3314	0.1010	0.02055	0.0031
$(K-0.61)^2$	0.3721	0.1521	1.9321	5.712	11.492
$(K-0.61)^2 Pr(k)$	0.202	0.0504	0.195	0.117	0.0360

$$Var[X] = \sigma^2 = E[(X-\lambda)^2] = \sum_{k=0}^{\infty} (k-\lambda)^2 p(k)$$

Cumulative Probability Mass Function

K	0	1	2	3	4
Pr(X<=k)	$\frac{0.61^0 e^{-0.61}}{0!}$	$\sum_0^1 \frac{0.61^k e^{-0.61}}{k!}$	$\sum_0^2 \frac{0.61^k e^{-0.61}}{k!}$	$\sum_0^3 \frac{0.61^k e^{-0.61}}{k!}$	$\sum_0^4 \frac{0.61^k e^{-0.61}}{k!}$
Pr(X=k)	0.54335	0.8747949	0.9758853	0.9964404	0.9995750



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Poisson Distributions

- *The mean and variance of a Poisson random variable with parameter* λ are both λ

$$E[X] = \lambda$$

$$Var[X] = \lambda$$

- This is an interesting property.

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Note about Notation

- $X \sim Poiss(\lambda)$
  - λ is the mean of the distribution

- In R
  - **dpoiss(x, lambda, log = FALSE)**: returns the value of the Poisson probability density function
  - **ppoiss(q, lambda, lower.tail = TRUE, log.p = FALSE)**: returns the value of the Poisson density function.
  - **qpoiss(p, lambda, lower.tail = TRUE, log.p = FALSE)**: returns the value of the Poisson cumulative density function.
  - **rpoiss(n, lambda)**: generates a vector of Poisson random variables.

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Rugby - Ireland vs New Zealand

Ireland vs New Zealand

1. The number of Irish wins in a 15-year period is 2.2 wins
2. Describe the Poisson distribution of the number of Irish wins in a 15-year period.

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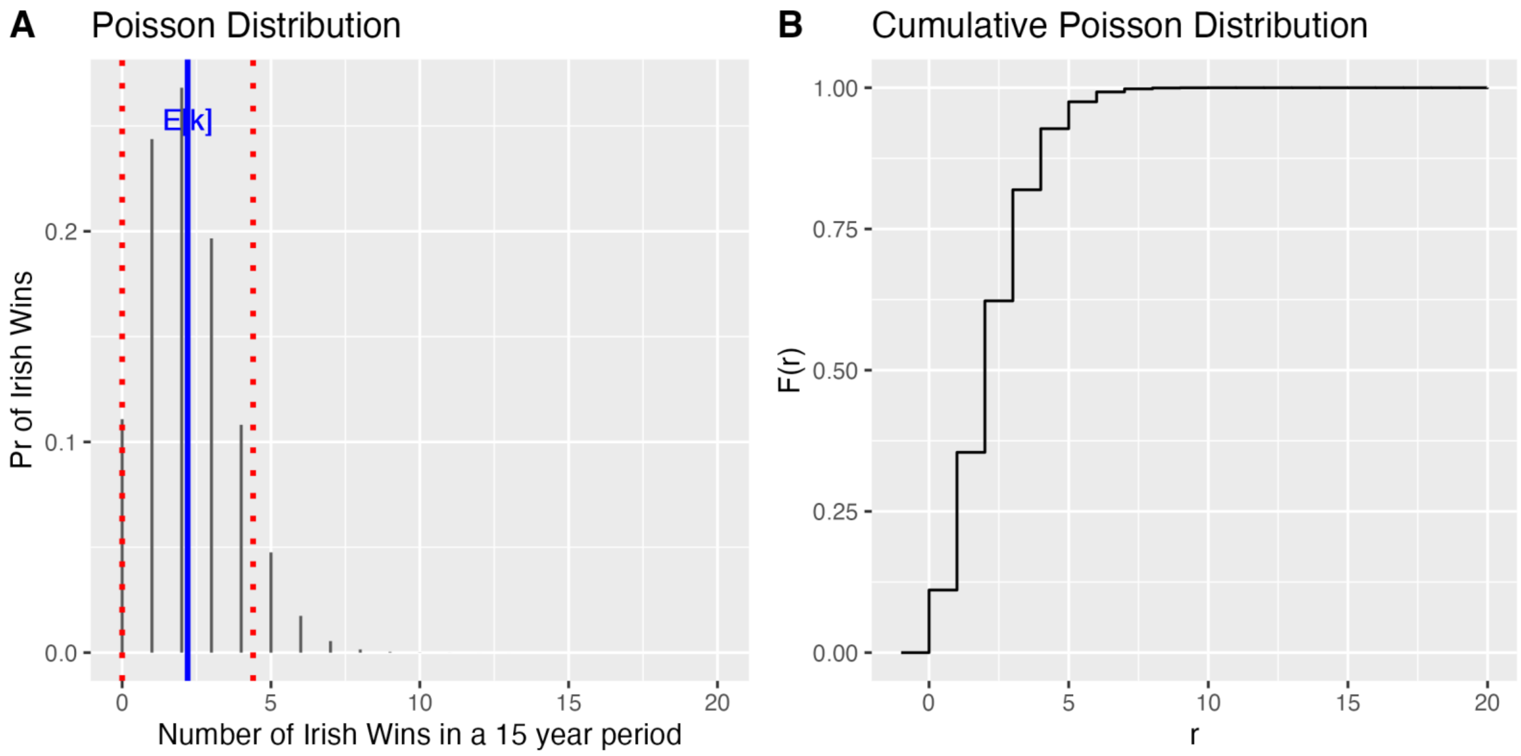
Ireland vs New Zealand

$$Pr(X = k) = f(k) = \frac{2.2^k e^{-2.2}}{k!}, k = 0, 1, 2, \ldots$$

- k is the number of Irish wins in a 15 year period

Ireland vs New Zealand



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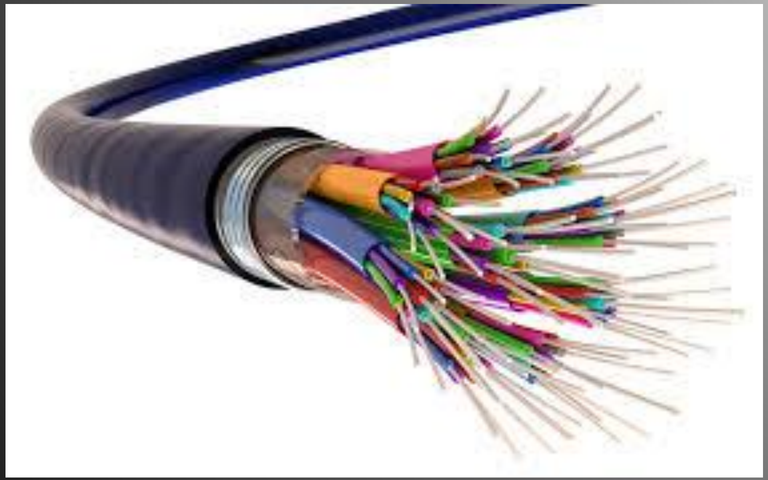
Example Poisson Distributions



Cable

The number of flaws in a fiber optic cable follows a Poisson Distribution. The average number of flaws in a 50m cable is 1.2

1. What is the probability of exactly two flaws in 50m of cable?
2. What is the probability of exactly three flaws in 150m of cable?
3. What is the probability of at least two flaws in 100ms of cable?
4. What is the probability of exactly one flaw in the first 50m and one flaw in the second 50m of cable?

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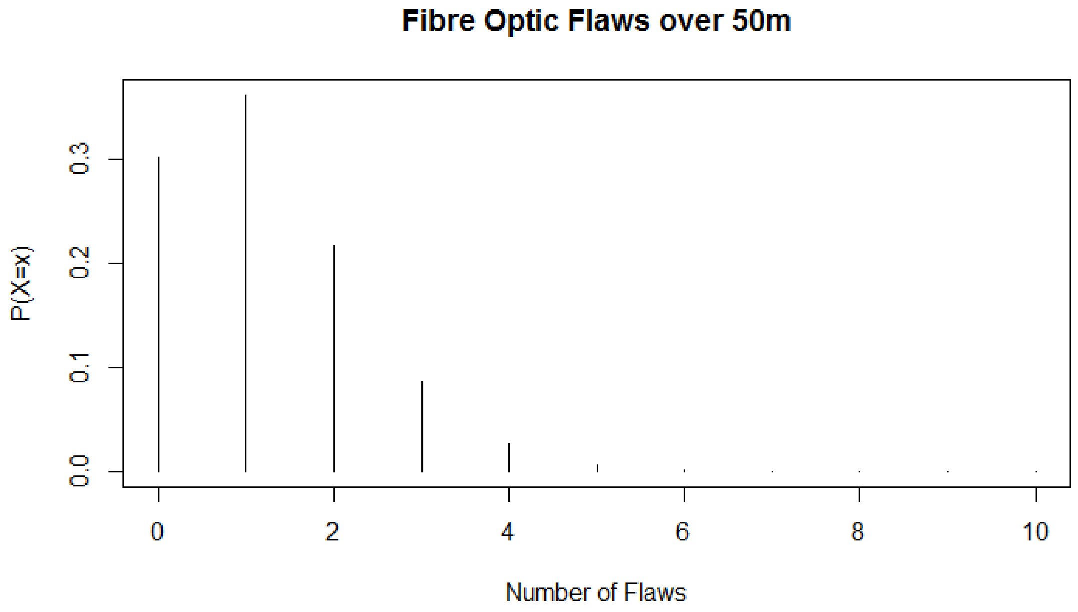
Example Poisson Distributions

- Formula for 50m

$$Pr(X = k) = f(k) = \frac{1.2^k e^{-1.2}}{k!}, k = 0, 1, 2, \ldots$$



Example Poisson Distributions

1. What is the probability of exactly two flaws in 50m of cable?

Example Poisson Distributions

1. What is the probability of exactly two flaws in 50m of cable?

$$Pr(X = 2) = f(2) = \frac{1.2^2 e^{-1.2}}{2!} = 0.216$$

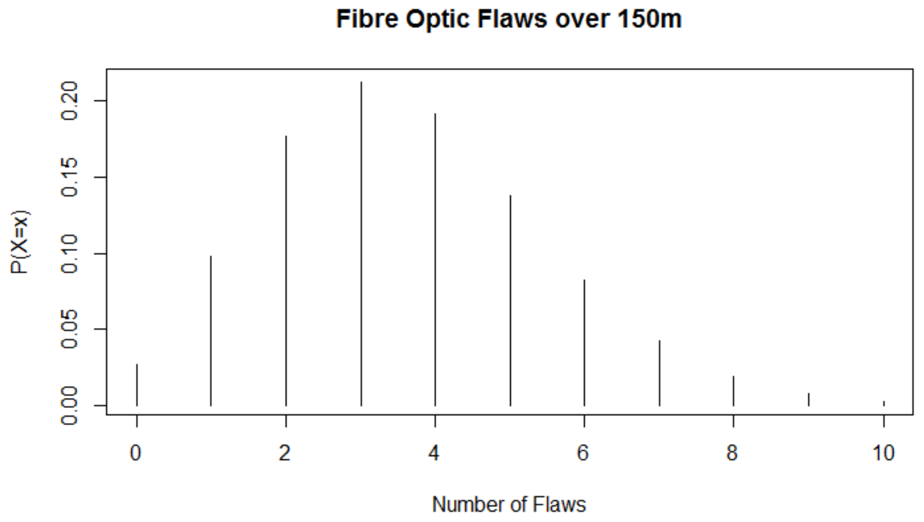
Example Poisson Distributions

2. What is the probability of exactly three flaws in 150m of cable?

Example Poisson Distributions

2. What is the probability of exactly three flaws in 150m of cable?

$$Pr(X = 3) = f(3) = \frac{3.6^3 e^{-3.6}}{3!} = 0.212$$



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Example Poisson Distributions

2. What is the probability of exactly three flaws in 150m of cable?

Average Number of flaws for 150m = Average number of flaw for 50m ×3

Average Number of flaws for 150m = 1.2 ×3=3.6

$$Pr(X=3)=f(3)=\frac{3.6^3e^{-3.6}}{3!}=0.212$$

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Example Poisson Distributions

3. What is the probability of at least two flaws in 100ms of cable?

Example Poisson Distributions

3. What is the probability of at least two flaws in 100ms of cable?

Average Number of flaws for 100m = 1.2 ×2=2.4

$$Pr(X \geq 2) = (1 - Pr(0) - Pr(1)) = 1 - \frac{2.4^{0}e^{-2.4}}{0!} - \frac{2.4^{1}e^{-2.4}}{1!} = 0.6915$$

Example Poisson Distributions

4. What is the probability of exactly one flaw in the first 50m and one flaw in the second 50m of cable?

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Example Poisson Distributions

4. What is the probability of exactly one flaw in the first 50m and one flaw in the second 50m of cable?

$$Pr(k = 1) \times Pr(k = 1) = \frac{1.2^1 e^{-1.2}}{1!} \times \frac{1.2^1 e^{-1.2}}{1!} = 0.13$$

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Example Poisson Distributions

4. What is the probability of exactly one flaw in the first 50m and one flaw in the second 50m of cable?

Is this the same as two faults in 100m?

$$Pr(k = 2) = \frac{2.4^2 e^{-2.4}}{2!}$$

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Linking the Binomial and the Poisson Distributions

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Binomial Distribution

$$\text{Pr(X=k)} = Pr(k) = \binom{n}{k} p^k q^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$k = 1,2,3, \dots, n, 0 < p < 1$$

The random variable X that counts the **k successes** in **n trials**

Poisson Distribution

X has a Poisson Distribution with parameter λ and

$$Pr(X = k) = f(k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \dots$$

Where X is the number of **k events (successes)**

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Poisson and Binomial Distribution

- When
- $n \to \infty$
- $p \to 0$
- Then $\lambda = np$

$$\lim_{\substack{n \to \infty \\ p \to 0 \\ np = \lambda}} \binom{n}{k} p^k q^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!}$$

This will work ok n>100 p<0.05 is an acceptable level

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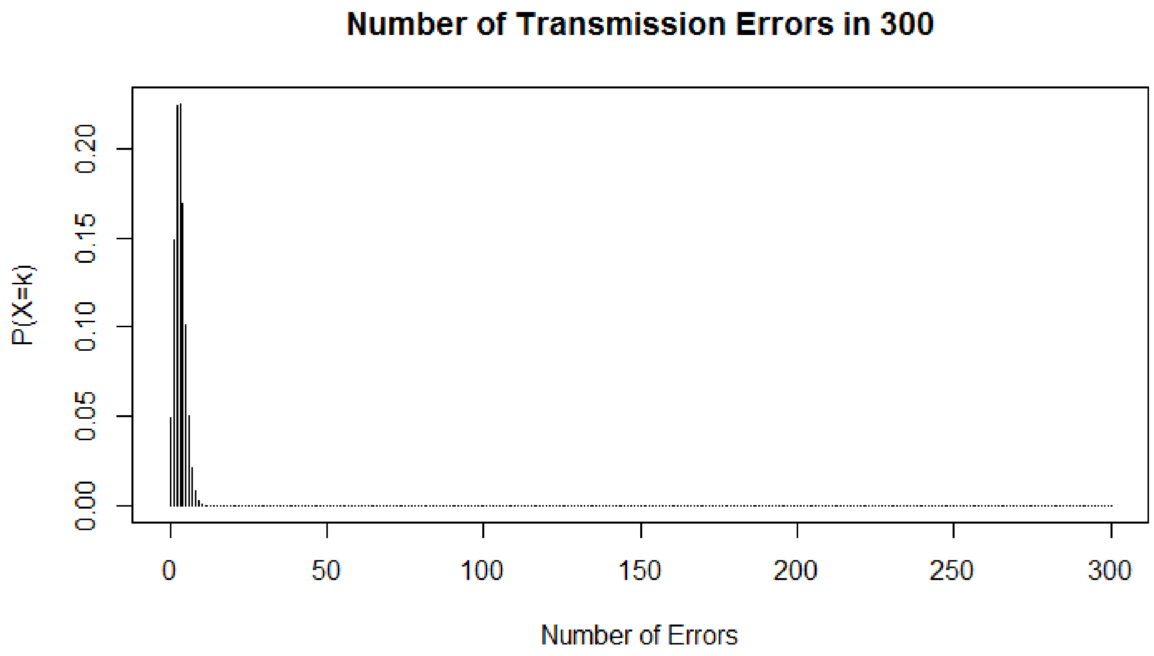
Example- Transmission Error

1. One percent of bits transmitted through a digital transmission are received in error.
2. Let k denote the bit errors in 300 transmission.

Example- Transmission Error

Binomial

$$\Pr(X{=}k) = Pr(k) = \binom{300}{k} 0.01^k 0.99^{n-k}, k = 0,1 \ldots 300$$

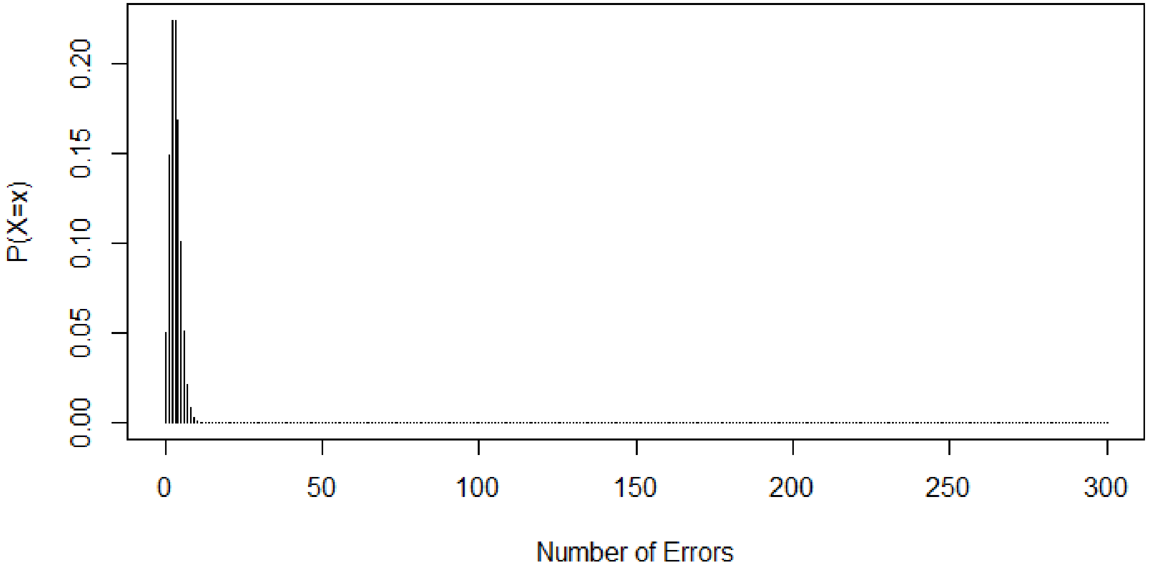


Example- Transmission Error

Poisson

$$\Pr(X{=}k) = Pr(k) = \frac{3^k e^{-3}}{k!}$$

Number of Errors



Recap of Discrete Distributions

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Probability distribution

- Suppose X is a discrete RV which can take the values

$$x_1, x_2, \ldots, x_k$$

- where $x_1 < x_2 < \ldots < x_k$ (k can be finite of infinite)
- The probability distribution can be written in tabular form

x	x_1	x_2	…	x_k
Pr(x)	$Pr(x_1)$	$Pr(x_2)$	…	$Pr(x_k)$

- The distribution can also be represented by a mathematical function that give the probability of each x occurring
- Pr(x) =function of x, known as the Probability mass function

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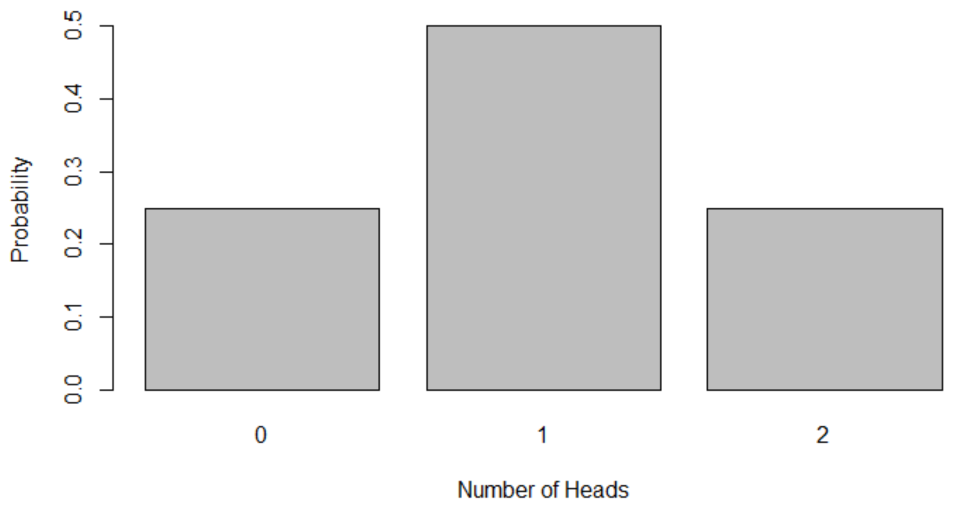
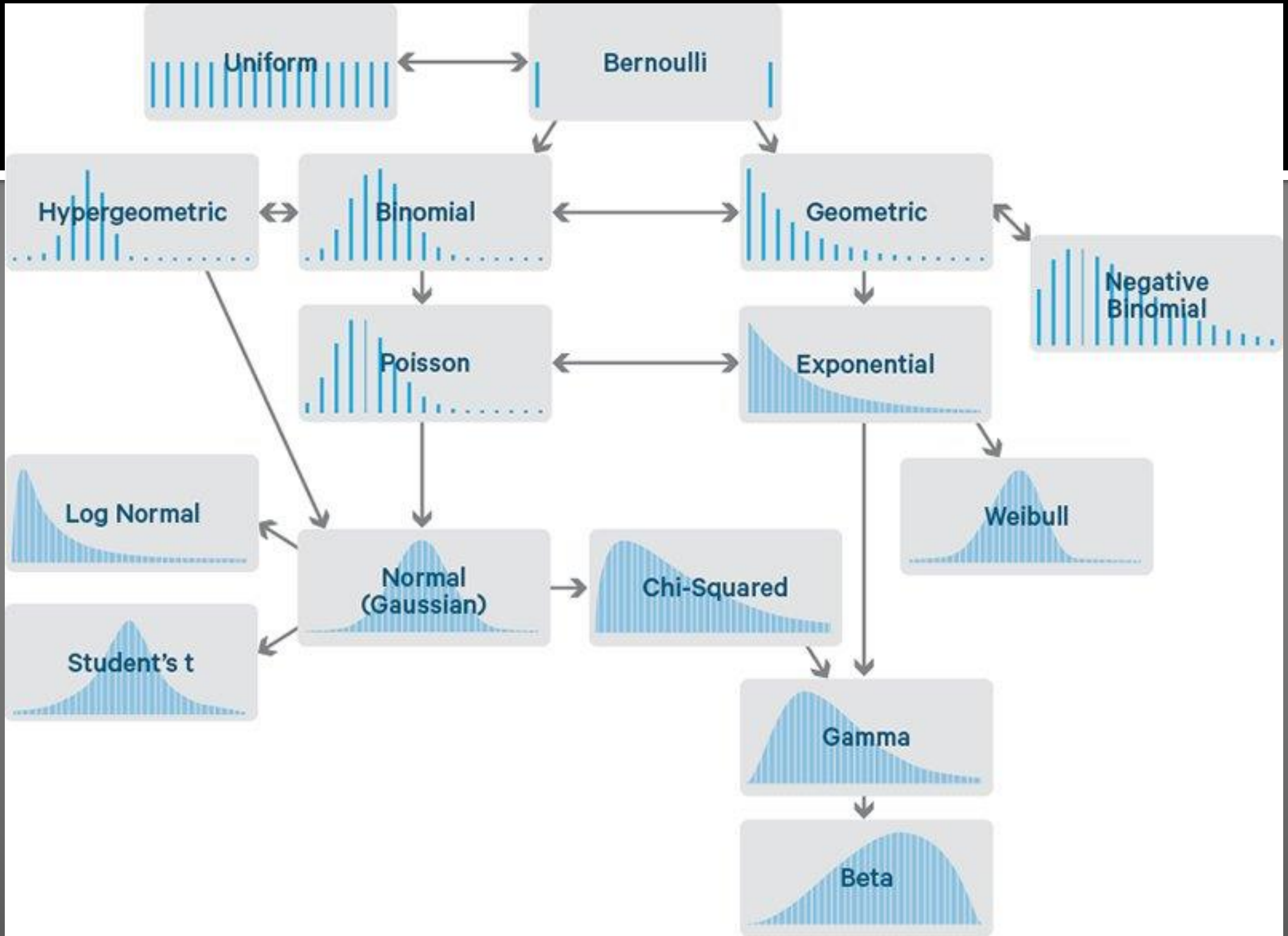
Tossing two coins

- The probability mass function

x	0	1	2
Pr(x)	0.25	0.5	0.25



A Map of Distributions



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Geometric Distribution

Pr(X=x) $= p(x) = q^{x-1}p = (1-p)^{x-1}p$

$x = 1,2,3, \dots, \infty, 0 < p < 1$

This is the probability mass function for a **geometric distribution**

ENDS WHEN YOU SUCCEED

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Binomial Distribution

$$\Pr(X=k) = Pr(k) = \binom{n}{k} p^k q^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$k = 0,1,2,3, \ldots, n, 0 < p < 1$$

This is the probability mass function for a **binomial distribution.** The random variable X that counts the **k successes** in

n trials

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Negative Binomial Distribution

$$\Pr(X=x) = Pr(x) = \binom{x-1}{r-1} p^r q^{x-r}$$

$$x = r, r+1, r+2, \dots, \infty, \qquad 0 < p, q < 1$$

This is the probability mass function for a **negative binomial distribution.** The random variable X that counts the **r specific successes** in **x trials**

ENDS WHEN YOU SUCCEED r times

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Poisson Distribution

We assume the following

- λ is the mean number of successes in the interval

X has a Poisson Distribution with parameter λ and

$$Pr(X = k) = f(k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \ldots$$

The random variable **X denotes the number of successes** in the whole interval

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Takeaway Point

- **The Poisson distribution** models the number of events occurring in a fixed interval of time or space, assuming events happen independently and at a constant average rate.

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