Mathematical Probability

Dr. John S. Butler

Mathematical Probability

- Basis of inferential statistics
- Modelling real-life stochastic systems

Experimental (simulated) data

- The type of experiments we are interested have the following characteristics

1. When the experiment is performed, several outcomes are possible
2. We cannot predict with certainty the outcome

Examples

- Toss of a coin
- Number of cars owned by a household
- The value of stocks in six minutes time
- Voting
- Reactions to drug trials
- Did you see the light?
- What is the likelihood you will have heard of synaesthesia?
- What is the likelihood you will have synaesthesia?

Definitions

- Define some event A that can be the outcome of an experiment

- p(A) is the probability of a given event A will happen

- p(A) is a number between 0 and 1, $0 \leq \mathrm{p}(A) \leq 1$

Definitions

- If p(A)=1 the event will definitely happen
- If p(A)=0 the event will definitely not happen
- If p(A)= 0.5 it is 50-50
- **If p(A)=0.05 it is considered not likely**

Algebra of Probability

- The **Sample Space** of an experiment is the universal set of all possible outcomes for that experiment, defined so, no two outcomes can occur simultaneously.

- The sample space is a function of the recording method

Examples

- Throwing a die S={1,2,3,4,5,6}

- Tossing a coin twice S={HH,TH,HT,TT} or S={0, 1, 2} if we report number of heads (or tails) observed

- Tossing a coin until you get a head
  S={H,TH,TTH,TTTH,TTTTH….}
  or the number of tosses required S={1,2,3}

- Recording the lifetime of a battery S={t|t≥0}

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Event

- An event is a subset of the sample space S
- Any event A defined on S is a subset of the space

- Example throwing the dice

$$A=\{2,4,6\} \quad A=\{2\} \cup \{4\} \cup \{6\}$$

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All other events

- If $A \subset S$ that the event that is not A ($\bar{A}$) is the set of all other outcomes, **the complement**
- The whole set S is defined as all the events in A and in $\bar{A}$

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All other events

Example

- If you need an even number to win, all odd number will loose

- If A={2,4,6} then $\bar{A} = \{1,3,5\}$

S={1,2,3,4,5,6}

Combination of Events

- Consider any two events $A, B \subset S$
- Then the following events are also defined on S
- Extend to more than 2 events

Combination of Events

Back to the Dice

- Given

$$\mathrm{A} = \{2,6\}, \mathrm{B} = \{1,3,6\}$$

Calculate

$$A \cup B = \{1,2,3,6\}$$

- A occurs or B occurs

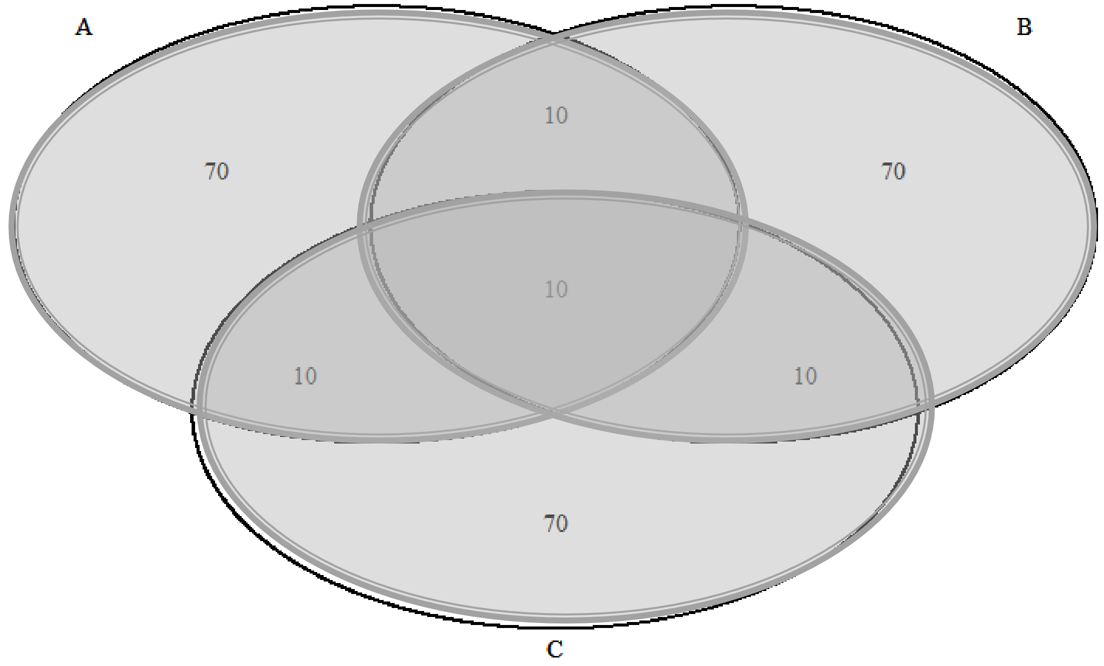
$$A \cap B = \{6\}$$

- Extend to more than 2 events

Union ∪

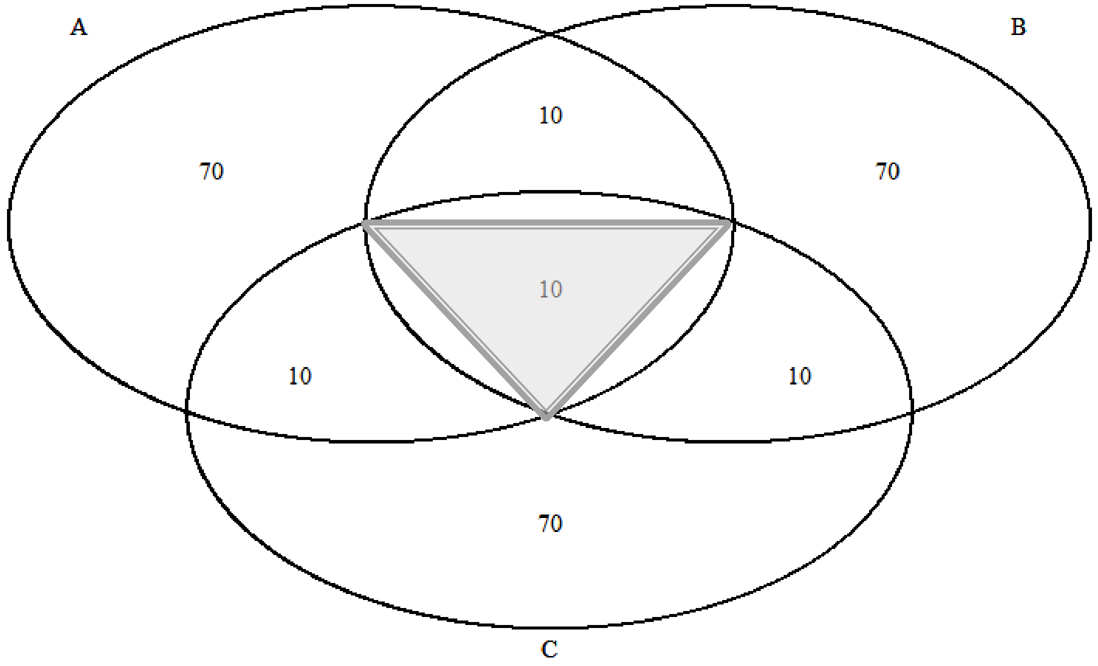
- Is the event consisting of all outcomes contained in one of more of A, B, C

- Occurs in at least one of A, B or C occurs



Intersection ∩

- Event consisting of all outcomes of A, B, C
- Occurs if the events occur in A, B and C

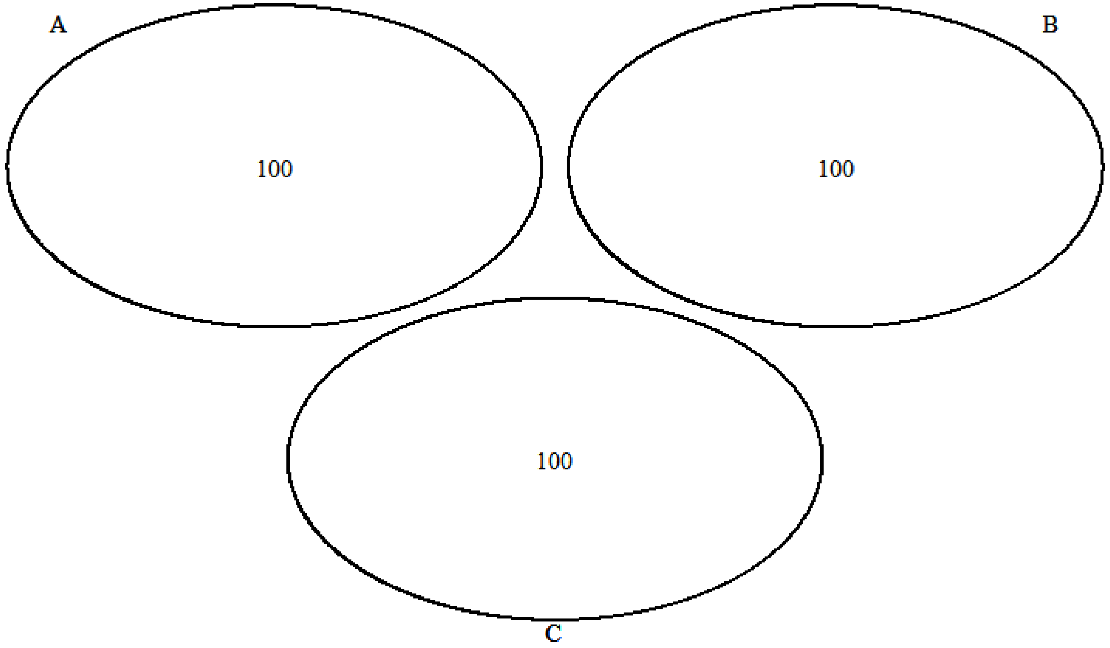


Mutually Exclusive

- If A and B cannot occur simultaneously, they are termed **mutually exclusive** (or **disjoint)**

$$A \cap B = 0$$

- If events A, B and C are such that the occurrence of 1 makes the occurrence of the other impossible
- A set of events A, B, C are termed exhaustive if $A \cup B \cup C$=S



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Axioms and basic properties of probabilities

For an event A subset S associated a number p(A), the probability of A, which must have the following properties

Axiom 1: $0 \leq p(A) \leq 1$

Axiom 2: $p(S) = 1$

Axiom 3: If $p(A \cap B) = 0$

$$p(A \cup B) = p(A) + p(B)$$

Axioms and basic properties of probabilities

And more generally for any sequence of mutually exclusive events A, B, C,…

$$p(A \cup B \cup \cdots) = p(A) + p(B) + \cdots$$

- If the set of events A, B, C, is exhaustive than

$$p(A \cup B \cup C) = p(A) + p(B) + P(C) = 1$$

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Axioms and basic properties of probabilities

- *Probability of the Null Event*

$$p(\emptyset) = 0$$

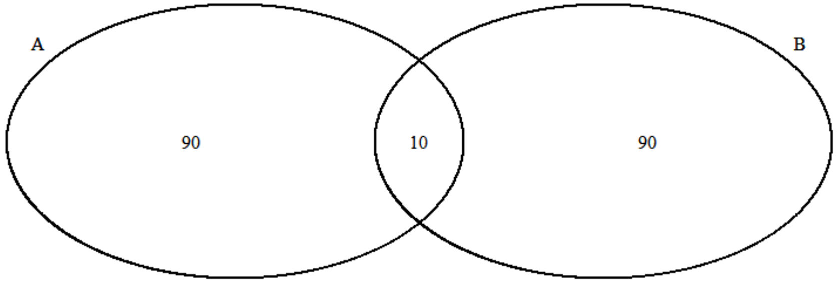
Complementary Rule

$$p(\bar{A}) = 1 - p(A)$$

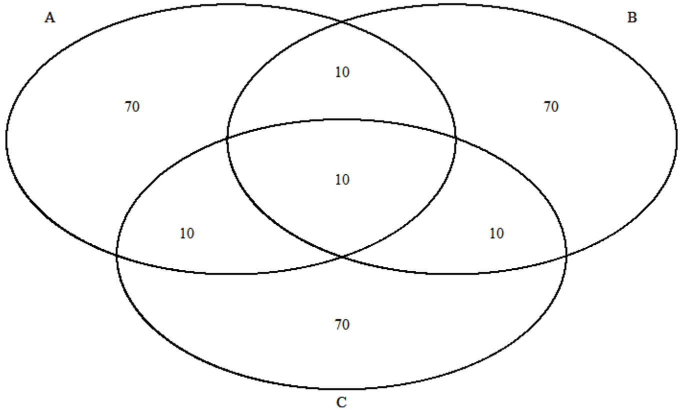
- If A is a subset of B
- The p(A) less than of equal to p(B)

Axioms and basic properties of probabilities

- **Addition Law**
- $p(A \cup B) = p(A) + p(B) - p(A \cap B)$



- **Extended**
- $p(A \cup B \cup C) = p(A) + p(B) + p(C) - p(A \cap B) - p(A \cap C) - p(B \cap C) + p(A \cap B \cap C)$



Example -Dice

- If A={2,4,6} then $B = \{2,3\}$
- $A \cap B = \{2\}$
- $A \cup B = \{2,4,5,6\}$
- {2,4,6} and $\{2,3\}$
- $$\begin{aligned}(A \cup B) &= (A) + (B) - (A \cap B) \\ &= \{2,4,6\} + \{2,3\} - \{2\}\end{aligned}$$
- $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

Classical Methods for Assigning Values to Probabilities

- The number of outcomes in the sample space is finite

- The outcomes are intuitively equally likely
  i.e. $S = \{E_1, \dots, E_N\}$ where the outcomes of elementary events $E_1, \dots, E_N$ are mutually exclusive and exhaustive and equally likely.
- In this situation, the classical definition of the probability of the Event is

$$E_i = \frac{1}{N}$$

Classical Methods for Assigning Values to Probabilities

- An event A can be expressed in the form

$$A = E_1 \cup \cdots \cup E_m$$

- Where these elementary events are exclusive.

- $p(A) = \dfrac{\text{number of outcomes in A}}{Total\ Events}$
- $= P(E_1) + \cdots + P(E_m) = \dfrac{m}{N}$

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Takeaway Point

- **Mathematical probability provides the framework for quantifying uncertainty. Understanding sample spaces, events, and probability rules is essential for modelling and analysing real-world stochastic systems.**

Examples

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Examples

1. Two fair dice are thrown. What is the probability that the sum is great than 5

Outcome table of two dice

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Example

1. Three coins are tossed. What is the probability of observing 3 heads or 3 tails