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Discrete Probability Distributions

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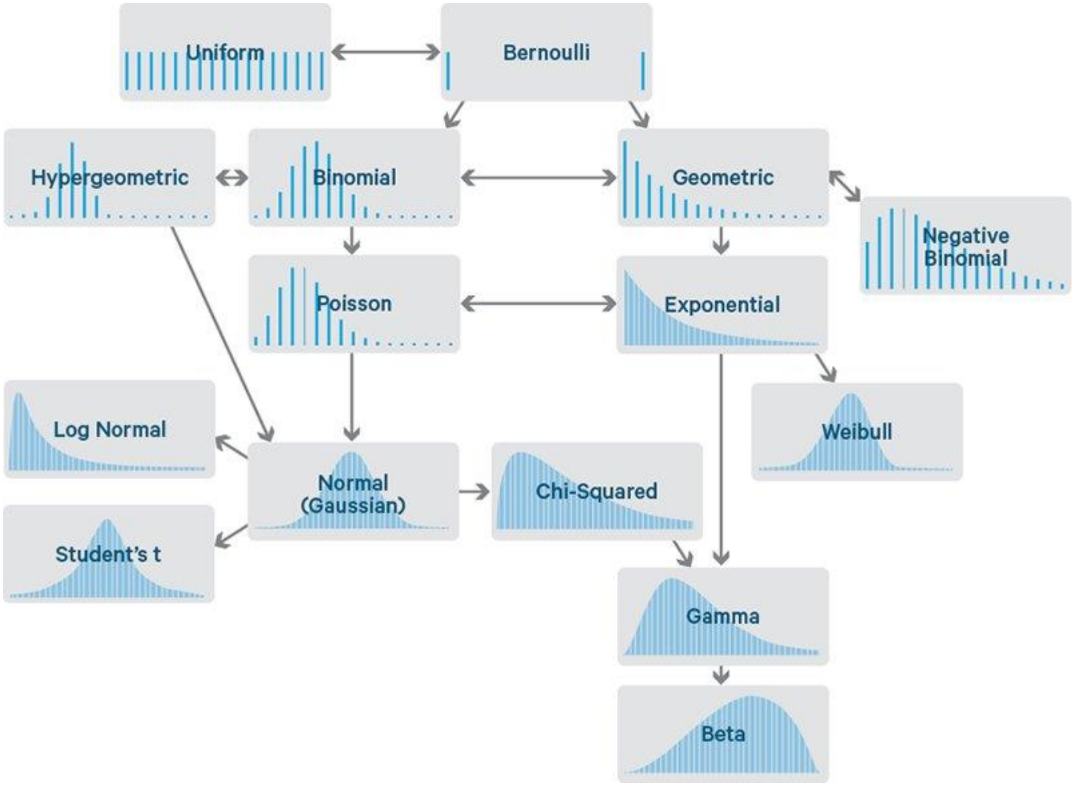
Types of Distributions



Discrete Random Variables and Probability Distributions

- A random variable (RV) is a real-valued function defined on the sample space S of a random experiment. A discrete RV is one can take a finite or countably infinite number of values (i.e. Values can be listed).

Discrete Random Variables and Probability Distributions

- Notation:
  - Capital letter, e.g., X, = a Random Variable
  - Corresponding small letter e.g. x, one of its values
  - Thus, x is an observation on X

Each possible value x of X represents an event i.e., a subset of the sample space, and hence has an associated probability

Example 1

- Toss 2 distinguishable coins, S has 4 outcomes

  E_1=HH, E_2=HT, E_3=TH, E_4=TT
- Let X=number of heads obtained

- Then
  - X=2, if E_1 occurs $E_1 \rightarrow 2$
  - X=1, if E_2 occurs $E_2 \rightarrow 1$
  - X=1, if E_3 occurs $E_3 \rightarrow 1$
  - X=0, if E_4 occurs $E_4 \rightarrow 0$

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Example

- Time to failure of an item of equipment $S=\{t|t>0\}$
- Suppose we define:

- $X=1$, if $0 < t < 100$
- $X=2$, if $100 < t < 500$
- $X=3$, if $500 > t$

- This is an example of a discrete random variable defined on a continuous sample space

Probability distribution

- Suppose X is a discrete RV which can take the values

$$x_1, x_2, \ldots, x_k$$

- where $x_1 < x_2 < \ldots < x_k$
- (k can be finite of infinite)
- The probability distribution can be written in tabular form

x	x_1	x_2	…	x_k
$p(x)$	$p(x_1)$	$p(x_2)$	…	$p(x_k)$

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Probability distribution

- The probability distribution can be written in tabular form

x	x_1	x_2	…	x_k
$p(x)$	$p(x_1)$	$p(x_2)$	…	$p(x_k)$

- The distribution can also be represented by a mathematical function that give the probability of each x occurring
- Pr(x) =function of x, known as the Probability mass function

Properties of a Probability Mass function

1. p(x)≥0 for x
2. The following must hold

$$\sum_{j=1}^{k} p(x_j) = 1$$

3. Each elementary event in S maps onto exactly one x_j Hence

$$\sum_{j=1}^{k} p(x_j) = p(S) = 1$$

http://www.itl.nist.gov/div898/handbook/eda/section3/eda361.htm

Probability Mass function

Any function p(x) satisfying these properties may be considered as a probability density mass function

Example - Coins

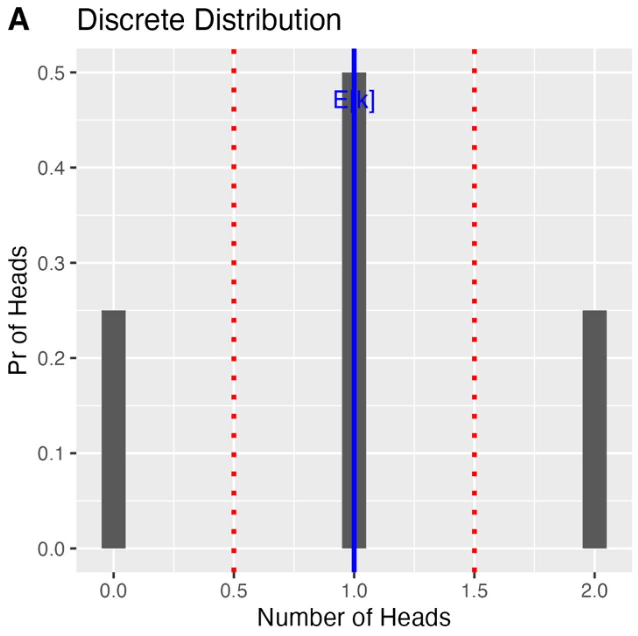
- Tossing 2 fair distinct coins
- Let X be the number of heads obtained

- What is the probability distribution?
- What is the cumulative probability distribution?

Tossing two coins

- The probability mass function

x	0	1	2
p(x)	0.25	0.5	0.25



Cumulative Distribution Function

The Function

$$F(r) = p(X \leq r), \qquad -\infty < r < \infty$$

is called the **Cumulative Distribution Function(CDF)** of the RV X

$$F(r) = p(X \leq r) = \sum_{x \leq r} p(X = x) = \sum_{x \leq r} p(x)$$

Where $\sum_{x \leq r}$ denotes summation over the values of x

Properties of a Cumulative Distribution Function

1. $F(r) = 0$ when r $< x_1$
2. $F(r) = 1$ when r $> x_{k}$`
3. $0 \leq F(r) \leq 1$
4. $F(\infty) = 1$
5. $F(a) < F(b)$ when $a < b$

Calculating a Distribution Function from CDF

We can derive the probability function from the CDF as we have:

- $p(x_1) = F(x_1)$
- $p(x_j) = F(x_j) - F(x_{j-1})$, for $\mathrm{j} = 2, \dots, \mathrm{k}$

$$p(a < X \leq b) = \sum_{a<X\leq b} p(x)$$

$$= \sum_{X\leq b} p(x) - \sum_{X\leq a} p(x) = F(b) - F(a)$$

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Calculating a Distribution Function from CDF

and

$$p(a \leq X \leq b) = p((X = a) \cup (a < X \leq b))$$
$$= p(a) + p(a < X \leq b)$$
$$= p(a) + F(b) - F(a)$$

Cumulative Distribution Function

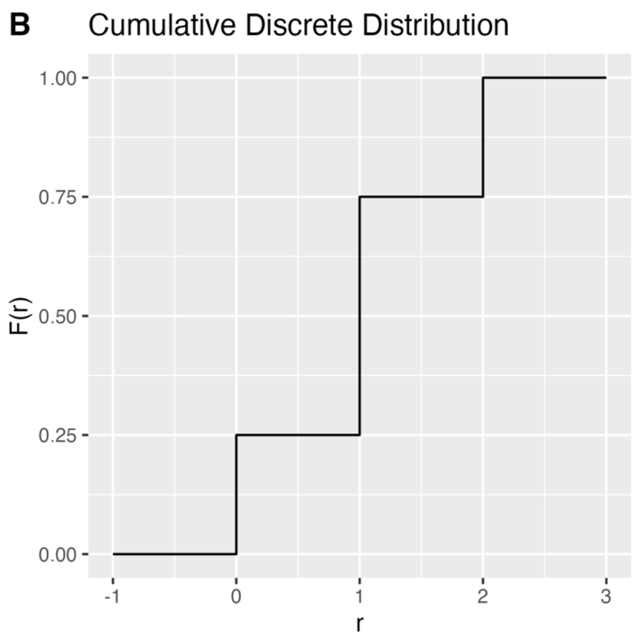
- The cumulative probability distribution can be written in tabular form

r	$<x_1$	$x_1<=r<x_2$	...	$r>X_k$
F(r)	0	$p(x_1)$	...	1

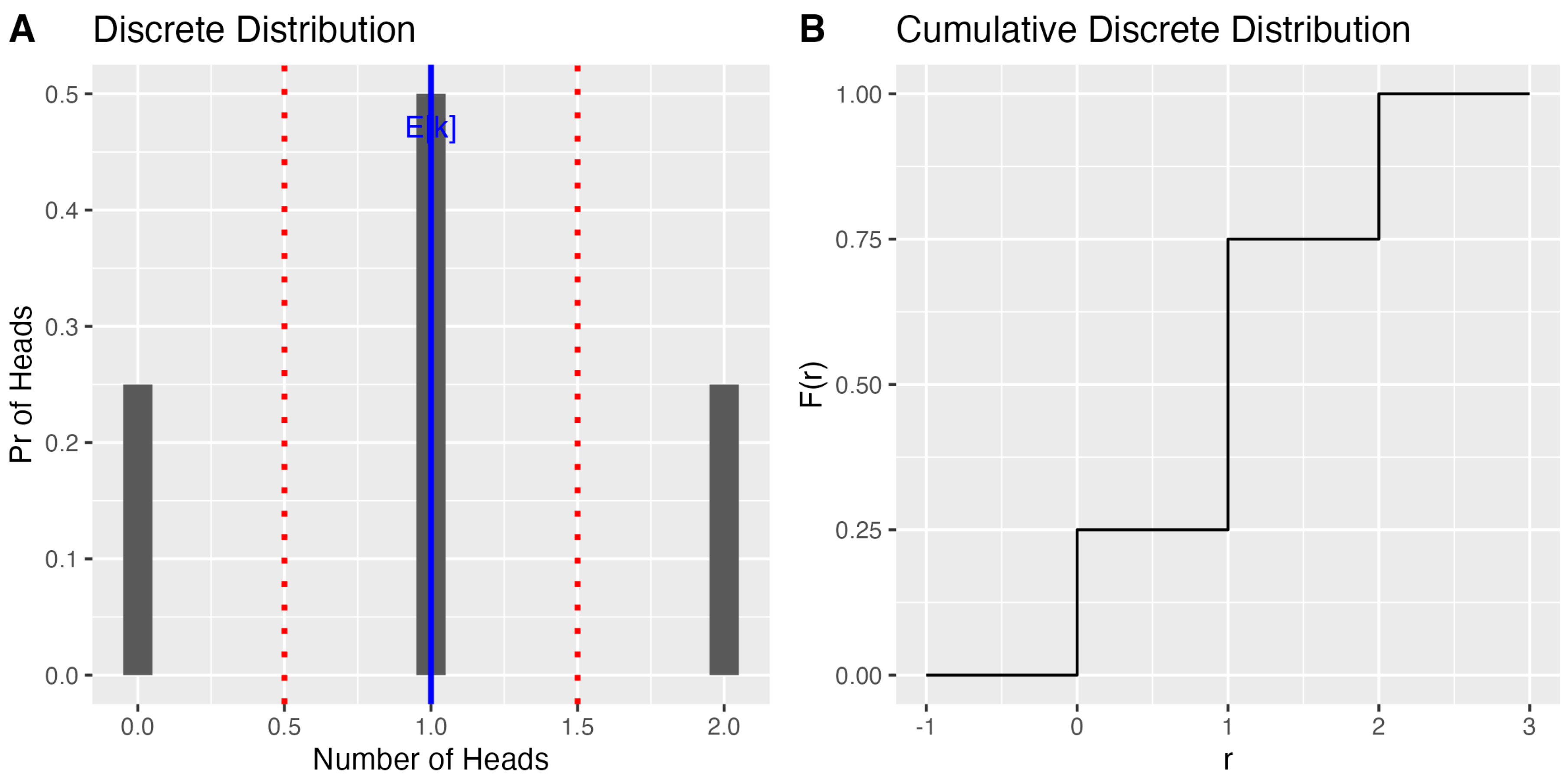
Example-Tossing two coins

- Cumulative Probability Function

r	<0	0<=r<1	1<=r<2	r<=2
F(r)	0	0.25	0.75	1



Tossing two coins



Example 2- Die roll

- Rolling a die let X be the number on the die
- What is the probability mass distribution
- What is the cumulative probability distribution

Expectation: Mean

- The expected value (or expectation) of a random variable (RV) X is denoted by E[X] or μ. For a discrete random variable the expected value is defined as

$$E[X] = \sum_{j=1}^{k} x_j p(x_j)$$

- Other terms for $E[X]$
- The mean of the probability distribution of X
- Mean value of X
- The population mean

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Expectation: Mean

- $E[X]$ gives some ideas of the “center” of the distribution of the values x_1, x_2,…, x_k taking into account the probability distribution

- E[X] does not have to be possible value in set.

Expectation: Variance

- The **variance** of a RV X (or the variance of the probability distribution of X or the population variance) is denoted by Var[X] or sigma^2 and defined as

$$Var[X] = \sigma^2 = E[(X-\mu)^2] = \sum_{j=1}^{k} (x_j - \mu)^2 p(x_j)$$

- It is a measure of a variation (or spread) of the values of X and the mean, considering the probability distribution of X.

Expectation: Variance

- Note that Var[X]>=0.
- The larger Var[X], the more spread out are its values
- Var[X] are in units squared
- A related measure of variation is the standard deviation (SD)

$$SD(X) = \sigma = \sqrt{Var[X]}$$

Expectation: Variance alternative calculation

- The **variance** of a RV X (or the variance of the probability distribution of X or the population variance) is denoted by Var[X] or sigma^2 and defined as

$$Var[X] = \sigma^2 = E[X^2] - E[X]^2 = E[X^2] - \mu^2$$

Properties of the Variance Operator Var[]

1. $g(X)=a+X$ where a is a constant then
   $Var[a+X]=Var[X]$
2. $g(X)=bX$ where b is a constant then
   $Var[bX]=b^2\,Var[X]$
3. $g(X)= a+bX$, where a and b are constants
   $Var[a+bX]=b^2\,Var[X]$

Properties of the Variance Operator Var[]

4. Two discrete Random Variables are termed independent if $p(x \cap y) = p(x)p(y)$ for all variables x and y

If X,Y are independent Random Variables, then

Var[X+Y]=Var[X]+Var[Y] and Var[X-Y]=Var[X]+Var[Y]

More generally for a linear combination, if $X_1 \ldots X_n$ are n random variables and $a_1 \ldots a_n$ are constants and $Y = \sum_{i=1}^{n} a_i X_i$ then

$$Var[Y] = Var\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} Var[a_i X_i] = \sum_{i=1}^{n} a_i^2 Var[X_i]$$

Example - Coins

- Tossing 2 fair distinct coins
- Let X be the number of heads obtained

x	0	1	2
p(x)	0.25	0.5	0.25

- What is the expected value?
- What is the Variance?

Example – Coins

What is the expected value?

x	0	1	2
p(x)	0.25	0.5	0.25
xp(x)	0×0.25=0	1×0.5=0.5	2×0.25=0.5

$$E[X] = \sum_{j=1}^{k} x_j p(x_j) = 0 \times 0.25 + 1 \times 0.5 + 2 \times 0.25 = 1$$

- What is the Variance?

Example – Coins

- What is the Variance?

x	0	1	2
p(x)	0.25	0.5	0.25
xp(x)	0	0.5	0.5
$(x-1)^2p(x)$	$(0-1)^2 \times 0.25=0.25$	$(1-1)^2 \times 0.5=0$	$(2-1)^2 \times 0.25=0.25$

$$Var[X] = \sum_{j=1}^{k}(x_j - \mu)^2 p(x_j) = 0.25 + 0 + 0.25 = 0.5$$

Example – Coins

- What is the Variance (alternative calculation)?

x	0	1	2
p(x)	0.25	0.5	0.25
x^2	0	1	4
$x^2p(x)$	$0^2 \times 0.25=0$	$1^2 \times 0.5=0.5$	$2^2 \times 0.25=1.0$

$$E[X^2] = \sum_{j=1}^{k} x^2 p\left(x_j\right) = 0 + 0.5 + 1 = 1.5$$

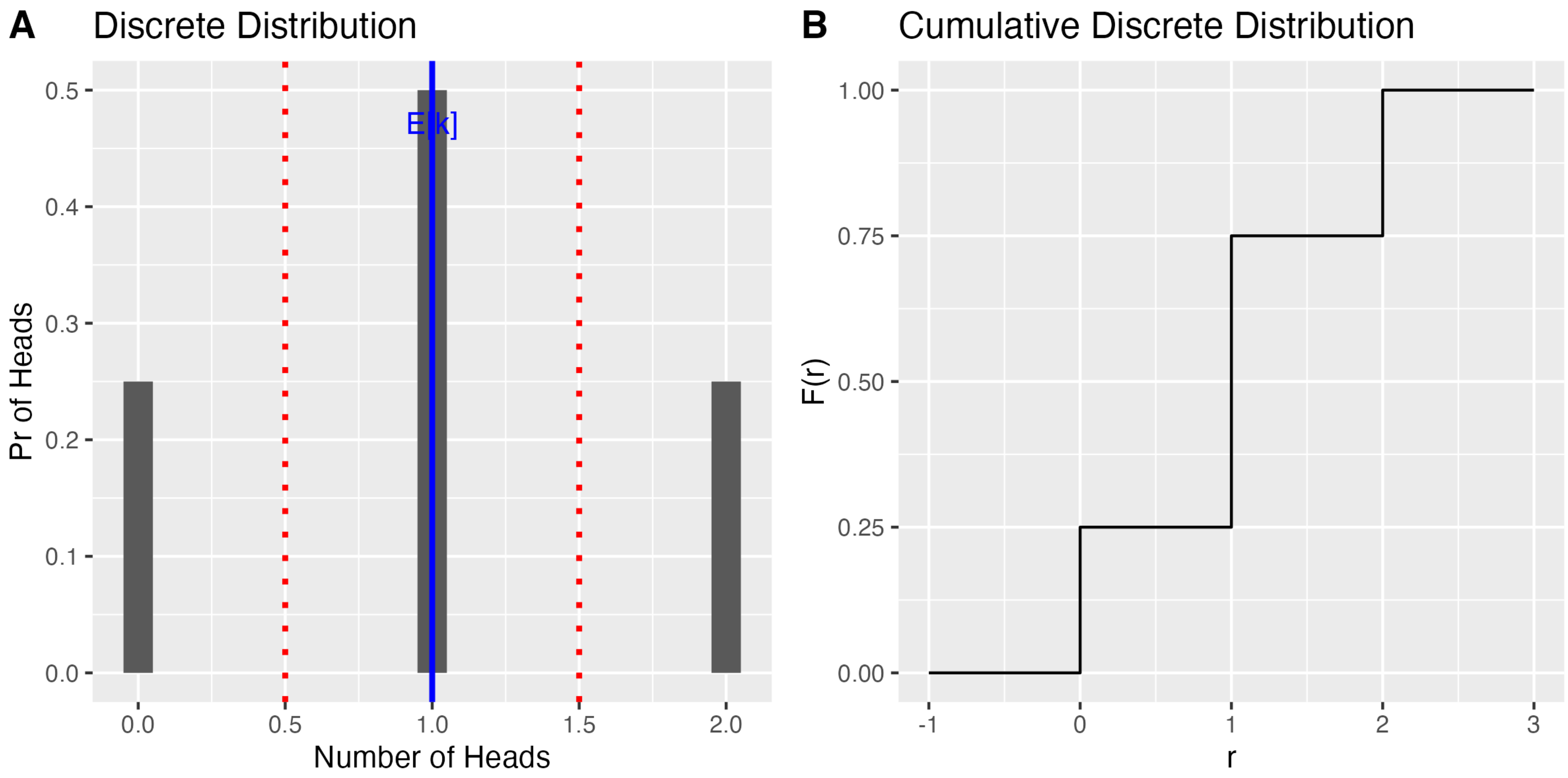
$$Var[X] = E[X^2] - \mu^2 = 1.5 - 1^2 = 0.5$$

Example – Coins

x	0	1	2	Sum
p(x)	0.25	0.5	0.25	1
xp(x)	0	0.5	0.5	1
$(x-1)^2p(x)$	0.25	0	0.25	5

$$E[X] = 1$$
$$Var[X] = 0.5$$

Tossing two coins

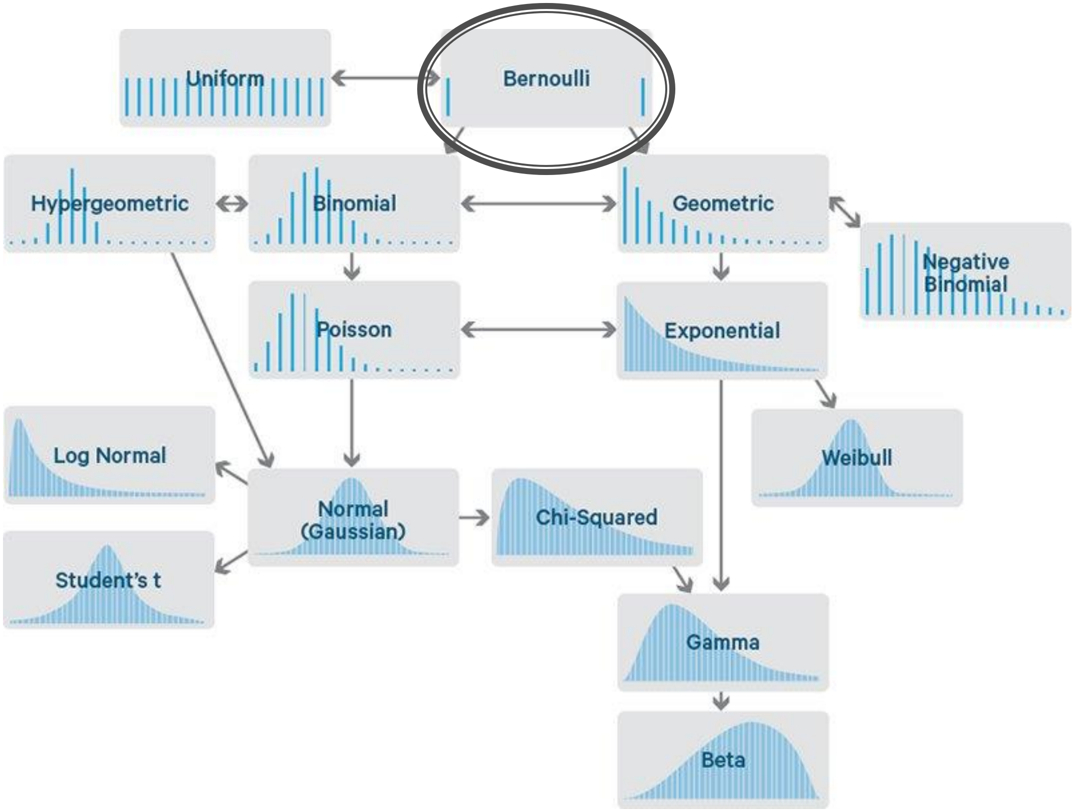


Example 2- Die

- Rolling a die
- Let X be the number on the die

- What is the expected value?
- What is the Variance?

Bernoulli Distribution



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