Bernoulli Distribution
Geometric Distribution

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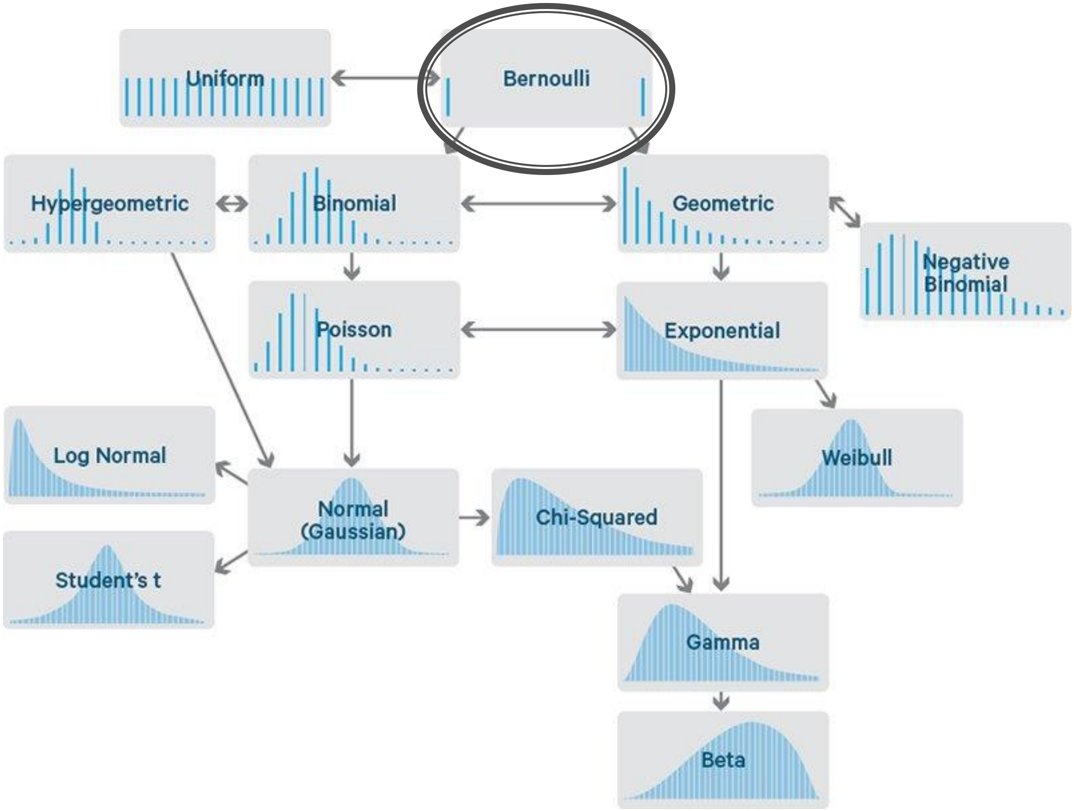
Bernoulli Distribution



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Bernoulli Distribution

- A Bernoulli trial generates one of two possible outcomes – “success” or “failure”

- Define the Random Variable
- $X=1$ if a success
- $X=0$ if a failure

- The Random Variable is termed a Bernoulli RV with the Bernoulli probability distribution

Bernoulli Distribution

- The Random Variable is termed a Bernoulli RV with the Bernoulli probability distribution

$$\Pr(X=x_j)=p(x_j)=\begin{cases} p & for\ x_2 = 1 \\ 1-p & for\ x_1 = 0 \end{cases}$$

i	1	2
x_i	0	1
$P(x_i)$	1-p=q	p

Bernoulli Distribution

- From this we can show that

$$E[X]=p$$

$$Var[X]=pq$$

- Proof?

Example Bernoulli Distribution

The example we shall use to illustrate the Bernoulli probability distribution distributions is the New Zealand vs Ireland World Cup Rugby Quarter Final.

$$Pr(X=x_j)=p(x_j)=\begin{cases} 0.15 & for\ x_2 = 1\ Ireland\ Wins \\ 1-0.15 & for\ x_1 = 0\ Ireland\ Loses \end{cases}$$

i	1	2
x_i	0	1
$P(x_i)$	1-0.15=0.85	0.15

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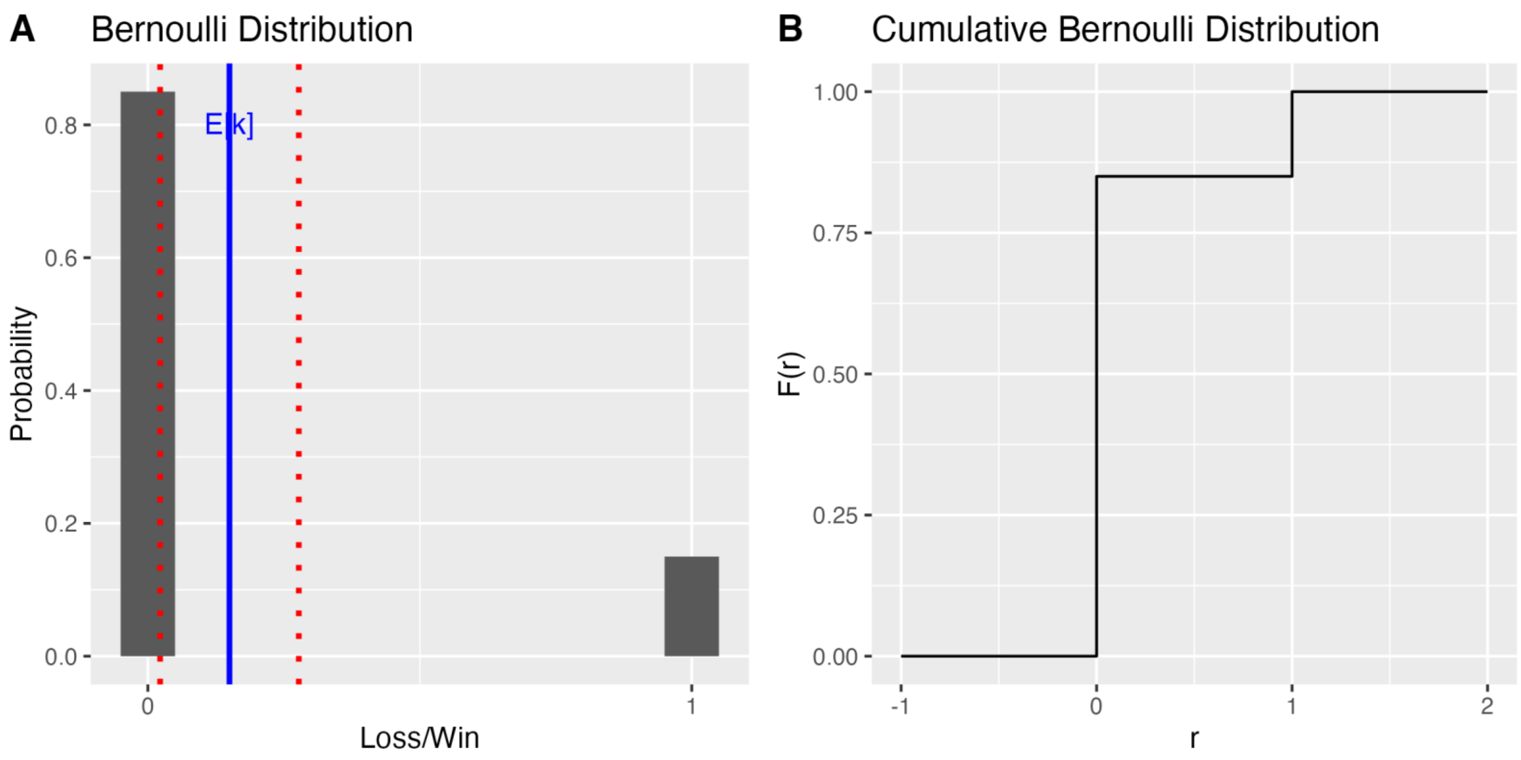
Example Bernoulli Distribution

$E[X]=0.15$

$Var[X]=0.125$



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Features of a Bernoulli Experiment

1. There are two possible outcomes arbitrary called success and failure
2. A success occurs with probability p and a failure occurs with probability q=1-p
3. The Random Variable is ordered as 1 if success and 0 if failure

Bernoulli Process

- What is the number of tosses of a fair coin required until the first Head?

- Let's take a specific situation.

- What is the probability that the fifth toss is a head

$$\Pr(\text{Fifth Toss is a Head})=\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right)\frac{1}{2}=\left(\frac{1}{2}\right)^4\frac{1}{2}$$

https://onlinecourses.science.psu.edu/stat414/node/76

Bernoulli Process

- What is the number of rolls of a fair die required until the first six?
- Let's take a specific situation.
- What is the probability that the fifth roll is a six

$$\text{Pr(Fifth Roll is a Six)}=\left(\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\right)\frac{1}{6} = \left(\frac{5}{6}\right)^4 \frac{1}{6}$$

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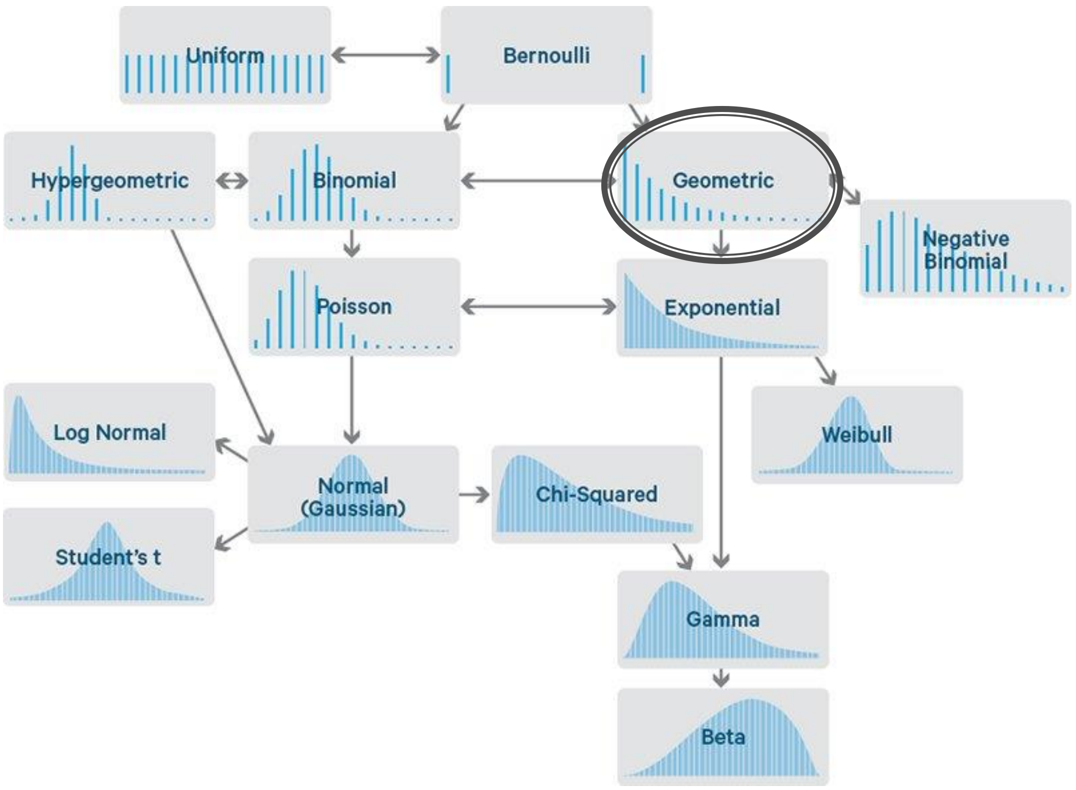
Geometric Distribution



Geometric Distribution

I play until I win

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Geometric Distribution

$$\Pr(X{=}x) = p(x) = q^{x-1}p = (1-p)^{x-1}p$$

$$x = 1,2,3,\ldots,\infty, 0 < p < 1$$

This is the probability mass function for a **geometric distribution**

Geometric Distribution

Is the sum of p all equal 1

$$\sum_{x=1}^{\infty} Pr(x) = p\sum_{x=1}^{\infty} q^{x-1} = p(1 + q + q^2 + \cdots)$$

$$\sum_{x=1}^{\infty} Pr(x) = p\sum_{x=1}^{\infty} q^{x-1} = \frac{p}{1-q} = \frac{p}{p} = 1$$

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Geometric Distribution

Is the sum of p all equal 1

$$E[X] = \sum_{x=1}^{\infty} xq^{x-1}p = \frac{1}{p}$$

$$\text{Var}[X] = \sum_{x=1}^{\infty}(x - \tfrac{1}{p})q^{x-1}p = \frac{q}{p^2}$$

Proof?

Coin Game

Toss a coin until you get a Tails

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Coin Toss - Geometric Distribution

q=Pr(H)=1-Pr(T)=0.5, p=Pr(T)=0.5

Pr(X=x) $= p(x) = 0.5^{x-1}0.5 =(1 - 0.5)^{x-1}0.5$

Number of times to play until a win, $x = 1,2,3, \ldots, \infty, 0 < p < 1$

$$E[X] = \sum_{x=1}^{\infty} x0.5^{x-1}0.5 = \frac{1}{0.5} = 2$$

Var$[X] = \sum_{x=1}^{\infty}\left(x - \frac{1}{0.5}\right)5 = \frac{0.5}{0.5^2} = 2$

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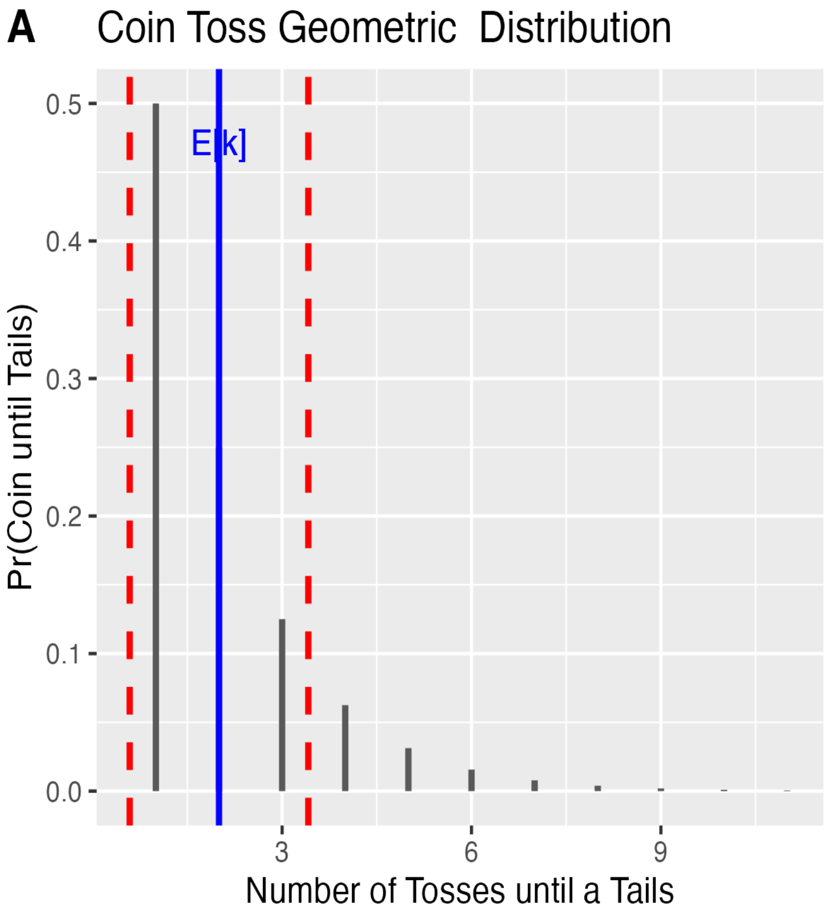
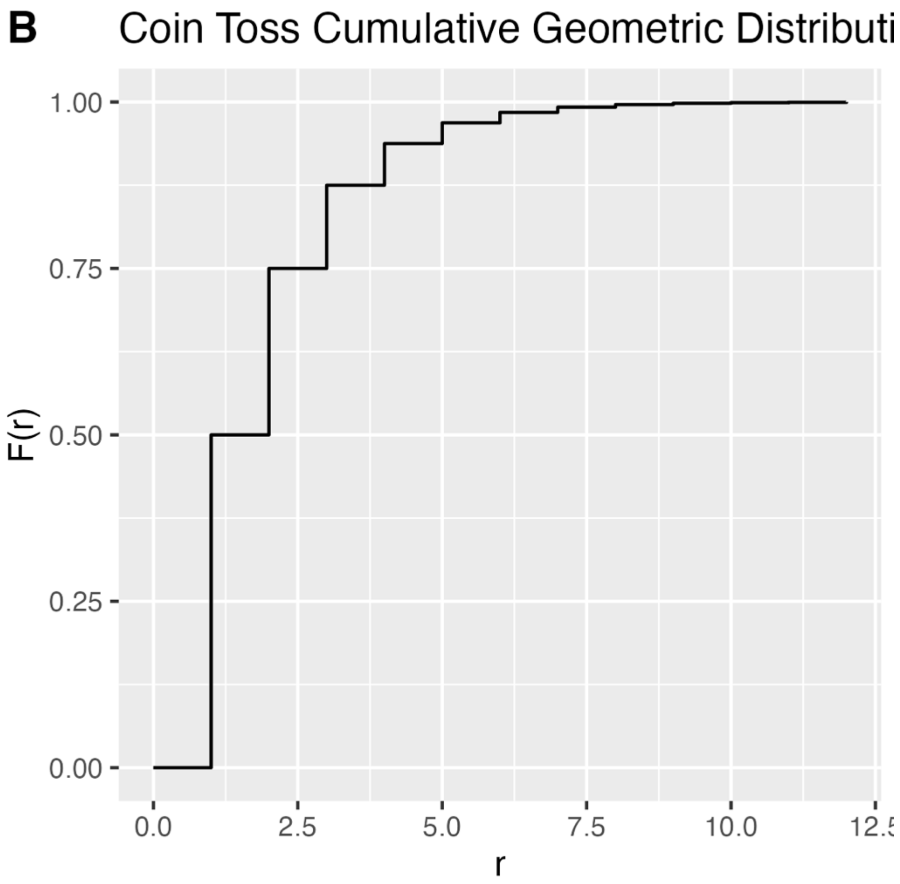
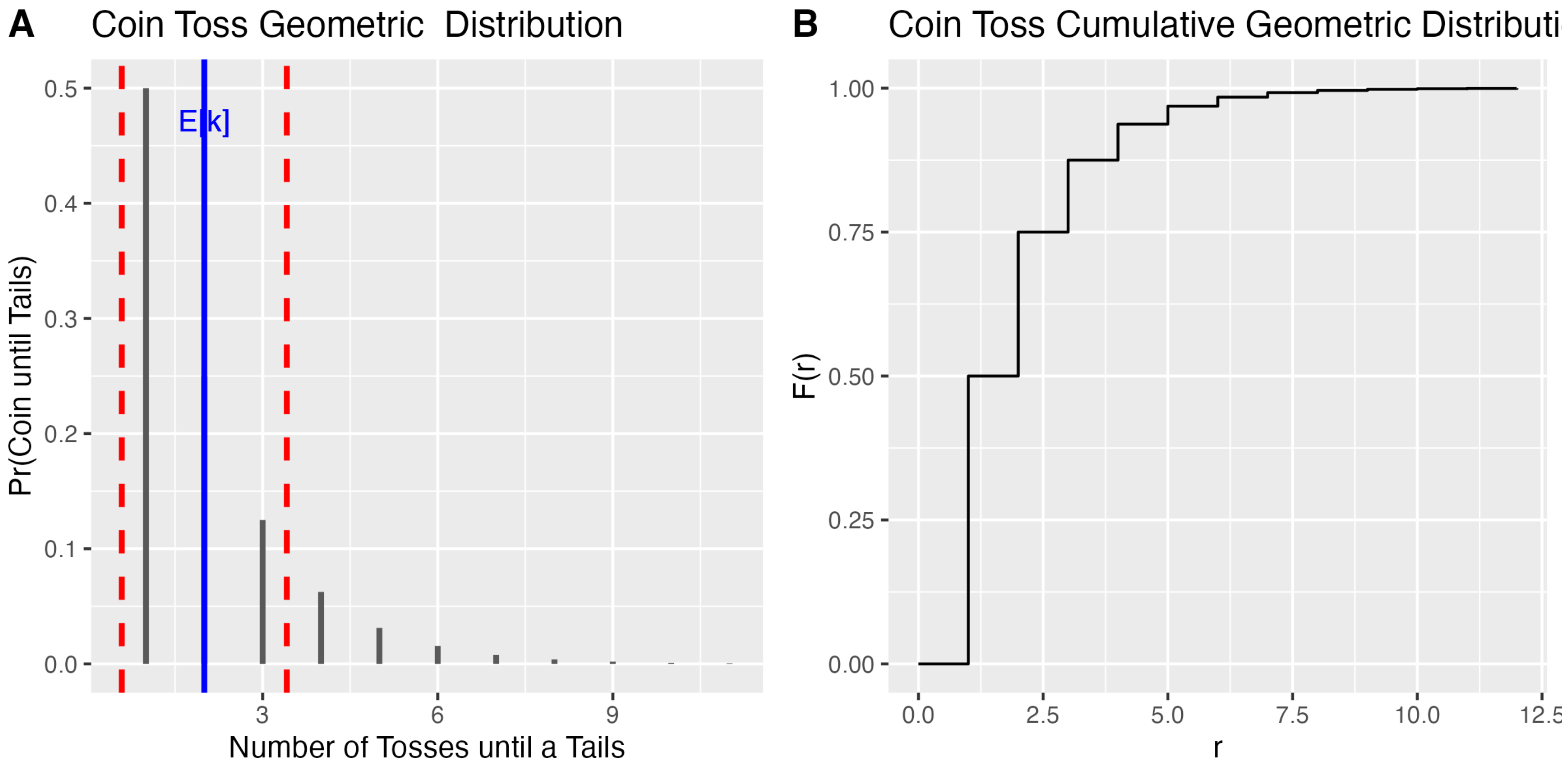
Coin Toss - Probability MASS Function



Coin Toss - Cumulative Distribution



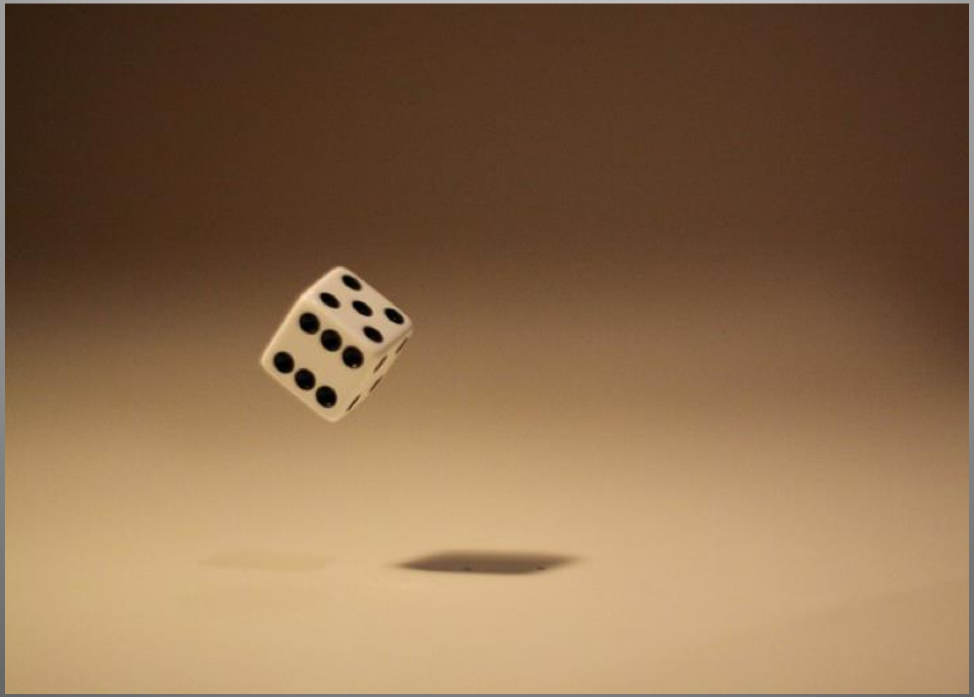
Coin Toss



A
Coin Toss Geometric Distribution
E[k]
Pr(Coin until Tails)
0.5
0.4
0.3
0.2
0.1
0.0
3
6
9
Number of Tosses until a Tails
B
Coin Toss Cumulative Geometric Distributi
F(r)
1.00
0.75
0.50
0.25
0.00
0.0
2.5
5.0
7.5
10.0
12.5
r

Dice Game

Roll Dice until you get a 6



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Dice Game-Geometric Distribution

q=Pr(1,2,3,4,5)=1-Pr(6)=$\frac{5}{6}$,

p=Pr(6)=$\frac{1}{6}$,

Pr(X=x) $= p(x) = \frac{5}{6}^{x-1}\frac{1}{6}$ $=(1-\frac{1}{6})^{x-1}\frac{1}{6}$

Number of times to play until a win, $x = 1,2,3,\ldots,\infty, 0 < p < 1$

$$E[X] = \frac{1}{\frac{1}{6}} = 6$$

$$\mathrm{Var}[X] = \frac{\frac{5}{6}}{\frac{1}{6}^{2}} = 30$$

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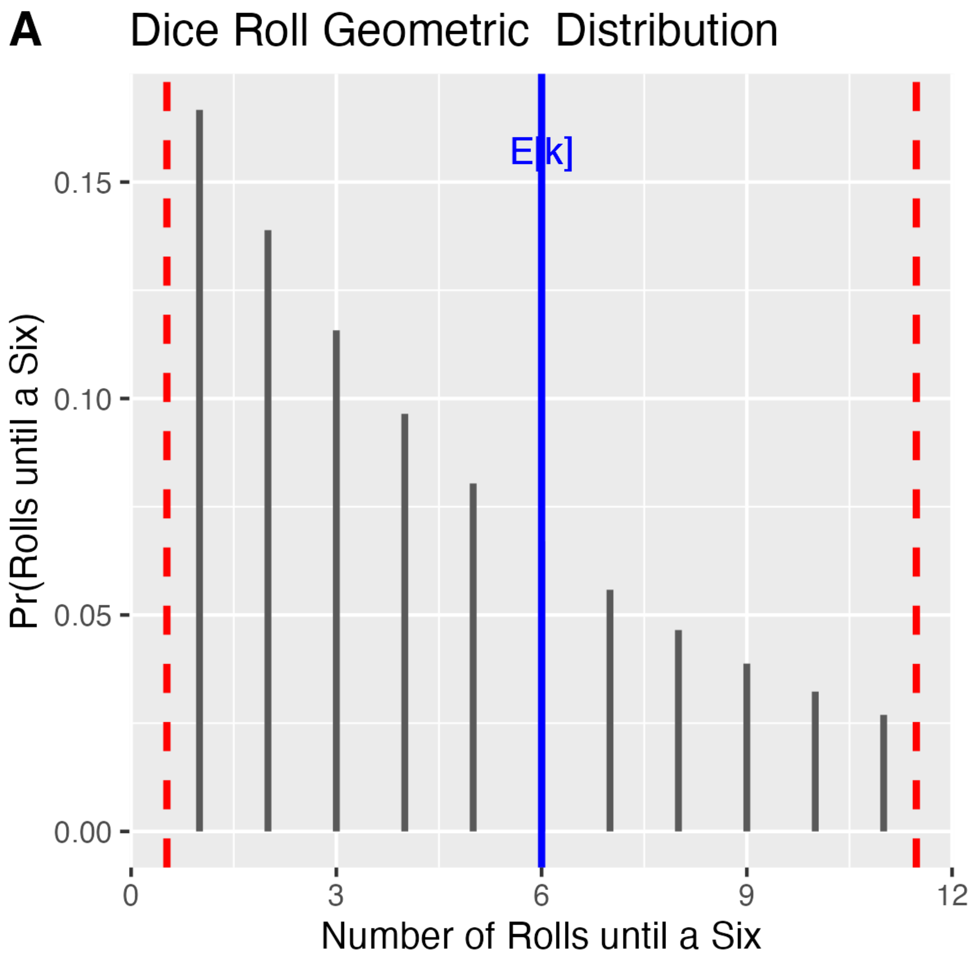
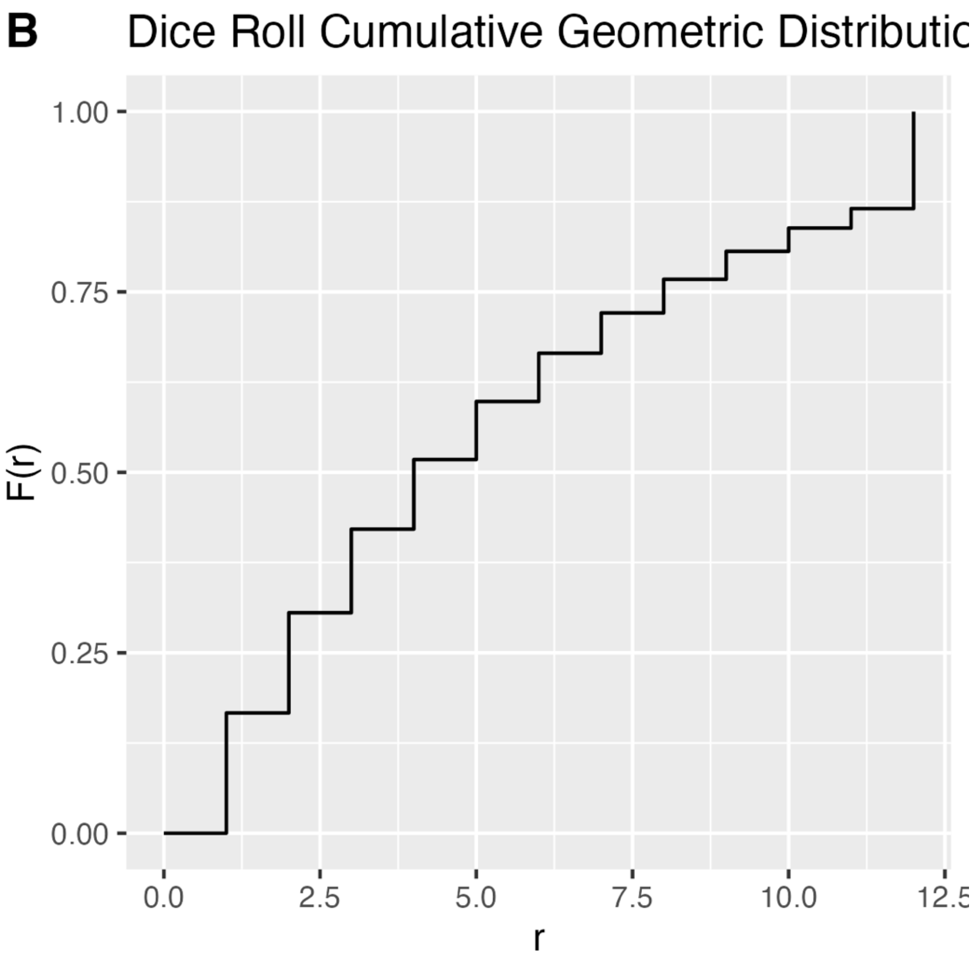
First Six - Probability Mass Distribution

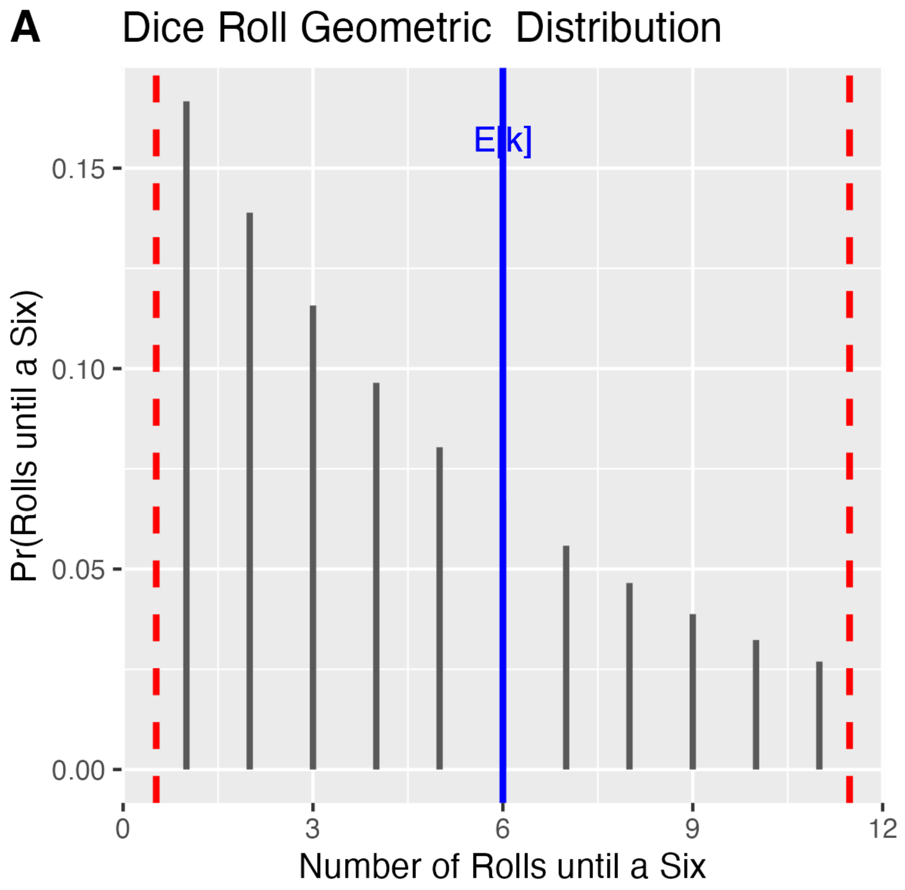


First Six -Cumulative Distribution

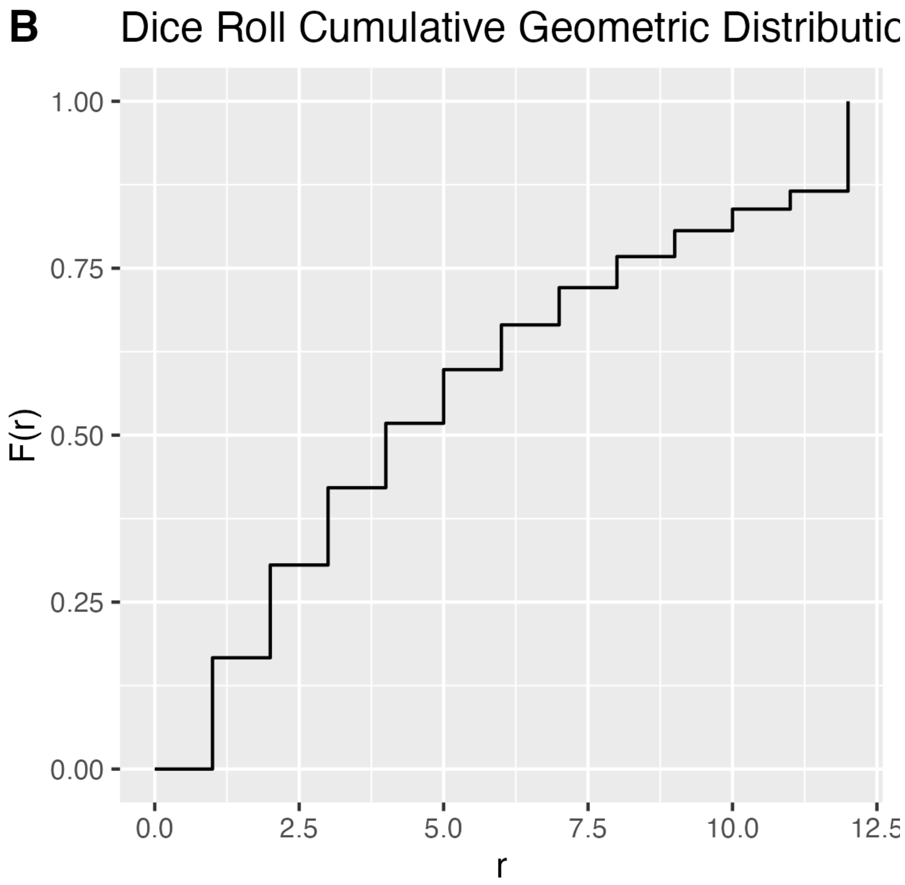


First Six

PROBABILITY MASS FUNCTION



CUMULATIVE DISTRIBUTION



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Note about Notation

- $X \sim G(p)$
  - P is the probability

- In R
  - **dgeom(x, prob, log = FALSE)**: returns the value of the geometric probability density function Note x starts a 0.
  - **pgeom(q, prob, lower.tail = TRUE, log.p = FALSE)**: returns the value of the geometric cumulative density function.
  - **qgeom(p, prob, lower.tail = TRUE, log.p = FALSE)**: returns the value of the inverse geometric cumulative density function.
  - **rgeom(n, prob)**: generates a vector of geometric distributed random variables.

Rugby - Ireland vs New Zealand

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Ireland vs New Zealand

1. The probability of Ireland beating New Zealand in a one off game is 0.15.
2. Describe the geometric distribution if Ireland played New Zealand until Ireland wins.

Ireland vs New Zealand

Pr(Ireland Win)=0.15, Pr(New Zealand Win)=1-0.15=0.85

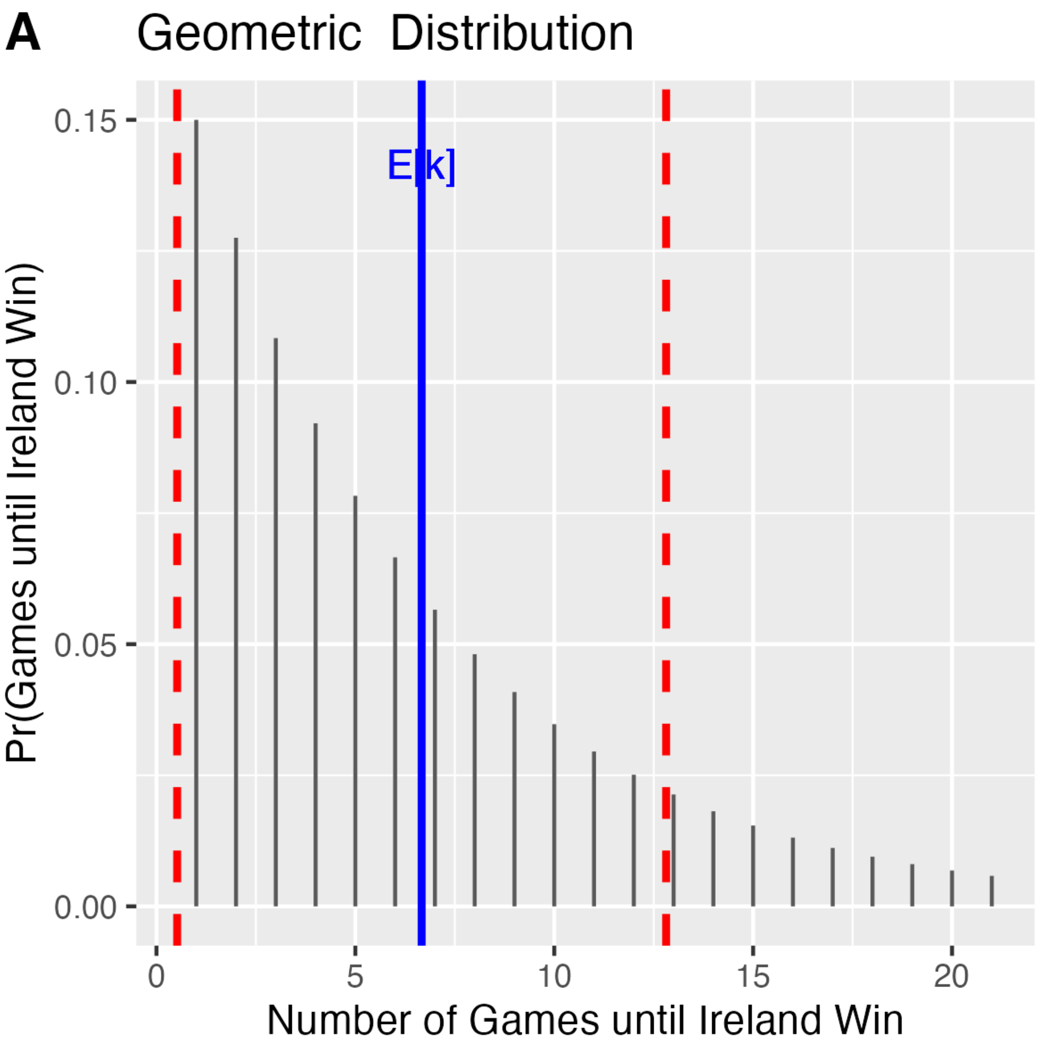
Pr(X=x) $= p(x) = 0.85^{x-1}0.15 =(1-0.15)^{x-1}0.15$

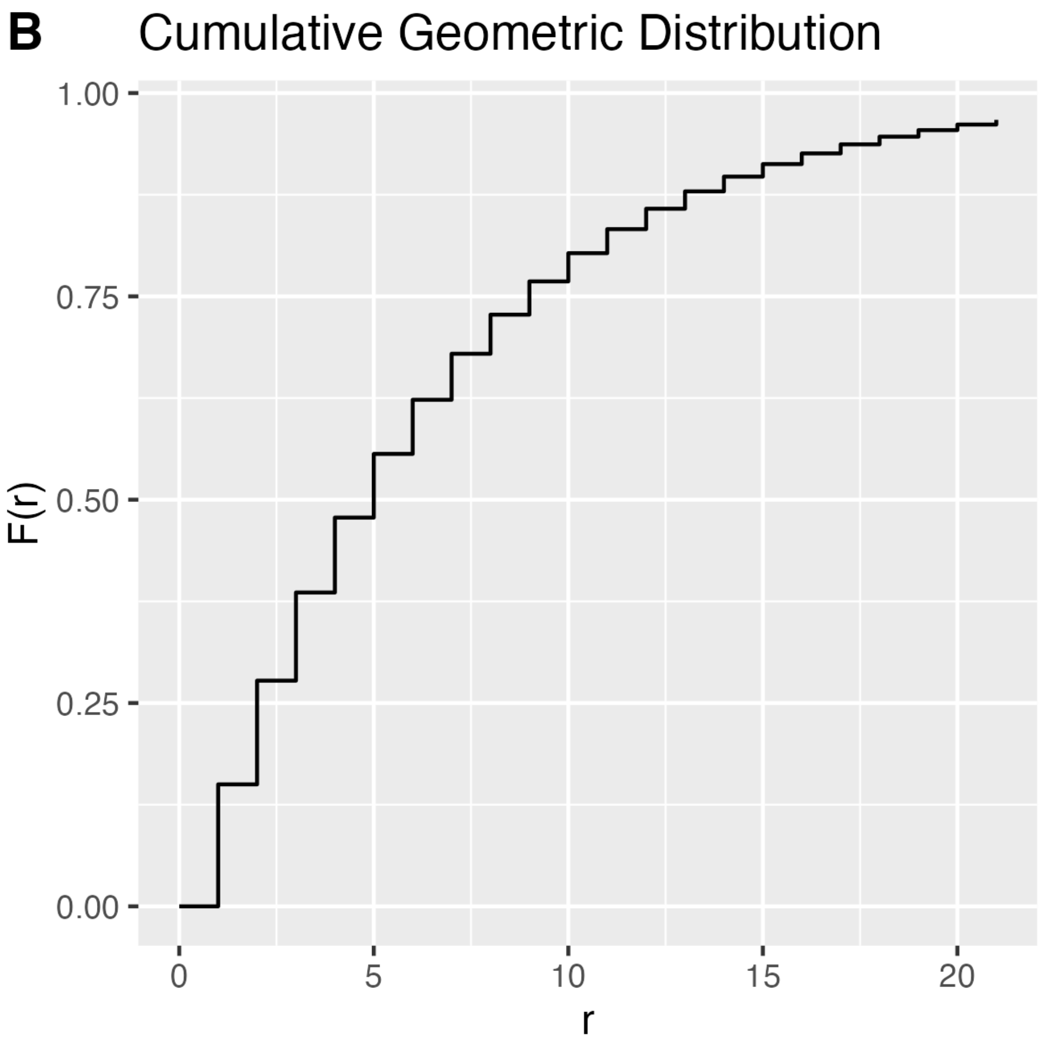
Number of times to play until a win, $x = 1,2,3, \ldots, \infty, 0 < p < 1$

$$E[X] = \frac{1}{0.15} = 6.66667$$

$$\text{Var}[X] = \frac{0.85}{0.15^2} = 37.778$$

Ireland vs New Zealand





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Transmission Error

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Example- Transmission Error

1. One percent of bits transmitted through a digital transmission are received in error. Bits are transmitted until the first error.
2. Let X denote the number of bits transmitted until the first error.

Product Error

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Example - Product Error

- It is known that 5% of smart phones on a production line are defective. Products are inspected until the first defective smart phone is encountered

- Let X number of inspections to obtain first defective

Example 2 - Product Error

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Example 2 - Product Error

- You go to the Factory and for 10 days you test until you find a defective product

Day	1	2	3	4	5	6	7	8	9	10
Freq of fault	15	13	16	14	11	7	9	6	5	4

- Do you believe the factory has a product error rate of 0.05?

Takeaway Point

- **Probability distributions** allow us to model uncertainty in discrete outcomes.

- A **Bernoulli trial** is a random experiment with exactly two possible outcomes—success or failure—where the probability of success remains constant across trials.

- **Geometric distributions** model binary outcomes and the number of trials until the first success.