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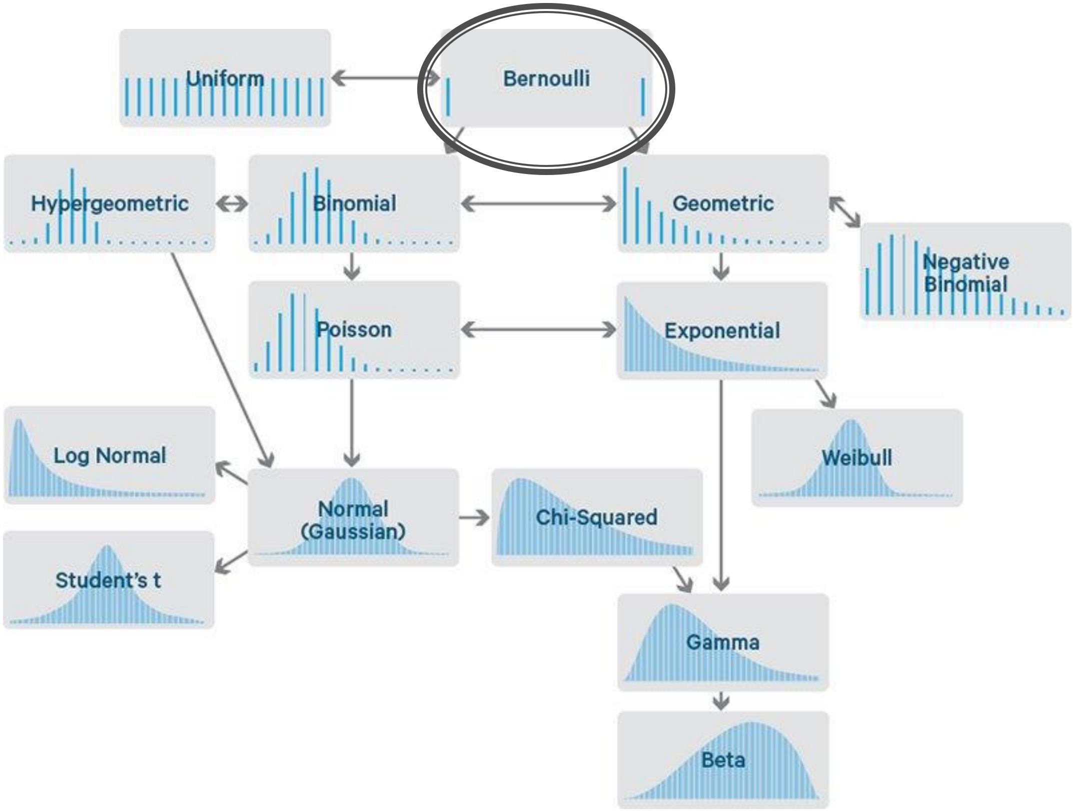
# Binomial Distributions

Dr. John S. Butler

# Bernoulli Distribution

We only play n games.

# Bernoulli Distribution



# Bernoulli Distribution

- A Bernoulli trial generates one of two possible outcomes – “success” or “failure”

- Define the Random Variable
- X=1 if a success
- X= 0 if a failure

- The Random Variable is termed a Bernoulli RV with the Bernoulli probability distribution

# Bernoulli Distribution

- The Random Variable is termed a Bernoulli RV with the Bernoulli probability distribution

$$\Pr(X=x_j)=\Pr(x_j)=\begin{cases} p & for\ x_2 = 1 \\ 1-p & for\ x_1 = 0 \end{cases}$$

# Bernoulli Distribution

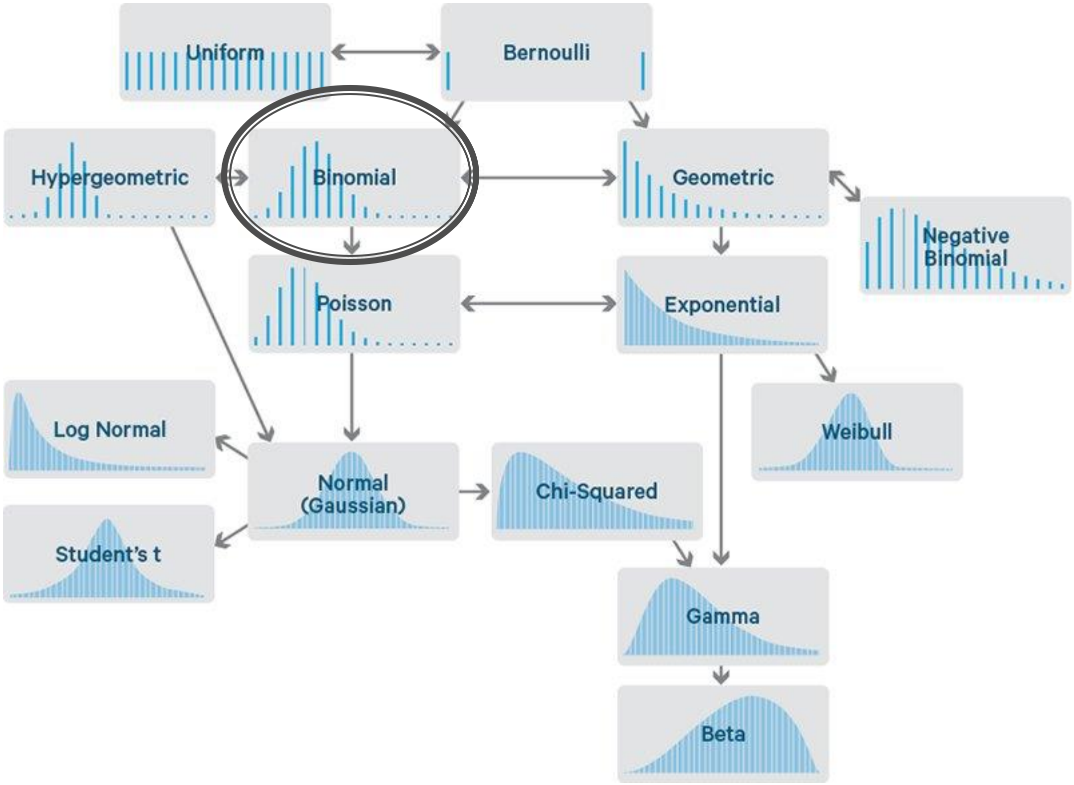
- From this we can show that

$$E[X]=p$$
$$Var[X]=pq$$

# Features of a Bernoulli Experiment

1. There are two possible outcomes arbitrary called success and failure
2. A success occurs with probability p and a failure occurs with probability q=1-p
3. The Random Variable is ordered as 1 if success and 0 if failure

# Binomial Distributions



# Binomial Distributions

We only play n games.

# Binomial Process

- What is the probability of X tail in 5 tosses of a fair coin?
- Let's take a specific situation.
- What is the probability of one tail in five tosses

$$\text{Pr(Fifth Toss is a Tail)}=(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})\frac{1}{2} = \left(\frac{1}{2}\right)^4 \frac{1}{2}$$

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# Binomial Process

What is the probability of one head in five tosses

1. Pr(First Toss is a Tail)= $\frac{1}{2}\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 \frac{1}{2}$
2. Pr(Second Toss is a Tail)= $(\frac{1}{2})\frac{1}{2}(\frac{1}{2}\frac{1}{2}\frac{1}{2}) = \left(\frac{1}{2}\right)^4 \frac{1}{2}$
3. Pr(Third Toss is a Tail)= $(\frac{1}{2}\frac{1}{2})\frac{1}{2}(\frac{1}{2}\frac{1}{2}) = \left(\frac{1}{2}\right)^4 \frac{1}{2}$
4. Pr(Fourth Toss is a Tail)= $(\frac{1}{2}\frac{1}{2}\frac{1}{2})\frac{1}{2}(\frac{1}{2}) = \left(\frac{1}{2}\right)^4 \frac{1}{2}$
5. Pr(Fifth Toss is a Tail)= $(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})\frac{1}{2} = \left(\frac{1}{2}\right)^4 \frac{1}{2}$

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# Binomial Process

What is the probability of one tail in five tosses

$$\text{Pr(one tails in five tosses)} = 5\left(\frac{1}{2}\right)^4 \frac{1}{2}$$

# Binomial Process

- What is the probability of X sixes in 5 tosses of a fair die?

- Let's take a specific situation.

- What is the probability of one six in 5 tosses

- Pr(Fifth Toss is a Six)=$\left(\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\right)\frac{1}{6} = \left(\frac{5}{6}\right)^4 \frac{1}{6}$

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# Binomial Process

What is the probability of one six in 5 tosses

1. Pr(First Toss is a Six)=$\frac{1}{6}\left(\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\right) = \left(\frac{5}{6}\right)^4 \frac{1}{6}$
2. Pr(Second Toss is a Six)=$\left(\frac{5}{6}\right)\frac{1}{6}\left(\frac{5}{6}\frac{5}{6}\frac{5}{6}\right) = \left(\frac{5}{6}\right)^4 \frac{1}{6}$
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# Binomial Process

What is the probability of one six in 5 tosses

$$\Pr(\text{one six in 5 tosses}) = 5\left(\frac{5}{6}\right)^4 \frac{1}{6}$$

# Features of the Binomial Experiment

1. The experiment consists of n repeated Bernoulli trials
2. The trials are independent
3. The probability of success in each trial is constant

# Binomial Distribution

$$\Pr(X=k) = Pr(k) = \binom{n}{k} p^k q^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$k = 0,1,2,3, \dots, n, 0 < p < 1$$

This is the probability mass function for a **binomial distribution.** The random variable X that counts the **k successes** in **n trials**

# Binomial Distribution

Is the sum of p all equal 1

$$\sum_{k=0}^{n} Pr(X = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \quad = 1$$

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# Binomial Distribution

Is the sum of p all equal 1

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$$

$$\text{Var}[X] = np(1\text{-}p)$$

# Example - Coin Toss


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# Coin Toss - Probability Distribution

p=Pr(T)=0.5, q=Pr(H)=1-Pr(T)=0.5

Pr(X=k) $= Pr(k) = \binom{n}{k} 0.5^k 0.5^{n-k}$

n-number of times you play the game (always the same)

k – the number of Tails (T) from 0 to n

$$E[X] = \sum_{k=0}^{n} \binom{n}{k} 0.5^k (1-0.5)^{n-k} = n0.5$$

Var$[X] = n0.5$(1-0.5)

# Coin Toss - Probability Distribution

Now let's play the game 10 times

Pr(T)=0.5 Pr(H)=1-Pr(T)=0.5

Pr(X=k) $= Pr(k) = \binom{10}{k} 0.5^k 0.5^{10-k}$

10-number of times you play the game (always the same)

k=0,1,…,10 – the number of Tails (T) from 0 to 10

$$E[X] = \sum_{k=0}^{n} \binom{10}{k} 0.5^k (1-0.5)^{n-k} = 10 * 0.5 = 5$$

Var$[X] = 10 * 0.5$(1-0.5)=2.5

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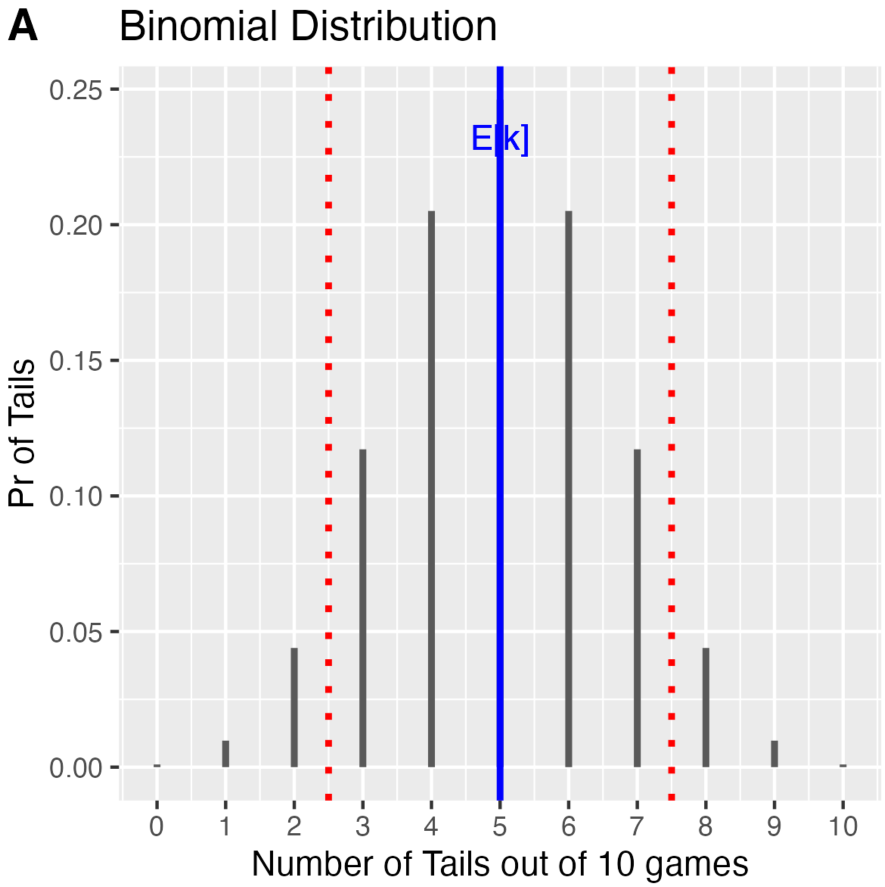
# Coin Toss - Probability Distribution




Play the game n=10 times

$$\Pr(X=k) = Pr(k) = \binom{10}{k} 0.5^k 0.5^{10-k}, \; k = 0,1,\ldots,n$$

# Expected Number of Heads

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Pr(X=k) | 0.00097 | 0.00976 | 0.0439 | 0.1171 | 0.205 | 0.246 | 0.205 | 0.117 | 0.0439 | 0.0097 | 0.00097 |

$$E[X] = \sum_{k=0}^{10} kPr(k)$$

# Variance of Heads

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Pr(X=k) | 0.00097 | 0.00976 | 0.0439 | 0.1171 | 0.205 | 0.246 | 0.205 | 0.117 | 0.0439 | 0.0097 | 0.00097 |
| $(x_k - 5)$ | 25 | 16 | 9 | 4 | 1 | 0 | 1 | 4 | 9 | 16 | 25 |
| | 0.024 | 0.1562 | 0.3955 | 0.468 | 0.205 | 0 | 0.205 | 0.468 | 0.3955 | 0.1562 | 0.024 |

$$Var[X] = \sigma^2 = E[(X-5)^2] = \sum_{k=0}^{10} (k-5)^2 Pr(k)$$

# Coin Toss

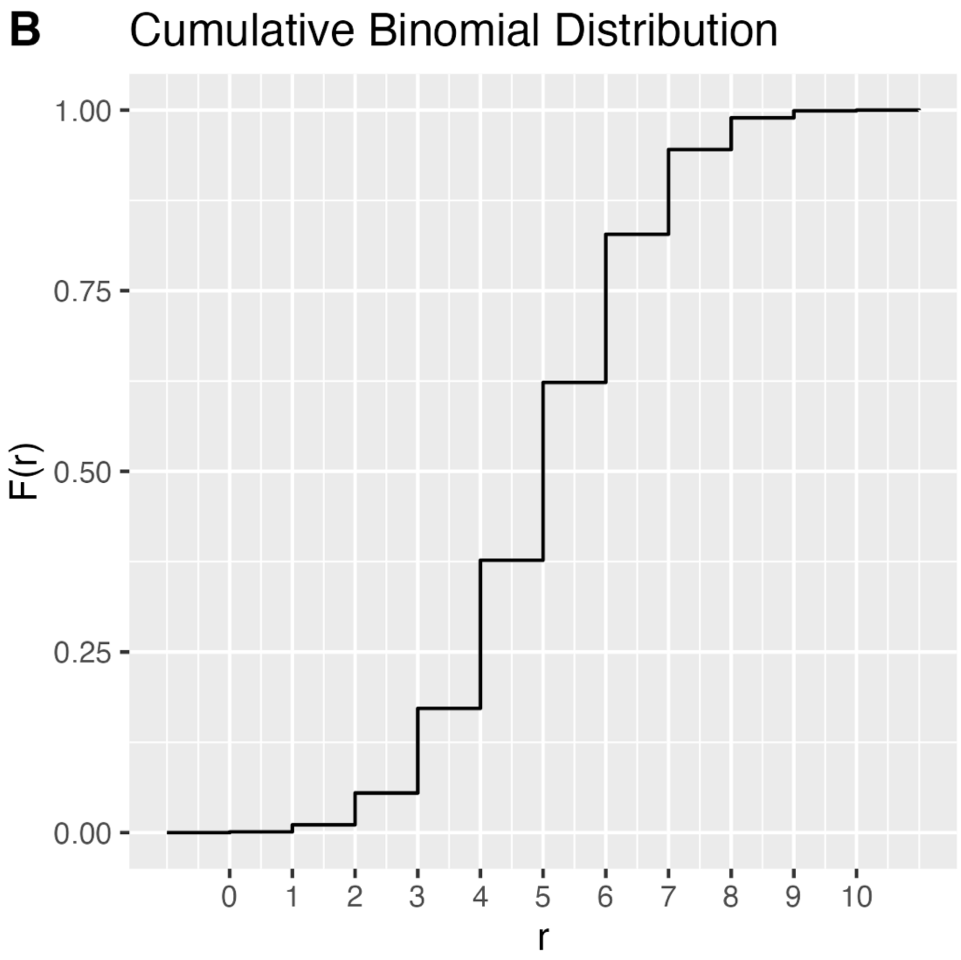
**EXPECTED NUMBER OF TOSSES**

$$E[X] = np = 10\frac{1}{2} = 5$$

**VARIANCE**

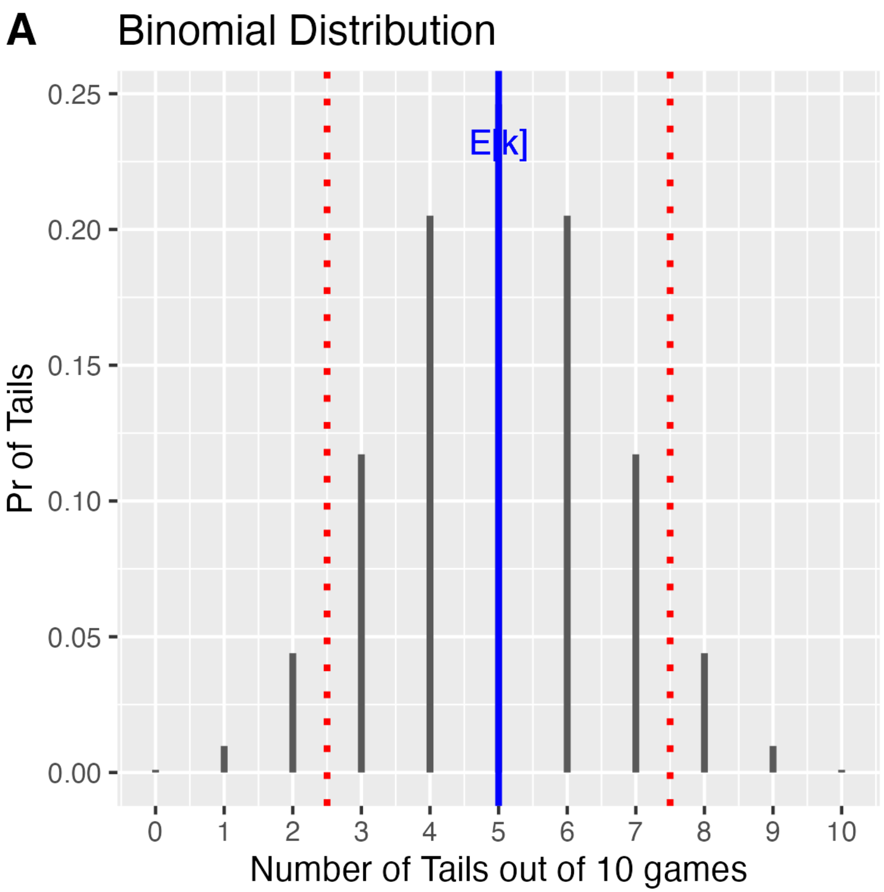
$$\mathrm{Var}[X] = 10\frac{1}{2}(1\text{-}\frac{1}{2}) = \frac{10}{4} = 2.5$$

# Coin Toss -Cumulative Distribution

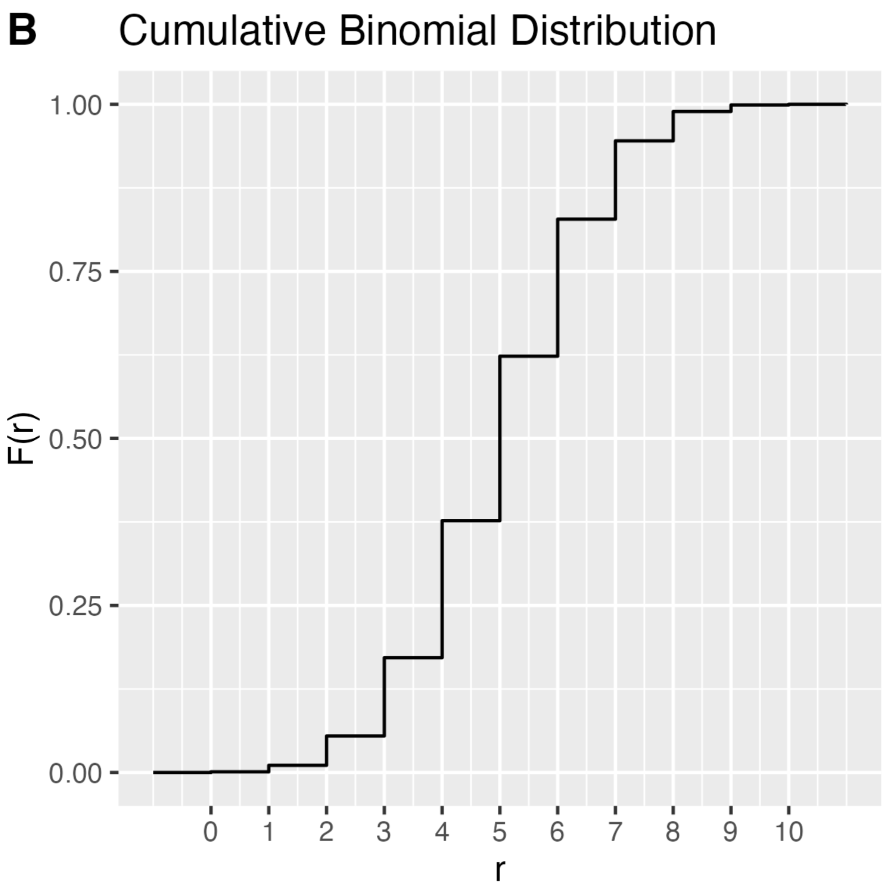


# Coin Toss

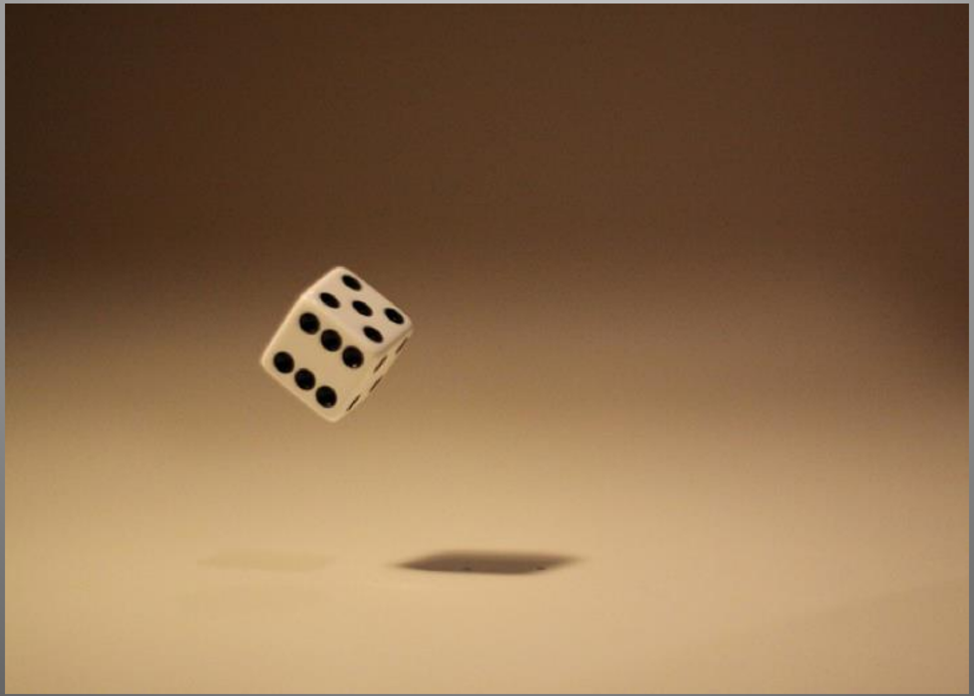
## PROBABILITY DISTRIBUTION




## CUMULATIVE DISTRIBUTION



# Example - Dice



# Dice Game – First Six

p=Pr(6)=$\frac{1}{6}$,        q=Pr(1,2,3,4,5)=1-Pr(6)=$\frac{5}{6}$

Pr(X=k) $= Pr(k) = \binom{n}{k}\frac{1}{6}^{k}\ \frac{5}{6}^{n-k}$

n-number of times you play the game (always the same)

k– the number of sixes 0 to n

$$E[X] = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{6}^{k}\ (1-\frac{1}{6})^{n-k} = n\frac{1}{6}$$

Var$[X] = n\frac{1}{6}$(1-$\frac{1}{6}$)

# Dice Game – First Six

Now let's play the game 10 times

Pr(6)=$\frac{1}{6}$ ,  Pr(1,2,3,4,5)=1-Pr(6)=$\frac{5}{6}$

Pr(X=k) $= Pr(k) = \binom{10}{k} \frac{1}{6}^{k} \frac{5}{6}^{10-k}$

10-number of times you play the game (always the same)

k =0,1,…,10 – the number of sixes 0 to 10

$$E[X] = \sum_{k=0} \binom{10}{k} \frac{1}{6}^{k} (1 - \frac{1}{6})^{n-k} = 10\frac{1}{6}$$

Var$[X] = 10\frac{1}{6}$(1-$\frac{1}{6}$)

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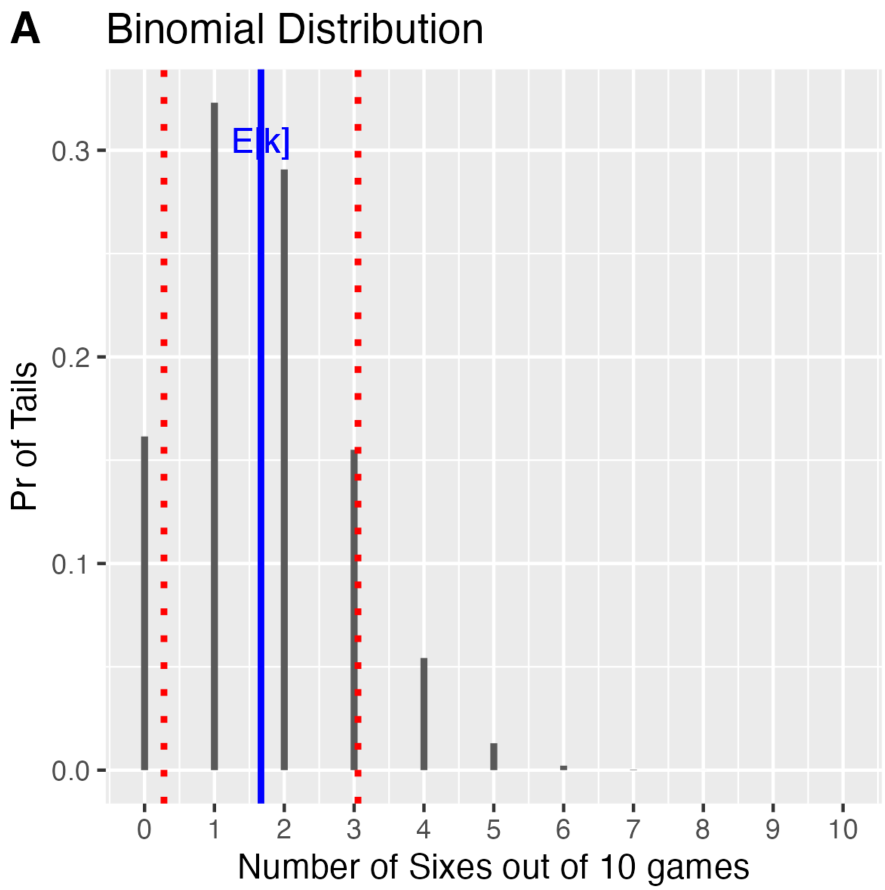
# First Six- Probability Distribution




$$\text{Pr(X=k)} = Pr(k) = \binom{10}{k}\frac{1}{6}^{k}\,\frac{5}{6}^{n-k} = \binom{10}{k}\frac{1}{6}^{k}\,(1-\frac{1}{6})^{10-k}$$

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# Expected Number of Heads

| K | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Pr(X=k) | 0.161 | 0.323 | 0.2907 | 0.155 | 0.054 | 0.013 | 0.002 | 0.0002 | ~0 | ~0 | ~0 |

$$E[X] = \sum_{k=0}^{10} kPr(k)$$

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# Variance of Heads

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Pr(X=k) | 0.161 | 0.32 | 0.2907 | 0.155 | 0.054 | 0.013 | 0.002 | 0.0002 | ~0 | ~0 | ~0 |
| $(x_k - 10/6)^2$ | 2.777 | 0.44 | 0.111 | 1.778 | 5.44 | 11.1 | 18.77 | 28.44 | 20.11 | 53.77 | 69.44 |
| | 0.448 | 0.14 | 0.032 | 0.275 | 0.29 | 0.14 | 0.04 | ~0 | ~0 | ~0 | ~0 |

$$Var[X] = \sigma^2 = E\left[\left(X - \frac{10}{6}\right)^2\right] = \sum_{k=0}^{10} (k - \frac{10}{6})^2 Pr(k)$$

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# Die Roll

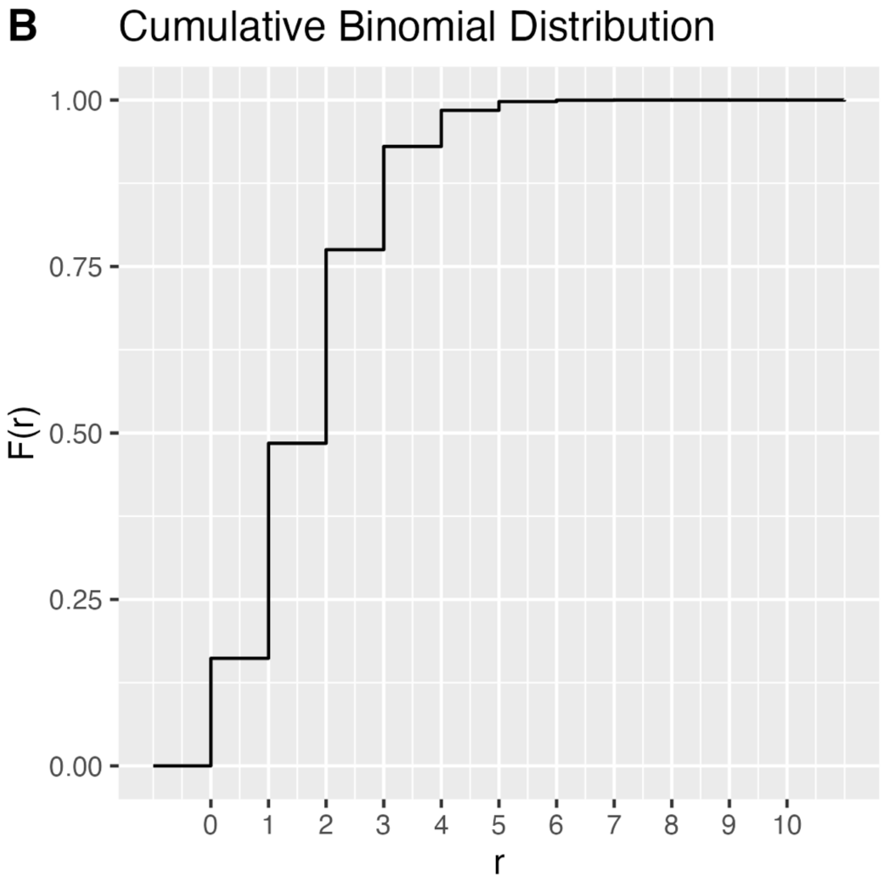
**EXPECTED NUMBER OF SIXES**

$$E[X] = np = 10\frac{1}{6} = \frac{10}{6}$$

**VARIANCE**

$$\mathrm{Var}[X] = np(1-p) = 10\frac{1}{6}\frac{5}{6} = \frac{50}{36}$$

# First Six- Cumulative Distribution

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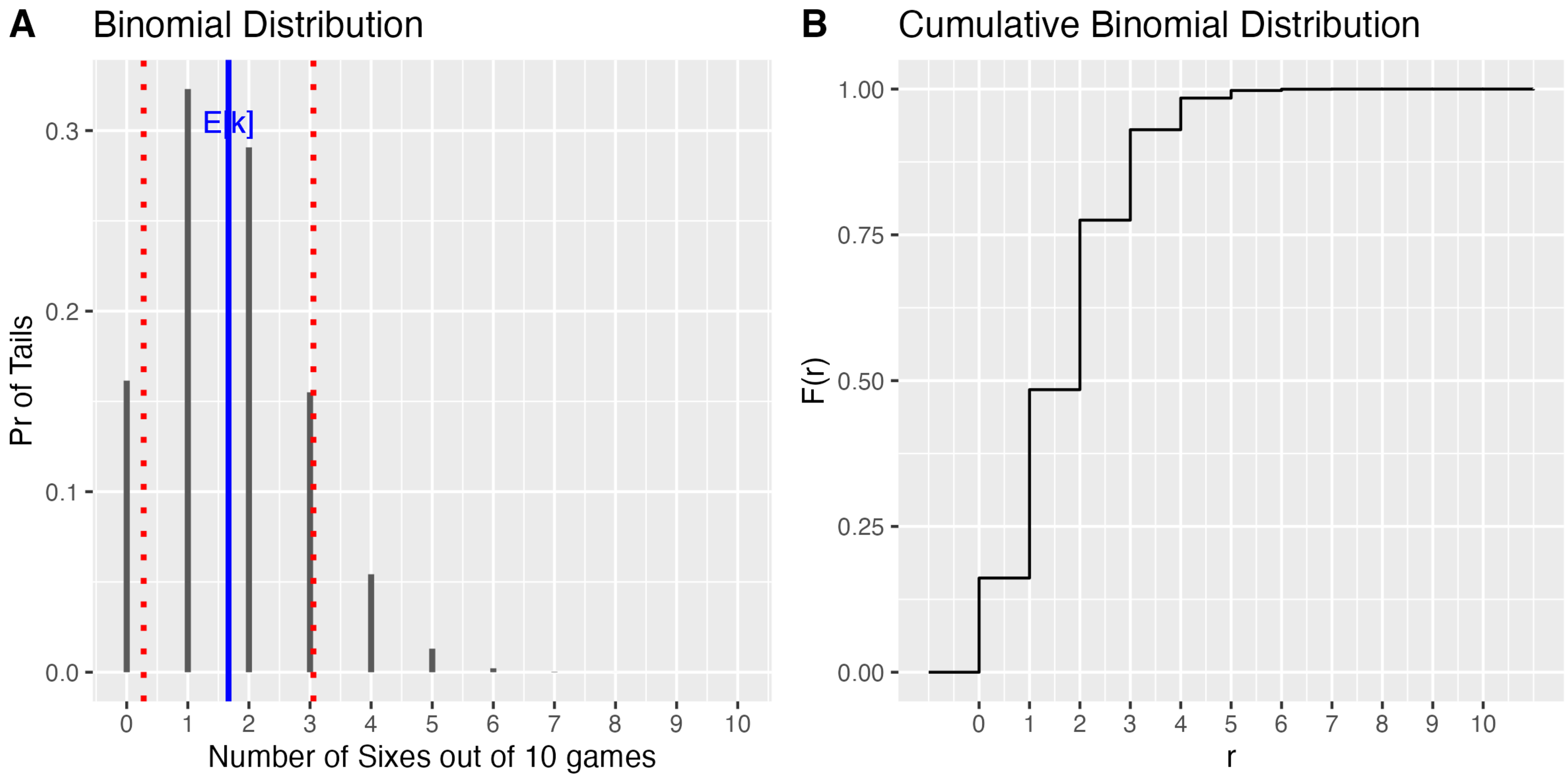
# First Six

**PROBA**




$$\Pr(X{=}k) = Pr(k) = \binom{10}{k}\frac{1}{6}^{k}\,\frac{5}{6}^{n-k} = \binom{10}{k}\frac{1}{6}^{k}\,\left(1-\frac{1}{6}\right)^{10-k}$$

# Note about Notation

- *X ~ B(n,p)*
  - n is size of distribution
  - p is the probability

- In R
  - **dbinom(x, size, prob, log = FALSE)**: returns the value of the binomial probability density function
  - **pbinom(q, size, prob, lower.tail = TRUE, log.p = FALSE)**: returns the value of the binomial cumulative density function.
  - **qbinom(p, size, prob, lower.tail = TRUE, log.p = FALSE)**: returns the value of the inverse binomial cumulative density function.
  - **rbinom (n, size, prob)**: generates a vector of binomial random variables.

# Rugby - Ireland vs New Zealand

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# Ireland vs New Zealand

1. The probability of Ireland beating New Zealand in a one off game is 0.15.

2. Describe the binomial distribution if Ireland played New Zealand 10 times for 10 Irish wins

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# Ireland vs New Zealand

Now let's play the game 10 times

Pr(Ireland Win)=0.15, Pr(New Zealand Win)=1-0.15=0.85

Pr(X=k) $= Pr(k) = \binom{10}{k} 0.15^k 0.85^{10-k}$

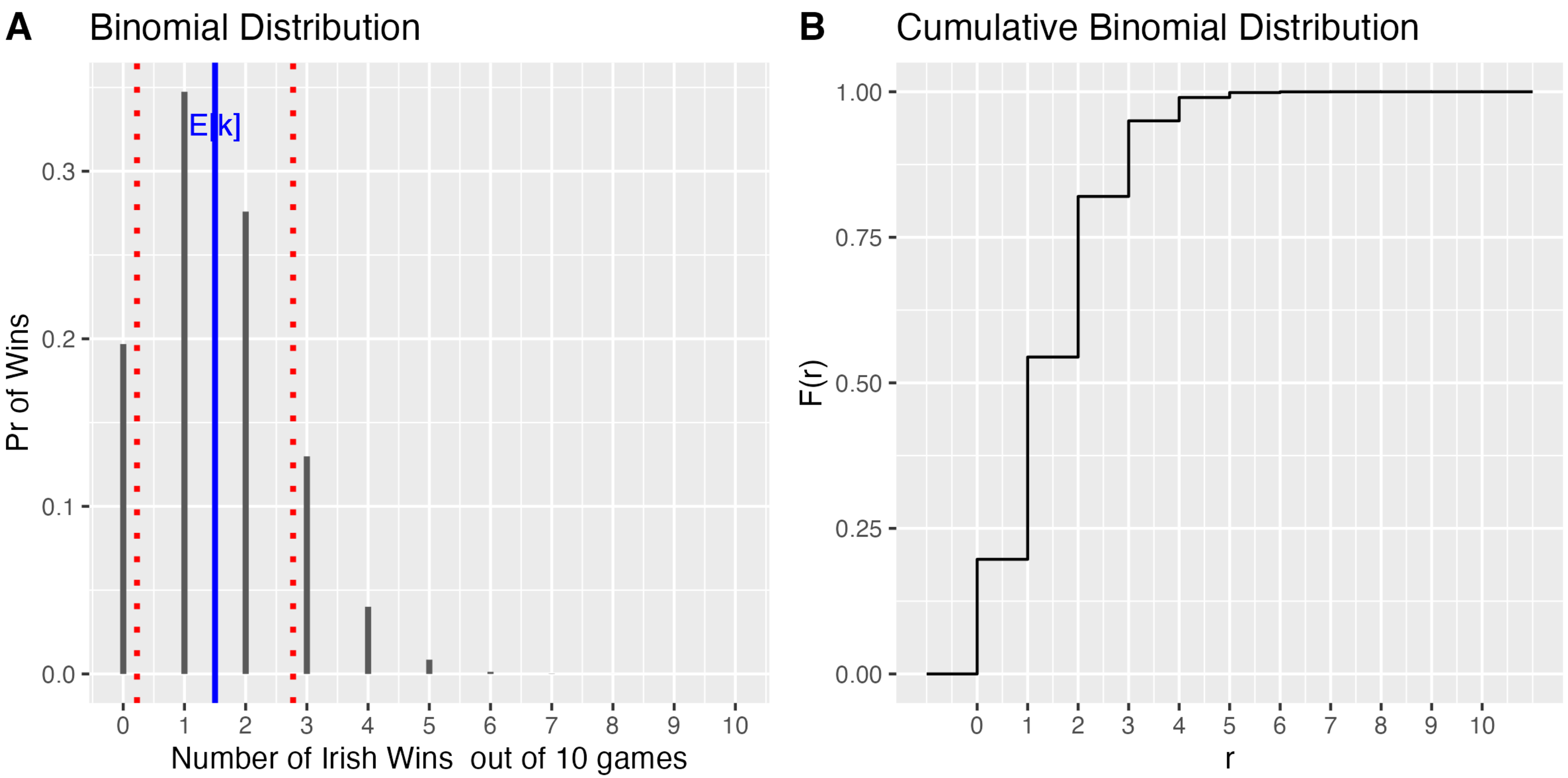
10-number of times you play the game (always the same)

k=0,1…,10 – the number of Ireland wins 0 to 10

$E[X] = 10(.15) = 1.5$

Var$[X] = 10(0.15)\,(1-0.15)$=1.35

# Ireland vs New Zealand



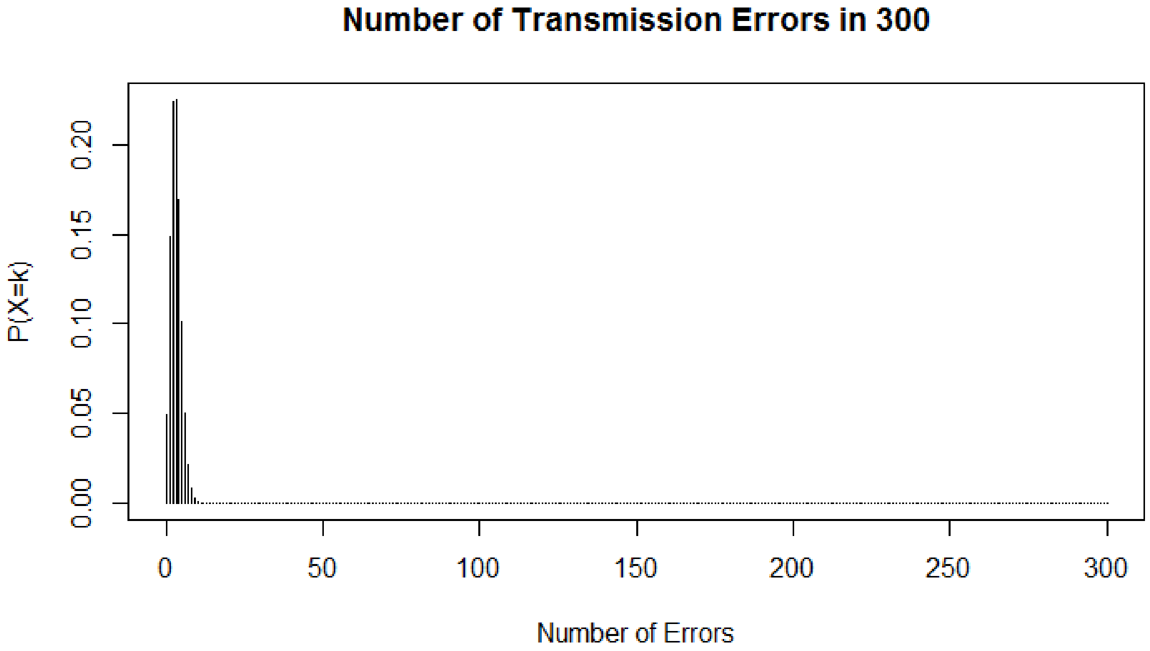
# Transmission Error

# Example- Transmission Error

1. One percent of bits transmitted through a digital transmission are received in error.
2. Let k denote the bit errors in 300 transmission.

# Example- Transmission Error

- Pr(X=k) $= Pr(k) = \binom{300}{k} 0.01^k 0.99^{200-k}$, $k$=0,…,300

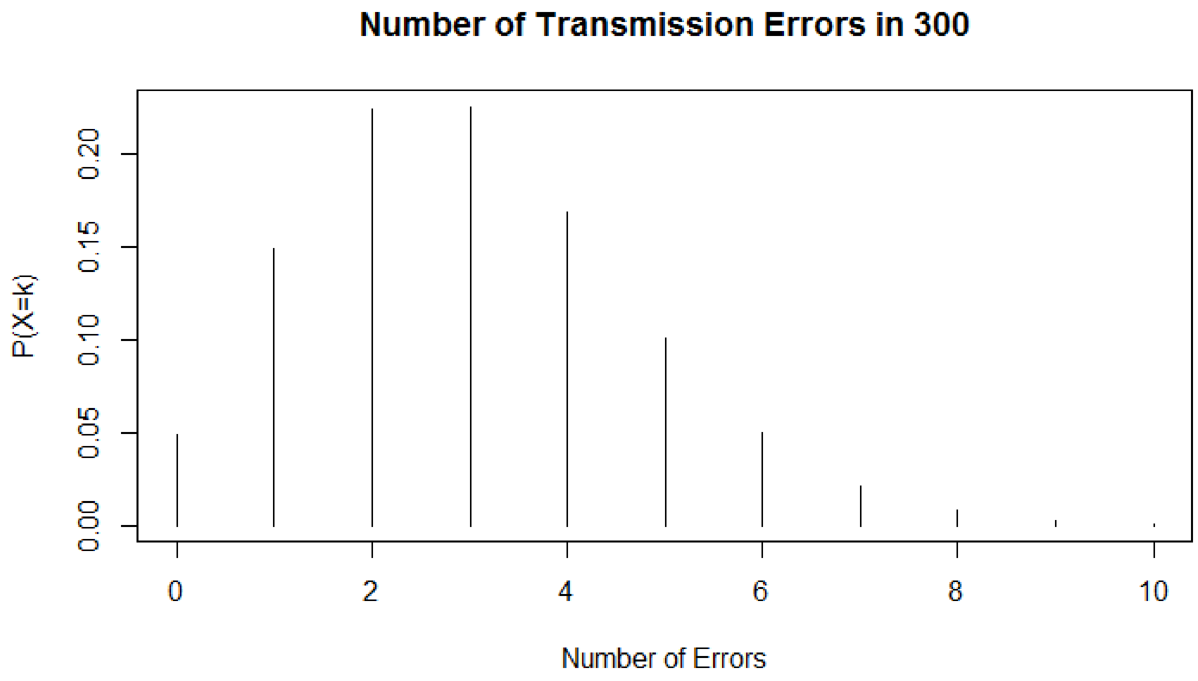



$E[X] = 300 * 0.01 = 3$
Var$[X] = 300 * 0.01$ (1-0.01)=2.97

# Example- Transmission Error

- Pr(X=k) $= Pr(k) = \binom{300}{k} 0.01^{k} 0.99^{300-k}$, $k$=0,…,300

$E[X] = 300 * 0.01 = 3$
Var$[X] = 300 * 0.01$ (1-0.01)=2.97



# Product Error

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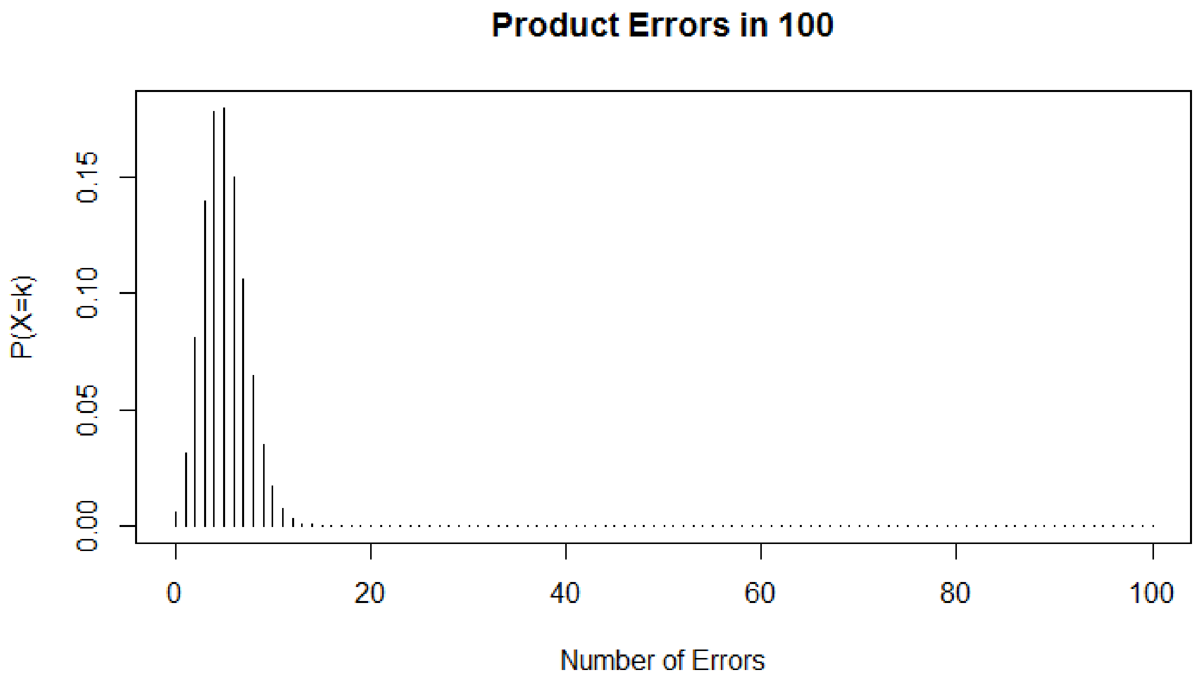
# Example - Product Error

- It is known that 5% of smart phones on a production line are defective. 100 Products are inspected and the number of defective products are counted.

- Let k be the number of defective devices in 100 tests

# Example - Product Error

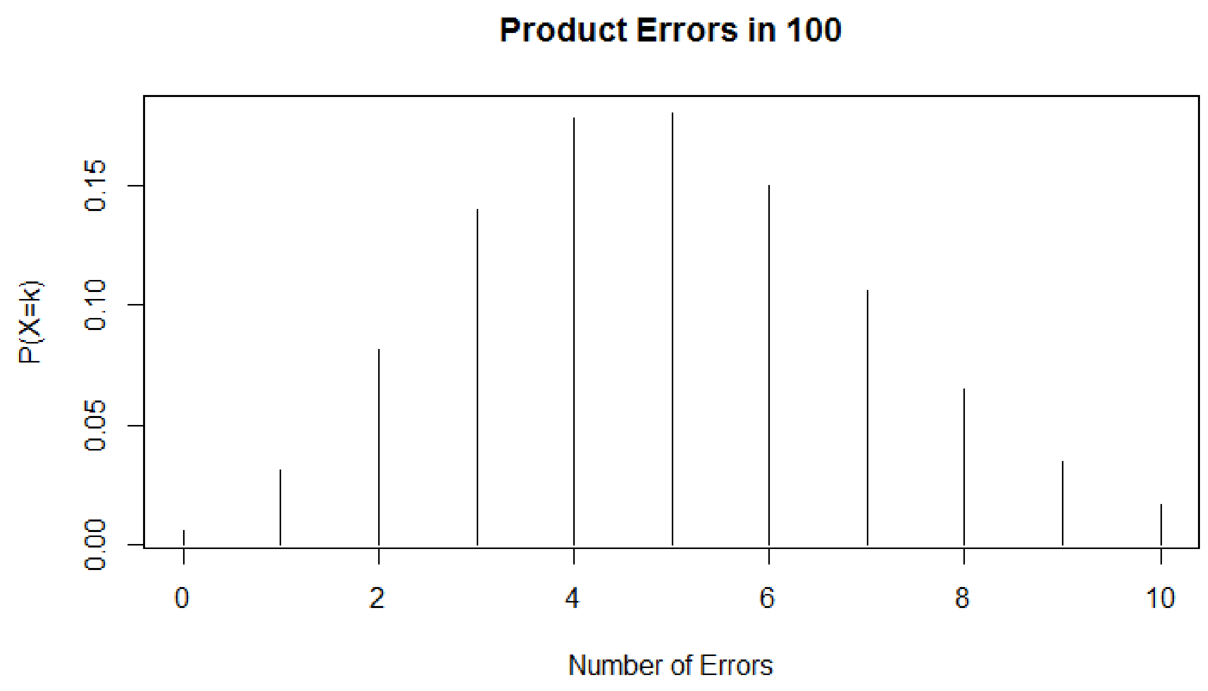
- Pr(X=k) $= Pr(k) = \binom{100}{k} 0.05^k 0.95^{100-k}$, $k$=0,…,100




$E[X] = 100 * 0.05 = 5$
Var$[X] = 100 * 0.05$ (1-0.05)=4.75

# Example - Product Error

- Pr(X=k) $= Pr(k) = \binom{100}{k} 0.05^{k} 0.95^{100-k}$, $k$=0,…,100



# Example - Product Error

- You go to the Factory and for 10 days you test 100 products each day and you get you this table:

| Day | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of Faults | 15 | 13 | 16 | 14 | 11 | 7 | 9 | 6 | 5 | 4 |

- Do you believe the factory has a product error rate of 0.05?

# Takeaway Point

- **The Binomial distribution** models the number of successes in a fixed number of independent Bernoulli trials.

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