1. 1) In the empty string (ε) both $N_a(s)$ & $N_b(s)$ are equal to 0 so ε is balanced, since $0+0=0$

2) $s=xy$ is balanced since $s=xy=s(x)+s(y)$. $s(x)$ is balanced so $N_a(x)-N_b(x)=0$ & $s(y)$ is balanced so $N_a(y)-N_b(y)=0$ since we can rewrite $s(xy)$ as $(N_a(x)-N_b(x))+(N_a(y)-N_b(y))$, $s=xy$ must be balanced

3) $s=azb$ is balanced since we can write it as the concatenation of 2 balanced strings ("ab" and z) since I proved in part 2 that the concatenation of 2 balanced strings is balanced $s=azb$ is balanced

2) base case: $n=2$

$10(2)\log_2(2) = 20$

With 2 teams the max amount of pushups is 10

Since $10 \leq 20$ the base case holds true

induction step:

$$\sum_{n=1}^{k+1} = 10n\log_2 n = 10(k+1)\log_2(k+1)$$

The number of players on both teams in the final match must equal $k+1$ so if the teams are a,b then $a+b=k+1$.

There are 2 cases

$a<b$ - since a is smaller that team will do $10a$

or

$a=b$ Since $a=b$ the amount of pushups is $10a$

$10a\log_2 a + 10b\log_2 b + 10a = 10(k+1)\log_2(k+1)$ is the formula for total pushups

$a < {k+1}/{2} \rightarrow 2a < k+1$ so,

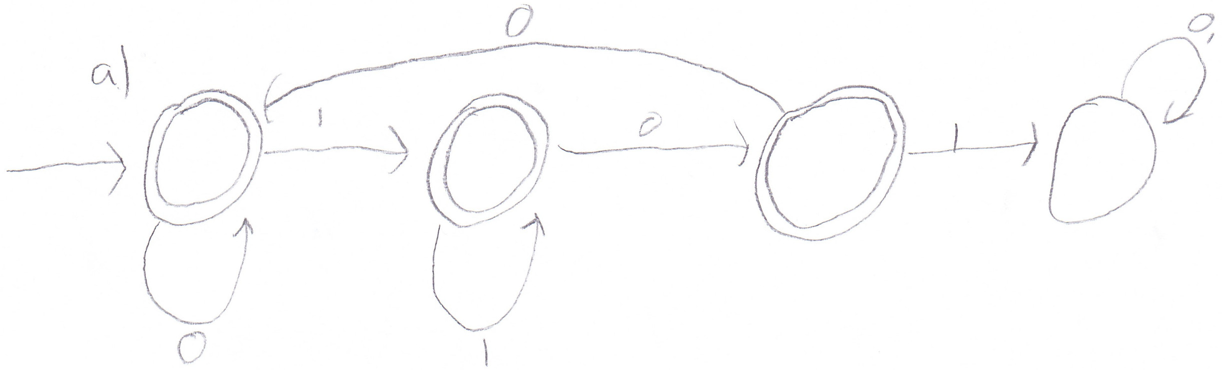
$10a\log_2(2a) + 10b\log_2 b < 10a\log_2(k+1) + 10b\log_2(k+1)$

$= 10(a+b)\log_2(k+1)$ since $a+b=k+1$

$= 10(k+1)\log_2(k+1)$

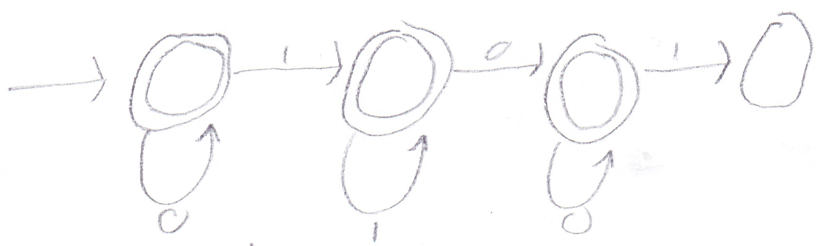
3

a)



This DFA accepts all except the final node that represents 101. All other states are accept states. 101 must be consecutive.

b)



This DFA will stay in an accept state until a 1, 0, 1 gets passed into the system. DFA ignores other digits meaning the 101 can be non-consecutive