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In this note, I will discuss some consequences of the fact that 'energy' is conserved. We will first work in one dimension, and then consider some analogues to our results in higher dimensions.

1 One dimension

For a one-dimensional system whose stochastic dynamics are governed by

$$\dot{x} = f(x) + g(x)\eta(t) \ ,$$

its least action path satisfies

$$E = \frac{g^2}{2}p^2 + fp = \text{const.}$$

for all points (x, p) in the trajectory.

Lemma. $E = 0$ if and only if $\dot{x} = \pm f$.

Proof. Suppose that $E = 0$. Then

$$\frac{g^2}{2}p^2 + fp = p\left[\frac{g^2}{2}p + f\right] = 0 \ ,$$

so we have either

$$p = 0 \quad \text{or} \quad \frac{g^2}{2}p + f = 0 \ .$$

If $p = 0$, then $\dot{x} = f$. If not, then

$$\frac{g^2}{2}p + f = 0 \quad \Longrightarrow \quad \frac{\dot{x} - f}{2} + f = 0 \quad \Longrightarrow \quad \dot{x} = -f \ .$$

Hence, the forward direction holds.

If $\dot{x} = +f$, then $p = 0$, so $E = 0$ trivially. If $\dot{x} = -f$, then

$$p = -2\frac{f}{g^2}$$

so that

$$E = \frac{g^2}{2}p^2 + fp = \frac{2f^2}{g^2} - 2\frac{f^2}{g^2} = 0 \ . \blacksquare$$

Lemma. If $E > 0$, then it must be true that $\dot{x} > |f|$ always or $\dot{x} < -|f|$ always. The former case corresponds to $p > 0$ always, while the latter case corresponds to $p < 0$ always.

Proof. Suppose that $E > 0$, i.e. that

$$\frac{g^2}{2}p^2 + fp > 0 \ .$$

Suppose that $p > 0$. Then, on the one hand,

$$\frac{\dot{x} - f}{g^2} > 0 \quad \Longrightarrow \quad \dot{x} > f \ .$$

On the other hand, we have

$$\frac{g^2}{2}p + f > 0 \quad \Longrightarrow \quad \frac{g^2}{2}\left(\frac{\dot{x} - f}{g^2}\right) + f > 0 \quad \Longrightarrow \quad \dot{x} > -f \ .$$

Hence, $\dot{x} > |f|$.

Suppose instead that $p < 0$. On the one hand,

$$\frac{\dot{x} - f}{g^2} < 0 \quad \Longrightarrow \quad \dot{x} < f \ .$$

On the other hand, we have

$$\frac{g^2}{2}p + f < 0 \quad \Longrightarrow \quad \frac{g^2}{2}\left(\frac{\dot{x} - f}{g^2}\right) + f < 0 \quad \Longrightarrow \quad \dot{x} < -f \ .$$

Hence, $\dot{x} < -|f|$ in this case.

Is it possible that we have $p > 0$ for a while, and then we have $p < 0$ some time later? The answer is *no*, because (by the continuity of p) it forces $p = 0$ at some intermediate point. If $p = 0$, then $\dot{x} = f$, which means that $E = 0$, contradicting our assumption.

Hence, if $p > 0$ at some point, then $p > 0$ and $\dot{x} > |f|$ at all points. If $p < 0$ at some point, then $p < 0$ and $\dot{x} < -|f|$ at all points. ■

Lemma. If $E > 0$, then $\dot{x}$ is either always positive or always negative. In particular, if $x_f > x_0$, then we must have $\dot{x_0} > 0$; similarly, if $x_f < x_0$, we must have $\dot{x_0} < 0$.

In fact, if $x_f > x_0$, we must have $\dot{x_0} > |f|_0$, since $\dot{x} > |f|$ always. Also, if $x_f < x_0$, we must have $\dot{x_0} < -|f|_0$, since $\dot{x} < -|f|$ always.

Proof. Use the previous lemma. ■