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Consider a Lagrangian

$$L = \sum_{j=1}^{N} \frac{(\dot{x_j} - f_j)^2}{2g_j^2} \tag{1}$$

The canonical momenta are

$$p_{x_i} := \frac{\partial L}{\partial \dot{x_i}} = \frac{\dot{x_i} - f_i}{g_i^2} \qquad\qquad i = 1, 2, ..., N$$

The total time derivatives of the canonical momenta are

$$\begin{aligned}
\dot{p_{x_i}} &= \frac{d}{dt}\left[\frac{\dot{x_i} - f_i}{g_i^2}\right] \\
&= \frac{(\ddot{x_i} - \dot{f_i})}{g_i^2} - \frac{(\dot{x_i} - f_i)}{g_i^3}\dot{g_i} \\
&= \frac{\left(\ddot{x_i} - \sum_{j=1}^{N} \frac{\partial f_i}{\partial x_j}\dot{x_j}\right)}{g_i^2} - \frac{(\dot{x_i} - f_i)}{g_i^3}\sum_{j=1}^{N}\frac{\partial g_i}{\partial x_j}\dot{x_j} \\
&= \frac{\ddot{x_i}}{g_i^2} - \left[\frac{1}{g_i^2}\sum_{j=1}^{N}\left(\frac{\partial f_i}{\partial x_j} + \frac{(\dot{x_i} - f_i)}{g_i}\frac{\partial g_i}{\partial x_j}\right)\dot{x_j}\right]
\end{aligned}$$

The partial derivatives of the Lagrangian are

$$\begin{aligned}
\frac{\partial L}{\partial x_i} &= \sum_{j=1}^{N}\frac{\partial}{\partial x_i}\left[\frac{(\dot{x_j} - f_j)^2}{2g_j^2}\right] \\
&= \sum_{j=1}^{N} -\frac{\partial f_j}{\partial x_i}\frac{(\dot{x_j} - f_j)}{g_j^2} - \frac{\partial g_j}{\partial x_i}\frac{(\dot{x_j} - f_j)^2}{g_j^3}
\end{aligned}$$

The Euler-Lagrange equation for x_i reads

$$\dot{p_{x_i}} = \frac{\partial L}{\partial x_i}$$

$$\implies \frac{\ddot{x_i}}{g_i^2} - \left[\frac{1}{g_i^2}\sum_{j=1}^{N}\left(\frac{\partial f_i}{\partial x_j} + \frac{(\dot{x_i} - f_i)}{g_i}\frac{\partial g_i}{\partial x_j}\right)\dot{x_j}\right] = \sum_{j=1}^{N} -\frac{\partial f_j}{\partial x_i}\frac{(\dot{x_j} - f_j)}{g_j^2} - \frac{\partial g_j}{\partial x_i}\frac{(\dot{x_j} - f_j)^2}{g_j^3}$$

Rearranging, we have

$$\ddot{x}_i = \sum_{j=1}^{N} \left(\frac{\partial f_i}{\partial x_j} + \frac{(\dot{x}_i - f_i)}{g_i} \frac{\partial g_i}{\partial x_j} \right) \dot{x}_j - g_i^2 \frac{\partial f_j}{\partial x_i} \frac{(\dot{x}_j - f_j)}{g_j^2} - g_i^2 \frac{\partial g_j}{\partial x_i} \frac{(\dot{x}_j - f_j)^2}{g_j^3}$$

1 Special cases

1.1 One-dimensional dynamics

In one dimension, the Lagrangian reads

$$L = \frac{(\dot{x} - f)^2}{2g^2} \tag{2}$$

and the equation of motion reads

$$\begin{aligned}
\ddot{x} &= \left(\frac{\partial f}{\partial x} + \frac{(\dot{x} - f)}{g} \frac{\partial g}{\partial x} \right) \dot{x} - g^2 \frac{\partial f}{\partial x} \frac{(\dot{x} - f)}{g^2} - g^2 \frac{\partial g}{\partial x} \frac{(\dot{x} - f)^2}{g^3} \\
&= \left(\frac{\partial f}{\partial x} + \frac{(\dot{x} - f)}{g} \frac{\partial g}{\partial x} \right) \dot{x} - \frac{\partial f}{\partial x}(\dot{x} - f) - \frac{\partial g}{\partial x} \frac{(\dot{x} - f)^2}{g} \\
&= \frac{\partial f}{\partial x}\dot{x} + \frac{(\dot{x} - f)}{g}(\dot{x} - f + f)\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}(\dot{x} - f) - \frac{\partial g}{\partial x} \frac{(\dot{x} - f)^2}{g} \\
&= \frac{\partial f}{\partial x}\dot{x} + \frac{f(\dot{x} - f)}{g} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}(\dot{x} - f) - \frac{\partial g}{\partial x} \frac{(\dot{x} - f)^2}{g} + \frac{(\dot{x} - f)^2}{g} \frac{\partial g}{\partial x} \\
&= \frac{f(\dot{x} - f)}{g} \frac{\partial g}{\partial x} + f \frac{\partial f}{\partial x}
\end{aligned}$$

In the case of additive noise (i.e. g =const.), we have

$$\begin{aligned}
\ddot{x} &= f \frac{\partial f}{\partial x} \\
\implies \quad 2\ddot{x}\dot{x} &= 2f \frac{\partial f}{\partial x} \dot{x} \\
\implies \quad \dot{x}^2 &= f^2 + C \\
\implies \quad \dot{x} &= \pm\sqrt{f^2 + C}
\end{aligned}$$

as we might expect. Using the plus sign and $C = 0$ corresponds exactly to deterministic dynamics, which the additive noise dynamics will coincide with when it is possible to go from x_0 to x_f in the specified amount of time T using only deterministic motion.

1.2 Two-dimensional dynamics

Assume that $g_1 = g_1(x)$ and $g_2 = g_2(y)$, so that $\frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x} = 0$. We have a Lagrangian

$$L = \frac{(\dot{x} - f_1)^2}{2g_1^2} + \frac{(\dot{y} - f_2)^2}{2g_2^2} \tag{3}$$

and equations of motion

$$\ddot{x} = \left(\frac{\partial f_1}{\partial x} + \frac{(\dot{x} - f_1)}{g_1}\frac{\partial g_1}{\partial x}\right)\dot{x} + \frac{\partial f_1}{\partial y}\dot{y} - g_1^2\frac{\partial f_1}{\partial x}\frac{(\dot{x} - f_1)}{g_1^2} - g_1^2\frac{\partial f_2}{\partial x}\frac{(\dot{y} - f_2)}{g_2^2} - g_1^2\frac{\partial g_1}{\partial x}\frac{(\dot{x} - f_1)^2}{g_1^3}$$
$$\ddot{y} = \left(\frac{\partial f_2}{\partial y} + \frac{(\dot{y} - f_2)}{g_2}\frac{\partial g_2}{\partial y}\right)\dot{y} + \frac{\partial f_2}{\partial x}\dot{x} - g_2^2\frac{\partial f_1}{\partial y}\frac{(\dot{x} - f_1)}{g_1^2} - g_2^2\frac{\partial f_2}{\partial y}\frac{(\dot{y} - f_2)}{g_2^2} - g_2^2\frac{\partial g_2}{\partial y}\frac{(\dot{y} - f_2)^2}{g_2^3}$$

Simplifying a little, they read

$$\ddot{x} = \left(\frac{\partial f_1}{\partial x} + \frac{(\dot{x} - f_1)}{g_1}\frac{\partial g_1}{\partial x}\right)\dot{x} + \frac{\partial f_1}{\partial y}\dot{y} - \frac{\partial f_1}{\partial x}(\dot{x} - f_1) - g_1^2\frac{\partial f_2}{\partial x}\frac{(\dot{y} - f_2)}{g_2^2} - \frac{\partial g_1}{\partial x}\frac{(\dot{x} - f_1)^2}{g_1}$$
$$\ddot{y} = \left(\frac{\partial f_2}{\partial y} + \frac{(\dot{y} - f_2)}{g_2}\frac{\partial g_2}{\partial y}\right)\dot{y} + \frac{\partial f_2}{\partial x}\dot{x} - g_2^2\frac{\partial f_1}{\partial y}\frac{(\dot{x} - f_1)}{g_1^2} - \frac{\partial f_2}{\partial y}(\dot{y} - f_2) - \frac{\partial g_2}{\partial y}\frac{(\dot{y} - f_2)^2}{g_2}$$