Some solutions to Kreyszig's *Introductory Functional Analysis with Applications*

2.1.6

Show that in an n-dimensional vector space X, the representation of any x as a linear combination of given basis vectors $e_1, \ldots, e_2$ is unique.

Suppose we have two representations for x:

$$\begin{aligned} x &= \alpha_1 e_1 + \ldots \alpha_n e_n \\ \text{and } x &= \beta_1 e_1 + \cdots + \beta_n e_n \\ \iff 0 &= x - x = (\alpha_1 - \beta_1)e_1 + \cdots + (\alpha_n - \beta_n)e_n \end{aligned}$$

Since the basis vectors must be linearly independent, this means that each $(\alpha_i - \beta_i) = 0$, i.e. $\alpha_i = \beta_i$.

2.1.10

If Y and Z are subspaces of a vector space X, show that $Y \cap Z$ is a subspace of X, but $Y \cup Z$ need not be one. Give examples.

For $Y \cap Z$, we show that this set is closed under scalar multiplication and addition. Given $x \in Y \cap Z$, we know $\alpha x \in Y$ since Y is a vector space, and $\alpha x \in Z$ since Z is a vector space, hence $\alpha x \in Y \cap Z$. Similarly, given any x and y in $Y \cap Z$, $x + y \in Y$ by definition of subspace for Y and $x + y \in Z$ by definition of subspace for Z, hence $x + y \in Y \cap Z$.

For $Y \cup Z$, we can use $\mathbf{R}^2$ as a counterexample. Let the "x-axis" or span$\{(1,0)\}$ act as Z and the "y-axis" or span$\{(0,1)\}$ act as Y. The union $Y \cup Z$ is not closed under vector addition (note that $(0, 1) \in Y$ and $(1, 0) \in Z$):

$$(1, 0) + (0, 1) = (1, 1) \notin Y \cup Z$$

Clearly the intersection of these two sets, the zero vector, is a subspace.

2.2.11

A subset A of a vector space X is said to be *convex* if $x, y \in A$ implies

$$M = \{z \in X : z = \alpha x + (1 - \alpha)y, 0 \leq \alpha \leq 1\} \subset A$$

M is called a closed segment with boundary points x and y; any other $z \in M$ is called an interior point of M. Show that the closed unit ball

$$\tilde{B}(0; 1) = \{x \in X : \|x\| \leq 1\}$$

in a normed space X is convex.

We take any two points x and y in the unit ball $\tilde{B}(0; 1)$ and show that any point in the segment joining x and y is also in $\tilde{B}(0; 1)$. Let $z \in \tilde{B}(0; 1)$. Then there exists $\alpha \in [0, 1]$ such that

$$\begin{aligned} && z &= \alpha x + (1 - \alpha)y \\ \text{so} && \|z\| &= \|\alpha x + (1 - \alpha)y\| \\ \text{triangle inequality:} && &\leq \|\alpha x\| + \|(1 - \alpha)y\| \\ \text{scalar invariance of norm:} && &= |\alpha|\|x\| + |(1 - \alpha)|\|y\| \\ \|x\|, \|y\| \leq 1: && &\leq |\alpha| + |1 - \alpha| \\ 0 \leq \alpha \leq 1: && &= \alpha + 1 - \alpha = 1 \end{aligned}$$

So $\|z\| \leq 1$ so it must be in the unit ball $\tilde{B}(0; 1)$.

This is easy enough to see for $\mathbf{R}^2$ but what about for more exotic spaces like C[0,1]? This means that if we combine any two continuous functions with a common bound in the manner above for z, the result will have the same bound. See jupyter notebook animation.