Some solutions to Kreyszig's *Introductory Functional Analysis with Applications*, chapter 2

2.1.6
Show that in an n-dimensional vector space X, the representation of any x as a linear combination of given basis vectors $e_1, \dots, e_n$ is unique.

Suppose we have two representations for x:

$$\begin{aligned} x &= \alpha_1 e_1 + \dots \alpha_n e_n \\ \text{and } x &= \beta_1 e_1 + \dots + \beta_n e_n \\ \iff 0 &= x - x = (\alpha_1 - \beta_1)e_1 + \dots + (\alpha_n - \beta_n)e_n \end{aligned}$$

Since the basis vectors must be linearly independent, this means that each $(\alpha_i - \beta_i) = 0$, i.e. $\alpha_i = \beta_i$.

2.1.10
If Y and Z are subspaces of a vector space X, show that $Y \cap Z$ is a subspace of X, but $Y \cup Z$ need not be one. Give examples.

For $Y \cap Z$, we show that this set is closed under scalar multiplication and addition. Given $x \in Y \cap Z$, we know $\alpha x \in Y$ since Y is a vector space, and $\alpha x \in Z$ since Z is a vector space, hence $\alpha x \in Y \cap Z$. Similarly, given any x and y in $Y \cap Z$, $x + y \in Y$ by definition of subspace for Y and $x + y \in Z$ by definition of subspace for Z, hence $x + y \in Y \cap Z$.
For $Y \cup Z$, we can use $\mathbf{R}^2$ as a counterexample. Let the "x-axis" or span$\{(1,0)\}$ act as Z and the "y-axis" or span$\{(0,1)\}$ act as Y. The union $Y \cup Z$ is not closed under vector addition (note that $(0,1) \in Y$ and $(1,0) \in Z$):

$$(1,0) + (0,1) = (1,1) \notin Y \cup Z$$

Clearly the intersection of these two sets, the zero vector, is a subspace.

2.2.11
A subset A of a vector space X is said to be *convex* if $x, y \in A$ implies

$$M = \{z \in X : z = \alpha x + (1-\alpha)y, 0 \le \alpha \le 1\} \subset A$$

M is called a closed segment with boundary points x and y; any other $z \in M$ is called an interior point of M. Show that the closed unit ball

$$\tilde{B}(0;1) = \{x \in X : \|x\| \le 1\}$$

in a normed space X is convex.

We take any two points x and y in the unit ball $\tilde{B}(0;1)$ and show that any point in the segment joining x and y is also in $\tilde{B}(0;1)$. Let $z \in M$, the line segment. Then there exists $\alpha \in [0,1]$ such that

$$\begin{aligned} z &= \alpha x + (1-\alpha)y \\ \text{so} \quad \|z\| &= \|\alpha x + (1-\alpha)y\| \\ \text{triangle inequality:} \quad &\le \|\alpha x\| + \|(1-\alpha)y\| \\ \text{scalar invariance of norm:} \quad &= |\alpha|\|x\| + |(1-\alpha)|\|y\| \\ \|x\|, \|y\| \le 1: \quad &\le |\alpha| + |1-\alpha| \\ 0 \le \alpha \le 1: \quad &= \alpha + 1 - \alpha = 1 \end{aligned}$$

So $\|z\| \le 1$ so it must be in the unit ball $\tilde{B}(0;1)$.

This is easy enough to see for $\mathbf{R}^2$ but what about for more exotic spaces like C[0,1]? This means that if we combine any two continuous functions with a common bound in the manner above for z, the result will have the same bound. See jupyter notebook `lineoffunctions.ipynb` for an illustrating animation.

2.3.2 Show that c_0, the space of all sequences of scalars converging to zero, is a closed subspace of l^∞.

We want to show that the complement of c_0 is open. That is, every element in the set of all non-convergent sequences has a neighborhood of only non-convergent sequences. First write what it means for an $x = \{x_1, x_2, \dots\} \in c_0^C$ not to converge to zero: there is an $\epsilon > 0$ such that for all N, there is an $n > N$ such that $|x_n - 0| \geq \epsilon$. Now let an $x' = \{x'_1, x'_2, ...\} \in B(x, \epsilon/2)$ for the same ϵ, so we have from the metric on l^∞ that $\sup |x_i - x'_i| < \epsilon/2$. Now we want to show that x' does not converge. From non-convergence of x,

$$
\begin{aligned}
& \epsilon \leq |x_n| = |x_n - x'_n + x'_n| \\
\text{triangle inequality:} \quad & \leq |x_n - x'_n| + |x'_n| \\
\text{by construction of } x'\text{:} \quad & < \epsilon/2 + |x'_n| \\
\text{subtract } \epsilon/2 \text{ from first and last expression:} \quad & \epsilon/2 < |x'_n|
\end{aligned}
$$

hence there is an $\epsilon' = \epsilon/2$ so that $\forall N$ there is an $n > N$ so that $|x'_n - 0| \geq \epsilon'$, which means that x' does not converge to zero. Since x' was an arbitrary element of an open ball of x, and $x \in c_0^C$ is arbitrary, c_0^C is open, which means c_0 is closed.

I had a crisis of faith. For some reason I thought that the monomials formed a Cauchy sequence. But their limit is discontinuous and not a polynomial. The limiting function equals zero on $[0, 1)$ and 1 at $t = 1$. Since we showed that $C[a, b]$ is complete, this is a contradiction, so it must not be Cauchy. Let's prove this directly just for fun.

Let $\epsilon = 1/2$ and take any positive integer N. We want to show that there exist $n, m > N$ with $\sup |t^n - t^m| \geq \epsilon$. It's sufficient to show that there exists $t \in [0, 1]$ such that $|t^n - t^m| \geq \epsilon$. For all $n > 1$, t^n takes all values in [0,1) for $t \in [0, 1)$. Let $n = N + 1$ and t_0 denote the input such that $t_0^n = 3/4$. Now we know that the monomials converge to zero pointwise on $[0, 1)$ so for $\epsilon' = 1/4$ there exists N' such that $m' > N'$ implies $|t_0^{m'}| < \epsilon'$. Let $m = \max\{N, N'\} + 1$ so we still have $|t_0^m| < \epsilon'$. Hence

$$
|t_0^m - t_0^n| = t_0^n - t_0^m = 3/4 - t_0^m > 3/4 - 1/4 = 1/2
$$

So for some positive ϵ, for arbitrary N there exists $n, m > N$ with $\sup |t^m - t^n| \geq \epsilon$, which means $\{t^n\}$ is not Cauchy under $\sup |t^n - t^m|$.

Alternative proof of Corollary 2.7-10 (b)

The text shows that the null space of a bounded linear operator is closed by showing that the closure of the null space is the same set. I would like a proof using the definition of a closed set, i.e. that the complement is open.

Let $x \in N(T)^C$, the complement of the null space. We want a neighborhood of x fully contained in $N(T)^C$. First, since T is bounded, it is also continuous. Hence for $\epsilon = \|Tx\|/2$, there exists a δ such that any x' satisfying $\|x - x'\| < \delta$ (i.e. $x' \in B(x, \delta)$) also has $\|Tx - Tx'\| < \epsilon$. Now take an arbitrary such x' and suppose it is in the null space (in order to show a contradiction), i.e. $Tx = 0$. We have

$$
\begin{aligned}
\frac{\|Tx\|}{2} & < \|Tx\| \\
& = \|Tx - Tx'\| \\
\text{and by continuity of } T\text{:} \quad & < \frac{\|Tx\|}{2}
\end{aligned}
$$

a contradiction.